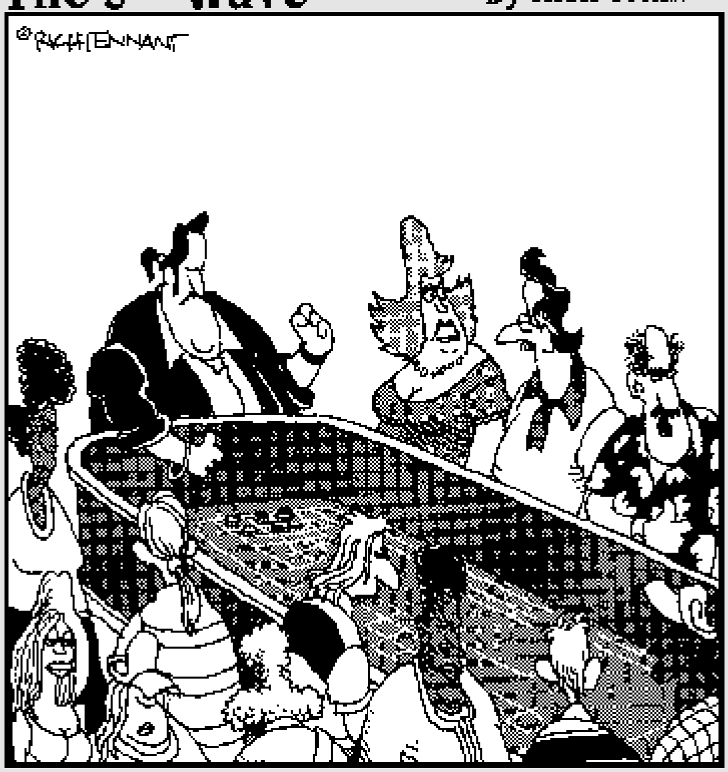
Part II

Counting on Probability and Betting to Win

The 5th Wave By Rich Tennant



"Laplace's rule of succession doesn't always stand up next to Bruno's law of rotten mojo."

In this part . . .

Part II takes you behind the scenes so you can see probability at work in three major areas: two-way tables, counting rules — including combinations and permutations — and games of chance. (If you win big-time at your local gaming establishment, you won't need the rest of this book; you'll need *Personal Finance For Dummies* . . .)

Chapter 4

Setting the Contingency Table with Probabilities

In This Chapter

- Classifying probabilities in contingency tables
- Using contingency tables to find joint, marginal, and conditional probabilities
- Identifying events as independent or dependent

A *contingency table* is a table that classifies outcomes and their probabilities into rows and columns, depending on which categories the outcomes fit into. Contingency tables are very popular for displaying joint probabilities (the probability that two events, such as A and B or A^c and B, occur at the same time; see Chapter 2) because you can use rows to indicate whether A occurred and columns to indicate whether B occurred. The intersections of the rows and columns indicate all the possible joint probabilities between A and B and their complements. You can also use contingency tables in lieu of Venn diagrams or tree diagrams (see Chapter 3) for problems where you have to find conditional, joint, or marginal probabilities.

In this chapter, you discover how contingency tables help you organize and figure probabilities and check for independence of events.

Organizing a Contingency Table

Information on probabilities may be hard to analyze without organizing it somehow. A contingency table is a great way to organize, especially if the information is in the form of intersections of events (see Chapter 2) — a contingency table's specialty. The reason a contingency table is so good with intersections is because the table is set up to intersect two events and show all the combinations easily and efficiently.

For example, suppose you watch a lot of basketball, and you especially enjoy watching players at the free-throw line shooting free throws. While watching your favorite player shoot pairs of free throws in a game, you notice that if he makes the first one, he seems less likely to make the second one. But if he misses the first free throw, he seems more likely to make the second one. Your observation leads you to wonder if the outcome of the first free throw and the outcome of the second free throw are related for basketball players.

Suppose that data you collect on a sample of your favorite player shooting 155 pairs of free throws gives you the following information:

- 40 times your favorite player made the first free throw and missed the second.
- 60 times your favorite player made both free throws.
- 10 times your favorite player missed both free throws.
- 45 times your favorite player missed the first free throw and made the second.

A contingency table is the best way to organize your information here, because the given information is in the form of intersections of events. In order to organize a contingency table, however, a few important steps are needed, as discussed in the following sections.

Defining the sample space

The first step in setting up a contingency table is defining the sample space and the possible outcomes of the experiment, using probability notation. A *sample space,* S, is the set of all possible outcomes of an experiment (which in most cases involves all the possible intersections that can occur). The probability notation you use is the same as you see for any other probability; for example, P(A) indicates the marginal probability of A, and $P(B^c)$ is the probability of the complement of B. (See Chapter 2 for more on probability notation.)

In the free-throw example from the start of this section, let Y_1 = made the first free throw and N_1 = missed the first free throw; similarly, let Y_2 and N_2 represent making and missing the second free throw, respectively. The sample space reflects the pairs of outcomes of this two-stage experiment — stage one is the outcome of the first free throw, and stage two is the outcome of the second free throw. The sample space contains four outcomes: $\{Y_1Y_2, Y_1N_2, N_1Y_2, \text{and } N_1N_2\}$.

Setting up the rows and columns

You can organize the outcomes from a sample space that reflects pairs of outcomes of a two-stage experiment into a contingency table by using rows to represent stage one (for example, the outcome of the first free throw shot by your favorite basketball player) and columns to represent stage two (the outcome of the second free throw). The following table shows what your newly created contingency table looks like for the free-throw example I introduce at the beginning of this section.

	Made Second Free Throw (Y_2)	**Missed Second Free Throw (N_2)**
Made First Free Throw (Y_1)	$Y_1 \cap Y_2$	$Y_1 \cap N_2$
Missed First Free Throw (N_1)	$N_1 \cap Y_2$	$N_1 \cap N_2$

Notice that the body of the table has $2 * 2 = 4$ entries. These entries are called the *cells* of the contingency table. Each cell represents an intersection of a row and column. For example, the cell in the upper-right-hand corner of the table represents the outcome where the player makes the first free throw and misses the second. In probability notation, this represents the intersection of the events Y_1 and N_2, written as $Y_1 \cap N_2$.

Inserting the data

After you define your sample space and set up the contingency table (see the previous two sections), you're ready to put your information into the contingency table and start calculating probabilities. Using the example laid out in the introduction to this section, you know that 40 times the player made the first free throw and missed the second; 60 times the player made both free throws; 10 times the player missed both; and 45 times the player missed the first and made the second. Therefore, you put 60 into the upper-left cell (represented by the event $Y_1 \cap Y_2$), 40 into the upper-right cell ($Y_1 \cap N_2$), 45 into the bottom-left cell ($N_1 \cap Y_2$), and 10 into the bottom-right cell ($N_1 \cap N_2$).

TIP

The number of individuals inside a cell in row i and column j of a contingency table is called the *cell count* for the (i, j)th cell. Most textbooks use the notation *i* to indicate the row of the table and *j* to indicate the column of the table. To remember whether i or j stands for the row or column, memorize this helpful hint: RC Cola — row and then column, just like i and then j!

Adding the row, column, and grand totals

After you place the cell counts into a contingency table (see the previous section), you should total the rows and columns and write those totals, aptly named *marginal totals,* in the margins of the table. You can see in the following table, which represents the free-throw example I present in the introduction to this section, that the total for the first row, 60 + 40 = 100, appears in the row-totals column of the first row, which means the total number of shots made on the first free-throw attempt is 100. Similarly, the row total for row two represents the total number of shots missed on the first free-throw attempt (45 + 10 = 55).

The column totals appear in the row at the bottom of the table. The first column total is 60 + 45 = 105, which represents the total number of shots made on the second free-throw attempt. The second column total represents the total number of shots missed on the second free-throw attempt (40 + 10 = 50). Notice that the row totals sum to a grand total of 155, the total number of shots attempted (first and second free-throw attempts). Similarly, the column totals sum to a grand total of 155.

	Made Second Free Throw (Y_2)	**Missed Second Free Throw (N_2)**	**Row Totals**
Made First Free Throw (Y_1)	60	40	60 + 40 = 100
Missed First Free Throw (N_1)	45	10	45 + 10 = 55
Column Totals	60 + 45 = 105	40 + 10 = 50	**Grand Total** = 155

Finding and Interpreting Probabilities within a Contingency Table

When you have a contingency table that's all set up, you can use it to calculate probabilities and answer important questions about the events illustrated by the data in the table. Using the example I lay out at the beginning of the section "Organizing a Contingency Table," you can answer questions such as: What's the probability that the player makes both free throws? What's the chance that the player makes the first? What's the probability that the player makes the second free throw given that he missed the first? And if he misses the first free throw, does that affect his chances of making the second? In this section, I discuss all these questions and the methods you use to answer them.

Figuring joint probabilities

A *joint probability* is the probability of the intersection of two sets or events (see Chapter 2 for more on this topic). For example, the probability that a basketball player makes both free throws (see the section "Organizing a Contingency Table" for this complete example) is a joint probability and is denoted $P(Y_1 \cap Y_2)$.

Finding joint probabilities is easy when you use a contingency table, because the cells of a contingency table already show the number of individuals in each intersection. To find the probability of any intersection, you take the number in the cell in question and divide by the grand total (found in the lower-right corner of the contingency table). For example, the probability that a player makes both free throws, $P(Y_1 \cap Y_2)$, is the number in the upper-left cell of the contingency table, 60, divided by the grand total, 155: $60 \div 155 = 0.39$, or 39 percent.

When given a probability question, the clue that the probability is a joint probability is the word "and" in the question, as in "A *and* B."

The general formula for finding a joint probability with a contingency table is $\frac{\text{count in cell}(i,j)}{\text{grand total}}$, where the cell in the *ith* row and the *jth* column is denoted by cell (i, j).

Calculating marginal probabilities

A *marginal probability* is the probability of one event happening by itself, whether or not any other event occurs (see Chapter 2 for more on this topic). Using the basketball example (see the section "Organizing a Contingency Table"), the probability that a basketball player makes his or her first free throw (regardless of what happens on the second shot) is the marginal probability of Y_1 and is denoted $P(Y_1)$. The probability that the player makes the second free throw (regardless of what happens on the first shot) is the marginal probability of Y_2 and is denoted $P(Y_2)$.

As with joint probabilities (see the previous section), finding marginal probabilities is easy if you use a contingency table, because the row and column totals show the number of individuals in each event separately. To find the marginal probability of any single event, you take the number in the corresponding row or column total and divide by the grand total. For example, look at the event that the basketball player makes his or her first free throw (regardless of what happens on the second shot). Row one of the contingency table, denoted Y_1, represents this event. So, you find the probability that the player makes the first free throw, $P(Y_1)$, by taking the row-one total, 100, and dividing by the grand total, 155: $100 \div 155 = 0.65$, or 65 percent.

TIP

When given a probability question, the clue that you're dealing with a marginal probability is that the question mentions only one event.

Now look at the event that the player makes the second free throw. (Notice that I don't mention the first shot; you know you're dealing with a marginal probability.) Column one of the contingency table, denoted Y_2, represents this event. So, you find the probability that the player makes the second free throw, $P(Y_2)$, by taking the column-one total, 105, and dividing by the grand total, 155: $105 \div 155 = 0.68$, or 68 percent.

The general formula for finding the marginal probability of an event in row i of a contingency table is $P(\text{Row } i \text{ event}) = \frac{\text{row } i \text{ total}}{\text{grand total}}$. The general formula for finding the marginal probability of an event in column j of a contingency table is $P(\text{Column } j \text{ event}) = \frac{\text{column } j \text{ total}}{\text{grand total}}$.

Identifying conditional probabilities

A *conditional probability* is the probability of one event happening given that another event has already happened (see Chapter 2 for more on this topic). Using the basketball free-throw example (see the section "Organizing a Contingency Table"), the probability that a player makes his second free throw given that he made his first is the conditional probability of Y_2 given Y_1 and is denoted $P(Y_2 | Y_1)$. The probability that the player misses his second free throw given that he missed his first is the conditional probability of N_2 given N_1 and is denoted $P(N_2 | N_1)$.

The formula for the conditional probability of A given B (see Chapter 2) is $P(A|B) = \frac{P(A \cap B)}{P(B)}$. You divide by P(B) because you know event B has already happened. However, because the denominators of $P(A \cap B)$ and P(B) are both equal to the grand total in the contingency table, you can find the conditional probability by taking the number in the cell representing $A \cap B$ and dividing by the row or column total for event B. (In other words, the denominators for these probabilities are the same — the grand total — and they cancel out when you divide the probabilities, so you don't have to include them in the formula.)

For example, look at the event that the player makes his second free throw given that he made the first, denoted $Y_2 | Y_1$. You find this probability, $P(Y_2 | Y_1)$, by taking the number in the cell representing $Y_2 \cap Y_1$, 60, and dividing by the row total representing Y_1, 100: $60 \div 100 = 0.60$, or 60 percent.

TIP

When given a probability question, the clue that you're working with a conditional probability is that you know one event has already happened. Words like "given," "knowing," and "of" are often used to signal conditional probability.

Now look at the event that the player misses the second free throw given that he missed the first, denoted $N_2 | N_1$. You find this probability, $P(N_2 | N_1)$, by taking the number in the cell representing $N_2 \cap N_1$, 10, and dividing by the row total representing N_1, 55: $10 \div 55 = 0.18$, or 18 percent.

The general formula for finding the conditional probability of an event in row i of a contingency table given an event in row j is $P\left(\text{Row i} \middle| \text{Column j}\right) = \frac{\text{count in cell(i, j)}}{\text{column j total}}$. The general formula for finding the conditional probability of an event in column j given an event in row i is $P\left(\text{Column j} \middle| \text{Row i}\right) = \frac{\text{count in cell(i, j)}}{\text{row i total}}$.

Here are the conditional probabilities you can calculate for the free-throw example:

- The probability of the player making the second shot given that he made the first, $P(Y_2 | Y_1)$, is $60 \div 100 = 0.60$, or 60 percent.
- The probability of the player missing the second shot given that he made the first, $P(N_2 | Y_1)$, is $1 - 0.60 = 0.40$, or 40 percent, by the complement rule (see Chapter 2).
- The probability of the player missing the second shot given that he missed the first, $P(N_2 | N_1)$, is $10 \div 55 = 0.18$, or 18 percent.
- The probability of the player making the second shot given that he missed the first, $P(Y_2 | N_1)$, is $1 - 0.18 = 0.82$, or 82 percent, by the complement rule.

If you know from the question that event B has happened, event A either happens or it doesn't, so you can say that $P(A | B) + P(A^c | B) = 1$. Therefore, the first two bullets in the previous list add up to one, and the last two bullets add up to one. Given that the player makes the first free throw (Y_1 occurs), he either makes the second free throw ($Y_2 | Y_1$) or doesn't ($N_2 | Y_1$).

WARNING!

However, it *isn't* true that $P(A | B) + P(A | B^c) = 1$ because in each term you're conditioning on a different event. This is a common mistake that you need to avoid. In the free-throw example, you can see that $P(Y_2 | Y_1) + P(Y_2 | N_1) = 0.60 + 0.82 \neq 1$.

Checking for Independence of Two Events

If two events A and B are *independent,* $P(A|B) = P(A)$ and $P(A|B^c) = P(A)$. In other words, if the knowledge that event B has happened doesn't change the probability of A happening, events A and B are independent. The definition also says that if events A and B are independent, $P(A|B) = P(A|B^c)$ because both of these terms must be equal to P(A) in that case.

You can show that events A and B are independent if the situation meets any of the following conditions:

- If $P(A|B) = P(A)$
- If $P(A|B^c) = P(A)$
- If $P(A|B) = P(A|B^c)$

If two events are not independent, they must be dependent. You can show that events A and B are dependent if the situation meets any of the following conditions:

- If $P(A|B) \neq P(A)$
- If $P(A|B^c) \neq P(A)$
- If $P(A|B) \neq P(A|B^c)$

TIP

The first bullet in each of the previous two lists is by far the most common way to show that two events are independent or dependent, respectively. However, it may be easier to show independence or dependence in certain situations by using the second and third bullets, respectively, so keep them in mind.

Now you're ready to answer the big question from the free-throw example introduced in the section "Organizing a Contingency Table." You want to know if the outcome of the first free-throw attempt influences the outcome of the second. In probability lingo, you want to know whether these two events are independent.

In the section "Identifying conditional probabilities," you calculate the following information:

- $P(Y_2|Y_1) = 0.60$, or 60 percent
- $P(N_2|Y_1) = 1 - 0.60$, or 40 percent
- $P(N_2|N_1) = 0.18$, or 18 percent
- $P(Y_2|N_1) = 1 - 0.18$, or 82 percent

To find out if the outcome of the first shot influences the second shot, you can check to see if the probability of making the second shot is the same given that the player made or missed the first shot — $P(Y_2|Y_1) = P(Y_2|N_1)$ or $P(A|B) = P(A|B^c)$. In other words, you check to see if the first and fourth probabilities from the previous list are equal. Because 0.60 isn't equal to 0.82, you know that these events are dependent.

To give you a range of perspectives, I run through the other methods for checking for independence (or dependence) in the following list:

- You can check to see if the probability of missing the second shot is the same given that the player made or missed the first shot. In other words, you check to see if the second and third probabilities from the previous list are equal: $P(N_2|Y_1) = P(N_2|N_1)$. Because 0.40 isn't equal to 0.18, you know that the outcome of the first shot depends on the second shot.
- You can check to see whether $P(Y_2) = P(Y_2|N_1)$ or $P(Y_2) = P(Y_2|Y_1)$. You know that $P(Y_2) = 0.68$, or 68 percent (because this is the marginal probability of making the second shot, you use the column total for Y_2 divided by the grand total). You also know that $P(Y_2|N_1) = 0.82$. Because 0.68 isn't equal to 0.82, you know that the events are dependent.

REMEMBER

You have multiple options for showing that two events are independent or dependent. Choose the method that works best for each situation, and be sure to keep the other options open because different methods prove to be more helpful at different times. The more tools you have available, the better off you are.

Chapter 5

Applying Counting Rules with Combinations and Permutations

In This Chapter

- Counting the number of ways an event can occur
- Crunching probabilities that have restrictions
- Distinguishing permutation problems from combination problems
- Sorting objects with the use of combinations
- Using poker as a combinations tool

All dreamers and gamblers want to know the odds of winning the lottery, the chance of getting a four of a kind in a poker hand, or the chance of not rolling a 7 or 10 on two throws in a row in craps. The odds all have to do with *counting rules,* which is what this chapter is all about. Counting rules provide a mathematical way to *enumerate,* or count, the number of ways that a certain outcome can occur. And they bring order to an otherwise seemingly chaotic process; for example, listing all the ways not to roll a 7 when throwing two dice. Without having an ordered process, this task would be difficult — not to mention rolling, say, five dice.

Although counting rules are very helpful and provide order to the process of finding probabilities, they can also be challenging. Determining when to use a *combination* and when to use a *permutation,* for example, is one of the biggest challenges people face. Making sure you've covered all possible situations and that you aren't counting certain outcomes more than once are other challenges.

In this chapter, you discover how to set up and solve problems that need counting rules. The goal is to be comfortable and confident with the different methods and when to use them. In other words, you always want to have the right tool for the right job.

Counting on Permutations

Probability is the number of ways that a certain outcome in a random experiment can occur divided by the total number of possible outcomes. Many times, figuring out the number to put in the numerator and/or denominator of a probability involves finding the number of ways to rearrange the outcomes, and this task involves figuring the number of permutations. A *permutation* is a rearrangement of a certain number of items that are chosen without replacement.

Unraveling a permutation

In general, the number of possible ways to rearrange k objects is $k * (k-1) * (k-2) * \ldots * (2) * (1)$. For example, the number of ways to rearrange six items is $6 * 5 * 4 * 3 * 2 * 1 = 720$. In mathematical shorthand, you express this product as k!, called *k factorial.* The exclamation point denotes multiplying by k, and then k – 1, and then k – 2, and on down to 2 times 1 for any k greater than or equal to 1. Note that 1! = 1. Zero factorial, 0!, is defined as 1. (It's defined this way to make the math work for other situations, but you can think of it this way: You have only one way to rearrange no items — don't rearrange them.)

In general, the number of ways to choose and rearrange k items from a total of n items is the number of permutations of k items from n items; in equation form, you see it written as P_k^n or ${}_nP_k$. (In this section, I use the first form of the notation, but you should be on the lookout for the other form in other books.)

The formula for counting the number of permutations of k items chosen from n items (without replacement) is $P_k^n = \frac{n!}{(n-k)!}$, where n is the total number of items in the population and k is the number of items being selected. Probabilists call this the *permutation formula.* Notice that in the permutation formula you take the number of ways to rearrange all n items and divide by the number of ways to rearrange the items you didn't select. Because you select k items, you leave n – k items unselected.

If you're rearranging all n items, you have a permutation of n items chosen from n items. This is equal to P_n^n, which becomes $\frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1}$, or just n!.

For example, suppose you have six people in the group going into a movie theater, but you can select only four to sit in a row together. How many ways do you have to select the four people *and* rearrange them in one row in the theater? This question requires figuring the number of permutations, but the added twist is that you have to choose the people first and then rearrange them.

You can take on the choosing and rearranging process with the following steps:

1. **Find the total number of ways to rearrange all the objects.**

   For the previous example, the total number of ways to rearrange the six people to put them all together is 6! = 720.

2. **Divide out the number of ways to rearrange the objects you can't select.**

   For this example, you leave out two people (2! = 2). Therefore, to get the number of ways to select and arrange 4 people (out of 6), you can take 720 ÷ 2 = 360. The reason you divide by 2! is because in the end, it doesn't matter what happens to the other two people you don't select, so you need to take them out of the equation.

   So, you have 360 ways to choose 4 people from the group of 6 and rearrange those 4 people in a row together. You've done what's known as a permutation of 4 items from 6 items chosen. The notation for this permutation is P_4^6, or ${}_6P_4$.

TIP

Factorials, especially for large numbers, are tedious to write out all the way during calculations. A shortcut you can use is to start doing the factorial for the numerator ($n * [n-1] * [n-2]$ and so on) and stop when you get to the number that equals what's in the denominator. Write that number with a factorial sign after it and then stop. Now you can cancel out that factorial with the denominator factorial. For the previous movie example, your work looks like this: $P_4^6 = \frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{6 * 5 * 4 * 3 * \cancel{2!}}{\cancel{2!}} = 360$.

Suppose the people you choose to go to the movie are Tim, Syd, Elena, and Mark. How many different ways can they sit together in one row (straight line) in the movie theater? You have four friends and four seats. You can have any of the four people sit in the first seat; and because you can't choose that same person again, you have three people for the second seat; two people for the third seat; and the fourth person automatically takes the last seat. You use the multiplication rule (see Chapter 2) to multiply all the possible ways together: 4 * 3 * 2 * 1 = 24. So, you have 24 different ways to seat 4 people in a movie theater.

Table 5-1 shows all the possible rearrangements of the four people. If you look at the way I order the list, you can see that I fixed the first person (Tim, for example). Then I fixed one of the remaining people for the second seat (Syd, for example). Then I fixed one of the remaining people for the third seat (Elena, for example). The last person, therefore, had to be Mark. Then I worked my way inside out, changing the person in the third seat and then changing the person in the second seat. This method gets you through six possibilities when Tim is in the first seat. Finally, you change the person in the first seat and repeat the process four times (because you have four people to choose from for the first seat). This gives you a total of 4 * 3 * 2 * 1 = 24 possibilities.

TIP

The number of ways to rearrange k items gets very large very quickly as the number of objects increases. For example, you have only 24 ways to rearrange 4 people, but if k increases to 5 people, the number of ways increases to $5! = 5 * 4 * 3 * 2 * 1 = 5 * 24 = 120$. Guess how many possible ways you have to rearrange 10 people: $10! = 3{,}628{,}800$. No wonder wedding planners work so hard to come up with suitable seating arrangements for wedding receptions. There are too many ways for people to seat themselves and get into trouble! Seriously, when you're working on counting-rule problems, try to develop a good system to enumerate or list the possibilities — at least for small scenarios — so that when the more difficult problems come along, you understand how the processes and formulas work.

Table 5-1 Possible Rearrangements for Four People Sitting in a Straight Line

Rearrangement #	*Listing*	*Rearrangement #*	*Listing*
1	Tim, Syd, Elena, Mark	13	Elena, Tim, Syd, Mark
2	Tim, Syd, Mark, Elena	14	Elena, Tim, Mark, Syd
3	Tim, Elena, Syd, Mark	15	Elena, Syd, Tim, Mark
4	Tim, Elena, Mark, Syd	16	Elena, Syd, Mark, Tim
5	Tim, Mark, Syd, Elena	17	Elena, Mark, Tim, Syd
6	Tim, Mark, Elena, Syd	18	Elena, Mark, Syd, Tim
7	Syd, Tim, Elena, Mark	19	Mark, Tim, Syd, Elena
8	Syd, Tim, Mark, Elena	20	Mark, Tim, Elena, Syd
9	Syd, Elena, Tim, Mark	21	Mark, Syd, Elena, Tim
10	Syd, Elena, Mark, Tim	22	Mark, Syd, Tim, Elena
11	Syd, Mark, Tim, Elena	23	Mark, Elena, Tim, Syd
12	Syd, Mark, Elena, Tim	24	Mark, Elena, Syd, Tim

You can also show the list of rearrangements by using a tree diagram (see Chapter 3 for more on these diagrams), shown in Figure 5-1. Follow through the 24 possible paths on the tree to get the same listings of possible rearrangements that I show in Table 5-1.

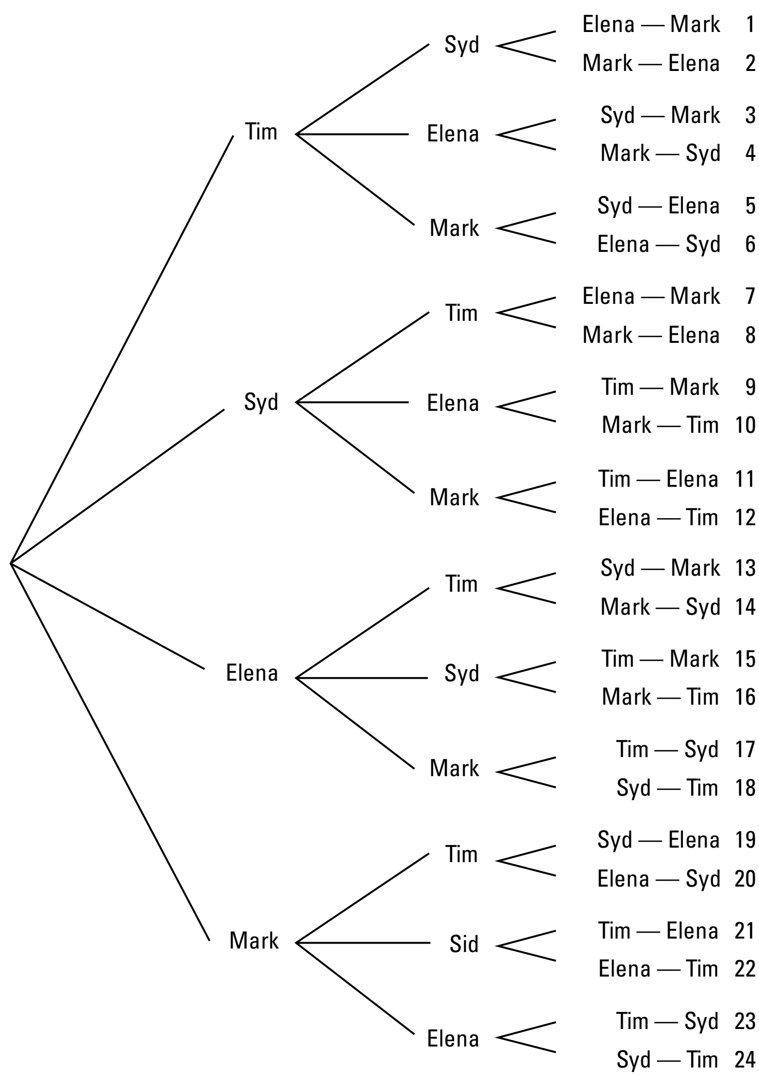


Figure 5-1: Tree diagram for the number of rearrangements of four people sitting in a straight line.

When doing a problem that involves counting the total number of possible outcomes, you have to ask the question, "Does the order of the selected items matter?" If the order matters, you need to use a permutation (see the previous pages for more on this topic). When the order doesn't matter, you use comibinations (see the section "Counting Combinations" for more). But how do you know if the order matters? Think about what happens after the items are selected and whether the order they appear in matters. For example, if you're picking three people from a class to win a prize, and the prizes are all the same, the order doesn't matter. If you're picking students to win three different prizes, however, the order does matter.

Permutation problems with added restrictions: Are we having fun yet?

When you understand the basic ideas behind permutations, you can use them to work more complicated problems that involve selecting and rearranging items into different groups under different conditions. These conditions then turn into restrictions that are placed on the problem to be solved. Here I deal with the common restrictions that professors use to take the problems up a notch. Think of these restrictions as part of a wedding planner's dilemma; there are always people who require some special considerations in the effort to keep the peace.

Don't try to fit every "number of arrangements" problem automatically into the permutation formula. When a problem asks you to figure the number of ways to do a certain process with restrictions, separate the stages of the process (each seat to be filled or each digit to be chosen, for example) and figure out how many ways you have at your disposal to do the first stage. Then you can move to the next stage, considering possibilities that can and can't occur based on the first stage. Continue on in this manner, figuring the possibilities for each stage, and then multiply them all altogether.

Certain items can't be placed next to each other

Suppose you have four friends named Jim, Arun, Soma, and Eric. How many ways can you rearrange the individuals in a row so that Soma and Eric don't sit next to each other? You have $4! = 24$ possible ways to rearrange all four of them. But certain scenarios among the 24 possibilities won't work because of the restriction on the problem. If you can figure out which possibilities include Soma and Eric sitting next to each other, you can subtract those from the total.

It may be easier to figure out the number of ways an outcome can occur by taking the total number and subtracting the number of ways that the outcome *can't* occur. This method is similar to the complement rule (see Chapter 2).

If you think of the four seats as slots, you can figure out how many ways you can fill the slots with Eric and Soma sitting next to each other. First, you can force Soma to sit on Eric's right by creating three slots for Eric to sit in: seat 1, 2, or 3. After Eric selects his seat, Soma automatically sits on his right. So, the total number of arrangements where Soma and Eric sit together with

Soma on Eric's right is $3 * 1 = 3$. (You also have the possibility of Eric sitting on Soma's right, but you can just multiply your answer by 2.)

Figure 5-2 shows all the possibilities, using slots to designate the seats. Of the 24 ways to arrange the four people, $3 * 2 = 6$ of those ways involve Eric and Soma sitting next to each other. Therefore, $24 - 6 = 18$ of the scenarios involve the two not sitting next to each other, which answers the original question.

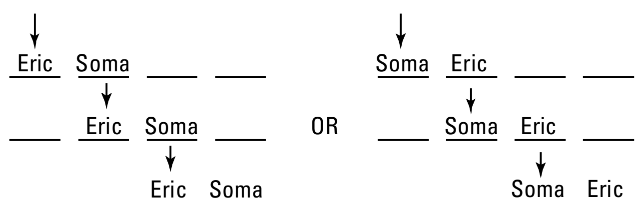


Figure 5-2: The number of ways that Eric and Soma can sit next to each other amongst four seats.

Certain items are distinct; others are not

How many ways can you rearrange the letters of the word "Mississippi?" The word contains 11 letters, but not all the letters are distinct — you see four Ss, two Ps, and four Is to go along with the M. Although you have 11! ways to arrange all 11 items, you have to account for arrangements where like letters are interchanged and the word is exactly the same; for example, changing the two Ps around doesn't do anything. So, although the order matters when the letters are distinct, the order doesn't matter when the letters are the same.

You handle this situation by taking the 11! and dividing by the number of rearrangements of the like letters, doing so for each letter. In the case of Mississippi, the answer is $11! \div (4! * 2! * 4!)$, because the word has four Ss, two Ps, and four Is whose individual rearrangements don't really count. This calculation gives you $\frac{11!}{4!\,2!\,4!} = \frac{11 * 10 * 9 * 8 * 7 * 6 * 5 * \cancel{4!}}{\cancel{4!}(2 * 1)(4 * 3 * 2 * 1)} = \frac{1{,}663{,}200}{48}$, which equals 34,650 possible distinguishable rearrangements of the letters in the word Mississippi.

Rearranging items in a circle versus a straight line

Suppose you have four friends — Tim, Syd, Elena, and Mark — who go to the ice cream shop after a movie, and they sit at a round table. How many ways can they arrange themselves in a circle? This situation differs from the theater situation (see the section "Unraveling a Permutation") because a row has a clear beginning and end, but a circle does not. Can you guess which arrangement has fewer possibilities given the same number of people involved?

To tackle this situation, choose any person from the group, and make that person your "starting point." For this example, your starting point is Tim. From here, you have to rearrange the remaining three people: Syd, Elena, and Mark. You have 3! = 6 ways to do this. Figure 5-3 shows all six possibilities. Note that if you change the starting point from Tim to Mark, for example, you still get the same six rearrangements just by rotating the table 90 degrees to the right. Because you don't count rearrangements that you get just by rotating the table, you have fewer arrangements in this situation than you would if the people sat in a straight line.

REMEMBER

In general, you have (k – 1)! possible ways to rearrange k items in a circle. Because you select one item or individual to be your starting point, and you don't care where the beginning of the circle is, you leave the other k – 1 individuals to sit in the remaining k – 1 positions.

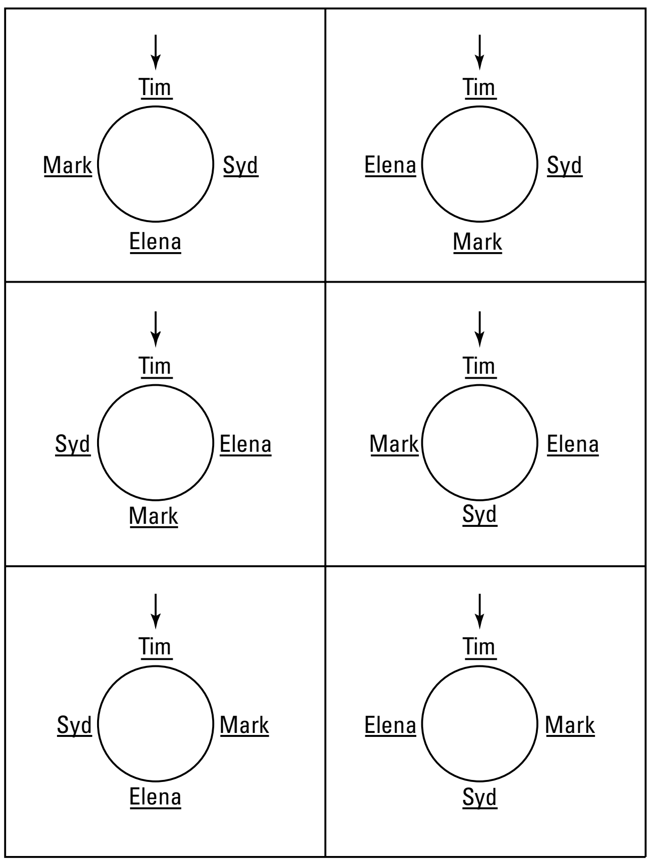


Figure 5-3: Possible rearrangements of four people sitting around a table discussing probability.

Choosing items with no repeats allowed when order matters

Choosing items with no repeats allowed and when order matters is pretty much a straight permutation problem setup. You basically choose the items without replacement because you are not allowed to repeat them again.

For example, how many ways can you choose five *distinct* numbers from zero through nine and put them together on a license plate? This problem is really asking you to find the number of ways to choose and rearrange five numbers from ten, which is P_5^{10}. The answer is $P_5^{10} = \frac{10!}{(10-5)!} = \frac{10 * 9 * 8 * 7 * 6 * \cancel{5!}}{\cancel{5!}}$, which equals 30,240. This is a permutation problem, because it samples without replacement.

Suppose you now have to find out how many odd five-digit numbers are available to put on a license plate with no repeating, except for the last digit. You can't answer this with a regular permutation formula because of the restrictions. You must figure each digit separately and multiply the results together. First, you know that a five-digit number can't start with zero, so you have nine choices for the first number. The second, third, and fourth digits can be anything except what was already chosen, so you have 9, 8, and 7 choices, respectively, for each (9 * 8 * 7). For the number to be odd, its last digit must be one, three, five, seven, or nine, so you have five choices for that. So, you use the multiplication rule from Chapter 2 to get the total number of five-digit numbers: 9 * 9 * 8 * 7 * 5 = 22,680. Figure 5-4 shows the situation where you choose 4, 3, 2, 1, and 5.

9	*	9	*	8	*	7	*	5
1		0		0		0		~~0~~
2		1		1		1		1
3		2		2		~~2~~		~~2~~
4		3		~~3~~		~~3~~		3
5		~~4~~		~~4~~		~~4~~		~~4~~
6		5		5		5		5
7		6		6		6		~~6~~
8		7		7		7		7
9		8		8		8		~~8~~
		9		9		9		9

Figure 5-4: Number of ways to choose a five-digit number without repeating digits.

Choosing numbers when order matters and repeats are allowed

When you allow for items to be chosen repeatedly, you basically multiply the total number of items over and over to obtain your total number of outcomes. Although it sounds similar, this is not a permutation problem at all; it's a counting problem that uses the multiplication rule. You can tell that because the sampling is done with replacement and the number of possible values doesn't change from one selection to the next.

How many ways can you choose five numbers from zero through nine and put them together on a license plate in which the numbers can be repeated? Using the multiplication rule from Chapter 2, you find that the answer is $10 * 10 * 10 * 10 * 10 = 100{,}000$. Remember, you have ten possible digits each time, because each one can be from zero through nine.

Finding probabilities involving permutations

To find the probability for a problem involving permutations, you typically find the number of permutations for the event you're interested in divided by the total number of ways to arrange all the items.

REMEMBER: If a problem asks you to find only the number of ways that a certain outcome can happen, you use counting rules such as the multiplication rule (see Chapter 2) and/or the permutation formula, and your answer should be a whole number. If a problem asks you to find a probability, you need to consider a numerator and a denominator, divide the numerator by the denominator, and get an answer between zero and one. Always be sure you answer the actual question being asked.

Choosing your words carefully: Splitting out the numerator and denominator

TIP: When figuring a probability that involves permutations, the best thing you can do is split the numerator and the denominator into two separate permutation problems, figure them separately, then put them together into a fraction, numerator divided by denominator, to get your final probability. Trying to do the problem all at once can get too confusing.

For example, suppose you choose four different letters from the alphabet and rearrange them into a four-letter "word." (Don't worry about whether the "word" makes sense or not.) What's the chance that the letters spell the word "sing?" To solve this problem, you think of the numerator and denominator separately, and look at the denominator first. You want the number of ways to choose four letters from the full alphabet (26). Because you're arranging the letters into a four-letter "word," that means the order is important, so you need a permutation, P_4^{26}. The number in the denominator then becomes $P_4^{26} = \frac{26!}{(26-4)!} = \frac{26!}{22!} = \frac{26 * 25 * 24 * 23 * \cancel{22!}}{\cancel{22!}}$, which equals 358,800.

The numerator is the number of ways to achieve the outcome of interest, which is to spell out the word "sing." You have only one way to do this, so the numerator is one. The probability, therefore, of choosing four different letters and spelling out the word "sing" is 1 out of 358,800, which is 0.000002787. (So, if you let a monkey type on a typewriter as long as he wants, what's the chance that he types out "Hamlet"? It all starts with this problem.)

If you want another challenge, you can figure out how many ways you can choose four different letters and have them involve only the letters a through m. The denominator is the same as in the previous example, a permutation of 4 items from 26, which is 358,800. The numerator is the number of ways to choose four different letters from a through m; the range involves 13 letters, and you need to choose and rearrange 4 of them, so you have P_4^{13}. The numerator then becomes $P_4^{13} = \frac{13!}{(13-4)!} = \frac{13!}{9!} = \frac{13 * 12 * 11 * 10 * \cancel{9!}}{\cancel{9!}} = 13 * 12 * 11 * 10$, which equals 17,160. So, the probability of making a four letter "word" with different letters from only a through m is $17{,}160 \div 358{,}800 = 0.0478$.

Getting people lined up: Looking for the hidden subtleties of each problem

When finding a probability involving a permutation, you have to think of all the possible scenarios that could work, after you know what it is you are trying to count in the numerator. For example, if you want the number of ways to seat four people in line so Bill and Bob sit next to each other, you have to look separately at the scenarios where Bill sits on the left and Bob sits on the right, and vice versa. Buried in each probability problem involving permutations are hidden situations that you have to root out as you figure out the problem.

Suppose, for example, you have eight friends line up in a row in the movie theater — four males and four females. What's the chance that your final arrangement alternates gender all the way down (male-female-male-female or female-male-female-male)?

The denominator of this probability is the number of ways to rearrange the eight people, denoted P_8^8, which equals $8! = 40{,}320$. The numerator is the number of ways to arrange the individuals under the restriction that they alternate by gender. You have two scenarios to look at: Alternating gender with a male first or alternating gender with a female first.

Say that you first look at the case where you want male-female. In the first spot, you have four males to choose from; in the second spot, you have four females to choose from; in the third spot, you have the three remaining males to choose from; and in the fourth spot, you have the three remaining females to choose from. Similarly, the fifth and sixth spots have the two remaining males and two remaining females to choose from, respectively. The seventh spot goes to your only remaining male, and your eighth spot goes to your only remaining female. This gives you a total of $4 * 4 * 3 * 3 * 2 * 2 * 1 * 1 = 576$ ways to rearrange the individuals so that you put them in a male-female order.

Now, for the case where the order is female-male, the number of ways is also 576 because the number of males and females is the same for this problem. So, the total number of ways to rearrange the group so that gender alternates is $576 * 2 = 1{,}152$. Therefore, the probability that you arrange eight people and end up alternating gender is $1{,}152 \div 40{,}320 = 0.029$.

Counting Combinations

Permutations focus on the number of ways to choose and then rearrange your k items, without replacement, from a group of n items. The bottom line is, order matters when it comes to permutations. But suppose that after you chose the k items without replacement, you didn't care about how they were rearranged. These kinds of problems require a related but different counting technique to solve. The technique is called combinations.

Many probability problems involve selecting a number of items, k, without replacement from a group of items, n, with no regard for the order in which the selected items appear. The number of ways to perform this task is called a *combination* of k items from n items; the notation is ${}_nC_k$, C^n_k, or $\binom{n}{k}$. When you talk about a combination of k items from n items, you call it a combination of "n choose k." (I use the second and third of these notations to indicate combinations in this book.)

Be careful when you write and use the notation for combinations. A combination is not a fraction; in other words, $\binom{n}{k} \neq \left(\frac{n}{k}\right)$.

A counting problem involves a combination if the problem meets two conditions:

- You select the items without replacement (so the results can't repeat).
- The order of the items selected doesn't matter.

With any combination problem, you have many different ways to look at it and solve it. If you have a different way of solving a problem, but you get the same answer, you should be okay. Just be sure to show your work so that readers know where you're coming from.

Solving combination problems

To solve a combination problem, you start with the same approach that you use to do a permutation problem (see the section "Counting Permutations" earlier in this chapter); the only difference is that the order of the k items selected doesn't matter. So, you take the total number of ways to rearrange all n items (denoted n!), and, in addition to dividing by the number of ways you rearrange the n – k items you don't select [(n – k)!], you divide by the number of ways to rearrange the k items you do select (k!).

Suppose, for example, that you have ten people in a class. You need to choose three students to win a prize, and all the prizes are the same (hence, the order in which you choose the three people isn't important). How many ways can you choose? You have $10! \div (10-3)! = 10 * 9 * 8 = 720$ ways to rearrange and select three people from the class of ten. But with combination problems, you don't care about the number of ways to arrange the three you select. So, you take 720 and divide it by $3! = 6$ to get 120. You have 120 ways to choose three people from a class of ten for identical prizes, compared to 720 ways to choose three people where the prizes are different.

In general, the number of ways to select k distinct items without replacement from n items, and do so in any order, is $\binom{n}{k} = \frac{n!}{(n-k)!k!}$. To confirm this, suppose you choose two letters at random from the group a, b, and c. If the order matters, you have a permutation of two items from three items, with $P_2^3 = \frac{3!}{(3-2)!} = \frac{3!}{1!} = \frac{3*2*1}{1} = 6$ ways to choose. If the order doesn't matter, you have a combination of two items from three items (stated as "3 choose 2"), with $C_2^3 = \binom{3}{2} = \frac{3!}{(3-2)!2!} = \frac{3!}{1!2!} = \frac{3*\cancel{2}*\cancel{1}}{1*\cancel{2}*\cancel{1}} = 3$ ways to choose. Essentially, if the order doesn't matter, you need to divide out every possible permutation of the two letters chosen — the $2! = 2$ permutations.

TIP

You can show that there are k! times more permutations than combinations when you choose k items from n items without replacement. That's because the formula for a permutation is $\frac{n!}{(n-k)!}$, while the formula for a combination is $\frac{n!}{(n-k)!k!}$. If you take k! times the combination formula, you get the permutation formula. That means it takes k! combinations to equal one permutation (which is true because with a combination, you don't care how those k items are arranged, and with a permutation, you do).

Combinations and Pascal's Triangle

You sometimes use an odd, even mystic mathematical form called *Pascal's Triangle* to find the number of combinations of "n choose k" items. Figure 5-5 shows Pascal's Triangle for all the values of n from zero through ten and for all possible values of k from zero to n. Each row shows all the possible combinations for a given n in order: "n choose zero" (which is one), "n choose 1," "n choose 2," and so on, all the way up to "n choose n" (which is one also). The triangle starts with a one at the very top (defining "0 choose 0" as one) and ones in the second row (because "1 choose 0" and "1 choose 1" both equal one).

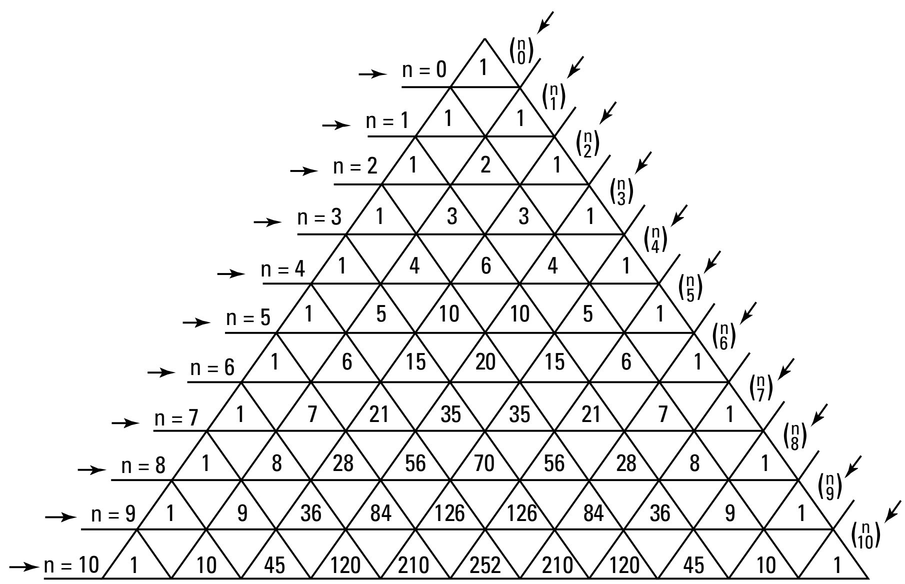


Figure 5-5: Finding the number of combinations using Pascal's Triangle.

Each row in Pascal's Triangle starts and ends with one (because "n choose 0" and "n choose n" both equal one). To find the value for any other number in the table, you take the sum of the two numbers in the previous row that fall directly above the number you're interested in. For example, in Row 5, you find all the combinations for n = four. The first is "4 choose 0," which is one; "4 choose 1," which is four (you find this by adding the one and the three that fall directly above), "4 choose 2," which is six (you find this by adding the three and the three directly above); "4 choose 3," which is four; and finally, "4 choose 4," which is one. You can continue adding rows to the triangle as long as you want, using the method to find the number for "n choose k" for any n and any k. (Of course, you could also use your calculator!)

Looking at Figure 5-5, you can see that the number of combinations is symmetric; in other words, when you go across a row, you see that the numbers start at one, go up, reach a peak, come back down (using those same numbers), and finally go back to one. For example, in Row 6, the number of ways to choose three items from five items ("5 choose 3") is $C^5_3 = \binom{5}{3} = \frac{5!}{(5-3)!3!} = \frac{5!}{2!3!} = \frac{5 * 4 * \cancel{3!}}{2 * 1 * \cancel{3!}} = \frac{20}{2}$, which is 10; and the number of ways to choose two items from five items ("5 choose 2") is $C^5_2 = \binom{5}{2} = \frac{5!}{(5-2)!2!} = \frac{5!}{3!2!} = \frac{5 * 4 * \cancel{3!}}{\cancel{3!} * 2 * 1} = \frac{20}{2}$, which is also 10. The symmetry isn't a coincidence. When you choose three items, you divide out the ways to rearrange the items not selected (5 – 3)!, and you divide out the ways to rearrange the items selected (3!). When you choose two items, you divide by (5 – 2!) = 3 and also by 2. In both cases, you divide by 2! * 3! = 12 or 3! * 2! = 12. That's where the symmetry comes in. Your calculations tell you that "n choose k" is equal to "n choose (n – k)."

Probability problems involving combinations

Probability problems involving combinations come in a variety of different scenarios; some of the most common are

- Splitting one big group into two subgroups
- Choosing items in any order, no repeats allowed
- Choosing items in any order, repeats allowed

Splitting objects or individuals into two groups

Suppose you have a group of ten friends going to a movie, and you need to split the individuals into two groups; one group will sit in the front row, and one group will sit in the back row. (Assume that you don't care who sits by whom within a row.) How many ways can you split the groups? Imagine taking this group of ten and choosing five people to sit in the front row. Suppose you figure out the number of ways to do just this part. What do you have left to figure out? Nothing! After you choose five people to sit in the front row, you automatically assign the remaining five people to the back row. So, instead of figuring out the ways to choose the two groups of five, you have to figure out only how to select the first group of five; the second group is automatic.

Because you don't care about who sits next to whom within a selected row, the order of the selected people doesn't matter. That tells you a combination is in order. You select five people from the ten without replacement when order doesn't matter, so you find $C^{10}_5 = \binom{10}{5} = \frac{10!}{(10-5)!(5!)} = \frac{10!}{5!5!} = \frac{10 * 9 * 8 * 7 * 6 * \cancel{5!}}{\cancel{5!}5 * 4 * 3 * 2 * 1} = \frac{30{,}240}{120}$, which equals 252.

Picking items in any order with no repeats allowed

Suppose you want to pick five lottery numbers without any numbers repeating. You have a choice of 1 through 42 for each number. How many ways can you choose your numbers? This problem is a combination problem because the order of the lottery numbers (unless otherwise stated) doesn't matter, and you're selecting without replacement because you can't repeat any numbers. Because you want to select 5 numbers from the group of 42 without replacement with no regard for order, the total number of ways to choose is "42 choose 5," which equals $C_5^{42} = \binom{42}{5} = \frac{42!}{(42-5)!5!} = \frac{42!}{37!5!} = \frac{42 * 40 * 39 * 38 * \cancel{37!}}{\cancel{37!}5 * 4 * 3 * 2 * 1} = \frac{102{,}080{,}160}{120}$, or 850,668. Now that's a lot of combinations! And those combinations are only the situations where the numbers don't repeat. If you allow the numbers to repeat, you'd have 42 * 42 * 42 * 42 * 42 = 130,691,232 possible combinations (using the multiplication rule outlined in Chapter 2).

Now your local lottery decides (as it every so often does) to increase the possible numbers to 43. How does that change affect the total possible combinations? If you don't allow numbers to repeat, the number of possible combinations of "43 choose 5" increases to 962,598. You see a 13-percent increase in the overall number of combinations (which lowers your chance of winning quite a bit).

Picking items in any order with repeats allowed

When you are asked to figure a probability for picking certain items when order doesn't matter, always think combination. The most commonly discussed probability scenario involving combinations is picking cards from a deck. Figure 5-6 shows a listing of the 52 cards in a standard deck. (For more on the standard deck, see Chapter 2.)

Probability can answer tons of different questions about a 52-card deck. For starters, suppose you're playing poker (a game involving five-card hands) — how many possible hands can you make? Because you don't care about the order in which the cards are dealt to you, you use a combination of 5 items selected from 52 items, or "52 choose 5." This combination gives you $C_5^{52} = \binom{52}{5} = \frac{52!}{(52-5)!5!} = \frac{52!}{47!5!} = \frac{52 * 51 * 50 * 49 * 48 * \cancel{47!}}{\cancel{47!}5 * 4 * 3 * 2 * 1} = \frac{311{,}875{,}200}{120}$, which equals 2,598,960 possible five-card hands from one deck. (And you wonder why you never get a good hand in poker the first time around?)

You can find many combinations within one deck of cards. If you pull out two cards from deck, for example, how many ways can they both be hearts? Because one deck features 13 different hearts, you find "13 choose 2," which equals $C_2^{13} = \binom{13}{2} = \frac{13!}{(13-2)!2!} = \frac{13!}{11!2!} = \frac{13 * 12 * \cancel{11!}}{\cancel{11!}2 * 1} = \frac{156}{2}$, or 78. How many ways can you get two 2s? Because a deck has four 2s total, you find "4 choose 2," which equals 6.

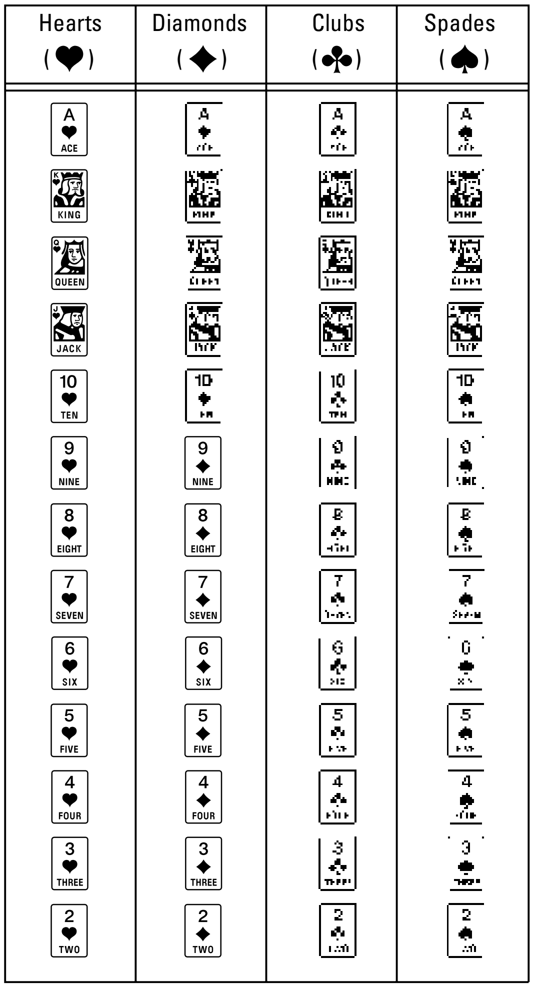


Figure 5-6: Cards in a standard 52-card deck.

Here's one way to think about selecting objects that have a certain condition (like hearts or 2s): Imagine that you're taking all the objects with that condition, setting them aside, and choosing from them. That's how you figure out the top number in the "n choose k" formula.

Studying more complex combinations through poker hands

Combinations provide a wide variety of situations where you reach into a group and select certain items without replacement. You can use this method to solve more complex problems that involve making various selections of items in a series of stages or that have a series of different attributes (poker hands, for example). You can also use the method when you divide a large group into more than two smaller subgroups. In this section, you see several examples, highlighting each situation, in the realm of poker hands.

Ranking poker hands

Poker is a very popular card game, so if you want to play or watch (or just use the game to work on your probability skills), you should develop an understanding of what kinds of hands are desirable and how many ways you can get these hands. But even if you're not into poker as a game, studying poker hands provides an excellent way to understand combinations and how they work. If you can get through all the different poker hands, you can work on plenty of different combination problems that come your way.

Figure 5-7 shows you all the possible hands in poker, ranked from highest to lowest. For people who aren't familiar with the different poker hands, here's a brief overview:

- A *straight flush* is five cards of the same suit that fall in sequence; ranked by the top card in the straight, the highest ranked straight flush is the *royal flush,* which starts with the high ace and ends with the 10.
- *Four of a kind* is four cards of the same denomination.
- A *full house* is three cards of one denomination with two of another denomination.
- A *flush* is five cards from the same suit.
- A *straight* is five cards in sequence.
- *Three of a kind* is three cards of one denomination; the other two cards have different denominations.
- *Two pair* is two different pairs of cards — two of one denomination and two of another. The fifth card is of a different denomination from the others.
- *One pair* is one pair of cards of one denomination; the other three cards are of totally different denominations from all the rest.
- In the event that the players have none of the previous hands, you go by who has the *highest card* in their hands, with ace as the highest.

Poker provides a distinct ranking among the five-card hands, and this ranking is directly related to the number of ways to get each type of hand. The highest-ranked hands occur less often than the lowest-ranked hands, with each hand getting progressively harder and harder to get as you move uphill.

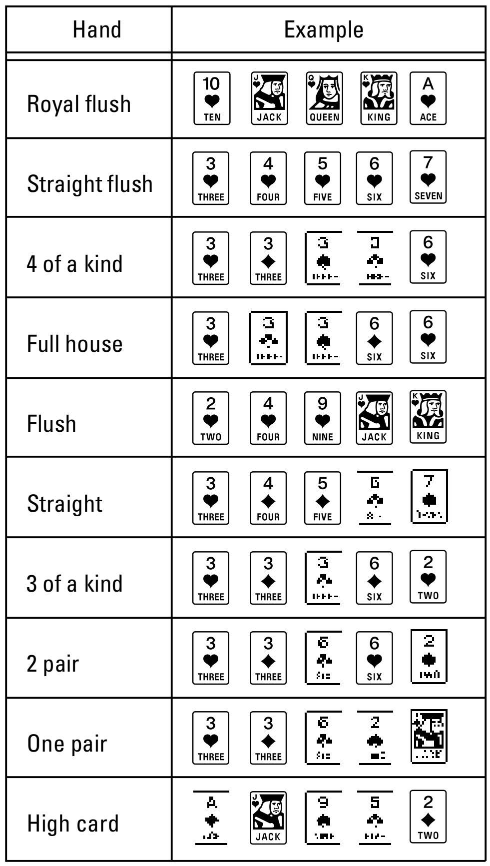


Figure 5-7: 5-card poker hands (ranked from high to low) from a 52-card deck.

Figuring the number of ways to draw each poker hand

To find the number of ways you can get a certain poker hand, it helps to visualize yourself taking charge of the problem and forcing the cards to do what you want.

In this section, I break down the number of ways to get each poker hand, starting with four of a kind. It's important to develop a thinking and organizing strategy for doing any combinations problem, and in the following pages, I explain how to develop this strategy. The big picture: Choose the big things first and then the small things. For example, start with the suits (if important to the hand), move on to choosing the denominations, and then the number of cards from each denomination you need. You'll get the idea as I go through each of the various poker hands in a regular game of poker.

Hitting four of a kind

To get a four of a kind, you need to first pick one denomination ("13 choose 1" ways to do this), and of that denomination, pick all four cards ("4 choose 4" ways to do this). Now you have to take care of the fifth card. There are 12 denominations left to choose from, so choose one; and from the four cards of that denomination, choose one. Now to find the total number of ways to get a four of a kind, multiply all these together to get $\left[\binom{13}{1}\binom{4}{4}\right]\left[\binom{12}{1}\binom{4}{1}\right] =$ $13 * 1 * 12 * 4$, which equals 624 possible hands.

Serving up three of a kind

To get three of a kind, you again select the denomination you want the three cards to come from ("13 choose 1" ways to do that). Then from the four cards of that denomination, select three of them ("4 choose 3" ways). Now choose your fourth and fifth cards so they have different denominations. First, choose the denominations ("12 choose 2" ways). For the fourth card, there are four cards available in the chosen denomination, so choose one. For the fifth card, do the same. The total number of ways to get three of a kind is $\left[\binom{13}{1}\binom{4}{3}\right]\left[\binom{12}{2}\binom{4}{1}\binom{4}{1}\right] = 13 * 4 * 66 * 4 * 4$, which is 54,912. Notice there are more ways to get three of a kind than four of a kind, which is why three of a kind is ranked lower on the list in Figure 5-8.

You might have thought about choosing those last two cards one at a time, with "48 choose 1" times "44 choose 1" ways (because after you pick the denomination for the three of a kind, there are 52 – 4 = 48 possibilities for the fourth card, and the fifth card can be any of the remaining 48 – 4 = 44 cards). But this gives you an answer that's double the correct answer. Why? Because it double counts all the combinations of the last two cards. It's easier to see the problem if you have a small case, like A, B, C. Say you want to choose two letters with no repeats. If you do it one at a time, you have "3 choose 1" times "2 choose 1" equals six ways to do it: AB, AC, BA, BC, CA, CB. But if you choose them both at the same time, you would have only three possible combinations: AB, AC, or BC, with the pairs of letters in any order. And notice that "3 choose 1" times "2 choose 1" is not the same as "3 choose 2." Breaking combinations into too small of stages tends to lead to the double counting problem; avoid this by making choices all at once for items that affect each other, like the two cards left after the three of a kind has been selected.

Working on a full house

To get a full house, note that you need to choose two denominations, and you also need to determine which one will get the three of a kind and which one will get the two of a kind. For example, 2-2-3-3-3 is different from 3-3-2-2-2, and you need to count that. To do this, you first choose your two denominations ("13 choose 2" ways), and then choose which of the two denominations will be the three of a kind (the other will be the two of a kind); there are "2 choose 1" ways to do this. For the three of a kind, you have four cards available, so choose three; for the two of a kind, you have "4 choose 2." All together then, there are $\binom{13}{2}\binom{2}{1}\binom{4}{3}\binom{4}{2} = 78 * 2 * 4 * 6$, or 3,744 ways to get a full house. A full house falls below a four of a kind but above a three of a kind in the hand rankings, which makes sense if you look at how many ways you can get these hands.

Suiting up for a flush

To get a flush (all five cards of the same suit), you can pick the suit first ("4 choose 1" ways); and then of the 13 cards available for that suit, you choose five of them ("13 choose 5" ways). In total, there are $\binom{4}{1}\binom{13}{5} = 4 * \frac{13!}{8!5!} = 4 * \frac{13 * 12 * 11 * 10 * 9}{120} = 5{,}148$ different ways to get a flush in poker. (***Note:*** This total contains all the 40 flushes that are special in the sense that they are straight flushes. More on this after I go through the straights.)

Straightening out a straight

To get a straight, you only need to get five cards in consecutive order; the suit doesn't matter. You can use the ace as either low or high here. To organize your thinking on this, you only have to think about what the starting card is (the lowest card in the straight), because after that is figured out, the rest falls into place. How many cards can start a straight? You can start with an ace (low), 2, 3, 4, 5, 6, 7, 8, 9, or 10. You can't start with a jack or above, because you need 5 in a row and there aren't enough cards that go that high. So, for a given suit, there are "10 choose 1" ways to pick the starting card. You have four suits to pick from for the starting card, so the total choices for the starting card is "10 choose 1" times "4 choose 1," which is 40. Now, after you pick the starting card (say it's a 7), the next card has to be of a certain denominator (here it has to be an 8). There are four possibilities to pick from, so choose one (there are "4 choose 1" ways to do that). The third, fourth, and fifth cards follow in the same way, so you multiply by "4 choose 1" three more times to take care of those cards. In total, there are $\left[\binom{10}{1}\binom{4}{1}\right]\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1} = 10 * 4^5$ or 10,240 possible straights. (This total includes the 40 straight flushes; notice they have a higher ranking because they're harder to get.)

Shooting for a straight flush

The straight flush is a special case of both straights and flushes, because the cards have to be in a row and of the same suit. To figure the number of straight flushes, you can look at how you calculate the number of straights and modify it slightly. After you pick the starting card and its suit ("10 choose 1" times "4 choose 1" ways to do this), you're done. That's because the second, third, fourth, and fifth cards have to be of that same suit, and they have to follow in a row from the starting card. That means you multiply by "1 choose 1" four times. (Say, for example, that you start with a 2 of hearts. The next card has to be a 3 of hearts, and only one of those exists in the deck; the third card has to be a 4 of hearts, and again only one of those exists in the deck.) So, the total number of straight flushes is determined by $\left[\binom{10}{1}\binom{4}{1}\right]\binom{1}{1}\binom{1}{1}\binom{1}{1}\binom{1}{1} = 10 * 4$, which is 40.

The straight flush sits at the top of the rankings ladder in terms of poker hands (see Figure 5-8). If there is a tie and two people have a straight flush, the one with the highest card in the flush wins. The top straight flush, using this high-card ranking, occurs when you have 10, J, Q, K, and Ace (high) of the same suit. This special straight flush is called a royal flush. There are only four royal flushes, one for each suit. So, if you separate the royal flush from the other straight flushes, you have four royal flushes and 40 – 4 = 36 straight flushes.

Sinking to the bottom: Two pair, one pair, and high card

At the bottom of the poker-hand-rankings ladder are the two pair, one pair, and high card situations. To get two pair, you need two different pairs of two cards. You can't distinguish one pair from the other, so choose their denominations at once ("13 choose 2" ways). Then from the four cards of one denomination, choose two, and from the four cards of the other denomination, choose two. Finally, choose the fifth card from amongst any card not of those two denominations. There are 52 – 4 – 4 = 44 cards to pick from; choose one. In total, there are $\left[\binom{13}{2}\binom{4}{2}\binom{4}{2}\right]\binom{44}{1} = 78 * 6 * 6 * 44 = 123{,}552$ ways to get two pair.

To get one pair, you use the same basic idea as for two pair. From the 13 denominations, choose one for your pair, and from the four cards available, choose two of them. Now, you can't distinguish between the three remaining cards, so you need to pick them all at once to avoid double counting. So, of the remaining 12 denominations, choose three for your last three cards. You have four cards from each denomination; you need to choose one card from each set of four to get your third, fourth, and fifth cards, respectively (there are "4 choose 1" ways to do this, multiplied three times). In total, there are $\left[\binom{13}{1}\binom{4}{2}\right]\left[\binom{12}{3}\binom{4}{1}\binom{4}{1}\binom{4}{1}\right] = 13 * 6 * 220 * 4 * 4 * 4 = 1{,}098{,}240$ ways to get one pair.

For the high-card category, you start by figuring out how many hands there are where every card is of a different denomination, and then subtract out the straights and/or flushes (because in all those hands, every card is of a different denomination as well). You choose five different denominations ("13 choose 5" ways to do this), and for each set of four cards available in each of those five denominations, you choose one for the first, second, third, fourth, and fifth cards, respectively. That means there are $\binom{13}{5}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1} =$ $1,287 * 4 * 4 * 4 * 4 * 4 = 1{,}317{,}888$ ways to do this. Now subtract out the straights (10,200), the flushes (5,108), straight flushes (36), and royal flushes (4); you get 1,302,540 total card hands that fall in the high-card ranking at the bottom of the barrel.

Figure 5-8 shows a summary of the total number of ways to get each type of poker hand. You find the total number of five-card poker hands by taking "52 choose 5," which gives you 2,598,960.

Hand	Number of Ways
Royal flush	4
Straight flush	(40–4) = 36
4 of a kind	624
Full house	3,744
Flush	(5,148–40) = 5,108
Straight	(10,240–40) = 10,200
3 of a kind	54,912
2 pair	123,552
One pair	1,098,240
High card	1,302,540
Total	2,598,960

Figure 5-8: The hierarchy of poker hands, and the number of ways to obtain each hand.

Finding probabilities involving combinations

You find probabilities involving combinations by concentrating on developing a numerator (the number of ways that the event of interest can occur) and a denominator (the total number of ways to do the experiment).

Many combination problems require you to find probabilities involving "and," "and/or," and "not" probabilities:

- If you have an "and" scenario, you use the multiplication rule (see Chapter 2) in the event that you want multiple outcomes to occur at once. The multiplication rule says that P(A and B) = P(A) * P(B|A), and if A and B are independent, you just have P(A) * P(B).
- If you have an "and/or" situation, you use the addition rule (see Chapter 2). The addition rule tells you that P(A or B) = P(A) + P(B) – P(A and B), and if A and B don't intersect, you just have P(A) + P(B).
- If you have a "not" problem, you use the complement rule (see Chapter 2), which says that the probability of not being in A is one minus the probability of being in A — in other words, $P(A^c) = 1 - P(A)$.

Choosing with restrictions

Suppose you have ten people in a class — six males and four females — and you want to choose a committee of four people. What's the chance of choosing a committee that contains

- Exactly one male?
- At least one male?

To answer these questions, you first find the denominator of the probability. How many ways can you choose a committee of four from ten people? The order isn't important, and you're choosing without replacement, so you have "10 choose 4," or 210 ways to choose. That figure is your denominator for both questions.

To choose exactly one male for the committee, you choose one male from the six available ("6 choose 1" = 6 ways). You have to make sure that the other three people on the committee are female, so of the four females, you choose three ("4 choose 3" = 4 ways). Because you need one male and three females, you multiply the number of ways to do this using the multiplication rule (Chapter 2). So, in total, you have 24 ways to choose exactly one male (which really means one male and three females). This is the numerator for the probability. So, the probability of choosing a committee of four people containing exactly one male is 24 ÷ 210 = 0.114, or 11 percent. (This particular problem uses the hypergeometric distribution, which I discuss in depth in Chapter 16.)

WARNING!

In the previous example, you may think that you have to worry only about the males, and that "4 choose 1" = 4 is the numerator. But in saying that you need exactly one male, you conclude that the rest of the committee can't be male, so you have to say that exactly one male really means one male *and* three females. Therefore, you need to use the multiplication rule and multiply those two combinations together. You always have to account for all the items you're choosing, or your counts won't be correct.

To choose a committee with at least one male, you have four possibilities: one male (and three females), two males (and two females), three males (and one female), or four males (and no females). Normally, you would find each of these probabilities with the method you use for the first question of this problem and then add them together, because the word "or" indicates the use of the addition rule. But it's easier for this problem to use the complement rule (see Chapter 2) to find the probability of the opposite and take one minus that value.

The opposite of "at least one male" is "no males." To choose no males for the committee, you need to choose four females ("4 choose 4" = 1 way) and zero males ("6 choose 0" = 1 way). So, the probability of choosing all females for the committee is 1 out of 210. That means the probability of choosing at least one male is one minus 1 out of 210, which is 1 – 0.005 = 0.995.

Calculating probabilities for poker hands

Notice that Figure 5-8 shows you all the different ways to get any of the poker hands (in the second column) *and* the total number of all poker hands (at the bottom of the second column). To find the probability of a poker hand you are interested in, you just take the number of ways to get that hand (corresponding value in column two) and divide by the total.

For example, the probability of being dealt four of a kind is the number of four-of-a-kind hands divided by the total number of poker hands. The total number of four-of-a-kind hands is "13 choose 1" times "4 choose 4" to get the four of one kind times "48 choose 1" for the fifth card, to get 624. The total number of five-card poker hands is "52 choose 5" = 2,598,960. So you take 624 ÷ 2,598,960. That probability is 0.00024, which makes the probability of being dealt a four of a kind very unlikely! That's why those gunslingers in the spaghetti westerns always suspected cheating when they saw someone come up with one during a poker game, leading to the big bar-room brawl and shootout.

TIP

Manufacturers of video poker machines use the probabilities you find with Figure 5-8 to set their winnings for the different hands that you can get. The manufacturers know the chance of being dealt a four of a kind right out of the box, and even though you get a chance to discard the cards you don't want and get new ones (adding another layer to the probability calculations), the manufacturers still know you're unlikely to get the hand. That's why they put the biggest pot on that combination. However, not all video poker machines give you the same winnings for the same hands; before you play, be sure to check the payouts for the various hands. If you have a choice, pick the one with the highest payout for the same outcome. (Casinos bank on the fact that people won't check.)

Grouping and regrouping

Suppose you have 18 comic books: 8 of excellent quality, 5 of good quality, 3 of poor quality, and 2 of no value. You randomly mix the comic books into a pile, and you take five off the top.

- What's the chance that all five comic books are of excellent or good quality?
- What's the chance that the five comic books chosen represent each of the four quality groups?

First, you notice that the denominator for these problems is the same; you choose 5 comic books from 18, without replacement and with no regard for the order, so you have a combination "18 choose 5," which is 8,568.

When figuring the numerators in each case, notice that this example forces you to regroup the comic books because of the questions I pose. In the first problem, you want all four comic books to be of excellent or good quality. To do this problem, you can put all the excellent- and good-quality comic books together ($8 + 5 = 13$ total) and choose the five from there. You have "13 choose 5" = 1,287 ways to choose, which is your numerator. So, the probability of choosing five comic books rated good to excellent is $1{,}287 \div 8{,}568$, which is 0.15, or 15 percent.

The second problem calls for representation from all four quality levels. You must distinguish the different comic books coming from each quality level, so you choose them separately. Of the eight of excellent quality, you choose one; of the five of good quality, you choose one; of the three of poor quality, you choose one; and of the two of no quality, you choose one. But that takes care of only four of the comic books. The fifth book can come from any category, so put all the remaining $18 - 4 = 14$ books together in one pile and choose one. This gives you a total of $\binom{8}{1}\binom{5}{1}\binom{3}{1}\binom{2}{1}\binom{14}{1} = 8 * 5 * 3 * 2 * 14 = 3{,}360$ ways to choose. Therefore, the probability of representing all four quality levels is $3{,}360 \div 8{,}568$, which is 0.39, or 39 percent.

REMEMBER

With combination or permutation problems, you need to visualize the process, break the parts down, consider if the order of the items chosen is important or not, and make sure you cover all possibilities. This strategy gives you direction and the organizational skills you need to face the next problem that comes along.

Chapter 6

Against All Odds: Probability in Gaming

In This Chapter

- Shuffling through the basic concepts of popular games of chance
- Considering expected value and probability when playing the lottery
- Dueling with probability on the slot machines
- Spinning the roulette wheel with probability in mind
- Yelling over the complicated mess that is BINGO
- Surviving the phenomenon called gambler's ruin

Casinos don't use the word "gambling" anymore — they prefer the term "gaming." It has a better, more positive sound to it, they figure. Probabilists use the subject of gaming as a vehicle to study any games that involve chance. The games may or may not involve a skill component, or a model for measuring your chances against various opponents, but gaming theory can get very involved and intricate — to the point of programming a computer that can actually "think" and beat any player at chess no matter how many games are played or strategies it faces.

Entire books focus on the subject of probability in gaming — how to win at blackjack, how to beat the odds at poker, the best moves to make in chess given any possible combination of pieces on the board, the strategies for most any game of chance out there, and so on. I don't go into all the nuances and details involved with all the various games of chance in this chapter, but I do hit the highlights regarding gaming probabilities and strategies. I give you an overview of how gaming bodies set up some of the more popular games of chance, how probability is involved in the games of chance, how to assess your overall chance of winning or losing, and some tips to help you win more (or at least lose less). I also share one of the most famous and intriguing probability problems: the birthday problem. Although this example doesn't actually cover a game (you won't find the birthday problem in a casino anywhere), you can use it to dazzle your friends (professors use the problem all the time to try to dazzle their students).

Knowing Your Chances: Probability, Odds, and Expected Value

The first rule of gaming, from the point of view of the player: Know the rules. The second rule of gaming: Know your chance of winning. Although most people who enjoy games of chance do take time to figure out the rules, few players take the time to really understand their chances of winning and losing. The topic isn't fun to think about, after all, because it makes the games appear as if who's playing them has no bearing on the outcome; chances suck all the fun out of the process. But that attitude is what the casino (also referred to as "the house") is betting on. It wants you to feel in control and lucky, like you can beat the odds, run the table, and buy the dealer a brand new car on your way out. And although I have seen this happen in a casino in Las Vegas, I've also seen plenty of people pour money into games with no payout, hour after hour after hour.

Some of the money people gamble with is fun money (my grandma always stopped when she lost a roll of nickels), and they like the entertainment value, but other losses are more profound. Players can avoid big losses if they understand and put into perspective the actual probabilities of winning and losing, as well as the overall expected amount of winnings and losses (in other words, the perspective that the casinos don't give you).

TIP

It's easier to think about how much you can win when you play a game of chance than to think about how much you can lose. In order to have a realistic assessment of how good a game is for your wallet or purse before you play, figure out what your expected value is per play. The *expected value* is the overall long-term average amount you'll win (or lose) per play. Use the ideas from Chapter 7, regarding expected value of a random variable, as the basis of your calculations.

Let X be the amount you can win at a game of chance. You indicate a loss, therefore, by a negative value of X. X is a discrete random variable (see Chapter 7) with a probability distribution p(x); that is, X can take on a finite number of possible values, with certain probabilities. The expected value of X is the weighted average of all the values of X taken over the long term. The notation is E(X). To find E(X), you take each value of X and multiply by its probability; do this for all values of X, and take the sum. The formula for the expected value of X is $E(X) = \sum xp(x)$.

In the following sections of this chapter, you discover how to figure your expected gain (or loss) per play for games of chance, and you find out how to use that information to make good decisions when gaming.

Playing the Lottery

Many different types of lotteries exist, but the big idea is that people choose combinations of three to six numbers from a selected range of numbers (usually 1 to 30 or 1 to 40). The people then bet on these numbers by buying lottery tickets for each combination they choose (the tickets usually cost one dollar each), and they wait for the drawing to occur. Sometimes lotteries allow for numbers to repeat (for example, 1-1-1-1-1-1), and sometimes they don't. The losing tickets sometimes have a second-chance drawing for a smaller prize, but most of the time they end up as garbage, and the cycle starts all over again.

In this section, you find out how to calculate probability for the lottery, and you examine your expected value per ticket purchased. With this knowledge, you can make informed decisions before you make your lottery purchases (even if the informed decision is to stay away from the lottery!).

Mulling the probability of winning the lottery

The probability of winning the lottery seems much higher than it actually is. How hard can it be to match only six balls? The problem is, with each added ball, the probability of winning becomes smaller; this is mainly due to the *multiplication rule* (see Chapter 2). The multiplication rule says that if the trials are independent (which they are with picking lottery numbers), the probability of having six events occur at the same time is the product of all their individual probabilities. For example, if you flip a coin six times, with the probability of getting heads being 0.50 each time, the chance of matching all six outcomes for the six flips is $0.50 * 0.50 * 0.50 * 0.50 * 0.50 * 0.50 = 0.0156$. So, it happens only 1.56 percent of the time. Because you multiply by $\frac{1}{2}$ each time, you cut down the probability by another factor of two each time you add another flip.

Pick three

With the pick-three lottery, you typically pick three numbers from zero to nine with repeats allowed. On each pick, you have ten possible choices; the total number of possible outcomes for pick-three numbers, using the multiplication rule (see Chapter 2), therefore, is $10 * 10 * 10 = 1{,}000$. The chance of getting all three numbers correct in the exact order in which the numbers are chosen is $\frac{1}{1,000} = 0.001$.

Powerball

In the Powerball lottery, you have to match five numbers in any order, from 1 to 55, with no repeats allowed. The sixth number you choose is a special ball, called the "Powerball," with numbers from 1 to 42; if you match it along with the other five numbers, your winnings go up dramatically. The total number of combinations of the first five numbers drawn is "55 choose 5", because you choose 5 numbers from 55 without replacement, and the order doesn't matter. (See Chapter 5 for more on combinations.)

The combination "55 choose 5" turns out to be a whopping 3,478,761 possible combinations, just for the first five numbers. Here's the work to show this combination: $C_5^{55} = \binom{55}{5} = \frac{55!}{(55-50)!5!} = \frac{55!}{50!5!} = \frac{55*54*53*52*51*\cancel{50!}}{\cancel{50!}*5*4*3*2*1} = \frac{55*54*53*52*51}{120} = 3{,}478{,}761.$

When you add on the Powerball number, you have to account for 42 possible numbers for the selection. You have "42 choose 1" = 42 ways to choose the Powerball. In total, using the multiplication rule (see Chapter 2), the amount of possible Powerball number combinations is 3,478,761 $*$ 42 = 146,107,962. Wow!

Now think about the probability of winning the big Powerball jackpot. You need to match all five numbers in the first set — "5 choose 5" = 1 way to do this. You have "1 choose 1" way to match the Powerball number. So, the number of ways to match all six numbers is only one. Therefore, your chance of winning the mega jackpot is $1/146{,}107{,}962$.

How do you put a number that small into perspective? Imagine a pile of papers that contains one lottery ticket for every single combination possible. In other words, you have a pile that contains 146,107,962 pieces of paper. One of those pieces is yours. You get one chance to reach into this pile and pull out your winning ticket — like finding a needle in a lottery paper stack. But if you want to play, more power (ball) to you!

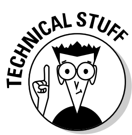
WARNING!

All kidding aside, you need to understand your odds of winning and losing before you play games of chance. With the Powerball lottery, many people like to increase their chances of winning by purchasing a bundle of tickets — 100, for example. Someone to whom $100 means a lot in terms of his paycheck may think that his odds really go up if he buys 100 lottery tickets rather than just 1. But what does this piling on really do? It increases the odds to $100/146{,}107{,}962$, which is 0.000000684, from the original number of $1/146{,}107{,}962$, which is 0.000000007. (Table 6-1 gives you the breakdown of the winning outcomes, their payouts, and their probabilities; don't look if you don't want to be depressed.) You're much better off buying only one ticket and spending the other $99 on the necessities of life. However, many people don't understand the true magnitude and small relative difference in these probabilities, and they can run into severe money trouble.

Figuring the odds

Speaking about the odds of an outcome happening isn't the same as speaking about the probability of an outcome happening, although the terms do get a great deal of misuse. You find a probability (fraction) by taking the number of ways to get the desired outcome divided by the total number of outcomes. Odds are ratios of the number of ways to lose (not get the desired outcome) to the total number of ways to win (get the desired outcome). Say, for example, the probability of getting a 1 when you roll a fair die is $\frac{1}{6}$. That means the odds of rolling a 1 are 5 to 1.

Going the other way, suppose the odds of a horse winning are 3 to 1. That means the probability of winning is $1 \div (3 + 1)$, or $\frac{1}{4}$. (Remember, the first number in the odds is the number of ways to lose, not the number of ways to win.) The denominator of a probability is the total number of outcomes, which includes both the number of ways to win and the number of ways to lose, so you sum both numbers in the odds ratio to find the denominator of the corresponding probability.



Odds are used most often in discussing horse races. A horse that has higher odds of winning is favored more than other horses by the people betting. In fact, the amount of money bet for/against a horse sets the odds for that horse. However, horses that have lower odds of winning — say, 70 to 1 (that is, the probability of the horses winning a race is predicted to be 1 out of 71 by the betters) — have much higher payouts when they do win. This is the element that balances the scales; you can go with the "sure thing" and win a few cents per dollar bet, or you can go for the "long shot" and cash in big every once in a (long) while.

Finding the expected value of a lottery ticket

Finding the expected value of a lottery ticket involves finding all the possible winning payouts (minus the cost per ticket), along with their probabilities, and multiplying them together, using the regular expected value formula for E(X) (see Chapter 7).

Expected value of a pick-three ticket

In the pick-three lottery, the lottery picks three numbers at random, with repeats allowed. Say, for example, that you win $500 for matching all three numbers in the right order. Therefore, if you play the pick-three to get all three numbers in the right order, your expected return is $\frac{1}{1{,}000} * \$500 + \frac{999}{1{,}000} * \$0 = \$0.50$. If you play the game many times, you'll win an average of $0.50 each time over the long term.

The pick-three tickets typically cost you one dollar each time, so your overall expected value (your long-term overall winnings) actually is $0.50 – $1.00 = –$0.50 per ticket purchased. Therefore, your overall expected winnings are really overall expected losses. And the more you play, the more you can expect to lose, in real dollars. What this means is that the house (the lottery folks) has an advantage (called a *house edge*) of 50 percent. In other words, in the long term, the house takes half of everyone's money (part of the profit pays off the winnings for the person who wins, but the house has plenty of money left). Considering that the payout is $500, and supposing 1,000 people play, the lottery walks away with $500.

Suppose the pick-three lottery doesn't allow repeats, but you can win with three numbers in any order. The chance of winning is $\frac{1}{1,000} * 3! = 6$, or $\frac{6}{1,000}$ (because you have six ways to rearrange, or permute, the three numbers chosen; for more on permutations, represented by the "!" symbol, see Chapter 5). For example, if you choose 123, you can win with 123, 132, 213, 231, 312, or 321. Assume that the payout for winning with three numbers in any order is $80. Your overall expected return is $\frac{6}{1,000} * \$80 + \frac{994}{1,000} * \$0 = \$0.48$ per ticket, on average. The tickets cost $1.00, so your expected value is $0.48 – $1.00 = –$0.52. The house expects to gain 52 percent.

Expected value of a Powerball ticket

A Powerball lottery ticket costs $1.00 and can be worth millions if it wins the big jackpot. If you buy one (or more) lottery ticket each week for years and years, what is your expected value of each ticket? That is, how much on average (not at any one time but over the long haul) are you expected to win or lose?

Table 6-1 shows the winning Powerball lottery outcomes, the chance of winning each one, the payout, and the expected payout over the long term for each outcome. You don't know how many millions the Powerball jackpot will be worth, because it increases each week that no one wins. However, all the other payouts are fixed. So, you can figure the overall expected value of a lottery ticket up to a point. Let k be the number of millions that the jackpot is worth.

The expected value of a lottery ticket is the probability of winning times the payout for each outcome, minus the dollar that you spent on the ticket. Because the multiplication part was already done to give you the last column of Table 6-1, just add those values up to get the overall expected value. This comes out to $0.196 + 0.007\,k - 1$ (where k is the number of millions of dollars that the jackpot is worth). Remember, you subtract the dollar that you spent on the ticket. The expected value for your lottery ticket is shown in Table 6-2 for different values of the jackpot.

Table 6-1 Payout Multiplied by Chance of Winning for Each Powerball Lottery Winning Outcome

Outcome	*Payout*	*Chance*	*Payout * Chance*
Match all 5 numbers + PB	k * $1 million	1 ÷ 146,107,962	k * 0.007
Match all 5 numbers only	$200,000	41 ÷ 146,107,962	0.056
Match 4 out of 5 numbers + PB	$10,000	250 ÷ 146,107,962	0.017
Match 4 out of 5 numbers only	$100	10,250 ÷ 146,107,962	0.007
Match 3 out of 5 numbers + PB	$100	12,250 ÷ 146,107,962	0.008
Match 3 out of 5 numbers only	$7	502,250 ÷ 146,107,962	0.024
Match 2 out of 5 numbers + PB	$7	196,000 ÷ 146,107,962	0.009
Match 1 out of 5 numbers + PB	$4	1,151,500 ÷ 146,107,962	0.032
Match 0 out of 5 numbers + PB	$3	2,118,760 ÷ 146,107,962	0.043

Table 6-2 Expected Value of a Lottery Ticket for Different Jackpot Amounts

Jackpot Size	*k (number of millions)*	*Expected Value of Ticket*
$1,000,000	1	0.196 + 0.007 * 1 – 1 = $–0.80
$2,000,000	2	0.196 + 0.007 * 2 – 1 = $–0.79
$5,000,000	5	0.196 + 0.007 * 5 – 1 = $–0.76
$10,000,000	10	0.196 + 0.007 * 10 – 1 = $–0.73
$50,000,000	50	0.196 + 0.007 * 50 – 1 = $–0.45
$100,000,000	100	0.196 + 0.007 * 100 – 1 = $–0.10
$114,857,143	114.857143	$0 (the break even amount)
$150,000,000	150	0.196 + 0.007 * 150 – 1 = $0.25
$200,000,000	200	0.196 + 0.007 * 200 – 1 = $0.60
$300,000,000	300	0.196 + 0.007 * 300 – 1 = $1.30

Why not buy all the lottery numbers?

You would win a lottery like the Powerball for sure if you bought tickets for all the possible combinations, right? So why doesn't anybody do it? First, you need at least $146,000,000 to cover all the combinations. (And if you have that kind of money to spend, do you really need to play the lottery? What would Oprah do?) But even if you have that kind of money, you need to figure out your expected value of that money; in other words, what are your expected winnings if you buy all the tickets? If the lottery jackpot isn't over $146,000,000, it would be pointless to cover all the numbers because you'd spend more than you'd bring in. But when the pot gets up in the 200- or 300-million-dollar range, you may start thinking, " What's to stop me?"

What should stop you is that you may have to split the jackpot with another player who also chose the winning combination. Have you noticed that as the lottery jackpot increases, more and more people start buying tickets? And when the pot reaches huge numbers, like 350 million, every dreamer and his brother start making a beeline for the borders of states that sell lottery tickets because everyone wants that jackpot. What the money chasers don't realize, though, is that because so many more people are out there buying tickets for the jackpot, the chance of multiple people buying the same ticket gets higher and higher — until the chance of at least two people drawing the same set of numbers is nearly one. And if a whole bunch of people pick your winning combination, you have to split the pot many more ways.

Your state's lottery will sell everyone the same set of numbers if that's what the like-minded people want; it won't make a set of numbers unavailable when another player picks it. Therefore, you always have a chance of splitting the jackpot. My advice is, if you want to buy a lottery ticket, buy one when the lottery is fairly large but not big enough to bring everyone out of the woodwork.

So, if the jackpot is a "measly" $1 million, the expected return on one ticket is –80 cents, or a loss of 80 cents. Even with a 100-million-dollar jackpot, your expected value per ticket is –10 cents. You can figure out the "break even point" — that is, where your expected value would be zero. To figure this, set the formula for the expected value equal to 0 and solve for k, the number of millions that the jackpot has to be. So, you have $0.196 + 0.007 * k - 1 = 0$. That means $0.007 * k = 1 - 0.196 = 0.804$. You find that $0.007k = 0.804$. Divide through by 0.007 to get $k = 114.8571429$. Remember, k is in millions, so the jackpot would have to be as high as $114,857,143 for the expected value of your ticket to be zero. From there, it's all uphill.

Why is the expected value of a lottery ticket so low? What you may not know is that for every dollar put into the lottery, at least 50 cents goes to the state's general fund, with some states taking as many as 70 cents on the dollar.

Note that these calculations for expected value assume you win the jackpot all by your lonesome, without having to split with anyone. The chance of this happening decreases as the size of the jackpot increases, and that makes your expected value decrease as well.

Expected value for a lottery ticket means the overall expected amount you will win (or lose) in the long term, if you buy tons and tons of tickets (actually, the probabilities given in Table 6-1 assume you are doing it into infinity). An expected value doesn't give you any perspective into the amount you'll win or lose in the short term, because you either lose your dollar, or you win a prize — that's all that can happen. So go ahead and spend that dollar if you want to buy a ticket; who knows? Like my dad says, "Somebody's gotta win; it might as well be me." Just keep your odds and expected value in mind and don't go overboard buying more tickets. After all, if your chance of winning the jackpot is 0.000000007 for one ticket, the chance of winning the jackpot is only twice that for two tickets.

Hitting the Slot Machines

I'll always remember the movie *National Lampoon's Vegas Vacation,* when gambling fever consumes Chevy Chase's character, Clark W. Griswold. He goes on a losing streak to beat all losing streaks while his son, Rusty, wins four cars by playing the slot machines. Maybe Clark would've done better if he had read this book! In this section, I cover the basic ideas behind slot machines and how they work so you can get past the myths and develop a strategy based on sound probability. And if you prefer not to gamble, you can always give out advice for a little fee! Figure 6-1 displays a typical slot machine.

Understanding average payout

When casinos advertise that their slot machines pay out an average of 90 percent, the fine print they don't want you to read says that you lose 10 cents from each dollar you put into the machines in the long term. (In probability terms, this advertisement means that your expected winnings are minus 10 cents on every dollar you spend every time the money goes through the machines.)

Suppose you start with \$100 and bet a dollar at a time, for example. After inserting all \$100 into the slot, 100 pulls later you'll end up on average with \$90, because you lose 10 percent of your money. If you run the \$90 back through the machine, you'll end up with 90 percent of it back, which is 0.90 * 90 = \$81. If you run that amount through in 81 pulls, you'll have \$72.90 afterward (0.90 * 81 = 72.90). If you keep going for 44 rounds, on average, the money will be gone, unless you have the luck of Rusty Griswold!

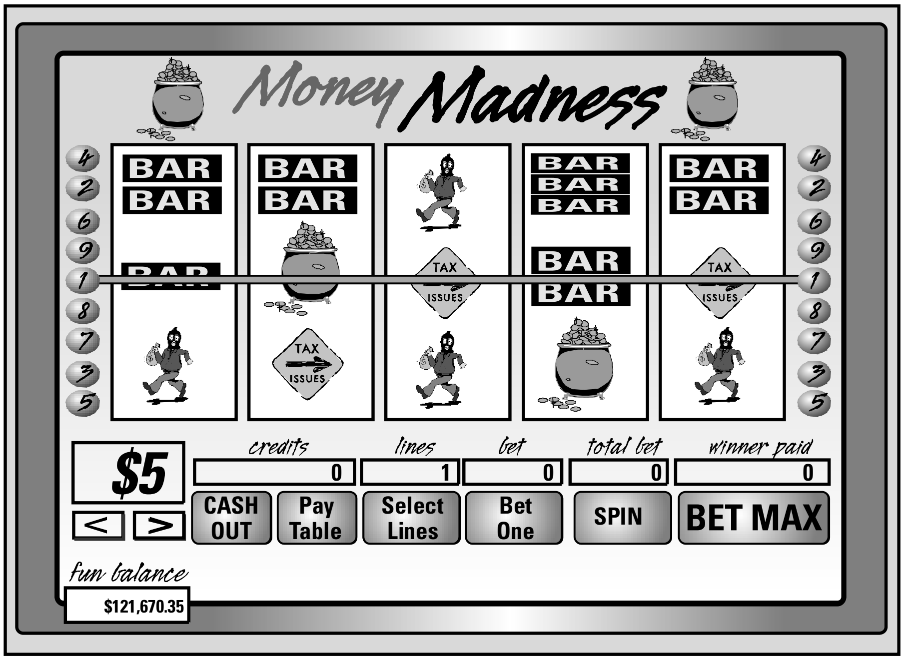


Figure 6-1: A Vegas-style slot machine (Clark W. Griswold, eat your heart out!).

How many pulls on the machine does your $100 give you at this rate? Each time you have less money to run through the machine, so you have fewer pulls left. If you insert $1 at a time, you can expect 972 total pulls in the long term with these average payouts (that's the total pulls in 44 rounds). But keep in mind that casinos are designing slot machines to go faster and faster between spins. Some are even doing away with the handles and tokens by using digital readouts on gaming cards that you put into the machines. The faster machines can play up to 25 spins per hour, and 972 spins divided by 25 spins per minute is 38.88 minutes. You don't have a very long time to enjoy your $100 before it's gone!

WARNING!

The worst part? Casinos often advertise that their "average payouts" are even as high as 95 percent. But beware: That number applies only to certain machines, and the casinos don't rush to tell you which ones. You really need to read or ask about the fine print before playing. You can also try to check the information on the machine to see if it lists its payouts. (Don't expect this information to be front and center.)

Unraveling slot machine myths

Slot-machine dwellers worldwide share many popular beliefs regarding slot machines; some are surely false, but others may or may not be true, depending on whom you ask (maybe the casino employee isn't your most reliable source). I address many popular myths in this section; it's up to you to decide on their reliability.

Slot machines stop on all possible outcomes with equal probability

Wouldn't it be nice if all the outcomes on a slot machine had the same probability of popping up in the windows? Alas, this belief isn't true. Most slot machines have 22 positions on which to stop. Each machine contains a computerized random-number generator that continuously generates a new random number every-so-many-thousandths of a second (whether or not anyone happens to be playing). When you hit the spin button (or pull the handle), the computer inside the machine selects the particular random number that the generator spits out at that precise moment. The computer then looks up that number on a table that tells the machine which position to stop on.

Suppose the slot machine uses a four-digit random number; the machine generates 10,000 possible random numbers and has only 22 possible stops, so this gives the house plenty of leeway in assigning those numbers. For example, it could decide that numbers 0000–5,550 stop on orange-banana-cherry; numbers 5,551–7,778 stop on cherry-cherry-7; and so on. Finally, at the end of the cycle, the casino could have 9,999 stop on 777. With this information, you can see that the 22 positions (or stops) on a slot machine aren't equally likely, but the outcomes are generated randomly in a fair way.

Someone immediately hits a jackpot after you leave a machine; you would've won that pot had you stayed

Regretting your decision to leave a machine after you see it hit a jackpot is unnecessary, because the machine's random number generator is constantly creating random numbers. The exact moment you hit spin, you're in essence determining your destiny. You probably wouldn't have hit the spin button at the exact same time as the next guy, but it still hurts, doesn't it?

Machines are programmed to "heat up" during certain times

Most people believe that machines can "go on a roll" and spit out winner after winner during certain times; because the machines use random-number generators behind the scenes, what's perceived as a hot streak is just probability happening before your eyes. Sometimes machines produce streaks just by chance. In fact, if people didn't experience hot streaks, they would spread more suspicion that the machines are "rigged." (See Chapter 21 for more probability misconceptions like this.) Plus, gaming facilities have to abide by certain rules and regulations that disallow this kind of outcome setting.

If you've been sitting at a machine that hasn't won in a long time, stay with it — it's due to hit soon

Gaming environments often cause people to assume that some "law of averages" should kick in after a while. But the law of averages gives this myth a kick in the pants. The *law of averages* says that over a long period of time, the average payout on a slot machine will be exactly what it's expected to be, given the odds of winning and the payouts offered. It doesn't say that you should hit a big jackpot every 10,000 spins. Games of chance have no prior memory of what players have recently done (even Blackjack now uses five to seven decks, all shuffled together, to avoid card counting). So, in the gaming world, there's no such thing as being "due for a hit" or being "on a roll." (However, casinos *want* you to believe these magical ideas so you walk out the doors with empty pockets!)

Machines set up in different places in the casino have better payouts

You may hear people preaching about the rewards of finding similar machines set up in different areas of the casino. I can't say for sure if these machines have better payouts, but I will say that I myself believe it to be true. Casinos have to compete for your attention and your money. And what better way to win your affections than by having the machines near the entryway winning when you walk by? Other popular places include the ends of the aisles or near the restrooms. Although I can't prove for sure that these machines have better payouts, I do know that every time I'm in a casino, I can't get a seat near the doorway, on the end of a row, or near the restroom.

TIP

One place to avoid, I've heard, is the machines by the blackjack tables. The blackjack players don't want to be distracted with all the winning going on, and the dealers want to keep players at their tables to lose money at blackjack, so the slot machines near blackjack tables may have fewer payouts. None of this is based on scientific theory, however. (Why would casinos let a probabalist into their establishments? They'd have to be crazy!)

Implementing a simple strategy for slots

Advice varies regarding whether you should play nickel, quarter, or dollar slot machines and whether you should max out the number of coins you bet or not (you usually get to choose between one and five coins to bet on a standard slot machine). In this section, I present a few tips for getting the most bang for your buck (or nickel) when playing slot machines.

Basically, when it comes to slot machines, strategy boils down to this: Know the rules, your probability of winning, and the expected payouts; dispel any myths (see the previous section); and quit while you're ahead. If you win $100, cash out $50 and play with the rest, for example. After you lose a certain amount (determined by you in advance), don't hesitate to quit. Go to the all-you-can-eat buffet and try your luck with the casino food; odds are it's pretty good!

Choosing among nickel, quarter, and dollar machines

The machines that have the higher denominations usually give the best payouts. So, between the nickel and quarter slots, for example, the quarter slots generally give better payouts. However, you run the risk of getting in way over your head in a hurry, so don't bet more than you can afford to lose. I find that the nickel slots are more fun because my two-buck limit lasts longer before I head over for another free margarita! The bottom line: Always choose a level that you have fun playing at and that allows you to play for your full set time limit.

Deciding how many coins to play at a time

When deciding on the number of coins you should play per spin, keep in mind that more is sometimes better. If the slot machine gives you more than two times the payout when you put in two times the number of coins, for example, you should max it out instead of playing single coins because you increase your chances of winning a bigger pot, and the expected value is higher. If the machine just gives you k times the payout for k coins, it doesn't matter if you use the maximum number of coins. You may as well play one at a time until you can make some money and leave so your money lasts a little longer.

For example, say a quarter machine pays 10 credits for the outcome 777 when you play only a single quarter, but if you play two quarters, it gives you 25 credits for the same outcome. And if you play the maximum number of quarters (say, four), a 777 results in 1,000 credits. You can see that playing four quarters at a time gives you a better chance of winning a bigger pot in the long run (if you win, that is) compared to playing a single quarter at a time for four consecutive tries.

The latest slot machine sweeping the nation is the so-called "penny slot machine." Although it professes to require only a penny for a spin, you get this rate only if you want to bet one penny at a time. The machines entice you to bet *way* more than one penny at a time; in fact, on some machines, you can bet more than 1,000 coins (called *lines*) on each spin — $10 a shot here, folks. Because these machines take any denomination of paper bill, as well as credit cards, your money can go faster on penny machines than on dollar machines because you can quickly lose track of your spendings. Pinching pennies may not be worth it after all!

Spinning the Roulette Wheel

The roulette wheels found in casinos often draw crowds with their flashy colors and seemingly nonstop excitement and action. And the casinos love the attention; because it's so hard to win money playing the roulette wheel, the casinos are more than happy to provide an exciting atmosphere in return for your money. In this section, you find out how the wheel works and what your chances of winning and losing really are.

Covering roulette wheel basics

The American version of the roulette wheel has 36 numbers from 1 to 36, made up of 18 black numbers and 18 red numbers. It also includes one 0 and one 00 ("double zero"), both colored green, yielding a total of 38 possible outcomes. Figure 6-2 shows an American-type roulette wheel. Other types of roulette wheels exist (including a European version), but they're all very similar.

Figure 6-2: An American-type roulette wheel.

The betting layout consists of every individual number as well as a host of "outside" combinations of numbers (see the upcoming section "Making outside and inside bets" for more). After the players make their bets, the dealer spins the wheel and drops a ball in after several seconds; the ball will eventually land in one of the numbered slots. The 38 outcomes are equally likely to come up.

REMEMBER

Players new to roulette need to be aware of certain etiquette standards associated with the casino game. You have to bet the minimum on the specific bet you make. For example, if the minimum at the roulette wheel is $10, and you want to bet on red, you have to bet at least $10 on red; you can't cheap out and put $5 on red and $5 on something else. Also, dealers get really annoyed if you touch your chips inside the betting area after they close the betting window and before they pass out your winnings (or take your chips away if you lost). No touching allowed, so sit back and relax (or bite your nails).

Making outside and inside bets

In roulette, the action comes in the form of different types of bets that players make on what outcome (or number) the ball will drop into on a given spin. The betting table (where you put your chips down prior to the spin) is shown in Figure 6-3. The roulette table is just that, a table, with 3 columns and 12 rows.

Placing an outside bet

The core of the roulette board contains the numbers 1 through 36, organized in 12 rows of 3 numbers each (row one being 1-2-3, row two being 4-5-6, and so on), plus 0 and 00 on top. On the edges of the table are places for outside bets. You can place an outside bet on a group of numbers or on some binding characteristic of the numbers. All outside bets lose if the ball drops on 0 or 00.

Here are the characteristics of the numbers you can bet on for outside bets; you make these bets by placing your chips in the marked box that represents the particular characteristic:

- **Red-Black:** A bet that the winning number will be of the color you bet on. This bet pays even money: 1 to 1.

  REMEMBER

  A 1 to 1 payout means that if you win, you get your bet back plus the amount of your bet. If the odds are 2 to 1, you get your bet back, plus two times your bet, and so on down the line.
- **Odd-Even:** A bet that the winning number will be either odd or even. This bet pays even money: 1 to 1. ***Note:*** 0 and 00 are neither odd nor even in this game.
- **Low-High:** A bet that the winning number will fall in the range of lower numbers (1–18) or the range of higher numbers (19–36). This bet pays even money: 1 to 1.
- **Columns:** A bet on the 12 numbers contained in any one of the three long columns on the layout. At the end of these columns, you find places (usually labeled 2 to 1) where you can place your chips. This bet pays 2 to 1; in other words, you get your bet back plus two times your bet.
- **Dozens:** A bet on either the first dozen numbers (1–12), the second dozen (13–24), or the third dozen (25–36). This bet pays 2 to 1.

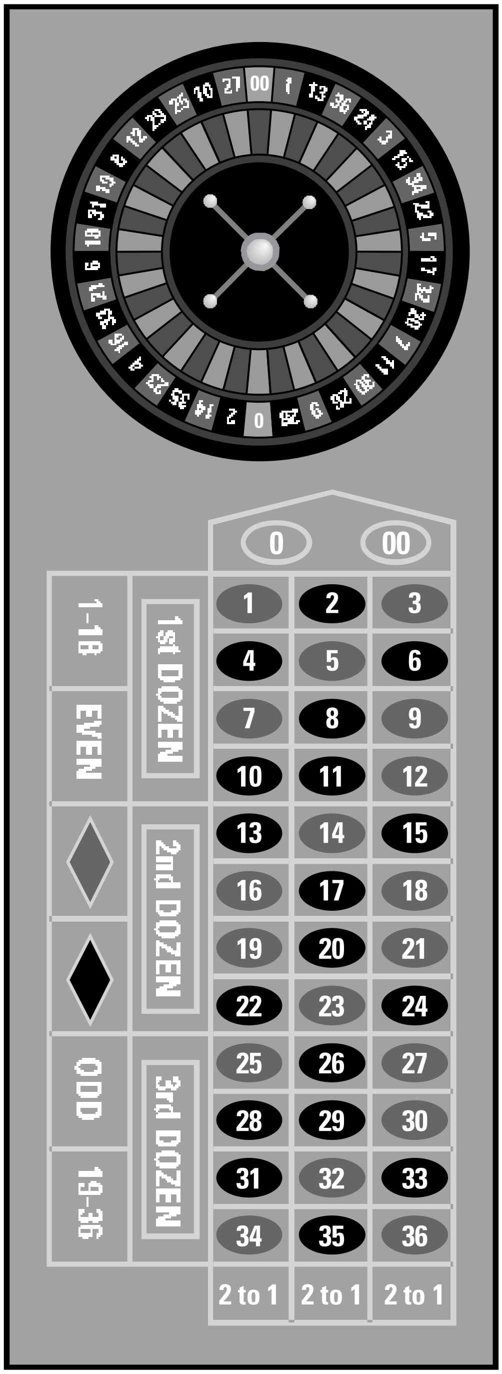


Figure 6-3: Diagram of the betting table for an American roulette wheel.

Placing an inside bet

You place inside bets on specific numbers or combinations of numbers either within the number layout or on the borders (the lines that separate numbers and inside/outside bets). Here are a few different types of inside bets that you can typically make in a casino:

- **Straight bet:** A bet made on a single number (including 0 or 00). You place this bet by putting your chips on the number you want. This bet pays 35 to 1 and is known by gamblers as a “long shot” — it rarely happens, but when it does, you win big.
- **Split bet:** A bet made on any two numbers that sit next to each other on the table (including 0 and 00). To place this bet, you put your chips on the line separating the two numbers. If one of the two numbers comes up on the spin, you win. This bet pays 17 to 1.
- **Street bet:** A bet that covers three numbers in a row. You place this bet by putting your chips on the outside line of the row you want to wager on. This bet pays 11 to 1.
- **Corner bet:** A bet you make on four adjoining numbers that form a square on the table. To place this bet, you put your chips at the spot where the four numbers intersect. This bet pays 8 to 1.
- **Five-number bet:** A bet you make on the set of numbers 0, 00, 1, 2, and 3. You place this bet by putting your chips on a designated line on the table that splits the 0 and the 1. This bet pays 6 to 1.

Chances and expected payouts on roulette bets

The casino presets the payoffs for the roulette wheel. The payoffs are related to the chance of winning — outcomes with a higher chance of winning have lower payouts, and vice versa. Table 6-3 displays the probability and expected payoff, using the American roulette wheel. The probability in each situation is just the amount of numbers bet on divided by the total amount of numbers (38).

REMEMBER

To find the expected value for any single one-dollar bet, you multiply the probability of winning times the payout and add the probability of losing times –1. For all possible bets, the overall expected value per spin is –0.0526, or –5.26 percent. This value means that the house has more than a 5-percent edge over you on every spin — per dollar bet. This is a pretty hefty edge for a casino situation. It’s better than the lottery but not as good as other games such as blackjack, which takes less money on average from you if you’re a good player. (When skill plays a role, you can do better than when your results are totally up to chance.)

Table 6-3 Roulette Probabilities and Expected Payouts

Type of Bet	*Probability*	*Payout*	*Amount Won per Dollar Bet (Payout minus Amount Bet)*
Red/Black	$\frac{18}{38}$	1 to 1	\$2 – \$1 = \$1
Odd/Even	$\frac{18}{38}$	1 to 1	\$1
Low/High	$\frac{18}{38}$	1 to 1	\$1
Dozens (of 12)	$\frac{12}{38}$	2 to 1	\$2
Columns (of 12)	$\frac{12}{38}$	2 to 1	\$2
Five-Number Bet	$\frac{5}{38}$	6 to 1	\$6
Corner (4-number) Bet	$\frac{4}{38}$	8 to 1	\$8
Street (3-numbers-in-a-row) bet	$\frac{3}{38}$	11 to 1	\$11
Split (2-number) bet	$\frac{2}{38}$	17 to 1	\$17
Straight (1-number) bet	$\frac{1}{38}$	35 to 1	\$35

Developing a roulette strategy

Many people think there's little to discuss when it comes to roulette strategy. This pessimism is in part true, because in almost all situations under the typical rules for roulette, the expected value is the same no matter what bet you make. So, the key is to find casinos that let you tweak the typical rules a bit.

Atlantic City casinos, for the most part, entertain a beneficial rule called "surrender" that applies only to outside bets that pay even money — red-black, even-odd, and high-low. If the ball lands on 0 or 00 (double zero), you lose only half your bet. The casinos edge drops to about 2.63 percent — still not in your favor but a heck of a lot better than 5.26 percent.

Another roulette strategy you can employ is to look for wheels that don't contain a double zero. European roulette wheels come this way standard, but American tables rarely exclude them because it decreases the house edge. If 00 is missing from the wheel, the house edge drops to $\frac{1}{37}$ = 2.7 percent.

Betting the farm against the advice of probability

Sometimes people do win big on the roulette table, and if they quit while they're ahead, they can walk away happy. Ashley Revell of London made international news in 2004 when he sold all his clothes and everything he owned — which totaled to $135,300 — brought it all in to the Plaza Hotel in Las Vegas, and bet it all on red at the roulette wheel in a double-or-nothing bet. The ball landed on "Red 7," and he walked away with his net-worth doubled to $270,600. (The key here is that he *walked away*.) Like Kenny Rogers sings, "You gotta know when to hold 'em, know when to fold 'em, know when to walk away, and know when to run."

And if all else fails, just walk away! Even Albert Einstein is noted to have given up on the game (and if this strategy is good enough for Albert Einstein, it's good enough for me). He's quoted as having said, "You cannot beat a roulette table unless you steal money from it."

Getting Your Chance to Yell "BINGO!"

You may remember experiences from growing up and playing BINGO for fun as a board game or to raise money for a local church or other charitable event. BINGO is a popular game amongst the older crowd, but it's also catching on with the new generation. One of my grandmothers played BINGO every chance she could. Come rain, sleet, hail, or snow, she was there, marking her numbers off her sheet of cards. She was a BINGO fanatic. This section gives you an overview of the game played by millions of grandmas and other crazed fans who proudly bring their good-luck trolls to BINGO night at the local club and promote "BINGO-ism" by sporting bumper stickers that say, "Outta my way, I'm headed to BINGO!"

Ways to win at BINGO

A typical BINGO card is a 5-x-5 card containing 24 numbers from 1 to 75 (with no repeats) and a "free space" directly in the middle (5 numbers in each row and column). The five columns are headed by the letters B-I-N-G-O, and each column can fall only within a certain range. The B column contains numbers from 1 to 15; the I column contains numbers from 16 to 30; the N column contains numbers from 31 to 45; the G column contains numbers from 46 to 60; and the O column contains numbers from 61 to 75. An example of a BINGO card is shown in Figure 6-4.

B	I	N	G	O
1	30	41	53	75
8	23	31	60	68
4	26		56	61
15	19	34	49	64
11	16	45	46	71

Figure 6-4: A BINGO card for your yelling pleasure.

When the game begins, the designated official starts calling randomly selected BINGO numbers one at a time. If your card has the called number on it, you cover the number with a chip or mark it with a special giant magic marker called a *dauber* (my grandmother had many of these and carried them in a special bag every time she played BINGO). As more numbers are called and covered, eventually a player will have a card that contains a straight line of five called numbers that form a corner to corner straight line (see Figure 6-5a); go down one whole column (see Figure 6-5b); or go across a whole row. (***Note:*** The “Free” center space can be part of the straight line.) Some BINGO parlors allow you to win if you hit all four corners, plus your free one in the middle (see Figure 6-5c). At this point, the player yells BINGO!

Now, I’ve heard it’s exciting to draw a great poker hand, and I know it’s fun to draw three 7s in a row on a slot machine, but the thrill of being able to jump up and shout BINGO to the rooftops is unparalleled in all gambling sport, in my opinion.

The officials check the winning card and if it’s legit, that player is declared the winner of the pot for that particular BINGO game (pots can range in size from $20 to thousands of dollars or more, depending on the number of players, the amount charged per card, and how generous the house decides to be in its payouts). If many people are playing at the same time, more than one person may yell BINGO at the same time. In this case, the pot is split up equally between the winners.

REMEMBER

Players have many ways to win at BINGO, depending on what type of game they’re playing at the time. The game is sort of like poker in that sense; people can play different versions. For example, the purpose of one game may be to not get a BINGO, but to get two diagonal BINGOs that criss-cross each other (see Figure 6-5d), making an X. Or the game may claim a winner when a player covers all the numbers on his or her card; in BINGO lingo, this is called a “blackout” (see Figure 6-5e). Clubs typically save a blackout round for the grand finale of BINGO night and give the game the biggest pot (because it requires the most endurance).

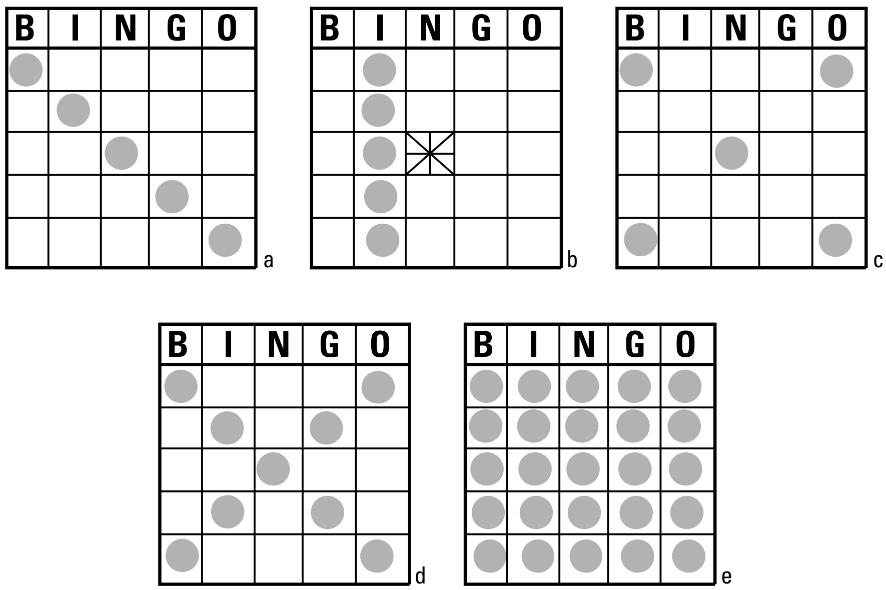


Figure 6-5: The many ways to win at BINGO — hope your voice can hold up!

The probability of getting BINGO — more complicated than you may think

The chances of winning at BINGO, unlike with other games, are quite hard to figure. The actual calculations are, in fact, so horribly messy and tedious that they require a computer to work through. However, it's worth noting that some probability calculations require computers (such as BINGO); others require combinations and permutations (such as poker and the lottery; see Chapter 5); and some require straight counting (such as roulette). It helps to see an example of a situation where computers are needed and why.

Finding the total cards possible in BINGO

Take a guess at how many different BINGO cards are out there. Remember, the same number located in two different places on two different cards gets marked off on both and has the same probability of being chosen as all the other numbers. However, *where* those numbers appear on the card makes a big difference in regard to whether you have a BINGO going somewhere and if that number in that location will help you. Every player will get a BINGO if the game drags on long enough, but the goal isn't to get a BINGO; you want to get one before the rest of the world does. So, each number placed in each different location within its given column needs to be counted differently.

To figure out the number of distinct BINGO cards, you use permutations because you sample without replacement and the order is important (see Chapter 5). For the first column, you select and rearrange 5 of the 15 possible numbers from 1 to 15 (because 5 numbers make up the column). You do the same for the second, fourth, and fifth columns (you multiply by permutations of 5 objects from 15 objects together four times). The third column has a free space, so you choose and rearrange only 4 of the 15 numbers for that column. In the end, you have the permutation of 5 items from 15, multiplied by itself four times, times the permutation of 4 items from 15. This calculation gives you the following:

$$P_5^{15} * P_5^{15} * P_5^{15} * P_5^{15} * P_4^{15} = \frac{15!}{(15-5)!} * \frac{15!}{(15-5)!} * \frac{15!}{(15-5)!} * \frac{15!}{(15-5)!} * \frac{15!}{(15-4)!} =$$

$$\left[\frac{15!}{10!}\right]^4 * \left[\frac{15!}{11!}\right] = \left[\frac{15 * 14 * 13 * 12 * 11 * \cancel{10!}}{\cancel{10!}}\right]^4 * \left[\frac{15 * 14 * 13 * 12 * \cancel{11!}}{\cancel{11!}}\right] =$$

$$[15 * 14 * 13 * 12 * 11]^4 [15 * 14 * 13 * 12] = 360{,}360^4 * 32{,}760 = 5.52447 * 10^{26}.$$

Believe it or not, the final total comes to $5.52447 * 10^{26}$ (5.52447 + E26 on your calculator). This total is more than 552 million billion billion possible BINGO cards that could exist. Now do you see why you need a computer to figure your odds of winning?

Finding the odds of getting BINGO

A computer runs through all the possible situations by using a technique called a *computer simulation.* The computer repeats tons and tons of BINGO games over and over, records all the results, and then sets up a giant table that shows all the possible cards at each possible stage of the game, along with the probability that any one of those cards will win by getting its first BINGO when a particular number is called. It's a big deal.

According to the Wizard of Odds Web site, the probability of getting a BINGO right away on the fourth number called (meaning the player gets all four corners covered right away) is only 0.000003. The chance of winning by the time the 12th number is called is 0.001995. On the 20th number, the chance of winning is 0.022874. By the 32nd number, the chance jumps to 0.188813; on the 48th number, it's about 75 percent; and the chance of winning before the 60th number is called is over 99 percent. Notice that it takes longer than you may think to get a BINGO; remember all the possibilities out there.

Now, according to my grandma, your probability goes up if you carry a lucky troll, have the right dauber, and go in on the purchase of your cards with your friend who drove you to the place (saving you on gas money). BINGO players are quite thrifty!

Knowing What You're Up Against: Gambler's Ruin

When you play any casino game that involves betting, you're playing against a house that has probability on its side. Studies will show that if you play any game long enough (without stopping), you eventually lose everything. This phenomenon is called *gambler's ruin.* Here's how you can test the theory. You and an opponent each start with a fixed number of fair pennies (you have n pennies, and your opponent has m pennies). One of you flips a penny, and the opponent "calls it" (he or she guesses on whether it will come up heads or tails while it's still in the air). If the person who calls loses, he or she gives one penny to the person who flipped. Repeat this process over and over until one player has all the pennies. If you repeat the process indefinitely, the probability that one player will eventually lose all his/her pennies must be 100 percent. And this is when the odds are even!

You can even figure out your chances of winning. Because you start with n pennies, and you designate a total of n + m pennies to be won, your chance of winning is n ÷ (n + m). That means your chance of losing (and your opponent's chance of winning) is m ÷ (n + m). Notice that these probabilities sum to one, meaning that someone has to win and someone has to lose. However, these probabilities aren't the same, unless the starting amounts for n and m are equal — the person who starts out with more pennies is more likely to win in the end.

Would you like to put your pennies up against the casino's pennies? At a casino, you're playing with denominations of 100 dollars (n) against a house that has a billion dollars (m). Your chance of going bankrupt (and the house's chance of winning) are equal to m ÷ (n + m), which equals the size of the house's pot (a billion dollars) divided by the house's pot plus your, say, $100. Ready to calculate? The probability that you'll be the one to lose is ${}^{1,000,000,000}/_{1,000,000,100} = 0.9999999$.

I didn't even talk about how long it will take to lose your money. It depends on how much you start with, how much you bet, what the odds are of winning each game, and how lucky the house is. Even if you have the edge over the casino (which never happens), you'll still go bankrupt eventually if you keep playing because of gambler's ruin and the probability that backs it up.

TIP

If you're going to gamble, the best strategy is to have a predetermined amount you're willing to lose and a predetermined amount you're willing to win and walk away with. Stick to those limits no matter what. And, of course, never set a losing amount to be more than you can afford.

The Famous Birthday Problem

There are games of chance that highlight the world of probability, and there are games you can play just for fun to show probability at work. The famous (or infamous) birthday problem is an example of the latter (and a far cheaper option). The birthday problem is a classic example of how probability and your intuition don't always mix, and it's a problem that instructors love to demonstrate in their probability classes. In this section, you discover how to find the answer to the birthday problem . . . it's very surprising!

Here's the set up. You're in a class of 25 students. What's the chance that at least two of those students share the same birthday (not the same year, necessarily, but the same day)? Do you think the probability is more than 50 percent? Before going any further, write down what you think this probability is.

The big idea is that each person in the class can compare his or her birthday with every other person in the class, so you want to count the total number of pairs of people because that gives you the total possible number of matches.

The chance of at least two people in the class who match is one minus the probability that nobody in the class matches. To find the probability that nobody in the class matches, you first find the probability that one pair of people don't match and use the multiplication rule (see Chapter 2) to multiply that probability over and over for each pair of people in the class. How many pairs of people are in the class? A class of size n has "n choose 2" pairs of people in it (using the formula for combinations from Chapter 5). This is equal to $C_2^n = \binom{n}{2} = \frac{n!}{(n-2)!2!} = n * (n-1) \div 2$.

What's the chance that a pair doesn't match? Counting 365 days in a year (ignoring leap years), the probability of finding a birthday match between any two people is $\frac{1}{365}$. You can say this because the first person can have any birthday, but the second person has to match the first person's birthday, and that person has only 1 out of 365 ways to do this. Therefore, the probability of no match is $[1 - (\frac{1}{365})]$, or $\frac{364}{365}$ (0.99726). You take this probability to the power "n choose 2" for any class size n, which gives you the chance that no student in the room matches. To get the probability that you have at least two matches, you take one minus that chance.

If your class size is 25, you find "25 choose 2" = $(25 * 24) \div 2 = 300$ pairs of people. The probability of having no birthday matches for these 25 students (assuming no twins, triplets, and so on) is $\frac{364}{365}$ to the 300th power, which is 0.4391. So, the probability of at least one match in a class of size n = 25 is $1 - 0.4391 = 0.56$, or 56 percent. This chance is much higher than most people think! However, in all my years of teaching, with class sizes from 20 to 100, the birthday problem has never let me down. (Just to make sure my students aren't "helping me out," I have them write down their birthdays on pieces of paper before we start checking to see if anyone matches.)

Table 6-4 shows the probabilities for at least one matching birthday for different class sizes.

Table 6-4 Probabilities for the Birthday Problem

Size of Class	*Probability*
10	0.12
20	0.41
25	0.56
30	0.70
50	0.97
100	Approximately 1.00