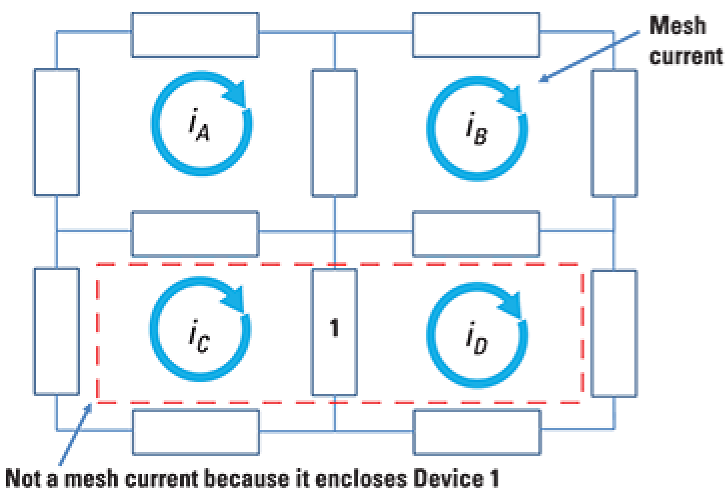
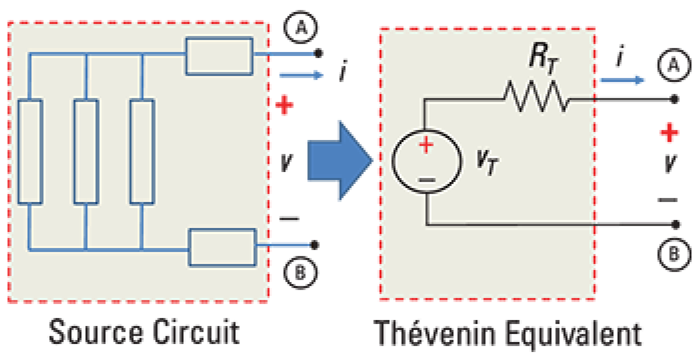
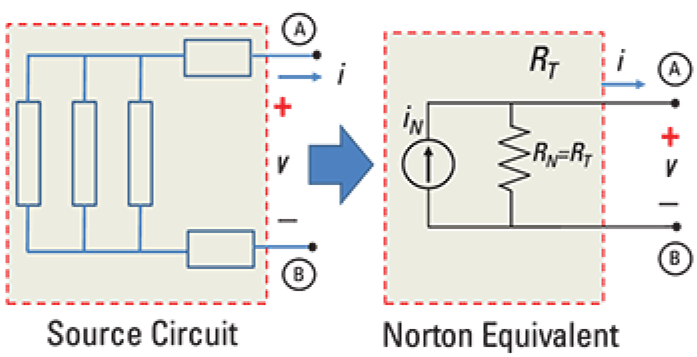
Part II

Applying Analytical Methods for Complex Circuits







Head to www.dummies.com/cheatsheet/circuitanalysis for an at-a-glance breakdown of useful circuit analysis techniques.

In this part . . .

- Practice node-voltage analysis in order to describe the voltages across each device in a circuit.
- Apply mesh-current analysis to circuits that have many devices connected in series.
- Deal with multiple current and voltage sources with the superposition technique.
- Simplify source circuits with the Thévenin and Norton theorems.

Chapter 5

Giving the Nod to Node-Voltage Analysis

In This Chapter

- Describing node-voltage analysis
- Applying Kirchhoff's current law to node-voltage analysis
- Putting node-voltage equations in matrix form

You can describe voltages across each device in a circuit by using node-voltage analysis (NVA), one of the major techniques in circuit analysis. Better yet, NVA reduces the number of equations you have to deal with. I tell you all about the key ingredients of NVA — node voltages and reference nodes — in this chapter. I also walk you through the technique, first with a basic example and then with more-complex ones.

Getting Acquainted with Node Voltages and Reference Nodes

A *node* is a particular junction or point on a circuit. To use node voltages, you need to select a reference point (or *ground point*) defined as 0 volts. *Node voltages* are voltages at circuit nodes measured with respect to that reference node.

Figure 5-1 shows you the notation for node voltage variables as well as the voltage across each device. The voltages V_A and V_B are the node voltages measured with respect to a reference node, which is identified by the ground symbol. These node voltages can describe the voltages v_1, v_2, and v_3 for the three devices in the circuit.

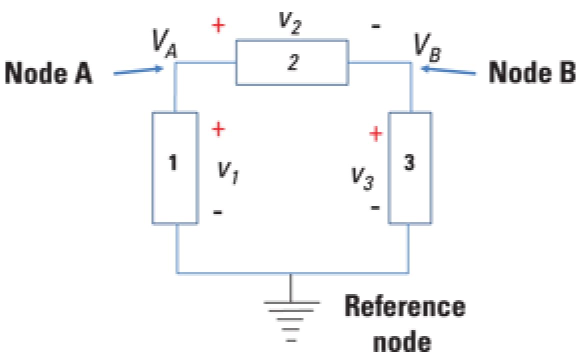


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Figure 5-1: Definition and notation for node voltages.

You calculate device voltage as the difference between two node voltages. Take the node voltage at the device's positive terminal minus the node voltage at the device's negative terminal.

Look at Figure 5-1. Because it takes two points to define a voltage, the device voltage v_1 is the difference between the node voltage V_A and the voltage of the reference node, 0 volts. Device 1 has its positive terminal connected to Node A and its negative terminal connected to the reference node, so here's the device voltage:

$$v_1 = V_A - 0 = V_A$$

Device 2 has its positive terminal connected to Node A and its negative terminal connected to Node B. Device 2's voltage v_2 is

$v_2 = V_A - V_B$

Device 3 has its positive terminal connected to Node B and its negative terminal connected to the reference node. Because Device 3 is connected to a reference node, its voltage v_3 is

$v_3 = V_B - 0 = V_B$

Testing the Waters with Node-Voltage Analysis

With node-voltage analysis, or NVA, the goal is to find the voltages across the devices in a circuit. You first apply Kirchhoff's current law (KCL), which states that the sum of incoming currents is equal to the sum of the outgoing currents at any node in the circuit. (See Chapter 3 for more on KCL.) With KCL, you can find a set of equations to determine the unknown node voltages. And when you know all the node voltages in the circuit, you can find the voltages across each device in terms of the node voltages.

In other words, node-voltage analysis involves the following steps:

1. **Select a reference (ground) node.**

   The reference node doesn't have to be actually connected to ground. You simply identify the node that way for the analysis.

   Because a reference node has 0 volts, you can simplify the analysis by choosing a node where a large number of devices are connected as your reference node.

2. **Formulate a KCL equation for each nonreference node.**
3. **Express the device currents in terms of node voltages by using device relationships such as Ohm's law.**
4. **Substitute the device equations from Step 3 into the KCL equations of Step 2.**

   Simplify the equations to put them in standard form.
5. **Solve the system of equations to find the unknown node voltages.**

   Rearrange the standard-form equations into matrix form and use matrix software to solve for the node voltages (or solve very simple systems of equations using other techniques from linear algebra).

Because Step 1 is easy, the next sections walk you through the rest of the steps of node-voltage analysis.

What goes in must come out: Starting with KCL at the nodes

After you choose a reference node, the first step in finding node voltage equations is to set up the Kirchhoff's current law (KCL) equations for a given circuit. I use the circuit in Figure 5-2 to show you how to develop these equations. The ground symbol at the bottom of the figure tells you which node is the reference node (the node having 0 volts).

This circuit has two node voltages, V_A and V_B, and four element currents, i_s, i_1, i_2, and i_3.

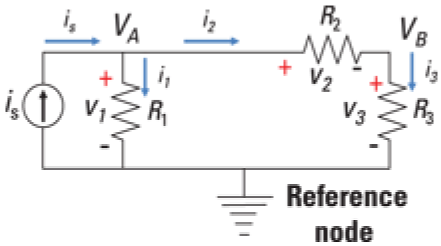

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Figure 5-2: A circuit with a reference node and two node voltages.

At Node A, the source current i_s splits into i_1 and i_2. Here's the KCL equation for the device currents at Node A:

$$\text{in} = \text{out} \quad \rightarrow \quad i_s = i_1 + i_2$$

And here's the KCL equation for Node B:

$$\text{in} = \text{out} \quad \rightarrow \quad i_2 = i_3$$

So now you know about the currents at play in Figure 5-2. How do you get the voltages? By applying Ohm's law, as I explain next.

Describing device currents in terms of node voltages with Ohm's law

Ohm's law expresses a linear relationship between voltage and current when the device in question is a resistor. You need Ohm's law to describe a device's current in terms of its node voltages. First, you determine what the node voltages are and find the device voltages. Then you substitute the node voltage expressions of the device currents into KCL and get the set of node voltage equations to be solved.

Look at the node voltages on either side of resistor R_1 in Figure 5-2. The device voltage is the difference in node voltages. Because the negative terminal is connected to a reference node, the voltage v_1 across resistor R_1 is

$$v_1 = V_A - 0$$
$$v_1 = V_A$$

The voltage v_2 for Device 2 is the difference between the node voltages at Nodes A and B. The device's positive terminal is connected to Node A, and its negative terminal is connected to Node B, so

$$v_2 = V_A - V_B$$

The negative terminal for Device 3 is connected to a reference node, so the voltage v_3 is simply

$$v_3 = V_B - 0$$
$$v_3 = V_B$$

Now apply Ohm's law ($i = v/R$) to express the device currents through R_1, R_2, and R_3 in terms of the node voltages. Using Ohm's law produces the following device currents:

$$i_1 = \frac{v_1}{R_1} = \left(\frac{1}{R_1}\right)V_A$$

$$i_2 = \frac{v_2}{R_2} = \left(\frac{1}{R_2}\right)(V_A - V_B)$$

$$i_3 = \frac{v_3}{R_3} = \left(\frac{1}{R_3}\right)V_B$$

You can now substitute these device-current expressions into the KCL equations at Nodes A and B (see the preceding section for the KCL equations). You wind up with:

$$\text{Node A:} \qquad i_s = i_1 + i_2 \quad \rightarrow \quad i_s = \left(\frac{1}{R_1}\right)V_A + \left(\frac{1}{R_2}\right)(V_A - V_B)$$

$$\text{Node B:} \qquad i_2 = i_3 \quad \rightarrow \quad \left(\frac{1}{R_2}\right)(V_A - V_B) = \left(\frac{1}{R_3}\right)V_B$$

Collect like terms and rearrange these two node equations to get the following:

Node A: $\left(\frac{1}{R_1}+\frac{1}{R_2}\right)V_A-\left(\frac{1}{R_2}\right)V_B=i_s$

Node B: $-\left(\frac{1}{R_2}\right)V_A+\left(\frac{1}{R_2}+\frac{1}{R_3}\right)V_B=0$

These two node voltage equations are said to be in standard form, but you can easily put this set of equations into matrix form.

Putting a system of node voltage equations in matrix form

The node voltage equations (see the preceding section) give you a system of linear equations, which you can solve using matrices. Of course, you can skip the matrices if the system is simple and you want to use other techniques from linear algebra, such as back substitution, to find the answers. But in most cases, using matrices is faster and easier, especially if you have a large and complicated circuit.

Here's how to transform node voltage equations from standard form to matrix form:

1. **Take the coefficients (of resistors or conductances) of the node voltages to form a square matrix.**

   

   Make sure the variable terms are in the same order in all your node voltage equations before setting up the matrix.

   A square matrix has the same number of columns and rows. Each column holds all the coefficients on a particular variable, and each row holds all the coefficients from a particular equation.

2. **Multiply the coefficient matrix from Step 1 by a column vector of the node voltages (the variables**

you want to solve for).

A column vector is a single-column matrix. The number of rows in the column vector should equal the number of columns in the square matrix.

In the column vector, write the variables in the order in which they appear in your node voltage equations.

3. **Write the right side of each node equation as a vector element to form a column vector of current sources when combining the system of node equations.**

   The column vector of current sources should have the same number of rows as the column vector of node voltages.

   In this column vector, write the current sources that appear to the right of the equal signs in your node voltage equations.

When you translate the set of node voltage equations from the preceding section into matrix form, you wind up with the following description of the circuit:

$$\begin{bmatrix} \frac{1}{R_1}+\frac{1}{R_2} & -\frac{1}{R_2} \\ -\frac{1}{R_2} & \frac{1}{R_2}+\frac{1}{R_3} \end{bmatrix} \begin{bmatrix} V_A \\ V_B \end{bmatrix} = \begin{bmatrix} i_s \\ 0 \end{bmatrix}$$

This matrix equation follows the form of $Ax = b$, where A consists of a matrix of coefficients of resistors or conductances, x is a vector of unknown node voltages, and b is vector of independent current sources.

Confirming diagonal symmetry in the square matrix is a useful way to check that your node voltage equations are right. For circuits with independent sources, you should see positive values along one diagonal and negative values along the opposite diagonal (the off-diagonal elements).

Solving for unknown node voltages

After you have your system of node voltage equations in matrix form, you're ready to solve for the unknown node voltages. You could solve simple matrices for the node voltages using Cramer's rule or other techniques from linear algebra. But for circuits with a large number of elements, use matrix software or a graphing calculator. For instance, you can find node voltages by multiplying the inverse of the coefficient matrix by the answer matrix (the column vector of current sources) on your graphing calculator: $A^{-1}b = x$.

Matrix software is great for doing calculations, but it doesn't develop the node voltage equations for you. Make sure you know how to set up the matrix problem to help you solve for the node voltages. Fortunately, some circuit analysis software does solve for the unknown voltages. You need to build the circuit graphically (depending on the software), and the software performs the required calculations.

Applying the NVA Technique

If you've reviewed the earlier sections in this chapter, then you're ready to set up some node voltage equations with numerical examples. When you have a voltage source with one of its terminals connected to a reference node, the node voltage is simply equal to the voltage source. Although doing so requires a little more work, if you're comfortable with current sources, you can always transform a voltage source into a current source. (If you're wondering what *NVA* and *node voltage equations* are, spend some time with the first part of this chapter before moving on.)

Solving for unknown node voltageswith a current source

Formulating the node voltage equations leads to a linear system of equations. You can see what I mean by working through the NVA process I outline in the earlier section "Testing the Waters with Node-Voltage Analysis." Try finding the voltages and currents for the devices in the circuit in Figure 5-3. (Note that Figure 5-3 is the same as Figure 5-2 but with numbers given for R_1, R_2, and R_3.)

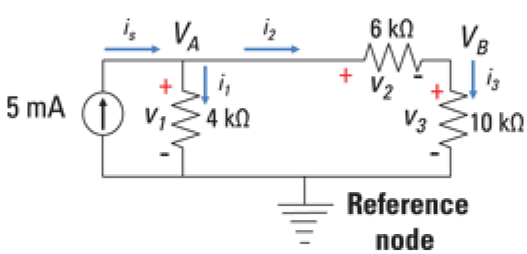

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Figure 5-3: Numerical example of node-voltage analysis.

Start by identifying your reference node. I marked my chosen reference node with the ground symbol in Figure

5-3. Now you can form the KCL equations for Nodes A and B:

$$\text{Node A: in} = \text{out} \quad \rightarrow \quad i_s = i_1 + i_2$$

$$\text{Node B: in} = \text{out} \quad \rightarrow \quad i_2 = i_3$$

Next, express the device currents in terms of node voltages by using Ohm's law (see the earlier section "Getting Acquainted with Node Voltages and Reference Nodes" for help writing the device currents). You wind up with the following equations:

$$i_1 = \frac{v_1}{R_1} = \left(\frac{1}{4\text{ k}\Omega}\right)V_A$$

$$i_2 = \frac{v_2}{R_2} = \left(\frac{1}{6\text{ k}\Omega}\right)(V_A - V_B)$$

$$i_3 = \frac{v_3}{R_3} = \left(\frac{1}{10\text{ k}\Omega}\right)V_B$$

Go ahead and substitute these current values into the KCL equations. Then rearrange the equations to put them in standard form. Here's the equation for Node A:

$$i_1 + i_2 = i_s$$

$$\left(\frac{1}{4\text{ k}\Omega}\right)V_A + \left(\frac{1}{6\text{ k}\Omega}\right)(V_A - V_B) = 5\text{ mA}$$

$$\left(\frac{1}{4\text{ k}\Omega} + \frac{1}{6\text{ k}\Omega}\right)V_A - \left(\frac{1}{6\text{ k}\Omega}\right)V_B = 5\text{ mA}$$

And here's the equation for Node B:

$$i_2 = i_3$$

$$\left(\frac{1}{6\text{ k}\Omega}\right)(V_A - V_B) = \left(\frac{1}{10\text{ k}\Omega}\right)V_B$$

$$-\left(\frac{1}{6\text{ k}\Omega}\right)V_A + \left(\frac{1}{6\text{ k}\Omega} + \frac{1}{10\text{ k}\Omega}\right)V_B = 0$$

You now have a system of linear equations for Node A and Node B — two equations with two variables. Write this system of equations in matrix form (for details, see the earlier section "Putting a system of node voltage equations in matrix form"). The resulting matrix looks like this:

$$\begin{bmatrix} \frac{1}{4\ \text{k}\Omega}+\frac{1}{6\ \text{k}\Omega} & -\frac{1}{6\ \text{k}\Omega} \\ -\frac{1}{6\ \text{k}\Omega} & \frac{1}{6\ \text{k}\Omega}+\frac{1}{10\ \text{k}\Omega} \end{bmatrix} \begin{bmatrix} V_A \\ V_B \end{bmatrix} = \begin{bmatrix} 5\ \text{mA} \\ 0 \end{bmatrix}$$

Use your calculator or matrix software to solve for V_A and V_B. You wind up with the following node voltages:

$$\begin{bmatrix} V_A \\ V_B \end{bmatrix} = \begin{bmatrix} 16\ \text{V} \\ 10\ \text{V} \end{bmatrix}$$

Now calculate the device voltages by finding the difference between two node voltages (see the first section in this chapter for details). Given these node voltages, the voltages across the resistors are

$$v_1 = V_A - 0 = 16\ \text{V}$$

$$v_2 = V_A - V_B = 16\ \text{V} - 10\ \text{V} = 6\ \text{V}$$

$$v_3 = V_B - 0 = 10\ \text{V}$$

To complete the analysis, use Ohm’s law to calculate the current for each resistor:

$$i_1 = \frac{v_1}{R_1} = \frac{16\ \text{V}}{4\ \text{k}\Omega} = 4\ \text{mA}$$

$$i_2 = \frac{v_2}{R_2} = \frac{6\ \text{V}}{6\ \text{k}\Omega} = 1\ \text{mA}$$

$$i_3 = \frac{v_3}{R_3} = \frac{10\ \text{V}}{10\ \text{k}\Omega} = 1\ \text{mA}$$

TIP After you finish the calculations, check whether your answers make sense. You can verify your results by applying Kirchhoff’s and Ohm’s laws. You can also verify that devices connected in series have the same current and that devices connected in parallel have the same voltage.

The answers make sense for this problem because the outgoing currents i_1 and i_2 at Node A add up to 5 milliamps. Also, resistor R_1, with a resistance of 4 kΩ, has four times the current of the series combination of R_2 and R_3, whose resistance totals 16 kΩ. And R_2 and R_3 have the same current, as is true for any series combination.

Dealing with three or more node equations

The node voltage approach is most useful when the circuit has three or more node voltages. You can use the same step-by-step process you use for circuits with two nodes, which I show you earlier in "Testing the Waters with Node-Voltage Analysis."

The circuit in Figure 5-4 has four nodes: A, B, C, and a reference node of 0 volts marked with a ground symbol. You want to find the voltage across each device in the circuit.

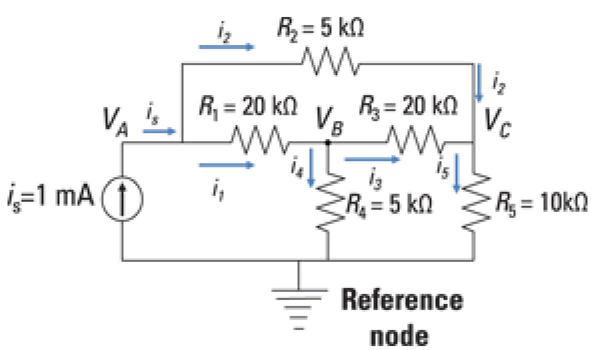

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Figure 5-4: A circuit with three nonreference nodes.

At Node A, you have incoming current i_s and outgoing currents i_1 and i_2. At Node B, you have incoming current i_1 and outgoing currents i_3 and i_4. And at Node C, you have incoming currents i_2 and i_3 and outgoing current i_5.

By applying the KCL equations at Nodes A, B, and C, you wind up with the following:

$$\begin{aligned}
&\text{Node A}: \quad \text{in} = \text{out}: \quad i_s = i_1 + i_2 \\
&\text{Node B}: \quad \text{in} = \text{out}: \quad i_1 = i_3 + i_4 \\
&\text{Node C}: \quad \text{in} = \text{out}: \quad i_2 + i_3 = i_5
\end{aligned}$$

Next, express the device currents in terms of node voltages using Ohm's law (see the earlier section "Getting Acquainted with Node Voltages and Reference Nodes" for info on writing the device currents):

$$i_1 = \frac{v_1}{R_1} = \left(\frac{1}{20\ \text{k}\Omega}\right)(V_A - V_B)$$

$$i_2 = \frac{v_2}{R_2} = \left(\frac{1}{5\ \text{k}\Omega}\right)(V_A - V_C)$$

$$i_3 = \frac{v_3}{R_3} = \left(\frac{1}{20\ \text{k}\Omega}\right)(V_B - V_C)$$

$$i_4 = \frac{v_4}{R_4} = \left(\frac{1}{5\ \text{k}\Omega}\right)V_B$$

$$i_5 = \frac{v_5}{R_5} = \left(\frac{1}{10\ \text{k}\Omega}\right)V_C$$

Substitute these device-current equations into the KCL equations. Then algebraically rearrange the equations to put them in standard form. Here's the equation for Node A:

$$\begin{gathered}
i_1 + i_2 = i_s \\
\left(\frac{1}{20\ \text{k}\Omega}\right)(V_A - V_B) + \left(\frac{1}{5\ \text{k}\Omega}\right)(V_A - V_C) = 1\ \text{mA} \\
\left(\frac{1}{5\ \text{k}\Omega} + \frac{1}{20\ \text{k}\Omega}\right)V_A - \left(\frac{1}{20\ \text{k}\Omega}\right)V_B - \left(\frac{1}{5\ \text{k}\Omega}\right)V_C = 1\ \text{mA}
\end{gathered}$$

Here's the equation for Node B:

$$\begin{gathered}
i_3 + i_4 = i_1 \\
\left(\frac{1}{20\ \text{k}\Omega}\right)(V_B - V_C) + \left(\frac{1}{5\ \text{k}\Omega}\right)V_B = \left(\frac{1}{20\ \text{k}\Omega}\right)(V_A - V_B) \\
\left(-\frac{1}{20\ \text{k}\Omega}\right)V_A + \left(\frac{1}{20\ \text{k}\Omega} + \frac{1}{20\ \text{k}\Omega} + \frac{1}{5\ \text{k}\Omega}\right)V_B - \left(\frac{1}{20\ \text{k}\Omega}\right)V_C = 0
\end{gathered}$$

And here's the equation for Node C:

$$i_2 + i_3 = i_5$$

$$\left(\frac{1}{5\text{ k}\Omega}\right)(V_A - V_C) + \left(\frac{1}{20\text{ k}\Omega}\right)(V_B - V_C) = \left(\frac{1}{10\text{ k}\Omega}\right)V_C$$

$$\left(-\frac{1}{5\text{ k}\Omega}\right)V_A - \left(\frac{1}{20\text{ k}\Omega}\right)V_B + \left(\frac{1}{5\text{ k}\Omega} + \frac{1}{20\text{ k}\Omega} + \frac{1}{10\text{ k}\Omega}\right)V_C = 0$$

Simplifying the coefficients gives you the following set of node voltage equations for Nodes A, B, and C:

$$\left(\frac{1}{4\text{ k}\Omega}\right)V_A - \left(\frac{1}{20\text{ k}\Omega}\right)V_B - \left(\frac{1}{5\text{ k}\Omega}\right)V_C = 1\text{ mA}$$

$$\left(-\frac{1}{20\text{ k}\Omega}\right)V_A + \left(\frac{3}{10\text{ k}\Omega}\right)V_B - \left(\frac{1}{20\text{ k}\Omega}\right)V_C = 0\text{ mA}$$

$$\left(-\frac{1}{5\text{ k}\Omega}\right)V_A - \left(\frac{1}{20\text{ k}\Omega}\right)V_B + \left(\frac{7}{20\text{ k}\Omega}\right)V_C = 0\text{ mA}$$

Now put this system of node voltage equations in matrix form. (I explain how to do this earlier in “Putting a system of node-voltage equations in matrix form.”)

$$\begin{bmatrix} \frac{1}{4\text{ k}\Omega} & -\frac{1}{20\text{ k}\Omega} & -\frac{1}{5\text{ k}\Omega} \\ -\frac{1}{20\text{ k}\Omega} & \frac{3}{10\text{ k}\Omega} & -\frac{1}{20\text{ k}\Omega} \\ -\frac{1}{5\text{ k}\Omega} & -\frac{1}{20\text{ k}\Omega} & \frac{7}{20\text{ k}\Omega} \end{bmatrix} \begin{bmatrix} V_A \\ V_B \\ V_C \end{bmatrix} = \begin{bmatrix} 1\text{ mA} \\ 0\text{ mA} \\ 0\text{ mA} \end{bmatrix}$$

The preceding matrix equation is of the form $Ax = b$. Notice that the square matrix is symmetrical along the diagonal the diagonal terms are positive, and the off-diagonal terms are negative, all of which suggests that you’ve converted to matrix form correctly.

Plug the matrix equation into your calculator or matrix software, giving you

$$\begin{bmatrix} V_A \\ V_B \\ V_C \end{bmatrix} = \begin{bmatrix} 8.723\text{ V} \\ 2.340\text{ V} \\ 5.319\text{ V} \end{bmatrix}$$

Now that you know the voltages for Nodes A, B, and C, you can determine the voltages across Devices 1 through 5:

$$v_1 = V_A - V_B = 6.383 \text{ V}$$
$$v_2 = V_A - V_C = 3.404 \text{ V}$$
$$v_3 = V_B - V_C = -2.979 \text{ V}$$
$$v_4 = V_B - 0 = 2.3404 \text{ V}$$
$$v_5 = V_C - 0 = 5.319 \text{ V}$$

To complete the analysis, find the current through each device:

$$i_1 = \frac{v_1}{R_1} = \frac{6.383 \text{ V}}{20 \text{ k}\Omega} = 0.3192 \text{ mA}$$

$$i_2 = \frac{v_2}{R_2} = \frac{3.404 \text{ V}}{5 \text{ k}\Omega} = 0.6807 \text{ mA}$$

$$i_3 = \frac{v_4}{R_4} = \frac{-2.979 \text{ V}}{20 \text{ k}\Omega} = -0.1490 \text{ mA}$$

$$i_4 = \frac{v_4}{R_4} = \frac{2.3404 \text{ V}}{5 \text{ k}\Omega} = 0.4681 \text{ mA}$$

$$i_5 = \frac{v_5}{R_5} = \frac{5.319 \text{ V}}{10 \text{ k}\Omega} = 0.5319 \text{ mA}$$

These results make sense because they satisfy the KCL equations at each of the three nodes.

Working with Voltage Sources in Node-Voltage Analysis

REMEMBER When a voltage source is connected to a node, you end up with fewer unknown node voltage equations because one of the node voltages is given in terms of the known voltage source. Here's how the node voltages compare if you have a voltage source:

- If the negative terminal of the voltage source is connected to a reference node, then the voltage of the node connected to the positive terminal of the voltage source has to be equal to the source voltage.

- If the voltage source terminals are connected to two nonreference nodes, then the difference between the two node voltages is simply the source voltage. So if you know one node voltage, you get the other by adding or subtracting the source voltage to or from the known node voltage.

If you're more comfortable dealing with current sources, you can perform a source transformation by replacing the voltage source and resistors connected in series with an equivalent current source and resistors connected in parallel. I show you how to transform independent sources in Chapter 4.

Figure 5-5 shows that the negative terminal of a voltage source is usually given as 0 volts. As you can see, Circuit A has two voltage sources and three nonzero nodes. Through source transformation, you can transform the circuit into Circuit B, which has only one nonreference node.

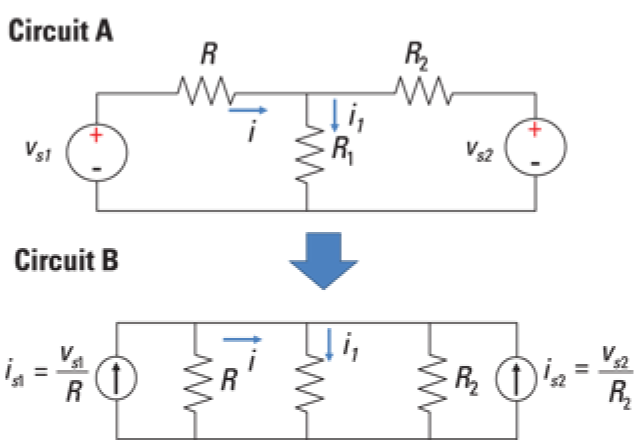

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Figure 5-5: Using source transformation of voltage sources for NVA.

When you apply node-voltage analysis in Circuit B, you wind up with the following equation:

$$\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)V_A=\frac{v_{s1}}{R_1}+\frac{v_{s2}}{R_2}$$

You get the same result using source transformation by noting that $V_A = V_{s1}$ and $V_C = v_{s2}$. The next example illustrates the technique by relating the node voltages to the voltage source.

Sometimes you encounter circuits with two voltage sources that don't have a common node. One voltage source is connected to a reference node, and the other voltage source has terminals connected to nonreference nodes, as in Figure 5-6.

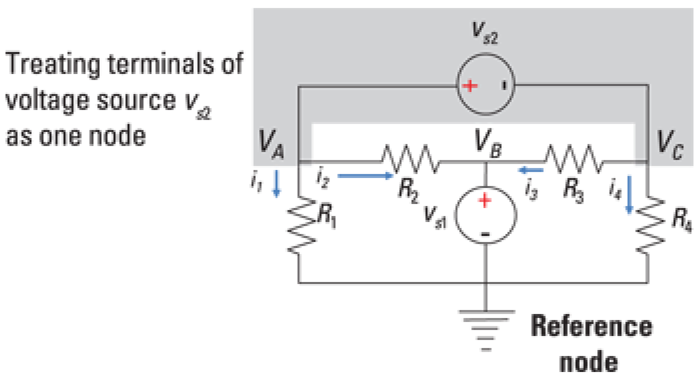

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Figure 5-6: Dealing with ungrounded voltage source for NVA.

Consider the voltage source at the top of Figure 5-6. Currents i_1 through i_4 leave and enter through the negative and positive terminals of v_{s2}, which leads to the following KCL equation:

$$i_1 + i_2 + i_3 + i_4 = 0$$

You can express these node voltages in the KCL equation in the following expression:

$$\frac{V_A}{R_1}+\frac{V_A-V_B}{R_2}+\frac{V_C-V_B}{R_3}+\frac{V_C}{R_4}=0$$

The source voltage v_{s1} at Node B is connected to a reference node, which means that

$$V_B = v_{s1}$$

Because v_{s2} is connected at Nodes A and C, the voltage across v_{s2} is the difference between the node voltages at these nodes:

$$V_A - V_C = v_{s2}$$
$$V_C = V_A - v_{s2}$$

Replace V_B and V_C in the KCL equation to get the following expression:

$$\frac{V_A}{R_1}+\frac{V_A-v_{s1}}{R_2}+\frac{V_A-v_{s2}-v_{s1}}{R_3}+\frac{V_A-v_{s2}}{R_4}=0$$

Put the source voltages on one side of the equation, which gives you

$$\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4}\right)V_A=\left(\frac{1}{R_2}+\frac{1}{R_3}\right)v_{s1}+\left(\frac{1}{R_3}+\frac{1}{R_4}\right)v_{s2}$$

This equation now has one node voltage term.

Now, suppose the desired output voltage is the voltage across resistor R_4, connected to V_C. Substitute $V_C + v_{s2}$ for V_A into the preceding equation:

$$\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4}\right)(V_C+v_{s2})=\left(\frac{1}{R_2}+\frac{1}{R_3}\right)v_{s1}+\left(\frac{1}{R_3}+\frac{1}{R_4}\right)v_{s2}$$

Now simplify the equation:

$$\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4}\right)V_C=\left(\frac{1}{R_2}+\frac{1}{R_3}\right)v_{s1}-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)v_{s2}$$

Now you have one equation with the node voltage V_C. This equation is easily solvable using algebra after you plug in some numbers for the resistors and voltage sources.

Chapter 6

Getting in the Loop on Mesh Current Equations

In This Chapter

- Describing mesh currents
- Applying Kirchhoff's voltage laws (KVL) to mesh-current analysis
- Analyzing a couple of circuits

M*esh-current analysis* (also known as *loop-current analysis*) can help reduce the number of equations you need to solve simultaneously when dealing with circuits that have many devices connected in multiple loops. This method is nothing but Kirchhoff's voltage law adapted for circuits with unique configurations.

In this chapter, I explain how to recognize meshes and assign mesh currents in order to calculate device currents and voltages.

Windowpanes: Looking at Meshes and Mesh Currents

To understand how mesh-current analysis works its magic, you need to know what a mesh is. Meshes occur in *planar circuits* — circuits that are drawn in a single plane or flat surface, without crossovers. The single plane is divided into a number of distinct areas, each of which looks like a windowpane, and the boundary of each windowpane is called a *mesh* of the circuit. The mesh can't enclose any devices — devices must fall on the boundary of the loop.

A *mesh current* is the current flowing around a mesh of the circuit. You get to choose the direction of the mesh current for your analysis. If the answer comes out negative, then the actual current is opposite the mesh current.

To help you distinguish between mesh currents and nonmesh currents, check out the planar circuit in Figure 6-1, which shows the notation of mesh current variables. The currents i_A, i_B, i_C, and i_D are all mesh currents, but the dashed rectangular box is not a mesh current because it encloses Device 1.

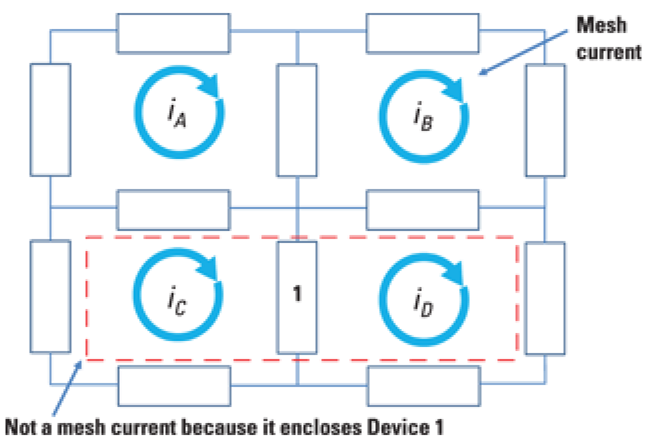


Illustration by Wiley, Composition Services Graphics

Figure 6-1: Definition and notation for mesh currents for planar circuits.

Relating Device Currents to Mesh Currents

So why should you care about mesh currents in the first place? Because using mesh currents to describe the currents flowing through each device in a circuit reduces the number of equations you need to solve simultaneously. To see the relationship between mesh currents and the device currents, consider Figure 6-2. In this figure, the mesh currents are i_A, i_B, and i_C. Currents i_1 through i_9 are the device currents.

TIP

You get to choose the direction of the mesh currents. I recommend making them all point in the same clockwise (or counterclockwise) direction so that it's easier to formulate the equations consistently. With a little practice, you'll be able to write the mesh equations simply by looking at the circuit diagram.

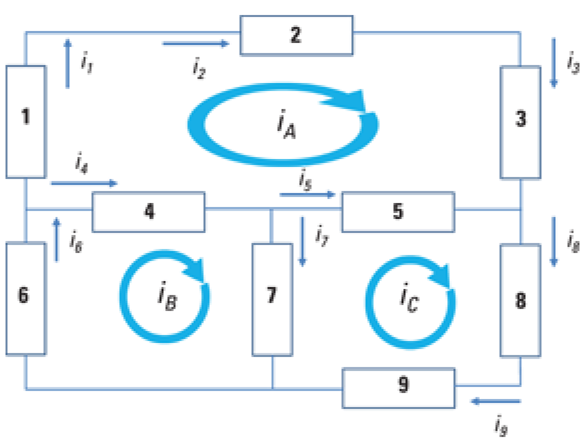


Illustration by Wiley, Composition Services Graphics

Figure 6-2: Relating device currents and mesh currents.

After choosing a current direction for each mesh, you can describe the currents for all the devices in the circuit. Here are the key points in describing device currents in terms of mesh currents:

- **If a device has only one mesh current flowing through it and the device current flows in the same direction as the mesh current, the currents are equal.** For example, because i_1, i_2, and i_3 flow in the same direction as mesh current i_A, you can express the device currents i_i, i_2, and i_3 as follows:

  $$i_1 = i_2 = i_3 = i_A$$

  You have similar situations for Devices 6, 8, and 9, where the device currents likewise flow in the same direction as their respective mesh currents:

  $$i_6 = i_B$$

  $$i_8 = i_9 = i_C$$

  If, however, the device current flows in the opposite direction of the mesh current, you put a negative sign on the mesh current.

- **If a device has two mesh currents flowing through it, the device current equals the algebraic sum of the mesh currents.** Remember that a mesh current is positive if it's in the same direction as the device current and negative if it's in the opposite direction.

  Consider Device 4, which has mesh currents i_A and i_B flowing through it. To describe i_4, find the sum of the mesh currents, making i_A negative because it flows in the opposite direction of i_4. Mathematically, you describe this device current as

  $$i_4 = -i_A + i_B$$

  Devices 5 and 7 similarly have multiple mesh currents flowing through each device, so you get the following equations:

  $$i_5 = -i_A + i_C$$

$$i_7 = i_B - i_C$$

Generating the Mesh Current Equations

Mesh-current analysis is straightforward when planar circuits have voltage sources because you can easily develop Kirchhoff’s voltage law (KVL) equations for each loop of the circuit. KVL says that the sum of the voltage rises and drops for any loop — or mesh — is equal to zero. (Check out Chapter 3 for more on KVL.) With KVL, Ohm’s law, and a few substitutions, you can find device currents and voltages given only the source voltages and resistors.

Here’s how it works: After choosing the direction of the current in each mesh, you look at the circuit diagram and describe the device currents in terms of the mesh currents. Substituting the device currents into Ohm’s law gives you device voltages in terms of the mesh currents. And substituting the device voltages into the KVL equations gives you the source voltages in terms of the mesh currents. At that point, you have a system of linear equations, and you can solve for the mesh currents using matrices. With the mesh currents, you finish analyzing the circuit, finding the currents and voltages for all your devices.

Here’s the step-by-step process:

1. **Select a current direction for each mesh.**

   See the preceding section for info on selecting the current direction.

2. **Formulate the KVL equations for each mesh.**

3. **Express the device voltages in terms of mesh currents using device relationships such as Ohm’s law.**

   First use Ohm’s law ($v = iR$) to relate device voltage to device current. Then replace the device current with its equivalent in terms of mesh current. I tell you how to express device current in terms of mesh currents in the preceding section.
4. **Substitute the device equations from Step 3 into the KVL equations from Step 2.**
5. **Put the equations in matrix form and solve the equations.**

   When you know the mesh currents, you can plug those values into the earlier equations to find the device currents and voltages.

The following sections walk you through Steps 2 through 5.

Finding the KVL equations first

To show you how to develop mesh current equations, I need a circuit with multiple loops. Enter Figure 6-3, which has two mesh currents (i_A and i_B) and five devices in the circuit. I decided to have both mesh currents flow clockwise.

KVL says that the sum of the voltage rises and drops for any loop is equal to zero. So for Mesh A, the KVL equation is

$$v_1 + v_2 = v_{s1}$$

For Mesh B, the KVL equation is

$$-v_2 + v_3 = -v_{s2}$$

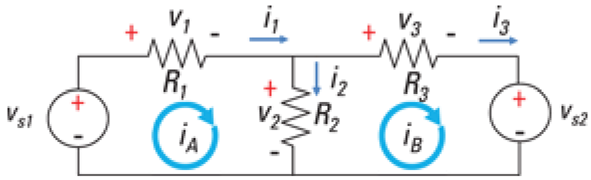


Illustration by Wiley, Composition Services Graphics

Figure 6-3: Demonstra-tion of mesh-current analysis.

Ohm's law: Putting device voltages in terms of mesh currents

Ohm's law relates voltage and current. After you write device currents in terms of mesh currents, you can use Ohm's law to express the device voltages in terms of mesh currents.

First express the device currents in terms of the mesh currents (see the earlier section "Relating Device Currents to Mesh Currents" for details). In Figure 6-3, because the device current i_1 and the mesh current i_A point in the same direction through resistor R_1 , the currents are equal:

$$i_1 = i_A$$

Now consider the current i_2 flowing through resistor R_2, which has two mesh currents flowing through it. In this case, i_A is in the same direction as i_2, but i_B is in the opposite direction of i_2, so i_B must be negative. You get the following expression for i_2:

$$i_2 = i_A - i_B$$

As for resistor R_3, you have its current i_3 equal to mesh current i_B:

$$i_3 = i_B$$

Now apply Ohm's law ($v = iR$) to relate current to voltage — and then do a little substitution. By replacing the device currents with their mesh current equivalents, you express the device voltages in terms of mesh currents. Here are the Ohm's law relationships for R_1, R_2, and R_3:

$$v_1 = i_1 R_1 = i_A R_1$$

$$v_2 = i_2 R_2 = (i_A - i_B) R_2$$

$$v_3 = i_3 R_3 = i_B R_3$$

Substituting the device voltages into the KVL equations

After you've expressed all the device currents in terms of the mesh currents, you're ready to substitute the device voltages v_1, v_2, and v_3 into the KVL equations for Meshes A and B. The result?

$$\text{Mesh A}: \quad v_1 + v_2 = v_{s1} \quad \rightarrow \quad i_A R_1 + (i_A - i_B) R_2 = v_{s1}$$

$$\text{Mesh B}: \quad -v_2 + v_3 = -v_{s2} \quad \rightarrow \quad -(i_A - i_B) R_2 + i_B R_3 = -v_{s2}$$

Then collect like terms and rearrange the preceding equations. Here are the equations for Meshes A and B in standard form:

$$\text{Mesh A}: \quad (R_1 + R_2) i_A - R_2 i_B = v_{s1}$$

$$\text{Mesh B}: \quad -R_2 i_A + (R_2 + R_3) i_B = -v_{s2}$$

Putting mesh current equations into matrix form

Together, the KVL equations for the meshes create a system of linear equations. The next step is to put the equations in matrix form so you can easily find the mesh currents using matrix software.

First, make sure your equations are in standard form (as they are in the preceding section). The standard form allows you to easily rearrange the mesh equations into matrix form: Resistors go in the coefficient matrix, mesh currents go in the variable vector, and source voltages go in the column vector of sources.

For Figure 6-3, the set of mesh equations in standard form (from the preceding section) becomes the following matrix form:

$$\begin{bmatrix} R_1+R_2 & -R_2 \\ -R_2 & R_2+R_3 \end{bmatrix} \begin{bmatrix} i_A \\ i_B \end{bmatrix} = \begin{bmatrix} v_{s1} \\ v_{s2} \end{bmatrix}$$

Notice the symmetry with respect to the main diagonal. The matrix has positive values along the main diagonal, and the off-diagonal terms have negative values.

> For circuits with independent sources, the matrix of resistors has a symmetry that can serve as a useful check when you're developing the mesh current equations. If you've set up the equations correctly, the off-diagonal terms will be symmetric with respect to the main diagonal. The terms along the main diagonal will be positive, and the off-diagonal terms will be negative or zero.

Solving for unknown currents and voltages

With the KVL equations in matrix form, you can solve for the mesh currents. When you know the mesh currents, the rest of the analysis is a snap. In deriving the KVL equations, you wrote some equations for the circuit devices that were in terms of the mesh currents (see the earlier section "Ohm's law: Putting device voltages in terms of mesh currents"). Go back to those equations, plug in the numbers, and do the math. I show you the whole process with some numbers in the next section.

Crunching Numbers: Using Meshes to Analyze Circuits

This section offers some numerical examples to show you how mesh-current analysis works. The first example involves two meshes, and the second example involves three.

Tackling two-mesh circuits

This section walks you through mesh-current analysis when you have two equations, one for Mesh A and one for Mesh B. In Figure 6-4, I decided to give both meshes a clockwise current. The next step is to apply KVL to Mesh A and B to arrive at the following mesh equations:

$$\text{Mesh A}: \quad v_{s1} = v_1 + v_2 \quad \rightarrow \quad v_1 + v_2 = 15 \text{ V}$$

$$\text{Mesh B}: \quad -v_2 + v_3 = -v_{s2} \quad \rightarrow \quad -v_2 + v_3 = -10 \text{ V}$$

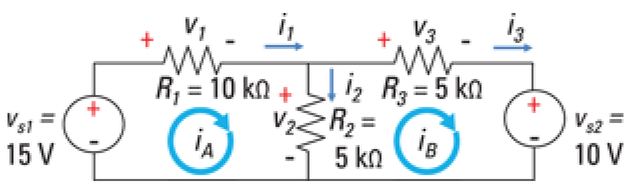


Illustration by Wiley, Composition Services Graphics

Figure 6-4: Mesh-current analysis with two meshes.

Next, write the device currents in terms of mesh currents. Then express the device currents in terms of the mesh currents using Ohm’s law:

$$\begin{aligned} v_1 &= i_1 R_1 \\ &= R_1 i_A = (10\ \text{k}\Omega) i_A \\ v_2 &= i_2 R_2 \\ &= R_2 (i_A - i_B) = (5\ \text{k}\Omega)(i_A - i_B) \\ v_3 &= i_3 R_3 \\ &= R_3 i_B = (5\ \text{k}\Omega) i_B \end{aligned}$$

Now you can substitute the preceding voltage values into the KVL equations you found earlier:

$$\text{Mesh A:} \quad \begin{aligned} v_1 + v_2 &= 15\ \text{V} \\ (10\ \text{k}\Omega) i_A + (5\ \text{k}\Omega)(i_A - i_B) &= 15\ \text{V} \end{aligned}$$

$$\text{Mesh B:} \quad \begin{aligned} -v_2 + v_3 &= -10\ \text{V} \\ -(5\ \text{k}\Omega)(i_A - i_B) + (5\ \text{k}\Omega) i_B &= -10\ \text{V} \end{aligned}$$

When you rearrange the preceding equations to put them in standard form, you get

$$\begin{aligned} &\text{Mesh A:} \quad (15\ \text{k}\Omega) i_A - (5\ \text{k}\Omega) i_B = 15\ \text{V} \\ &\text{Mesh B:} \quad (-5\ \text{k}\Omega) i_A + (10\ \text{k}\Omega) i_B = -10\ \text{V} \end{aligned}$$

Converting these mesh equations into matrix form results in

$$\begin{bmatrix} 15\ \text{k}\Omega & -5\ \text{k}\Omega \\ -5\ \text{k}\Omega & 10\ \text{k}\Omega \end{bmatrix} \begin{bmatrix} i_A \\ i_B \end{bmatrix} = \begin{bmatrix} 15\ \text{V} \\ -10\ \text{V} \end{bmatrix}$$

The preceding equation has the form $Ax = b$, where matrix A is the coefficients of resistors, x is a vector of unknown mesh currents, and b is a vector of independent voltage sources.

You can use your graphing calculator or matrix software to give you the mesh currents:

$$\begin{bmatrix} i_A \\ i_B \end{bmatrix} = \begin{bmatrix} 0.8\ \text{mA} \\ -0.6\ \text{mA} \end{bmatrix}$$

With these calculated mesh currents, you can find the device currents:

$$i_1 = i_A = 0.8 \text{ mA}$$
$$i_2 = i_A - i_B = 0.8 \text{ mA} - (-0.6 \text{ mA}) = 1.4 \text{ mA}$$
$$i_3 = i_B = -0.6 \text{ mA}$$

To complete the analysis, plug the device currents and resistances into the Ohm's law equations. You find the following device voltages:

$$v_1 = i_1 R_1 = (0.8 \text{ mA})(10 \text{ k}\Omega) = 8 \text{ V}$$
$$v_2 = i_2 R_2 = (1.4 \text{ mA})(5 \text{ k}\Omega) = 7 \text{ V}$$
$$v_3 = i_3 R_3 = (-0.6 \text{ mA})(5 \text{ k}\Omega) = -3 \text{ V}$$

The preceding device voltages make sense because they satisfy KVL for each mesh.

Analyzing circuits with three or more meshes

You can apply mesh-current analysis when dealing with circuits that have three or more meshes. The process is the same as for circuits with only two mesh currents. To see what I mean, consider Figure 6-5, which shows voltages and currents for each of the devices as well as the mesh currents i_A, i_B, and i_C. I chose to have all the mesh currents flow clockwise.

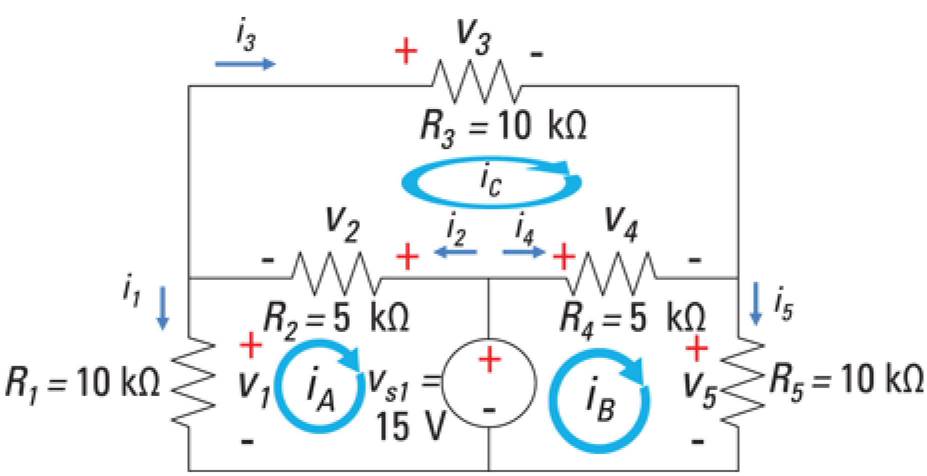


Illustration by Wiley, Composition Services Graphics

Figure 6-5: Demonstra-tion of mesh-current analysis with three meshes.

The KVL equations for Meshes A, B, and C are

$$\text{Mesh A}: \; v_1 + v_2 = 15 \text{ V}$$
$$\text{Mesh B}: \; v_4 + v_5 = 15 \text{ V}$$
$$\text{Mesh C}: \; v_2 + v_3 = v_4$$

Now express the device currents in terms of mesh currents (see the earlier section "Relating Device Currents to Mesh Currents"). Then apply Ohm's law to get the element voltages in terms of the mesh currents:

$$i_1 = -i_A \quad \rightarrow \quad v_1 = (10 \text{ k}\Omega) i_1 = -(10 \text{ k}\Omega) i_A$$
$$i_2 = i_C - i_A \quad \rightarrow \quad v_2 = (5 \text{ k}\Omega) \mathrm{i}_2 = (5 \text{ k}\Omega)(\mathrm{i}_C - i_A)$$
$$i_3 = i_C \quad \rightarrow \quad v_3 = (10 \text{ k}\Omega) i_3 = (10 \text{ k}\Omega) i_C$$
$$i_4 = i_B - i_C \quad \rightarrow \quad v_4 = (5 \text{ k}\Omega) i_4 = (5 \text{ k}\Omega)(i_B - i_C)$$
$$i_5 = i_B \quad \rightarrow \quad v_5 = (10 \text{ k}\Omega) i_5 = (10 \text{ k}\Omega) i_B$$

When you substitute the preceding device voltages into the KVL equations found earlier, you wind up with

$$\text{Mesh A}: \; v_1 + v_2 = v_{s1} \quad \rightarrow \quad -(10 \text{ k}\Omega)\mathrm{i}_A + (5 \text{ k}\Omega)(\mathrm{i}_C - i_A) = v_{s1}$$
$$\text{Mesh B}: \; v_4 + v_5 = v_{s1} \quad \rightarrow \quad (5 \text{ k}\Omega)(\mathrm{i}_B - i_C) + (10 \text{ k}\Omega) i_B = v_{s1}$$
$$\text{Mesh C}: \; v_2 + v_3 = v_4 \quad \rightarrow \quad (5 \text{ k}\Omega)(i_C - i_A) + (10 \text{ k}\Omega) i_C = (5 \text{ k}\Omega)(i_B - i_C)$$

Rearrange the equations to put them in standard form. I've inserted some zeros as placeholder terms to help you set up the matrices in the next step:

$$\text{Mesh A:} \quad (10 \text{ k}\Omega + 5 \text{ k}\Omega) i_A + 0 i_B - (5 \text{ k}\Omega) i_c = -v_{s1}$$
$$\text{Mesh B:} \quad 0 i_A + (10 \text{ k}\Omega + 5 \text{ k}\Omega) i_B - (5 \text{ k}\Omega) i_c = v_{s1}$$
$$\text{Mesh C:} \quad -(5 \text{ k}\Omega) i_A - (5 \text{ k}\Omega) i_B + (5 \text{ k}\Omega + 10 \text{ k}\Omega + 5 \text{ k}\Omega) i_c = 0$$

And you can translate these standard-form equations into matrix form to get

$$\begin{bmatrix} 10 \text{ k}\Omega + 5 \text{ k}\Omega & 0 & -5 \text{ k}\Omega \\ 0 & 5 \text{ k}\Omega + 10 \text{ k}\Omega & -5 \text{ k}\Omega \\ -5 \text{ k}\Omega & -5 \text{ k}\Omega & 5 \text{ k}\Omega + 10 \text{ k}\Omega + 5 \text{ k}\Omega \end{bmatrix} \begin{bmatrix} i_A \\ i_B \\ i_C \end{bmatrix} = \begin{bmatrix} -15 \text{ V} \\ 15 \text{ V} \\ 0 \end{bmatrix}$$

Simplify the elements in the resistor matrix:

$$\begin{bmatrix} 15\text{ k}\Omega & 0 & -5\text{ k}\Omega \\ 0 & 15\text{ k}\Omega & -5\text{ k}\Omega \\ -5\text{ k}\Omega & -5\text{ k}\Omega & 20\text{ k}\Omega \end{bmatrix} \begin{bmatrix} i_A \\ i_B \\ i_C \end{bmatrix} = \begin{bmatrix} -15\text{ V} \\ 15\text{ V} \\ 0 \end{bmatrix}$$

Notice that in the resistor matrix, the main-diagonal values are all positive, the off-diagonal values are all negative or zero, and the off-diagonal values are symmetric. For a circuit with an independent source, that symmetry with respect to the main diagonal is a good sign that you've set up the problem correctly.

You can use your graphing calculator or matrix software to find the mesh currents:

$$\begin{bmatrix} i_A \\ i_B \\ i_C \end{bmatrix} = \begin{bmatrix} -1.0\text{ mA} \\ 1.0\text{ mA} \\ 0.0\text{ mA} \end{bmatrix}$$

The current $i_C = 0$ makes sense due to the circuit symmetry. With these calculated values for mesh currents, you find the following device currents:

$$\begin{aligned} i_1 &= -i_A = -(-1.0\text{ mA}) = 1.0\text{ mA} \\ i_2 &= i_C - i_A = 0 - (-1.0\text{ mA}) = 1.0\text{ mA} \\ i_3 &= i_C = 0.0\text{ mA} \\ i_4 &= i_B - i_C = 1.0\text{ mA} - 0 = 1.0\text{ mA} \\ i_5 &= i_B = 1.0\text{ mA} \end{aligned}$$

To complete the analysis, calculate the device voltages using Ohm's law, relating the device currents and voltages:

$$\begin{aligned} v_1 &= R_1 i_1 = (10\text{ k}\Omega)\cdot(1\text{ mA}) &\rightarrow\quad v_1 &= 10\text{ V} \\ v_2 &= R_2 i_2 = (5\text{ k}\Omega)\cdot(1\text{ mA}) &\rightarrow\quad v_2 &= 5\text{ V} \\ v_3 &= R_3 i_3 = (10\text{ k}\Omega)\cdot(0\text{ mA}) &\rightarrow\quad v_3 &= 0\text{ V} \\ v_4 &= R_4 i_4 = (5\text{ k}\Omega)\cdot(1\text{ mA}) &\rightarrow\quad v_4 &= 5\text{ V} \\ v_5 &= R_5 i_5 = (10\text{ k}\Omega)\cdot(1\text{ mA}) &\rightarrow\quad v_5 &= 10\text{ V} \end{aligned}$$

The preceding results make sense because they satisfy the KVL equations for the three meshes.

Chapter 7

Solving One Problem at a Time Using Superposition

In This Chapter

- Describing the superposition method
- Taking care of sources one at a time
- Getting a handle on analyzing a circuit with two independent sources
- Solving a circuit with three independent sources

The method of circuit analysis known as *superposition* can be your best friend when you're faced with circuits that have lots of voltage and current sources. Superposition allows you to break down complex linear circuits composed of multiple independent sources into simpler circuits that have just one independent source. The total output, then, is the algebraic sum of individual outputs from each independent source. In this chapter, I show you just how superposition works. I also walk you through focusing on a single independent source when you have multiple sources in a circuit and using superposition to analyze circuits with two or three independent sources.

Discovering How Superposition Works

Superposition states that the output (or response) in any device of a linear circuit having two or more independent sources is the sum of the individual outputs resulting from each input source with all other sources turned off.

To use the superposition method, you need to understand the additive property of linearity. Linearity allows you to predict circuit behavior when applying an independent input source such as a battery. The circuit outputs (either a current or voltage for a particular device) are simply linear combinations of the independent input sources. Two properties are needed to describe linearity: proportionality and addivity.

Making sense of proportionality

The following equation describes proportionality mathematically, with *x* as the input transformed into some output *y* by a mathematical operation described as *T(x)* scaled by a constant *K:*

$$y = KT(x) \quad (\text{Proportionality Property})$$

A transformation *T* means a mathematical function like multiplication, division, differentiation, or integration. In terms of circuits, this equation means that if you have a newove an ideal voltage source by rep input voltage that is doubled in amplitude from the original, the new circuit output will also be doubled. For example, suppose output y_1 is related to input x_1 by the following transformation

$$y_1 = T(x_1)$$

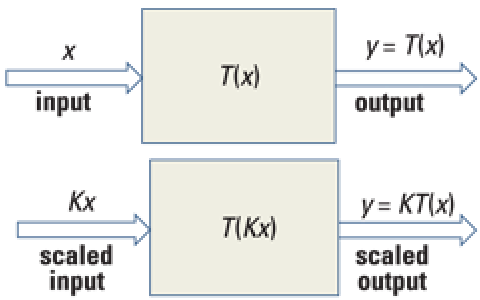
When you apply a new input $x = 2x_1$, which is twice as big as x_1, the new output y is also twice as big as the original y_1:

$$\text{New input, } x = 2x_1: \quad y = T(x)$$

$$y = T(2x_1) = 2\underbrace{T(x_1)}_{=y_1}$$

$$\text{New output:} \quad y = 2y_1$$

Figure 7-1 diagrams the proportionality concept. The top diagram illustrates how the output results from a transformed input. The bottom diagram shows that scaling your input by a constant *K* results in an output scaled by the same amount as the original output.



Multiplying the input by an amount *K* multiplies your output by *K*

Illustration by Wiley, Composition Services Graphics

Figure 7-1: Diagram of the proportionality property.

Voltage and current division techniques (see Chapter 4) rely on the concept of proportionality. Voltage division involves a series circuit with a voltage source. In the voltage divider method, you proportionally multiply the input voltage source v_s according to the value of a resistor and divide by the total resistance found in the series circuit to get the output voltage v_o of the resistor. Current division involves a parallel circuit with a current source. In the current division method, you proportionally multiply the input current source i_s according to the value of a conductance (or resistance) and divide by the total conductance found for the parallel circuit to get the output current i_o of the resistor.

Figure 7-2 illustrates the voltage divider technique as a proportionality concept. The top diagram illustrates a series circuit, and the bottom diagram corresponds to its block diagram viewed from an input-output system perspective. The input voltage is v_s, and the output voltage v_o is the voltage across resistor R_2 in series with resistor R_1. The output voltage is given as

$$v_o = \underbrace{\left(\frac{R_2}{R_1 + R_2}\right)}_{=K} v_s$$

$$v_o = K v_s$$

The preceding equation shows that the output voltage v_o is proportional to the voltage source v_s scaled by the constant K.

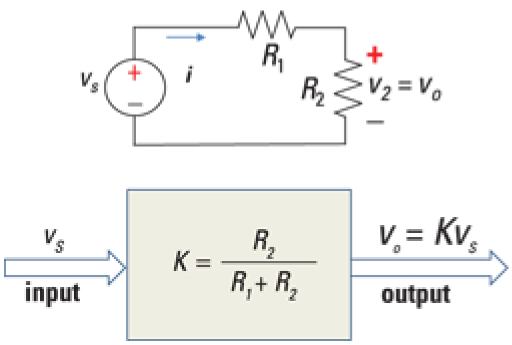


Illustration by Wiley, Composition Services Graphics

Figure 7-2: A voltage divider circuit and its associated system block diagram.

Figure 7-3 illustrates the current divider technique as a proportionality concept. The top diagram shows a parallel circuit, and the bottom diagram corresponds to its block diagram from an input-output system perspective. The input current source is i_s, and the output current i_o is the current through resistor R_2. Using current division techniques, you wind up with the following output current:

$$i_o = \underbrace{\left(\frac{R_1}{R_1 + R_2}\right)}_{=K} i_s$$

$$i_o = K i_s$$

The preceding equation shows that the output current i_o is proportional to the current source i_s scaled by the constant K.

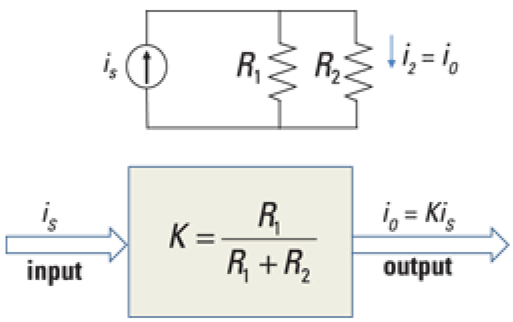


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Figure 7-3: A current divider circuit and its associated system block diagram.

Applying superposition in circuits

You can apply the superposition (additive) property to circuits. For circuits, this property states that you can express the output current or voltage of a linear circuit having multiple inputs as a linear combination of these inputs.

Here are the steps for superposition in plain English:

1. **Find the individual output of the circuit resulting from a single source acting alone by "turning off" all other independent sources.**

   To turn off an independent source, you replace it with something that has equivalent current and voltage (*i*-*v*) characteristics:

   - **Remove an ideal voltage source by replacing it with a short circuit.** You can make this replacement because the voltage is constant in both cases, so the resistance is zero:

$$R = \frac{\Delta v}{\Delta i} = \frac{0}{\Delta i} = 0$$

- **Replace an ideal current source with an open circuit.** You can make this replacement because the current is constant in both cases, so the resistance is infinite:

  $$R = \frac{\Delta v}{\Delta i} = \frac{\Delta v}{0} = \infty$$

2. **Repeat Step 1 for each independent source to find each source's output contribution when the other sources are turned off.**
3. **Algebraically add up all the individual outputs from the sources to get the total output.**

To illustrate superposition, suppose you have an input x consisting of inputs x_1, x_2, and x_3 added together:

$$x = x_1 + x_2 + x_3$$

The superposition property states that for some transformation T operating on an input x, you obtain an output y as the sum of individual outputs due to each input, x_1, x_2, and x_3. You can mathematically describe the superposition concept as follows:

$$y = T(x_1 + x_2 + x_3)$$
$$y = \underbrace{T(x_1)}_{=y_1} + \underbrace{T(x_2)}_{=y_2} + \underbrace{T(x_3)}_{=y_3}$$
$$y = y_1 + y_2 + y_3 \quad \text{(Superposition Property)}$$

In the preceding equation, the total output y is a result of three outputs: Output y_1 is due to input x_1, output y_2 is due to input x_2, and output y_3 is due to input x_3.

If you're a visual learner, take a look at Figure 7-4, which illustrates the superposition concept. In this figure, you have three inputs resulting in three outputs. Each input is transformed by the transformation T to produce an individual output. Adding up the individual outputs, you wind up with the total output y.

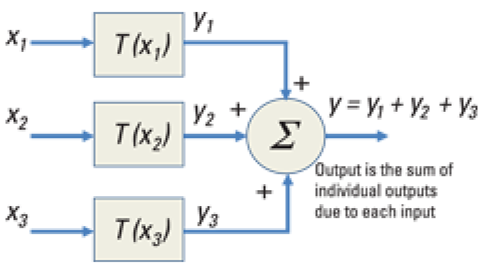

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Figure 7-4: Diagram of the superposition concept.

Adding the contributions of each independent source

To understand the property of superposition, consider analyzing a circuit with two independent sources: one current source and one voltage source. This example shows that the output consists of a linear combination of the current source and the voltage source.

You can see a typical circuit with two sources in Figure 7-5. Apply Kirchhoff's current law (KCL), which says that the sum of the incoming currents is equal to the sum of the outgoing currents at any node. Applying KCL at Node A produces the following result:

$$\text{in} = \text{out}: i_1 + i_s = i_2 \quad \rightarrow \quad i_1 + i_s - i_2 = 0$$

Using Ohm's law ($v = iR$), the current i_1 through R_1 is

$$i_1 = \frac{v_1}{R_1} = \frac{v_s - v_A}{R_1}$$

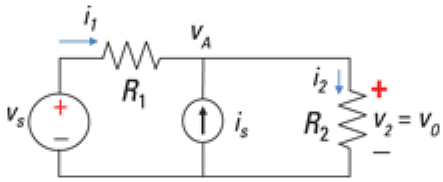

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Figure 7-5: Circuit demonstrating the superposition technique.

In Figure 7-5, because i_s and R_2 are connected in parallel, you have the output voltage v_o equal to the voltage v_2 (across resistor R_2). The current i_2 through R_2 using Ohm's law is

$$i_2 = \frac{v_2}{R_2} = \frac{v_o}{R_2}$$

and $v_o = v_A$. Substituting expressions for the element currents i_1 and i_2 into the KCL equation produces

$$\frac{v_s - v_o}{R_1} + i_s - \frac{v_o}{R_2} = 0$$

Algebraically move the input sources v_s and i_s to the right side of the equation, which results in

$$\left(\frac{1}{R_1} + \frac{1}{R_2}\right) v_o = \frac{v_s}{R_1} + i_s$$

Let $1/R_{eq} = 1/R_1 + 1/R_2$, where R_{eq} is the equivalent resistance for parallel resistors, and then solve for v_o. (In this case, the resistors are viewed as a parallel connection only when you turn off the two independent sources.) You wind up with the following output voltage v_o:

$$v_o = \underbrace{\frac{R_{eq}}{R_1}}_{=K_1} v_s + \underbrace{R_{eq}}_{=K_2} i_s$$

$$v_o = K_1 v_s + K_2 i_s$$

This equation shows output v_o as a linear combination of two input sources, v_s and i_s.

Getting Rid of the Sources of Frustration

To apply the superposition technique, you need to turn off independent sources so you can look at the output contribution from each input source. To see how to turn off independent sources, you look at the current and voltage (*i*-*v*) characteristics of a voltage source and a current source, as I explain in the next sections.

Although *i*-*v* graphs are typically drawn with *i* as the vertical axis and *v* as the horizontal axis, as I show you in Chapter 2, I've reversed that pattern in the graphs in the following sections simply to show that the slope is zero for a resistor of zero.

Short circuit: Removing a voltage source

To remove an ideal voltage source, replace it with a short circuit, because both a short circuit and an ideal voltage source have zero resistance. To see why an ideal voltage source has zero resistance, look at the *i*-*v* characteristic of an ideal voltage source in Figure 7-6.

The ideal voltage source has a constant voltage, regardless of the current supplied by the voltage source. Because the voltage source has constant voltage, the slope of the resistance is zero. The slope is given as

$$\text{slope} = \frac{\Delta v}{\Delta i} = R$$

An ideal voltage source doesn't change in the voltage for a given change in the supplied current, so $\Delta v = 0$, which implies that the slope is zero. Mathematically,

$$\text{Slope } R: \quad R = \frac{\Delta v}{\Delta i} = \frac{0}{\Delta i} \quad \rightarrow \quad R = 0 \quad (\text{short circuit})$$

In other words, turning off the voltage source means replacing it with a short circuit. Using a short circuit ensures that no voltage is present from the voltage source.

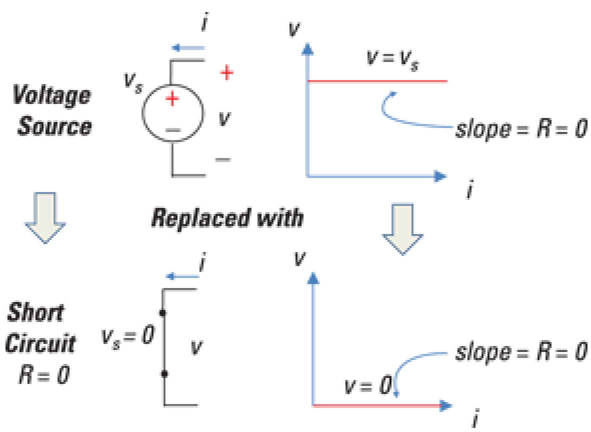

Illustration by Wiley, Composition Services Graphics

Figure 7-6: Turning off an ideal voltage source with a short circuit (zero resistance).

Open circuit: Taking out a current source

REMEMBER To remove an ideal current source, replace it with an open circuit, because both devices have infinite resistance. To see why a current source has infinite resistance, look at its current and voltage (*i*-*v*) characteristic.

The ideal current source provides constant current, regardless of the voltage across the current source; that is, $\Delta i = 0$. With constant current, the slope of the resistance is infinite. With R as the slope of the line in Figure 7-7, the slope is mathematically

$$\text{Slope } R\text{:}\quad R = \frac{\Delta v}{\Delta i} = \frac{\Delta v}{0} = \infty \quad \rightarrow \quad R = \infty \quad (\text{open circuit})$$

In other words, turning off the current source means removing the current source from the circuit or, equivalently, replacing the current source with an open circuit. Using an open circuit ensures that no current flows.

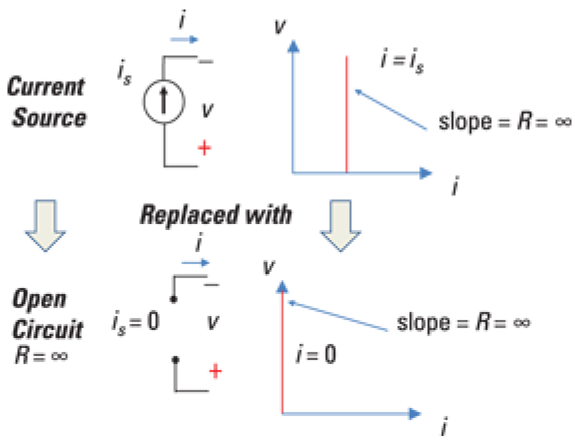

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Figure 7-7: Turning off an ideal current source with an open circuit (infinite resistance).

Analyzing Circuits with Two Independent Sources

The simplest way to understand superposition is to tackle circuits that have just two independent sources. That's why the following sections focus on such circuits. One circuit has two independent voltage sources, another circuit has two independent current sources, and the last has both voltage and current sources.

Knowing what to do when the sources are two voltage sources

With the help of superposition, you can break down the complex circuit in Figure 7-8 into two simpler circuits that have just one voltage source each. To turn off a voltage source, you replace it with a short circuit.

Circuit A contains two voltage sources, v_{s1} and v_{s2}, and you want to find the output voltage v_o across the 10-kΩ resistor. The next diagram shows the same circuit with one voltage source turned off: Circuit B contains one voltage source, with v_{s2} turned off and replaced by a short circuit. The output voltage due to v_{s1} is v_{o1}. Similarly, Circuit C is Circuit A with the other voltage source turned off. Circuit C contains one voltage source, with v_{s1} replaced by a short circuit. The output voltage due to voltage source v_{s2} is v_{o2}.

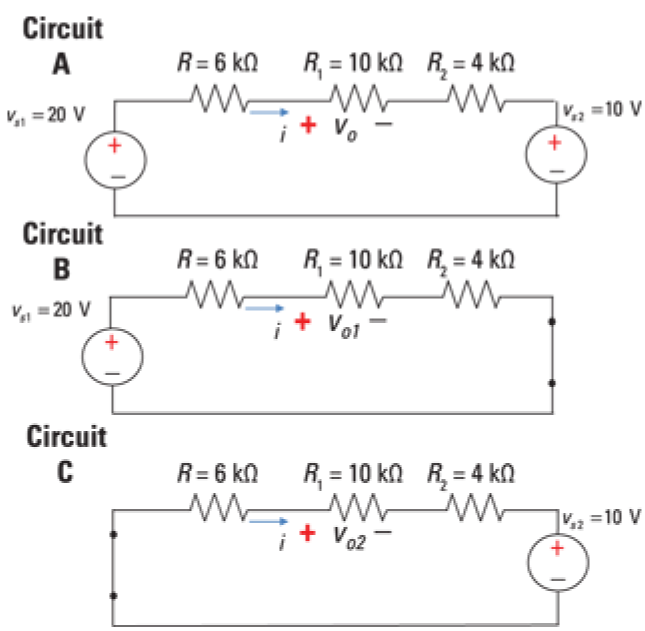

Illustration by Wiley, Composition Services Graphics

Figure 7-8: Using superposition for a circuit with two independent voltage sources.

Summing up the two outputs due to each voltage source, you wind up with the following output voltage:

$$v_o = v_{o1} + v_{o2}$$

To find the output voltages for Circuits B and C, you use voltage divider techniques. That is, you use the idea that a circuit with a voltage source connected in series with resistors divides its source voltage proportionally according to the ratio of a resistor value to the total resistance. (For the full scoop on the voltage divider technique, see Chapter 4.)

In Circuit B, you simply find the output voltage v_{o1} due to v_{s1} with a voltage divider equation:

$$v_{o1} = v_{s1}\left(\frac{R_1}{R+R_1+R_2}\right)$$

$$v_{o1} = (20\text{ V})\left(\frac{10\text{ k}\Omega}{6\text{ k}\Omega+10\text{ k}\Omega+4\text{ k}\Omega}\right) = 10\text{ V}$$

In Circuit C, finding the output voltage v_{o2} due to v_{s2} also requires a voltage divider equation, with the polarities of v_{o2} opposite v_{s2}. Using the voltage divider method produces the output voltage v_{o2} as follows:

$$v_{o2} = -v_{s2}\left(\frac{R_1}{R+R_1+R_2}\right)$$

$$v_{o2} = (-10\text{ V})\left(\frac{10\text{ k}\Omega}{6\text{ k}\Omega+10\text{ k}\Omega+4\text{ k}\Omega}\right) = -5\text{ V}$$

Adding up the individual outputs due to each source, you wind up with the following total output for the voltage across the 10-kΩ resistor:

$$v_o = v_{o1} + v_{o2} = (10\text{ V} - 5\text{ V}) = 5\text{ V}$$

Proceeding when the sources are two current sources

The plan in this section is to reduce the circuit in Figure 7-9 to two simpler circuits, each one having a single current source, and add the outputs using superposition. You consider the outputs from the current sources one at a time, turning off a current source by replacing it with an open circuit.

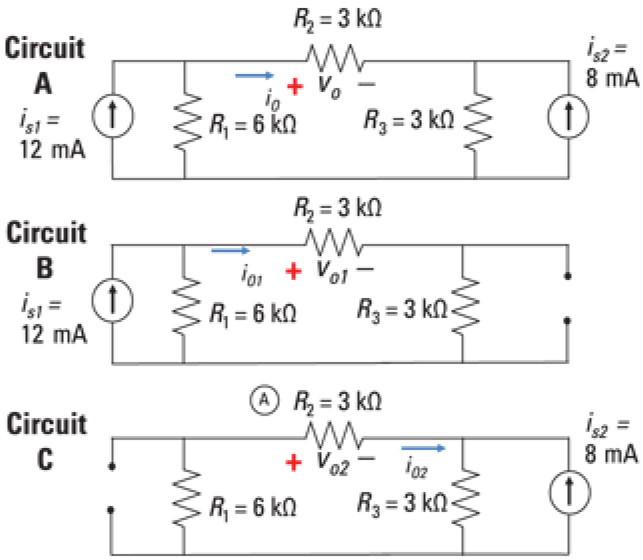

Illustration by Wiley, Composition Services Graphics

Figure 7-9: Using superposition for a circuit with two independent current sources.

Circuit A consists of two current sources, i_{s1} and i_{s2}, and you want to find the output current i_o flowing through resistor R_2. Circuit B is the same circuit with one current source turned off: Circuit B contains one current source, with i_{s2} replaced by an open circuit. The output voltage due to i_{s1} is i_{o1}. Similarly, Circuit C is Circuit A with only current source, with i_{s1} replaced by an open circuit. The output current due to current source i_{s2} is i_{o2}.

Adding up the two current outputs due to each source, you wind up with the following net output current through R_2:

$$i_o = i_{o1} + i_{o2}$$

To find the output currents for Circuits B and C, you use current divider techniques. That is, you use the idea that for a parallel circuit, the current source connected in parallel with resistors divides its supplied current proportionally according to the ratio of the value of the conductance to the total conductance. (See Chapter 4 for more on the current divider technique.)

For Circuit B, you find the output current i_{o1} due to i_{s1} using a current divider equation. Note that there are two 3-kΩ resistors connected in series in one branch of the circuit, so use their combined resistance in the equation. Given R_{eq1} = 3 kΩ + 3 kΩ and R_1 = 6 kΩ, here's output current for the first current source:

$$i_{o1} = \frac{\left(\frac{1}{R_{eq1}}\right)}{\left(\frac{1}{R_1} + \frac{1}{R_{eq1}}\right)} \cdot i_{s1}$$

$$i_{o1} = \left(\frac{\frac{1}{(3\text{ k}\Omega + 3\text{ k}\Omega)}}{\frac{1}{6\text{ k}\Omega} + \frac{1}{(3\text{ k}\Omega + 3\text{ k}\Omega)}}\right) \cdot (12\text{ mA}) = 6\text{ mA}$$

In Circuit C, the output current i_{o2} due to i_{s2} also requires a current divider equation. Note the current direction between i_{o2} and i_{s2}: i_{s2} is opposite in sign to i_{o2}. Given R_{eq2} = 6 kΩ + 3 kΩ and R_3 = 3 kΩ, the output current from the second current source is

$$i_{o2} = \frac{\left(\frac{1}{R_{eq2}}\right)}{\left(\frac{1}{R_{eq2}} + \frac{1}{R_3}\right)} \cdot i_{s2}$$

$$i_{o2} = \left(\frac{\frac{1}{(6\text{ k}\Omega + 3\text{ k}\Omega)}}{\frac{1}{(6\text{ k}\Omega + 3\text{ k}\Omega)} + \frac{1}{3\text{ k}\Omega}}\right) \cdot (-8\text{ mA}) = -2\text{ mA}$$

Adding up i_{o1} and i_{o2}, you wind up with the following total output current:

$$i_o = i_{o1} + i_{o2} = 6\text{ mA} - 2\text{ mA} = 4\text{ mA}$$

Dealing with one voltage source and one current source

You can use superposition when a circuit has a mixture of two independent sources, with one voltage source and one current source. You need to turn off the independent sources one at a time. To do so, replace the current source with an open circuit and the voltage source with a short circuit.

Circuit A of Figure 7-10 has an independent voltage source and an independent current source. How do you find the output voltage v_o as the voltage across resistor R_2?

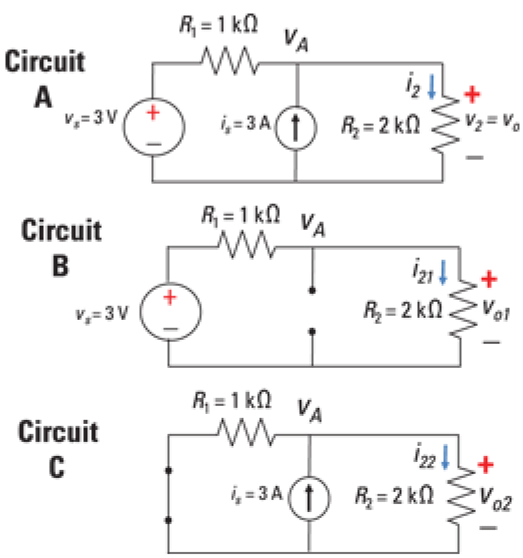

Illustration by Wiley, Composition Services Graphics

Figure 7-10: Using superposition for a circuit with a mixture of independent sources.

Circuit A (with its two independent sources) breaks up into two simpler circuits, B and C, which have just one source each. Circuit B has one voltage source because I replaced the current source with an open circuit. Circuit C has one current source because I replaced the voltage source with a short circuit.

For Circuit B, you can use the voltage divider technique because its resistors, R_1 and R_2, are connected in series with a voltage source. So here's the voltage v_{o1} across resistor R_2:

$$v_{o1} = \left(\frac{R_2}{R_1 + R_2}\right)v_s$$
$$v_{o1} = \left(\frac{2\ \text{k}\Omega}{1\ \text{k}\Omega + 2\ \text{k}\Omega}\right)(3\ \text{V}) = 2\ \text{V}$$

For Circuit C, you can use a current divider technique because the resistors are connected in parallel with a current source. The current source provides the following current i_{22} flowing through resistor R_2:

$$i_{22} = \left(\frac{R_1}{R_1 + R_2}\right)i_s$$
$$i_{22} = \left(\frac{1\ \text{k}\Omega}{1\ \text{k}\Omega + 2\ \text{k}\Omega}\right)(3\ \text{A}) = 1\ \text{mA}$$

You can use Ohm's law to find the voltage output v_{o2} across resistor R_2:

$$v_{o2} = i_{22}R_2$$
$$v_{o2} = (1\ \text{mA})(2\ \text{k}\Omega) = 2\ \text{V}$$

Now find the total output voltage across R_2 for the two independent sources in Circuit C by adding v_{o1} (due to the source voltage v_s) and v_{o2} (due to the source current i_s). You wind up with the following output voltage:

$$v_o = v_{o1} + v_{o2}$$
$$v_o = 2\ \text{V} + 2\ \text{V} = 4\ \text{V}$$

Solving a Circuit with Three Independent Sources

You can use superposition when faced with a circuit that has three (or more) independent sources. With three independent sources, you find the output voltage of three simplified circuits, where each circuit has one source working and the others turned off. Then add the outputs due to the three power sources.

Circuit A in Figure 7-11 has two voltage sources and one current source. Suppose you want to find the output voltage across the current source i_s.

To help you follow the analysis, I identified the voltage v_{AB} by labeling Terminals A and B. This voltage is equal to the output voltage v_o across the current source. The voltage across the current source is equivalent to the voltage across resistor R_3 connected in series with voltage source v_{s2}.

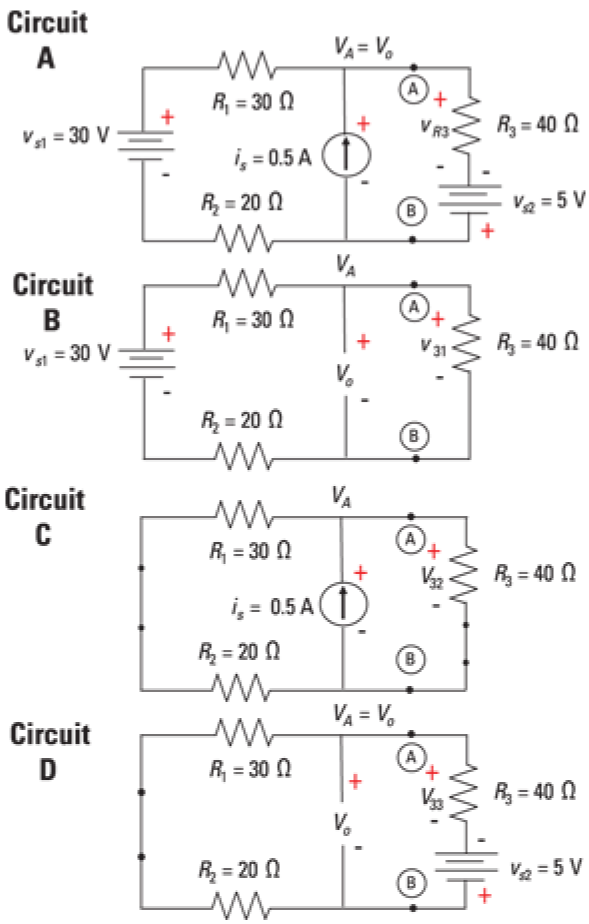

Figure 7-11: Using superposition for a circuit with three independent sources.

In Circuit A, the voltage across the current source i_s is connected in parallel with the series combination of R_3 and v_{s2}. You can find the voltage across R_3 and v_{s2}, which is equal to the output voltage v_o.

Applying Kirchhoff's voltage law (KVL) to describe this situation, you wind up with

$$v_o = v_{R3} + v_{s2} = v_{R3} - 5\text{ V}$$

Essentially, finding v_o involves finding the voltage across resistor R_3. When you know this voltage, you can easily calculate the output voltage, v_o, with the preceding equation.

You can break down Circuit A, with three independent sources, into simpler Circuits B, C, and D, each having a single independent source with the other sources removed or turned off. To analyze the simpler circuits with one source, you apply voltage and current divider techniques, which I introduce in Chapter 4.

You need to first find the voltage across R_3 due to each independent source. Here's how it works:

- **Source 1: Circuit B, first voltage source:** You calculate the voltage across R_3 due to v_{s1} by first removing the voltage source v_{s2} and replacing it with a short. You also remove the current source i_s by replacing it with an open circuit.

  After removing two independent sources, you have Circuit B, a series circuit driven by a single voltage source, v_{s1}. Consequently, the voltage divider technique applies, yielding a voltage v_{31} across resistor R_3 due to v_{s1}:

  $$v_{31} = (30\text{ V})\left(\frac{40\ \Omega}{30\ \Omega + 20\ \Omega + 40\ \Omega}\right) = 13.33\text{ V}$$

- **Source 2: Circuit C, current source:** You calculate the voltage across R_3 due to i_s by first removing the voltage sources v_{s1} and v_{s2} and replacing them with shorts.

  After removing two independent voltage sources, you have Circuit C, a parallel circuit driven by a single current source i_s. As a result, the current divider technique applies. This produces a current i_{32} through resistor R_3, resulting from current source i_s. Also not that the voltage polarity of V_{s2} is opposite that of v_{33}.

Using the current divider for Circuit C yields the following:

$$i_{32} = (0.50\text{ A})\left(\frac{50\ \Omega}{50\ \Omega + 40\ \Omega}\right) = 0.2778\text{ A}$$

Next, use Ohm's law to find the voltage across R_3 due to current source i_s:

$$v_{32} = (0.278\text{ A})(40\ \Omega) = 11.11\text{ V}$$

- **Source 3: Circuit D, second voltage source:** You calculate the voltage across R_3 due to v_{s2} by first removing the voltage source v_{s1} and replacing it with a short circuit. Also remove the current source i_s by replacing it with an open circuit.

  After removing two independent sources, you have Circuit D, a series circuit driven by a single voltage source, v_{s2}. Because this is a series circuit, the voltage divider technique applies, producing a voltage v_{33} across resistor R_3 due to v_{s2}. Also note that the voltage polarity of v_{s2} is opposite that of v_{33}. Using the voltage divider technique produces the following output:

$$v_{33} = (5\text{ V})\left(\frac{40\ \Omega}{30\ \Omega + 20\ \Omega + 40\ \Omega}\right) = 2.222\text{ V}$$

To find v_{R3}, add up the voltages across resistor R_3 due to each independent source:

$$v_{R3} = v_{31} + v_{32} + v_{33}$$
$$v_{R3} = 13.33\text{ V} + 11.11\text{ V} + 2.222\text{ V}$$
$$v_{R3} = 26.67\text{ V}$$

Here's the total output voltage (v_o + v_{AB}) across the current source (or voltage v_{AB} across Terminals A and B):

$$V_o = V_{AB} = V_{R3} - V_{s2} = 26.67\text{ V} - 5\text{ V} = 21.67\text{ V}$$

Chapter 8

Applying Thévenin’s and Norton’s Theorems

In This Chapter

- Simplifying source circuits with Thévenin’s and Norton’s theorems
- Using the Thévenin and Norton approach with superposition
- Delivering maximum power transfer

Thévenin and Norton equivalent circuits are valuable tools when you’re connecting and analyzing two different parts of a circuit. In this chapter, one part of the circuit, called the *source circuit,* delivers signals and interacts with another part, dubbed a *load circuit.* The interaction between the source and load circuits offers a major challenge when analyzing circuits.

Fortunately, Thévenin’s theorem and Norton’s theorem simplify the analysis. Each theorem allows you to replace a complicated array of independent sources and resistors, turning the source circuit into a single independent source connected with a single resistor. As a result, you don’t have to reanalyze the entire circuit when you want to try different loads — you can just use the same source circuit.

This chapter reveals just how Thévenin’s and Norton’s theorems work. It also shows you how to use the superposition technique (which I present in Chapter 7) to find equivalent Thévenin and Norton circuits when a circuit has multiple sources. Last but not least, I explain how to apply the Thévenin or Norton equivalent to show how to deliver maximum power to a load circuit.

Showing What You Can Do with Thévenin's and Norton's Theorems

Trying out and replacing devices in a circuit can be dull and dreary work. But you can minimize the toil by replacing part of the circuit with a simpler but equivalent circuit using Thévenin's or Norton's theorem.

These theorems come in handy when you're faced with circuits like the one in Figure 8-1. Note the following parts of the circuit:

- The circuit to the left of Terminals A and B is the *source circuit.* It's a linear circuit with an array of voltage sources and resistors.
- The source circuit delivers a signal to the *load circuits,* which are to the right of Terminals A and B.
- The terminals at A and B make up the *interface* between the source circuit and the load circuits.

To find the voltage across each device in the load circuit, you'd usually have to connect Devices 1, 2, and 3 one at a time to get three different answers for three different loads. Talk about tedious!

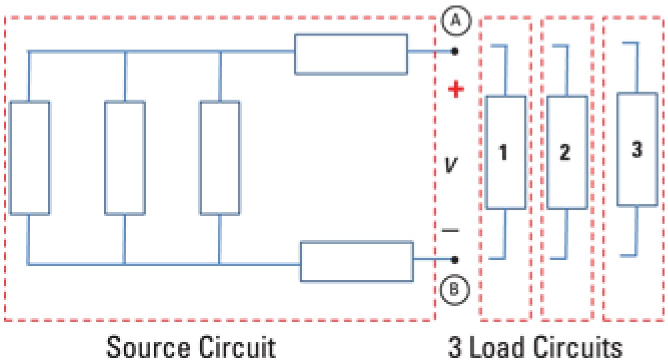

Illustration by Wiley, Composition Services Graphics

Figure 8-1: A source circuit with multiple load circuits.

Here's where Thévenin's theorem comes to the rescue. *Thévenin's theorem* lets you replace the source circuit — a linear array of devices having multiple independent sources and resistors — with a single voltage source connected in series with a single resistor. Figure 8-2 replaces the source circuit from Figure 8-1 with a simplified circuit, which has a Thévenin voltage source, v_T, connected in series with a Thévenin resistor, R_T. The Thévenin equivalent is useful when the devices in the load circuits are connected in series.

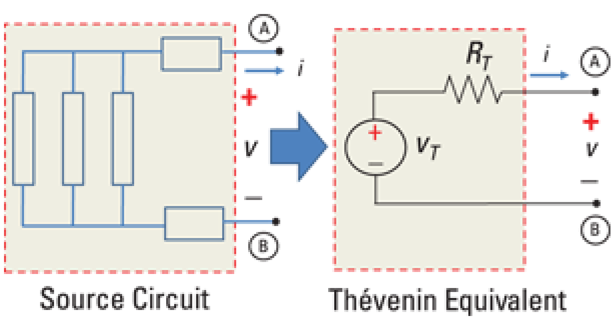

Illustration by Wiley, Composition Services Graphics

Figure 8-2: Transform-ing the source circuit into a Thévenin equivalent.

Norton's theorem likewise allows you to simplify the source circuit. Specifically, *Norton's theorem* says you can replace a linear array of devices having multiple independent sources and resistors with a single current source in parallel with a single resistor. Check out Figure 8-3 for an example. It shows the linear source circuit being replaced with a simplified circuit that has one current source, i_N, and one resistor, R_N, connected in parallel. The Norton equivalent is useful when you want to try loads that have devices connected in parallel.

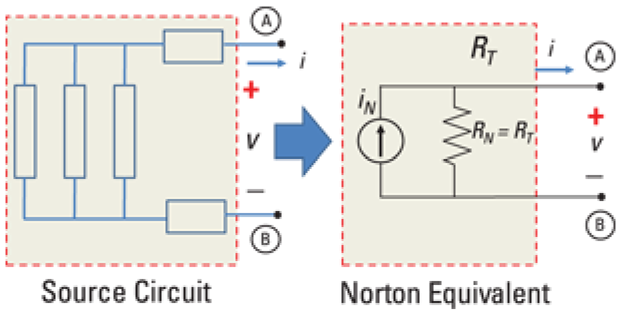

Illustration by Wiley, Composition Services Graphics

Figure 8-3: Replacing the source circuit with a Norton equivalent.

Finding the Norton and Thévenin Equivalents for Complex Source Circuits

Norton's theorem and Thévenin's theorem say essentially the same thing. The Norton equivalent is the Thévenin equivalent with a source transformation (I cover source transformation in Chapter 4).

To find the Thévenin or Norton equivalent of a linear source circuit, calculate the following variables at interface Terminals A and B:

- **Thévenin voltage source, v_T:** This equals open-circuit voltage:

$v_T = v_{oc}$

- **Norton current source, i_N:** This equals short-circuit current:

  $i_N = i_{sc}$

- **Thévenin resistance, R_T, or Norton resistance, R_N:** This resistance equals open-circuit voltage divided by short-circuit current:

  $$R_T = R_N = \frac{v_{OC}}{i_{SC}}$$

So where do these equivalents come from? Resistor loads can have a wide variety of resistor values, ranging from a short circuit having zero resistance to an open circuit having infinite resistance. These extreme ends of the resistance spectrum are convenient when you're analyzing circuits because you can easily find the Thévenin voltage v_T by having an open-circuit load, and you can get the Norton current i_N by having a short-circuit load.

Figure 8-4 shows a source circuit and its Thévenin equivalent. The top diagram shows the open circuit load you use to find the open-circuit voltage, v_{oc}, across Terminals A and B. The bottom diagram shows the short-circuit load you use to find the short-circuit current, i_{sc}, through Terminals A and B. With the open-circuit voltage, v_{oc}, and the short-circuit current, i_{sc}, you can find the Thévenin or Norton resistance ($R_T = R_N$).

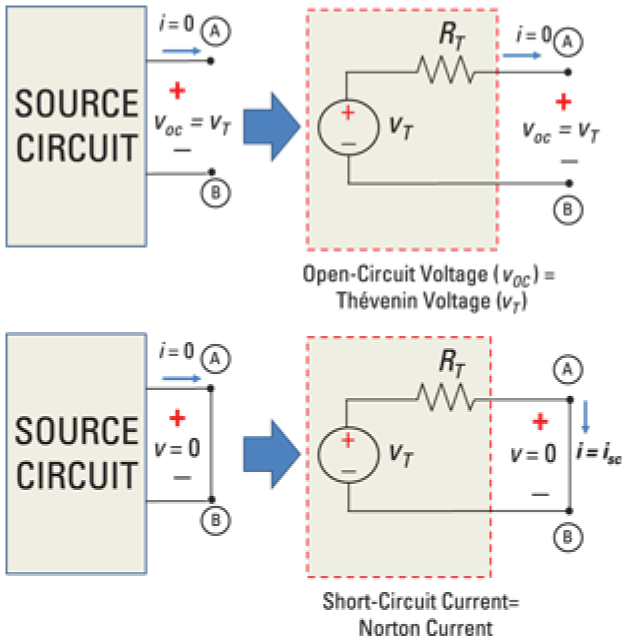

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Figure 8-4: Finding the Thévenin equivalent using open-circuit loads (top) and short-circuit loads (bottom).

In the following sections, I show you how to apply Thévenin's and Norton's theorems, and I show you how to use source transformation to go from one equivalent to another. I also offer an alternate way of finding R_T or R_N: finding the total resistance between Terminals A and B by removing all the independent sources of the source circuit.

Applying Thévenin's theorem

To simplify your analysis when interfacing between source and load circuits, the Thévenin method replaces a complex source circuit with a single voltage source in series with a single resistor. To obtain the Thévenin equivalent, you need to calculate the open-circuit voltage v_{oc} and the short-circuit current i_{sc}.

Finding the Thévenin equivalent of a circuit with a single independent voltage source

Circuit A in Figure 8-5 is a source circuit with an independent voltage source connected to a load circuit. Circuit B shows the same circuit, except I've replaced the load circuit with an open-circuit load. You use the open-circuit load to get the Thévenin voltage, v_T, across Terminals A and B. The Thévenin voltage equals the open-circuit voltage, v_{oc}.

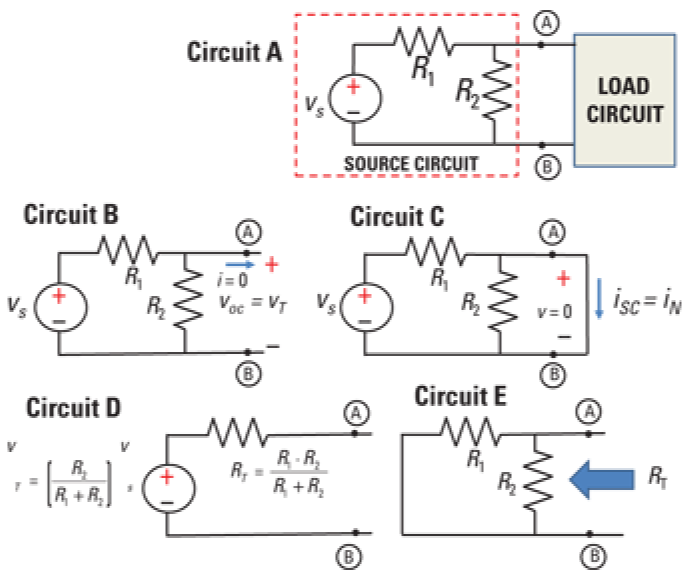

Illustration by Wiley, Composition Services Graphics

Figure 8-5: Using the Thévenin equivalent.

The voltage is driven by a voltage source for this series circuit, so use the voltage divider technique (from Chapter 4) to get v_{oc}:

$$v_{oc} = \left(\frac{R_2}{R_1 + R_2}\right) v_s \qquad \text{(open-circuit voltage)}$$

$$v_{oc} = v_T$$

Solving for v_{oc} gives you the Thévenin voltage, v_T.

Circuit C shows the same source circuit as a short-circuit load. You use the short-circuit load to get the Norton current, i_N, through Terminals A and B. And you find the Norton current by finding the short-circuit current, i_{sc}.

In Circuit C, the short circuit is in parallel with resistor R_2. This means that all the current coming out from resistor R_1 will flow through the short because the short has zero resistance. In other words, the short bypasses R_2. You can find the current through Terminals A and B using Ohm's law, producing the short-circuit current:

$$i_{sc} = \frac{v_s}{R_1}$$
$$i_{sc} = i_N$$

This short-circuit current, i_{sc}, gives you the Norton current, i_N.

Finally, to get the Thévenin resistance, R_T, you divide the open-circuit voltage by the short-circuit current. You then wind up with the following expression for R_T:

$$R_T = \frac{v_{oc}}{i_{sc}}$$
$$R_T = \frac{\left(\frac{R_2}{R_1 + R_2}\right)v_s}{\frac{v_s}{R_1}}$$

Simplify that equation to get the Thévenin resistance:

$$R_T = \frac{R_1 \cdot R_2}{R_1 + R_2}$$

Circuit D shows the Thévenin equivalent for the source circuit in Circuit A.

The preceding equation looks like the total resistance for the parallel connection between resistors R_1 and R_2 when you short (or remove) the voltage source and look back from Terminals A and B.

When looking to the left from the Terminals A and B, you can find the Thévenin resistance R_T by removing all independent sources by shorting voltage sources and replacing current sources with open circuits. After getting rid of the independent sources, you can find the total resistance between Terminals A and B, shown in Circuit E of Figure 8-5. (Note that this tactic only works when there are no dependent sources.)

Applying Norton's theorem

To see how to use the Norton approach for circuits with multiple sources, consider Circuit A in Figure 8-6. Because it doesn't matter whether you find the short-circuit current or the open-circuit voltage first, you can begin by determining the open-circuit voltage. Putting an open load at Terminals A and B results in Circuit B. The following analysis shows you how to obtain i_{s1} and R_N in Circuit B.

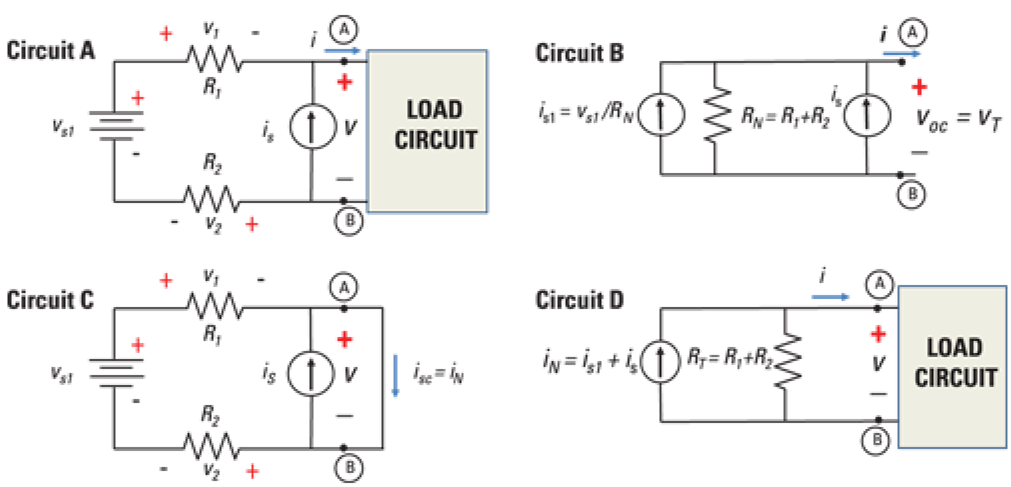

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Figure 8-6: Applying the Norton equivalent.

Applying Kirchhoff's voltage law (KVL) in Circuit A lets you determine the open-circuit voltage, v_{oc}. KVL says that the sum of the voltage rises and drops around the loop is zero. Assuming an open circuit load for Circuit A, you get the following KVL equation (where the load is an open circuit, $v = v_{oc}$):

$$-v_{s1} + v_1 + v_2 + v_{oc} = 0$$

Algebraically solve for v_{oc} to get the open-circuit voltage:

$$v_{oc} = v_{s1} - v_1 - v_2$$

The current supplied by the voltage source v_s goes through resistors R_1 and R_2 because the current going through an open circuit load is zero. In Circuit B, you can view the current source i_s as a device having an infinite resistance (that is, as an open circuit). However, all the current provided by the current source i_s will go through R_1 and R_2, and none of the current from i_s will go through the open-circuit load. Applying Ohm's law ($v = iR$), you have the following voltages across resistors R_1 and R_2:

$$v_1 = -i_s R_1$$
$$v_2 = -i_s R_2$$

The minus sign appears in these equations because the current from i_s flows opposite in direction to the assigned voltage polarities across the resistors.

Substitute v_1 and v_2 into the expression for v_{oc}, and you wind up with the following open-circuit voltage:

$$v_{oc} = v_{s1} + i_s(R_1 + R_2)$$

The open-circuit voltage is equal to the Thévenin equivalent voltage, $v_{oc} = v_T$.

Next, find the short-circuit current in Circuit C of Figure 8-6. The current i_{s1} supplied by the voltage source will flow only through resistors R_1 and R_2, not through the current source i_s, which has infinite resistance. Because of the short circuit, the resistors R_1 and R_2 are connected in series, resulting in an equivalent resistance of $R_1 + R_2$. Applying Ohm's law to this series combination gives you the following expression for i_{s1} provided by the voltage source v_{s1}:

$$i_{s1} = \frac{v_{s1}}{R_1 + R_2}$$

Kirchhoff's current law (KCL) says that the sum of the incoming currents is equal to the sum of the outgoing currents at a node. Applying KCL at Node A, you get

$$i_{s1} + i_s = i_{sc}$$

Substituting the expression for i_{s1} into the preceding KCL equation gives you the short-circuit current, i_{sc}:

$$i_{sc} = \frac{v_{s1}}{R_1 + R_2} + i_s = i_N$$

The Norton current i_N is equal to the short-circuit current: $i_N = i_{sc}$.

Finally, divide the open-circuit voltage by the short-circuit current to get the Norton resistance, R_N:

$$R_N = R_T = \frac{v_{oc}}{i_{sc}}$$

Plugging in the expressions for v_{oc} and i_{sc} gives you the Norton resistance:

$$R_N = \frac{v_{s1} + i_s\left(R_1 + R_2\right)}{\dfrac{v_{s1}}{R_1 + R_2} + i_s}$$

Adding the terms in the denominator requires adding fractions, so rewrite the terms so they have a common denominator. Algebraically, the equation simplifies as follows:

$$R_N = \frac{v_{s1} + i_s(R_1 + R_2)}{\left(\frac{v_{s1} + i_s(R_1 + R_2)}{R_1 + R_2}\right)}$$

$$R_N = \left(v_{s1} + i_s(R_1 + R_2)\right)\left(\frac{R_1 + R_2}{v_{s1} + i_s(R_1 + R_2)}\right)$$

$$R_N = R_1 + R_2$$

When you look left from the right of Terminals A and B, the Norton resistance is equal to the total resistance while removing all the independent sources. You see the Norton equivalent in Circuit D of Figure 8-6, where $R_T = R_N$.

Using source transformation to find Thévenin or Norton

In this section, I show you how to apply Thévenin’s and Norton’s theorems to analyze complex circuits using source transformation.

REMEMBER You commonly use the Thévenin equivalent when you want circuit devices to be connected in series and the Norton equivalent when you want devices to be connected in parallel with the load. You can then use the voltage divider technique for a series circuit to obtain the load voltage, or you can use the current divider technique for a parallel circuit to obtain the load current.

A shortcut: Finding Thévenin or Norton equivalents with source transformation

To transform a circuit using the Thévenin or Norton approach, you need to know both the Thévenin voltage (open-circuit voltage) and the Norton current (short-circuit current). But you don't need to find the Thévenin and Norton equivalent circuits separately. After you figure out the Thévenin equivalent for a circuit, you can find the Norton equivalent using source transformation. Or if you figure out the Norton equivalent first, source transformation lets you find the Thévenin equivalent. (I cover source transformations in Chapter 4.)

For example, if you already have the Thévenin equivalent circuit, then obtaining the Norton equivalent is a piece of cake. You perform the source transformation to convert the Thévenin voltage source connected in series with the Thévenin resistance into a current source connected in parallel with the Thévenin resistance. The result is the Norton equivalent: The current source is the Norton current source, and the Thévenin resistance is the Norton resistance.

Finding the Thévenin equivalent of a circuit with multiple independent sources

You can use the Thévenin approach for circuits that have multiple independent sources. In some cases, you can use source transformation techniques to find the Thévenin resistor R_T without actually computing v_{oc} and i_{sc}.

For example, consider Circuit A in Figure 8-7. In this circuit, the voltage source v_s and resistors R_1 and R_2 are connected in series. When you remove independent sources v_{S1} and i_s in Circuit A, this series combination of resistors produces the following total Thévenin resistance:

$$R_T = R_1 + R_2$$

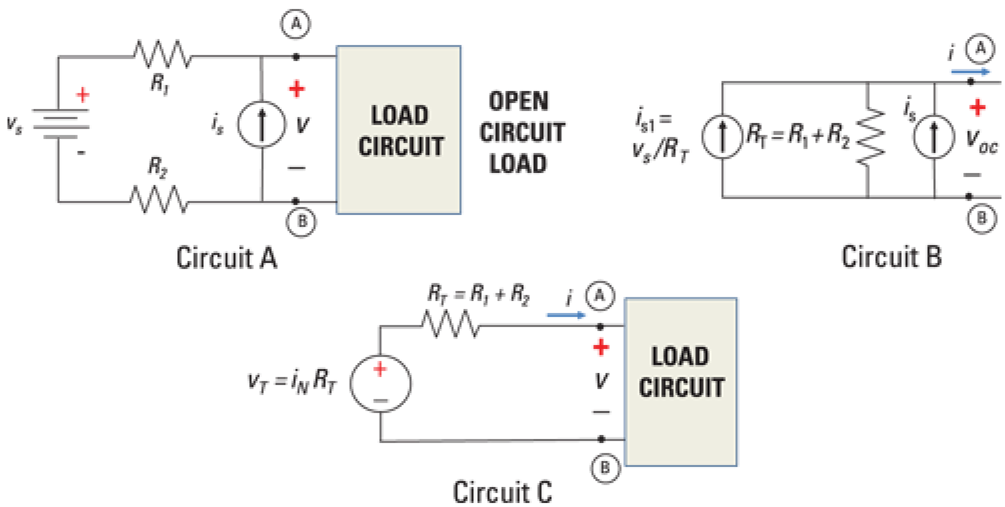

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Figure 8-7: Application of a Thévenin equivalent circuit with multiple sources.

You can then use source transformation to convert the Thévenin voltage source, which is connected in series to Thévenin resistance R_T, into a current source that's connected in parallel with R_T. Here's your current source:

$$i_{s1} = \frac{v_s}{R_T}$$

Circuit B shows the transformed circuit with two independent current sources.

Because independent current sources are in parallel and point in the same direction, you can add up the two source currents, which produces the equivalent Norton current, i_N:

$$i_N = i_s + i_{s1}$$

$$i_N = i_s + \frac{v_s}{R_T}$$

Circuit B shows the combination of the two current sources. When you combine the two current sources into one single current source connected in parallel with one resistor, you have the Norton equivalent.

You can convert the current source i_N in parallel with R_T to a voltage source in series with R_T using the following source transformation equation:

$$v_T = i_N R_T$$

Circuit C is the Thévenin equivalent consisting of one voltage source connected in series with a single equivalent resistor, R_T.

Finding Thévenin or Norton with superposition

When a complex circuit has multiple sources, you can use superposition to obtain either the Thévenin or Norton equivalent. As I explain in Chapter 7, superposition involves determining the contribution of each independent source while turning off the other sources. After determining the contribution of each source, you add up the contributions of all the sources.

This section shows you how to use superposition to find the Thévenin equivalent, but the process for finding the Norton equivalent is essentially the same. You simply find the Thévenin equivalent — consisting of one voltage source connected in series with a resistor — and get the Norton equivalent through a source transformation.

To see how superposition can help you obtain the Thévenin equivalent, consider Circuit A of Figure 8-8. To find the open-circuit voltage due to only the voltage source v_s, you turn off the current source i_s by removing it from Circuit A. Circuit B is the resulting circuit.

Because of the open-circuit load in Circuit B, no current will flow through resistors R_1 and R_2. And because there's no current flow, the voltage drop across each of these two resistors is equal to zero, according to Ohm's law ($v = iR$). The open-circuit voltage due to v_s, denoted as v_{oc1}, is therefore equal to v_s. Mathematically, you can write

$$v_{oc1} = v_s$$

To find the open-circuit voltage contribution due to the current source i_s, you turn off the voltage source v_s by replacing it with a short circuit, which has zero resistance. You see the resulting circuit in Circuit C.

Because no current flows through the open-circuit load, i_s flows through resistors R_1 and R_2. The open-circuit voltage across Terminals A and B, denoted as v_{oc2}, is equal to the voltage drop across the two resistors. Using Ohm's law, you have the following expression for v_{oc2}:

$$v_{oc2} = i_s\left(R_1 + R_2\right)$$

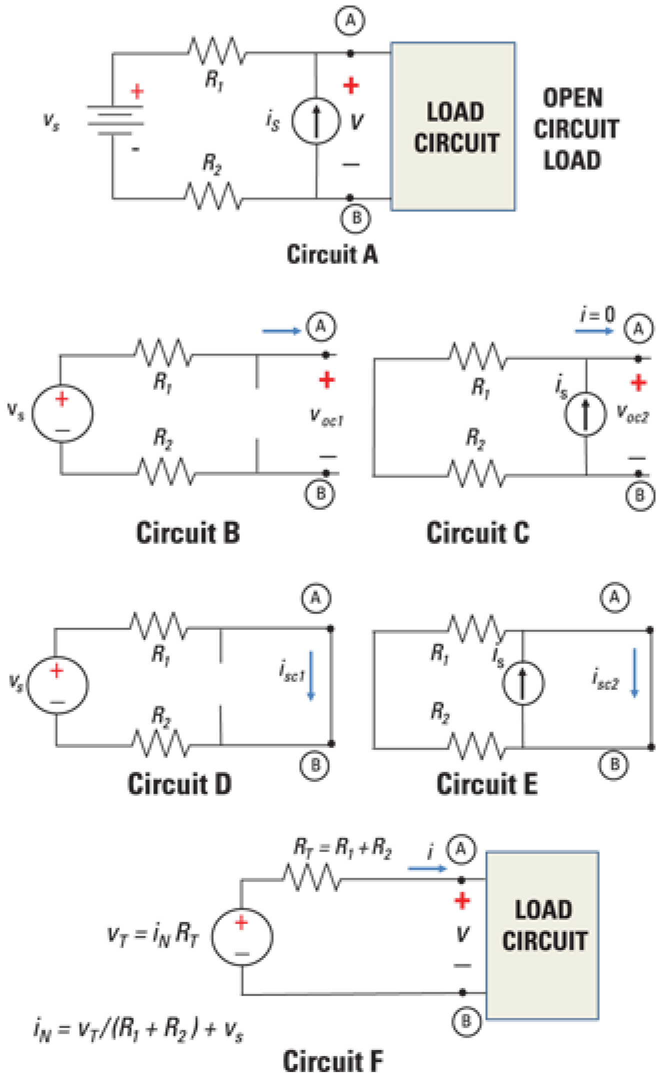

Illustration by Wiley, Composition Services Graphics

Figure 8-8: Using the superposition method to get the Thévenin equivalent.

Adding up v_{oc1} and v_{oc2} gives you the total open-circuit output contribution due to v_s and i_s:

$$
\begin{aligned}
v_{oc} &= v_{oc1} + v_{oc2} \\
&= v_s + i_s(R_1 + R_2)
\end{aligned}
$$

The open-circuit voltage equals the Thévenin voltage source ($v_{oc} = v_T$).

To find the short-circuit current due to only voltage source v_s, you turn off the current source i_s by removing it from Circuit A. The result is Circuit D.

Current flows through resistors R_1 and R_2 because the short connects the resistors in series. The series combination gives you an equivalent resistance of $R_1 + R_2$. Applying Ohm's law, you get the following expression for i_{sc1}, which is the short-circuit current due to v_s:

$$i_{sc1} = \frac{v_s}{R_1 + R_2}$$

To find the short-circuit current contribution due to only current source i_s, you turn off the voltage source v_s by replacing it with a short circuit, which has zero resistance. The result is Circuit E. Because of the short circuit, all the current provided by i_s flows through Terminals A and B. In other words, i_{sc2}, which is the short-circuit current resulting from i_s, equals the source current i_s:

$$i_{sc2} = i_s$$

Adding up i_{sc1} and i_{sc2} gives you the total contribution due to v_s and i_s:

$$\begin{aligned} i_{sc} &= i_{sc1} + i_{sc2} \\ &= \frac{v_s}{R_1 + R_2} + i_s \end{aligned}$$

This short-circuit current equals the Norton current source ($i_N = i_{sc}$).

Divide the expression for v_{oc} by the expression for i_{sc}, and you get the Thévenin resistance:

$$R_T = \frac{v_{oc}}{i_{sc}} = \frac{v_s + i_s(R_1 + R_2)}{\left(\dfrac{v_s}{R_1 + R_2} + i_s\right)}$$

This equation simplifies as follows:

$$R_T = \frac{v_s + i_s(R_1 + R_2)}{\left(\frac{v_s + i_s(R_1 + R_2)}{R_1 + R_2}\right)}$$

$$R_T = \left(v_s + i_s(R_1 + R_2)\right)\left(\frac{R_1 + R_2}{v_s + i_s(R_1 + R_2)}\right)$$

$$R_T = R_1 + R_2$$

The Thévenin resistance is equal to the total resistance of the series combination when you're looking left from the right of Terminals A and B while removing all the independent sources.

You can see the Thévenin equivalent in Circuit F of Figure 8-8. The Thévenin equivalent reduces the source circuit of Circuit A to one voltage source in series with one resistor.

With the superposition method, I get the same expressions for v_{oc}, i_{sc}, and R_N as I get using source transformation (see the preceding section). You can use whichever method works best for you.

Gauging Maximum Power Transfer: A Practical Application of Both Theorems

The power *p* coming from the source circuit to be delivered to the load depends on both the current *i* flowing through the load circuit and the voltage *v* across the load circuit at the interface between the two circuits.

The *maximum power theorem* states that for a given source with a fixed Thévenin resistance R_T, the maximum power delivered to a load resistor R_L occurs when the R_L is matched or equal to R_T:

p_{max} when $R_L = R_T$

Mathematically, the power is given by the following expression:

$p = iv$

The source circuit delivers maximum voltage when you have an open-circuit load. Because zero current flows through the open-circuit load, zero power is delivered to the load. Mathematically, the power p_{oc} delivered to the open-circuit load is

$p_{oc} = iv = 0 \cdot v = 0$

On the other hand, the source circuit delivers maximum current when you have a short-circuit load. Because zero voltage occurs across the short-circuit load, zero power is delivered to the load. Mathematically, the power p_{sc} delivered to the short-circuit load is

$p_{sc} = iv = i \cdot 0 = 0$

So what's the maximum power punch delivered for a given load resistance? Using either the Thévenin or Norton approach allows you to find the maximum power delivered to the load circuit.

To see how to determine the maximum power, look at the resistor arrangements for both the source and load circuits in Figure 8-9. In this figure, the source circuit is the Thévenin equivalent, and the load resistor is a simple but adjustable resistor.

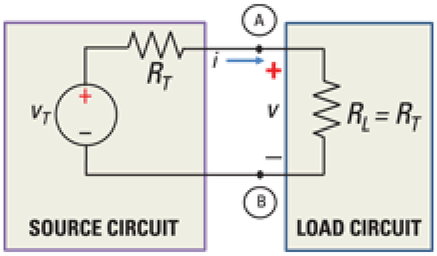

Figure 8-9: Determining maximum power delivery.

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Intuitively, you know that maximum power is delivered when both the current and voltage are maximized at the interface Terminals A and B.

Using voltage division (see Chapter 4), the voltage v across the interface at A and B is

$$v = \frac{R_L}{R_T + R_L} \cdot v_T$$

In Figure 8-9, the connected circuit between the source and load is a series circuit. The current i flows through each of the resistors, so

$$i = \frac{v_T}{R_T + R_L}$$

Substituting the values of v and i into the power equation, you wind up with the following power equation:

$$p = iv$$

$$p = \left(\frac{v_T}{R_T + R_L}\right)\left(\frac{R_L}{R_T + R_L} \cdot v_T\right)$$

$$p = \frac{R_L}{(R_T + R_L)^2} \cdot v_T^2$$

Determining the maximum power delivered to the load means taking the derivative of the preceding equation with respect to R_L and setting the derivative equal to zero. Here's the result:

$$\frac{dp}{dR_L} = \frac{R_T - R_L}{(R_T + R_L)^3} \cdot v_T^2 = 0$$

This equation equals zero when the numerator is zero. This occurs when $R_L = R_T$. Therefore, maximum power occurs when the source and load resistances are equal or matched.