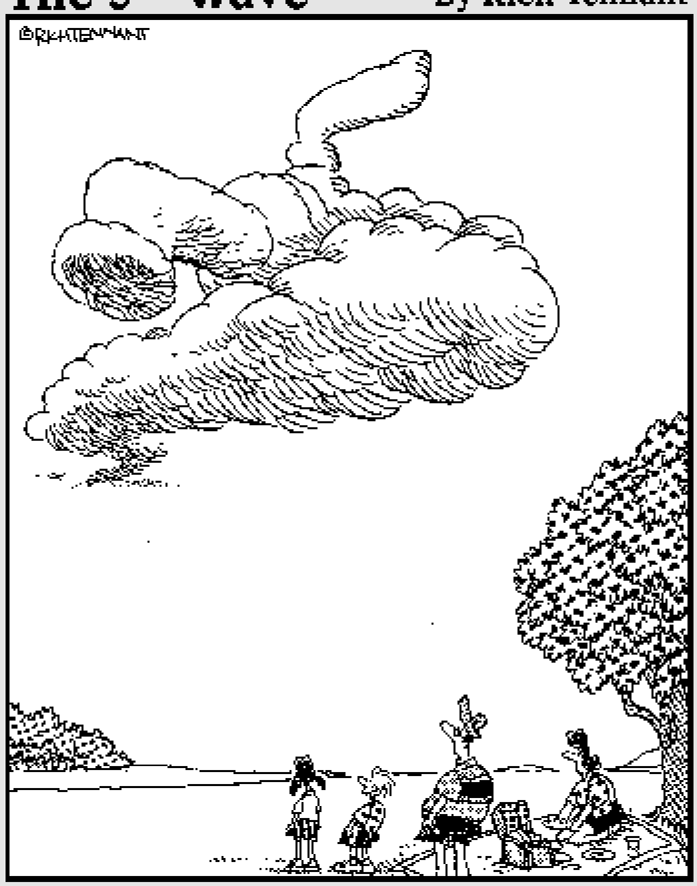
Part V

For the Hotshots: Continuous Probability Models

The 5th Wave By Rich Tennant



"So, what are our chances of seeing some rain today?"

In this part . . .

Part V paves the way for you to work with continuous probability models — situations that involve measurements — which have tons of real world uses. For example, haven't you always wanted to know how long that Energizer bunny will last? Use an exponential distribution. Want to calculate the chance of beating out your nemesis on the next exam? Use a normal distribution. Want to give every possibility an equal chance? Use the uniform distribution. You can use calculus, predetermined formulas, tables, or geometry to solve these problems.

Chapter 17

Staying in Line with the Continuous Uniform Distribution

In This Chapter

- Exploring the uniform probability model for the continuous case
- Calculating the density function of the continuous uniform
- Finding less-than, greater-than, and between-values probabilities
- Using the cdf to find cumulative probabilities
- Determining the expected value and variance for the continuous uniform

A *continuous random variable,* X, is a random variable that has an uncountably infinite number of possible values. For example, X could represent the length of time it takes to do a task (measurable to any number of decimal places); in this case, any value greater than zero is a possible value of X. The variable could also represent heights, weights, grade point averages, or any value that represents a measurement rather than a count. The *probability density function* (pdf) is the function that tells you how dense, or heavy, the concentration of probability is for X at any particular point and is denoted by f(x) (see Chapter 9 for full details).

The most basic continuous random variable is the *continuous uniform distribution*. For the continuous uniform f(x) has a special shape: a rectangle. All the values of the probability density function are the same. You use the continuous uniform when you know that the density function takes on this rectangle shape. You can also use it in situations where you don't have an idea for the density function, and you want to give all the values equal density as sort of an "uninformed guess" as to what's really going on.

In this chapter, you determine the density function for the continuous uniform probability model and find probabilities under this model by using integration (if you know calculus) or geometry (if you don't). You also calculate the cumulative distribution function (cdf), the expected value, and the variance and standard deviation of a continuous uniform distribution.

Understanding the Continuous Uniform Distribution

The continuous uniform distribution is characterized by having equal density for all possible values of a continuous random variable X, where X is between two values a and b. You can use a uniform to find probabilities for completing a task where distinct endpoints exist, for example, or in any situation where you have a continuous random variable and you want to give equal density to each possible value of X. Because a uniform distribution is continuous, its values of X are uncountable. Therefore, you can't give any probability to any single value of X. (How can you sum an uncountably infinite number of values and get a total probability of one? You can't.) So, rather than find a probability for a single value of X when it's continuous, you find the probability for an interval of values for X. For example, you could find the probability that X is between two and three — which means all real numbers between two and three — written as $P(2 \le x \le 3)$; the probability that X is more than three, written as $P(X > 3)$; or the probability that X is less than two, written as $P(X < 2)$.

The interval that represents the possible values of X is called the *domain* of X. For the uniform distribution, the domain of X is the interval a to b, written as $[a, b]$ or $a \le x \le b$.

Because you can't find a probability for any single value of X when X is continuous, you don't use a probability distribution function (pdf; see Chapter 7) to find probabilities for values of X. Instead, you look at the probability as the area under a curve — a curve called the *density function* of X, written as f(x). The function tells you how dense the probability is around X. To find the probability that X is between two and three, for example, you find the area under the curve f(x) where X is between two and three. If you graph the density function, f(x), for the uniform distribution, you get a straight, horizontal line.

Suppose you decide that the time a teacher needs to grade a quiz, in minutes, has a continuous uniform distribution where X is between 0 and 5 minutes (a sort of worst-case scenario for teachers in terms of time!). (***Note:*** Zero represents the amount of time for someone who didn't turn the quiz in, and five represents the maximum time the teacher will spend grading any one quiz.) For this particular example, the domain space of X is [0, 5] (all real numbers from 0 to 5, including 0 and 5).

Determining the Density Function for the Continuous Uniform Distribution

You can take advantage of the unique shape of a continuous uniform distribution, a rectangle, when you're finding probabilities for the uniform. Because probabilities represent the area under the probability density function, and the area of a rectangle is the base of the rectangle times the height, you just need to figure out the length of the base and the height. The base represents the overall distance that covers the values of X that you want to find probabilities for, and the height represents the value of the probability density function f(x). So, you need to determine f(x) for your uniform before you can find any probabilities.

Before you dive into figuring the density function, however, you need to keep in mind two important properties of a density function:

- Any value of f(x) has to be greater than or equal to zero.
- The total area under the curve, f(x), over all possible values of X is equal to one.

Building the general form of f(x)

In general terms, if X is a continuous random variable on the interval [a, b], $f(x) = \frac{1}{b-a}$ for $a \le x \le b$, and zero otherwise. As long as you're given the values of a and b, you can find f(x) for the continuous uniform distribution.

TIP

You can grasp the formula to find f(x) intuitively by looking at pictures of rectangles and using the given information about density functions. Figure 17-1 shows a picture of f(x) for the grading example I present in the previous section. The values of X are on the X-axis (the horizontal), and the values of f(x) are on the Y-axis (the vertical). Y = f(x) is the way to write a function when using mathematical notation, so I use f(x) to represent the value of the density function from now on rather than Y. Notice that f(x) is a horizontal line for values of X between zero and five. The area under f(x), in this case, represents the area of a rectangle.

The value of f(x) for the grading example is $\frac{1}{5-0} = \frac{1}{5}$. This means that you write the density function for X as $f(x) = \frac{1}{5}$, for $0 \le x \le 5$ and 0 otherwise.

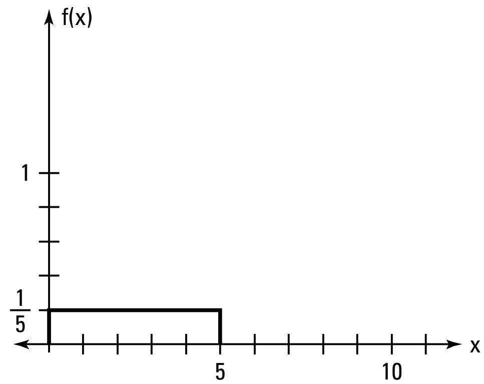


Figure 17-1: Graph of f(x) where X is uniform on the domain [0, 5].

The "0 otherwise" part of the density function means that you find no density anywhere outside the interval [0, 5]. It's customary to include "0 otherwise" whenever you define f(x). Also, notice that I could've used two less-than signs for the domain of X rather than two less-than or equal-to signs. It doesn't matter in the continuous case whether you include a specific endpoint because the probability of any single point is zero anyway.

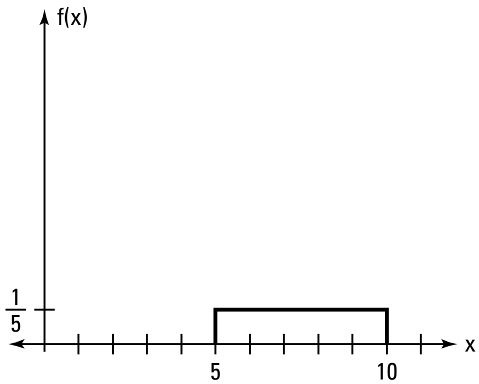
Don't get density and probability mixed up (it's easy to do). Notice the *density* of $f(x) = \frac{1}{5}$ for all X between 0 and 5 in the previous grading example. That doesn't mean the *probability* is $\frac{1}{5}$ at each of those values of X. In other words, $f(x) \neq P(x)$. If $\frac{1}{5}$ at each value were true, it would have to be true for an uncountable number of values for X between 0 and 5 (including numbers like 2.56, 3.8, 1.1, and so on), and all those probabilities of $\frac{1}{5}$ would sum past 1 in a hurry. However, the total area under the graph between 0 and 5 *is* equal to 1, which is why you have to deal with area, rather than sums of probabilities, when you deal with continuous random variables. (To understand why this seemingly strange paradox works requires knowledge of calculus and Riemann sums, for those who are dying to know.)

Finding f(x) given a and b

You can come up with the value of the density function, f(x), by using the area of a rectangle. You may be asked to do this on an exam, and the good news is, you can. One of the important properties of f(x) is that the total area under the curve for all possible values of X must equal one; so, sticking with the grading example from the previous section, the area under the line between X = 0 and X = 5, the domain [a, b], has to equal one. This represents a rectangle whose length is 5 – 0 = 5 and whose height is equal to f(x), which is constant because f(x) is a straight line. Therefore, the total area of the rectangle from zero to five is length * height = (5 – 0) * f(x) = 1. If you solve for f(x), you get 5 * f(x) = 1, so $f(x) = \frac{1}{5}$ where X is between 0 and 5 and 0 otherwise. In general, $f(x) = \frac{1}{(b-a)}$ for all x between a and b.

Suppose that X is uniform on the interval [5, 10]. Therefore, $f(x) = \frac{1}{b-a} = \frac{1}{10-5} = \frac{1}{5}$, where X is between 5 and 10 and 0 otherwise. The graph of f(x) is shown in Figure 17-2. The graph is simply shifted over five units from the density function shown in Figure 17-1. This example illustrates that not all continuous uniform random variables start at X = 0. To find the length of this rectangle, you need to subtract the right endpoint for X (10) minus the left endpoint for X (5).

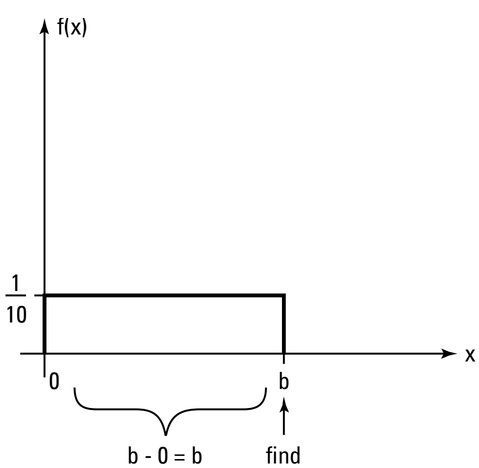
Figure 17-2: Graph of f(x) where X is uniform on the domain [5, 10].



Finding the value of b given f(x)

What's nice about the continuous uniform distribution is that because f(x) is a straight line, you can use the areas of rectangles to answer all kinds of questions about X. Here's another question you can answer by using geometry: Suppose you know that X is uniform on [0, b], and you know that f(x) is equal to ⅒; what's the value of b? Figure 17-3 shows a picture of this situation.

Figure 17-3: Graph of f(x) where X is uniform on the domain [0, b].



Even though this question is different from finding f(x) when you have both a and b, you can figure it out by using the same technique. You know that the area of the rectangle must equal one. The length is equal to b – 0 = b. (You don't know what b is, but that's okay for now.) The height is $\frac{1}{10}$. Multiply the length * height to get b * $\frac{1}{10}$ = 1. If you solve for b, you get b = 10. So, X is a continuous uniform distribution on the interval [0, 10].

The graph of a density function, f(x), for a continuous random variable is a continuous smooth curve that doesn't take any separate jumps when X changes from one number to the next like the graph of a probability mass function (pmf; see Chapter 7). Also, you can't look at any one value of f(x) and say that it's a probability.

For those of you who know calculus, the density function for the uniform distribution represents a straight line, which you can write as f(x) = c, where *c* is a constant. The total area under f(x) from a to b is the integral of the constant c from a to b. And this total area must equal one; you use this to find f(x). Here's the integration that gets you to the same place in terms of defining f(x):

$$\int_a^b f(x)\,dx = 1 \rightarrow \int_a^b c\,dx = 1 \rightarrow c\int_a^b 1\,dx = 1 \rightarrow cx\Big|_a^b = 1 \rightarrow c(b-a) = 1 \rightarrow c = \frac{1}{b-a}.$$

Remember throughout your calculations that c is a constant, and you're integrating with respect to X. (For the inside scoop on calculus, see *Calculus For Dummies,* by Mark Ryan [Wiley].)

Drawing Up Probabilities for the Continuous Uniform Distribution

The wonderful thing about the continuous uniform is that it's easy to draw up and work with, compared to the other distributions out there, yet it sets the stage for a good understanding of the more complicated continuous distributions because the same big ideas, such as finding probability as area under a curve, apply. The main difference is that with the continuous uniform, a rectangle represents the area. And you remember from geometry how to work with rectangles, right? No? Well, you take length times width, or base times height, to get the area. The best way to work with this info is to draw up the distribution before you start finding probabilities.

In general, I always draw a picture of the distribution first before attempting any calculations involving a probability distribution. Pictures help you to stay focused and to check your final answers.

Finding less-than probabilities

To find a less-than probability for a uniform distribution, you look at the picture of the density function, f(x), and shade in the area where the base is all the numbers less than X and the height is the value of f(x). You then find the area by taking base times height. In equation terms, to find $P(X < x)$, you take $(x - a) * f(x) = (x - a) * \frac{1}{b-a} = \frac{x-a}{b-a}$ to get $\frac{x-a}{b-a}$ for any x between a and b. The important thing to remember with a less-than probability is that you want the area to the left of x. Also, "at least" doesn't mean "less than" — it means greater than or equal to (see the section "Finding greater-than probabilities").

Suppose, for example, that time grading a quiz has a uniform distribution on the interval 0 to 5 minutes. Say you want to know the probability of a teacher spending less than 4 minutes grading a quiz — $P(X < 4)$. Figure 17-4 shows a picture of this situation. In terms of the uniform distribution, you want to find the area of the rectangle where X is between zero and four. To find the area of this rectangle, you take length * height = $(4 - 0) * \frac{1}{5} = \frac{4}{5} = 0.80$. This value should make sense with respect to Figure 17-4, because the shaded area represents 80 percent of the full rectangle.

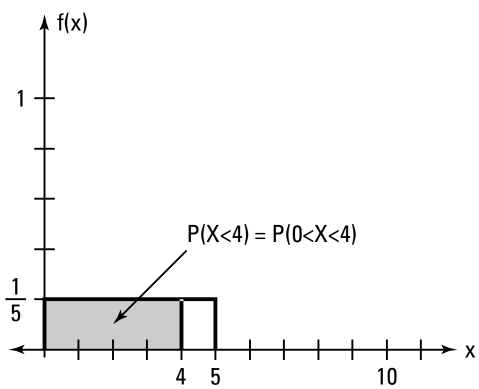


Figure 17-4: The probability of $(X < 4)$ where X is uniform on the domain [0, 5].

Now look at the same uniform distribution on the interval [5, 10]. Suppose you want to find $P(X < 9)$. Figure 17-5 shows a picture of this situation. You already know that X has to be greater than five, so you want the area of the rectangle between five and nine. This area is equal to the length $(9 - 5)$ times height $(\frac{1}{5}) = 4 * \frac{1}{5} = \frac{4}{5} = 0.80$. The area you find also represents 80 percent of the rectangle. Notice that to get the length, you took the value of x (9) minus the lower endpoint (5), and you multiplied that by f(x) to get the probability.

In Figure 17-5, you see that $P(X < 9)$ is really the probability that X is between 5 and 9, written as $P(5 < X < 9)$, because all the probability below 5 is 0, and the actual probability starts at 5. You immediately know that when you draw the picture, so you can just write it as $P(X < 9)$, but drawing the picture is important to keep this clear.

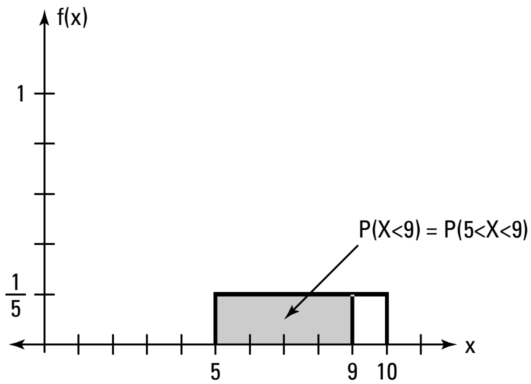


Figure 17-5: The probability of $(X < 9)$ where X is uniform on the domain [5, 10].

For those of you who know calculus, you can find $P(X < x)$ by integrating the density function f(x) from a to x, where a is the smallest possible value of x. This method gives you the same result.

Finding greater-than probabilities

Greater-than probabilities appear in problems when you're asked for the probability that it takes at least a certain amount of time or more than a certain amount of time. Greater-than probabilities require you to find the area to the right of x on the continuous uniform distribution. In general, to find $P(X > x)$ if X is a continuous uniform distribution, you find $\frac{b-x}{b-a}$ for any value of x between the domain a and b.

It also helps to remember that greater-than probabilities are the complements of less-than probabilities, meaning that you can take one minus a less-than probability to get a greater-than probability, and vice versa.

For example, suppose that the time grading quizzes, X, is uniform on the domain [0, 5], and you want the probability that a teacher will take more than 4 minutes to grade a paper. In other words, you want $P(X > 4)$. You can use the complement rule (see Chapter 2) to find $1 - P(X < 4)$, or you can find the area of the rectangle directly. Figure 17-6 shows a picture of the situation.

Note: Although the equal sign doesn't matter because the probability of being equal to any value is zero with a continuous distribution, it's good form to be consistent.

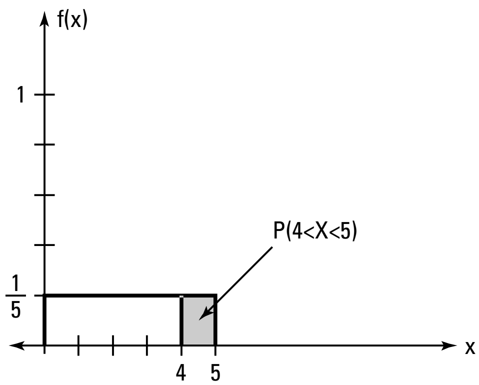


Figure 17-6: The probability of $(X > 4)$ where X is uniform on the domain [0, 5].

Make sure you look carefully at the probability the problem asks for. Suppose, for example, that X is uniform on the domain [0, 5], and you want $P(X > 6)$. Because six is outside the domain of X (beyond the highest possible value), the probability is zero. Don't plug in six for X in the formula for $P(X > x)$; it won't work. You get a negative number, which doesn't make sense. Examples such as this illustrate why drawing a picture and being aware of the domain of X are very critical steps.

Because the problem asks for a greater-than probability, the right side of the rectangle is shaded. You find the length by taking the right endpoint (b) minus x, and you multiply by the height of the rectangle, represented by $f(x) = \frac{1}{b-a}$ to get the area. So, $P(X > 4) = (5 - 4) * \frac{1}{5-0} = 1 * \frac{1}{5} = \frac{1}{5} = 0.20$. You can see this probability represented in Figure 17-6, because the shaded area is 20 percent of the rectangle.

For those of you who know calculus, you can find $P(X > x)$ by integrating the density function $f(x)$ from x to b. This method gives you the same result.

Finding probabilities between two values

When a problem asks you to find the probability that X is between two values, it helps first to understand exactly what the question means and how you can write it out with probability symbols. If the problem asks for, say, the probability that X is between two and three, you can say that X is greater than two and less than three at the same time. That means you combine two expressions, $P(X > 2)$ and $P(X < 3)$, into one expression. To do this, you put the smaller value on the left, the larger value on the right, X in the middle (because it's between those values), and less-than signs into the expression. Notice that $P(X > 2)$ is the same as $P(2 < X)$, so when you put $P(2 < X)$ together with $P(X < 3)$, you get $P(2 < X < 3)$.

In general terms, to get $P(x_1 < X < x_2)$, you use the formula $\frac{x_2 - x_1}{b - a}$ for any two values x_1 and x_2 between a and b where x_1 is the smaller of the two values. For example, say you want to find the probability that it will take a teacher between 2 and 3 minutes to grade a quiz on the domain [0, 5]. You want $P(2 < X < 3)$, where x_1 represents 2 and x_2 represents 3. Figure 17-7 shows a picture of this situation. The area of the rectangle is equal to $(3 - 2) * \frac{1}{5} = 1 * \frac{1}{5}$, which is $\frac{1}{5} = 0.20$. You can see that the shading in Figure 17-7 represents this probability.

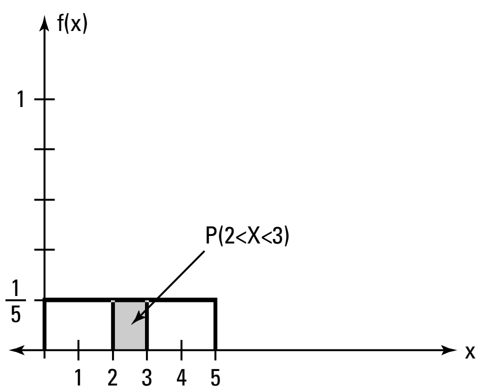


Figure 17-7: The probability of $(2 < X < 3)$ where X is uniform on the domain [0,5].

For those of you who know calculus, you can find $P(x_1 < X < x_2)$ by integrating the density function f(x) from x_1 to x_2. This method gives you the same result. You also work with this same integral to get the probability that X is between x_1 and x_2, including x_1 and/or x_2.

Corralling Cumulative Probabilities, Using F(x)

Sometimes you have to find many less-than, greater-than, or between-values probabilities for the same distribution, and rather than doing every probability again from scratch (either by drawing the picture to find the areas of the rectangles or integrating with calculus), you can use a special function that finds the probability accumulated up to a given value of X and work with that. This function is called the *cumulative distribution function,* or cdf of X. You write it as F(x) to distinguish it from the density function f(x). In other words, the value of the cumulative distribution function at x is $F(x) = P(X \leq x)$, which is the same as $P(X < x)$ for continuous random variables. (See Chapter 7 for more information on the cumulative distribution function F[x].)

In the case of the continuous uniform distribution, $P(X < x)$ equals $\frac{x-a}{b-a}$ for any value of x between a and b (see the previous section for more on this equation). A graph of F(x) for the continuous uniform is shown in Figure 17-8.

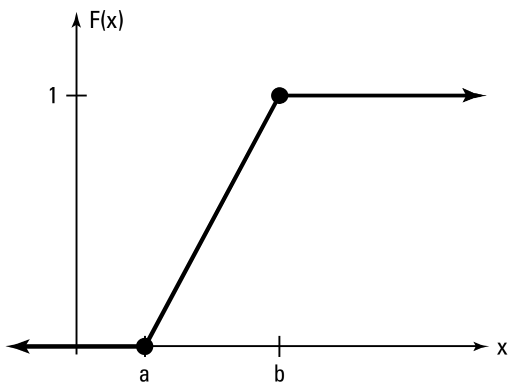


Figure 17-8: The graph of F(x) for the continuous uniform distribution on the domain [a, b].

Notice that F(x) is defined for all real numbers, which is customary. No probability accumulates until you get to a, so F(x) is zero when X is less than a. When you pass that value, probability starts accumulating between a and b. The line shown in Figure 17-8 between a and b represents the formula for F(x) for a continuous uniform distribution, namely $\frac{x-a}{b-a}$. You can rewrite this as $\frac{x-a}{b-a} = \frac{x}{b-a} - \frac{a}{b-a} = \frac{1}{b-a}(x) - \frac{a}{b-a}$, which is the equation of a line (because a and b are constants). When X is equal to b, all the probability has accumulated, because $P(X \leq b) = 1$, with b being the right endpoint.

However, if you want $P(X < x)$ for any number beyond b, the answer to that probability is still equal to one, because if you want all the area below x, and x is beyond the domain of X, that area still equals one. Nothing new has been added. Similarly, $P(X < x)$ for any value below a is equal to zero. For example, if X is uniform on the interval [2, 4] and you want $P(X < 10)$, the probability is one, and $P(X < 1)$ is zero. Even though these probabilities may not be interesting, they're still well defined, and you can still provide answers.

For the grading example I use in the section "Drawing Up Probabilities for the Continuous Uniform," you find $P(X < 4)$, $P(X > 4)$, and $P(2 < X < 3)$ by finding the areas of the corresponding rectangles. Now you can use the cdf to calculate them:

- First, $P(X < 4)$ equals $F(4)$. So, let $x = 4$, $a = 0$, and $b = 5$ in the formula for $F(x)$ to get $\frac{4-0}{5-0} = \frac{4}{5} = 0.80$.
- You have to rewrite $P(X > 4)$ as a less-than probability, because that's how probability defines $F(x)$. By the complement rule (see Chapter 2), $P(X > 4) = 1 - P(X < 4) = 1 - F(4)$. You find in the previous bullet that $F(4) = 0.80$, so $1 - F(4) = 1 - 0.80 = 0.20$.
- You also have to rewrite $P(2 < X < 3)$. Because the area between two and three is equal to the area from zero to three minus the area from zero to two (see Chapter 2), you can rewrite $P(2 < X < 3)$ as $P(X < 3) - P(X < 2)$. This is the same as $F(3) - F(2)$. Calculate $F(3) = \frac{3-0}{5-0} = \frac{3}{5}$, which equals 0.60, and calculate $F(2) = \frac{2-0}{5-0} = \frac{2}{5}$, which equals 0.40. Now, subtract to get $0.60 - 0.40 = 0.20$.

TIP

If you have to do many calculations of probabilities for X, you can use F(x) to do them. If you have only a few do to, I suggest drawing a picture and using areas of rectangles to do them. For the uniform distribution, this method is even faster than using the cdf, F(x).

Figuring the Expected Value and Variance of the Continuous Uniform

Because you work with a rectangle when dealing with the probability of a continuous uniform distribution (when you aren't using the cdf; see the previous section), the expected value is easy to spot: It's the value right smack in the middle, also known as the midpoint between a and b. The variance of the random variable X, which measures the amount of spread in the values of X, also has something to do with a and b: The distance between them is in the formula. That fact makes sense, because the wider the distance between a and b, the larger you should expect the spread in the values to be, over the long term.

In this section, you find and work with formulas for the expected value, variance, and standard deviation of the continuous uniform distribution.

The expected value of the continuous uniform

The *expected value* of a random variable, E(X), is the expected average value of X. In the case of the continuous uniform distribution, because the density function, f(x), is a flat line (see the section "Determining the Density Function for the Continuous Uniform Distribution"), the expected overall average value of X is the point that lies exactly halfway between a and b. So, $E(X) = \frac{a+b}{2}$ for the continuous uniform distribution.

In the grading example I use in previous sections, you try to find the probability of a teacher grading quizzes in a time period of 0 to 5 minutes. X is a continuous uniform on the domain [0, 5], and you have a = 0 and b = 5, so $E(X) = \frac{a+b}{2} = \frac{0+5}{2} = \frac{5}{2} = 2.5$. On average, the teacher will take 2.5 minutes per student to grade quizzes.

For those of you who know calculus, you find the formula for E(X) by taking the integral of x * f(x) over the domain of X (the interval [a, b]). This is similar to the formula $E(X) = \sum_x xp(x)$ for the expected value when X is discrete (when it has a finite or countably infinite number of possible values; see Chapter 7), because you're taking X times the probability for each value of X and summing. In the continuous uniform case, you take X times the density for X, over the domain of X, and integrate. Here's the actual integration:

$$\int_a^b xf(x)\,dx = \int_a^b x * \frac{1}{b-a}\,dx = \frac{1}{b-a}\int_a^b x\,dx = \frac{1}{b-a}\left.\frac{x^2}{2}\right|_a^b = \frac{1}{b-a}\left[\frac{b^2}{2} - \frac{a^2}{2}\right] =$$

$$\frac{1}{b-a}\,\frac{(b^2-a^2)}{2} = \frac{1}{\cancel{(b-a)}} * \frac{\cancel{(b-a)}(b+a)}{2} = \frac{b+a}{2} = \frac{a+b}{2}.$$

The variance and standard deviation of the continuous uniform

The *variance,* V(X), of the continuous uniform distribution is $\frac{(b-a)^2}{12}$. If a = 0 and b = 5 — which is true in the grading papers example I use in previous sections in this chapter, where you have a time period of 0 to 5 minutes for a teacher grading papers — $V(X) = \frac{(5-0)^2}{12} = \frac{25}{12}$, which equals 2.08.

PROBABILITY IN ACTION

Using the uniform distribution in a random-number generator

Have you ever played a computer game that has different outcomes at different times, and you can't totally predict what's going to happen? How does the video game determine what will happen in a given situation? It sets up a probability for each outcome first — for example, the chance of you getting a pot of gold is $\frac{10}{100}$, the chance of you falling through a hole in the floor is $\frac{40}{100}$, and the chance of you continuing on without anything happening is $\frac{50}{100}$. Now the computer chooses a random number between 0 and 100. If it comes up with a number from 1 to 10, you win the pot of gold. If it comes up with a number from 11 to 50, you fall through the floor; and if it comes up with a number from 51 to 100, you just go on without anything happening.

Researchers do many computer studies and experiments by using random-number generators, which generate random numbers they use to select samples and determine outcomes. The continuous uniform distribution is the basis for coming up with random numbers in a random number generator, because it gives equal density to every possible value of X. The most commonly used continuous uniform distribution for generating random numbers is the uniform on the domain [0, 1], because you can find values to any number of decimal places that you want and multiply by the necessary powers of ten afterward to get the desired numbers. For example, if you need to draw a number between 1 and 100, you can have the computer generate a number from the uniform [0, 1] distribution to two decimal places (say, 0.55) and multiply by 100 to get the number (55). If you want to generate a random number between 1 and 1,000,000, you can have the computer choose a random number from the continuous uniform on [0, 1] to six decimal places (say, 0.000123) and multiply by 1,000,000 to get the number (123).

The standard deviation, SD(X), of the continuous uniform distribution is the square root of the variance: $\sqrt{\frac{(b-a)^2}{12}}$. In the previous example, the standard deviation is equal to $\sqrt{2.08}$, which equals 1.44. So, the average deviation in grading time between one quiz and another is 1.44 minutes.

REMEMBER

You write the distance between a and b as b – a because b is larger than a. This distance appears in the formula for V(X). As the distance between a and b gets larger, the variance of X increases. The increase makes sense because if the distance between a and b gets larger, more and more possible values of X appear, and they all have equal density. So, the variability in the results from trial to trial will differ by more in that situation than it would if a and b were close together to begin with.

Chapter 18

The Exponential (and Its Relationship to Poisson) Exposed

In This Chapter

- Recognizing the density functions for exponentials
- Pinpointing probabilities for an exponential distribution
- Calculating the expected value, variance, and standard deviation of the exponential
- Comparing the exponential and Poisson distributions

The *exponential distribution* is a model used for a continuous distribution whose probability density function (pdf), f(x) (the function telling you how dense or tightly packed the probability is for any point x; see Chapter 17), has the shape of an exponential function. The distribution crosses the Y-axis at some positive value (called λ) and then slopes down and to the right in a curve, decreasing toward zero as the values of X, the random variable, go to infinity. It's based on the exponential function f(x) (f[x] equals "e" to a power of x) used for birth and death models, decay rates, and interest rates. Other real-world examples that call for the exponential model include lifetimes of products, times between phone calls, and the time you spend waiting in line.

Figure 18-1 shows an example of an exponential probability density function e^{-x} for $x \geq 0$. The amount of slope in the curve is determined by a constant that's different for each exponential. That constant is equal to λ, the place where the probability density function crosses the Y-axis. (In this example, $\lambda = 1$.)

In this chapter, you determine the pdf for the exponential as well as the cumulative distribution function (cdf), F(x), which gives you the total probability accumulated up to and including x (see Chapter 7 for more). You also find probabilities below, above, or between two values and calculate the expected value and the variance of the exponential. Finally, you discover an interesting and important relationship between the exponential distribution and the Poisson distribution (see Chapter 13 for full information on the Poisson distribution).

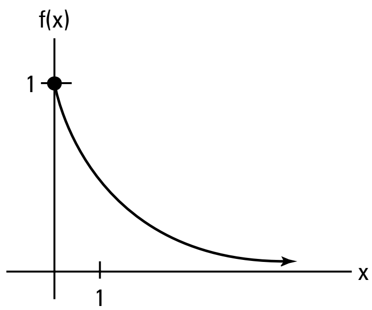


Figure 18-1: Graph of an exponential pdf where λ equals one: $f(x) = e^{-x}$, $x \geq 0$.

Identifying the Density Function for the Exponential

The exponential distribution doesn't have a specified set of conditions that you can check to see if you're using the correct distribution. Rather, it is a model that probalists fit to certain types of data, and under that assumed model, you'll be asked to find probabilities, expected values, and variances. What you do know is that exponential distributions are used when you're measuring times, such as time between customers, or time until an event occurs. So, when you're given a probability problem involving the exponential, you will most likely be told you have an exponential distribution. That is, you're simply given its density function and are asked to work with it. Therefore, identifying the density function for an exponential is particularly important.

The *density function* for the exponential distribution is $f(x) = \lambda e^{-\lambda x}$, where $x \geq 0$ (and 0 otherwise) and λ is constant and is called the *parameter* of the exponential distribution. The purpose of λ is to allow the different functions within the exponential family to cross the Y-axis in a variety of different places and to have differing amounts of downward slope in their density functions.

The constant λ is also the reciprocal of the mean for the exponential (λ is one over the mean; see the section "The Expected Value of the Exponential"). So, if the probability density function crosses the Y-axis at a high value of λ, it has a small mean and drops very quickly toward zero. If the function crosses at a small value of λ, it has a large mean and drops very slowly toward zero.

Figure 18-2 shows several different graphs of what an exponential pdf looks like for different values of λ.

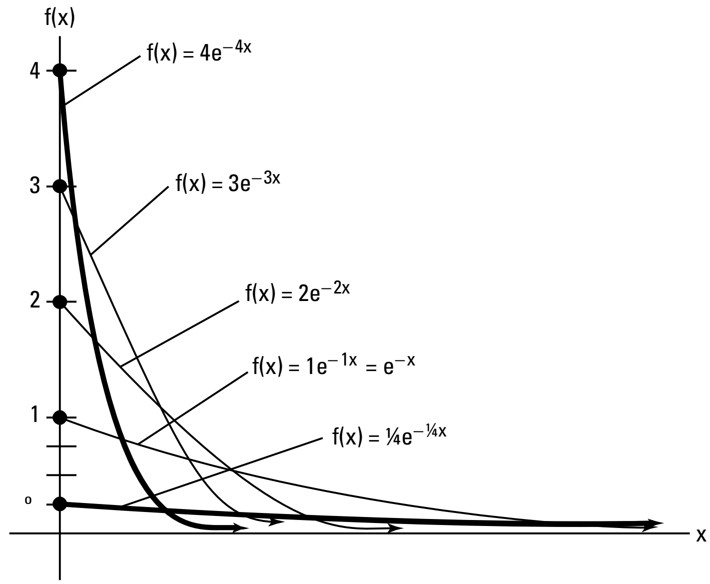


Figure 18-2: Exponential density functions for different values of λ.

REMEMBER

You can attribute the trade-off between the Y-intercept and the amount of downward slope in the density function to the fact that the total area under the curve always has to equal one for any exponential density function.

TIP

If a problem asks you where the probability density function for any random variable crosses the Y-axis, just plug in 0 for X. When you put X = 0 into any exponential function, you get λ multiplied by *e* to the 0 power, which equals 1. So, the Y-intercept for any exponential distribution is equal to λ.

TECHNICAL STUFF

For people who know calculus, the total area under f(x) over the domain of X has to equal one, which means the integral of $\lambda e^{-\lambda x}$ from zero to infinity is equal to one. Here's the integration: $\int_0^\infty \lambda e^{-\lambda} dx = -e^{-\lambda x}\Big|_0^\infty = -\left(\lim_{x\to\infty} e^{-\lambda x} - e^{-0}\right) = -\left(\lim_{x\to\infty} \frac{1}{e^{\lambda x}} - 1\right) = -\lim_{x\to\infty} \frac{1}{e^{\lambda x}} + 1 = -0 + 1 = 1$. Remember throughout your calculations of exponential functions that λ is a constant, and you're integrating with respect to X. As part of these calculations, you use a *u substitution.* (For information on calculus, check out *Calculus For Dummies,* by Mark Ryan [Wiley].)

Determining Probabilities for the Exponential

You figure the probabilities for the exponential distribution by finding the area under the curve represented by the probability density function, f(x). Probability problems for the exponential (as well as for any other continuous random variable) fall into three basic categories: the probability of being less than x, greater than x, and between two values a and b:

- The probability of being less than x, denoted $P(X < x)$, is the same as the probability of being less than or equal to x, denoted $P(X \leq x)$, because the probability that X equals any a single point is zero when X is continuous (see Chapter 7). For the exponential, $P(X < x)$ is equal to $1 - e^{-\lambda x}$.
- The probability of being greater than x, denoted $P(X > x)$, is the same as the probability of being greater than or equal to x, denoted $P(X \geq x)$, because the probability that X equals any single point is zero when X is continuous (see Chapter 7). For the exponential, $P(X > x)$ is equal to $e^{-\lambda x}$.
- The probability of being between two values a and b, where a is the lower of the two values and b is the higher of the two values, is denoted $P(a < X < b)$, which covers the probability for all values of X that are greater than a and less than b at the same time. Changing the < sign to a ≤ sign or changing the > sign to a ≥ sign doesn't change the probability because the probability that X equals any a single point is zero when X is continuous (see Chapter 7). For the exponential, $P(a < X < b) = e^{-\lambda a} - e^{-\lambda b}$.

Finding a less-than probability for an exponential

You have a general formula at your disposal to find $P(X < x)$ if X is an exponential distribution with parameter λ. This formula is $1 - e^{-\lambda x}$, for any value of x greater than zero that you want to find a less-than probability for.

Say, for example, that you want to know the probability of spending less than 15 minutes waiting in line at the supermarket, or $P(X < 0.25)$; when X is the waiting time you have in the supermarket checkout line and has an exponential distribution with a parameter, λ, equal to 4 hours. How did you get the 0.25 here? Because you define X in hours, you need to take 15 minutes and convert it into hours. Using the proportion $\frac{60 \text{ min}}{1 \text{ hour}} = \frac{15 \text{ min}}{x \text{ hours}} \rightarrow \frac{60}{1} = \frac{15}{x}$ and solving for x, you find that $60x = 15$, so $x = 0.25$ hours. So, you have an exponential distribution with λ equal to 4, and you want $P(X < 0.25)$. Figure 18-3 shows a picture of the situation.

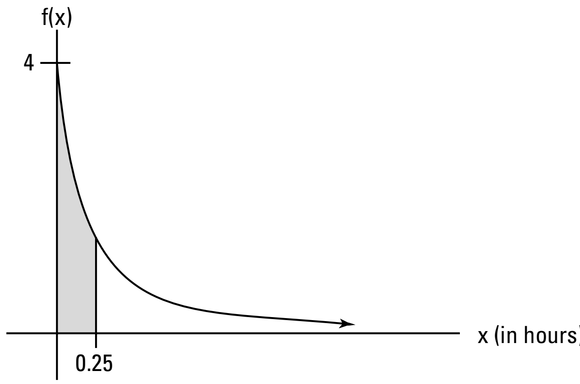


Figure 18-3: The probability of waiting less than 15 minutes when your wait time has an exponential distribution with λ equal to 4.

To find P(X < 0.25), you need to find the area under the curve between 0 and 0.25. Because the density function for the exponential is in the shape of a curve (see the section "Identifying the Density Function for the Exponential"), you can't use any nice formulas from geometry (like you do for the uniform, where area is equal to length times height; see Chapter 17). You have to use calculus to get the probability, or you have to use the formula that has already been developed to find less-than probabilities for any exponential distribution.

Here are the steps for finding a less-than probability for an exponential:

1. **Write down the expression you want to find the probability for.**

   For this example, P(X < 0.25).

2. **Write down the value of λ. The mean of the exponential is $1/\lambda$, so if you're given the mean, you need to find the reciprocal for the value of λ.**

   For this example, λ is 4 hours.

3. **Write down the formula for the less-than probability for the exponential in its general form: $1 - e^{-\lambda x}$.**

4. **Take the formula from Step 3 and substitute your value of λ and the desired value of X that you want the probability for.**

5. **Solve the probability; you should get a number between zero and one.**

   Using the general formula for $P(X < x) = 1 - e^{-\lambda x}$, the probability of waiting less than 15 minutes in line is $P(X < 0.25) = 1 - e^{-4(0.25)}$, which equals 0.632. So, if you're waiting in this particular line, the chance that you'll have to wait less than 15 minutes is 63.2 percent.

Note that $P(X < x)$ is equal to the cumulative distribution function, $F(x)$, also known as the cdf (see Chapter 7). So, the formula in the previous example equals the cdf of X for the exponential. If you plug in very large values for X (for you calculus folks, that means taking the limit as X goes to infinity), you get one. And if you plug in very small (positive) values of X that are close to zero, you get zero.

For people who know calculus: To get the area of the shaded region in Figure 18-3, you integrate the probability density function, $f(x)$, from 0 to 0.25, putting in 4 for λ. You get the same answer. (For those who don't know calculus, check out *Calculus For Dummies,* by Mark Ryan [Wiley].)

Finding a greater-than probability for an exponential

The general formula for finding a greater-than probability for the exponential, $P(X > x)$, is $e^{-\lambda x}$. Sticking with the grocery-line example from the previous section, where X is the waiting time you have in the supermarket checkout line and has an exponential distribution with a parameter, λ, equal to 4 hours, suppose you want the probability that you'll have to wait in line at the supermarket for more than 15 minutes — $P(X > 0.25)$. Figure 18-4 shows you a picture of this situation.

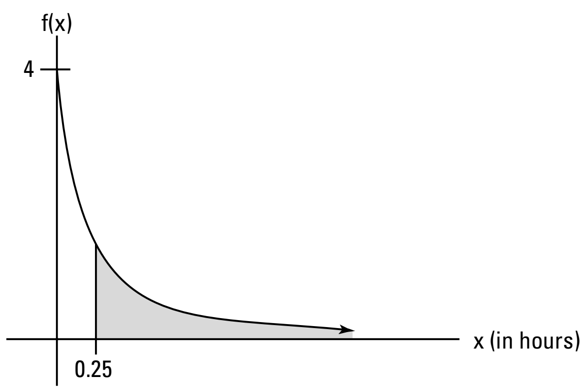


Figure 18-4: The probability of waiting more than 15 minutes when your waiting time is exponential with $\lambda = 4$.

To find this probability (the shaded area in Figure 18-4), you can use the formula for greater-than probabilities. Here are the steps for finding a greater-than probability for the exponential:

1. **Write down the expression you want to find the probability for.**

   For this example, $P(X > 0.25)$. Be sure your time units are correct. If you're dealing in hours, 15 minutes equals 1/4th of an hour, or 0.25 hours.

2. **Write down the value of λ for the problem. The mean of the exponential is $1/\lambda$, so if you're given the mean, you need to find the reciprocal for the value of λ.**

   For this example, λ equals 4 hours.

3. **Write down the formula for finding a greater-than probability for the exponential: $e^{-\lambda x}$.**

4. **Take the formula from Step 3 and substitute your value of λ and the desired value of X that you want the probability for.**

5. **Solve the probability, keeping in mind that you should get a number between zero and one.**

   For this example, where $X = 0.25$ and $\lambda = 4$, you get $e^{-4(0.25)}$, which equals 0.368. So, if you're waiting in this particular line, the chance that you'll have to wait more than 15 minutes is 36.8 percent.

TECHNICAL STUFF

To get the area of the shaded region in Figure 18-4 by using calculus, you integrate $f(x)$ from 0.25 to infinity (or take one minus the integral from 0 to 0.25 because the total area under the curve must be one). (For more information on calculus, look to *Calculus For Dummies,* by Mark Ryan [Wiley].)

TIP

Notice that because the probability of being greater than x is the complement (the opposite) of the probability of being less than x, the formula for $P(X > x)$ is one minus $P(X < x)$ (which you find in the previous section). And because $P(X < x)$ is equal to the cumulative distribution function, $F(x)$, also known as the cdf (see Chapter 7), $P(X > x)$ is equal to $1 - F(x)$. For the example in this section, you can use the complement rule (see Chapter 2) to find $1 - P(X < 0.25) = 1 - 0.632 = 0.368$. With calculus, to avoid a situation where you have to deal with infinity in the limits of integration, consider taking the complement rule when you need a greater-than probability. The rule allows you to get one minus the probability of being less than x, which is the integral of $f(x)$ from zero to x — much easier to work with.

Finding a between-values probability for an exponential

If you want to find the probability of being between two values on an exponential distribution — call them a and b, where a is the lower and b is the higher of the two values — you want $P(a < X < b)$. In other words, you want the probability that X is greater than a and less than b at the same time (between a and b). The general formula for $P(a < X < b)$ for an exponential distribution is $e^{-\lambda a} - e^{-\lambda b}$.

Sticking with the grocery-line example you see in the less-than and greater-than sections, suppose you want the probability that you'll have to wait in line at the supermarket between 15 and 30 minutes: $P(0.25 < X < 0.50)$ hours. Here, X is the waiting time you have in the supermarket checkout line and has an exponential distribution with a parameter, λ, equal to 4 hours. Figure 18-5 shows a picture of the situation.

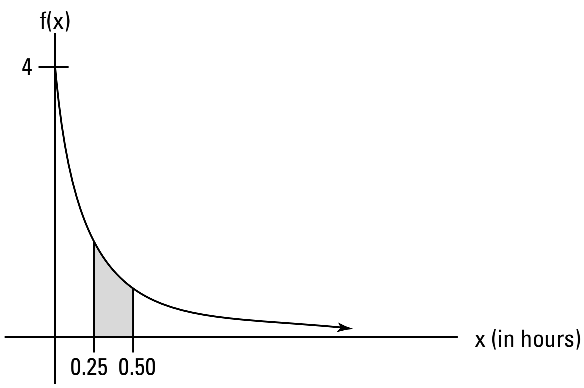


Figure 18-5: The probability of waiting between 15 and 30 minutes when λ is equal to 4.

Here are the steps for finding a between-values probability for the exponential:

1. **Write down the expression that you want to find the probability for.**

   For this example, $P(0.25 < X < 0.50)$. Be sure your time units are correct. If you're dealing in hours, 15 minutes equals 1/4th of an hour, or 0.25 hours, and 30 minutes equals 1/2 hour, or 0.50 hours.

2. **Write down the value of λ for the problem. The mean of the exponential is $1/\lambda$, so if you're given the mean, you need to find the reciprocal for the value of λ.**

   For this example, $\lambda = 4$ hours.

3. **Write down the formula for a between-values probability for the exponential: $e^{-\lambda a} - e^{\lambda b}$.**

4. **Take the formula from Step 3 and substitute your value of λ and the desired values of X that you want the probability for; a is the lowest value and b is the highest value.**

   For this example, $a = 0.25$, $b = 0.50$, and $\lambda = 4$.

5. **Solve the probability, remembering that you should get a number between zero and one.**

   For this example, you have a = 0.25 and b = 0.50, so $P(0.25 < X < 0.50) = e^{-4(0.25)} - e^{-4(0.50)} = e^{-1} - e^{-2} = 0.368 - 0.135 = 0.233$. So, 23.3 percent of the time you'll have to wait between 15 and 30 minutes in this line.

TIP

To get $P(a < X < b)$ when thinking about Figure 18-5, you can take all the area up to b represented by $P(X < b)$ and subtract off the part you don't want, which is all the area up to a, represented by $P(X < a)$. So, $P(a < X < b) = P(X < b) - P(X < a)$. Technically, the only formula you really need here is $P(X < a)$; all the other probability formulas (greater than, between values, and so on) fall from that one.

TECHNICAL STUFF

You can find $P(a < X < b)$ with calculus by integrating the density function f(x) from a to b; you get the same result. (For more useful information about calculus, see *Calculus For Dummies,* by Mark Ryan [Wiley].)

Figuring Formulas for the Expected Value and Variance of the Exponential

In this section, you find and work with formulas for the expected value, variance, and standard deviation of the exponential distribution. These topics are important because they answer some practical questions about lifetimes, or times between events. For example, what's the average amount of time you spend on hold when you call a help line? What's the amount of variability in the time between customers walking into a bank?

The expected value of the exponential

The expected value of an exponential distribution is $E(X) = \frac{1}{\lambda}$. So, if $\lambda = 4$ hours waiting in line at the supermarket, the expected waiting time is $E(X) = \frac{1}{4} = 0.25$ hours ($* 60 = 15$ minutes, using the proportion that compares hours to minutes).

Notice that as λ gets larger, the expected value of X, which is $\frac{1}{\lambda}$, gets smaller, and vice versa. Looking at Figure 18-2, you can see why: λ is the Y-intercept for the density function for the exponential, and the higher the intercept is, the more quickly the curve drops and the smaller the mean will be. Alternatively, for small values of λ, the opposite happens: The Y-intercept is low, and the function drops slowly, pushing the mean of X out along with it.

The mean isn't equal to λ; it's equal to one over λ. This fact also works the other way around: λ isn't equal to the mean; it's equal to one over the mean. You may be asked to come up with the density function, f(x), for a given value of λ or for a given value of the mean (which is the same thing). Be careful to set up the density function correctly, using the information given to you in the problem. For example, if X has an exponential with a mean 4, $f(x) = \frac{1}{4}e^{-\frac{1}{4}x}, x \geq 0$, as you see in Figure 18-2. But if X has an exponential with a mean ¼, $f(x) = 4e^{-4x}, x \geq 0$.

For people who know calculus, you find the formula for E(X) by taking the integral of x * f(x) over the domain of X (in this case, the interval zero to infinity). That means you want to find the following integral: $\int_0^{\infty} x\lambda e^{-\lambda x}\,dx$. Its derivation, using calculus, goes beyond the scope of this text; however, if you want to try it, it involves *integration by parts* where u = x and $dv = \lambda e^{-\lambda x}\,dx$. You use L'Hopital's Rule to find the limit of the first ("uv") term. Have fun! (For more on calculus, see *Calculus For Dummies,* by Mark Ryan [Wiley].)

The variance and standard deviation of the exponential

The variance of the exponential distribution is $V(X) = \frac{1}{\lambda^2}$. If, for example, you have to wait in line at a supermarket for 4 hours, $\lambda = 4$, $V(X) = \frac{1}{4^2} = \frac{1}{16}$, which equals 0.0625, in hours squared. (This can't be interpreted.)

The standard deviation is the square root of the variance, so $SD(X) = \sqrt{\frac{1}{\lambda^2}} = \frac{1}{\lambda}$. For the waiting-in-line example, the standard deviation is $\sqrt{\frac{1}{16}}$ = ¼ hours, or 15 minutes. So, the expected amount of deviation in waiting times with repeated visits to this supermarket checkout line is 15 minutes.

The expected value and standard deviation are the same for the exponential, which means as one goes up, so does the other. In other words, if you expect to wait a long time, the amount of variability from person to person is expected to increase. If you expect to wait a small amount of time, you see less variation from person to person.

The derivation of the formula for V(X) for the exponential, using calculus, is beyond the scope of this text. However, the least painful way to get it is by using the formula $V(X) = E(X^2) - [E(X)]^2$ (see Chapter 7). Because the last term is just the mean squared (which you can find by using the formula for E[X] and squaring it), you need to integrate "only" the function $x^2 * f(x)$. So, your job is to find $E(X^2) = \int_0^\infty x^2 \lambda e^{-\lambda x} dx$. Because you have an x-squared term involved, you need to do integration by parts — not once, but twice.

But before you break out into hives about integrating by parts twice, know that most professors don't expect you to be able to do this level of integration (after all, probability isn't a math class). For most situations, the most important task is recognizing how and when to use the formulas for V(X) and SD(X) and to relate them to the value of λ.

Relating the Poisson and Exponential Distributions

You use the Poisson distribution to model the number of outcomes that occur within a fixed period of time (see Chapter 13 for the full scoop on the Poisson distribution). This has a direct and interesting relationship with the exponential distribution (which models times between occurrences or time that passes until an occurrence).

For example, suppose that X represents the number of phone calls coming in to a computer-support line. Assume that X has a Poisson distribution with a mean equal to ten calls per hour. What do you think is the expected time that passes between calls? If ten calls come in during a one-hour period, you can flip this fact around to say that in one hour, you should expect ten calls, or one call approximately every six minutes. (Hours per call is equal to $1 \div 10 = 0.10$. And one-tenth of an hour is equal to $0.10 * 60 = 6$ minutes.)

PROBABILITY IN ACTION

Lining up to see the exponential and Poisson distributions at work

Statisticians use the exponential distribution a great deal in a field of statistics called *queuing theory.* A *queue* is a line of individuals waiting to be served. You find yourself in queues all the time: at the drive-through, at the bank, in a traffic jam, on the phone when you're put on hold — "We'll take your call in the order in which we received it. Estimated time is XX minutes . . ." — and so on.

Experts also use exponential and Poisson distributions to model population size, using something called a *birth and death process.* In a birth and death process, you allow individuals to come into (immigrate) and leave (emigrate) a population, which means you have two different values of λ operating simultaneously. Population models can also predict how long it will take a certain population to die out or to explode out of control.

So, if X = number of calls per hour is a Poisson distribution with a mean of ten calls per hour, Y = time between calls is an exponential distribution with a mean of $\frac{1}{10}$ hours, or 6 minutes.

In general, if X counts the number of occurrences in a fixed time period and has a Poisson distribution with mean λ, Y = the time between those occurrences as an exponential distribution with mean $\frac{1}{\lambda}$.

Notice that the Poisson distribution counts occurrences, so it's a discrete random variable; Y measures times between occurrences, so it's a continuous random variable (see Chapter 7 to find out more about this distinction). Just another way for you to keep the distributions straight!

Part VI
The Part of Tens

"I always get a good night's sleep the day before a probability test, so I'm relaxed and alert the next morning. Then I grab my pen, eat a banana, and I'm on my way."

In this part . . .

In Part VI, you find my top-ten probability formulas, along with explanations and examples of each; ten steps to a better grade on your probability exam; and ten common probability misconceptions to avoid.

This part is designed to be your "bottom line" for doing well in probability. I base it on my years (and I won't tell you how many) of experience writing homework and exam questions and grading plenty of student papers. Think of it as your inside track to probability success!

Chapter 19

Ten Steps to a Better Probability Grade

In This Chapter

- Practicing strategies you can use to be successful in probability
- Discovering ways to add points to your total score
- Feeling more confident about starting and finishing problems correctly

I remember my first probability class; it was a bear. All those formulas floated around in the air, my eyes glazed over well before the lecture ended, and I felt like every single problem was completely different from all the others I'd ever seen. Yes, I've been there before. How did I survive to write this book? That's a long story, but here's the short version: I hung in there, kept working problems, and asked my instructors plenty of questions.

Over all my years of working out, designing, and grading probability problems, I've developed some strategies and tips that I fall back on when times get rough. I present these tips and strategies to you in this chapter. They're hard fought, and I still cling to them today. I hope they can save you some time, lessen your headaches, give you more confidence, and help you score better on your probability exams.

REMEMBER

Solving probability problems is a process that starts with good organization and translation skills, moves into picking up cues to get directions for solving the problems, builds up to the calculations, and ends with the proper interpretations and conclusions (and with you checking your work, of course). I use the same example problem throughout this chapter to simulate the actual problem-solving process from beginning to end. I suggest that you read this chapter from beginning to end, incorporate the ideas you deem useful, and put them into practice. You can return to this chapter again later to add more tips and ideas to perfect your problem-solving skills.

Get Into the Problem

A helpful first step to take when dealing with a probability problem is to get into the problem. When I say get into the problem, I mean that you shouldn't rush to the question and start multiplying some numbers together to get the answer. I know it feels better to write something down right away, but try to resist the urge and spend a few moments understanding the problem first. Trust me, patience pays off.

Try getting into the head of the person in the problem (no matter how strange, contrite, or downright silly the situation may be), and visualize yourself in that person's situation. You can even try to explain the problem in your own words; if you can do this, you show that you really understand the problem.

Here's an example of how to "get into" a test problem. Suppose that an exam has ten multiple-choice questions on it, with four possible answers each, and a student decides to guess on every problem. You're asked to figure out the number of questions that the student should get right just by guessing. Putting yourself in the student's position, you want to know the probability of guessing a single answer right first. Each problem has four possible answers, so the probability of getting any one problem right is 1 in 4.

You're now engaged in really thinking about the situation, and you understand and can explain the situation that makes up the problem. Why does this help? Because in order to solve a word problem, you first have to completely understand its context. You never know what information you may need to bring up on your own in order to make some headway. (Plus, grasping the problem focuses your concentration, and you're less likely to freak out, get distracted, or freeze up.)

Understand the Question

After you have a good idea of the scenario of the problem, I suggest going to the end of the problem (which is where the questions usually are) and reading the question again. Focus on it, and make sure you understand it. Try translating the question into a probability problem that involves calculations rather than words. Doing so should calm you down, and you'll start to get an idea of where you need to go.

Sticking with the test example I present in the previous section, suppose the question at the end states: How many questions out of ten should a student get right just by guessing, with four possibilities for each question? In other words, you want to know how many questions the student should get right if he or she guesses on all the problems.

Addressing the nuts and bolts of probability problems

Here's a story that teachers can relate to students and students can keep in mind to hammer home the importance of understanding the question. A company was having problems on its production line, so it called in an expert for help. The expert walked in, and amongst all the complicated machinery, he went straight to a specific spot and tightened a single bolt. Everything started working again, and he walked out of the building. A few weeks later, the company received a bill for $10,000. Executives asked the expert, "Why did you charge so much? All you did was tighten a single bolt." He responded, "Oh, the bill for the bolt was only a dollar. Knowing which bolt to tighten, however, cost you $9,999."

Knowing how to set up a probability problem is 99 percent of the work, so it's worth the extra time understanding and translating the problem. The rest is just arithmetic (well, almost).

TIP

"How many" pertains to X, because in every single probability model in this book, you use X to represent a count or a measure. In the example problem, you count the number of questions the student gets correct, so you let X represent that number.

One part of the question may jump out at you: How many questions the student *should* get right. If you have a good understanding of expected value (to gain this understanding, see Chapter 2), you immediately say, "What you should get" sounds like "What do you expect to get," and then you think expected value. But expected value of what? Expected value of the number of correct answers, which is X. So, in the end, the question "How many questions should the student get right just by guessing" translates to "Find the expected value, E(X)."

Now you may be asking, "Why didn't the problem just *say that?* If I had known it wanted *that*, I would've been able to do it." Or, "After I set up the problem, I can solve it." That's right, but your instructor expects that anyone can do that part. The hard part is figuring out what you have to solve.

Organize the Information

When you have the question at hand translated into probability terms (see the previous section), the next step is to write down everything you know — in other words, organize the information that you have. Using the test example I present in the previous sections, you know that the student taking the exam must answer ten questions total. Because the variable n usually represents the total number of anything, let $n = 10$. Next, you're interested in the expected value, $E(X)$ (see Chapter 2), so you have to discover more about the random variable X. In this case, X is counting the number of questions the student guesses correct.

The chance of the student getting any single problem correct is 1 out of 4, which is 0.25, or 25 percent. Because the letter p usually represents the probability of success, and success here is getting the problem right, you have $p = \frac{1}{4}$. (Notice that the problem I present doesn't give you any value of p, which is why thinking it through really helps.)

The following list presents the way I recommend organizing the information in a probability problem:

1. **Write down what you need to find in terms of the question.**

   For the test example, you can write down "need E(X)," where X is the number of correct answers.

2. **Go through and find all the numbers in the problem, and try to label them.**

   You can write down $n = 10$ and $p = \frac{1}{4}$.

3. **Try to think of anything else you need in order to solve the problem.**

   In this case, you haven't decided what type of distribution X has: Is it binomial (see Chapter 8), negative binomial (see Chapter 15), geometric (see Chapter 14), and so on?

   Try looking at your review sheet or thinking back (see the section "Make a Review Sheet" later in this chapter). The binomial distribution has a fixed number of trials, but the negative binomial and geometric don't. Bingo! At this point, you should have a big picture outline of all the information you need to solve the problem, as well as a direction for solving it.

Write Down the Formula You Need

Whenever a problem calls on you to use a formula of any kind, you should first write down the formula in its general form. This makes you feel much more confident, and it shows your instructor exactly what you're doing and where you're going with the problem (which ensures that you'll get the maximum partial credit if something goes wrong down the road).

For the test example I use in previous sections of this chapter, you would write $E(X) = np$, because X has a binomial distribution (see Chapter 8). When you have the general form down, write down each letter involved in the formula and what it's equal to. For example, $n = 10$ and $p = \frac{1}{4}$.

If the formula you choose has a letter in it for which you can't find or come up with a value, take it as a clue that you may be using the wrong formula. Now's the time to reevaluate the situation instead of trying to plow forward by moving numbers around and getting frustrated.

Writing down formulas takes more time, but it helps you remember the formulas and makes you feel more familiar with them (bonuses if you have to memorize formulas in your class). It's worth the time; try it!

Check the Conditions

You need to make sure you check any conditions that the problem needs to meet before you proceed with the solution you want to employ. For example, if you think that X is binomial (see Chapter 8), you need to make sure you see n independent trials, each with two possible outcomes (success and failure), and that the probability of success, p, is the same for each trial.

A tip for the binomial distribution is to use the mnemonic code BINS — Binary (meaning two outcomes); Independent; N is fixed; and Same p for all trials — to remember the conditions.

In the case of the test example you see in previous sections of this chapter, you have $n = 10$ fixed questions (which is what tips you off to the binomial distribution over the negative binomial or geometric, where the trials aren't fixed). Because the student is guessing the answers, the guesses are probably independent of each other. You also know that $p = \frac{1}{4}$ stays the same for each problem because the student is just guessing, so each answer has an equal chance of selection. Therefore, it appears that the problem meets the conditions for the binomial.

If your instructor is big on checking conditions, write down the conditions that you need to meet and check off each as you meet it, including a brief explanation. It could be something as simple as "Independent trials — yes, because the person is guessing." Now you know that you're going in the right direction, and you let your instructor know that you took the time to actually check before you used a formula. Trust me, he or she will be impressed.

Calculate with Confidence

After you set up the problem and organize all your info, the time comes to "plug and chug"; "grind it out"; or "crunch the numbers." Notice first, though, how much work you've done in the previous sections just to get to this point and how small of a step calculating actually is in the big scheme of things. Calculations are important, no doubt, but if you pick the wrong distribution — even if all your calculations are letter perfect — you may get zero points. Having the right idea is 99 percent of the problem.

Note: Instructors vary in terms of the amount of partial credit they give in situations where students use the wrong setup, but put it this way: If you have the right formula, the calculations are more likely to take care of themselves.

Here are some tips I've found useful for making calculations:

- **Don't round off until the last step.** Keep all the calculations in your calculator. Write down the work as you go along.
- **If you can't keep all the work in your calculator the whole time, keep at least three significant digits after the decimal point, and round off to two decimal places at the very last step.** Or check with your instructor to see exactly what he or she wants you to do.
- **Use parentheses.** For example, if you need to calculate $\frac{20-9}{3+8}$, type it in as $(20 - 9) \div (3 + 8)$; otherwise, you get the wrong answer.
- **Remember the order of operations.** You can use PEMDAS (also known as Please Excuse My Dear Aunt Sally) to help you remember that P = parentheses; E = exponents; M = multiplication; D = division; A = addition; and S = subtraction.
- **Don't skip steps (even the small ones).** This is probably the most important factor in determining whether you get the right or wrong answer. People who skip steps often run into trouble, and because they don't have complete work to show, they lose points from possible partial credit.

I often hear students say, "Man, I set up everything right, but I made a DUMB math mistake." If you follow the previous tips and pay close attention to your work, you'll make fewer and fewer calculation errors. Just think of all the points you can save here and there on an exam!

For the question involving a student guessing on a test, you have $E(X) = n * p = 10 * \frac{1}{4} = 2.5$.

Show Your Work

Showing your work seems to be a theme throughout this chapter, but, as an instructor, I can't overemphasize it. Picture yourself in my shoes, looking through students' homework where they're trying to figure the probability of getting a straight flush in poker (to find this probability, see Chapter 5). Some students have scribbling all over the pages — some in the margins, with cross-outs and arrows all over the place. These students include no clear indication of where they're going, what they're doing, or even when (or if) they stopped working on the problem and came up with a final answer. And some people jot down a number with no work at all. How would you feel?

Okay, enough with the sob story; jump back into your student shoes. How do you feel when faced with a tough probability problem? I'm guessing you feel frustrated, confused, and unsure of yourself. That's not the attitude I want to build here. Just like with a messy house, a little cleaning up can do a lot of good in terms of how your work looks from the outside and how you feel on the inside. Think of it this way: If you have to do homework problems, you may as well use good habits so they come naturally to you in exam situations.

REMEMBER

To show your work, you just have to write down your steps *clearly,* one after the other, in an organized and orderly fashion. The two big dos and don'ts to keep in mind are

- Don't skip steps or do steps in your head (unless they're quick arithmetic items).
- Do circle your final answers. Your instructor will love you for it, and you can find your answers much faster when you go back over your work later on.

When you follow my advice, showing your work actually saves you time, because you can go back over your work later and find any errors much faster. When you're studying, you can use your previous work as a quick review to build your confidence. You can remember problems and recreate solutions if you're asked to.

Check Your Answer

After you finish all the calculations in one problem, you probably feel like moving on to the next problem. But not so fast my friend — did you check your answer? What do I mean by checking your answer? I mean going over everything again and making sure you did the work right. But even before you do that, you should check to make sure your answer even makes sense.

You may think that I'm asking, "Did you go to the back of your book to see if the answer matches yours?" I don't mean that. In fact, I recommend that you resist the urge to flip to the back until you double-check your work yourself. If you get in the habit of letting the back of your book tell you when you're wrong and when to fix your answer, you'll need the back of your book during an exam because you'll be lost without it. Not only that, but you'll have a false sense of security going into the exam.

What should you check for in your answer? First, make sure you actually answered the question asked by the problem. For example, if X represents the number of successes in 10 trials, and the problem asks for the probability of 3 failures, did you remember that X = 7 and not 3? Here are some other things to check:

- Probabilities should be between zero and one.
- Expected values should be between the minimum and maximum values of X.
- The standard deviation can never be negative.
- The units should remain consistent (for example, if X is in minutes, the mean and standard deviation should also be in minutes).

Writing down a list of the items to check and watch out for can be very helpful (see the section "Make a Review Sheet" later in this chapter).

One of the most disappointing things an instructor can see is a student writing down an answer for a probability that crosses her desk as –2 or 23.18 — in other words, something outside the possible values of zero and one. Sometimes, instructors take this so seriously that they don't give any points for partial credit, even if the errors were based on small calculation mistakes. Why? Because the student didn't check the answer to see if it made sense.

With that said, I have to tell you that the head of my department keeps a paper he turned in many moons ago when he was in school. One of his answers gave a probability of two. He says he keeps that paper around because it keeps him humble. The moral of the story: It happens to everyone. But you can minimize the chance that it will happen to you on an exam by always checking your answers.

Imagine that you're taking an exam, and you're working on a problem that asks for a probability. You keep getting $\frac{13}{6}$, and you know the answer is wrong because probabilities can't be greater than one, but you can't find your mistake. Here's some advice for this situation: Instead of working and working on the problem, circle your answer and write a little note to the instructor, saying something like, "I know this answer is wrong because probabilities can't be more than one, but I can't find my mistake." The instructor will realize that you recognized the problem, and that will likely mean a lot (it does to me).

Wasting 15 minutes trying to find a calculation error that would give you only a couple more points will cost you precious time that you need to work on other problems.

Interpret Your Results

After you calculate, check, and circle your final answer, be ready to interpret your results in the context of the problem if your instructor asks you to. The important thing is to answer the original question in the same language the instructor used to ask the question. Don't just give a number, in other words, but be able to explain it.

Suppose a question asks you to decide which of two betting games you should play: game A or game B. You find the expected value and variance of each game, write the values down, circle "A," and move on. Your instructor likely expects more than that. Answer the question as if your instructor asked you out loud.

For example, you can write, "Game A has a higher expected value than game B, so I would play game A." Or, you can write, "Game B has a lower expected value but a higher variance, so I would choose game B because I may win more." In this case, the justification is important because your reasoning could lead you to choose game A or B. Both answers could be right as long as you justify them properly.

Make a Review Sheet

Some instructors (myself included) allow students to bring one-page review sheets (or "cheat sheets") with them to their exams. The sheet gives you easy-access to resource material so you don't have to page through all your books and notes. It also reduces the amount of rote memorization that you have to do (although some instructors argue that memorization is a good thing).

Whether or not your instructor allows you to use a review sheet, I strongly urge you to make one anyway (just don't bring it to the exam if it isn't allowed). Here are the huge advantages a review sheet provides:

- It forces you to sit down and summarize all your knowledge.
- It makes you think through the ideas, put them together, compare and contrast, and organize.
- It helps you figure out what questions you still need to ask and realize which concepts you know well and which you need more work on.
- It helps you see the BIG picture (or the forest through all the trees).

If you decide to make a sheet, take the time to sit down and really make one (instead of just copying all the formulas onto a piece of paper). Believe me, instructors know the difference. (The Cheat Sheet at the front of this book gives you a look at an example review sheet.)

So, how do you make a good review sheet? Every review sheet is different, but here are some things that I typically do:

- Start at the beginning and work through the material in order — in your notes and on the review sheet.
- Separate the chapters with straight lines.
- Write down definitions and important results and theorems. I also write down *how* the definitions may be used and *why* the theorems are important.
- Write down any important properties that the instructor covered — for example, the standard deviation can never be negative.
- Write down plenty of examples of each concept — examples are critical.
- Write down notes to yourself so you can avoid some of the mistakes you've made in the past and to remind you of things you often forget to check. (Be sure to include notes on how to identify and start a problem.)

TIP

After you finish your review sheet, try to work some problems with the review sheet beside you (and nothing else) to see how well it works. If you find that something is missing, add it. It also helps to keep your review sheet clean and clear. If you make it so messy and unorganized that you can't find anything (or you can't read it without a magnifying class — this actually happened to a student of mine), your review sheet will be useless. Consider making a rough outline before finalizing the information. And always ask your instructor what he or she thinks of it; that's a good way to see if you're on the right track.

WARNING!

One common mistake students make is leaving something off their review sheets, thinking that "I already know this." In an exam situation, you can forget your name under certain conditions. Be safe and comprehensive; you'll be glad in the end.

Chapter 20

Top Ten (Plus One) Probability Mistakes

In This Chapter

- Avoiding common mistakes made on exams or in a casino
- Correcting the problems that plague you and your grades

In probability, your intuition can get you into a lot of trouble. For example, a woman with three daughters may think that her chance of having a boy next time *has* to be higher than ½, so why not give it a go? Wrong. In other words, probability and intuition don't always mix. In this chapter, I outline the most common probability mistakes students (and teachers) make that I've seen in my many years of teaching. And, more importantly, I discuss how you can avoid making these mistakes.

Forgetting a Probability Must Be Between Zero and One

Above all things, remember this: A probability must be between zero and one. Forgetting this is the mistake that teachers dread most. Teachers ask students to work a problem that calls for the probability of some event, and inevitably, it shows up. A paper that has tons of work flowing down the page, around to the back, up and over the side, with plenty of stops and starts marked out. Finally, through much searching, the teacher finds the answer circled way off in a tiny corner: 2.588742. She collapses in despair.

Yes, you can lose track of the forest for the trees; yes, you can forget what you were working on because you had to use so many formulas; and yes, probability problems can get mean, nasty, and complicated. But never forget that a probability must be between zero and one.

If you want any credit for all the work you've done, stop and make sure your answer falls in the necessary range. If it doesn't, you have two choices. You can go back and try to fix your error. If that doesn't work, or if you don't have time, leave a note for your teacher that says, "I know the answer is supposed to be between zero and one, but I can't find my mistake." It could mean the difference between getting some points and no points.

REMEMBER

Always check your answers for *any* problems you work in probability to make sure they make sense. All standard deviations must be greater than or equal to zero; all expected values should be within the possible values of X; and all probabilities must be between zero and one.

Misinterpreting Small Probabilities

The chance of being struck and killed by lightning is about 80 in 300,000,000 (because about 80 people die from lightning strikes each year in the United States, and as of this writing, the U.S. Census Bureau Web site gives the estimated population count creeping ever closer to 300 million). This probability, in decimal form, is 0.000000267. The chance of winning the Powerball lottery is about 7 out of one billion. (To win the Powerball lottery, you typically must match five numbers from 1 to 49, with repeats allowed, plus match the Powerball, an additional number between 1 and 49.) This probability in decimal form is 0.000000007; see Chapter 6. According to these calculations, you're about 38 times more likely to die from a lightning strike than you are to win the Powerball lottery. But, because getting struck by lightning is a bad thing, people tend to say, "Oh it will NEVER happen to me!" However, winning the Powerball lottery is a different story. "Somebody's gotta win; it may as well be me, right?"

It's important to interpret probabilities as numbers between zero and one, with small probabilities meaning that the situation is less and less likely to happen to an individual person. But in the long term, over the whole country — or all the people buying lottery tickets — someone is bound to end up in that boat. So, you can say with near certainty that out of a billion lottery tickets, someone (even up to 7 people) will win. But even though it may as well be you, it probably won't be.

It also helps your understanding to make comparisons when you examine probabilities. For example, the chance of your child having complications while the doctor puts ear tubes in his or her ears, according to my little boy's surgeon, is less than the chance of getting struck by lightning. Maybe I need to go out and buy a lottery ticket! (For more information on interpreting probability, see Chapter 1.)

Using Probability for Short-Term Predictions

Probability is based on the long-term percentage of times an outcome will occur; it doesn't do well at predicting short-term results. (For more information on interpreting probabilities, see Chapter 1.)

Say, for example, that you flip a coin five times and get heads every time. You know that the chance of flipping five heads in a row is small, so shouldn't it be more likely that the next toss will be a tail? No. Although it may be unusual to flip five heads in a row with a fair coin, it does happen (the chance is 3.125 percent; see Chapter 8). But more importantly, the fact that you just flipped a string of five heads is something that the coin (and the whole coin-flipping process) doesn't realize or keep track of. Because you assume that you're flipping the coin independently each time, you have to say that the chance of getting a tail next time is still ½, just like it was for the previous five tosses.

Here's something really wild: If your teacher gives your class a homework assignment to flip a fair coin ten times and record your results, he or she can actually figure out, with a high probability, whether students faked their results because they forgot to do the assignment. The teacher does it by examining how many runs the students put in their results (if any). For example, HTHTHTHTHT happens with the same chance of any other outcome of ten flips, but if it reportedly happens to 20 people in a class of 70, "Houston, we have a problem."

Thinking That 1-2-3-4-5-6 Can't Win

When I wrote *Statistics For Dummies* (Wiley), I included a short chapter on probability that gave readers some tips for winning the Powerball lottery. I reminded readers that every possible combination of numbers from 1 through 49 on those six balls (repeats allowed) has the same chance of occurring. However, people bet on certain combinations more than others; for example, people often choose numbers that represent months or days of the month because of birthdays. People ignore other combinations because they "appear" to have a smaller chance of occurring; for example, people shy away from the combination 1-2-3-4-5-6. So, I told my readers that to have an edge on the lottery, they should pick the combo 1-2-3-4-5-6, because the combo has the same chance of winning as any other set of numbers, but if you win, you're less likely to have to split your winnings with anyone.

Well, people must have read that book! I recently heard a very popular morning talk-show program announcer saying that the most frequently chosen number combination for the latest lottery (worth a record $325 million) was — you guessed it — 1-2-3-4-5-6! So, now I urge you to choose six consecutive numbers — just don't choose those. See where it takes you from there! The moral of the story is that just because a result doesn't appear to be as "random" and "equally likely" as other outcomes, it *is* just as likely as the others. In fact, it shows you just how small your chance of winning the lottery is in the first place. (For more on the lottery and games of chance, see Chapter 6.)

"Keep 'em Coming . . . I'm on a Roll!"

If you've ever set foot in a casino, you've seen them. The people who hover at the craps tables, thinking that Lady Luck is on their side, that they can do no wrong, and that they're on a roll. Next time you run into such a poor sap, make a mental note to return to that same place a few minutes later to get an update. You often find that the party has broken up, and everyone has gone home.

The buzzkill occurs because when it comes to games of chance, there's no such thing as being on a roll. Outcomes from one trial to the next are independent — they don't affect each other. So, just because you go three rolls in a row without rolling a 7 in craps doesn't mean you should roll again. That 7 can come up at any time. In fact, a sum of 7 is the most common roll you can get on two dice. The probability of getting a total of 7 when rolling two dice is $\frac{6}{36} = 0.166$, or 16.7 percent — 1 out of 6 rolls over the long term.

Casinos don't want you to think about this fact, however; they want you to think that some magic is involved in gambling, and that skill is a factor when spinning a roulette wheel or rolling dice. The skill actually comes in when you decide what to bet on, how much to bet, and, more importantly, what not to bet on and when to quit. These are skills that the casinos want you to pay a lot less attention to. (For more information on independence and how it applies to gaming, see Chapter 6.)

Giving Every Situation a 50-50 Chance

Probabilities are defined by the long-term percentage of time that a certain event occurs (for more information on this topic, see Chapter 1). Every year I have an argument with at least one of my students about this. For example, I ask the student: If you stand at the free-throw line and shoot a basketball at the hoop, what's the chance you'll make a basket? The student says, "50-50," meaning that the chance of making the basket is 50 percent, and the chance of missing the basket is 50 percent. The reasoning? You either make it or you don't.

This reasoning makes sense, in a way, because you have only two outcomes, and society drums into our heads the idea of the fair coin with a 50-50 outcome and assigns a 50-50 chance to most anything. "But if the chance in this example is 50-50," I ask the student, "why do professional basketball players practice making free throws? Why do certain players get teased about their low free-throw percentages?" If all situations are 50-50, practice shouldn't matter, right? Wrong.

The only time you can use a 50-50 connotation for probability is if an event poses only two possible outcomes and both are equally likely to occur. The only time this happens is in situations that mimic the toss of a fair coin. Situations like free throws and traffic lights don't work this way. I live in Columbus, Ohio, where no one slows down on a yellow, so there are really two colors of traffic light: red and green. But just because we recognize only two colors doesn't mean that they both occur with the same frequency. For example, I know the stoplight by my house is red 90 percent of the time, because my street is much smaller than the one it feeds into. And with free throws, a player's skill level plays a large part in whether he or she makes free throws regularly.

Switching Conditional Probabilities Around

A common problem I see is that when a conditional probability question uses the word "given," people are fine, but the minute a problem uses a different phrase or description for the same conditional probability, people can't see it. In order to truly understand how to set up and solve a conditional probability, you need to sift out two items. First, what is *known?* That is, what do you know about the people when you read the question? This event goes after the symbol "|" in the conditional probability — the A in $P(B|A)$. Second, you need to find out what you want the probability of. This event goes in the front of the conditional probability — the B in $P(B|A)$. No matter how the question is worded, if you can pick out these two pieces of information, you'll be okay.

Suppose, for example, that you want to examine the voting patterns in the last election, including the voters' gender and which candidate they voted for (Republican, Democrat, Other). The information from 600 voters is given in the following table.

	Democrat	***Republican***	***Other***	***Total***
Male	120	170	10	300
Female	150	125	25	300
Total	270	295	35	600

The question asks you to find the chance that a voter selected at random voted Democrat, given that he's male. This means you want P(D | M). To find this probability, you look at the total number of males, 300, and find the Democrats (120) among the males. Now you take 120 ÷ 300 = 0.40. The chance of voting Democrat in the last election given that you're male was 40 percent. Now, what's the chance of being male given that you voted Democrat? This presents a conditional probability in the reverse order. You know that the person voted Democrat, so you want P(M | D). The denominator is 270, the total number of Democrats, and the numerator is the number of males out of that group, 120. So, if you voted Democrat, the chance that you're male is 120 ÷ 270 = 0.44, or 44 percent.

Always interpret what the question actually wants from you. For the voting example, asking for P(M | D) can come in many forms:

- What's the chance that a Democrat voter is male?
- What percentage of the Democrat voters are male?
- What's the chance of being male if you voted Democrat?
- How did the Democrat voters break down in terms of gender? What percent are males?

If you have trouble with conditional probabilities and want more information, see Chapter 4.

Applying the Wrong Probability Distribution

One of the big challenges you face in a probability course is determining which probability model to use in certain situations. You may have a final exam where one question involves counting heads on 100 coin flips, the next measures the time between phone calls, and the next counts the number of chocolate chips in a cookie. I suggest you address the "which formula do I use when" issue by backing up the train a little bit and coming up with a series of questions about the problem to help you sort things out.

Here are some questions you can ask, in order, to gain control of a probability problem instead of letting the problem control you:

1. **Is the data discrete or continuous?**
    - If you're measuring time or distance, like time between phone calls, the data is continuous.
    - If you're counting an amount, like the number of phone calls that come in, the data is discrete.

2. **If the data is discrete, do you see a fixed total number of trials?**
   - If you do, you use either the binomial (Chapter 8) or the hypergeometric (Chapter 16) distribution. If the problem samples with replacement (you return the item to the group, allowing it to possibly be selected again), use the binomial. If the problem samples without replacement (you don't return the item to the group, preventing it from being selected again), use the hypergeometric.
   - If you don't, you have to ask if the number of successes is fixed. If so, you use the geometric (Chapter 14) or negative binomial (Chapter 15) model. If you want the number of trials until the first success, use the geometric; otherwise, use the negative binomial. If the number of successes isn't fixed, you use the Poisson model (Chapter 13), which counts the number of successes in a fixed period of time or space.
3. **If the data is continuous, what shape is the density function (see Chapter 7)?**
   - If it's a constant, you use the uniform probability model (Chapter 17), where a rectangle represents how the probability is distributed evenly among the possible values.
   - If it has a bell shape (a mound in the middle with tails sloping down toward zero on either side), you use the normal distribution (Chapter 9).
   - If it's an exponential curve, sloping down toward zero for larger and larger values of X, use the exponential probability model (Chapter 18), which measures the time between events, lifetimes of products, and so on.

TIP: The Cheat Sheet in the front of this book contains two tables that help you answer these questions; one is for discrete, the other is for continuous. Use these tables to get a good idea of what kind of probability model you should be using in any situation.

Leaving Probability Model Conditions Unchecked

Every probability model has a set of characteristics that determine it. But just as important, every probability model has a set of assumptions, or *conditions,* that a situation must meet in order to use it. Sometimes you'll have ways of checking those conditions to determine whether a certain probability model is appropriate (for example, the binomial probability model in Chapter 8). Other times, the model is just assumed to hold and you are told which probability model it is (like the exponential model in Chapter 18).

For example, the binomial probability model (see Chapter 8) has the following conditions: You have n independent trials that each result in success or failure with a constant probability of success (p) on each trial, and you're counting the number of successes (X) out of the total number of trials. The geometric probability model (see Chapter 14) is similar, except it doesn't have a fixed number of trials. Instead, the conditions are the following: You have a series of independent trials, each resulting in success or failure with a constant probability (p) of success on each trial, and you're counting the total number of trials up to and including the first success.

Verifying conditions is important when you do any probability problem, and many instructors take off points for not doing so. Checking conditions can also help you determine or identify which model to use when. For example, if you have to find a fixed number of trials and you see none, chances are the problem isn't binomial; perhaps it calls for a geometric model.

The Cheat Sheet at the front of this book gives you tables of all the distributions and some important characteristics of each to help you sort out the information.

Confusing Permutations and Combinations

Permutations (the number of ways to choose objects without replacement when order matters) and combinations (the number of ways to choose objects without replacement when order doesn't matter) are perhaps the most challenging and trying topics in probability. You may understand each one separately, but when you're looking at a group of problems that could use one or the other, how do you know when to use a permutation and when to use a combination?

Chapter 5 tells you plenty more about this issue, but the bottom line is this: In both situations, you sample or select k items without replacement (you don't return them to the group, preventing them from being selected again) from a total of n possible items. The difference is in whether the order of the items you select matters. If the order of the k items selected matters, you have a permutation problem. If the order of the k items selected doesn't matter, you have a combination problem.

For example, consider the numbers 1, 2, 3. If a problem asks you to choose two numbers in the correct order to win a prize, with no repeats allowed, it's a permutation problem because you're selecting without replacement, and the order of the selected items matters. Here are all the possible outcomes: 1-2, 1-3, 2-1, 2-3, 3-1, 3-2. Using permutations, you want the total permutations or rearrangements of two objects from three objects, which is $P_2^3 = \frac{3!}{(3-2)!} = \frac{3!}{1!} = \frac{3 * 2 * 1}{1}$, or 6.

Now, if you have to choose two numbers in any order to win, with no repeats allowed, you have a combination problem because you're selecting without replacement, and the order of the selected items doesn't matter. Here are the total possibilities: 1-2 (in any order); 1-3 (in any order); or 2-3 (in any order). Using combination notation, you want the total number of ways to select two items from three items, which is $C_2^3 = \binom{3}{2} = \frac{3!}{(3-2)!2!} = \frac{3 * \cancel{2!}}{1! * \cancel{2!}} = \frac{3}{1}$, or 3. Each combination has two permutations, because you can arrange each pair in the combination 2! = two different ways.

The ! sign doesn't mean excitement (however, counting can be exciting); it stands for *factorial,* which is shorthand notation for saying that you start at the number given, multiply by one less than the number, multiply by one number less than that, and so on, all the way down to one and stop. So, 2! is equal to 2 * 1 = 2. (See Chapter 5 for more on this topic.)

In general, there are always k! times as many permutations than combinations when you choose k items from n items. If the order matters, there are more routes to account for.

Assuming Independence

Independence of two events A and B is a situation probalists always want to have, but in the real world, it really doesn't happen that often unless you set things up that way (like in a binomial experiment; see Chapter 8). Perhaps, as a result, independence is something students often assume in problems when they shouldn't, and that's where the problem comes in.

The basic definition of independence is that if events A and B are independent, P(A|B) = P(A). In other words, knowing that B has occurred does nothing to affect the probability of A. It also means that P(B|A) = P(B). Now, if A and B are independent, you can say — by the multiplication rule for independence (see Chapter 2) — that P(A and B) = P(A) * P(B), which is what you want to say anyway because it looks good and makes sense. The problem comes in when A and B are *not* independent. In that case, to find P(A and B), you have to use the general multiplication rule — P(A and B) = P(A) * P(B|A) — which involves conditional probabilities (something most people don't like to deal with, but have no fear; see Chapter 4).

For example, if you have ten black balls and ten red balls mixed in a bowl, and you sample two balls with replacement (meaning that you put the balls back to give them a chance at reselection), what's the chance of getting two red balls? You calculate $P(\text{Red on one and Red on two}) = P(\text{Red on one}) * P(\text{Red on two} \mid \text{Red on one}) = (10 \div 20) * (10 \div 20) = 100 \div 400 = 0.250$. Because you put the first ball back after you replace it, the probability didn't change. You could've just multiplied $(10 \div 20) * (10 \div 20)$ right away because the selections were independent. Now, if you sample without replacement (meaning that you take the balls out, preventing reselection), you get $P(\text{Red on one and Red on two}) = P(\text{Red on one}) * P(\text{Red on two} \mid \text{Red on one}) = (10 \div 20) * (9 \div 19) = 90 \div 380 = 0.237$. Because you didn't return the first ball, the total balls decreased to 19, and because the first ball is red, you have only 9 red balls left for the second draw. The selections are *not* independent, so using $(10 \div 20) * (10 \div 20)$ would be wrong here.

TIP

Often, instructors have questions that catch you if you use the wrong form of the multiplication rule. You can avoid the mistake by always using the general version of the multiplication rule, and if A and B happen to be independent, $P(B \mid A)$ just equals $P(B)$ anyway, so you'll be fine.

Appendix

Tables for Your Reference

This Appendix includes commonly used tables for finding probabilities for three important distributions: the binomial distribution, the normal distribution, and the Poisson distribution.

Binomial Table

Table A-1 shows the *cumulative distribution function* (cdf) for the binomial distribution (refer to Chapter 7). The cumulative probability is the total probability up to and including any given point. To use Table A-1, you need three pieces of information from the particular problem you're working on:

- The sample size, n
- The probability of success, p
- The value of X for which you want the cumulative probability

After you have this information, find the portion of Table A-1 that's devoted to your n and look at the row for your x and the column for your p. Intersect that row and column, and you'll see the probability that X is less than or equal to your x. To get the probability of being strictly less than, greater than, greater than or equal to, or between two values of X, you manipulate the values of Table A-1, using the steps found in Chapter 8.

Table A-1 The cdf of the Binomial Distribution

Numbers in the table represent $P(X \leq x)$

n=5

x	.01	.05	.10	.20	.25	.30	.40	.50	.60	.70	.75	.80	.90	.95	.99
0	.951	.774	.590	.328	.237	.168	.078	.031	.010	.002	.001	.000	.000	.000	.000
1	.999	.977	.919	.737	.633	.528	.337	.187	.087	.031	.016	.007	.000	.000	.000
2	1.000	.999	.991	.942	.896	.837	.683	.500	.317	.163	.104	.058	.009	.001	.000
3	1.000	1.000	1.000	.993	.984	.969	.913	.812	.663	.472	.367	.263	.081	.023	.001
4	1.000	1.000	1.000	1.000	.999	.998	.990	.969	.922	.832	.763	.672	.410	.226	.049
5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

n=6

x	.01	.05	.10	.20	.25	.30	.40	.50	.60	.70	.75	.80	.90	.95	.99
0	.941	.735	.531	.262	.178	.118	.047	.016	.004	.001	.000	.000	.000	.000	.000
1	.999	.967	.886	.655	.534	.420	.233	.109	.041	.011	.005	.002	.000	.000	.000
2	1.000	.998	.984	.901	.831	.744	.544	.344	.179	.070	.038	.017	.001	.000	.000
3	1.000	1.000	.999	.983	.962	.930	.821	.656	.456	.256	.169	.099	.016	.002	.000
4	1.000	1.000	1.000	.998	.995	.989	.959	.891	.767	.580	.466	.345	.114	.033	.001
5	1.000	1.000	1.000	1.000	1.000	.999	.996	.984	.953	.882	.822	.738	.469	.265	.059
6	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

n=7

x	.01	.05	.10	.20	.25	.30	.40	.50	.60	.70	.75	.80	.90	.95	.99
0	.932	.698	.478	.210	.133	.082	.028	.008	.002	.000	.000	.000	.000	.000	.000
1	.998	.956	.850	.577	.445	.329	.159	.063	.019	.004	.001	.000	.000	.000	.000
2	1.000	.996	.974	.852	.756	.647	.420	.227	.096	.029	.013	.005	.000	.000	.000
3	1.000	1.000	.997	.967	.929	.874	.710	.500	.290	.126	.071	.033	.003	.000	.000
4	1.000	1.000	1.000	.995	.987	.971	.904	.773	.580	.353	.244	.148	.026	.004	.000
5	1.000	1.000	1.000	1.000	.999	.996	.981	.937	.841	.671	.555	.423	.150	.044	.002
6	1.000	1.000	1.000	1.000	1.000	1.000	.998	.992	.972	.918	.867	.790	.522	.302	.068
7	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

n=8

x	.01	.05	.10	.20	.25	.30	.40	.50	.60	.70	.75	.80	.90	.95	.99
0	.923	.663	.430	.168	.100	.058	.017	.004	.001	.000	.000	.000	.000	.000	.000
1	.997	.943	.813	.503	.367	.255	.106	.035	.009	.001	.000	.000	.000	.000	.000
2	1.000	.994	.962	.797	.679	.552	.315	.145	.050	.011	.004	.001	.000	.000	.000
3	1.000	1.000	.995	.944	.886	.806	.594	.363	.174	.058	.027	.010	.000	.000	.000
4	1.000	1.000	1.000	.990	.973	.942	.826	.637	.406	.194	.114	.056	.005	.000	.000
5	1.000	1.000	1.000	.999	.996	.989	.950	.855	.685	.448	.321	.203	.038	.006	.000
6	1.000	1.000	1.000	1.000	1.000	.999	.991	.965	.894	.745	.633	.497	.187	.057	.003
7	1.000	1.000	1.000	1.000	1.000	1.000	.999	.996	.983	.942	.900	.832	.570	.337	.077
8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table A-1 *(continued)*

Numbers in the table represent P(X ≤ x)

n=9

	p														
x	.01	.05	.10	.20	.25	.30	.40	.50	.60	.70	.75	.80	.90	.95	.99
0	.914	.630	.387	.134	.075	.040	.010	.002	.000	.000	.000	.000	.000	.000	.000
1	.997	.929	.775	.436	.300	.196	.071	.020	.004	.000	.000	.000	.000	.000	.000
2	1.000	.992	.947	.738	.601	.463	.232	.090	.025	.004	.001	.000	.000	.000	.000
3	1.000	.999	.992	.914	.834	.730	.483	.254	.099	.025	.010	.003	.000	.000	.000
4	1.000	1.000	.999	.980	.951	.901	.733	.500	.267	.099	.049	.020	.001	.000	.000
5	1.000	1.000	1.000	.997	.990	.975	.901	.746	.517	.270	.166	.086	.008	.001	.000
6	1.000	1.000	1.000	1.000	.999	.996	.975	.910	.768	.537	.399	.262	.053	.008	.000
7	1.000	1.000	1.000	1.000	1.000	1.000	.996	.980	.929	.804	.700	.564	.225	.071	.003
8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.998	.990	.960	.925	.866	.613	.370	.086
9	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

n=10

	p														
x	.01	.05	.10	.20	.25	.30	.40	.50	.60	.70	.75	.80	.90	.95	.99
0	.904	.599	.349	.107	.056	.028	.006	.001	.000	.000	.000	.000	.000	.000	.000
1	.996	.914	.736	.376	.244	.149	.046	.011	.002	.000	.000	.000	.000	.000	.000
2	1.000	.988	.930	.678	.526	.383	.167	.055	.012	.002	.000	.000	.000	.000	.000
3	1.000	.999	.987	.879	.776	.650	.382	.172	.055	.011	.004	.001	.000	.000	.000
4	1.000	1.000	.998	.967	.922	.850	.633	.377	.166	.047	.020	.006	.000	.000	.000
5	1.000	1.000	1.000	.994	.980	.953	.834	.623	.367	.150	.078	.033	.002	.000	.000
6	1.000	1.000	1.000	.999	.996	.989	.945	.828	.618	.350	.224	.121	.013	.001	.000
7	1.000	1.000	1.000	1.000	1.000	.998	.988	.945	.833	.617	.474	.322	.070	.012	.000
8	1.000	1.000	1.000	1.000	1.000	1.000	.998	.989	.954	.851	.756	.624	.264	.086	.004
9	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.999	.994	.972	.944	.893	.651	.401	.096
10	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

n=15

	p														
x	.01	.05	.10	.20	.25	.30	.40	.50	.60	.70	.75	.80	.90	.95	.99
0	.860	.463	.206	.035	.013	.005	.000	.000	.000	.000	.000	.000	.000	.000	.000
1	.990	.829	.549	.167	.080	.035	.005	.000	.000	.000	.000	.000	.000	.000	.000
2	1.000	.964	.816	.398	.236	.127	.027	.004	.000	.000	.000	.000	.000	.000	.000
3	1.000	.995	.944	.648	.461	.297	.091	.018	.002	.000	.000	.000	.000	.000	.000
4	1.000	.999	.987	.836	.686	.515	.217	.059	.009	.001	.000	.000	.000	.000	.000
5	1.000	1.000	.998	.939	.852	.722	.403	.151	.034	.004	.001	.000	.000	.000	.000
6	1.000	1.000	1.000	.982	.943	.869	.610	.304	.095	.015	.004	.001	.000	.000	.000
7	1.000	1.000	1.000	.996	.983	.950	.787	.500	.213	.050	.017	.004	.000	.000	.000
8	1.000	1.000	1.000	.999	.996	.985	.905	.696	.390	.131	.057	.018	.000	.000	.000
9	1.000	1.000	1.000	1.000	.999	.996	.966	.849	.597	.278	.148	.061	.002	.000	.000
10	1.000	1.000	1.000	1.000	1.000	.999	.991	.941	.783	.485	.314	.164	.013	.001	.000
11	1.000	1.000	1.000	1.000	1.000	1.000	.998	.982	.909	.703	.539	.352	.056	.005	.000
12	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.996	.973	.873	.764	.602	.184	.036	.000
13	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.995	.965	.920	.833	.451	.171	.010
14	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.995	.987	.965	.794	.537	.140
15	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table A-1 *(continued)*

Numbers in the table represent $P(X \leq x)$

n=20

	p														
x	.01	.05	.10	.20	.25	.30	.40	.50	.60	.70	.75	.80	.90	.95	.99
0	.818	.358	.122	.012	.003	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000
1	.983	.736	.392	.069	.024	.008	.001	.000	.000	.000	.000	.000	.000	.000	.000
2	.999	.925	.677	.206	.091	.035	.004	.000	.000	.000	.000	.000	.000	.000	.000
3	1.000	.984	.867	.411	.225	.107	.016	.001	.000	.000	.000	.000	.000	.000	.000
4	1.000	.997	.957	.630	.415	.238	.051	.006	.000	.000	.000	.000	.000	.000	.000
5	1.000	1.000	.989	.804	.617	.416	.126	.021	.002	.000	.000	.000	.000	.000	.000
6	1.000	1.000	.998	.913	.786	.608	.250	.058	.006	.000	.000	.000	.000	.000	.000
7	1.000	1.000	1.000	.968	.898	.772	.416	.132	.021	.001	.000	.000	.000	.000	.000
8	1.000	1.000	1.000	.990	.959	.887	.596	.252	.057	.005	.001	.000	.000	.000	.000
9	1.000	1.000	1.000	.997	.986	.952	.755	.412	.128	.017	.004	.001	.000	.000	.000
10	1.000	1.000	1.000	.999	.996	.983	.872	.588	.245	.048	.014	.003	.000	.000	.000
11	1.000	1.000	1.000	1.000	.999	.995	.943	.748	.404	.113	.041	.010	.000	.000	.000
12	1.000	1.000	1.000	1.000	1.000	.999	.979	.868	.584	.228	.102	.032	.000	.000	.000
13	1.000	1.000	1.000	1.000	1.000	1.000	.994	.942	.750	.392	.214	.087	.002	.000	.000
14	1.000	1.000	1.000	1.000	1.000	1.000	.998	.979	.874	.584	.383	.196	.011	.000	.000
15	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.994	.949	.762	.585	.370	.043	.003	.000
16	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.999	.984	.893	.775	.589	.133	.016	.000
17	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.996	.965	.909	.794	.323	.075	.001
18	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.999	.992	.976	.931	.608	.264	.017
19	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.999	.997	.988	.878	.642	.182
20	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

n=25

	p														
x	.01	.05	.10	.20	.25	.30	.40	.50	.60	.70	.75	.80	.90	.95	.99
0	.778	.277	.072	.004	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
1	.974	.642	.271	.027	.007	.002	.000	.000	.000	.000	.000	.000	.000	.000	.000
2	.998	.873	.537	.098	.032	.009	.000	.000	.000	.000	.000	.000	.000	.000	.000
3	1.000	.966	.764	.234	.096	.033	.002	.000	.000	.000	.000	.000	.000	.000	.000
4	1.000	.993	.902	.421	.214	.090	.009	.000	.000	.000	.000	.000	.000	.000	.000
5	1.000	.999	.967	.617	.378	.193	.029	.002	.000	.000	.000	.000	.000	.000	.000
6	1.000	1.000	.991	.780	.561	.341	.074	.007	.000	.000	.000	.000	.000	.000	.000
7	1.000	1.000	.998	.891	.727	.512	.154	.022	.001	.000	.000	.000	.000	.000	.000
8	1.000	1.000	1.000	.953	.851	.677	.274	.054	.004	.000	.000	.000	.000	.000	.000
9	1.000	1.000	1.000	.983	.929	.811	.425	.115	.013	.000	.000	.000	.000	.000	.000
10	1.000	1.000	1.000	.994	.970	.902	.586	.212	.034	.002	.000	.000	.000	.000	.000
11	1.000	1.000	1.000	.998	.989	.956	.732	.345	.078	.006	.001	.000	.000	.000	.000
12	1.000	1.000	1.000	1.000	.997	.983	.846	.500	.154	.017	.003	.000	.000	.000	.000
13	1.000	1.000	1.000	1.000	.999	.994	.922	.655	.268	.044	.011	.002	.000	.000	.000
14	1.000	1.000	1.000	1.000	1.000	.998	.966	.788	.414	.098	.030	.006	.000	.000	.000
15	1.000	1.000	1.000	1.000	1.000	1.000	.987	.885	.575	.189	.071	.017	.000	.000	.000
16	1.000	1.000	1.000	1.000	1.000	1.000	.996	.946	.726	.323	.149	.047	.000	.000	.000
17	1.000	1.000	1.000	1.000	1.000	1.000	.999	.978	.846	.488	.273	.109	.002	.000	.000
18	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.993	.926	.659	.439	.220	.009	.000	.000
19	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.998	.971	.807	.622	.383	.033	.001	.000
20	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.991	.910	.786	.579	.098	.007	.000
21	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.998	.967	.904	.766	.236	.034	.000
22	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.991	.968	.902	.463	.127	.002
23	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.998	.993	.973	.729	.358	.026
24	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.999	.996	.928	.723	.222
25	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Normal Table

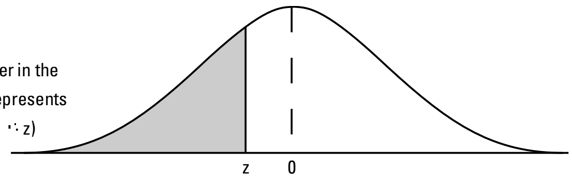
Table A-2 shows the cdf for the normal distribution (refer to Chapter 9). To use Table A-2, you need three pieces of information from the problem you're working on:

- The mean of X (the given normal distribution), which is μ
- The standard deviation of X, which is σ
- The value of X that you want the cumulative probability for

After you have this information, transform your value of X to a z value by taking your value of X, subtracting the mean, and dividing by the standard deviation, using the formula $Z = \frac{X - \mu}{\sigma}$ (refer to Chapter 9). Then look up this value of Z on Table A-2 by finding the row corresponding to the leading digit before the decimal point and the first digit after the decimal point of Z, and the column corresponding to the second digit after the decimal point of Z. The probability you find represents the probability that Z is less than or equal to that value of Z. For example, for Z = 1.23, go to the "1.2" row and the "0.03" column, and you'll find the probability that Z is less than or equal to 1.23 (which is 0.8907). To get the probability of being greater than Z or between two values of Z, you manipulate the values of Table A-2, using the steps found in Chapter 9.

Table A-2 The cdf of the Z Distribution (the Z Table)

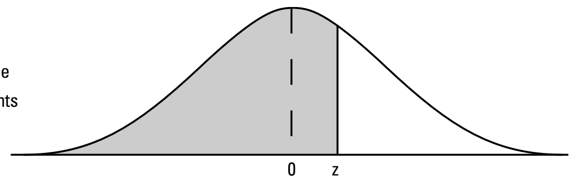
Number in the table represents P(z ·: z)



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
–3.6	.0002	.0002	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
–3.5	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002
–3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002	.0002
–3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0003	.0003
–3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	0005	.0005
–3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
–3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
–2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
–2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
–2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
–2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
–2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
–2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
–2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
–2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
–2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
–2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
–1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
–1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
–1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
–1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
–1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
–1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
–1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
–1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
–1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
–1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
–0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
–0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
–0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
–0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
–0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
–0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
–0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	3483
–0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
–0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
–0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

Table A-2 *(continued)*

Number in the table represents P(z ∵ z)



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998
3.5	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998
3.6	.9998	.9998	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999

Poisson Table

Table A-3 shows the cdf for the Poisson distribution (refer to Chapter 13). To use Table A-3, you need two pieces of information from the problem you're working on:

- The mean of X (the given Poisson distribution), which is equal to λ
- The value of X that you want the cumulative probability for

To use Table A-3, you find the column devoted to your value of λ and the row that represents your value of X. Intersect that row and column to find the probability that X is less than or equal to your value of X. To get the probability of being strictly less than X, greater than X, greater than or equal to X, or between two values of X, you manipulate the values of Table A-3, using the steps found in Chapter 13.

Table A-3 The Poisson cdf

Numbers in the table represent $P(X \le x)$

		λ									
		.1	.2	3	.4	.5	.6	.7	.8	.9	1.0
	0	.905	.819	.741	.670	.607	.549	.497	.449	.407	.368
	1	.995	.982	.963	.938	.910	.878	.844	.809	.772	.736
	2	1.000	.999	.996	.992	.986	.977	.966	.953	.937	.920
x	3		1.000	1.000	.999	.998	.997	.994	.991	.987	.981
	4				1.000	1.000	1.000	.999	.999	.998	.996
	5							1.000	1.000	1.000	.999
	6										1.000

		λ										
		2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0	15.0	20.0
	0	.135	.050	.018	.007	.002	.001	.000	.000	.000	.000	.000
	1	.406	.199	.092	.040	.017	.007	.003	.001	.000	.000	.000
	2	.677	.423	.238	.125	.062	.030	.014	.006	.003	.000	.000
	3	.857	.647	.433	.265	.151	.082	.042	.021	.010	.000	.000
	4	.947	.815	.629	.440	.285	.173	.100	.055	.029	.001	.000
	5	.983	.916	.785	.616	.446	.301	.191	.116	.067	.003	.000
	6	.995	.966	.889	.762	.606	.450	.313	.207	.130	.008	.000
	7	.999	.988	.949	.867	.744	.599	.453	.324	.220	.018	.001
	8	1.000	.996	.979	.932	.847	.729	.593	.456	.333	.037	.002
	9		.999	.992	.968	.916	.830	.717	.587	.458	.070	.005
	10		1.000	.997	.986	.957	.901	.816	.706	.583	.118	.011
	11			.999	.995	.980	.947	.888	.803	.697	.185	.021
	12			1.000	.998	.991	.973	.936	.876	.792	.268	.039
	13				.999	.996	.987	.966	.926	.864	.363	.066
	14				1.000	.999	.994	.983	.959	.917	.466	.105
	15					.999	.998	.992	.978	.951	.568	.157
	16					1.000	.999	.996	.989	.973	.664	.221
	17						1.000	.998	.995	.986	.749	.297
	18							.999	.998	.993	.819	.381
x	19							1.000	.999	.997	.875	.470
	20								1.000	.998	.917	.559
	21									.999	.947	.644
	22									1.000	.967	.721
	23										.981	.787
	24										.989	.843
	25										.994	.888
	26										.997	.922
	27										.998	.948
	28										.999	.966
	29										1.000	.978
	30											.987
	31											.992
	32											.995
	33											.997
	34											.999
	35											.999
	36											1.000