Any partial Collatz trajectory is realizable

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Abstract

It is easy to show that for the modified Collatz map $T : \mathbb{Z}^+ \to \mathbb{Z}^+$ defined by

$$T(n) = \begin{cases} \frac{3n+1}{2}, & n \text{ is odd} \\ \frac{n}{2}, & n \text{ is even} \end{cases}$$

any given order of ups and downs is produced by infinite number of starting points (*seeds*). These seeds can be defined as $n_s = 2^L r - c$, where L is the length of the given up-down sequence, $c : \ 0 \leq c < 2^L$ is a constant, and $r = 1, 2, 3, \ldots$. The corresponding ending points are $n_e = 3^{N_1} r - d$, where N_1 is the number of ups in the given sequence, and $d : \ 0 \leq d < 3^{N_1}$ is another constant.

1 Introduction

The Collatz conjecture can be formulated in many ways. For the purpose of this note we choose the one based on the modified Collatz map $T : \mathbb{Z}^+ \to \mathbb{Z}^+$:

$$T(n) = \begin{cases} \frac{3n+1}{2}, & n \equiv 1 \pmod 2 \\ \frac{n}{2}, & n \equiv 0 \pmod 2 \end{cases} \tag{1}$$

The conjecture states that for every n, $T^k(n) = 1$ for some k. It is believed to be one of the hardest open problems in mathematics.

Every time the current iterate is odd, T increases it about 1.5 times, and every time it is even, T decreses the iterate by halving. So, the trajectory of Collatz iterations is a certain alternation of ups and downs based on the iterate parity. We discuss possible up-down sequences and enumeration of the starting and ending points producing a given sequence.

2 Collatz iterations for univariate linear forms

We will apply Collatz iterations to univariate linear forms with integer coefficients $n = ar + b$, where $a \in \mathbb{Z}^+$ and $b \in \mathbb{Z}$ are known and $r \in \mathbb{Z}^+$ is a parameter. Notice that when a is even, the parity of n is determined by b and does not depend on r. On the other hand, if a is odd, r must have the same parity as b to make n even (and have the opposite parity to make n odd.) In this case, to apply a Collatz iteration, we must split n into two cases, the first one is when we make r even, and the second one when we make r odd:

$$n = \begin{cases} 2ar' + b, & \text{if } r = 2r' \\ 2ar' + (b-a), & \text{if } r = 2r' - 1 \end{cases} \tag{2}$$

Plugging the linear form into (1), we get

$$T(n) = \begin{cases} \frac{3ar+(3b+1)}{2}, & ar+b \text{ is odd,} \\ \frac{ar+b}{2}, & ar+b \text{ is even} \end{cases} \tag{3}$$

Combining (2) and (3), we arrive at the following rules for a single modified Collatz map:

b parity	r parity	r	n parity	n	$T(n)$	Traj. step
even	even	$2r'$	even	$2ar' + b$	$ar' + \frac{b}{2}$	down
even	odd	$2r' - 1$	odd	$2ar' + (b-a)$	$3ar' + \frac{3b-3a+1}{2}$	up
odd	even	$2r'$	odd	$2ar' + b$	$3ar' + \frac{3b+1}{2}$	up
odd	odd	$2r' - 1$	even	$2ar' + (b-a)$	$ar' + \frac{b-a}{2}$	down

If we apply the modified Collatz map repeatedly m times, instead of $T(n)$ we have $T^m(n) = T(T^{m-1}(n))$. Let $T^{m-1}(n) = \alpha r + \beta$. Then the rules become

b parity	r parity	r	n parity	n	$T^m(n)$	Traj. step
even	even	$2r'$	even	$2ar' + b$	$\alpha r' + \frac{\beta}{2}$	down
even	odd	$2r' - 1$	odd	$2ar' + (b-a)$	$3\alpha r' + \frac{3\beta-3\alpha+1}{2}$	up
odd	even	$2r'$	odd	$2ar' + b$	$3\alpha r' + \frac{3\beta+1}{2}$	up
odd	odd	$2r' - 1$	even	$2ar' + (b-a)$	$\alpha r' + \frac{\beta-\alpha}{2}$	down

Table 1: Collatz iteration rules for univariate linear forms

3 Seeds and ending points of up-down sequences

TODO: describe the algorithm obtaining all seeds and ending points and prove its correctness.