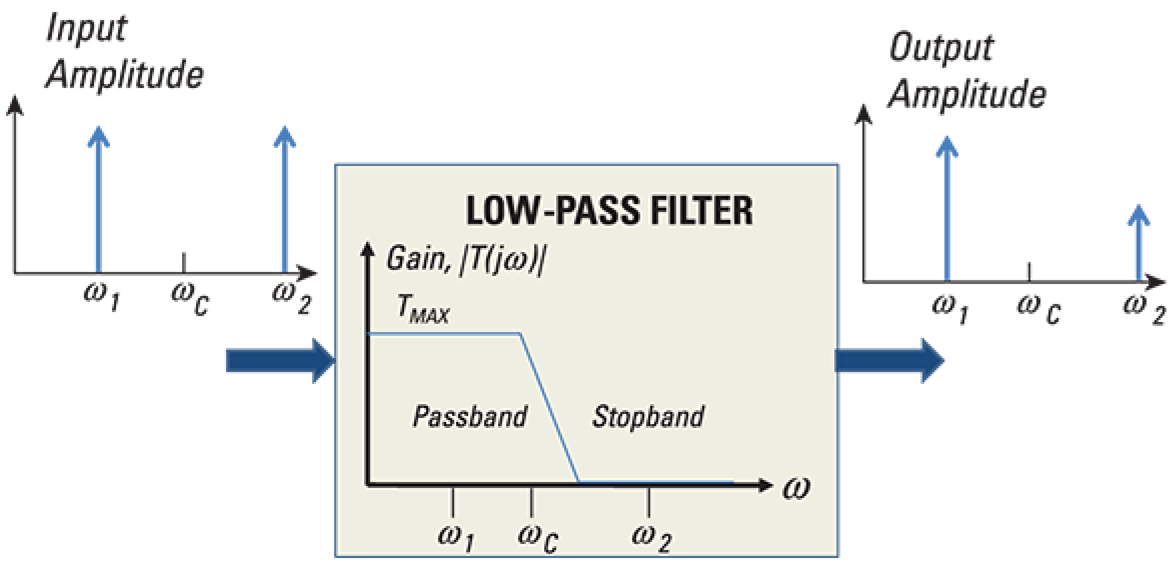
Part V

Advanced Techniques and Applications in Circuit Analysis



web extras Design a filter to improve the sound quality of a speaker system at www.dummies.com/extras/circuitanalysis.

In this part . . .

- Use phasors to describe a sinusoidal signal.
- Transform functions using the Laplace technique so you can solve problems algebraically.
- Analyze circuits that have voltage and current signals that change with time by using Laplace transforms.
- Explore filters and frequency response.

Chapter 15

Phasing in Phasors for Wave Functions

In This Chapter

- Describing circuit behavior with phasors
- Mixing phasors with impedance and Ohm's law
- Applying phasor techniques to circuits

P*hasors* — not to be confused with the phasers from *Star Trek* — are rotating vectors you can use to describe the behavior of circuits that include capacitors and inductors. Phasors make the analysis of such circuits easier because instead of dealing with differential equations, you just have to work with complex numbers. I don't know about you, but I'd take working with complex numbers to solve circuits any day over using differential equations.

Phasor analysis applies when your input is a sine wave (or sinusoidal signal). A phasor contains information about the amplitude and phase of the sinusoidal signal. Frequency isn't part of phasor form because the frequency doesn't change in a linear circuit.

This chapter introduces phasors and explains how they represent a circuit's *i*-*v* characteristics. I then show you how phasors let you summarize the complex interactions among resistors, capacitors, and inductors as a tidy value called *impedance*. Finally, you see how phasors let you analyze circuits with storage devices algebraically, in the same way you analyze circuits with only resistors.

Taking a More Imaginative Turn with Phasors

A phasor is a complex number in polar form. When you plot the amplitude and phase shift of a sinusoid in a complex plane, you form a phase vector, or phasor.

> As I'm sure you're well aware from algebra class, a complex number consists of a real part and an imaginary part. For circuit analysis, think of the real part as tying in with resistors that get rid of energy as heat and the imaginary part as relating to stored energy, like the kind found in inductors and capacitors.

You can also think of a phasor as a rotating vector. Unlike a vector having magnitude and direction, a phasor has *magnitude* V_A and *angular displacement* ϕ. You measure angular displacement in the counterclockwise direction from the positive x-axis.

Figure 15-1 shows a diagram of a voltage phasor as a rotating vector at some frequency, with its tail at the origin. If you need to add or subtract phasors, you can convert the vector into its x-component ($V_A \cos \phi$) and its y-component ($V_A \sin \phi$) with some trigonometry.

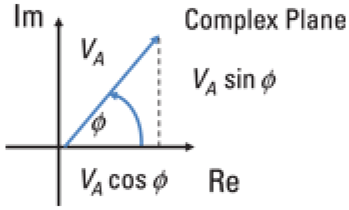


Illustration by Wiley, Composition Services Graphics

Figure 15-1: A phasor is a rotating vector in the complex plane.

The following sections explain how to find the different forms of phasors and introduce you to the properties of phasors.

Finding phasor forms

Phasors, which you describe with complex numbers, embody the amplitude and phase of a sinusoidal voltage or current. The phase is the angular shift of the sinusoid, which corresponds to a time shift t_0. So if you have $\cos[\omega(t - t_0)]$, then $\omega t_0 = \phi_O$, where ϕ_O is the angular phase shift.

To establish a connection between complex numbers and sine and cosine waves, you need the complex exponential $e^{j}\theta$ and Euler's formula:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

where $j = \sqrt{-1}$.

The left side of Euler's formula is the *polar* phasor form, and the right side is the *rectangular* phasor form. You can write the cosine and sine as follows:

$$\cos\theta = \mathrm{Re}\left[e^{j\theta}\right]$$
$$\sin\theta = \mathrm{Im}\left[e^{j\theta}\right]$$

REMEMBER Re[ ] denotes the real part of a complex number, and Im[ ] denotes the imaginary part of a complex number.

Figure 15-2 shows a cosine function and a shifted cosine function with a phase shift of π/2. In general, for the sinusoids in Figure 15-2, you have an amplitude V_A, a radian frequency ω, and a phase shift of ϕ given by the following expression:

$$v(t) = V_A \cos(\omega t + \phi)$$

$$v(t) = V_A \,\mathrm{Re}\left\{e^{j(\omega t+\phi)}\right\} = \mathrm{Re}\left[\underbrace{V_A e^{j\phi}}_{\mathbf{V}} e^{j\omega t}\right]$$

Because the radian frequency ω remains the same in a linear circuit, a phasor just needs the amplitude V_A and the phase ϕ to get into polar form:

$$\mathbf{V} = V_A e^{j\phi}$$

To describe a phasor, you need only the amplitude and phase shift (not the radian frequency). Using Euler's formula, the rectangular form of the phasor is

$$\mathbf{V} = V_A \cos\phi + jV_A \sin\phi$$

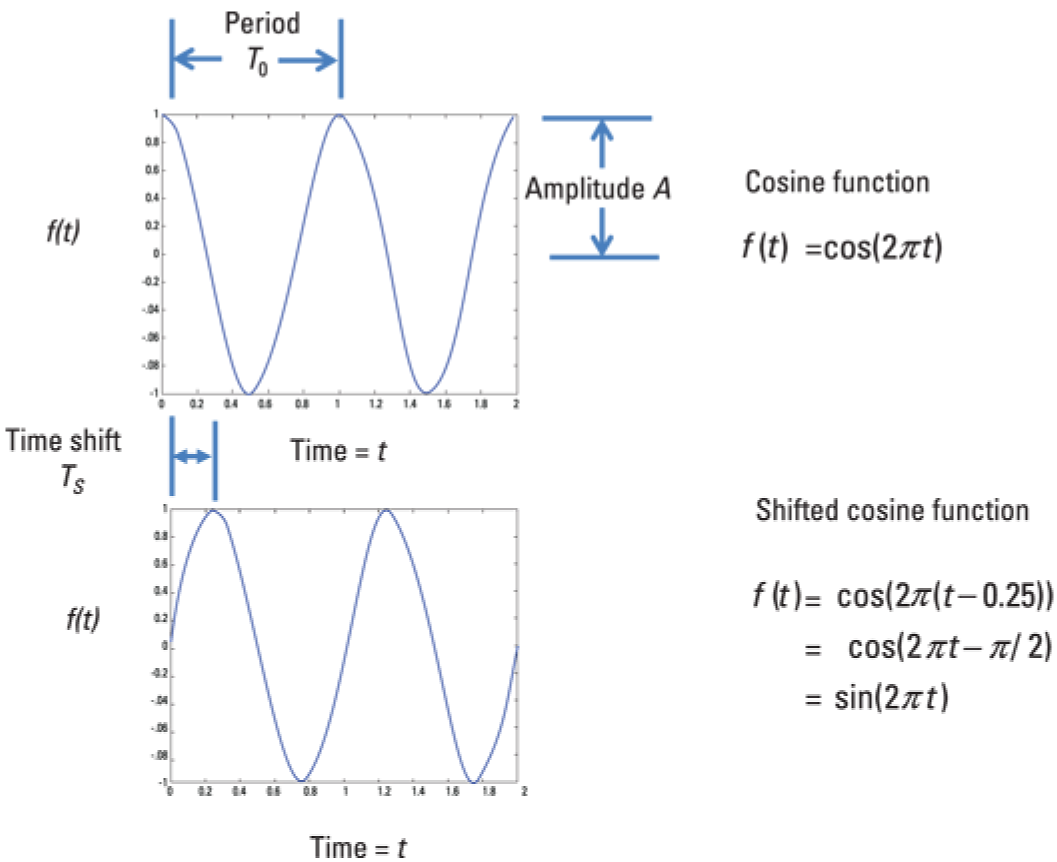


Illustration by Wiley, Composition Services Graphics

Figure 15-2: Cosine functions.

Examining the properties of phasors

One key phasor property is the additive property. If you add sinusoids that have the *same frequency,* then the resulting phasor is simply the vector sum of the phasors — just like adding vectors:

$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 + \cdots \mathbf{V}_N$$

For this equation to work, phasors $\mathbf{V}_1$, $\mathbf{V}_2$, ..., $\mathbf{V}_N$ must have the same frequency. You find this property useful when using Kirchhoff's laws.

Another vital phasor property is the time derivative. The time derivative of a sine wave is another scaled sine wave with the same frequency. Taking the derivative of phasors is an algebraic multiplication of $j\omega$ in the phasor domain. First, you relate the phasor of the original sine wave to the phasor of the derivative:

$$\begin{aligned}\frac{dv(t)}{dt} &= \frac{d}{dt}\left(\mathbf{V}e^{j\omega t}\right)\\ &= \mathbf{V}\frac{d}{dt}\left(e^{j\omega t}\right)\end{aligned}$$

But the derivative of a complex exponential is another exponential multiplied by $j\omega$:

$$\frac{dv(t)}{dt} = \left(j\omega\mathbf{V}\right)e^{j\omega t}$$

Based on the phasor definition, the quantity ($j\omega\mathbf{V}$) is the phasor of the time derivative of a sine wave phasor $\mathbf{V}$. Rewrite the phasor $j\omega\mathbf{V}$ as

$$\begin{aligned}j\omega\mathbf{V} &= \left(\omega e^{j90^\circ}\right)\left(V_A e^{j\theta}\right)\\ &= \omega V_A e^{j\left(\theta+90^\circ\right)}\end{aligned}$$

When taking the derivative, you multiply the amplitude V_A by ω and shift the phase angle by 90°, or equivalently, you multiply the original sine wave by $j\omega$. See how the imaginary number j rotates a phasor by 90°?

TIP Working with capacitors and inductors involves derivatives because things change over time. For capacitors, how quickly a capacitor voltage changes directs the capacitor current. For inductors, how quickly an inductor current changes controls the inductor voltage.

Using Impedance to Expand Ohm's Law to Capacitors and Inductors

The concept of impedance is very similar to resistance. You use the concept of impedance to formulate Ohm's law in phasor form so you can apply and extend the law to capacitors and inductors. After describing impedance, you use phasor diagrams to show the phase difference between voltage and current. These diagrams show how the phase relationship between the voltage and current differs for resistors, capacitors, and inductors.

Understanding impedance

For a circuit with only resistors, Ohm's law says that voltage equals current times resistance, or $V = IR$. But when you add storage devices to the circuit, the *i*-*v* relationship is a little more, well, complex. Resistors get rid of energy as heat, while capacitors and inductors store energy. Capacitors resist changes in voltage, while inductors resist changes in current. *Impedance* provides a direct relationship between voltage and current for resistors, capacitors, and inductors when you're analyzing circuits with phasor voltages or currents.

Like resistance, you can think of impedance as a proportionality constant that relates the phasor voltage **V** and the phasor current **I** in an electrical device. Put in terms of Ohm's law, you can relate **V, I,** and impedance *Z* as follows:

$$\mathbf{V} = \mathbf{I}Z$$

The impedance *Z* is a complex number:

$$Z = R + jX$$

Here's what the real and imaginary parts of *Z* mean:

- **The real part *R* is the resistance from the resistors.** You never get back the energy lost when current flows through the resistor. When you have a resistor connected in series with a capacitor, the initial capacitor voltage gradually decreases to 0 if no battery is connected to the circuit. Why? Because the resistor uses up the capacitor's initial stored energy as heat when current flows through the circuit. Similarly, resistors cause the inductor's initial current to gradually decay to 0.
- **The imaginary part *X* is the *reactance,* which comes from the effects of capacitors or inductors.** Whenever you see an imaginary number for impedance, it deals with storage devices. If the imaginary part of the impedance is negative, then the imaginary piece of the impedance is dominated by

capacitors. If it’s positive, the impedance is dominated by inductors.

When you have capacitors and inductors, the impedance changes with frequency. This is a big deal! Why? You can design circuits to accept or reject specific ranges of frequencies for various applications. When capacitors or inductors are used in this context, the circuits are called *filters*. You can use these filters for things like setting up fancy Christmas displays with multicolored lights flashing and dancing to the music.

The reciprocal of impedance Z is called the *admittance* Y:

$$Y = \frac{1}{Z} = G + jB$$

The real part G is called the *conductance,* and the imaginary part B is called *susceptance.*

Looking at phasor diagrams

Phasor diagrams explain the differences among resistors, capacitors, and inductors, where the voltage and current are either in phase or out of phase by 90°. A resistor’s voltage and current are in phase because an instantaneous change in current corresponds to an instantaneous change in voltage. But for capacitors, voltage doesn’t change instantaneously, so even if the current changes instantaneously, the voltage will lag the current. For inductors, current doesn’t change instantaneously, so when there’s an instantaneous change in voltage, the current lags behind the voltage.

Figure 15-3 shows the phasor diagrams for these three devices. For a resistor, the current and voltage are in phase because the phasor description of a resistor is $\mathbf{V_R} = \mathbf{I_R}R$. The capacitor voltage lags the current by 90° due to $-j/(\omega C)$, and the inductor voltage leads the current by 90° due to $j\omega L$.

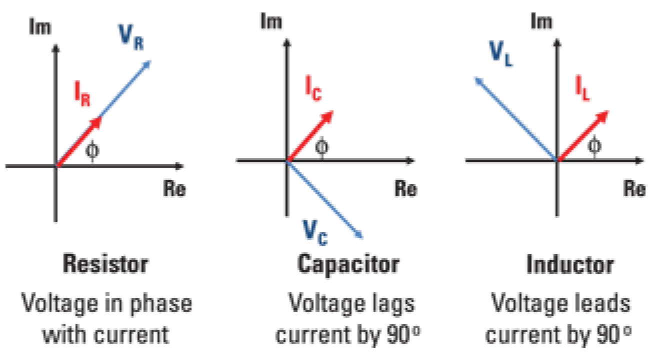


Illustration by Wiley, Composition Services Graphics

Figure 15-3: Phasor diagram of a resistor, capacitor, and inductor.

Putting Ohm’s law for capacitors in phasor form

For a capacitor with capacitance *C,* you have the following current:

$$i = C\frac{dv}{dt}$$

Because the derivative of a phasor simply multiplies the phasor by $j\omega$, the phasor description for a capacitor is

$$\mathbf{I} = j\omega C\mathbf{V} \text{ or } \mathbf{V} = \mathbf{I}\underbrace{\left(\frac{1}{j\omega C}\right)}_{Z_C}$$

The phasor description for a capacitor has a form similar to Ohm's law, showing that a capacitor's impedance is

$$Z_C = \frac{1}{j\omega C} = -j\frac{1}{\omega C}$$

Figure 15-3 shows the phasor diagram of a capacitor. The capacitor voltage lags the current by 90°, as you can see from Euler's formula:

$$-j = e^{-j90^\circ} = \cos\left(-90^\circ\right) + j\sin\left(-90^\circ\right)$$

Think of the imaginary number *j* as an operator that rotates a vector by 90° in the counterclockwise direction. A –*j* rotates a vector in the clockwise direction. You should also note j^2 rotates the phasor by 180° and is equal to –1.

TIP The imaginary component for a capacitor is negative. As the radian frequency ω increases, the capacitor's impedance goes down. Because the frequency for a battery is 0 and a battery has constant voltage, the impedance for a capacitor is infinite. The capacitor acts like an open circuit for a constant voltage source.

Putting Ohm’s law for inductors in phasor form

For an inductor with inductance *L*, the voltage is

$$v = L\frac{di}{dt}$$

The corresponding phasor description for an inductor is

$$\mathbf{V} = \underbrace{j\omega L}_{Z_L}\mathbf{I}$$

The impedance for an inductor is

$$Z_L = j\omega L$$

Figure 15-3 shows the phasor diagram of an inductor. The inductor voltage leads the current by 90° because of Euler’s formula:

$$j = e^{j90^\circ} = \cos(90^\circ) + j\sin(90^\circ)$$

> The imaginary component is positive for inductors. As the radian frequency ω increases, the inductor’s impedance goes up. Because the radian frequency for a battery is 0 and a battery has constant voltage, the impedance is 0. The inductor acts like a short circuit for a constant voltage source.

Tackling Circuits with Phasors

Phasors are great for solving steady-state responses (assuming zero initial conditions and sinusoidal inputs). Under the phasor concept, everything I cover in earlier chapters can be reapplied here. You can take functions of voltages $v(t)$ and currents $i(t)$ described in time to the phasor domain as **V** and **I**. With phasor methods, you can algebraically analyze circuits that have inductors and capacitors, similar to how you analyze resistor-only circuits.

When analyzing circuits in the phasor domain for sine or cosine wave (sinus-oidal signal) inputs, use these steps:

1. **Transform the circuit into the phasor domain by putting the sinusoidal inputs and outputs in phasor form.**
2. **Transform the resistors, capacitors, and inductors into their impedances in phasor form.**
3. **Use algebraic techniques to do circuit analysis to solve for unknown phasor responses.**
4. **Transform the phasor responses back into their time-domain sinusoids to get the response waveform.**

Using divider techniques in phasor form

In a series circuit with resistors, capacitors, inductors, and a voltage source, you can use phasor techniques to obtain the voltage across any device in the circuit. You can generalize the series circuit and voltage divider concept in Chapter 4 by replacing the resistors, inductors, and capacitors with impedances.

Remember that in a series circuit, you have the same current flowing through each device. When a series circuit is driven by a voltage source, you can find the voltage across each device using voltage divider techniques. This involves multiplying a voltage source by the ratio of the desired device impedance to the total impedance of the series circuit.

The top diagram of Figure 15-4 shows an RLC (resistor, inductor, capacitor) series circuit to illustrate the voltage divider concept and series equivalence:

$$\begin{aligned}\mathbf{V} &= \mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_3 \\ &= Z_1\mathbf{I} + Z_2\mathbf{I} + Z_3\mathbf{I} \\ &= \underbrace{(Z_1 + Z_2 + Z_3)}_{Z_{EQ}}\mathbf{I}\end{aligned}$$

You have an equivalent impedance Z_{EQ} from the three devices:

$$Z_{EQ} = Z_1 + Z_2 + Z_3$$

Here's the equivalent impedance for the RLC series circuit in Figure 15-4:

$$Z_{EQ} = j\omega L + R + \frac{1}{j\omega C} = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

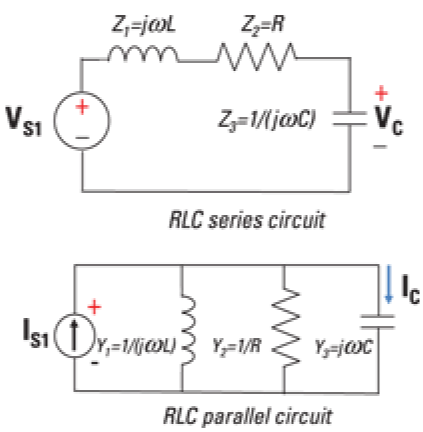


Illustration by Wiley, Composition Services Graphics

Figure 15-4: Voltage and current divider techniques in the phasor domain.

To get the voltage $\mathbf{V}_3 = \mathbf{V_C}$, use the voltage divider technique:

$$\mathbf{V}_3 = \frac{Z_3}{Z_{EQ}}\mathbf{V}$$

Now plug in the values for Z_3 and Z_{EQ} to get the capacitor voltage ($\mathbf{V}_3 = \mathbf{V_C}$):

$$\mathbf{V}_3 = \frac{\left(\frac{1}{j\omega C}\right)}{R + j\omega L + \left(\frac{1}{j\omega C}\right)}\mathbf{V}_{S1}$$

You can also obtain the equivalent impedance for parallel circuits and use the current divider method. (To see how to derive the equivalent resistance and current divider equations, see Chapter 4.) Parallel devices have the same voltage, which helps you get the total admittance (the reciprocal of impedance *Z*):

$$Y_{EQ} = Y_1 + Y_2 + Y_3$$

For the RLC parallel circuit in Figure 15-4, the equivalent admittance is

$$Y_{EQ} = \frac{1}{j\omega L} + \frac{1}{R} + j\omega C$$

To find the capacitor current $\mathbf{I}_3 = \mathbf{I_C}$, use the current divider technique:

$$\mathbf{I_3} = \mathbf{I_C} = \frac{Y_3}{Y_{EQ}}\mathbf{I}_{S1}$$

Plugging in the values for Y_1 and Y_{EQ}, the capacitor current $\mathbf{I}_3 = \mathbf{I_C}$ is

$$\mathbf{I}_3 = \mathbf{I}_C = \mathbf{I}_{S1}\frac{j\omega C}{\left(\frac{1}{j\omega L} + \frac{1}{R} + j\omega C\right)}$$

Adding phasor outputs with superposition

Superposition (see Chapter 7) says you can find the phasor output due to one source by turning off other sources; you then get the total output by adding up the individual phasor outputs.

You can use the superposition technique in phasor analysis only if all the independent sources have the same frequency. Superposition doesn’t work when you have different frequencies in the independent sources — you treat each source separately to get its steady-state output contribution to the total output.

To see how superposition works with phasors, first look at the top circuit in Figure 15-5. The middle diagram turns off $\mathbf{V}_{S2}$, leaving $\mathbf{V}_{S1}$ as the only voltage source. Use the voltage divider method to get the capacitor voltage $\mathbf{V}_{C1}$ due to $\mathbf{V}_{S1}$:

$$\mathbf{V}_{C1} = \frac{R // \left(\frac{1}{j\omega C}\right)}{j\omega L + R // \left(\frac{1}{j\omega C}\right)} \mathbf{V}_{S1}$$

where // denotes the parallel connection of capacitor C and resistor R.

The parallel combination of R and C has an equivalent impedance of

$$R // \left(\frac{1}{j\omega C}\right) = \frac{R\left(\frac{1}{j\omega C}\right)}{R + \left(\frac{1}{j\omega C}\right)}$$

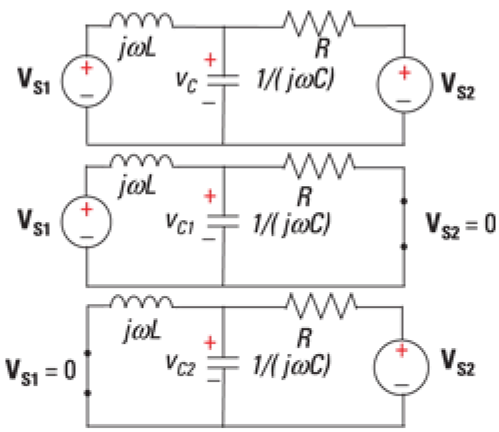


Illustration by Wiley, Composition Services Graphics

Figure 15-5: Super-position in the phasor domain.

The bottom diagram of Figure 15-5 turns off $\mathbf{V}_{S1}$, leaving only $\mathbf{V}_{S2}$ turned on. You use the voltage divider technique with capacitor C and inductor L connected in parallel to get the capacitor voltage due to $\mathbf{V}_{S2}$:

$$\mathbf{V}_{C2} = \frac{j\omega L \,//\left(\frac{1}{j\omega C}\right)}{R + j\omega L\,//\left(\frac{1}{j\omega C}\right)}\mathbf{V}_{S2}$$

The parallel combination of L and C has an equivalence impedance:

$$j\omega L\,//\left(\frac{1}{j\omega C}\right) = \frac{j\omega L\left(\frac{1}{j\omega C}\right)}{j\omega L + \left(\frac{1}{j\omega C}\right)}$$

The total output voltage is the sum of $\mathbf{V}_{C\mathbf{1}}$ and $\mathbf{V}_{\mathrm{C}\mathbf{2}}$ due to each source:

$$\mathbf{V}_C = \mathbf{V}_{C1} + \mathbf{V}_{C2}$$

Simplifying phasor analysis with Thévenin and Norton

You can use the Thévenin and Norton equivalents — which I first discuss in Chapter 8 with resistive circuits

— in the phasor domain as well. The Thévenin equivalent simplifies a complex array of impedances and independent sources to one voltage source connected in series with one impedance value (a complex number in general). The Norton equivalent simplifies a complex array of impedances and independent sources to one current source connected in parallel with one impedance value. The two equivalents are related by a source transformation. You use the Thévenin and Norton equivalents when you're analyzing different loads to a source circuit.

The Thévenin and Norton equivalents in Figure 15-6 follow the same approach as the one you'd use for resistive circuits. You simply calculate the open-circuit phasor voltage $\mathbf{V}_{OC}$ and short-circuit phasor current $\mathbf{I}_{SC}$ for each equivalent circuit.

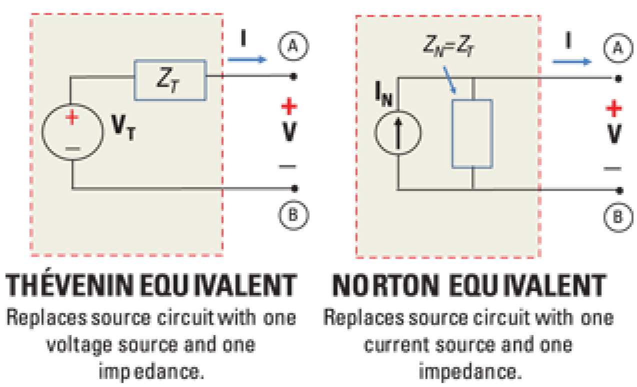


Illustration by Wiley, Composition Services Graphics

Figure 15-6: Thévenin and Norton equivalents in phasor domain.

The following phasor equations are similar to corresponding equations for resistive circuits:

$$\mathbf{V}_{OC} = \mathbf{V}_T = \mathbf{I}_N Z_T$$

$$\mathbf{I}_{SC} = \mathbf{I}_N = \frac{\mathbf{V}_T}{Z_T}$$

Using $\mathbf{V}_{OC}$ and $\mathbf{I}_{SC}$, you find Thévenin impedance Z_T as follows:

$$Z_T = \frac{\mathbf{V}_{OC}}{\mathbf{I}_{SC}}$$

Alternatively, you can calculate the impedance Z_T by looking back to the source circuit between Terminals A and B with all independent sources turned off, as described in Chapter 8.

Figure 15-7 shows a circuit to illustrate the Thévenin equivalent between Terminals A and B. Because you have an open-circuit load, no current flows through resistor *R*. You can find the open-circuit voltage using the voltage divider technique:

$$\mathbf{V}_{OC} = \frac{\left(\frac{1}{j\omega C}\right)}{j\omega L + \left(\frac{1}{j\omega C}\right)}\mathbf{V}_{S1}$$

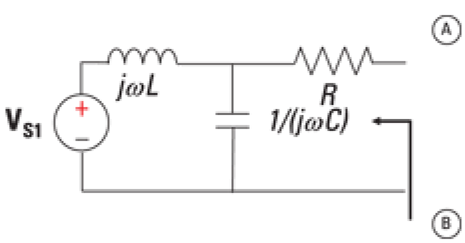


Illustration by Wiley, Composition Services Graphics

Figure 15-7: Example of the Thévenin equivalent in the phasor domain.

Putting a short across Terminals A and B implies that the resistor *R* and capacitor *C* are connected in parallel. The current flowing through this combination is

$$\mathbf{I}_1 = \frac{\mathbf{V}_{S1}}{j\omega L + R\,//\left(\frac{1}{j\omega C}\right)}$$

The short-circuit current $\mathbf{I}_{SC}$ flows through **R**. Using the current divider technique, you get

$$\mathbf{I}_{SC} = \mathbf{I}_1 \left(\frac{\frac{1}{j\omega C}}{R + \left(\frac{1}{j\omega C} \right)} \right)$$

$$= \frac{\mathbf{V}_{S1}}{j\omega L + R // \left(\frac{1}{j\omega C} \right)} \left(\frac{\frac{1}{j\omega C}}{R + \left(\frac{1}{j\omega C} \right)} \right)$$

You find the impedance Z_T by taking the ratio of $\mathbf{V}_{OC}/\mathbf{I}_{SC}$:

$$Z_T = R + j\omega L // \left(\frac{1}{j\omega C} \right)$$

Getting the nod for nodal analysis

When the circuit is large and complex, node-voltage analysis allows you to reduce the number of equations you need to deal with simultaneously. From the smaller set of node voltages, you can find any voltage or current for any device in the circuit. The node-voltage analysis technique I describe in Chapter 5 also works in the algebraic phasor domain. Figure 15-8 shows an op-amp circuit where you can use node-voltage analysis techniques. (For the scoop on op amps, see Chapter 10.)

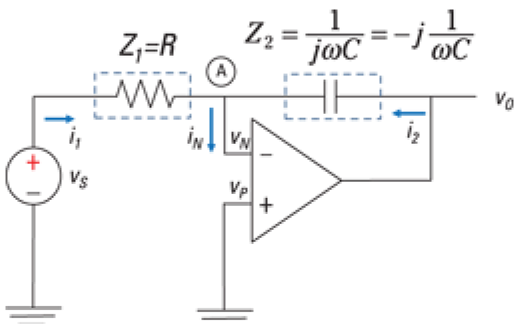


Illustration by Wiley, Composition Services Graphics

Figure 15-8: Op-amp node analysis in phasor form.

At Node A, you have the following KCL equation:

$$\text{in=out} \quad \rightarrow \quad \frac{\mathbf{V}_S - \overbrace{\mathbf{V}_N}^{=0}}{Z_1} + \frac{\mathbf{V}_O - \overbrace{\mathbf{V}_N}^{=0}}{Z_2} = \overbrace{\mathbf{I}_N}^{=0}$$

For ideal op amps with negative feedback, you have the inverting current $\mathbf{I}_N = 0$ and $\mathbf{V}_N = \mathbf{V}_P = 0$. Solve for the output $\mathbf{V}_O$ in terms of the input $\mathbf{V}_S$:

$$\mathbf{V}_o = -\frac{Z_2}{Z_1}\mathbf{V}_s$$

The output is an inverted input multiplied by the ratio of impedances. If the input impedance Z_1 is due to a resistor and feedback impedance Z_2 is due to a capacitor, then the phasor output $\mathbf{V}_O$ is

$$\mathbf{V}_o = -\frac{Z_2}{Z_1}\mathbf{V}_s = -\frac{\left(\frac{1}{j\omega C}\right)}{R}\mathbf{V}_s = -\left(\frac{1}{j\omega RC}\right)\mathbf{V}_s$$

This equation should look familiar, because it’s the integrator of the function waveform $v_S(t)$. You see that the $1/j\omega$ term describes the phasor for an integrator. That’s how an integrator is done electronically with op amps — beautiful!

Using mesh-current analysis with phasors

Mesh-current analysis is useful when a circuit has several loops. From the smaller set of mesh currents, you can find any voltage or current for any device in the circuit.

You can open up the mesh analysis approach in Chapter 6 to the phasor domain. You simply replace each device with its phasor impedance and apply KVL for each mesh to develop the mesh current equations. Figure 15-9 helps show the phasor analysis of circuits using mesh current techniques.

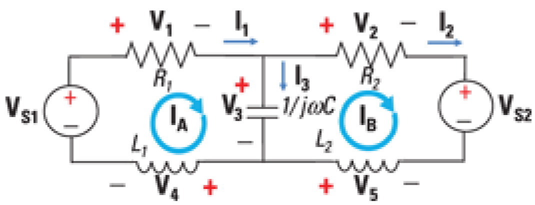


Illustration by Wiley, Composition Services Graphics

Figure 15-9: Mesh-current analysis using phasors.

The circuit has two mesh currents, $\mathbf{I}_A$ and $\mathbf{I}_B$, and five devices. For Meshes A and B, KVL produces the following:

$$\text{Mesh A:} \quad \mathbf{V}_1 + \mathbf{V}_3 + \mathbf{V}_4 = \mathbf{V}_{S1}$$

$$\text{Mesh B:} \quad -\mathbf{V}_2 + \mathbf{V}_3 + \mathbf{V}_5 = \mathbf{V}_{S2}$$

Replace the phasor voltages with the corresponding mesh currents and impedances:

$$\text{Mesh A:} \quad \mathbf{I}_A\left(R_1 + j\omega L_1\right) + \left(\mathbf{I}_A - \mathbf{I}_B\right)\frac{1}{j\omega C} = \mathbf{V}_{S1}$$

$$\text{Mesh B:} \quad -\left(\mathbf{I}_A - \mathbf{I}_B\right)\frac{1}{j\omega C} + \mathbf{I}_B\left(R_2 + j\omega L_2\right) = \mathbf{V}_{S2}$$

You then collect like terms and rearrange the mesh current equations to put them in standard form:

$$\text{Mesh A:} \quad \mathbf{I}_A\left(R_1 + j\omega L_1 + \frac{1}{j\omega C}\right) - \mathbf{I}_B\frac{1}{j\omega C} = \mathbf{V}_{S1}$$

$$\text{Mesh B:} \quad -\mathbf{I}_A\frac{1}{j\omega C} + \mathbf{I}_B\left(R_2 + j\omega L_2 + \frac{1}{j\omega C}\right) = \mathbf{V}_{S2}$$

Convert the equations to matrix form:

$$\begin{bmatrix} R_1 + j\omega L_1 + \frac{1}{j\omega C} & -\frac{1}{j\omega C} \\ -\frac{1}{j\omega C} & R_2 + j\omega L_2 + \frac{1}{j\omega C} \end{bmatrix} \begin{bmatrix} \mathbf{I}_A \\ \mathbf{I}_B \end{bmatrix} = \begin{bmatrix} \mathbf{V}_{S1} \\ \mathbf{V}_{S2} \end{bmatrix}$$

Note the symmetry along the diagonal of the first matrix. For circuits with independent sources, this symmetry is a useful check to verify that your mesh current equations are correct.

You can then use matrix software to solve for the unknown mesh currents $\mathbf{I}_A$ and $\mathbf{I}_B$, which you use to find the device currents and voltages.

Chapter 16

Predicting Circuit Behavior with Laplace Transform Techniques

In This Chapter

- Switching domains with the Laplace and inverse Laplace transforms
- Defining poles and zeros
- Working out a circuit response with Laplace methods

Analyzing the behavior of circuits consisting of resistors, capacitors, and inductors can get complicated because it involves differential equations. Although the classical differential equation approach using calculus is straightforward, the Laplace approach has the advantage of using simpler algebraic techniques. Also, the Laplace transform uncovers properties of circuit behavior you don't normally see using calculus.

In this chapter, I introduce you to the Laplace transform, show you how to find the inverse Laplace transform, and explain how to use the Laplace transform to predict a circuit's behavior.

Getting Acquainted with the Laplace Transform and

Key Transform Pairs

The Laplace transform allows you to change a tough differential equation requiring calculus into a simpler problem involving algebra in the *s*-domain (also known as the *Laplace domain*). After finding the transform solution in the *s*-domain, you use the inverse Laplace transform to find the time-domain solution to your original differential equation. In this chapter, finding the inverse Laplace transform basically requires you to look up a transform pair using a table.

In the following equation, the Laplace transform takes a function $f(t)$, described in the time-domain, and transforms it into another function $F(s)$, described in the *s*-domain.

$$F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt = \mathscr{L}[f(t)]$$

The Laplace transform of $f(t)$, defined as $F(s)$, is a function of the complex frequency variable *s,* which is defined as

$$s = \sigma + j\omega$$

The preceding equation has a real part σ and an imaginary part ω*.* The complex variable *s* is an independent variable in the complex frequency domain, similar to the independent variable *t* in the time-domain.

Based on the preceding discussion, Figure 16-1 shows the process of applying the Laplace and inverse Laplace transform techniques to solve a problem algebraically.

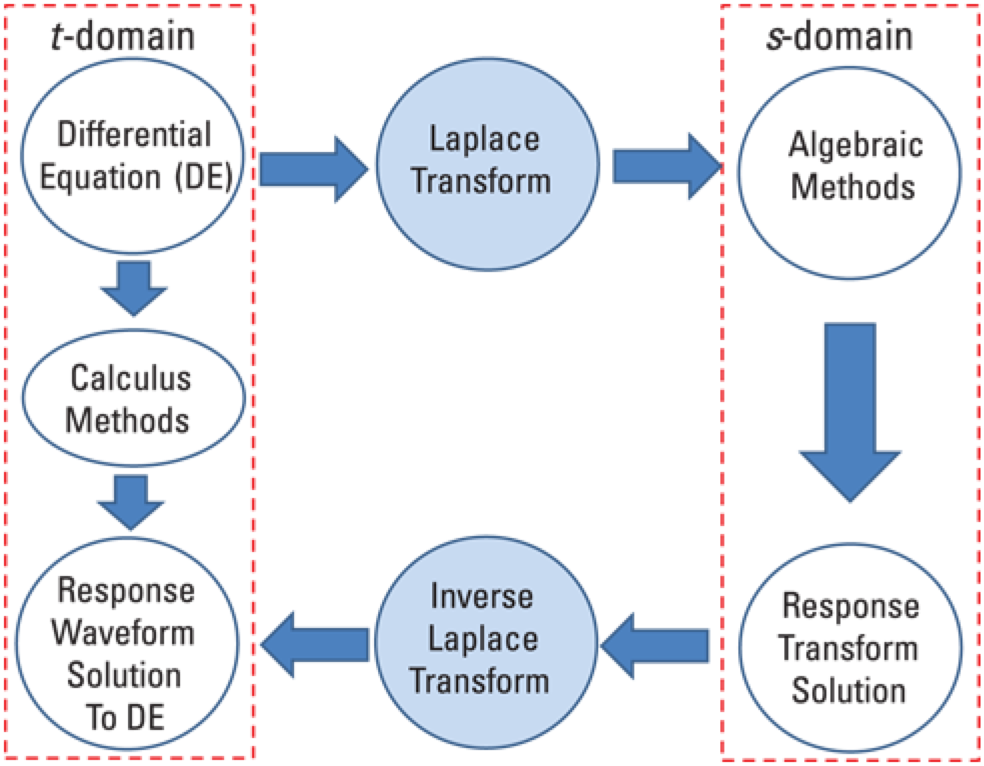


Illustration by Wiley, Composition Services Graphics

Figure 16-1: Flowchart comparing methods of solving circuits.

Table 16-1 lists the Laplace transform pairs that you'll find most helpful when working with circuits.

Table 16-1	Key Laplace Transform Pairs	
Signal Description	*Time-Domain Waveform, f(t)*	*s-Domain Waveform, F(s)*
Step	$u(t)$	$\frac{1}{s}$
Exponential	$\left[e^{-\alpha t}\right]u(t)$	$\frac{1}{s+\alpha}$
Impulse	$\delta(t)$	1
Ramp, *r(t)*	$tu(t)$	$\frac{1}{s^2}$
Sine	$[\sin\beta t]u(t)$	$\frac{\beta}{s^2+\beta^2}$
Cosine	$[\cos\beta t]u(t)$	$\frac{s}{s^2+\beta^2}$
Damped Pairs		
Damped ramp	$te^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^2}$
Damped sine	$\left[e^{-\alpha t}\sin\beta t\right]u(t)$	$\frac{\beta}{(s+\alpha)^2+\beta^2}$
Damped cosine	$\left[e^{-\alpha t}\cos\beta t\right]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\beta^2}$

Here are some key properties you may find helpful when analyzing circuits using the Laplace transform approach:

- **Linearity property:**

  $$af_1(t)+bf_2(t)\rightarrow a\mathscr{L}\{f_1(t)\}+b\mathscr{L}\{f_2(t)\}=aF_1(s)+bF_2(s)$$

  You find the linearity property useful when dealing with partial fraction expansion in later sections.

- **Integration property:**

  $$\int_0^t f(\tau)d\tau\rightarrow\frac{F(s)}{s}$$

- **Differentiation property:**
  - First-order:

  $$\frac{df(t)}{dt}\rightarrow sF(s)-f(0^-)$$

  - Second-order:

$$\frac{d^2f(t)}{dt^2} \to s^2F(s) - sf(0^-) - f'(0^-)$$

You find the integration and differentiation properties useful when dealing with derivatives and integral relationships of element constraints for capacitors and inductors.

Getting Your Time Back with the Inverse Laplace Transform

Say you're given the transform $F(s)$ in the s-domain. You now need to get back to the time-domain solution $f(t)$, which you get through the inverse Laplace transform of $F(s)$. When you have the simpler transforms, you just find the transform pair that has a form similar to the ones in Table 16-1. When you can't find a transform pair in the table, you need to break up the transform $F(s)$ into simpler transforms using a technique called *partial fraction expansion*. The following sections explain the basic partial fraction expansion method and how to modify the method when you have equations with complex or multiple poles.

Rewriting the transform with partial fraction expansion

When a transform $F(s)$ doesn't match those in Table 16-1, you can use partial fraction expansion to separate it. This method reduces the degree of the denominator of $F(s)$. You find the inverse Laplace transform $f(t)$ by rewriting the ratio of polynomials of $F(s)$ as the sum of simpler fractions, finding the inverse Laplace transform for each fraction, and adding the inverse Laplace transforms together. Here are the basic steps:

1. **Factor the numerator and denominator of *F*(*s*).**

   Consider the following transform $F(s)$:

   $$F(s) = 10 \cdot \frac{s^2 + 3s + 2}{s^3 + 17s^2 + 92s + 160}$$

   $$F(s) = \frac{10(s+1)(s+2)}{(s+4)(s+5)(s+8)}$$

   Putting the equation in factored form helps you figure out how to break $F(s)$ into simpler transforms.

2. **Rewrite the factored equation as the sum of fractions, using *A, B, C,* and so on as placeholders for the numerators.**

   Use each pole factor of $F(s)$ as the denominator of a new fraction. Write A, B, and C as placeholders in the numerators.

   Looking at the poles of the denominator of $F(s)$, you can separate $F(s)$ as follows:

   $$F(s) = \frac{10(s+1)(s+2)}{(s+4)(s+5)(s+8)} = \frac{A}{s+4} + \frac{B}{s+5} + \frac{C}{s+8}$$

   This equation is a partial fraction expansion of $F(s)$.

3. **Find the numerators by solving for the constants.**

   One way to find the constants is to get rid of the denominators. To do so, multiply both sides of the equation by $(s + 4)(s + 5)(s + 8)$:

   $$10(s+1)(s+2) = \left(\frac{A}{s+4} + \frac{B}{s+5} + \frac{C}{s+8}\right)(s+4)(s+5)(s+8)$$

   $$10(s+1)(s+2) = A(s+5)(s+8) + B(s+4)(s+8) + C(s+4)(s+5)$$

   To find A, plug in $s = -4$, which gets rid of the terms that contain B and C:

   $$10(-4+1)(-4+2) = A(-4+5)(-4+8) + 0 + 0$$

   $$10(-3)(-2) = A(1)(4)$$

   $$A = 15$$

To find B, substitute s = –5, which gets rid of the A and C terms:

$$\begin{aligned}10(-5+1)(-5+2) &= 0+B(-5+4)(-5+8)+0\\ 10(-4)(-3) &= B(-1)(3)\\ B &= -40\end{aligned}$$

To find C, substitute s = –8, which gets rid of the A and B terms:

$$\begin{aligned}10(-8+1)(-8+2) &= 0+0+C(-8+4)(-8+5)\\ 10(-7)(-6) &= C(-4)(-3)\\ C &= 35\end{aligned}$$

4. Plug the values of the constants into the partial fraction expansion form of *F*(*s*) and find each term’s transform pair.

Using the values of *A, B,* and *C,* you can express the original transform $F(s)$ in the following partial fraction expansion:

$$F(s)=\frac{15}{s+4}-\frac{-40}{s+5}+\frac{35}{s+8}$$

In this equation, each of the three simpler terms of the transform follows the mathematical form of an exponential. Using Table 16-1, the terms have the following transform pairs:

$$\frac{15}{s+4} \quad \leftrightarrow \quad 15e^{-4t}$$

$$\frac{-40}{s+5} \quad \leftrightarrow \quad -40e^{-5t}$$

$$\frac{35}{s+8} \quad \leftrightarrow \quad 35e^{-8t}$$

5. Write the inverse Laplace transform.

Based on these pairs, the inverse Laplace transform for $F(s)$ leads to the following transform pair:

$$F(s)=10\cdot\frac{s^2+3s+2}{s^3+17s^2+92s+160} \quad \leftrightarrow \quad f(t)=15e^{-4t}-40e^{-5t}+35e^{-8t}$$

Expanding Laplace transforms with complex poles

When you have complex poles in the denominator, the Laplace transform function $F(s)$ corresponds to a combination of damped sinusoids. Because $F(s)$ corresponds to damped sinusoids, you can write its partial fraction expansion as follows:

$$F(s)=\frac{As}{(s+\alpha)^2+\beta^2}+\frac{B}{(s+\alpha)^2+\beta^2}$$

You need to determine the constants A and B and then use Table 16-1 to obtain the transform pair. The following steps show how to put the preceding equation in the appropriate form so you can use Table 16-1:

$$F(s)=\frac{As+A\alpha-A\alpha}{(s+\alpha)^2+\beta^2}+\frac{B}{(s+\alpha)^2+\beta^2} \qquad \text{(add } A\alpha-A\alpha\text{)}$$

$$F(s)=\frac{As+A\alpha}{(s+\alpha)^2+\beta^2}+\frac{B-A\alpha}{(s+\alpha)^2+\beta^2} \qquad \text{(put } -A\alpha \text{ in 2nd term)}$$

$$F(s)=\frac{A(s+\alpha)}{(s+\alpha)^2+\beta^2}+\frac{[(B-A\alpha)/\beta]\beta}{(s+\alpha)^2+\beta^2} \qquad \text{(factor out } A \text{ in 1st term \& multiply by } \beta/\beta \text{ in 2nd term)}$$

Now you can use Table 16-1 to get the following transform pair:

$$F(s)=A\frac{(s+\alpha)}{(s+\alpha)^2+\beta^2}+\left(\frac{B-A\alpha}{\beta}\right)\frac{\beta}{(s+\alpha)^2+\beta^2} \leftrightarrow$$

$$f(t)=Ae^{-\alpha t}\cos(\beta t)+\left(\frac{B-A\alpha}{\beta}\right)e^{-\alpha t}\sin(\beta t)$$

I know what you’re thinking: Enough with the variables already! Your wish is my command. Consider the following transform $F(s)$ and its partial fraction expansion form. (Notice how the numbers are plugged in? You’re welcome.)

$$F(s)=\frac{20(s+3)}{(s+1)\left(s^2+2s+5\right)}=\frac{As}{s^2+2s+5}+\frac{B}{s^2+2s+5}+\frac{C}{s+1}$$

This equation has a complex pair of poles of 1 + 2*j* and 1 – 2*j* along with a real pole at –1. To match the form found in Table 16-1, you can take the denominators in the first two terms and manipulate them into a perfect square:

$$\frac{20(s+3)}{(s+1)(s^2+2s+5)} = \frac{As}{(s^2+2s+1)+2^2} + \frac{B}{(s^2+2s+1)+2^2} + \frac{C}{s+1}$$

$$\frac{20(s+3)}{(s+1)(s^2+2s+5)} = \frac{As}{(s+1)^2+2^2} + \frac{B}{(s+1)^2+2^2} + \frac{C}{s+1}$$

Clearing out the denominators generates the following equations:

$$20(s+3) = As(s+1) + B(s+1) + C(s^2+2s+5)$$

$$0s^2+20s+60 = (A+C)s^2 + (A+B+2C)s + (B+5C)$$

By equating the coefficients of s^2, s, and the constants on the left and right sides of the preceding equation, you get the following three equations and three unknowns:

$$0 = A+C$$

$$20 = A+B+2C$$

$$60 = B+5C$$

Solving for *A*, *B*, and *C* produces the following values: $A = -10$, $B = 10$, and $C = 10$.

You can verify that these values are correct by substituting them into the preceding equations. To apply a transform pair from Table 16-1, substitute the preceding values into the partial fraction expansion form of $F(s)$ to get the following series of algebraic manipulations:

$$F(s)=\frac{-10s}{s^2+2s+5}+\frac{10}{s^2+2s+5}+\frac{10}{s+1}$$

$$=\frac{\overbrace{-10}^{A=-10}\overbrace{(s+1)}^{s+\alpha}}{\underbrace{\left(s^2+2s+1^2\right)}_{(s+1)^2=(s+\alpha)^2}+\underbrace{2^2}_{\beta^2}}+\frac{\overbrace{\left[\left(10-(-10)\cdot 1\right)/2\right]}^{\left(\frac{B-A\alpha}{\beta}\right)=10}\cdot\overbrace{2}^{\beta=2}}{\underbrace{\left(s^2+2s+1^2\right)}_{(s+1)^2=(s+\alpha)^2}+\underbrace{2^2}_{\beta^2}}+\frac{\overbrace{10}^{C=10}}{s+1}$$

$$=\frac{-10(s+1)}{(s+1)^2+2^2}+\frac{10(2)}{(s+1)^2+2^2}+\frac{10}{s+1}$$

You can now use Table 16-1 to produce the following inverse Laplace transform of $F(s)$:

$$f(t)=-10e^{-t}\cos(2t)+10e^{-t}\sin(2t)+10e^{-t}$$

Dealing with transforms with multiple poles

When you have multiple *poles* — that is, roots in the denominator of $F(s)$ — you need to slightly modify the partial fraction expansion method. With multiple roots, you need to form unique partial fractions with the same poles. To make each fraction unique, you raise the power of the denominator to a specific power. The number of fractions you need with the same poles is equal to the number of poles that have the same value.

You start off with a fraction with the denominator raised to a power of 1. You form another fraction with the denominator raised to the power by incrementing the power (exponent) by 1. You keep forming fractions until you end up with the power that's the same as the number of poles that are equal. So if you have two poles that are the same, then you have one fraction with the polynomial in the denominator raised to a power of 1 and another fraction with the denominator raised to a power of 2.

For example, say you're given the following $F(s)$ with a double pole:

$$F(s)=\frac{8(s+6)}{s(s+4)^2}$$

The double pole is at s = –4, and the single pole is at s = 0. In this case, the partial fraction expansion for $F(s)$ is

$$F(s)=\frac{8(s+6)}{s(s+4)^2}=\frac{A}{s}+\frac{B}{s+4}+\frac{C}{(s+4)^2}$$

You need to determine the constants *A, B,* and *C*. Note that the right side of the equation has a single pole at –4 for the term having *B* in the numerator and that the third term has a double pole at –4 with *C* in the numerator. You can easily extend this setup of the partial fraction expansion for more than two poles.

Clearing out the denominators leads to the following expression:

$$8(s+6)=A(s+4)^2+Bs(s+4)+Cs$$

Substitute s = 0 in the preceding equation to find *A:*

$$\begin{aligned}8(0+6)&=A(0+4)^2+B(0)(0+4)+C(0)\\48&=16A+0+0\\A&=3\end{aligned}$$

Plug in s = –4 to find *C:*

$$\begin{aligned}8(-4+6)&=A(-4+4)^2+B(-4)(-4+4)+C(-4)\\16&=0+0-4C\\C&=-4\end{aligned}$$

To find *B,* you can't use s = –4 again. Because you already know *A* = 3 and *C* = –4, you can try any value of s to solve for *B*. Letting s = 1 produces the following expression and value for *B:*

$$8(1+6)=\underbrace{3}_{A}(1+4)^2+B(1)(1+4)+\underbrace{(-4)}_{C}1$$
$$56=75+5B-4$$
$$B=-3$$

Substituting *A, B,* and *C* into *F*(*s*) gives you the following expression:

$$F(s)=\frac{8(s+6)}{s(s+4)^2}=\frac{3}{s}+\frac{-3}{s+4}+\frac{-4}{(s+4)^2}$$

Based on Table 16-1, you wind up with the following inverse Laplace transform of *F*(*s*):

$$f(t)=3u(t)-3e^{-4t}-4te^{-4t}$$

Understanding Poles and Zeros of F(s)

You can view the Laplace transforms *F*(*s*) as ratios of polynomials in the *s*-domain. If you find the real and complex roots of these polynomials, you can use Table 16-1 to get a general idea of what the waveform *f*(*t*) will look like. For example, if the roots are real, then the waveform is exponential. If they're imaginary, then it's a combination of sines and cosines. And if they're complex, then it's a damping sinusoid.

The roots of the polynomial in the numerator of *F*(*s*) are *zeros,* and the roots of the polynomial in the denominator are *poles.* The poles result in *F*(*s*) blowing up to infinity or being undefined — they're the vertical asymptotes and holes in your graph.

Usually, you create a *pole-zero* diagram by plotting the roots in the *s*-plane (real and imaginary axes). The pole-zero diagram provides a geometric view and general interpretation of the circuit behavior.

For example, consider the following Laplace transform $F(s)$:

$$F(s) = 10 \cdot \frac{s^2 + 3s + 2}{s^3 + 17s^2 + 92s + 160}$$

This expression is a ratio of two polynomials in *s*. Factoring the numerator and denominator gives you the following Laplace description $F(s)$:

$$F(s) = 10 \cdot \frac{(s+1)(s+2)}{(s+4)(s+5)(s+8)}$$

The *zeros,* or roots of the numerator, are s = –1, –2. The *poles,* or roots of the denominator, are s = –4, –5, –8.

Both poles and zeros are collectively called *critical frequencies* because crazy output behavior occurs when $F(s)$ goes to zero or blows up. By combining the poles and zeros, you have the following set of critical frequencies: {–1, –2, –4, –5, –8}.

Figure 16-2 plots these critical frequencies in the *s*-plane, providing a geometric view of circuit behavior. In this pole-zero diagram, X denotes poles and O denotes the zeros.

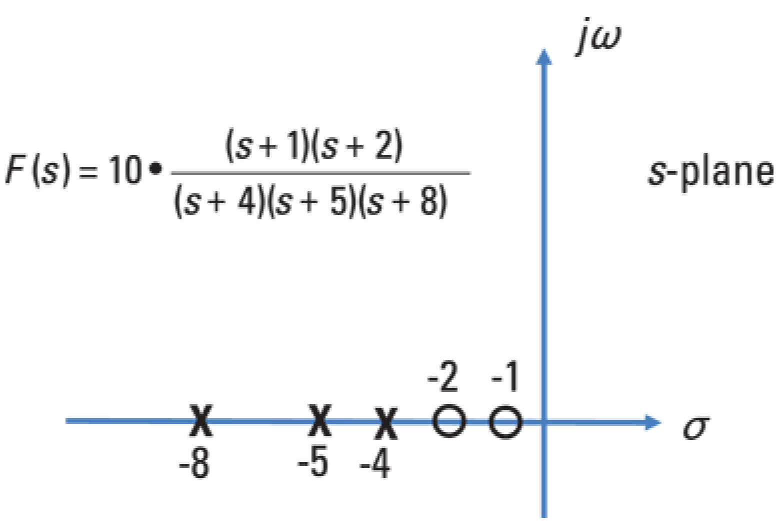


Illustration by Wiley, Composition Services Graphics

Figure 16-2: Pole-zero diagram in the complex plane *s.*

Here are some examples of the poles and zeros of the Laplace transforms, $F(s)$, that you see in Table 16-1. I then follow the examples with pole-zero diagrams — plots of their poles and zeros in the *s*-plane — in Figure 16-3.

The Laplace transform $F_1(s)$ for a damping exponential has a transform pair as follows:

$$f_1(t) = e^{-\alpha t}u(t) \quad \leftrightarrow \quad F_1(s) = \frac{1}{s+\alpha}$$

The exponential transform $F_1(s)$ has one pole at $s = -\alpha$ and no zeros. Diagram A of Figure 16-3 shows the pole of $F_1(s)$ plotted on the negative real axis in the left half plane.

The sine function has the following Laplace transform pair:

$$f_3(t) = \sin(\beta t) \quad \leftrightarrow \quad F_3(s) = \frac{\beta}{s^2+\beta^2}$$

The preceding equation has no zeros and two imaginary poles — at $s = +j\beta$ and $s = -j\beta$. Imaginary poles always come in pairs. These two poles are *undamped,* because whenever poles lie on the imaginary axis $j\omega$, the function $f(t)$ will oscillate forever, with nothing to damp it out. Diagram B of Figure 16-3 shows a plot of the pole-zero diagram for a sine function.

A ramp function has the following Laplace transform pair:

$$f_2(t) = tu(t) \quad \leftrightarrow \quad F_2(s) = \frac{1}{s^2}$$

The ramp function has double poles at the origin ($s = 0$) and has no zeros.

Here's a transform pair for a damped cosine signal:

$$f_4(t) = e^{-\alpha t}\cos(\beta t) \quad \leftrightarrow \quad F_4(s) = \frac{s+\alpha}{(s+\alpha)^2+\beta^2}$$

The preceding equation has two complex poles at $s = \alpha + j\beta$ and $s = \alpha - j\beta$ and one zero at $s = -\alpha$.

REMEMBER

Complex poles, like imaginary poles, always come in pairs. Whenever you have a complex pair of poles, the function has oscillations that will be damped out to zero in time — they won't go on forever. The damped sinusoidal behavior consists of a combination of an exponential (due to the real part α of the complex number) and sinusoidal oscillator (due to the imaginary part β of the complex number). Diagram C depicts the pole-zero diagram for a damped cosine.

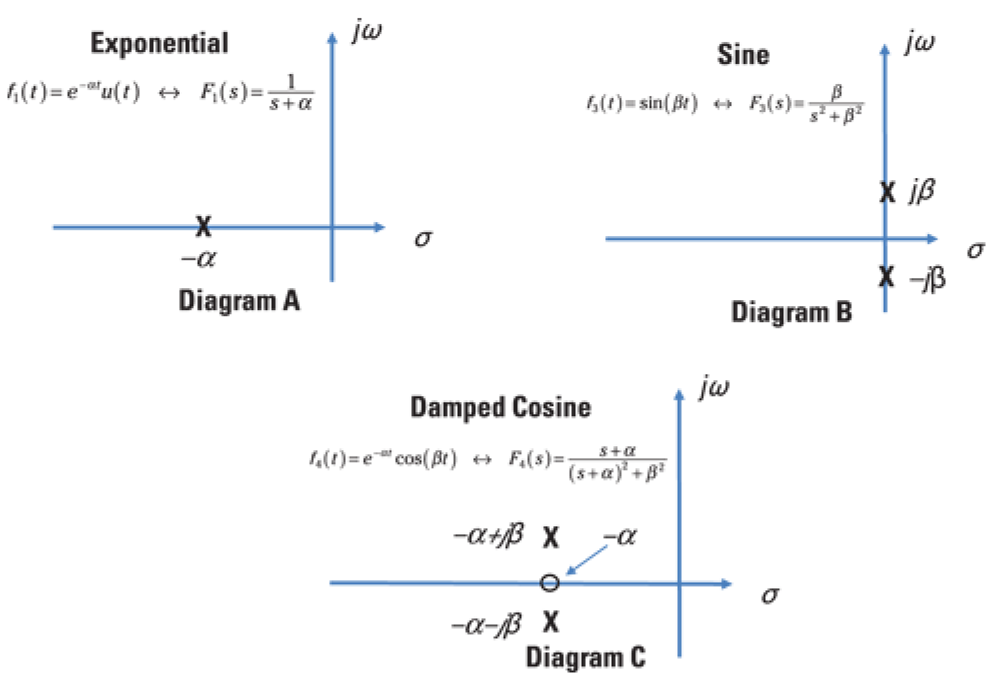


Illustration by Wiley, Composition Services Graphics

Figure 16-3: Pole-zero diagrams of Laplace transforms $F(s)$.

Predicting the Circuit Response with Laplace Methods

Using the Laplace transform as part of your circuit analysis provides you with a different point of view on circuit behavior. One benefit is that the poles of the Laplace transform give you a general idea of the output behavior. Real poles, for instance, indicate exponential output behavior.

All the concepts concerning transient response, frequency response, and the phasor approach developed in Chapters 14 and 15 come together with the Laplace transform. Following are the basic steps for analyzing a circuit using Laplace techniques:

1. **Develop the differential equation in the time-domain using Kirchhoff's laws and element equations.**
2. **Apply the Laplace transformation of the differential equation to put the equation in the *s*-domain.**
3. **Algebraically solve for the solution, or response transform.**
4. **Apply the inverse Laplace transformation to produce the solution to the original differential equation described in the time-domain.**

To get comfortable with this process, you simply need to practice applying it to different types of circuits. That's why the following sections walk you through each step for three circuits: an RC (resistor-capacitor) circuit, an

RL (resistor-inductor) circuit, and an RLC (resistor-inductor-capacitor) circuit.

Working out a first-order RC circuit

Consider the simple first-order RC series circuit in Figure 16-4. To set up the differential equation for this series circuit, you can use Kirchhoff's voltage law (KVL), which says the sum of the voltage rises and drops around a loop is zero. This circuit has the following KVL equation around the loop:

$$-v_s(t)+v_r(t)+v_c(t)=0$$

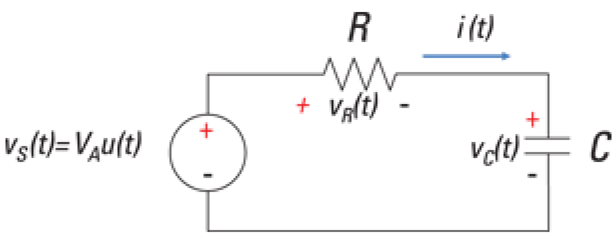


Illustration by Wiley, Composition Services Graphics

Figure 16-4: Analyzing a first-order RC series circuit in the *s*-domain.

Next, formulate the element equation (or *i*-*v* characteristic) for each device. The element equation for the source is

$$v_S(t)=V_Au(t)$$

Use Ohm's law to describe the voltage across the resistor:

$$v_R(t)=i(t)R$$

The capacitor's element equation is given as

$$i(t)=C\frac{dv_C(t)}{dt}$$

Substituting this expression for $i(t)$ into $v_R(t)$ gives you the following expression:

$$v_R(t) = i(t)R = RC\frac{dv_C(t)}{dt}$$

Substituting $v_R(t)$, $v_C(t)$, and $v_S(t)$ into the KVL equation leads to

$$-v_S(t) + v_R(t) + v_C(t) = 0$$
$$-V_A u(t) + RC\frac{dv_C(t)}{dt} + v_C(t) = 0$$

Now rearrange the equation to get the desired first-order differential equation:

$$RC\frac{dv_C(t)}{dt} + v_C(t) = V_A u(t)$$

Now you're ready to apply the Laplace transformation of the differential equation in the *s*-domain. The result is

$$\mathscr{L}\left[RC\frac{dv_C(t)}{dt} + v_C(t)\right] = \mathscr{L}\left[V_A u(t)\right]$$
$$\mathscr{L}\left[RC\frac{dv_C(t)}{dt}\right] + \mathscr{L}\left[v_C(t)\right] = \mathscr{L}\left[V_A u(t)\right]$$

On the left, I used the linearity property (from the first section in this chapter) to take the Laplace transform of each term.

For the first term on the left side of the equation, you use the differentiation property (also from the first section), which gives you

$$\mathscr{L}\left[RC\frac{dv_C(t)}{dt}\right] = RC\left[sV_C(s) - V_0\right]$$

This equation uses $V_C(s) = \mathscr{L}\left[v_C(t)\right]$, and V_0 is the initial voltage across the capacitor.

Using Table 16-1, the Laplace transform of a step function provides you with

$$\mathscr{L}\left[V_A u(t)\right] = \frac{V_A}{s}$$

Based on the preceding expressions for the Laplace transforms, the differential equation becomes the following:

$$RC\left[sV_C(s)-V_0\right]+V_C(s)=\frac{V_A}{s}$$

Next, rearrange the equation:

$$\left[s+\frac{1}{RC}\right]V_C(s)=\frac{V_A}{RC}\left(\frac{1}{s}\right)+V_0$$

Solve for the output $V_c(s)$ to get the following transform solution:

$$V_C(s)=\frac{V_A}{RC}\left[\frac{1}{s\left(s+\frac{1}{RC}\right)}\right]+\frac{V_0}{s+\frac{1}{RC}}$$

By performing an inverse Laplace transform of $V_C(s)$ for a given initial condition, this equation leads to the solution $v_C(t)$ of the original first-order differential equation.

On to Step 3 of the process. To get the time-domain solution $v_C(t)$, you need to do a partial fraction expansion for the first term on the right side of the preceding equation:

$$\frac{V_A}{RC}\left[\frac{1}{s\left(s+\frac{1}{RC}\right)}\right]=\frac{A}{s}+\left(\frac{B}{s+\frac{1}{RC}}\right)$$

You need to determine constants A and B. To simplify the preceding equation, multiply both sides by $s(s + 1/RC)$ to get rid of the denominators:

$$\frac{V_A}{RC}=A\left(s+\frac{1}{RC}\right)+Bs$$

Algebraically rearrange the equation by collecting like terms:

$$(A+B)s+\frac{1}{RC}(A-V_A)=0$$

In order for the left side of the preceding equation to be zero, the coefficients must be zero ($A + B = 0$ and $A - V_A = 0$). For constants A and B, you wind up with $A = V_A$ and $B = -V_A$. Substitute these values into the following equation:

$$\frac{V_A}{RC}\left[\frac{1}{s\left(s+\frac{1}{RC}\right)}\right]=\frac{A}{s}+\left(\frac{B}{s+\frac{1}{RC}}\right)$$

The substitution leads you to:

$$\frac{V_A}{RC}\left(\frac{1}{s\left(s+\frac{1}{RC}\right)}\right)=\frac{V_A}{s}+\frac{-V_A}{s+\frac{1}{RC}}$$

Now substitute the preceding expression into the $V_C(s)$ equation to get the transform solution:

$$\begin{aligned}V_C(s)&=\frac{V_A}{RC}\left(\frac{1}{s\left(s+\frac{1}{RC}\right)}\right)+\frac{V_0}{s+\frac{1}{RC}}\\&=\frac{V_A}{s}+\frac{-V_A}{s+\frac{1}{RC}}+\frac{V_0}{s+\frac{1}{RC}}\\&=\frac{V_A}{s}+\frac{-V_A}{s+\frac{1}{RC}}+\frac{V_0}{s+\frac{1}{RC}}\end{aligned}$$

That completes the partial fraction expansion. You can then use Table 16-1 to find the inverse Laplace transform for each term on the right side of the preceding equation. The first term has the form of a step function, and the last two terms have the form of an exponential, so the inverse Laplace transform of the preceding equation leads you to the following solution $v_C(t)$ in the time-domain:

$$v_C(t)=V_A u(t)-V_A e^{-\left(\frac{t}{RC}\right)}u(t)+V_0 e^{-\left(\frac{t}{RC}\right)}u(t)$$

$$v_C(t)=V_A\left(1-e^{-\left(\frac{t}{RC}\right)}\right)u(t)+V_0 e^{-\left(\frac{t}{RC}\right)}u(t)$$

The result shows as time t approaches infinity, the capacitor charges to the value of the input V_A. Also, the initial voltage of the capacitor eventually dies out to zero after a long period of time (about 5 time constants, RC).

Working out a first-order RL circuit

Analyzing an RL circuit using Laplace transforms is similar to analyzing an RC series circuit, which I cover in the preceding section. Figure 16-5 shows you a circuit that has a switch that's been in Position A for a long time. The switch moves to Position B at time $t = 0$.

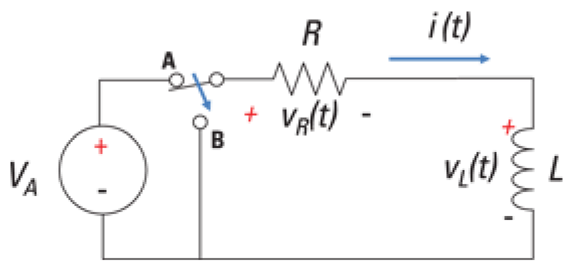


Illustration by Wiley, Composition Services Graphics

Figure 16-5: Analyzing a first-order RL circuit in the *s*-domain.

For this circuit, you have the following KVL equation:

$$v_R(t)+v_L(t)=0$$

Next, formulate the element equation (or *i*-*v* characteristic) for each device. Using Ohm's law to describe the voltage across the resistor, you have the following relationship:

$$v_R(t)=i_L(t)R$$

The inductor's element equation is

$$v_L(t)=L\frac{di_L(t)}{dt}$$

Substituting the element equations, $v_R(t)$ and $v_L(t)$, into the KVL equation gives you the desired first-order differential equation:

$$L\frac{di_L(t)}{dt}+i_L(t)R=0$$

On to Step 2: Apply the Laplace transform to the differential equation:

$$\mathscr{L}\left[L\frac{di_L(t)}{dt}+i_L(t)R\right]=0$$

$$\mathscr{L}\left[L\frac{di_L(t)}{dt}\right]+\mathscr{L}\left[i_L(t)R\right]=0$$

The preceding equation uses the linearity property (see the first section of the chapter), which says you can take the Laplace transform of each term. For the first term on the left side of the equation, you use the differentiation property:

$$\mathscr{L}\left[L\frac{di_L(t)}{dt}\right]=L\left[sI_L(s)-I_0\right]$$

This equation uses $I_L(s)=\mathscr{L}\left[i_L(t)\right]$, and I_0 is the initial current flowing through the inductor.

The Laplace transform of the differential equation becomes

$$I_L(s)R+L\left[sI_L(s)-I_0\right]=0$$

Solve for $I_L(s)$:

$$I_L(s)=\frac{I_0}{s+\frac{R}{L}}$$

For a given initial condition, this equation provides the solution $i_L(t)$ to the original first-order differential equation. You simply perform an inverse Laplace transform of $I_L(s)$ — or look for the appropriate transform pair in Table 16-1 — to get back to the time-domain.

The preceding equation has an exponential form for the Laplace transform pair. You wind up with the following solution:

$$I_L(s) = \frac{I_0}{s + \frac{R}{L}} \quad \leftrightarrow \quad i_L(t) = I_0 e^{-\left(\frac{R}{L}\right)t}$$

The result shows as time t approaches infinity, the initial inductor current eventually dies out to zero after a long period of time — about 5 time constants (L/R).

Working out an RLC circuit

Analyzing an RLC series circuit using the Laplace transform is similar to analyzing an RC series circuit and RL circuit, which I cover in the preceding sections. Figure 16-6 shows you an RLC circuit in which the switch has been open for a long time. The switch is closed at time t = 0.

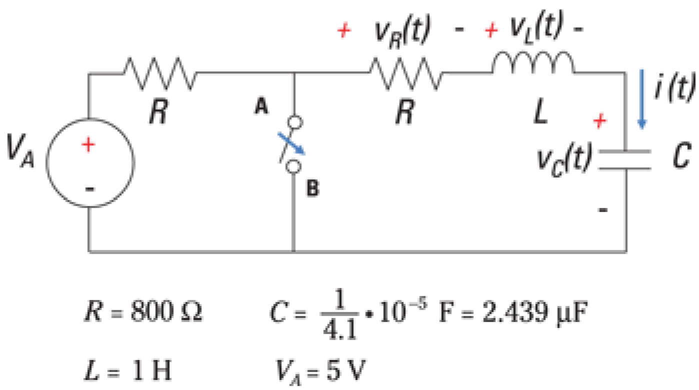


Illustration by Wiley, Composition Services Graphics

Figure 16-6: Analyzing a second-order RLC series circuit in the *s*-domain.

In this circuit, you have the following KVL equation:

$$v_R(t) + v_L(t) + v(t) = 0$$

Next, formulate the element equation (or *i*-*v* characteristic) for each device. Ohm's law describes the voltage across the resistor (noting that $i(t) = i_L(t)$ because the circuit is connected in series, where $I(s) = I_L(s)$ are the Laplace transforms):

$$v_R(t) = i(t)R$$

The inductor’s element equation is given by

$$v_L(t)=L\frac{di_L(t)}{dt}$$

And the capacitor’s element equation is

$$v_C(t)=\frac{1}{C}\int_0^t i(\tau)d\tau+v_C(0)$$

Here, $v_C(0) = V_0$ is the initial condition, and it’s equal to 5 volts.

Substituting the element equations, $v_R(t)$, $v_C(t)$, and $v_L(t)$, into the KVL equation gives you the following equation (with a fancy name: the *integro-differential equation*):

$$L\frac{di_L(t)}{dt}+i_L(t)R+\frac{1}{C}\int_0^t i(\tau)d\tau+v_C(0)=0$$

The next step is to apply the Laplace transform to the preceding equation to find an $I(s)$ that satisfies the integro-differential equation for a given set of initial conditions:

$$\mathscr{L}\left[L\frac{di_L(t)}{dt}+i(t)R+\frac{1}{C}\int_0^t i(\tau)d\tau+V_0\right]=0$$

$$\mathscr{L}\left[L\frac{di(t)}{dt}\right]+\mathscr{L}\left[i(t)R\right]+\mathscr{L}\left[\frac{1}{C}\int_0^t i(\tau)d\tau+V_0\right]=0$$

The preceding equation uses the linearity property (from the first section in this chapter), allowing you to take the Laplace transform of each term.

For the first term on the left side of the equation, you use the differentiation property to get the following transform:

$$\mathscr{L}\left[L\frac{di(t)}{dt}\right]=L\left[sI(s)-I_0\right]$$

This equation uses $I_L(s)=\mathscr{L}\left[i(t)\right]$, and I_0 is the initial current flowing through the inductor. Because the switch is open for a long time, the initial condition I_0 is equal to zero.

For the second term of the KVL equation dealing with resistor *R,* the Laplace transform is simply

$$\mathscr{L}[i(t)R] = I(s)R$$

For the third term in the KVL expression dealing with capacitor *C,* you have

$$\mathscr{L}\left[\frac{1}{C}\int_0^t i(\tau)d\tau + V_0\right] = \frac{I(s)}{sC} + \frac{V_0}{s}$$

The Laplace transform of the integro-differential equation becomes

$$L[sI(s) - I_0] + I(s)R + \frac{I(s)}{sC} + \frac{V_0}{s} = 0$$

Rearrange the equation and solve for $I(s)$:

$$I(s) = \frac{sI_0 - \frac{V_0}{L}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

To get the time-domain solution $i(t)$, use Table 16-1 and notice that the preceding equation has the form of a damping sinusoid. Plugging in $I_0 = 0$ and some numbers from Figure 16-6 into the preceding equation gives you

$$\begin{aligned} I(s) &= -\frac{5}{s^2 + 800s + 4.1 \cdot 10^5} \\ &= -\frac{5}{500}\left[\frac{500}{(s+400)^2 + (500)^2}\right] \end{aligned}$$

You wind up with the following solution:

$$i(t) = \left[-0.01e^{-400t}\sin 500t\right]u(t)$$

For this RLC circuit, you have a damping sinusoid. The oscillations will die out after a long period of time. For this example, the time constant is 1/400 and will die out after 5/400 = 1/80 seconds.

Chapter 17

Implementing Laplace Techniques for Circuit Analysis

In This Chapter

- Starting with basic constraints in the *s*-domain
- Looking at voltage and current divider techniques in the *s*-domain
- Using superposition, Thévenin, Norton, node voltages, and mesh currents in the *s*-domain

This chapter is all about applying Laplace transform techniques in order to study circuits that have voltage and current signals changing with time. That may sound complex, but it's really no more difficult than analyzing resistor-only circuits. You see, the Laplace method converts a circuit to the *s*-domain so you can study the circuit's action using only algebraic techniques (rather than the calculus techniques I show you in Chapters 13 and 14). The algebraic approach in the *s*-domain follows along the same lines as resistor-only circuits, except in place of resistors, you have *s*-domain impedances.

If you need a refresher on impedance or the Laplace transform in general, see Chapters 15 and 16, respectively. Otherwise, I invite you to dive into this chapter, which first has you describe the element and connection constraints in the *s*-domain. You then see how the *s*-domain approach works when you apply voltage and current divider methods, Thévenin and Norton equivalents, node-voltage analysis, and mesh-current analysis.

Starting Easy with Basic Constraints

Connection constraints are those physical laws that cause element voltages and currents to behave in certain ways when the devices are interconnected to form a circuit. You also have constraints on the individual devices themselves, where each device has a mathematical relationship between the voltage across the device and the current through the device. The following sections show you what connection constraints, device constraints, impedances, and admittances wind up looking like in the *s*-domain.

Connection constraints in the s-domain

Transforming the connection constraints to the *s*-domain is a piece of cake. Kirchhoff's current law (KCL) says the sum of the incoming and outgoing currents is equal to 0. Here's a typical KCL equation described in the time-domain:

$$i_1(t)+i_2(t)-i_3(t)=0$$

Because of the linearity property of the Laplace transform (Chapter 16), the KCL equation in the *s*-domain becomes the following:

$$I_1(s)+I_2(s)-I_3(s)=0$$

You transform Kirchhoff's voltage law (KVL) in the same way. KVL says the sum of the voltage rises and drops is equal to 0. Here's a classic KVL equation described in the time-domain:

$$v_1(t)+v(t)+v_3(t)=0$$

Because of linearity, the KVL equation in the *s*-domain produces

$$V_1(s)+V_2(s)+V_3(s)=0$$

The basic form of KVL remains the same. Piece of cake!

Device constraints in the s-domain

You can easily transform the *i*-*v* constraints of devices such as independent and dependent sources, op amps, resistors, capacitors, and inductors to algebraic equations in the *s*-domain. After converting the device constraints, all you need is algebra. I show you how to translate current and voltage relationships to the *s*-domain in the following sections.

Independent and dependent sources

Transforming independent sources is a no-brainer because the *s*-domain has the same form as the time-domain:

$$v_S(t) \quad \rightarrow \quad V_S(s)$$
$$i_S(t) \quad \rightarrow \quad I_S(s)$$

Converting dependent sources is easy, too. Here are the equations for voltage-controlled voltage sources (VCVS), voltage-controlled current sources (VCCS), current-controlled voltage sources (CCVS), and current-controlled current sources (CCCS):

$$\text{VCVS:} \quad v_2(t)=\mu v_1(t) \quad \rightarrow \quad V_2(s)=\mu V_1(s)$$
$$\text{VCCS:} \quad i_2(t)=g v_1(t) \quad \rightarrow \quad I_2(s)=g V_1(s)$$
$$\text{CCVS:} \quad v_2(t)=r i_1(t) \quad \rightarrow \quad V_2(s)=r i_1(s)$$
$$\text{CCCS:} \quad i_2(t)=\beta i_1(t) \quad \rightarrow \quad I_2(s)=\beta I_1(s)$$

The constants μ, g, r, and β relate the dependent output sources $V_2(s)$ and $I_2(s)$ controlled by input variables $V_1(s)$ and $I_1(s)$. (For more information on dependent sources, see Chapter 9.)

Passive elements: Resistors, capacitors, and inductors

For resistors, capacitors, and inductors, you convert their *i*-*v* relationships to the *s*-domain using Laplace transform properties, such as the integration and derivative properties (which you find in Chapter 16):

$$\text{Resistor:}\quad v_R(t) = Ri_R(t) \quad\rightarrow\quad V_R(s) = RI_R(s)$$

$$\text{Capacitor:}\quad v_C(t) = \int_0^t i_C(\tau)\,d\tau \quad\rightarrow\quad V_C(s) = \frac{1}{sC}I_C(s) + \frac{v_C(0)}{s}$$

$$\text{Inductor:}\quad v_2(t) = L\frac{di_L(t)}{dt} \quad\rightarrow\quad V_L(s) = sLI_L(s) - Li_L(0)$$

The preceding three equations on the right are *s*-domain models that use voltage sources for the initial capacitor voltage $v_C(0)$ and initial inductor current $i_L(0)$.

You can rewrite these equations in the *s*-domain to model the initial conditions, $v_C(0)$ and $i_L(0)$, as current sources:

$$\text{Resistor:}\quad V_R(s) = RI_R(s) \rightarrow I_R(s) = \left(\frac{1}{R}\right)V_R(s)$$

$$\text{Capacitor:}\quad V_C(s) = \frac{1}{sC}I_C(s) + \frac{v_C(0)}{s} \rightarrow I_C(s) = (sC)V_C(s) - Cv_C(0)$$

$$\text{Inductor:}\quad V_L(s) = sLI_L(s) - Li_L(0) \rightarrow I_L(s) = \left(\frac{1}{sL}\right)V_L(s) + \frac{i_L(0)}{s}$$

You see there are no integrals or derivatives in the *s*-domain.

The middle column of Figure 17-1 shows the constraints of the passive devices in the time-domain being converted to the *s*-domain. The left column shows initial conditions modeled as voltage sources in the *s*-domain, and the right column shows initial conditions modeled as current sources in the *s*-domain.

Taking the initial conditions into account in the *s*-domain analysis for capacitors and inductors is a big deal because it expedites the analysis. When you transform differential equations into the *s*-domain, you deal with input sources and initial conditions simultaneously.

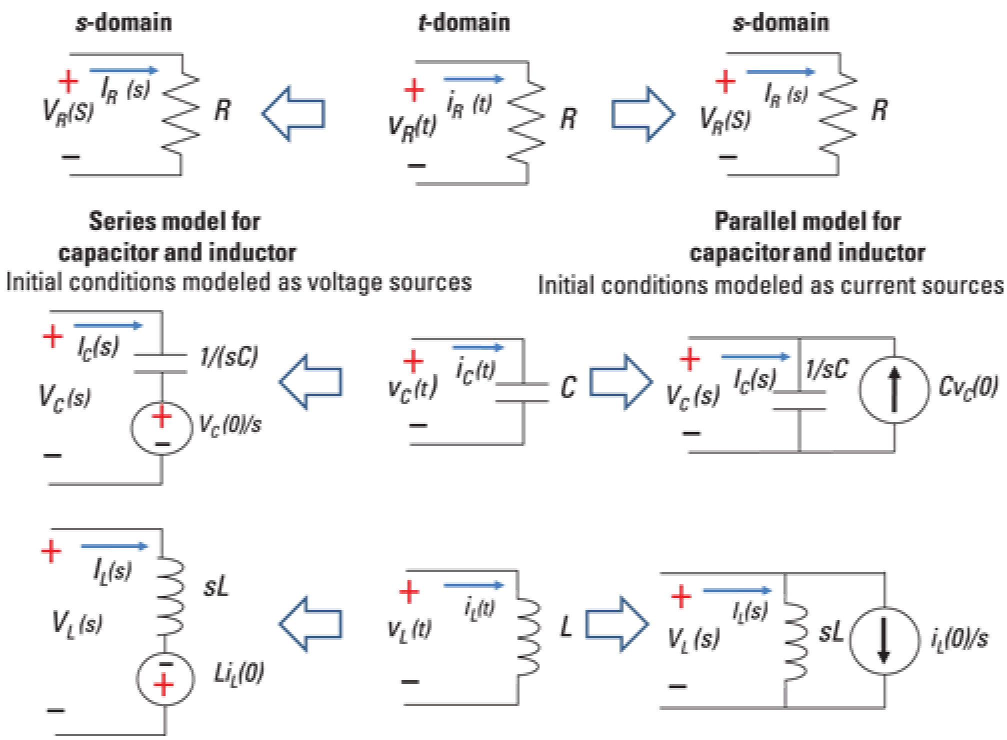


Illustration by Wiley, Composition Services Graphics

Figure 17-1: The *s*-domain models of passive devices.

Op-amp devices

The constraints of ideal operational amplifiers are unchanged in form in the *s*-domain:

Voltage constraint: $v_P(t) = v_N(t) \rightarrow V_P(s) = V_N(s)$

Current constraint: $i_P(t) = i_N(t) = 0 \rightarrow I_P(s) = I_N(s) = 0$

Impedance and admittance

Impedance Z (see Chapter 15) relates the voltage and current described in the s-domain when initial conditions are set to 0. The following algebraic form of the i-v relationship describes impedance in the s-domain:

$$V(s) = Z(s)I(s)$$

Admittance Y is the reciprocal of the impedance; it's useful when you're analyzing parallel circuits:

$$Y(s) = \frac{1}{Z(s)}$$

In the s-domain for zero initial conditions, the element constraints, impedances $Z(s)$, and admittances $Y(s)$ for the passive devices are as follows:

Resistor: $V_R(s) = RI_R(s) \to Z_R(s) = R$ or $Y_R(s) = \frac{1}{R}$

Capacitor: $V_C(s) = \frac{1}{sC} I_C(s) \to Z_C(s) = \frac{1}{sC}$ or $Y_C(s) = sC$

Inductor: $V_L(s) = sLI_L(s) \to Z_L(s) = sL$ or $Y_L(s) = \frac{1}{sL}$

Now you're ready to start analyzing circuits in the s-domain — without having to rely on calculus.

Seeing How Basic Circuit Analysis Works in the s-Domain

Circuit analysis techniques in the s-domain are powerful because you can treat a circuit that has voltage and current signals changing with time as though it were a resistor-only circuit. That means you can analyze the circuit algebraically, without having to mess with integrals and derivatives. In the following sections, you see how to apply voltage and current divider methods in the *s*-domain.

Applying voltage division with series circuits

You can put voltage divider techniques to work when dealing with series circuits, as Chapter 4 explains. To use voltage division in the *s*-domain, you simply replace the resistors with the impedances of devices connected in series. The following voltage divider equation is for three passive devices in a series circuit:

$$v_1(t)=v_s(t)\left(\frac{R_1}{R_1+R_2+R_3}\right)\rightarrow V_1(s)=V_S(s)\left(\frac{Z_1(s)}{Z_1(s)+Z_2(s)+Z_3(s)}\right)$$

The output voltage $V_1(s)$ is based on the voltage source $V_S(s)$ and on the ratio of the desired impedance $Z_1(s)$ to the total impedance.

Figure 17-2 illustrates the voltage divider for a series circuit for zero initial conditions: $i_L(0) = 0$ and $v_C(0) = 0$. You can find the output transform of the capacitor voltage using the voltage divider equation:

$$V_C(s)=V_S(s)\left(\frac{\frac{1}{sC}}{R+sL+\frac{1}{sC}}\right)$$

In a similar way, the voltage transform across the inductor is

$$V_L(s)=V_S(s)\left(\frac{sL}{R+sL+\frac{1}{sC}}\right)$$

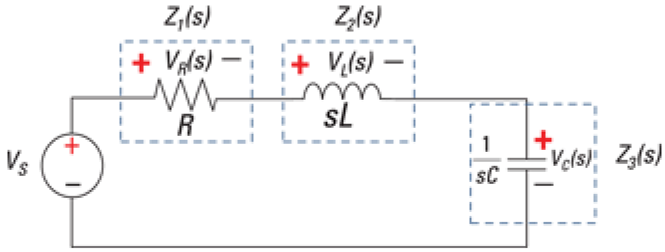


Illustration by Wiley, Composition Services Graphics

Figure 17-2: A series circuit and the voltage divider technique in the *s*-domain.

And the voltage transform across the resistor is

$$V_R(s) = V_S(s)\left(\frac{R}{R + sL + \frac{1}{sC}}\right)$$

That’s all there is to it. You may need to do more algebraic gymnastics to simplify other circuits, but you still don’t need calculus. To get back to a time-domain description, you need to do a partial fraction expansion; then you look up the inverse Laplace transforms in the table in Chapter 16.

In many cases, you just want to predict what the output is when you’re given a particular input. When you know the *transfer function,* which is the ratio between the output transform and the input transform, you can multiply the transfer function by the input voltage to find the output. As a result, you can rewrite the transform of the capacitor voltage as a ratio of polynomials:

$$V_C(s) = V_S(s)\left(\frac{1}{LCs^2 + RCs + 1}\right) = V_S(s)\left(\frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}\right)$$

The denominator is simply a quadratic equation, and the roots of the equation shape the circuit behavior.

Similarly, you can rewrite the transform of the resistor and inductor voltages as a ratio of polynomials.

Turning to current division for parallel circuits

To use current division for parallel circuits having passive devices, all you have to do in the *s*-domain is replace the conductances with admittances. The following current divider equation is for three passive devices connected in parallel:

$$i_1(t)=i_s(t)\left(\frac{G_1}{G_1+G_2+G_3}\right)\rightarrow I_1(s)=I_S(s)\left(\frac{Y_1(s)}{Y_1(s)+Y_2(s)+Y_3(s)}\right)$$

The output current $I_1(s)$ is based on the current source $I_S(s)$ and the ratio of the desired admittance $Y_1(s)$ to the total admittance.

Figure 17-3 illustrates the current divider technique for a parallel circuit for zero initial conditions: $i_L(0) = 0$ and $v_C(0) = 0$. You can find the output transform of the inductor current using the current divider equation:

$$I_L(s)=I_S(s)\left(\frac{\frac{1}{sL}}{G+sC+\frac{1}{sL}}\right)$$

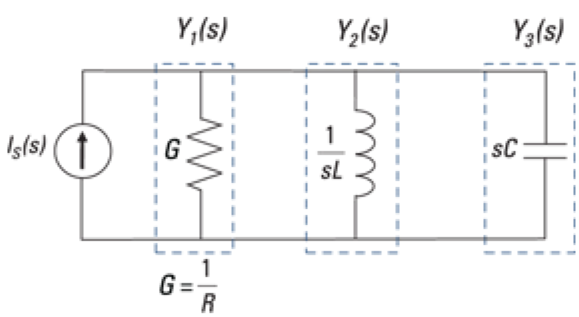


Illustration by Wiley, Composition Services Graphics

Figure 17-3: Parallel circuits and the current divider technique in the *s*-domain.

In the same way, you get the transform of the capacitor and conductance (or resistor) currents using the current divider technique:

$$I_C(s) = I_S(s)\left(\frac{sC}{G + sC + \frac{1}{sL}}\right)$$

$$I_R(s) = I_S(s)\left(\frac{G}{G + sC + \frac{1}{sL}}\right)$$

Note that the results resemble the form for series circuits using voltage divider techniques. Neat and simple in the *s*-domain — thank you, Pierre Laplace!

Conducting Complex Circuit Analysis in the s-Domain

In the time-domain, analyzing circuits with resistors, inductors, and capacitors involves integrals and derivatives. You use a simpler algebraic approach by describing and analyzing such circuits in the *s*-domain, as I show you next. The following sections cover node-voltage analysis, mesh-current analysis, superposition, and Norton and Thévenin equivalents in the *s*-domain.

Using node-voltage analysis

In the *s*-domain, node-voltage analysis works the same way as it does for resistor-only circuits, but this time you replace a device with its impedance. Node-voltage analysis (see Chapter 5) allows you to work with a smaller set of equations and unknowns that you need to deal with simultaneously. The unknown variables are called *node voltages*. After you find the unknown voltages, you can find the voltages and currents for each device.

Look at Figure 17-4, which shows a circuit at zero state using an op amp, resistor, and capacitor. You need to find the transfer function $V_O(s)/V_S(s)$. Applying KCL at Node A produces the following:

$$\frac{V_N - V_S}{R_1} + \frac{V_N - V_O}{R_2 + \frac{1}{sC}} + I_N = 0$$

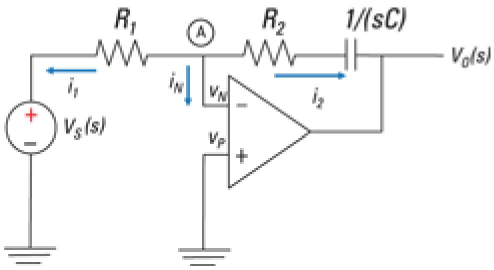


Illustration by Wiley, Composition Services Graphics

Figure 17-4: Op-amp node-voltage analysis in the *s*-domain.

For an ideal op amp, $I_N = 0$ and $V_N = V_P = 0$ because V_P is connected to ground. The KCL equation becomes

$$\frac{V_S}{R_1} = -\frac{V_O}{R_2 + \frac{1}{sC}}$$

After some algebra, you have the transfer function $V_O(s)/V_S(s)$:

$$\frac{V_O(s)}{V_S(s)} = -\frac{R_2 + \frac{1}{sC}}{R_1} = -\left(\frac{R_2}{R_1}\right)\left(\frac{s + \frac{1}{R_2C}}{s}\right)$$

If the input is a step input $u(t)$ and its transform is $V_S(s) = 1/s$, the output transform becomes

$$V_O(s) = -\left(\frac{R_2}{R_1}\right)\left(\frac{s + \frac{1}{R_2C}}{s^2}\right) = -\left(\frac{R_2}{R_1}\right)\left(\frac{1}{s} + \frac{1}{R_2Cs^2}\right)$$

Use the table in Chapter 16 to get the inverse Laplace transform:

$$v_O(t) = -\left(\frac{R_2}{R_1} + \frac{1}{R_1 C} r(t)\right) u(t)$$

The output $v_O(t)$ is a combination of a ramp and a step input. You get this when the circuit acts like an inverting amplifier and an integrator. You have a ramp resulting from the integration of a step input. The inverting amplifier comes into play when the capacitor acts like a short circuit, which occurs at high frequencies for sinusoidal inputs.

Using mesh-current analysis

Mesh-current analysis (see Chapter 6), which is useful when a circuit has several loops, works the same way in the *s*-domain as it does in resistor-only circuits. You simply use a device's impedance to work the problem. After you solve for the mesh currents, you can find the voltage and current for each device.

Consider the circuit in Figure 17-5. You want to formulate the mesh current equation and solve for the zero-input and zero-state responses. The circuit is transformed into the *s*-domain.

For mesh-current analysis, you need to use the voltage source model of initial conditions; this will give you a circuit with voltage sources. You have the following mesh equations for Loops A and B:

$$\text{Mesh A:} \quad \left(R_1 + \frac{1}{sC}\right) I_A - \left(\frac{1}{sC}\right) I_B = V_S(s)$$

$$\text{Mesh B:} \quad -\left(\frac{1}{sC}\right) I_A + \left(R_2 + sL + \frac{1}{sC}\right) I_B = 0$$

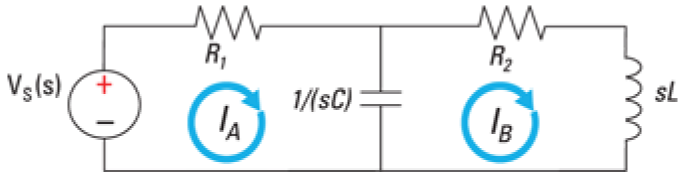


Illustration by Wiley, Composition Services Graphics

Figure 17-5: Mesh-current analysis in the *s*-domain.

The matrix software should give you the following result, the algebraic equivalent for $I_B(s)$:

$$I_B(s) = \frac{V_S(s)}{sC\left[\left(R_1 + \frac{1}{sC}\right)\left(R_2 + sL + \frac{1}{sC}\right) - \left(\frac{1}{sC}\right)^2\right]}$$

Using superposition and proportionality

The superposition concept basically says you can take an output *v* as a combination of weighted inputs. When applied to resistor circuits, the superposition concept (presented in Chapter 7) is described as

$$v_o(t) = K_1 v_1 + K_2 v_2 + \cdots + K_n v_n$$

You can apply the same concept to linear circuits in the *s*-domain just by replacing the weighted constants with rational functions of *s*. Then you can look for the response as a sum of the zero-input response due to initial conditions with inputs turned off and the zero-state response due to external sources (inputs) with initial conditions turned off, which means no energy is stored. (You can review these two concepts in Chapters 13 and 14.) You turn off voltage sources by replacing them with short circuits and turn off current sources by replacing them with open circuits.

To see how to use superposition in the *s*-domain, check out Figure 17-6 where $v_s(t)$ is a step input $u(t)$. The upper-left diagram describes an RC series circuit in the time domain, and the bottom-left diagram shows the same circuit described in the *s*-domain. I use this example to kill two birds with one stone: I apply superposition to find the zero-state $V_{ZS}(s)$ or $I_{ZS}(s)$ and the zero-input $V_{ZI}(s)$ or $I_{ZI}(s)$ transform responses, and I show you how to solve the problem by converting a differential equation or integral equation into the Laplace transform.

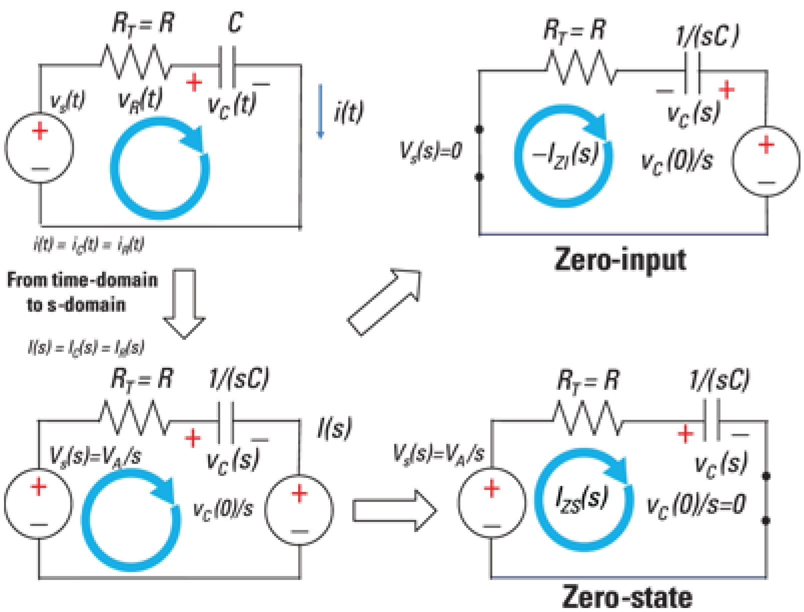


Illustration by Wiley, Composition Services Graphics

Figure 17-6: Zero-state and zero-input transforms using superposition.

First, you need to turn off the input source by replacing the voltage source with a short circuit, as the top-right diagram in Figure 17-6 shows. The result is the zero-input response:

$$I_{ZI}(s) = -\frac{\frac{v_C(0)}{s}}{R + \frac{1}{sC}} = -\frac{\frac{v_C(0)}{R}}{s + \frac{1}{RC}}$$

The minus sign appears because the current is opposite to the assigned current direction in Figure 17-6. The pole at $s = -1/(RC)$ comes from the circuit. Next, you need to turn off the initial condition modeled as a voltage source by replacing it with a short circuit. You see the zero-state diagram in the lower right of Figure 17-6. You now have the zero-state response for a step input:

$$I_{ZS}(s) = \frac{\frac{V_A}{s}}{R + \frac{1}{sC}} = \frac{\frac{V_A}{R}}{s + \frac{1}{RC}}$$

The pole for the zero-state response is $s = -1/(RC)$ from the circuit. Now use superposition to get the total response. Superposition says the total response is the sum of the zero-state and zero-input outputs:

$$\begin{aligned} I(s) &= I_{ZI}(s) + I_{ZS}(s) \\ &= -\frac{v_C(0)/R}{s + \frac{1}{RC}} + \frac{V_A/R}{s + \frac{1}{RC}} \end{aligned}$$

The table in Chapter 16 tells you that the inverse Laplace transform is an exponential. The inverse Laplace transform of $I(s)$ gives you the time response $i(t)$:

$$i(t) = -\frac{v_C(0)}{R} e^{-t/RC} u(t) + \frac{V_A}{R} e^{-t/RC} u(t)$$

This Laplace stuff really works! Calculus doesn't come into play at all — all you need to do is look up transform pairs in a table to get the time response.

Now take a look at the lower-right circuit in Figure 17-6, which describes the circuit in zero-state in the s-domain. The differential equation for this circuit is based on KVL, given the capacitor voltage as an output variable and replacing forcing function $v_T(t)$ with a step input $V_A u(t)$:

$$v_T(t) = RC\frac{dv(t)}{dt} + v(t)$$

$$V_A u(t) = RC\frac{dv_C(t)}{dt} + v_C(t)$$

Taking the Laplace transform of this equation gives you

$$\frac{V_A}{s} = RC\left(sV_C(s) - v_C(0)\right) + V_C(s)$$

Solve for the transform of the capacitor voltage $V_C(s)$:

$$V_C(s)(RCs+1)=\frac{V_A}{s}+RCv_C(0)$$

$$V_C(s)=\left[\frac{1}{s(RCs+1)}\right]V_A+\left[\frac{RC}{RCs+1}\right]v_C(0)$$

The preceding equation shows you how the forcing function $V_A u(t)$ and the initial condition $v_C(0)$ are taken into account with one step based on the *s*-domain techniques. Performing a partial fraction expansion on the preceding equation gives you

$$V_C(s)=\frac{V_A}{s}-\frac{V_A}{s+\frac{1}{RC}}+\frac{v_C(0)}{s+\frac{1}{RC}}$$

Now take the inverse Laplace transform using the table in Chapter 16 to get the capacitor voltage response $v_C(t)$ in the time-domain:

$$v_C(t)=V_A\left(1-e^{-\left(\frac{t}{RC}\right)}\right)+v_C(0)e^{-\left(\frac{t}{RC}\right)}$$

Taking the derivative of $v_C(t)$ leads you to the capacitor current:

$$i_C(t)=C\frac{dv_C(t)}{dt}$$
$$=\frac{V_A}{R}e^{-\left(\frac{t}{RC}\right)}-\frac{v_C(0)}{R}e^{-\left(\frac{t}{RC}\right)}$$

You get the same capacitor current $i_C(t)$, whether you transform the circuit or transform the differential equation. If you don't like to take the derivative, you can start describing the circuit as an integral equation that just involves the capacitor current $i_C(t)$.

If you use the capacitor current $i_C(t)$ as the output variable, then the KVL equation becomes

$$V_A u(t)=Ri_C(t)+\frac{1}{C}\int i_C(t)dt+v_C(0)$$

Next, perform a Laplace transformation of the preceding equation:

$$\frac{V_A}{s} = RI_C(s) + \frac{I_C(s)}{sC} + \frac{v_C(0)}{s}$$

Solve for the capacitor current $I_C(s)$:

$$\left(R + \frac{1}{sC}\right) I_C(s) = \frac{V_A(s)}{s} - \frac{v_C(0)}{s}$$

$$I_C(s) = \frac{V_A(s)}{R\left(s + \frac{1}{RC}\right)} - \frac{v_C(0)}{R\left(s + \frac{1}{RC}\right)}$$

Again, see how the forcing transform $V_A(s)$ and initial condition $v_C(0)$ are neatly separated components for the capacitor current $I_C(s)$.

Finally, take the inverse Laplace transform of the preceding equation:

$$i_C(t) = \frac{V_A}{R} e^{-\left(\frac{t}{RC}\right)} - \frac{v_C(0)}{R} e^{-\left(\frac{t}{RC}\right)}$$

Using the Thévenin and Norton equivalents

The Thévenin equivalent I present in Chapter 8 simplifies a circuit to one voltage source $v_T\,(t)$ and one single resistor R_T. Extending the concept to circuits described in the s-domain means replacing the Thévenin resistance R_T with an impedance $Z_T(s)$.

Similarly, the Norton equivalent replaces a complex circuit with a single current source $i_N(t)$ in parallel with the Norton resistor $R_N = R_T$. Extending the Norton concept to the s-domain means replacing the Norton resistance R_N with the impedance $Z_N(s) = Z_T(s)$. Figure 17-7 gives you the visual of how the Thévenin and Norton equivalents reduce circuits in the s-domain.

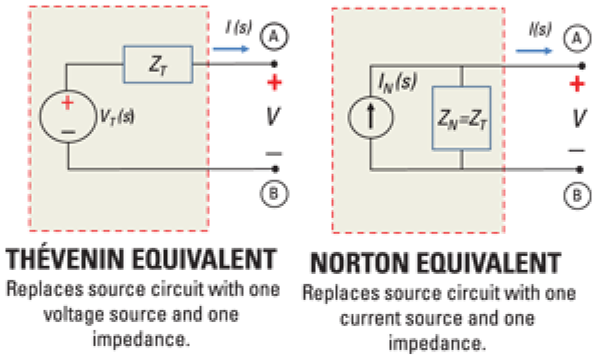


Illustration by Wiley, Composition Services Graphics

Figure 17-7: The *s*-domain Thévenin and Norton equivalents.

Take a look at the circuit in its zero state in Figure 17-8. The Thévenin and Norton equivalents are related by a source transformation, so use a source transformation to the left of Points A and B. The source transformation converts the Norton source circuit consisting of the independent current source $I_N = I_1(s)$ in parallel with an impedance $Z_N = R$ to a Thévenin equivalent. The Thévenin equivalent consists of a voltage source $V_T = I_N Z_N = RI_1(s)$ in series with $Z_T = Z_N = R$.

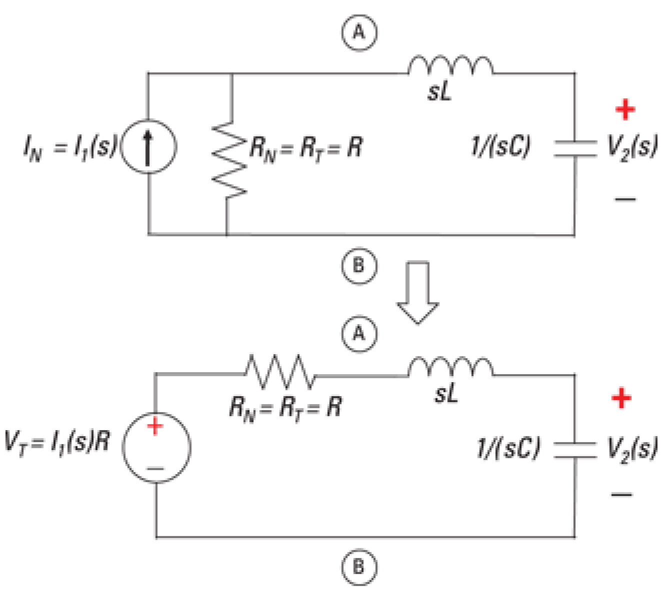


Illustration by Wiley, Composition Services Graphics

Figure 17-8: The *s*-domain source transformation and Thévenin equivalent.

You use voltage division to find the relationship between the output $V_2(s)$ and the input $I_1(s)$ in the s-domain:

$$V_2(s) = \left[\frac{\frac{1}{sC}}{R + sL + \frac{1}{sC}} \right] \underbrace{RI_1(s)}_{V_T(s)}$$
$$= \left[\frac{R}{LCs^2 + RCs + 1} \right] I_1(s)$$

Factoring out the coefficient LC in the denominator gives you

$$V_2(s) = \left[\frac{R/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \right] I_1(s)$$

To emphasize the Thévenin equivalent when you have circuits with capacitors and inductors, take a look at the bottom diagram of Figure 17-8. The equivalent Thévenin impedance looking to the left from the capacitor terminals is simply the series connection of resistor R and the inductor impedance sL (or mathematically, $ZT = R + sL$).

Chapter 18

Focusing on the Frequency Responses

In This Chapter

- Understanding frequency response and types of filters
- Interpreting Bode plots
- Using circuits to create high-pass, low-pass, band-pass, and band-reject filters

When you hear your favorite music coming from various instruments and melodic voices, the unique sounds you hear consist of many frequencies. In a stereo system, you can adjust the low-frequency and high-frequency sounds by adjusting a stereo equalizer. Equalizers adjust the volume of a specific band of frequencies relative to others. They’re often used to boost the bass guitar on bass-hungry speakers or to bring out the vocals of a favorite singer.

With a combination of resistors, capacitors, and inductors, you can select or reject a range of frequencies. As a result, you can pick out frequencies to boost or cut. For audio applications, you can adjust the bass, treble, or midrange frequencies to get the sound quality you like best. You also find wide applications of frequency response and filtering in communication, control, and instrumentation systems.

How is this all possible? A major component found in older entertainment systems is an electronic filter that shapes the frequency content of signals. You can describe low-pass filters, high-pass filters, band-pass filters, and band-reject filters based on simple circuits. This serves as a foundation for more-complex filters to meet more stringent requirements.

What happens when you want to study a range of frequencies? You use Bode plots. Bode plots help you visualize how poles and zeros affect the frequency response of a circuit.

This chapter shows you what different filters do, explains how Bode plots work, and shows how you can create filters by connecting resistors, inductors, and capacitors.

Describing the Frequency Response and Classy Filters

You find the sinusoidal steady-state output of the filter by evaluating the transfer function $T(s)$ at $s = j\omega$. The transfer function relates the input and output signals in the s-domain and assumes zero initial conditions. The radian frequency ω is a variable that stands for the frequency of the sinusoidal input. After you substitute the $s = j\omega$ into $T(s)$, the transfer function becomes a ratio of complex numbers $T(j\omega)$.

REMEMBER Because the function $T(j\omega)$ is a complex number for all frequencies, you can determine the gain $|T(j\omega)|$ and phase $\theta(j\omega)$. Here are the gain and phase relationships:

$$|T(j\omega)| = \frac{\text{Output amplitude}}{\text{Input amplitude}}$$

$$\theta(j\omega) = \angle T(j\omega)$$
$$= \text{Output phase} - \text{Input phase}$$

You can present the gain and phase as a function of frequency ω graphically, as in Figure 18-1. This figure shows an approximation of a typical filter. In a *passband* region, the gain function has nearly constant gain for a range of frequencies. In the *stopband* region, the gain is significantly reduced for a range of frequencies.

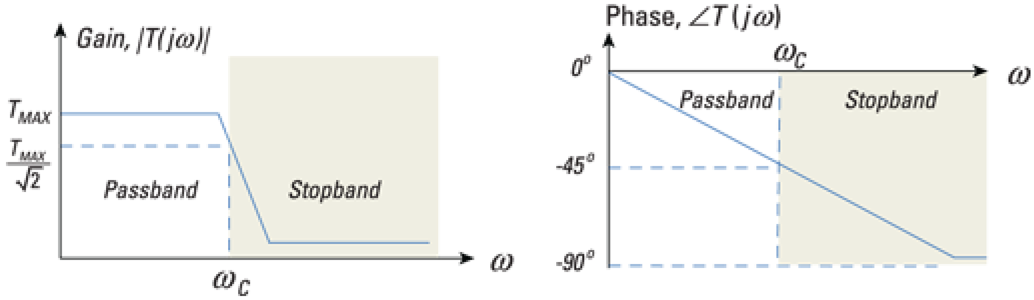


Illustration by Wiley, Composition Services Graphics

Figure 18-1: Gain and phase plots of the frequency response.

For nonideal filters, a *transition* region occurs between adjacent passband and stopband regions. The cutoff frequency ω_C occurs within the transition region, according to a prescribed definition. One widely used definition says the *cutoff* frequency occurs when the passband gain is decreased by a factor of 0.707 from a maximum value T_{MAX}. The mathematical condition for ω_C is therefore

$$|T(j\omega)| = \frac{1}{\sqrt{2}} T_{MAX} = 0.707 \cdot T_{MAX}$$

At the cutoff frequency, the output power has dropped to one half of its maximum passband value. Here, the passband includes those frequencies where the relative power is greater than the half-power point (0.707 of the maximum value of the transfer function). Frequencies that are less than the half-power point fall in the stopband.

The following sections introduce you to four types of filters. The filters differ in whether they block the frequencies above or below the cutoff frequencies or allow them to pass.

Low-pass filter

The low-pass filter has a gain response with a frequency range from zero frequency (DC) to ω_C. Any input that has a frequency below the cutoff frequency ω_C gets a pass, and anything above it gets attenuated or rejected. The gain approaches zero as frequency increases to infinity.

Figure 18-2 shows the frequency response of a low-pass filter. The input signal has equal amplitudes at frequencies ω_1 and ω_2. After passing through the low-pass filter, the output amplitude at ω_1 is unaffected because it's below the cutoff frequency ω_C. However, at ω_2, the signal amplitude is significantly decreased because it's above ω_C.

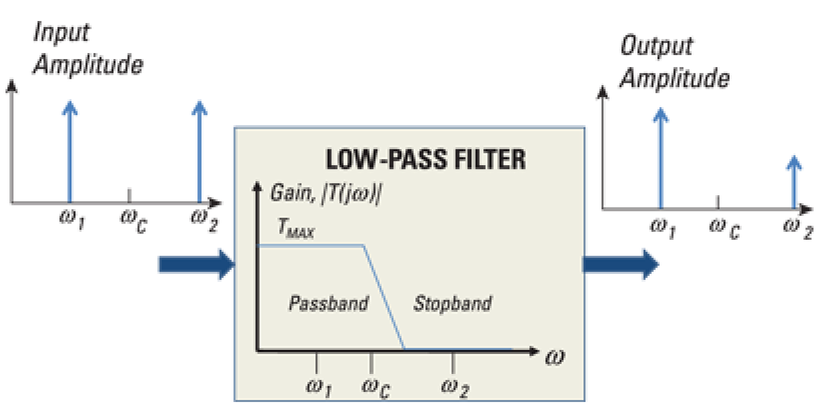


Illustration by Wiley, Composition Services Graphics

Figure 18-2: Gain response of a low-pass filter.

High-pass filter

The high-pass filter has a gain response with a frequency range from ω_C to infinity. Any input having a frequency below the cutoff frequency ω_C gets attenuated or rejected. Anything above ω_C passes through unaffected.

Figure 18-3 shows the frequency response of a high-pass filter. The input signal has equal amplitude at frequencies ω_1 and ω_2. After passing through the high-pass filter, the output amplitude at ω_1 is significantly decreased because it's below ω_C, and at ω_2, the signal amplitude passes through unaffected because it's above ω_C.

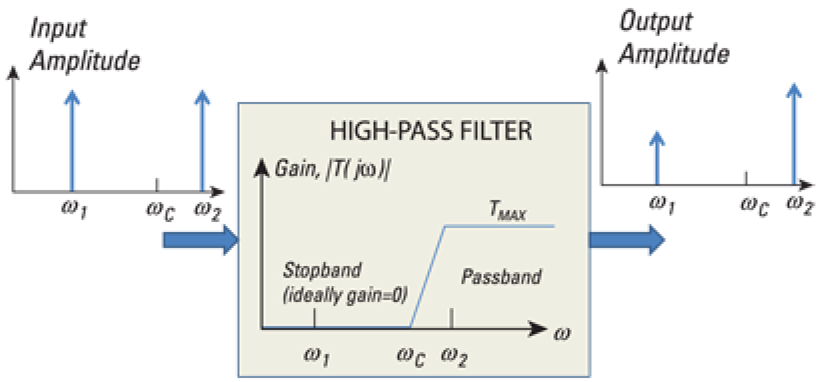


Illustration by Wiley, Composition Services Graphics

Figure 18-3: Gain response of a high-pass filter.

Band-pass filters

The *band-pass filter* has a gain response with a frequency range from ω_{C1} to ω_{C2}. Any input that has frequencies between ω_{C1} and ω_{C2} gets a pass, and anything outside this range gets attenuated or rejected.

Figure 18-4 shows the frequency response of a band-pass filter. The input signal has equal amplitude at frequencies ω_1, ω_2, and ω_3. After passing through the band-pass filter, the output amplitudes at ω_1 and ω_3 are significantly decreased because they fall outside the desired frequency range, while the frequency at ω_2 is within the desired range, so its signal amplitude passes through unaffected.

TIP

You can think of the band-pass filter as a series or cascaded connection of a low-pass filter with frequency ω_{C2} and a high-pass filter with frequency ω_{C1}. The bottom diagram of Figure 18-4 shows how the cascade connection of a low-pass filter and high-pass filter forms a band-pass filter. Although the figure shows the low-pass filter before the high-pass filter, the order of the filters doesn't matter.

WARNING!

If you're going to do a quick-and-dirty design of a band-pass filter based on a low-pass filter and high-pass filter, make sure you select the right cutoff frequencies. In Figure 18-4, if you give the low-pass filter a lower cutoff frequency of ω_{C1} and the high-pass filter an upper cutoff frequency of ω_{C2}, you'll get a very small signal at the output. What you'll design in that case is a *no-pass* filter — everything gets rejected.

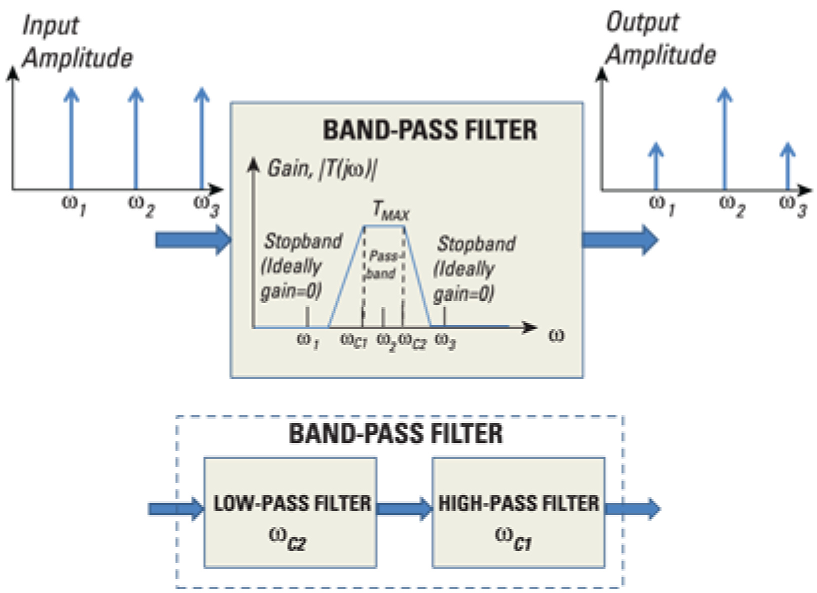


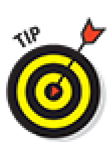
Illustration by Wiley, Composition Services Graphics

Figure 18-4: Gain response of a band-pass filter.

Band-reject filters

The *band-reject filter,* or *bandstop filter,* has a gain response with a frequency range from zero to ω_{C1} and from ω_{C2} to infinity. Any input that has frequencies between ω_{C1} and ω_{C2} gets significantly attenuated, and anything outside this range gets a pass.

Figure 18-5 shows the frequency response of a band-reject filter. The input signal has equal amplitude at frequencies ω_1, ω_2, and ω_3. After passing through the band-reject filter, the output amplitude at ω_1 and ω_3 is unaffected because those frequencies fall outside the range of ω_{C1} to ω_{C2}. But at ω_2, the signal amplitude gets attenuated because it falls within this range.



You can think of the band-pass filter as a parallel connection of a low-pass filter with cutoff frequency

ω_{C1} and a high-pass filter with cutoff frequency ω_{C2}.with their outputs added together. The bottom diagram of Figure 18-5 shows the parallel connection of a low-pass filter and high-pass filter to form a band-reject filter.

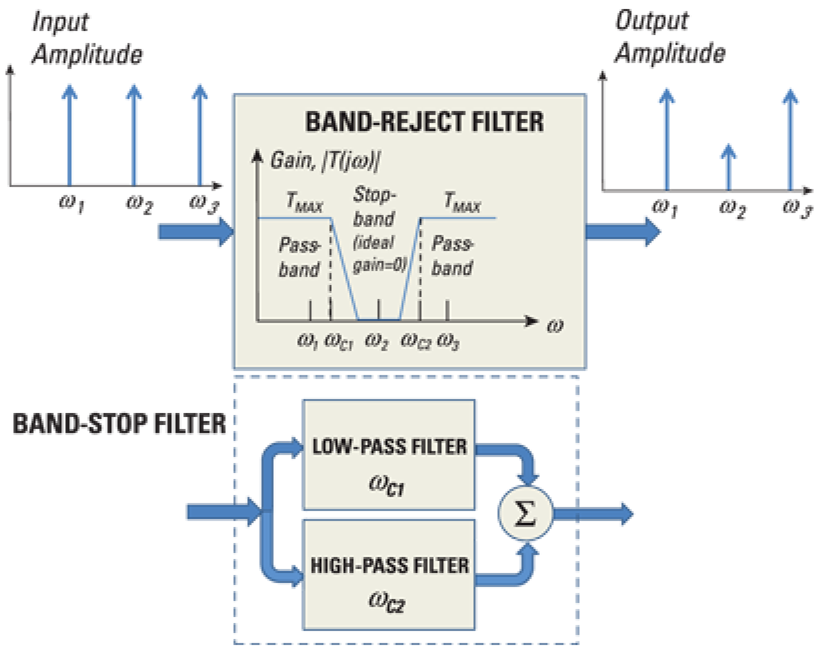


Illustration by Wiley, Composition Services Graphics

Figure 18-5: Gain response of a band-reject filter.

WARNING!

Make sure you select the right cutoff frequencies when you do a quick-and-dirty design of a band-reject filter based on a low-pass filter and high-pass filter connected in parallel. In Figure 18-5, if you give the low-pass filter a lower cutoff frequency of ω_{C2} and the high-pass filter an upper cutoff frequency of ω_{C1}, you'll have signals of all frequencies passing through the filter — not good for a band-reject filter. What you'll design instead is an *all-pass* filter. It's like using a coffee filter with a big, fat hole in it —

everything passes through, including the coffee grounds.

Plotting Something: Showing Frequency Response à la Bode

You can express the frequency response gain $|T(j\omega)|$ in terms of decibels. Using decibels compresses the magnitude and the frequency in a logarithmic scale so you don't need more than 10 feet of paper for your plots. *Decibels* are defined as

$$|T(j\omega)|_{dB} = 20\log_{10}|T(j\omega)|$$

For example, if the gain is $|T(j\omega)| = 100$, the gain in decibels is 40 dB. Also, a gain of 1 is 0 dB.

REMEMBER At the cutoff frequency ω_C, which is commonly defined as $T_{MAX}/\sqrt{2}$, you have the following gain:

$$\begin{aligned}|T(j\omega_C)|_{dB} &= 10\log_{10}\left|\frac{T_{MAX}}{\sqrt{2}}\right|^2 \\ &= 20\log_{10}\left|\frac{T_{MAX}}{\sqrt{2}}\right| = -3\text{ dB}\end{aligned}$$

Therefore, the cutoff frequency is also referred to as the *–3 dB point* or the *half-power point.* Why? Because the previous set of equations involving a transfer function can be viewed as the square of either the voltage or the current transfer function. Squaring the transfer function gives you the power ratio between the output and input signal transforms because the square of the voltage or current is proportional to power. To jog your memory and give you further insight into the –3 db point as a half-power point, see Chapter 2's section on calculating the power dissipated by resistors.

The log-frequency plots of the gain $|T(j\omega)|$ and phase $\theta(\omega)$ are called *Bode plots,* or *Bode diagrams.* In the following sections, I introduce you to basic Bode plots and help you interpret them.

Looking at a basic Bode plot

Bode plots come in pairs to describe the frequency response of circuits. Usually, you have

- A log-frequency gain plot in decibels given in the top diagram
- A log-frequency phase plot in degrees given in the bottom diagram

Figure 18-6 shows a sample Bode plot.

The horizontal axis usually comes in one of the following log-frequency scales, usually decades:

- **Octaves:** An octave has a frequency range whose upper limit is twice the lower limit (2:1 ratio). For example, the voice usually ranges from 2 kHz to 4 kHz, spanning about 1 octave.
- **Decades:** A decade has a range with a 10:1 ratio. For example, human hearing usually ranges from 20 Hz to 20 kHz (20×10^3 Hz), so it spans 3 decades.

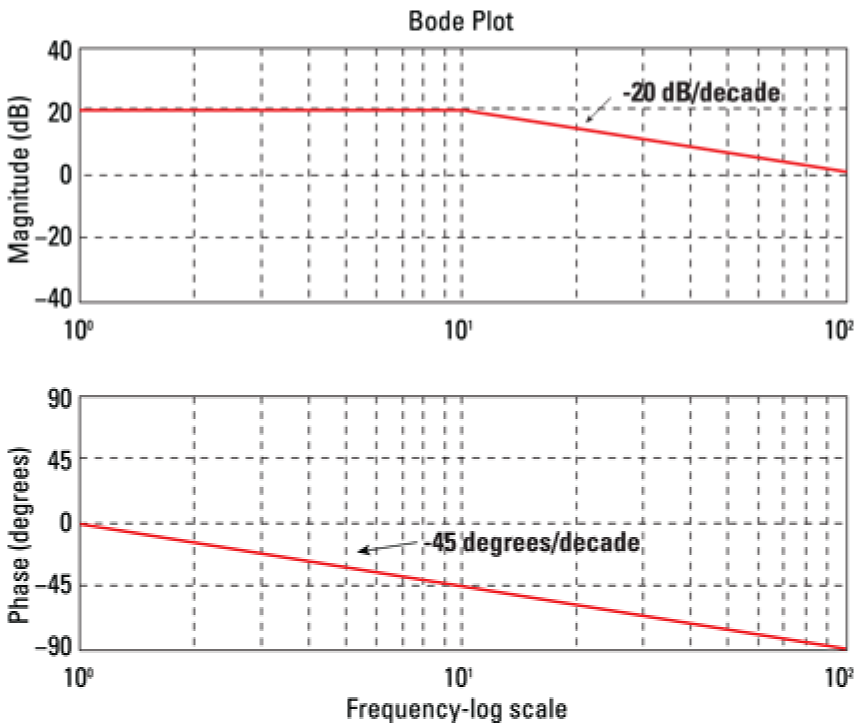


Illustration by Wiley, Composition Services Graphics

Figure 18-6: A sample Bode plot.

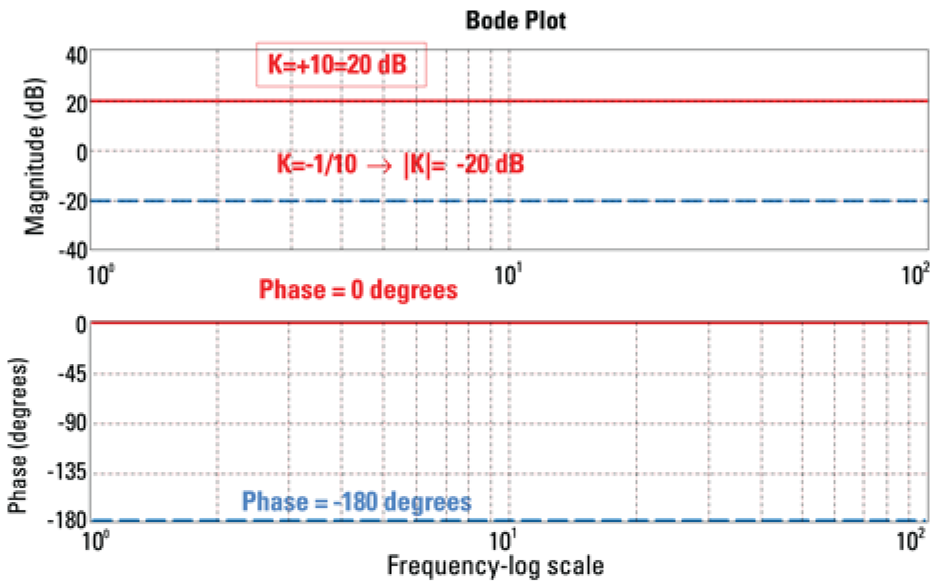
Poles, zeros, and scale factors: Picturing Bode plots from transfer functions

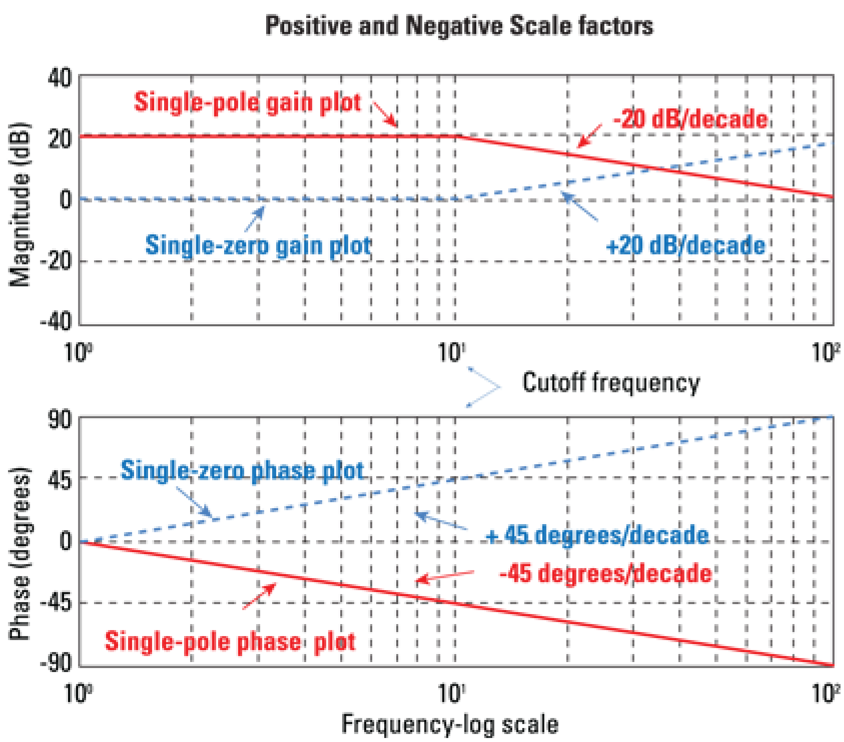
Most of the time, you use engineering software to draw Bode plots. But you can approximate Bode plots by hand — or at least notice when the computer-generated plot is messed up — if you understand how the transfer function’s poles and zeros shape the frequency response. The *poles,* of course, are the roots of the transfer function’s denominator, and *zeros* are the roots of its numerator.

Table 18-1 shows some basic, approximate rules to bear in mind when examining transfer functions and Bode plots. Figure 18-7 shows the graphical interpretation for each of the items in Table 18-1.

Table 18-1 Relating Bode Plots to a Transfer Function

Characteristic of the Transfer Function, $T(j\omega)$	*Effects on the Gain Plot,* $\lvert T(j\omega)\rvert_{dB}$	*Effects on the Phase Plot,* $j\theta_{dB}$
Scale factor (gain)	Shifts the entire gain plot up or down without changing the cutoff (corner) frequencies	The phase Bode plot is unaffected if the scale factor is positive. If the scale factor is negative, the phase Bode plot shifts by ±180°.
Real pole	Introduces a slope of –20 dB/decade to the gain Bode plot, starting at the pole frequency	The phase Bode plot rolls off at a slope of –45°/decade. The phase at the pole is –45°. For frequencies greater than 10 times the pole frequency, the phase angle contributed by a single pole is approximately –90°.
Real zero	Introduces a slope of +20 dB/decade to the gain Bode plot, starting at the zero frequency	The phase Bode plot rolls off at a slope of +45°/decade. The phase at the zero is +45°. For frequencies greater than 10 times the zero frequency, the phase angle contributed by a single real zero is approximately +90°.
Integrator	Introduces a real pole at the origin; a real pole at the origin (an integrator 1/s) has a gain slope of –20 dB/decade passing through 0 dB at $\omega = 1$	The angle contributed by an integrator is –90° at all frequencies.
Differentiator	Introduces a real zero at the origin; a zero at the origin (a differentiator) has a gain slope of +20 dB/decade passing through at 0 dB at $\omega = 1$	The angle contributed by a differentiator is +90° at all frequencies.
Complex pair of poles	Provides a slope of –40 dB/decade	The phase Bode plot has a slope of –90°/decade. The phase at the complex pole frequency is –90°. For frequencies greater than 10 times the cutoff frequency, the phase angle contributed by a complex pair of poles is approximately –180°.
Complex pair of zeros	Provides a slope of +40 dB/decade	The phase Bode plot has a slope of +90°/decade. The phase at the complex zero frequency is +90°. For frequencies greater than 10 times the cutoff frequency, the phase angle contributed by a complex pair of zeros is approximately +180°.


Bode Plot
K=+10=20 dB
K=-1/10 → |K|= -20 dB
Magnitude (dB)
Phase = 0 degrees
Phase (degrees)
Phase = -180 degrees
Frequency-log scale


Positive and Negative Scale factors
Single-pole gain plot
-20 dB/decade
Magnitude (dB)
Single-zero gain plot
+20 dB/decade
Cutoff frequency
Single-zero phase plot
+ 45 degrees/decade
-45 degrees/decade
Phase (degrees)
Single-pole phase plot
Frequency-log scale

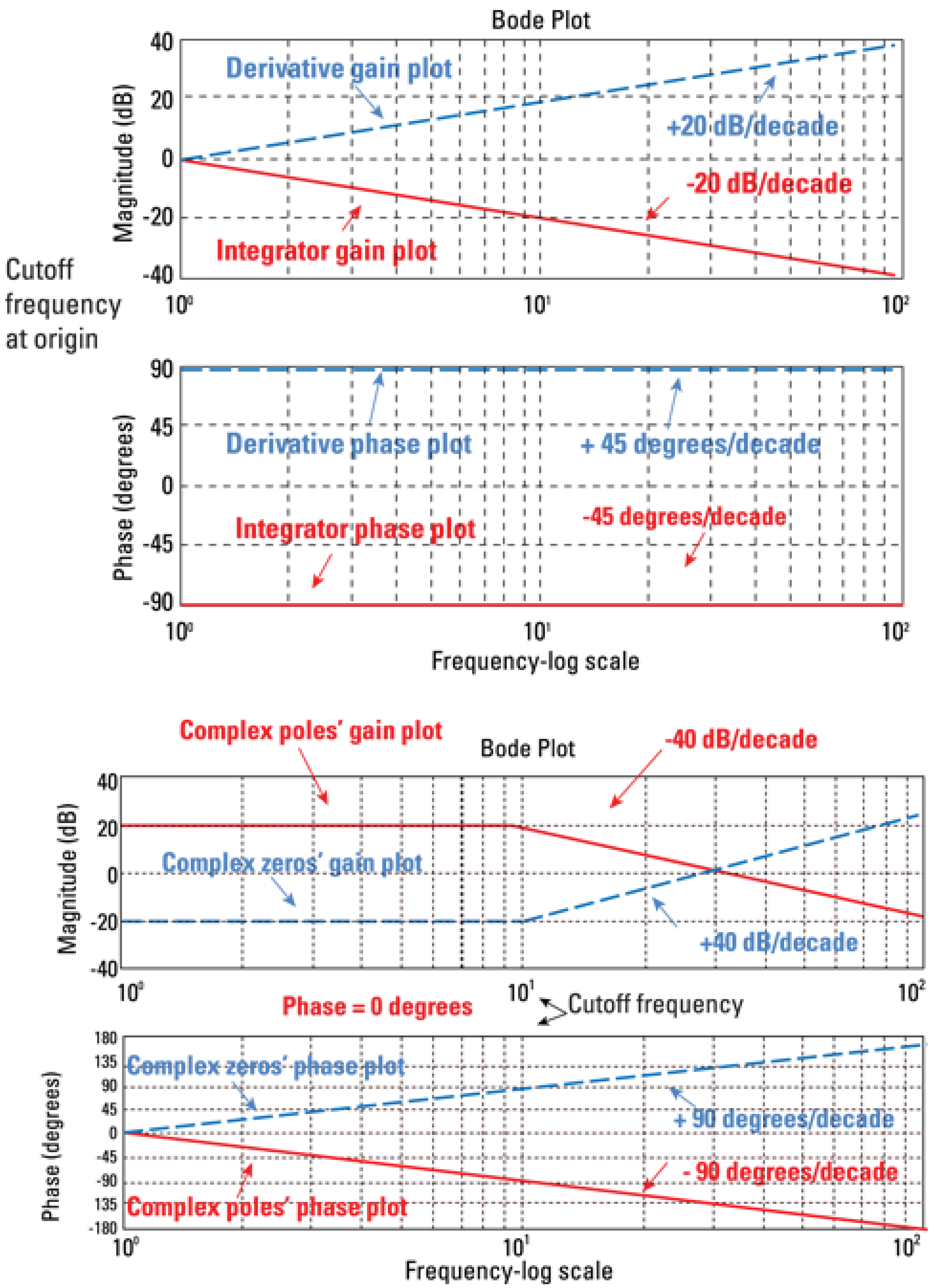


Illustration by Wiley, Composition Services Graphics

Figure 18-7: Bode diagrams of scale factors, poles, and zeros.

Turning the Corner: Making Low-Pass and

High-Pass Filters with RC Circuits

With simple first-order circuits, you can build low-pass and high-pass filters. These simple circuits can give you a foundational understanding of how filters work so you can build more-complex filters. In the following sections, I show you how to use RC circuits to build both low-pass and high-pass filters. Later in the chapter, I show you how to build band-pass and band-reject filters based on the low-pass and high-pass filters.

First-order RC low-pass filter (LPF)

Figure 18-8 shows an RC series circuit — a circuit with a resistor and capacitor connected in series. You can get a low-pass filter by forming a transfer function as the ratio of the capacitor voltage $V_C(s)$ to the voltage source $V_S(s)$.

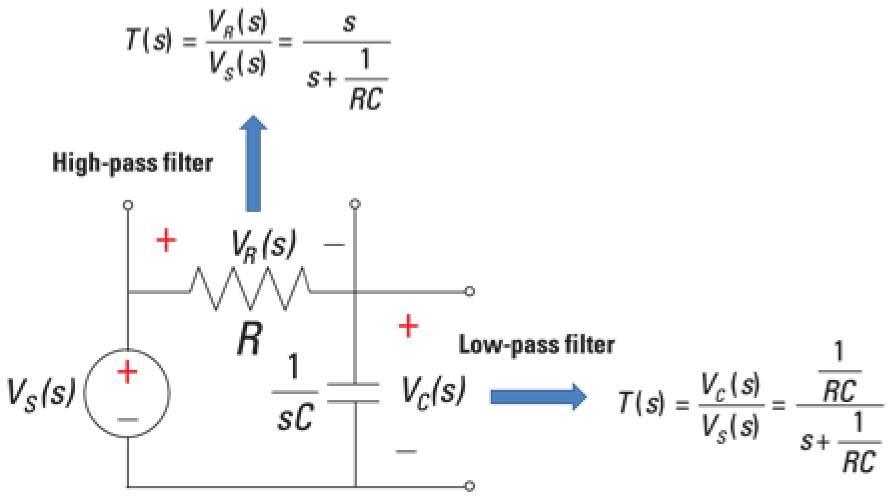


Illustration by Wiley, Composition Services Graphics

Figure 18-8: RC series circuits as a low-pass and high-pass filter.

You start with the voltage divider equation:

$$V_C(s) = V_S(s)\frac{\frac{1}{sC}}{R+\frac{1}{sC}}$$

The transfer function $T(s)$ equals $V_C(s)/V_S(s)$. With some algebra (including multiplying the numerator and denominator by s/R), you get a transfer function that looks like a low-pass filter:

$$T(s) = \frac{V_C(s)}{V_S(s)} = \frac{\frac{1}{RC}}{s+\frac{1}{RC}}$$

You have a pole or corner (cutoff) frequency at s = –1/(RC), and you have a DC gain of 1 at s = 0. The frequency response starts at s = 0 with a flat gain of 0 dB. When it hits $1/(RC)$, the frequency response rolls off with a slope of –20 dB/decade.

For circuits with only passive devices, you never get a gain greater than 1.

First-order RC high-pass filter (HPF)

To form a high-pass filter, you can use the same resistor and capacitor connected in series from Figure 18-8, but this time, you measure the resistor voltage $V_R(s)$. You start with the voltage divider equation for the voltage across the resistor $V_R(s)$:

$$V_R(s) = \left(\frac{R}{R + \frac{1}{sC}}\right) V_S(s)$$

With some algebraic manipulation (including multiplying the numerator and denominator by s/R), you can find the transfer function $T(s) = V_R(s)/V_S(s)$ of a high-pass filter:

$$T(s) = \frac{V_R(s)}{V_S(s)} = \frac{s}{s + \frac{1}{RC}}$$

You have a zero at $s = 0$ and a pole at $s = -1/(RC)$. You start off the frequency response with a zero with a positive slope of 20 dB/decade, and then the response flattens out starting at $1/(RC)$. You have a constant gain of 1 at high frequencies (or at infinity) starting at the pole frequency.

Speaker stuff: Feeding the woofer and tweeter with one RC circuit

If you're an audiophile, you know that speakers play an important role in getting high-fidelity music on your entertainment system. Because no one speaker can handle all frequencies contained in rich musical arrangements, you have an array of speakers. A simple two-speaker system consists of a small speaker, called a *tweeter,* to handle the high audio frequencies and a larger speaker, called a *woofer,* to handle the lower audio frequencies.

An RC series circuit can be either a low-pass filter when you measure the capacitor voltage or a high-pass filter when you measure the voltage across the resistor. The terminals across the capacitor can feed the woofer with low frequencies, and the terminals across the resistor form a high-pass filter to feed the tweeter with higher frequencies. At the *crossover frequency,* the low-pass filter and high-pass filter have the same cutoff frequency. The crossover frequency determines how you split the audio frequency range into two parts to feed the two-speaker system.

Creating Band-Pass and Band-Reject Filters with RLC or RC Circuits

The following sections show you how series and parallel RLC circuits form band-pass and band-reject filters. I also show you some quick-and-dirty bandpass and band-reject filters you can make using only capacitors, resistors, and op amps. These circuits come in handy when you don’t have inductors lying around, though you do need an external power source to make the op amps work. These filters are built around basic RC circuits.

Getting serious with RLC series circuits

With a circuit that has a resistor, inductor, and capacitor connected in series, you can form a band-pass filter or band-reject filter.

RLC series band-pass filter (BPF)

You can get a band-pass filter with a series RLC circuit by measuring the voltage across the resistor $V_R(s)$ driven by a source $V_S(s)$. Start with the voltage divider equation:

$$V_R(s) = \left(\frac{R}{sL + \frac{1}{sC} + R} \right) V_S(s)$$

With some algebraic manipulation, you obtain the transfer function, $T(s) = V_R(s)/V_S(s)$, of a band-pass filter:

$$T(s) = \frac{V_R(s)}{V_S(s)} = \left(\frac{R}{L} \right) \frac{s}{\left[s^2 + \left(\frac{R}{L} \right) s + \frac{1}{LC} \right]}$$

Plug in $s = j\omega$ to get the frequency response $T(j\omega)$:

$$T(j\omega) = \frac{V_R(j\omega)}{V_S(j\omega)} = \left(\frac{R}{L} \right) \frac{j\omega}{\left[(j\omega)^2 + \left(\frac{R}{L} \right) j\omega + \frac{1}{LC} \right]}$$

$$= \left(\frac{R}{L} \right) \frac{j\omega}{\left[\left(\frac{1}{LC} - \omega^2 \right) + \left(\frac{R}{L} \right) j\omega \right]}$$

The $T(j\omega)$ reaches a maximum when the denominator is a minimum, which occurs when the real part in the denominator equals 0. In math terms, this means that

$$\frac{1}{LC} - \omega^2 = 0 \quad \rightarrow \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

The frequency ω_0 is called the *center frequency*.

The cutoff frequencies are at the –3 dB half-power points. The –3 dB point occurs when the real part in the denominator is equal to $R\omega/L$:

$$\frac{1}{LC} - \omega^2 = \pm \frac{R}{L}\omega \quad \rightarrow \quad \omega^2 \pm \frac{R}{L}\omega - \frac{1}{LC} = 0$$

You basically have a quadratic equation, which has four roots due to the plus-or-minus sign in the second term.

The two appropriate roots of this equation give you cutoff frequencies at ω_{C1} an ω_{C2}:

$$\omega_{C1} = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$\omega_{C2} = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

The *bandwidth BW* defines the range of frequencies that pass through the filter relatively unaffected. Mathematically, it’s defined as

$$BW = \omega_{C2} - \omega_{C1} = \frac{R}{L}$$

Another measure of how narrow or wide the filter is with respect to the center frequency is the *quality factor Q.* The quality factor is defined as the ratio of the center frequency to the bandwidth:

$$Q = \frac{\omega_0}{BW} = \frac{1/\sqrt{LC}}{R/L}$$

$$= \frac{1}{R}\sqrt{\frac{L}{C}}$$

The RLC series circuit is *narrowband* when $Q >> 1$ (high Q) and *wideband* when $Q << 1$ (low Q). The separation between the narrowband and wideband responses occurs at $Q = 1$. Figure 18-9 shows the series band-pass circuit and gain equation for an RLC series circuit.

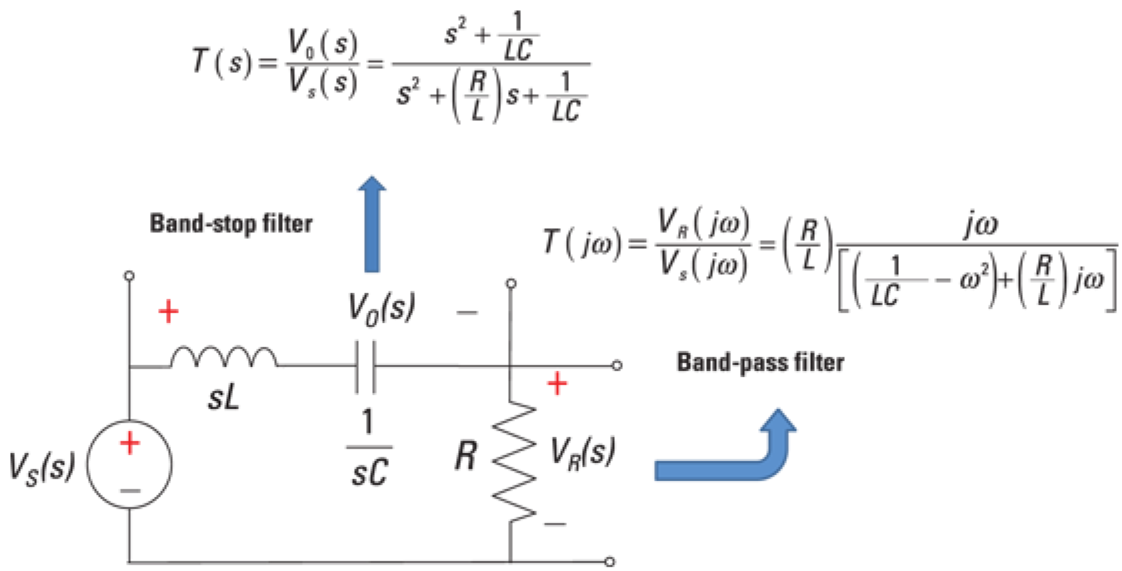


Illustration by Wiley, Composition Services Graphics

Figure 18-9: An RLC series circuit as a band-pass filter and a band-reject filter.

The frequency response is shaped by poles and zeros. For this band-pass filter, you have a zero at $\omega = 0$. You start with a gain slope of +20 dB. You hit a cutoff frequency at ω_{C1}, which flattens the frequency response until you hit another cutoff frequency above ω_{C2}, resulting in a slope of –20 dB/decade.

RLC series band-reject filter (BRF)

You form a band-reject filter by measuring the output across the series connection of the capacitor and inductor. You start with the voltage divider equation for the voltage across the series connection of the inductor and capacitor:

$$V_0(s) = \left(\frac{sL + \frac{1}{sC}}{sL + \frac{1}{sC} + R}\right)V_S(s)$$

You can rearrange the equation with some algebra to form the transfer function of a band-reject filter:

$$T(s) = \frac{V_0(s)}{V_S(s)} = \frac{s^2 + \frac{1}{LC}}{s^2 + \left(\frac{R}{L}\right)s + \frac{1}{LC}}$$

When you plug in $s = j\omega$, you have poles and zeros shaping the frequency response. For the band-reject filter, you have a double zero at $1/\sqrt{LC}$. Starting at $\omega = 0$, you have a gain of 0 dB. You hit a pole at ω_{C1}, which rolls off at –20 dB/decade until you hit a double zero, resulting in a net slope of +20 dB/decade. The frequency response then flattens out to a gain of 0 dB at the cutoff frequency ω_{C2}. You see how the poles and zeros form a band-reject filter.

Climbing the ladder with RLC parallel circuits

You can get a transfer function for a band-pass filter with a parallel RLC circuit, like the one in Figure 18-10.

$$T(j\omega) = \frac{I_R(s)}{I_s(s)} = \left(\frac{1}{RC}\right)\frac{j\omega}{\left[\left(\frac{1}{LC} - \omega^2\right) + \left(\frac{1}{RC}\right)j\omega\right]}$$

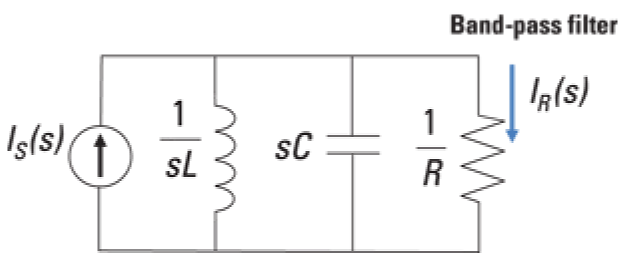


Illustration by Wiley, Composition Services Graphics

Figure 18-10: An RLC parallel circuit as a band-pass filter.

You can use current division to find the current transfer function of the parallel RLC circuit. By measuring the current through the resistor $I_R(s)$, you form a band-pass filter. Start with the current divider equation:

$$I_R(s) = \left(\frac{\frac{1}{R}}{sC + \frac{1}{sL} + \frac{1}{R}} \right) I_S(s)$$

A little algebraic manipulation gives you a current transfer function, $T(s) = I_R(s)/I_S(s)$, for the band-pass filter:

$$T(s) = \frac{I_R(s)}{I_S(s)} = \left(\frac{1}{RC}\right)\left(\frac{s}{s^2 + \left(\frac{1}{RC}\right)s + \frac{1}{LC}} \right)$$

Plug in $s = j\omega$ to get the frequency response $T(j\omega)$:

$$T(j\omega) = \frac{I_R(j\omega)}{I_S(j\omega)} = \left(\frac{1}{RC}\right) \frac{j\omega}{\left[\left(\frac{1}{LC} - \omega^2\right) + \left(\frac{1}{RC}\right)j\omega\right]}$$

This equation has the same form as the RLC series equations (see the earlier section "Getting serious with RLC series circuits" for details). For the rest of this problem, you follow the same process as for the RLC series circuit.

The transfer function is at a maximum when the denominator is minimized, which occurs when the real part of the denominator is set to 0. The cutoff frequencies are found when their gains $|T(j\omega_c)| = 0.707|T(j\omega_0)|$ or the –3 dB point. Therefore, ω_0 is

$$\frac{1}{LC} - \omega^2 = 0 \quad \rightarrow \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

The center frequency, the cutoff frequencies, and the bandwidth have equations indentical to the ones for the RLC series band-pass filter.

Your cutoff frequencies are ω_{C1} and ω_{C2}:

$$\omega_{C1} = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

$$\omega_{C2} = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

The bandwidth *BW* and quality factor *Q* are

$$BW = \omega_{C2} - \omega_{C1} = \frac{1}{RC}$$

$$Q = \frac{\omega_0}{BW} = R\sqrt{\frac{C}{L}}$$

RC only: Getting a pass with a band-pass and band-reject filter

Using simple first-order low-pass and high-pass filters based on the RC series circuit, you can form quick-and-dirty band-pass and band-reject filters with gain. You use a noninverting amplifier filter to provide circuit isolation between the low-pass filter and the high-pass filter.

The top diagram in Figure 18-11 shows this technique for a band-pass filter. The dashed lines indicate the RC series low-pass filter and high-pass filter and the noninverting amplifier. To form a band-reject filter, you can take the outputs of an RC series low-pass filter and high-pass filter with an inverting adder. The bottom of Figure 18-11 points out the key components of the quick-and-dirty band-reject filter: a low-pass filter, a high-pass filter, and an inverting adder. For this band-reject filter design, you need to choose values to prevent loading effects.

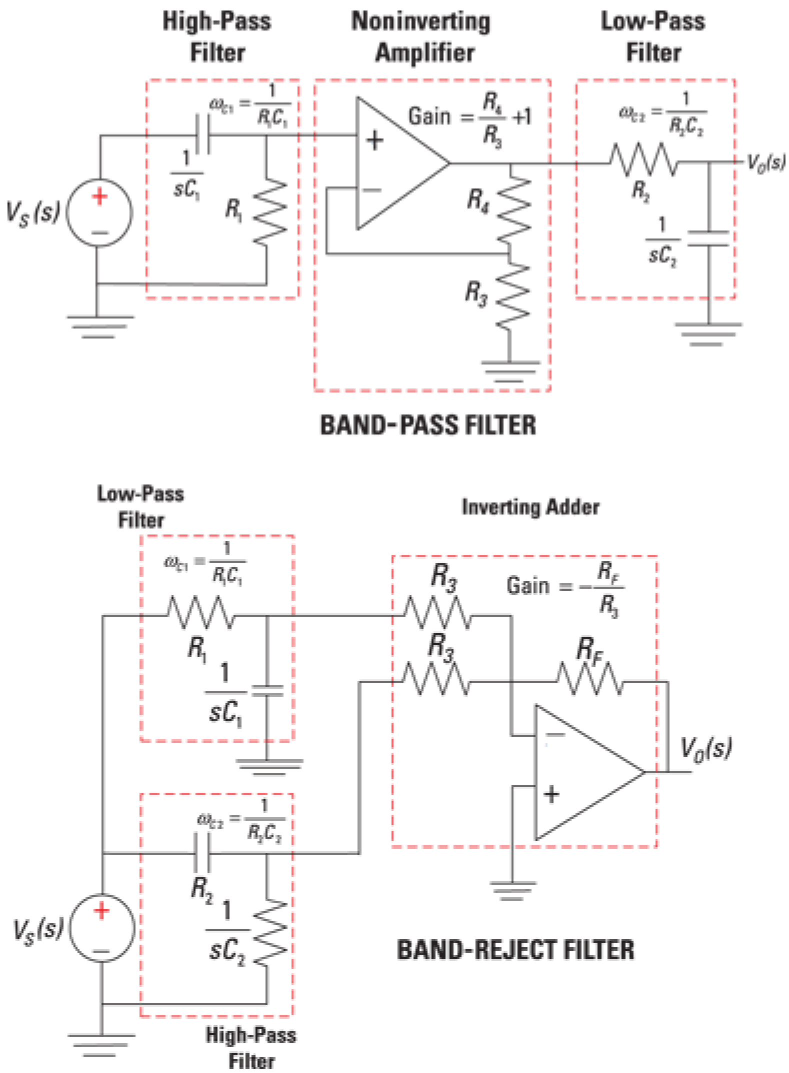


Illustration by Wiley, Composition Services Graphics

Figure 18-11: Quick-and-dirty band-pass and band-reject filters.