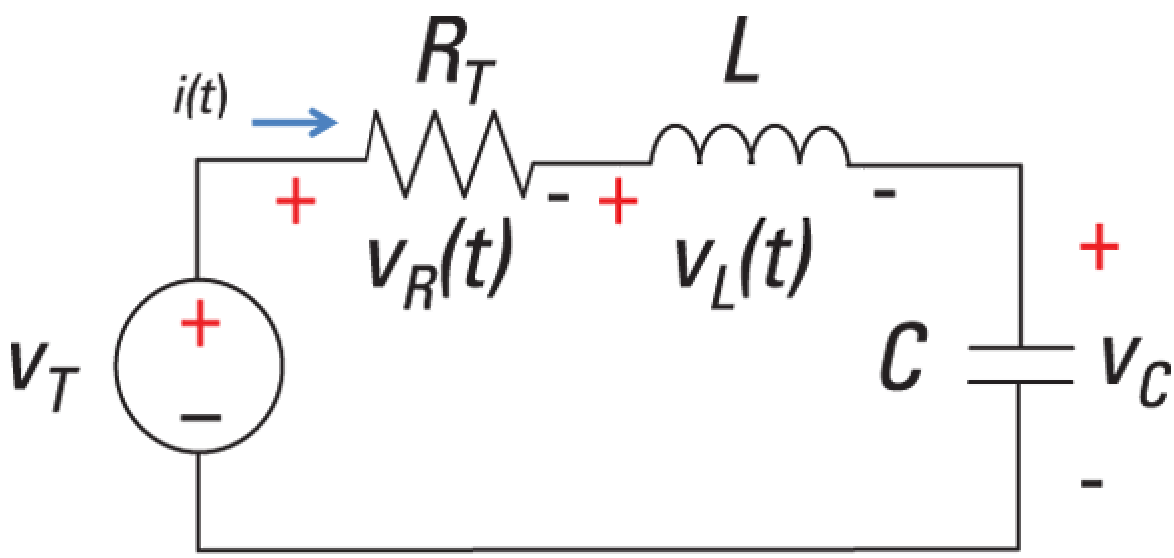
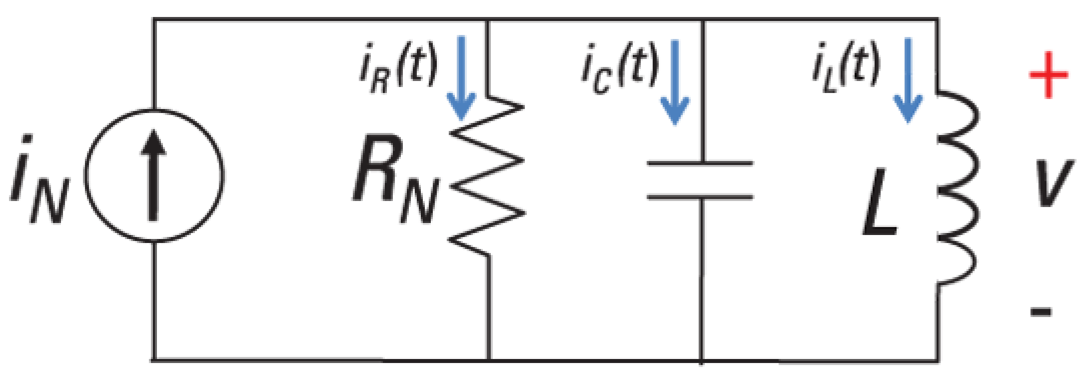
Part IV

Applying Time-Varying Signals to First- and Second-Order Circuits



RLC series circuit



RLC parallel circuit

Explore a timing circuit that detects rectangular pulses at www.dummies.com/extras/circuitanalysis.

In this part . . .

- Look at functions that describe AC signals, such as the step function and the exponential function.
- Get acquainted with capacitors and inductors and the roles they play in circuits.
- Find out how to analyze first-order circuits (circuits with a single storage element connected to a single resistor or a resistor network).
- Practice analyzing second-order circuits, which consist of capacitors, inductors, and resistors.

Chapter 11

Making Waves with Funky Functions

In This Chapter

- Observing spikes with the impulse function
- Creating step functions
- Rising or falling with the exponential function
- Cycling with sinusoidal functions

DC signals don't change with time . . . kinda boring, right? More interesting signals change in time like music. Such signals may spike, jump around, or rise and fall. They may build or decline steadily, or they may shoot up or plummet, picking up speed. They may repeat in cycles, continuing on and on.

Electric signals that change in time are useful because they can carry information about the real world, like temperature, pressure, and sound. This chapter covers basic time-varying signals commonly found in circuit analysis, including info on their key properties.

A word of warning: This chapter doesn't meet the high benchmarks of the Grand Poobah of precision math, but it's good enough to play with some funky functions.

Spiking It Up with the Lean, Mean Impulse

Function

The first funky function is one you may have never heard of, but it occurs frequently in real life. It’s called an *impulse function,* also known as a *Dirac delta function*. Just think of the impulse as a single spike that occurs in one instant of time. You can view this spiked function as one that’s infinitely large in magnitude and infinitely thin in time, having a total area of 1.

You can visualize the impulse as a limiting form of a rectangular pulse of unit area. Specifically, as you decrease the duration of the pulse, its amplitude increases so that the area remains constant at unity. The more you decrease the duration, the closer the rectangular pulse comes to the impulse function. The bottom diagram of Figure 11-1 shows the limiting form of the rectangular pulse approaching an impulse. (Check out the nearby sidebar “Identifying impulse functions in the day-to-day” if you’re having trouble wrapping your head around impulse functions.)

So what’s the practical use of the impulse function? By using the impulse as an input signal to a system, you can reveal the output behavior or character of a system. After you know the behavior of the system for an impulse, you can describe the system’s output behavior for any input. Why is that? Because any input is modeled as a series of impulses shifted in time with varying heights, amplitudes, or strengths.

Here’s the fancy pants description of the impulse function:

$$\delta(t)=1 \qquad t=0$$
$$\delta(t)=0 \qquad t\neq 0$$

Identifying impulse functions in the day-to-day

Some physical phenomena come very close to being modeled with impulse functions. One example is lightning. Lightning has lots of energy and occurs in a short amount of time. That fits the description of an impulse function. An ideal impulse has an infinitely high amplitude (high energy) and is infinitely thin in time. As you drive through a lightning storm, you may hear a popping noise if you’re tuned in to a radio weather station. This noise occurs when the energy of the lightning interferes with the signal coming from the radio weather station.

Another example of a real-world impulse function is a bomb. A powerful bomb has lots of energy occurring in a short amount of time. Similarly, fireworks, including cherry bombs, produce loud noises — audio energy — that occur as a series of popping noises having short durations.

This mathematical description says that the impulse function occurs at only one point in time; the function is zero elsewhere. The impulse here occurs at the origin of time — that is, when you decide to let $t = 0$ (not at the beginning of the universe or anything like that).

The top-left diagram of Figure 11-1 shows an ideal unit impulse function having a large amplitude with a short duration. You can describe the area of the impulse function as the strength of the impulse:

$$\int_{-\infty}^{t}\delta(t)dx=1u(t)$$

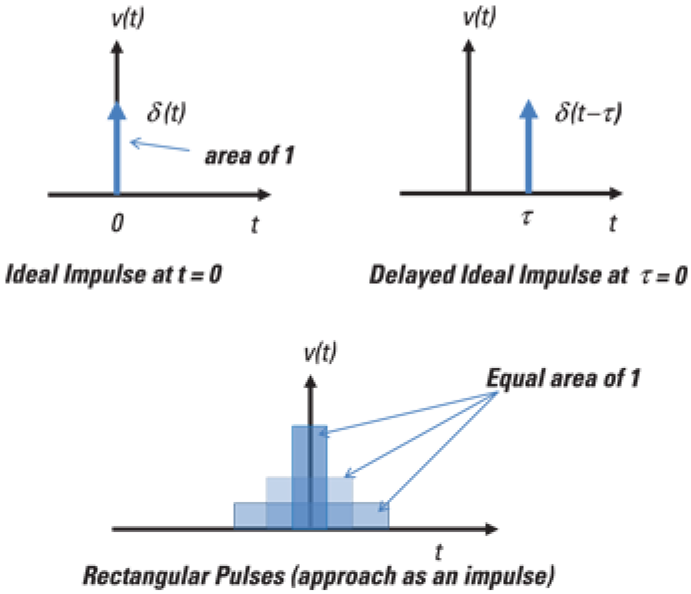


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Figure 11-1: The impulse function, delayed impulse function, and rectangular pulse.

At time $t = 0$, the area is a constant having a value of 1; and before $t = 0$, the area is equal to 0. The integration of the impulse results in another funky function, $u(t)$, called a *step function,* which I cover in the later section "Stepping It Up with a Step Function." You can view the impulse as a derivative of the step function $u(t)$ with respect to time:

$$\delta(t) = \frac{d}{dt}[u(t)]$$

What these two equations tell you is that if you know one function, you can determine the other function.

In the following sections, I tell you how to change the strength of the impulse, delay the impulse, and evaluate an integral with an impulse function.

Changing the strength of the impulse

Figure 11-1 shows an impulse with an area (or strength) equal to 1. To have a different area or strength *K,* you can modify the impulse:

$$v(t) = K\delta(t)$$

$$\int_{-\infty}^{t} v(x)\, dx = Ku(t)$$

The area under the curve is given by strength *K*. The result of integrating the impulse leads you to another step function with amplitude or strength *K.*

Delaying an impulse

Impulses can be delayed. Analytically, you can describe a delayed impulse that occurs later, say, at time *τ:*

$$\delta(t-\tau) = 1 \quad t = \tau$$

$$\delta(t-\tau) = 0 \quad t \neq \tau$$

This equation says the impulse occurs only at a time later *τ* and nowhere else, or it's equal to 0 at time not equal to *τ*. You see a delayed impulse in the top-right diagram of Figure 11-1.

For a numerical example, let an impulse having a strength of 10 occur at delayed time *τ* = 5. You can describe the delayed impulse as

$$10\delta(t-5) = 10 \qquad t = \tau = 5$$

$$10\delta(t-5) = 0 \qquad t \neq 5$$

The equation says that the impulse, which has strength *K* = 10, occurs only at a time *τ* = 5 later and that the impulse occurs nowhere else. In other words, the impulse is equal to 0 when time is not equal to 5.

Evaluating impulse functions with integrals

Assuming *x*(*t*) is a continuous function that's multiplied by a time-shifted (or delayed) impulse, the integral of the product is expressed and evaluated as follows:

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0)u(t-t_0)$$

You do this evaluation only where the impulse occurs — at only one point and nowhere else. The preceding equation sifts out or selects the value of $x(t)$ at time equal to t_0. This integration is one of the easiest integrations you'll encounter.

Here's a simple numerical example with $x(t) = 5t^2 + 3t + 6$ and $t_0 = 5$:

$$\begin{aligned} x(5) &= \int_{-\infty}^{\infty} \underbrace{\left[5t^2 + 3t + 6\right]}_{x(t)} \delta(t - \underbrace{5}_{t_0})dt \\ &= \left[5(5)^2 + 3(5) + 6\right]u(t-5) \\ &= \left[125 + 15 + 6\right]u(t-5) \\ &= 146u(t-5) \end{aligned}$$

Pretty funky way to integrate analytically, huh? The integration leads to a delayed (or time-shifted) step function (or constant) starting at a delayed time of t_0 = 5. I introduce step functions in the next section.

You can model any smooth function $x(t)$ as a series of delayed and time-shifted impulses in the following way:

$$x(t) = \int_{-\infty}^{t} x(\tau)\delta(t-\tau)d\tau \qquad \text{where } x(t) = x(t)u(t)$$

This equation says you can break up any function $x(t)$ into a sum of a whole bunch of delayed impulse functions with different strengths. The value of the strength is simply the function $x(t)$ evaluated where the shifted impulse occurs at time τ or t.

Stepping It Up with a Step Function

The step function is a funky function that looks like, well, a step. Practical step functions occur daily, like each time you turn mobile devices, stereos, and lights on and off. Here's the general definition of the unit step function:

$$u(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ 1 & \text{for} \quad t \geq 0 \end{cases}$$

So this step function is equal to 0 when time t is negative and is equal to 1 when time t is 0 or positive. Alternatively, you can say there's a jump in the function value at time $t = 0$. Math gurus call this jump a *discontinuity*.

Although you can't generate an ideal step function, you can approximate a step function. Figure 11-2 shows what a step function looks like, along with a circuit that's roughly a step function.

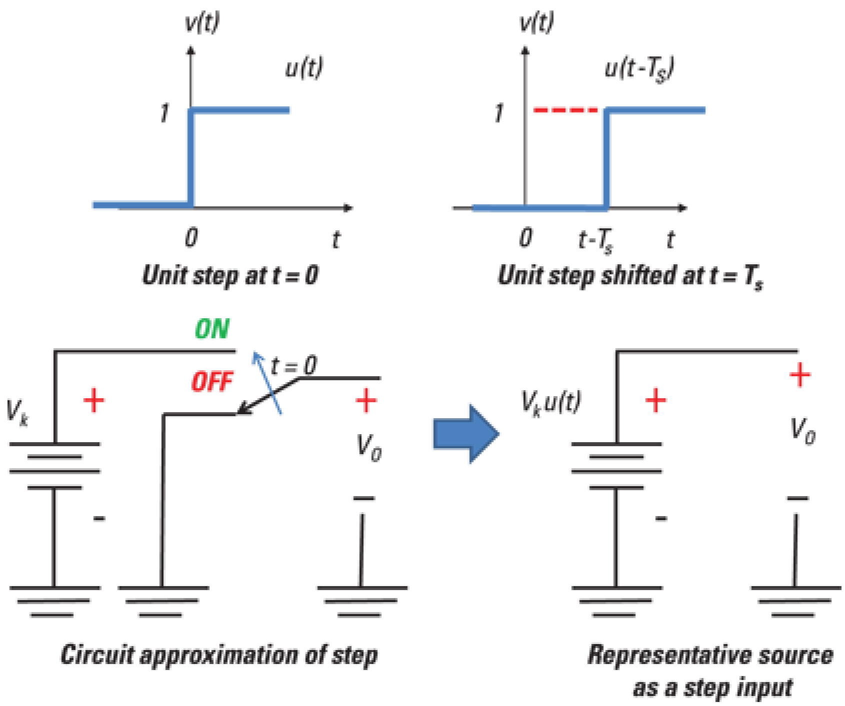

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Figure 11-2: The step function and its circuit approximation.

The following sections cover some operations for shifting and weighting step functions.

Creating a time-shifted, weighted step function

The circuit approximation of the step function in Figure 11-2 assumes you can quickly change from *off* to *on* at time $t = 0$ when the switch is thrown.

Although the unit step function appears not to do much, it's a versatile signal that can build other waveforms. In a graph, you can make the step shrink or stretch. You can multiply the step function $u(t)$ by a constant amplitude V_k to produce the following waveform:

$$V_k u(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ V_k & \text{for} \quad t \geq 0 \end{cases}$$

The scale or weight of the unit input is V_k. The amplitude V_k measures the size of the jump in function value.

You can move the step function in time with a shift of T_s, leading you to a shifted, weighted waveform:

$$V_k u(t - T_s) = \begin{cases} 0 & \text{for} \quad T_s < 0 \\ V_k & \text{for} \quad T_s \geq 0 \end{cases}$$

This equation says the function equals 0 before time T_s and that the value of the function jumps to V_k after time T_s. Figure 11-3 shows the step function weighted by V_k with a time shift of T_s.

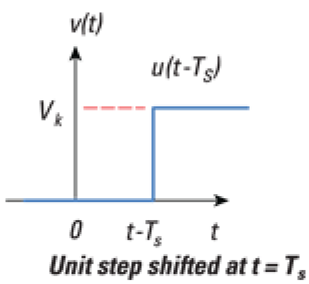

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Figure 11-3: A time-shifted step function.

You can add two step functions together to form a pulse function, as I show you in the next section.

Being out of step with shifted step functions

Step functions can dance around, but it's not the fancy twist-and-shout kind of dancing. The function can become bigger or smaller and move to the left or right. You can add those modified step functions to make even more funky step functions.

For example, you can generate a rectangular pulse as a sum of two step functions. To get a visual of this concept, see Figure 11-4, which shows a rectangular pulse that consists of the sum of two step functions in time. Before 1 second, the value of the pulse is 0. Then the amplitude of the pulse jumps to a value of 3 and stays at that value between 1 and 2 seconds. The pulse then returns to 0 at time t = 2 seconds. You wind up with the rectangular pulse $p(t)$ described as the sum of two step functions:

$$p(t) = 3u(t-1) - 3u(t-2)$$

This expression says that you create a pulse with a time-shifted step function starting at 1 second with an amplitude of 3 and add it to another time-shifted step function starting at 2 seconds with an amplitude of –3. You can view the pulse as a *gating function* for electronic switches to allow or stop a signal from passing through.

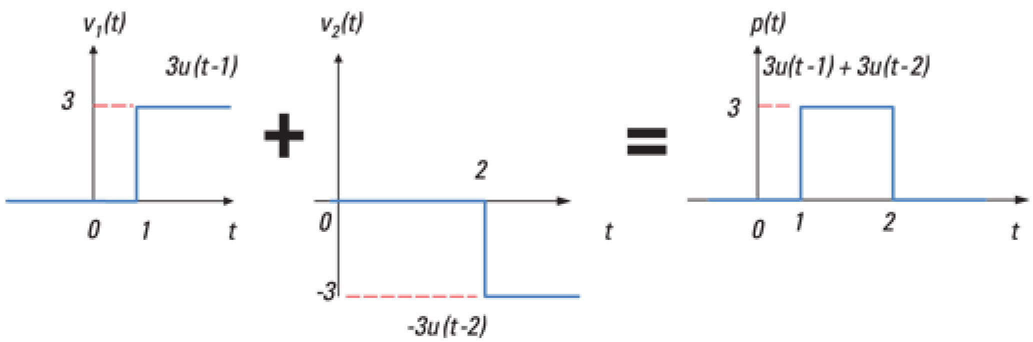

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Figure 11-4: Building a rectangular pulse with step functions.

Building a ramp function with a step function

The integral of the step function generates a ramp function, which consists of two functions multiplied together:

$$r(t) = \int_{-\infty}^{t} u(x)\,dx$$
$$= tu(t)$$

The time function $tu(t)$ is simply a ramp function with a slope (or strength) of 1, and the unit step function serves as a convenient mathematical tool to start the ramp at time $t = 0$. You can add a strength K to the ramp and shift the ramp function in time by T_S as follows:

$$v(t) = Kr(t - T_S)$$

The ramp doesn't start until T_S. Before the time shift T_S, the ramp function is 0. After time T_S, the ramp has a value equal to $Kr(t - T_S)$.

With ramp functions, you can create triangular and sawtooth functions (or waveforms). Figure 11-5 shows a ramp of unit strength, a ramp of strength K with a time shift of 1, a triangular waveform, and a sawtooth waveform. Building such waveforms from other functions is useful when you're breaking the input into recognizable pieces and applying superposition.

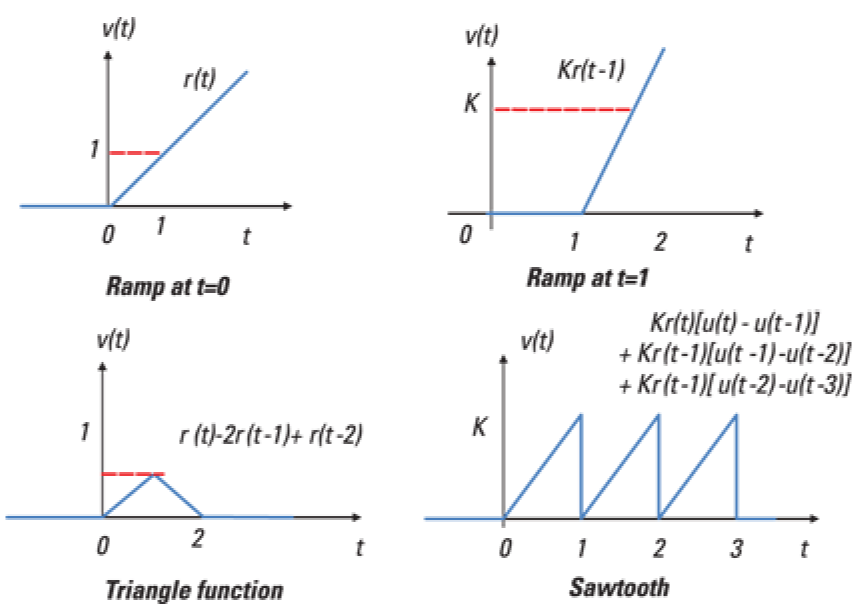

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Figure 11-5: The ramp function, its weighted and shifted version, and triangle and sawtooth variations.

Here's how to build the triangle function in Figure 11-5 using ramp functions:

1. **Turn on a ramp with a slope of 1 starting at time $t = 0$.**
2. **Add a ramp that has a slope of –2 and starts at $t = 1$.**

   At $t = 1$, you see the function start to decrease with a slope of –1. But before that, the slope of the function (from the first ramp) is 1; adding a ramp with a slope of –2 to the first ramp results in a ramp with a slope of –1.
3. **Turn off the second ramp by adding another delayed ramp that has a slope of 1 and starts at time $t = 2$.**

   Adding a ramp with a slope of 1 brings the slope back to 0.

Here's the math behind what I just said:

$$v(t) = r(t) - 2r(t-1) + r(t-2)$$

Here's how to build a sawtooth function like the one in Figure 11-5 using ramp and step functions:

1. **Start with a ramp of slope (or strength) K multiplied by a rectangular pulse of unit height.**

   The pulse consists of two step functions. Mathematically, you have a ramp with a specific time duration:

   $r_1(t) = Kr(t)[u(t) - u(t - 1)]$
2. **Apply a time delay of 1 to the ramp pulse $r_1(t)$ to get another ramp pulse $r_2(t)$ that's time shifted.**

   You get the following:

$r_2(t) = Kr_1(t - 1) = Kr(t - 1)[u(t - 1) - u(t - 2)]$

3. **Repeat Step 2 to get more delayed ramp pulses starting at 2, 3, 4, and so on.**
4. **Add up all the functions to get the sawtooth $s_t(t)$.**

   Here's the sawtooth function:

   $s_t(t) = K\{r(t)[u(t) - u(t - 1)] + r(t - 1)[u(t - 1) - u(t - 2)]+\ldots+\}$

Pushing the Limits with the Exponential Function

The *exponential function* is a step function whose amplitude V_k gradually decreases to 0. Exponential functions are important because they're solutions to many circuit analysis problems in which a circuit contains resistors, capacitors, and inductors.

The exponential waveform is described by the following equation:

$$v(t)=V_k e^{-\left(\frac{t}{T_C}\right)}u(t)$$

The time constant T_C provides a measure of how fast the function will decay or grow. Using the step function means that the function starts at $t = 0$.

REMEMBER A minus sign on the exponent indicates a decaying exponential, whereas a positive sign indicates a growing exponential. When you have a growing exponential, the circuit can't handle the input, and nothing works after exceeding the supplied voltage. In academia terms, the system goes *unstable.*

Here’s the time-shifted version of a decaying exponential starting at time t_0:

$$v(t) = V_k e^{-\left(\frac{t-t_0}{T_C}\right)} u(t - t_0)$$

Figure 11-6 shows a decaying exponential, its time-shifted version, and a growing exponential.

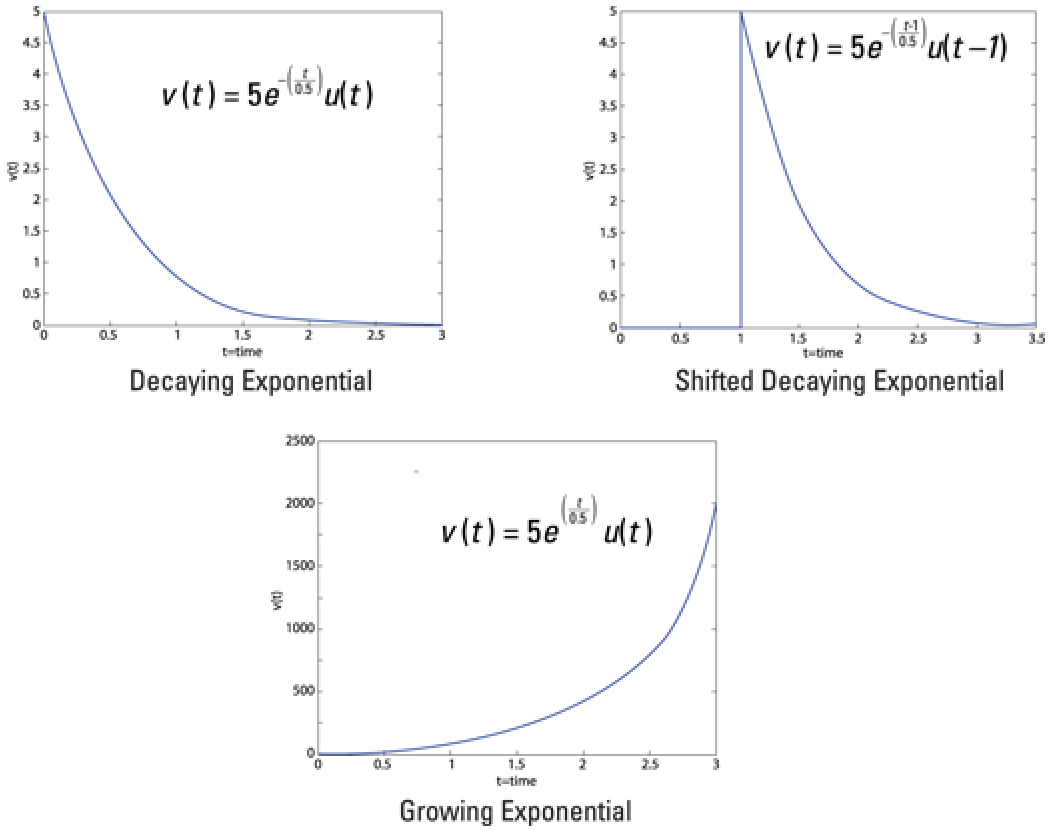

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Figure 11-6: The exponential function, its shifted version, and a growing exponential.

Seeing the Signs with Sinusoidal Functions

The sinusoidal functions (sine and cosine) appear everywhere, and they play an important role not only in electrical engineering but in many branches of science and engineering. In circuit analysis, the sinusoid serves as a good approximation to describe a circuit's input and output behavior.

The sinusoidal function is periodic, meaning its graph contains a basic shape that repeats over and over indefinitely. The function goes on forever, oscillating through endless peaks and valleys in both negative and positive directions of time. Here are some key parts of the function:

- The amplitude V_A defines the maximum and minimum peaks of the oscillations.
- Frequency f_0 describes the number of oscillations in 1 second.
- The period T_0 defines the time required to complete 1 cycle.

The period and frequency are reciprocals of each other, governed by the following mathematical relationship:

$$f_0 = \frac{1}{T_0}$$

In this book, I define the following cosine function as the reference signal:

$$\begin{aligned} v(t) &= V_A \cos(2\pi f_0 t) \\ &= V_A \cos\left(\frac{2\pi t}{T_0}\right) \end{aligned}$$

You can move sinusoidal functions left or right with a time shift as well as increase or decrease the amplitude. You can also describe a sinusoidal function with a phase shift in terms of a linear combination of sine and cosine functions. Figure 11-7 shows a cosine function and a shifted cosine function with a phase shift of π/2.

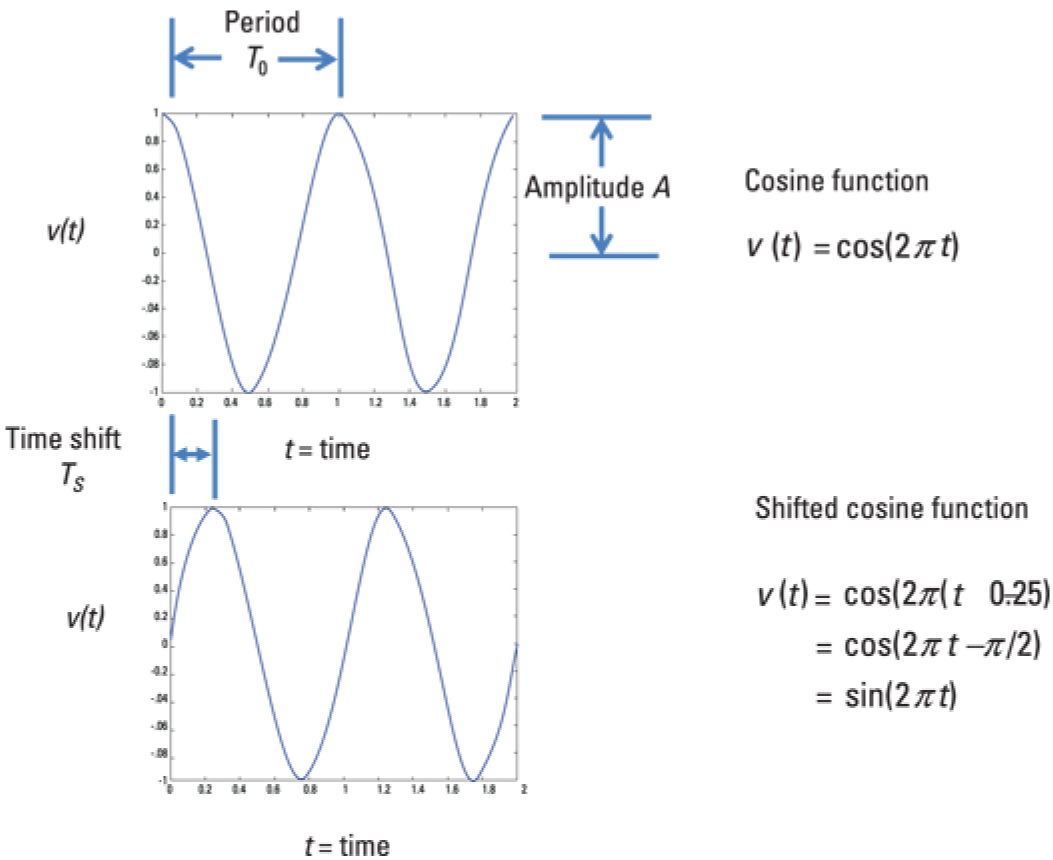

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Figure 11-7: A standard cosine function and a cosine function with a shift of π/2.

Giving wavy functions a phase shift

A signal that’s *out of phase* has been shifted left or right when compared to a reference signal:

- **Right shift:** When a function moves right, then the function is said to be *delayed.* The delayed cosine has its peak occur after the origin. A delayed signal is also said to be a *lag signal* because the signal arrives later than expected.
- **Left shift:** When the cosine function is shifted left, the shifted function is said to be *advanced.* The peak of the advanced signal occurs just before the origin.

An advanced signal is also called a *lead signal* because the lead signal arrives earlier than expected.

Figure 11-8 shows unshifted, lagged, and lead cosine functions.

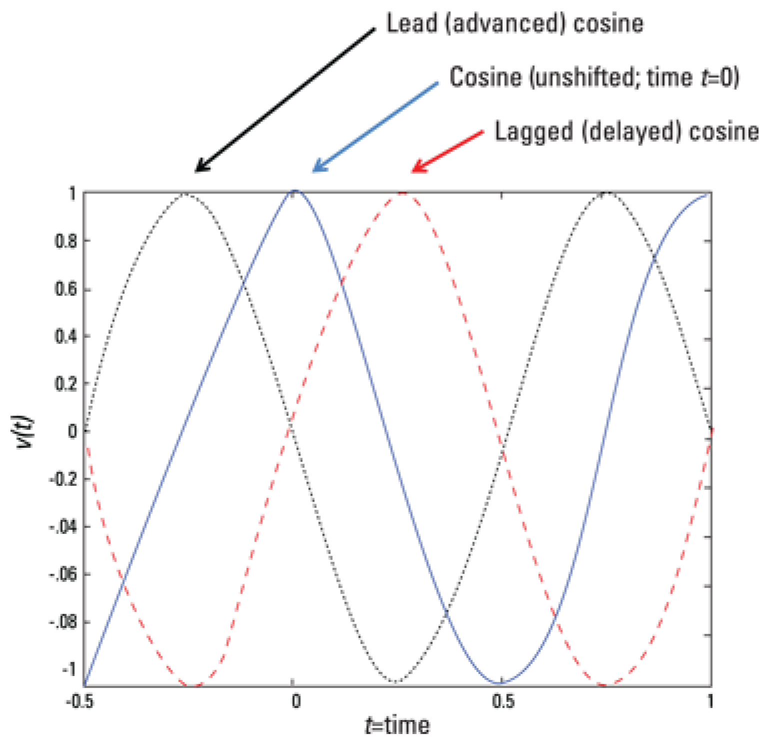

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Figure 11-8: Unshifted, lag, and lead cosine functions.

To see what a phase shift looks like mathematically, first take a look at the reference signal:

$$\begin{aligned} v(t) &= V_A \cos(2\pi f_0 t) \\ &= V_A \cos\left(\frac{2\pi t}{T_0}\right) \end{aligned}$$

At $t = 0$, the positive peak V_A serves as a reference point. To move the reference point by time shift T_S, replace the t with $(t - T_S)$:

$$
\begin{aligned}
v(t) &= V_A \cos\left[\frac{2\pi}{T_0}(t - T_S)\right] \\
&= V_A \cos\left[\frac{2\pi t}{T_0} - \phi\right]
\end{aligned}
$$

where $\phi = 2\pi\left(\frac{T_S}{T_0}\right) = 360^\circ\left(\frac{T_S}{T_0}\right)$.

The factor ϕ is the phase shift (or angle). The phase shift is the angle between $t = 0$ and the nearest positive peak. You can view the preceding equation as the polar representation of the sinusoid. When the phase shift is π/2, then the shifted cosine is a sine function.

Express the phase angle in radians to make sure it's in the same units as the argument of the cosine ($2\pi t/T_0 - \phi$). ***Note:*** Angles can be expressed in either radians or degrees; make sure you use the right setting on your calculator.

When you have a phase shift ϕ at the output when compared to the input, it's usually caused by the circuit itself.

Expanding the function and finding Fourier coefficients

The general sinusoid $v(t)$, which I introduce in the preceding section, involves the cosine of a difference of angles. In many applications, you can expand the general sinusoid using the following trigonometric identity:

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Expanding the general sinusoid $v(t)$ leads to

$$v(t) = V_A \cos\left[\underbrace{\frac{2\pi}{T_0}}_{\alpha} t - \underbrace{\phi}_{\beta}\right]$$

$$= \underbrace{\left[V_A \cos\phi\right]}_{c} \cos\left[\frac{2\pi t}{T_0}\right] + \underbrace{\left[V_A \sin\phi\right]}_{d} \sin\left[\frac{2\pi t}{T_0}\right]$$

The terms *c* and *d* are just special constants called *Fourier coefficients.* You can express the waveform as a combination of sines and cosines as follows:

$$v(t) = c\cos\left[\frac{2\pi t}{T_0}\right] + d\sin\left[\frac{2\pi t}{T_0}\right]$$

The function $v(t)$ describes a sinusoidal signal in rectangular form.

If you know your complex numbers going between polar and rectangular forms, then you can go between the two forms of the sinusoids. The Fourier coefficients *c* and *d* are related by the amplitude V_A and phase ϕ:

$$c = V_A \cos\phi$$
$$d = V_A \sin\phi$$

If you go back to find V_A and ϕ from the Fourier coefficients *c* and *d,* you wind up with these expressions:

$$V_A = \sqrt{c^2 + d^2}$$
$$\phi = \tan^{-1}\left(\frac{d}{c}\right)$$

TIP The inverse tangent function on a calculator has a positive or negative 180° (or π) phase ambiguity. You can figure out the phase by looking at the signs of the Fourier coefficients *c* and *d*. Draw the points *c* and *d* on the rectangular system, where *c* is the *x*-component (or *abscissa*) and *d* is the *y*-component (or *ordinate*). The ratio of *d/c* can be negative in Quadrants II and IV. Using the rectangular system helps you determine the angles when taking the arctangent, whose range is from $-\pi/2$ to $\pi/2$.

Connecting sinusoidal functions to exponentials with Euler's formula

Euler's formula connects trig functions with complex exponential functions (see the earlier section "Pushing the Limits with the Exponential Function" for details on exponentials). The formula states that for any real number θ, you have the following complex exponential expressions:

$$e^{j\theta} = \cos\theta + j\sin\theta$$
$$e^{-j\theta} = \cos\theta - j\sin\theta$$

The exponent $j\theta$ is an imaginary number, where $j = \sqrt{-1}$. (The imaginary number j is the same as the number i from your math classes, but all the cool people use j for imaginary numbers because i stands for current.)

You can add and subtract the two preceding equations to get the following relationships:

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$
$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

These equations say that the cosine and sine functions are built as a combination of complex exponentials. The complex exponentials play an important role when you're analyzing complex circuits that have storage devices such as capacitors and inductors.

Chapter 12

Spicing Up Circuit Analysis with Capacitors and Inductors

In This Chapter

- Using capacitors to store electrical energy
- Storing magnetic energy with inductors
- Using op amps to do your calculus

If you've previously analyzed circuits consisting of only resistors and batteries, you may be happy to hear that more-interesting circuits do exist. The addition of two passive devices — capacitors and inductors — help spice up the functioning of circuits by storing energy for later use. You couldn't have electronic multimedia devices or entertainment gear without capacitors and inductors.

The addition of capacitors and inductors also lets you use circuits to do some calculations for you. With these devices, you can perform mathematical operations that are usually done by hand, such as integration and differentiation, electronically. Yep, you read right. You can build on-the-spot calculus operations using capacitors and resistors, along with your life-long friend, the operational amplifier (see Chapter 10 for the scoop on op amps).

In this chapter, I introduce you to capacitors and inductors, and I help you find quantities such as voltage, current, power, energy, capacitance, and inductance in circuits that contain these storage devices. I then show you how to do a little calculus with op amps.

Storing Electrical Energy with Capacitors

Interesting things happens when capacitors come into play in circuit analysis. They allow you to build voltage dividers that depend on the frequency content of the signals. What use is that? Well, with capacitors and resistors, you can emphasize the frequencies produced by specific instruments in your favorite music, like the high-frequency beats from a snare drum or the low-frequency bass sounds of a cello. Or you can filter out the voices in a song to create your own karaoke soundtrack.

Other uses for capacitors include filtering and storing energy by bypassing or coupling capacitors to make circuits work properly.

The following sections give you insight into capacitors and the relationship between voltage and current in a capacitor. They also explain how to find the amount of energy stored in a capacitor, whether you're dealing with a single capacitor or multiple capacitors in a parallel or series construction.

Describing a capacitor

A capacitor consists of two parallel conducting plates like silver or aluminum separated by an insulator. Unlike resistors, which waste energy, capacitors store energy for later use. Here's the property that applies to capacitors: $q = Cv$. C is the capacitance, q is the amount of stored charge, and v is the voltage across the capacitor.

The *capacitance,* which is measured in *farads* (F), relates the amount of charge stored in a capacitor to the applied voltage. The formula shows that the larger the voltage across a capacitor, the larger the amount of stored charges. How much larger depends on the capacitance value. Because voltage is the amount of energy per unit charge, capacitance also measures a capacitor’s ability to store energy. The larger the capacitance, the more energy a capacitor can store. You can vary the amount of charge stored in a capacitor by changing certain physical properties of a capacitor, like the area of the conducting plates or their distance apart.

Figure 12-1 shows the schematic symbol for a capacitor: two parallel lines of equal length, separated by a gap. If you see a plus sign by the symbol, the capacitor is polarized. Polarized capacitors show distinct polarities; they’re touchy in how you should connect the voltage polarities to the circuit.

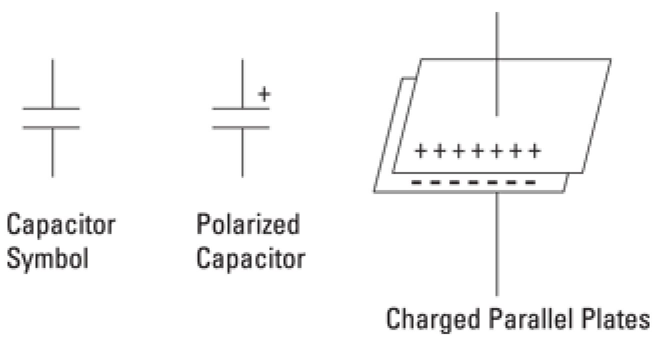

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Figure 12-1: Circuit symbols of capacitors and parallel plates

Charging a capacitor (credit cards not accepted)

When you connect a battery to a capacitor, the negative side of the battery pushes negative charges on one of the plates. These electrons form an electric field, repelling electrons on the other plate and leaving a positive charge. The electrons build up according to the amount of applied voltage. If the applied voltage remains constant, then the electrons build up until there’s no current flow. You now have a *charged* capacitor.

If you disconnect the charged capacitor from the battery, the capacitor saves the same voltage. Even more magic occurs when you connect a charged capacitor to a circuit with resistors. The voltage across the capacitor releases charges (or current) to discharge the capacitor. Eventually, the capacitor discharges to 0 volts for a circuit with no voltage sources. This charging and discharging action occurs over time, and because the action takes time, you can use capacitors in timing applications, like triggering an alarm to remind you to take a break to do 100 push-ups.

Relating the current and voltage of a capacitor

The voltage and current of a capacitor are related. To see this, you need to take the derivative of the capacitance equation $q(t) = Cv(t)$, which is

$$\frac{dq(t)}{dt} = C\frac{dv(t)}{dt}$$

Because $dq(t)/dt$ is the current through the capacitor, you get the following *i*-*v* relationship:

$$i(t) = C\frac{dv(t)}{dt}$$

This equation tells you that when the voltage doesn’t change across the capacitor, current doesn’t flow; to have current flow, the voltage must change. For a constant battery source, capacitors act as open circuits because there’s no current flow.

The voltage across a capacitor changes in a smooth fashion (and its derivatives are also smoothly changing functions), so there are no instantaneous jumps in voltages.

Just as you don't have gaps in velocities when you accelerate or decelerate your car, you don't have gaps in voltages. The mass of the car causes a smooth transition when going from 55 miles per hour to 60 miles per hour. In a similar and analogous way, you can think of the capacitance *C* as the mass in the circuit world that causes a smooth transition when changing voltages from one value to another.

To express the voltage across the capacitor in terms of the current, you integrate the preceding equation as follows:

$$v(t) = \frac{1}{C}\int_0^t i(\tau)\,d\tau + v_c(0)$$

The second term in this equation is the initial voltage across the capacitor at time $t = 0$.

You can see the *i*-*v* characteristic in Figure 12-2. The left diagram defines a linear relationship between the charge *q* stored in the capacitor and the voltage *v* across the capacitor. The right diagram shows a current relationship between the current and the derivative of the voltage, $dv_C(t)/dt$, across the capacitor with respect to time *t*.

Think of capacitance C as a proportionality constant, like a resistor acts as a constant in Ohm's law.

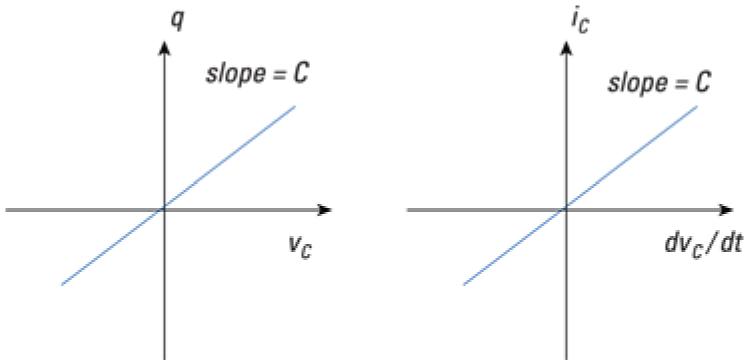

Illustration by Wiley, Composition Services Graphics

Figure 12-2: Linear relationships of capacitors.

Finding the power and energy of a capacitor

To find the instantaneous power of the capacitor, you need the following power definition, which applies to any device:

$$p_C(t) = \underbrace{i_C(t)}_{C\frac{dv_C(t)}{dt}} v_C(t)$$

The subscript *C* denotes a capacitance device (surprise!). Substituting the current for a capacitor (from the preceding section) into this equation gives you the following:

$$p_C(t) = C\frac{dv_C(t)}{dt}v_C(t)$$

$$\frac{dw_C(t)}{dt} = \frac{d}{dt}\left[\frac{1}{2}Cv_C^2(t)\right]$$

Assuming zero initial voltage, the energy $w_C(t)$ stored per unit time is the power. Integrating that equation gives you the energy stored in a capacitor:

$$w_C(t) = \frac{1}{2}Cv_C^2(t)$$

The energy equation implies that the energy stored in a capacitor is always positive. The capacitor absorbs power from a circuit when storing energy. The capacitor releases the stored energy when delivering energy to the circuit.

For a numerical example, look at the top-left diagram of Figure 12-3, which shows how the voltage changes across a 0.5-μF capacitor. Try calculating the capacitor's energy and power.

The slope of the voltage change (time derivative) is the amount of current flowing through the capacitor. Because the slope is constant, the current through the capacitor is constant for the given slopes. For this example, you calculate the slope for each time interval in the graph as follows:

$$\frac{dv_C(t)}{dt} = \frac{(10-0)\text{ V}}{(0.002-0)\text{ s}} = 5{,}000\text{ V/s} \qquad 0 \le t < 2\text{ ms}$$

$$\frac{dv_C(t)}{dt} = \frac{(10-10)\text{ V}}{(0.004-0.002)\text{ s}} = 0\text{ V/s} \qquad 2\text{ ms} \le t < 4\text{ ms}$$

$$\frac{dv_C(t)}{dt} = \frac{(0-10)\text{ V}}{(0.006-0.004)\text{ s}} = -5{,}000\text{ V/s} \qquad 4\text{ ms} \le t$$

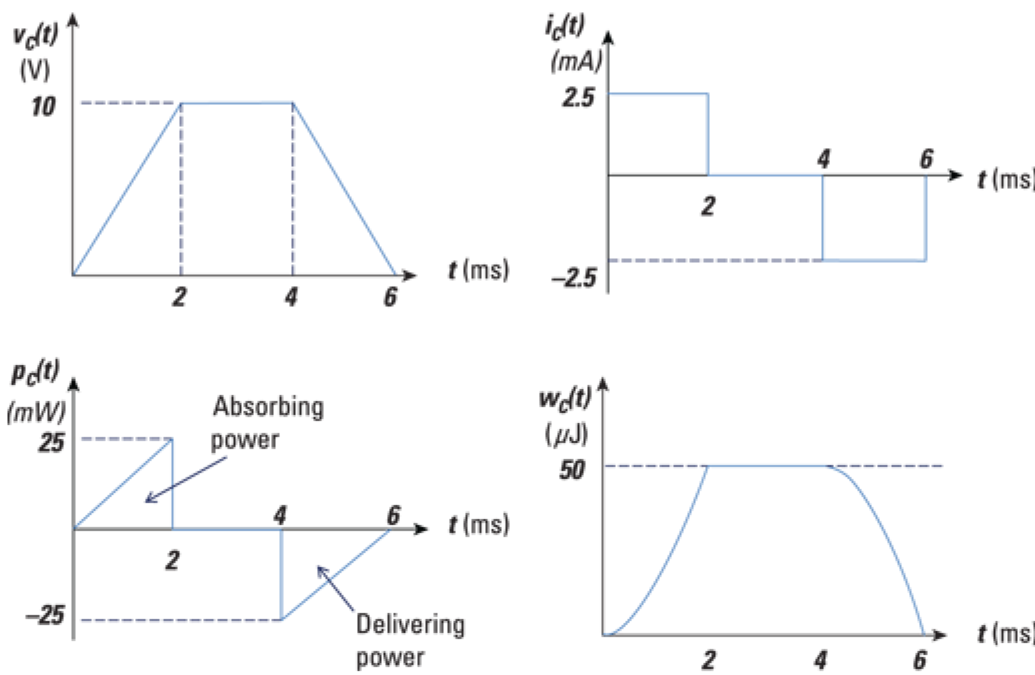

Illustration by Wiley, Composition Services Graphics

Figure 12-3: Power and energy of a capacitor.

Multiply the slopes by the capacitance (in farads) to get the capacitor current during each interval. The capacitance is 0.5 μF, or 0.5×10^{-6} F, so here are the currents:

$$i_C(t) = C\frac{dv(t)}{dt}$$

$$i_C(t) = \left(0.5\times 10^{-6}\ \mathrm{F}\right)\left(5{,}000\ \mathrm{V/s}\right) = 2.5\ \mathrm{mA} \qquad 0 \le t < 2\ \mathrm{ms}$$

$$i_C(t) = \left(0.5\times 10^{-6}\ \mathrm{F}\right)\left(0\ \mathrm{V/s}\right) = 0\ \mathrm{mA} \qquad 2\ \mathrm{ms} \le t < 4\ \mathrm{ms}$$

$$i_C(t) = \left(0.5\times 10^{-6}\ \mathrm{F}\right)\left(-5{,}000\ \mathrm{V/s}\right) = -2.5\ \mathrm{mA} \qquad 4\ \mathrm{ms} \le t$$

You see the graph of the calculated currents in the top-right diagram of Figure 12-3.

You find the power by multiplying the current and voltage, resulting in the bottom-left graph in Figure 12-3. Finally, you can find the energy by calculating $(1/2)C[v_C(t)]^2$. When you do this, you get the bottom-right graph of Figure 12-3. Here, the capacitor's energy increases when it's absorbing power and decreases when it's delivering power.

Calculating the total capacitance for parallel and series capacitors

You can reduce capacitors connected in parallel or connected in series to one single capacitor. This section shows you how.

Finding the equivalent capacitance of parallel capacitors

Consider the first circuit in Figure 12-4, which contains three parallel capacitors. Because the capacitors are connected in parallel, they have the same voltages:

$$v_1(t) = v_2(t) = v_3(t) = v(t)$$

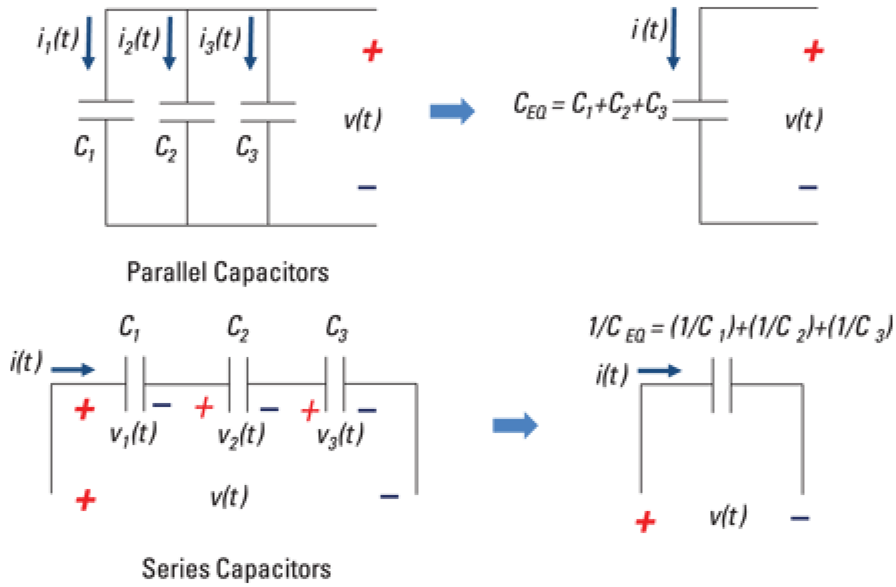

Illustration by Wiley, Composition Services Graphics

Figure 12-4: Parallel and series connection of capacitors.

Adding the current from each parallel capacitor gives you the net current $i(t)$:

$$i(t) = \underbrace{C_1\frac{dv(t)}{dt}}_{i_1(t)} + \underbrace{C_2\frac{dv(t)}{dt}}_{i_2(t)} + \underbrace{C_3\frac{dv(t)}{dt}}_{i_3(t)}$$
$$= \underbrace{(C_1 + C_2 + C_3)}_{C_{EQ}}\frac{dv(t)}{dt}$$

For parallel capacitors, the equivalent capacitance is

$$C_{EQ} = C_1 + C_2 + C_3$$

Finding the equivalent capacitance of capacitors in series

For a series connection of capacitors, apply Kirchhoff's voltage law (KVL) around a loop in the bottom diagram of Figure 12-4. KVL says the sum of the voltage rises and drops around a loop is 0, giving you

$$v(t) = \underbrace{v_1(t)}_{v_1(0)+\frac{1}{C_1}\int_0^t i_1(\tau)d\tau} + \underbrace{v_2(t)}_{v_2(0)+\frac{1}{C_2}\int_0^t i_2(\tau)d\tau} + \underbrace{v_3(t)}_{v_3(0)+\frac{1}{C_3}\int_0^t i_3(\tau)d\tau}$$

A series current has the same current $i(t)$ going through each of the series capacitors, so

$$v(t)=\left[v_1(0)+v_2(0)+v_3(0)\right]+\underbrace{\left(\frac{1}{C_1}+\frac{1}{C_2}+\frac{1}{C_3}\right)}_{\frac{1}{C_{EQ}}}\int_0^t i(\tau)\,d\tau$$

The preceding equation shows how you can reduce the series capacitance to one single capacitance:

$$\frac{1}{C_{EQ}}=\frac{1}{C_1}+\frac{1}{C_2}+\frac{1}{C_3}$$

Storing Magnetic Energy with Inductors

Inductors find heavy use in radiofrequency (RF) circuits. They serve as RF "chokes," blocking high-frequency signals. This application of inductor circuits is called *filtering.* Electronic filters select or block whichever frequencies the user chooses.

In the following sections, you discover how inductors resist instantaneous changes in current and store magnetic energy. You also find the equivalent inductance when inductors are connected in series or in parallel.

Describing an inductor

Unlike capacitors, which are electrostatic devices, inductors are electromagnetic devices. Whereas capacitors avoid an instantaneous change in voltage, inductors prevent an abrupt change in current. Inductors are wires wound into several loops to form coils. In fact, the inductor's symbol looks like a coil of wire (see Figure 12-5).

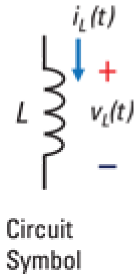

Illustration by Wiley, Composition Services Graphics

Figure 12-5: The circuit symbol for an inductor.

Current flowing through a wire creates a magnetic field, and the magnetic field lines encircle the wire along its axis. The concentration, or density, of the magnetic field lines is called *magnetic flux*. The coiled shape of inductors increases the magnetic flux that naturally occurs when current flows through a straight wire. The greater the flux, the greater the inductance. You can get even larger inductance values by inserting iron into the wire coil.

Here's the defining equation for the inductor:

$$v_L(t) = L\frac{di_L(t)}{dt}$$

where the inductance L is a constant measured in *henries* (H). You see this equation in graphical form in Figure 12-6. The figure shows the *i*-*v* characteristic of an inductor, where the slope of the line is the value of the inductance.

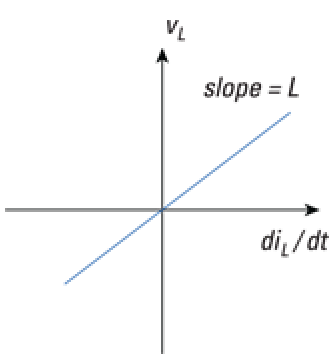

Illustration by Wiley, Composition Services Graphics

Figure 12-6: Linear relationship of inductors.

The preceding equation says that the voltage across the inductor depends on the time rate of change of the current. In other words, no change in inductor current means no voltage across the inductor. To create voltage across the inductor, current must change smoothly. Otherwise, an instantaneous change in current would create one humongous voltage across the inductor.

Think of inductance *L* as a proportionality constant, like a resistor acts as a constant in Ohm's law. This notion of Ohm's law for inductors (and capacitors) becomes useful when you start working with phasors (see Chapter 15).

To express the current through the inductor in terms of the voltage, you integrate the preceding equation as follows:

$$i_L(t) = \frac{1}{L}\int_0^t v(\tau)d\tau + i_L(0)$$

The second term in this equation is the initial current through the inductor at time $t = 0$.

Finding the energy storage of an attractive inductor

To find the energy stored in the inductor, you need the following power definition, which applies to any device:

$$p_L(t) = \underbrace{v_L(t)}_{L\frac{di_L(t)}{dt}} i_L(t)$$

The subscript *L* denotes an inductor device. Substituting the voltage for an inductor (from the preceding section) into the power equation gives you the following:

$$p_L(t) = L\frac{di_L(t)}{dt}i_L(t)$$

$$\frac{dw_L(t)}{dt} = \frac{d}{dt}\left[\frac{1}{2}Li_L^2(t)\right]$$

The energy $w_L(t)$ stored per unit time is the power. Integrating the preceding equation gives you the energy stored in an inductor:

$$w_L(t) = \frac{1}{2}Li_L^2(t)$$

The energy equation implies that the energy in the inductor is always positive. The inductor absorbs power from a circuit when storing energy, and the inductor releases the stored energy when delivering energy to the circuit.

To visualize the current and energy relationship, consider Figure 12-7, which shows the current as a function of time and the energy stored in an inductor. The figure also shows how you can get the current from the inductor relationship between current and voltage.

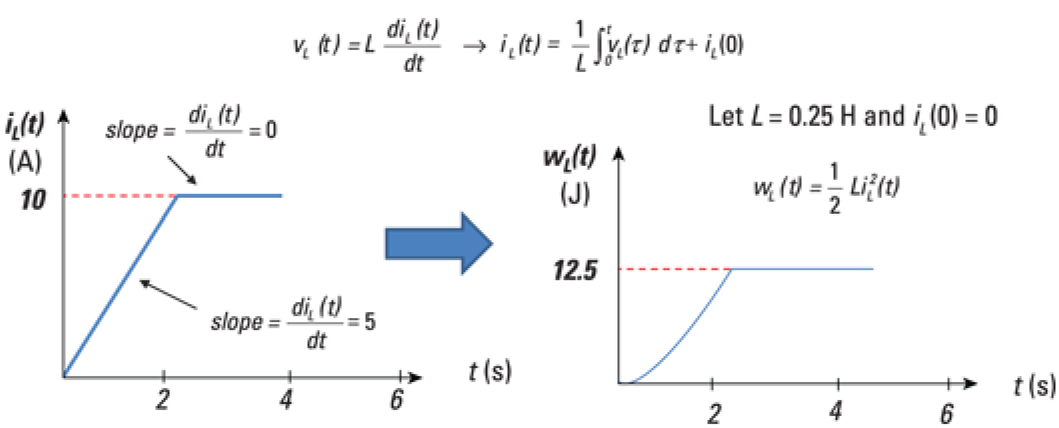

Illustration by Wiley, Composition Services Graphics

Figure 12-7: Energy storage of inductors.

Calculating total inductance for series and parallel inductors

Inductors connected in series or connected in parallel can be reduced to one single inductor, as I explain next.

Finding the equivalent inductance for inductors in series

Take a look at the circuit with three series inductors shown in the top diagram of Figure 12-8. Because the inductors are connected in series, they have the same currents:

$$i_1(t) = i_2(t) = i_3(t) = i(t)$$

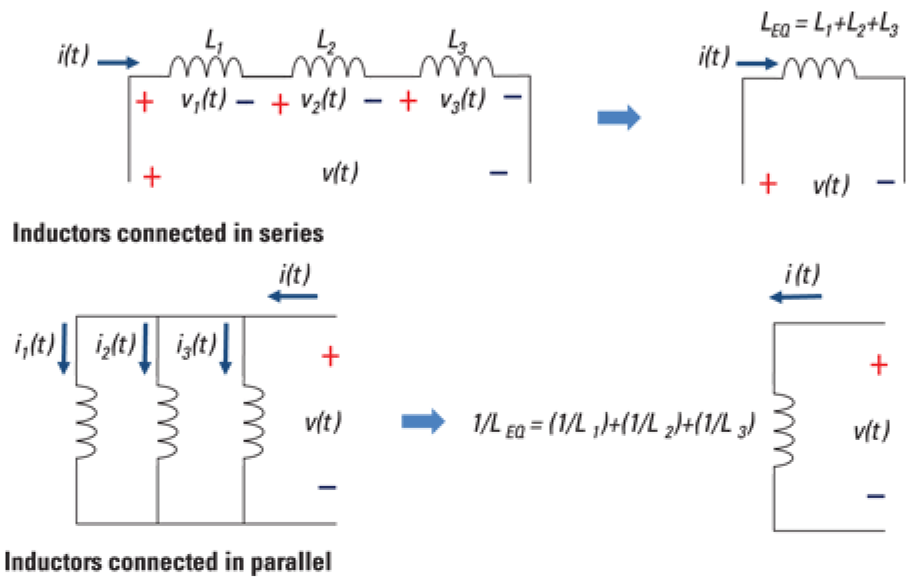

Illustration by Wiley, Composition Services Graphics

Figure 12-8: Total inductance for inductors connected in series and parallel.

Add up the voltages from the series inductors to get the net voltage $v(t)$, as follows:

$$v(t) = \underbrace{L_1\frac{di(t)}{dt}}_{v_1(t)} + \underbrace{L_2\frac{di(t)}{dt}}_{v_2(t)} + \underbrace{L_3\frac{di(t)}{dt}}_{v_3(t)}$$
$$= \underbrace{(L_1 + L_2 + L_3)}_{L_{EQ}}\frac{di(t)}{dt}$$

For a series inductors, you have an equivalent inductance of

$$L_{EQ} = L_1 + L_2 + L_3$$

Finding the equivalent inductance for inductors in parallel

For a parallel connection of inductors, apply Kirchhoff's current law (KCL) in the bottom diagram of Figure 12-8. KCL says the sum of the incoming currents and outgoing current at a node is equal to 0, giving you

$$i(t) = \underbrace{i_1(t)}_{i_1(0)+\frac{1}{L_1}\int_0^t v_1(\tau)d\tau} + \underbrace{i_2(t)}_{i_2(0)+\frac{1}{L_2}\int_0^t v_2(\tau)d\tau} + \underbrace{i_3(t)}_{i_3(0)+\frac{1}{L_3}\int_0^t v_3(\tau)d\tau}$$

Because you have the same voltage $v(t)$ across each of the parallel inductors, you can rewrite the equation as

$$i(t) = \left[i_1(0) + i_2(0) + i_3(0)\right] + \underbrace{\left(\frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}\right)}_{\frac{1}{L_{EQ}}}\int_0^t v(\tau)d\tau$$

This equation shows how you can reduce the parallel inductors to one single inductor:

$$\frac{1}{L_{EQ}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}$$

Calculus: Putting a Cap on Op-Amp Circuits

In this section, you add a capacitor to an operational-amplifier (op-amp) circuit. Doing so lets you use the circuit to do more-complex mathematical operations, like integration and differentiation. Practically speaking, you use capacitors instead of inductors because inductors are usually bulkier than capacitors.

Creating an op-amp integrator

Figure 12-9 shows an op-amp circuit that has a feedback element as a capacitor. The circuit is configured similarly to an inverting amplifier. (Check out Chapter 10 if you want to brush up on op amps.)

The cool thing about this op-amp circuit is that it performs integration. The circuit electronically calculates the integral of any input voltage, which is a lot simpler (and less painful!) than banging your head on the table as you try to integrate a weird function by hand.

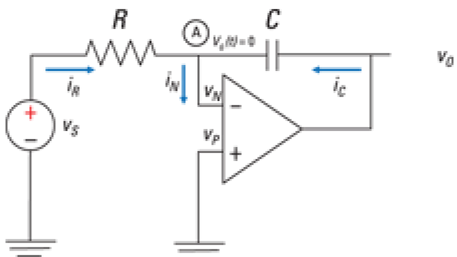

Illustration by Wiley, Composition Services Graphics

Figure 12-9: An op-amp integrator.

I walk you through the analysis so you can see how this circuit performs this incredible feat called integration. First, you use a KCL equation at Node A:

$$i_R(t) + i_C(t) = i_N(t)$$

Ohm's law ($i = v/R$) gives you the current through the resistor:

$$i_R(t) = \frac{1}{R}\left[v_S(t) - v_G(t)\right]$$

You get the current through the capacitor using the *i*-*v* relationship of a capacitor:

$$i_C(t) = C\frac{d\left[v_O(t) - v_G(t)\right]}{dt}$$

For ideal op-amp devices (see Chapter 10), the circuit gives you $v_G(t) = 0$ (virtual ground) and $i_N = 0$ (infinite input resistance). Substituting these op-amp constraints for $i_R(t)$ and $i_C(t)$ into the KCL equation gives you

$$\frac{v_S(t)}{R}+C\frac{dv_O(t)}{dt}=0$$

Then integrate both sides of the preceding equation. You wind up with the following output voltage $v_o(t)$:

$$v_O(t)=-\frac{1}{RC}\int_0^t v_s(\tau)d\tau\ +v_o(0)$$

The initial output voltage $v_o(0)$ across the capacitor — that, is the voltage at t = 0 — is 0. If $v_o(0)$ = 0, then the output-voltage equation reduces to

$$v_O(t)=-\frac{1}{RC}\int_0^t v_S(\tau)d\tau$$

The op-amp circuit accepts an input voltage and gives you an inverted output that's proportional to the integral of the input voltage.

Deriving an op-amp differentiator

With op-amp circuits where the resistor is the feedback element and the capacitor is the input device (like the one in Figure 12-10), you can perform differentiation electronically.

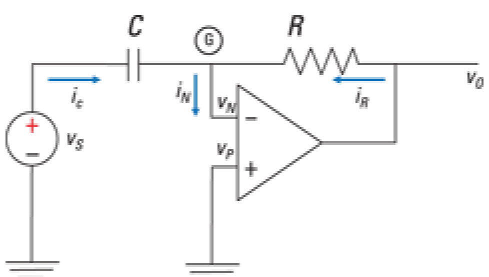

Illustration by Wiley, Composition Services Graphics

Figure 12-10: An op-amp differentiator.

You follow the same process as the one you use to find the relationship for an op-amp integrator (see the preceding section for details). Begin with a KCL equation at Node G:

$$i_R(t)+i_C(t)=i_N(t)$$

The current through the resistor is given by Ohm's law ($i = v/R$):

$$i_R(t) = \frac{1}{R}\left[v_O(t) - v_G(t)\right]$$

The current through the capacitors is given by the *i*-*v* relationship of a capacitor:

$$i_C(t) = C\frac{d\left[v_S(t) - v_G(t)\right]}{dt}$$

For ideal op-amp devices (see Chapter 11), the circuit gives you $v_G = 0$ (virtual ground) and $i_N = 0$ (infinite input resistance). Substituting these op-amp constraints for $i_R(t)$ and $i_C(t)$ into the KCL equation gives you the following:

$$\frac{v_O(t)}{R} + C\frac{dv_S(t)}{dt} = 0$$

Solving for v_o, you wind up with the following output voltage $v_o(t)$:

$$v_O(t) = -RC\frac{dv_S(t)}{dt}$$

So if you give me an input voltage, I say no sweat in getting its derivative as an output. The inverted output is simply proportional to the derivative of the input voltage.

Using Op Amps to Solve Differential Equations Really Fast

The op-amp circuit can solve mathematical equations fast, including calculus problems. The intent of this section is to give you a basic idea of how to implement various op-amp configurations and how they can be tied together.

Say you want to solve a differential equation by finding $v(t)$, a function that's a solution to a differential equation. In the following example, I show you how to use various op-amp configurations to find the output voltage $v_o(t) = v(t)$.

To simplify the problem, assume zero initial conditions: zero initial capacitor voltage for each integrator in Figure 12-11. To solve a differential equation, you need to develop a block diagram for the differential equation (which is represented by the dashed boxes in the figure), giving the input and the output for each dashed box. Then use the block diagram to design a circuit. On the far left of Figure 12-11 is a forcing function of 25 volts derived from the following steps, and the output voltage $v_o(t) = v(t)$ is on the far right of the figure.

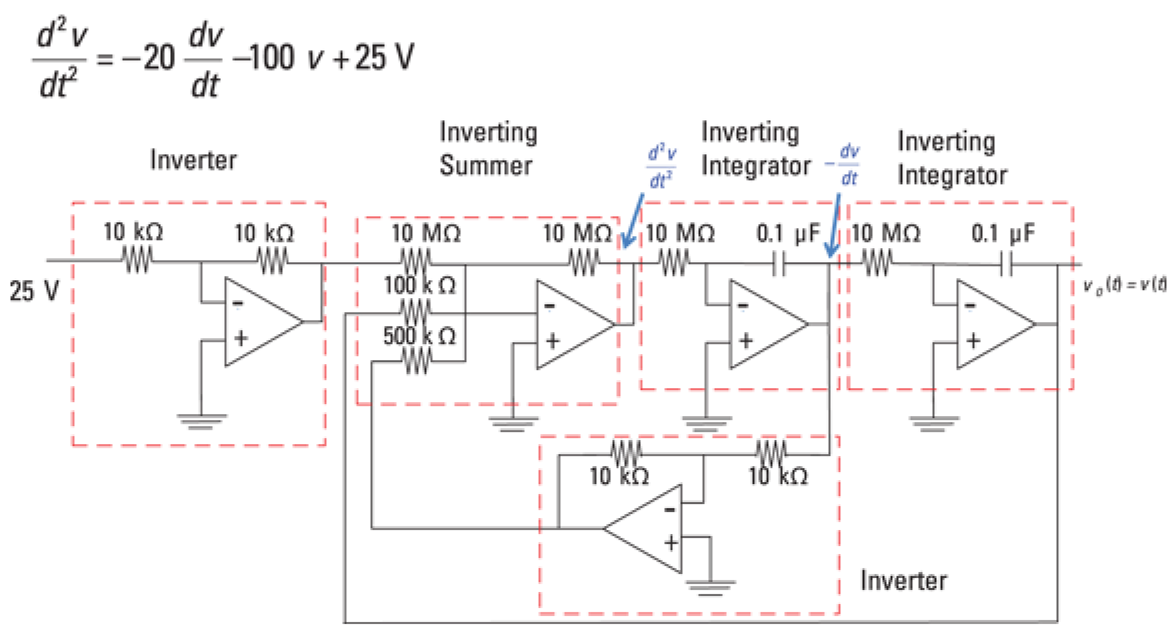

Illustration by Wiley, Composition Services Graphics

Figure 12-11: Solving differential equations with op amps.

Here are the basic steps for designing the circuit:

1. Solve for the highest-order derivative, showing that it consists of a sum of the lower derivatives.

   Suppose you want to solve the following second-order differential equation:

   $$10\frac{d^2v(t)}{dt^2}+200\frac{dv(t)}{dt}+1000v(t)=250\text{ V}$$

   The first step is to algebraically solve for the highest-order derivative, *d*2*v/dt*2:

   $$\frac{d^2v}{dt^2}=-20\frac{dv}{dt}-100v+25\text{ V}$$

   The highest-order derivative is a combination or sum of lower derivatives and the smaller input voltage: *dv/dt, v,* and 25. Therefore, you need an inverting summer to add the three terms, and these terms are forcing functions (or inputs) to the inverting summer.

2. Use integrators to help implement the block diagram, because the integral of the higher-order derivative is the derivative that's one order lower.

   For this example, integrate the second derivative, *d*2*v/dt*2, to give you the first derivative, *dv/dt*. As Figure 12-11 shows, the output of the inverting summing amplifier is the second derivative (which is also the input to the first integrator). The output of the first inverting integrator is the negative of the first derivative *dv/dt* and serves as the input to the second inverting integrator. With the second inverting integrator in Figure 12-11, integrate the negative of the first derivative, –*dv/dt*, to give you the desired output, *v*(*t*).

3. Take the outputs of the integrators, scale them, and feed them back to a summer (summing amplifier).

   The second derivative consists of a sum of three terms, so this is where the op-amp inverting summer comes in:

1. One of the inputs is a constant of 25 volts to the summer and will be an input voltage (or driving) source. The 25 volts at the input is fed to one of the inputs to the summer with a gain of 1.
2. The output of the first integrator is the first derivative of $v(t)$, which has a weight of 20 and is fed to the second input of the inverting summer.
3. The output of the second integrator is fed to the third input to the inverting summer with a weight of 100.

This completes the block diagram.

For this example, multiply the first derivative dv/dt by –10 and multiply v by –100. Sum them as shown in the block diagram of Figure 12-11.

4. Design the circuit to implement the block diagram.

To simplify the design, give each integrator a gain of –1. You need two more inverting amplifiers to make the signs come out right. Use the summer to achieve the gains of –10 and –100 found in Step 3. Figure 12-11 is one of many possible designs.

Chapter 13

Tackling First-Order Circuits

In This Chapter

- Focusing on first-order differential equations with constant coefficients
- Analyzing a series circuit that has a single resistor and capacitor
- Analyzing a parallel circuit that has a single resistor and inductor

Building more exciting circuit functions requires capacitors or inductors. These storage devices — which I introduce in Chapter 12 — tell other parts of the circuit to slow down and take time when things are about to change. Nothing happens instantaneously with capacitors and inductors. You can think of these devices as little bureaucracies slowing things down in the life of circuit city.

This chapter focuses on circuits with a single storage element connected to a single resistor or a resistor network. In math mumbo jumbo, a circuit with a single storage device is described with first-order differential equations; hence the name *first-order circuit.* The analysis helps you understand timing circuits and time delays if that's what's needed to achieve specific tasks. (A circuit with two storage devices is a second-order circuit, which I cover in Chapter 14.)

If your head is cloudy on the calculus, check out a diff EQ textbook or pick up a copy of *Differential Equations For Dummies* by Steven Holzner (Wiley) for a refresher.

Solving First-Order Circuits with Diff EQ

To find out what's happening in circuits with capacitors, inductors, and resistors, you need differential equations. Why? Because generating current through a capacitor requires a change in voltage, and generating voltage across an inductor requires a change in current. Differential equations take the rate of change into account.

You have a *first-order circuit* when a first-order differential equation describes the circuit. A resistor and capacitor connected in series (an *RC series circuit*) is one example of a first-order circuit. A capacitor's version of Ohm's law with capacitance *C* is described by a first-order derivative:

$$i_C(t) = C\frac{dv_C(t)}{dt}$$

Another first-order circuit is a resistor and inductor connected in parallel (an *RL parallel circuit*). An inductor with inductance value *L* has an *i*-*v* relationship also expressed by a first-order derivative:

$$v_L(t) = L\frac{di_L(t)}{dt}$$

Because the capacitance *C* and inductance *L* are constant and connected to a constant resistor, circuits with these devices lead to differential equations that have constant coefficients.

So when analyzing a circuit with an inductor or a capacitor with a resistor driven by an input source, you have a first-order differential equation. Both types of first-order circuits have only one energy storage device and one resistor, which converts electricity to heat. To get a complete solution to the first-order differential equation, you need to know a circuit's initial condition. An *initial condition* is simply the initial state of the circuit, such as the inductor current or the capacitor voltage at time $t = 0$.

You can solve differential equations in numerous ways, but because the circuits you encounter in this book have only constant values of resistors, inductors, and capacitors — leading to differential equations with constant coefficients — I give you just one approach to solving first-order differential. (This approach works for solving second-order differential equations, too, but that's the subject of Chapter 14.) The best part about this approach is that it converts a problem involving a differential equation to one that only involves algebra.

REMEMBER Here's how to solve a differential equation that has constant coefficients for first-order circuits, given an initial condition and *forcing function* (an input signal or function):

1. **Find the zero-input response by setting the input source to 0.**

   You want the output to be due to initial conditions only.

2. **Find the zero-state response by setting the initial conditions equal to 0 and adding together the solutions to the homogeneous equation and differential equation to a particular input.**

You want the output to be due to the input signal, or forcing function, only. In first-order circuits, you have 0 initial capacitor voltage or 0 initial inductor current.

To get the *zero-state response,* you have to find the following:

- **The homogeneous solution:** You get the solution to the homogeneous differential equation when you first set the input signal or forcing function equal to 0. This solution is for the zero initial condition.
- **The particular solution:** The particular solution is the solution to the differential equation with a particular input source. This means turning the input signal back on, so the solution depends on the type of input signal. For example, if your input is a constant, then your particular solution is also a constant. When you have a sine or cosine function as an input, the output is a combination of sine and cosine functions.

3. **Add up the zero-input and zero-state responses to get the total response.**

   Because you're dealing with linear circuits, you can add up the two solutions based on the superposition concept, which I cover in Chapter 7.

The following sections show you how to find the solution to a first-order differential equation. You start off with one circuit having a zero-input source and then look at circuits with a particular input source like a step input.

Guessing at the solution with the natural exponential function

Say you want to solve a homogeneous differential equation with constant coefficients having a zero-input source. The solution results from only the initial state (or initial condition) of the circuit. This response is called the *zero-input response.*

Consider the following homogeneous differential equation with zero forcing function $v_s(t) = 0$:

$$10\frac{dv(t)}{dt} + 20v(t) = v_s(t) = 0$$

The function $v_h(t)$ is the solution to the homogeneous differential equation.

You need to guess the function $v(t)$ to get 0 for the differential equation. Pssst . . . try $v(t) = e^{kt}$. Why? Because each time you take its derivative, you get the same exponential multiplied by constant k. When you substitute $v(t)$ into the differential equation, adding up the combination of exponentials leads to 0. The exponential equation is your best friend when solving this type of differential equation with constant coefficients.

The integral of an exponential is also an exponential multiplied by some constant or scale factor. This property makes exponentials useful in circuit analysis and many other applications.

Using the characteristic equation for a first-order equation

You can convert a first-order differential equation to a problem that involves algebra. Here's how you do it. You start with the zero forcing function (which I introduce in the preceding section):

$$10\frac{dv(t)}{dt}+20v(t)=0$$

Substitute your best-guess solution, $v(t) = e^{kt}$ (also from the preceding section), into the differential equation. With a little factoring, you get

$$10\frac{d}{dt}\left(e^{kt}\right)+20e^{kt}=0$$
$$(10k+20)e^{kt}=0$$

The coefficient of e^{kt} must be 0. Use that idea to find k*:*

$$10k+20=0$$
$$k=-\frac{1}{2}$$

Setting the algebraic equation to 0 gives you a *characteristic equation.* Why? Because solving for the root k determines the features of the solution $v(t)$. Here, the characteristic root found as $k = –1/2$ gives you the homogeneous solution:

$$v(t)=Ae^{-\left(\frac{1}{2}\right)t}$$

You determine the constant A by applying the initial condition or state $v(0)$ when $t = 0$. Guessing at a reasonable solution to the differential equation leads to a simpler, algebraic characteristic equation.

Analyzing a Series Circuit with a Single Resistor and Capacitor

A first-order *RC series circuit* has one resistor (or network of resistors) and one capacitor connected in series. You can see an example of one in Figure 13-1. In the following sections, I show you how to find the total response for this circuit.

If your RC series circuit has a capacitor connected with a network of resistors rather than a single resistor, you can use the same approach to analyze the circuit. You just have to find the Thévenin equivalent first, reducing the resistor network to a single resistor in series with a single voltage source. See Chapter 8 for details on the Thévenin approach.

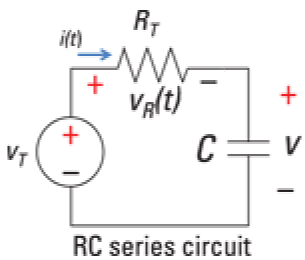

Illustration by Wiley, Composition Services Graphics

Figure 13-1: A first-order RC series circuit.

Starting with the simple RC series circuit

The simple RC series circuit in Figure 13-1 is driven by a voltage source. Because the resistor and capacitor are connected in series, they must have the same current $i(t)$. For Figure 13-1 and what follows next, let $R=R_T$.

To find the voltage across the resistor $v_R(t)$, you use Ohm's law for a resistor device:

$$v_R(t) = Ri(t)$$

The element constraint for a capacitor (found in Chapter 12) is given as

$$i(t) = C\frac{dv(t)}{dt}$$

where $v(t)$ is the capacitor voltage.

Generating current through a capacitor takes a changing voltage. If the capacitor voltage doesn't change, the current in the capacitor equals 0. Zero current implies infinite resistance for constant voltage across the capacitor.

Now substitute the capacitor current $i(t) = Cdv(t)/dt$ into Ohm's law for resistor R, because the same current flows through the resistor and capacitor. This gives you the voltage across the resistor, $v_R(t)$:

$$v_R(t) = RC\frac{dv(t)}{dt}$$

Kirchhoff's voltage law (KVL) says the sum of the voltage rises and drops around a loop of a circuit is equal to 0. Using KVL for the RC series circuit in Figure 13-1 gives you

$$v_T(t) = v_R(t) + v(t)$$

Now substitute $v_R(t)$ into KVL:

$$v_T(t) = RC\frac{dv(t)}{dt} + v(t)$$

You now have a first-order differential equation where the unknown function is the capacitor voltage. Knowing the voltage across the capacitor gives you the electrical energy stored in a capacitor.

In general, the capacitor voltage is referred to as a *state variable* because the capacitor voltage describes the state or behavior of the circuit at any time.

An easy way to remember that state variables — such as the capacitor voltage $v_C(t)$ and inductor current $i_L(t)$ — describe the present situation of the circuit is to think of your car's position and instantaneous velocity as your car's state variables. If you're racing along the majestic road of Rocky Mountain National Park, your GPS position and car's speed describe the current state of your driving.

The RC series circuit is a first-order circuit because it's described by a first-order differential equation. A circuit reduced to having a single equivalent capacitance and a single equivalent resistance is also a first-order circuit. The circuit has an applied input voltage $v_T(t)$.

To find the total response of an RC series circuit, you need to find the zero-input response and the zero-state response and then add them together. Figure 13-2 shows an RC series circuit broken up into two circuits. The top-right diagram shows the zero-input response, which you get by setting the input to 0. The bottom-right diagram shows the zero-state response, which you get by setting the initial conditions to 0.

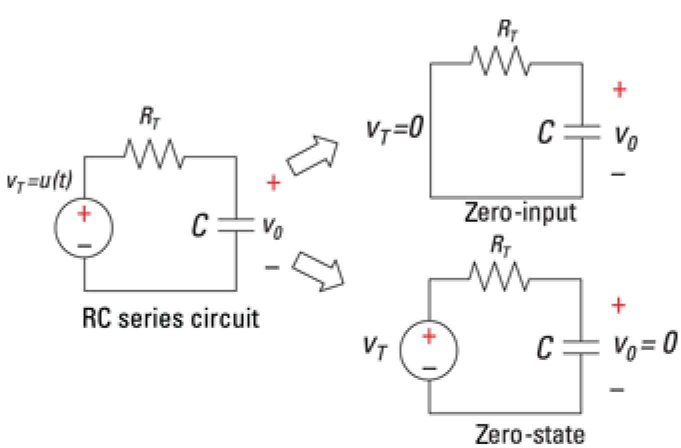

Illustration by Wiley, Composition Services Graphics

Figure 13-2: Analyzing a simple first-order RC series circuit.

Finding the zero-input response

You first want to find the zero-input response for the RC series circuit. The top-right diagram of Figure 13-2 shows the input signal $v_T(t)$ equal to 0. Zero-input voltage means you have zero . . . nada . . . zip . . . input for all time. The output response is due to the initial condition V_0 (initial capacitor voltage) at time $t = 0$. The first-order differential equation reduces to

$$v_T(t) = 0 = RC\frac{dv_{ZI}(t)}{dt} + v_{ZI}(t) \quad \text{or} \quad v_{ZI}(t) = -RC\frac{dv_{ZI}(t)}{dt}$$

Here, $v_{ZI}(t)$ is the capacitor voltage. For an input source set to 0 volts in Figure 13-2, the capacitor voltage is called a *zero-input response* or *free response.* No external forces (such as a battery) are acting on the circuit, except for the initial state of the capacitor voltage.

You can reasonably guess that the solution is the exponential function (you can check and verify the solution afterward). You try an exponential because the time derivative of an exponential is also an exponential (as I explain earlier in "Guessing at the solution with the natural exponential function"). Substitute that guess into the RC first-order circuit equation:

$$v_{ZI}(t) = Ae^{kt}$$

The A and k are arbitrary constants of the zero-input response.

Now substitute the solution $v_{ZI}(t) = Ae^{kt}$ into the differential equation:

$$
\begin{aligned}
v_{ZI}(t) &= -RC\frac{dv_{ZI}(t)}{dt} \\
Ae^{kt} &= -RC\frac{d\left(Ae^{kt}\right)}{dt} \\
Ae^{kt} &= -RC\left(kAe^{kt}\right)
\end{aligned}
$$

You get an algebraic characteristic equation after setting the equation equal to 0 and factoring out Ae^{kt}:

$$Ae^{kt}(1+RCk)=0$$

The characteristic equation gives you a much simpler problem. The coefficient of e^{kt} has to be 0, so you just solve for the constant k:

$$
\begin{aligned}
1+RCk &= 0 \\
k &= -\frac{1}{RC}
\end{aligned}
$$

When you have k, you have the zero-input response $v_{ZI}(t)$. Using $k = -1/RC$, you can find the solution to the differential equation for the zero input:

$$v_{ZI}(t) = Ae^{-\left(\frac{t}{RC}\right)}$$

Now you can find the constant A by applying the initial condition. At time $t = 0$, the initial voltage is V_0, which gives you

$$
\begin{aligned}
v_{ZI}(0) &= Ae^{-\left(\frac{0}{RC}\right)} \\
&= A = V_0
\end{aligned}
$$

The constant A is simply the initial voltage V_0 across the capacitor.

Finally, you have the solution to the capacitor voltage, which is the zero-input response $v_{ZI}(t)$:

$$v_{ZI}(t) = V_0e^{-\left(\frac{t}{RC}\right)} = v(0)e^{-\left(\frac{t}{RC}\right)} \quad \text{where } V_0 = v(0)$$

The constant term RC in this equation is called the *time constant.* The time constant provides a measure of how long a capacitor has discharged or charged. In this example, the capacitor starts at some initial state of voltage V_0 and dissipates quietly into oblivion to another state of 0 volts.

Suppose RC = 1 second and initial voltage V_0 = 5 volts. Figure 13-3 plots the decaying exponential, showing that it takes about 5 time constants, or 5 seconds, for the capacitor voltage to decay to 0.

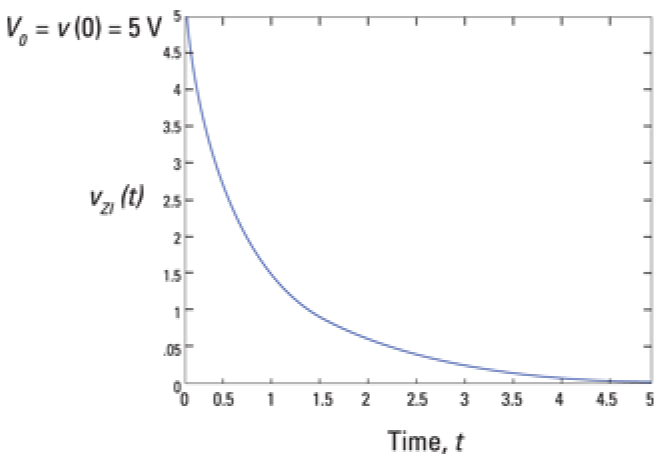

Illustration by Wiley, Composition Services Graphics

Figure 13-3: Zero-input response and the natural exponential.

Finding the zero-state response by focusing on the input source

Zero-state response means zero initial conditions, and it requires finding the capacitor voltage when there's an input source, $v_T(t)$. You need to find the homogeneous and particular solutions to get the zero-state response. To find zero initial conditions, you look at the circuit when there's no voltage across the capacitor at time t = 0.

The circuit at the bottom right of Figure 13-2 has *zero initial conditions* and an input voltage of $V_T(t) = u(t)$, where $u(t)$ is a unit step input. Mathematically, you can describe step function $u(t)$ as

$$u(t)=\begin{cases} 0 & t<0 \\ 1 & t\geq 0 \end{cases}$$

The input signal is divided into two time intervals. When $t < 0$, $u(t) = 0$. The first-order differential equation becomes

$$u(t)=0=RC\frac{dv_h(t)}{dt}+v_h(t) \qquad t<0$$

You've already found the solution before time $t = 0$, because $v_h(t)$ is the solution to the homogeneous equation:

$$v_h(t)=c_1e^{-\frac{t}{RC}}$$

You determine the arbitrary constant c_1 after finding the particular solution and applying the initial condition V_0 of 0 volts.

Now find the particular solution $v_p(t)$ when $u(t) = 1$ after $t = 0$.

After time $t = 0$, a unit step input describes the transient voltage behavior across the capacitor. The capacitor voltage reacting to a step input is called the *step response.*

For a step input $v_T(t) = u(t)$, you have a first-order differential equation:

$$u(t)=RC\frac{dv(t)}{dt}+v(t)$$

You already know that the value of the step $u(t)$ is equal to 1 after $t = 0$. Substitute $u(t) = 1$ into the preceding equation:

$$1 = RC\frac{dv_p(t)}{dt} + v_p(t) \qquad t \geq 0$$

Solve for the capacitor voltage $v_p(t)$, which is the particular solution. The particular solution always depends on the actual input signal.

Because the input is a constant after $t = 0$, the particular solution $v_p(t)$ is assumed to be a constant V_A as well.

The derivative of a constant is 0, which implies the following:

$$\frac{d}{dt}(V_A) = 0$$

Now substitute $v_p(t) = V_A$ and its derivative into the first-order differential equation:

$$1 = RC\underbrace{\frac{dv_p(t)}{dt}}_{=0} + \underbrace{v_p(t)}_{=V_A} \qquad \rightarrow \qquad v_p(t) = 1 = V_A \qquad t \geq 0$$

After a relatively long period of time, the particular solution follows the unit step input with strength $V_A = 1$. In general, a step input with strength V_A or $V_A u(t)$ leads to a capacitor voltage of V_A.

After finding the homogeneous and particular solutions, you add up the two solutions to get the zero-state response $v_{ZS}(t)$. You find c_1 by applying the initial condition that's equal to 0.

Adding up the homogeneous solution and the particular solution, you have $v_{ZS}(t)$:

$$v_{ZS}(t) = v_h(t) + v_p(t)$$

Substituting in the homogeneous and particular solutions gives you

$$v_{ZS}(t) = c_1 e^{-\left(\frac{t}{RC}\right)} + V_A$$

At t = 0, the initial condition is $v_c(0) = 0$ for the zero-state response. You now calculate $v_{ZS}(0)$ as

$$\begin{aligned} v_{ZS}(0) &= c_1 e^{-\left(\frac{0}{RC}\right)} + V_A \\ 0 &= c_1 + V_A \end{aligned}$$

Next, solve for c_1:

$$c_1 = -V_A$$

Substitute c_1 into the zero-state equation to produce the complete solution of the zero-state response $v_{ZS}(t)$:

$$v_{ZS}(t) = V_A\left(1 - e^{-\frac{t}{RC}}\right)$$

Adding the zero-input and zero-state responses to find the total response

You finally add up the zero-input response $v_{ZI}(t)$ and the zero-state response $v_{ZS}(t)$ to get the total response $v(t)$:

$$\begin{aligned} v(t) &= v_{ZI}(t) + v_{ZS}(t) \\ &= V_0 e^{-\left(\frac{t}{RC}\right)} + V_A\left(1 - e^{-\left(\frac{t}{RC}\right)}\right) \end{aligned}$$

Time to verify whether the solution is reasonable. When t = 0, the initial voltage across the capacitor is

$$\begin{aligned} v(0) &= V_0 e^{-\left(\frac{0}{RC}\right)} + V_A\left(1 - e^{-\left(\frac{0}{RC}\right)}\right) \\ &= V_0 \end{aligned}$$

You bet this is a true statement! But you can check out when the initial conditions die out after a long period of time if you feel unsure about your solution. The output should just be related to the input voltage or step voltage.

After a long period of time (or after 5 time constants), you get the following:

$$v(\infty) = V_0 e^{-\left(\frac{\infty}{RC}\right)} + V_A\left(1 - e^{-\left(\frac{\infty}{RC}\right)}\right)$$
$$= V_A$$

Another true statement. The output voltage follows the step input with strength V_A after a long time. In other words, the capacitor voltage charges to a value equal to the strength V_A of the step input after the initial condition dies out in about 5 time constants.

Try this example with these values: V_0 = 5 volts, V_A = 10 volts, and RC = 1 second. You should get the capacitor voltage charging from an initial voltage of 5 volts and a final voltage of 10 volts after 5 seconds (5 time constants). Using the given values, you get the plot in Figure 13-4. The plot starts at 5 volts, and you end up at 10 volts after 5 time constants (5 seconds = 5RC). So this example shows how changing voltage states takes time. Circuits with capacitors don’t change voltages instantaneously. A large resistor also slows things down. That’s why the time constant RC takes into account how the capacitor voltage will change from one voltage state to another.

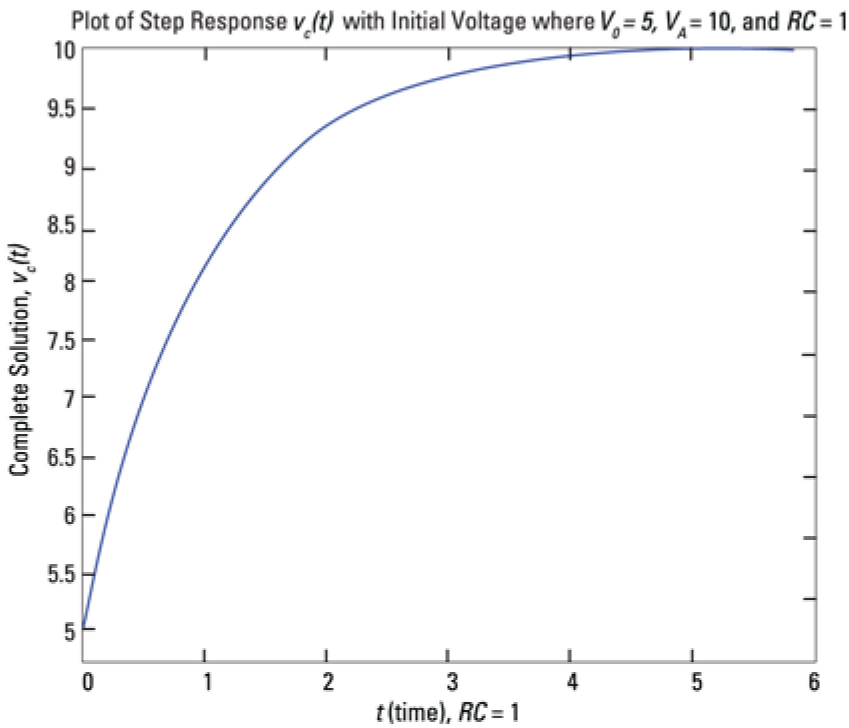

Illustration by Wiley, Composition Services Graphics

Figure 13-4: Total response of a simple first-order RC series circuit.

The total capacitor voltage consists of the zero-input response and a zero-state response:

$$v(t) = \underbrace{5e^{-\frac{t}{RC}}}_{\substack{\text{zero-input}\\ \text{response}}} + \underbrace{10\left(1 - e^{-\left(\frac{t}{RC}\right)}\right)}_{\substack{\text{zero-state}\\ \text{response}}}$$

The RC time constant

The following table shows the various output values of the capacitor voltage of a homogeneous RC series circuit given by multiple time constants. After 5 time constants, the output voltage decays to less than 1 percent of the initial voltage V_0.

Time t	Solution v(t) $v(t)=V_0e^{-\left(\frac{t}{RC}\right)}$	v(t) with Evaluation of the Natural Exponential Function
$t=0$	$v(0)=V_0e^{-\left(\frac{0}{RC}\right)}=V_0$	$v(0) = V_0$
$t=RC$	$v(RC)=V_0e^{-\left(\frac{RC}{RC}\right)}=V_0e^{-1}$	$v(RC) = 0.3679V_0$
$t=2RC$	$v(2RC)=V_0e^{-\left(\frac{2RC}{RC}\right)}=V_0e^{-2}$	$v(2RC) = 0.1353V_0$
$t=3RC$	$v(3RC)=V_0e^{-\left(\frac{3RC}{RC}\right)}=V_0e^{-3}$	$v(3RC) = 0.0498V_0$
$t=4RC$	$v(4RC)=V_0e^{-\left(\frac{4RC}{RC}\right)}=V_0e^{-4}$	$v(4RC) = 0.0183V_0$
$t=5RC$	$v(5RC)=V_0e^{-\left(\frac{5RC}{RC}\right)}=V_0e^{-5}$	$v(5RC) = 0.0067V_0$

This equation shows that the total response is a combination of two outputs added together: one output due only to the initial voltage V_0 = 5 volts (at time t = 0) and the other due only to the step input with strength V_A = 10 volts (after time t = 0).

Analyzing a Parallel Circuit with a Single Resistor and Inductor

One type of first-order circuit consists of a resistor (or a network of resistors) and a single inductor. Analyzing such a parallel RL circuit, like the one in Figure 13-5, follows the same process I describe for analyzing an RC series circuit. I walk you through each of the steps in the following sections.

If your RL parallel circuit has an inductor connected with a network of resistors rather than a single resistor, you can use the same approach to analyze the circuit. But you have to find the Norton equivalent first, reducing the resistor network to a single resistor in parallel with a single current source. I cover the Norton approach in Chapter 8.

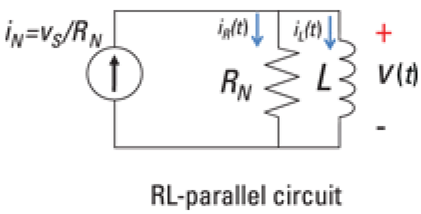

Figure 13-5: A first-order RL parallel circuit.

Starting with the simple RL parallel circuit

Because the resistor and inductor are connected in parallel in Figure 13-5, they must have the same voltage $v(t)$. The resistor current $i_R(t)$ is based on Ohm's law:

$$i_R(t) = \frac{v(t)}{R}$$

The element constraint for an inductor (see Chapter 12) is given as

$$v(t) = L\frac{di(t)}{dt}$$

where $i(t)$ is the inductor current and L is the inductance.

You need a changing current to generate voltage across an inductor. If the inductor current doesn't change, there’s no inductor voltage, which implies a short circuit.

Now substitute $v(t) = Ldi(t)/dt$ into Ohm’s law because you have the same voltage across the resistor and inductor:

$$i_R(t) = \left(\frac{L}{R}\right)\frac{di(t)}{dt}$$

Kirchhoff’s current law (KCL) says the incoming currents are equal to the outgoing currents at a node. Use KCL at Node A of Figure 13-5 to get

$$i_N(t) = i_R(t) + i(t).$$

Substitute $i_R(t)$ into the KCL equation to give you

$$i_N(t) = \left(\frac{L}{R}\right)\frac{di(t)}{dt} + i(t)$$

The RL parallel circuit is a first-order circuit because it’s described by a first-order differential equation, where the unknown variable is the inductor current $i(t)$. A circuit containing a single equivalent inductor and an equivalent resistor is a first-order circuit.

Knowing the inductor current gives you the magnetic energy stored in an inductor.

In general, the inductor current is referred to as a *state variable* because the inductor current describes the behavior of the circuit.

Calculating the zero-input response for an RL parallel circuit

Figure 13-6 shows how the RL parallel circuit is split up into two problems: the zero-input response and the zero-state response. This section starts off with the zero-input response, and the next section analyzes the zero-state response.

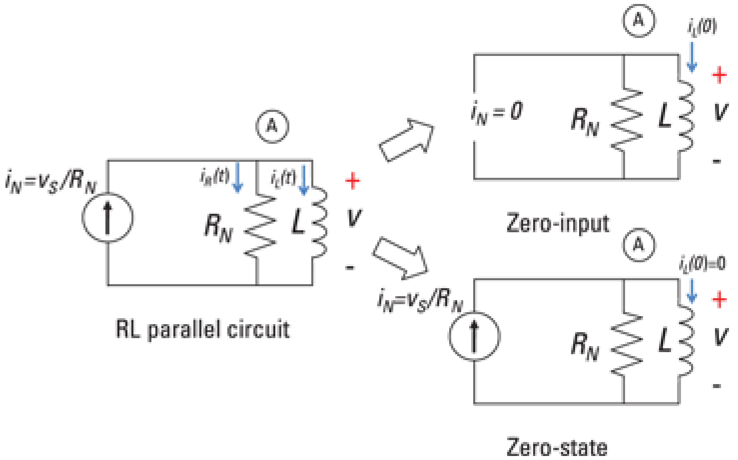

Figure 13-6: Zero-input and zero-state response for an RL parallel circuit.

To simplify matters, you set the input source (or forcing function) equal to 0: $i_N(t)$ = 0 amps. This means no input current for all time — a big, fat zero. The first-order differential equation reduces to

$$i_N(t) = 0 = \left(\frac{L}{R}\right)\frac{di_{ZI}(t)}{dt} + i_{ZI}(t) \quad \text{or} \quad i(t) = -\left(\frac{L}{R}\right)\frac{di_{ZI}(t)}{dt}$$

For an input source of no current, the inductor current i_{ZI} is called a *zero-input response*. No external forces are acting on the circuit except for its initial state (or inductor current, in this case). The output is due to some initial inductor current I_0 at time t = 0.

You make a reasonable guess at the solution (the natural exponential function!) and substitute your guess into the RL first-order differential equation. Assume the inductor current and solution to be

$$i_{ZI}(t) = Be^{kt}$$

This is a reasonable guess because the time derivative of an exponential is also an exponential. Like a good friend, the exponential function won’t let you down when solving these differential equations.

You determine the constants B and k next. Substitute your guess $i_{ZI}(t) = Be^{kt}$ into the differential equation:

$$i_{ZI}(t) = -\left(\frac{L}{R}\right)\frac{di_{ZI}(t)}{dt}$$

Replacing $i_{ZI}(t)$ with Be^{kt} and doing some math gives you the following:

$$\begin{aligned} Be^{kt} &= -\left(\frac{L}{R}\right)\frac{d}{dt}\left(Be^{kt}\right) \\ &= -\left(\frac{L}{R}\right)\left(kBe^{kt}\right) \end{aligned}$$

You have the characteristic equation after factoring out Be^{kt}:

$$Be^{kt}\left[1+\left(\frac{L}{R}\right)k\right]=0$$

The characteristic equation gives you an algebraic problem to solve for the constant k*:*

$$\begin{aligned} 1+\left(\frac{L}{R}\right)k &= 0 \\ k &= -\frac{R}{L} \end{aligned}$$

Use $k = –R/L$ and the initial inductor current I_0 at $t = 0$. This implies that $B = I_0$, so the zero-input response $i_{ZI}(t)$ gives you the following:

$$i_{ZI}(t) = I_0 e^{-\left(\frac{R}{L}\right)t}$$

The constant L/R is called the *time constant.* The time constant provides a measure of how long an inductor current takes to go to 0 or change from one state to another.

Calculating the zero-state response for an RL parallel circuit

Zero-state response means zero initial conditions. For the zero-state circuit in Figure 13-6, zero initial conditions means looking at the circuit with zero inductor current at $t < 0$. You need to find the homogeneous and particular solutions to get the zero-state response.

Next, you have zero initial conditions and an input current of $i_N(t) = u(t)$, where $u(t)$ is a unit step input.

When the step input $u(t) = 0$, the solution to the differential equation is the solution $i_h(t)$:

$$i_h(t) = c_1 e^{-\left(\frac{R}{L}\right)t}$$

The inductor current $i_h(t)$ is the solution to the homogeneous first-order differential equation:

$$u(t) = 0 = \left(\frac{L}{R}\right)\frac{di_h(t)}{dt} + i_h(t) \qquad t < 0$$

This solution is the general solution for the zero input. You find the constant c_1 after finding the particular solution and applying the initial condition of no inductor current.

After time $t = 0$, a unit step input describes the transient inductor current. The inductor current for this step input is called the *step response.* Not very creative, I know, but it does remind you of the step input.

You find the particular solution $i_p(t)$ by setting the step input $u(t)$ equal to 1. For a unit step input $i_N(t) = u(t)$, substitute $u(t) = 1$ into the differential equation:

$$u(t) = 1 = \left(\frac{L}{R}\right)\frac{di_p(t)}{dt} + i_p(t) \qquad t \geq 0$$

The particular solution $i_p(t)$ is the solution for the differential equation when the input is a unit step $u(t) = 1$ after $t = 0$.

Because $u(t) = 1$ (a constant) after time $t = 0$, assume a particular solution $i_p(t)$ is a constant I_A .

Because the derivative of a constant is 0, the following is true:

$$\frac{d}{dt}(I_A) = 0$$

Substitute $i_p(t) = I_A$ into the first-order differential equation:

$$1 = \left(\frac{R}{L}\right)\underbrace{\frac{di_p(t)}{dt}}_{=0} + \underbrace{i_p(t)}_{=I_A} \qquad \rightarrow \qquad i(t) = 1 = I_A \qquad t \geq 0$$

The particular solution eventually follows the form of the input because the zero-input (or free response) diminishes to 0 over time. You can generalize the result when the input step has strength I_A or $I_A u(t)$.

You need to add the homogeneous solution $i_h(t)$ and the particular solution $i_p(t)$ to get the zero-state response:

$$i_{ZS}(t) = i_h(t) + i_p(t)$$
$$i_{ZS}(t) = c_1 e^{-\left(\frac{R}{L}\right)t} + I_A$$

At $t = 0$, the initial condition is 0 because this is a zero-state calculation. To find c_1, apply $i_{ZS}(0) = 0$:

$$i_{ZS}(0) = c_1 e^{-\left(\frac{R}{L}\right)0} + I_A$$
$$0 = c_1 + I_A$$

Solving for c_1 gives you

$$c_1 = -I_A$$

Substituting c_1 into the zero-state response $i_{ZS}(t)$, you wind up with

$$i_{ZS}(t) = -I_A e^{-\left(\frac{R}{L}\right)t} + I_A$$
$$i_{ZS}(t) = I_A\left(1 - e^{-\left(\frac{R}{L}\right)t}\right)$$

Adding the zero-input and zero-state responses to find the total response

To obtain the total response for the RL parallel circuit, you need to add up the two solutions, the zero-input and zero-state responses:

$$i(t) = i_{ZI}(t) + i_{ZS}(t)$$

Substitute the zero-input and zero-state responses from the preceding sections into this equation, which gives you

$$i(t) = I_0 e^{-\left(\frac{R}{L}\right)t} + I_A\left(1 - e^{-\left(\frac{R}{L}\right)t}\right)$$

Check out the total response to verify the solution $i(t)$. When $t = 0$, the initial inductor current is

$$i(0) = I_0 e^{-\left(\frac{R}{L}\right)0} + I_A\left(1 - e^{-\left(\frac{R}{L}\right)0}\right)$$
$$i(0) = I_0$$

This is a true statement — for sure, for sure. If you're still not convinced, figure out when the initial condition dies out. The output should just be related to the input current or step current for this example.

After a long period of time (5 time constants), you get the following:

$$i(\infty) = I_0 e^{-\left(\frac{R}{L}\right)\infty} + I_A\left(1 - e^{-\left(\frac{R}{L}\right)\infty}\right)$$
$$i(\infty) = 0 + I_A(1 - 0)$$
$$i(\infty) = I_A$$

The output inductor current is just the step input having a strength of I_A. In other words, the inductor current reaches a value equal to the step input's strength I_A after the initial condition dies out in about 5 time constants of L/R, or $5L/R$. You see inductor currents don't change instantaneously. With inductors, currents change gradually in going from one state to another. A parallel resistor slows things down. That's why the time constant L/R takes into account how fast inductor currents change from one state to another.

The complete response of the inductor current follows the same shape of the capacitor voltage in Figure 13-4. The shape starts at some initial current and goes to another current state after 5 time constants.

The *L/R* time constant

For zero-input and initial current I_0, the output inductor current for a parallel RL circuit is

$$i(t) = I_0 e^{-\left(\frac{R}{L}\right)t}$$

The time constant is $t = L/R$. After 5 time constants, the output inductor current decays to less than 1 percent of the initial current I_0. The inductor current follows the same shape as the capacitor voltage given in Figure 13-3.

Chapter 14

Analyzing Second-Order Circuits

In This Chapter

- Focusing on second-order differential equations
- Analyzing an RLC series circuit
- Analyzing an RLC parallel circuit

Second-order circuits consist of capacitors, inductors, and resistors. In math terms, circuits that have both an inductor and a capacitor are described by second-order differential equations — hence the name *second-order circuits.* This chapter clues you in to what's unique about analyzing second-order circuits and then walks you through the analysis of an RLC (resistor, inductor, capacitor) series circuit and an RLC parallel circuit.

For a refresher on second-order differential equations, refer to your textbook or *Differential Equations For Dummies* by Steven Holzner (Wiley).

Examining Second-Order Differential Equations with Constant Coefficients

If you can use a second-order differential equation to describe the circuit you're looking at, then you're dealing with a second-order circuit. Circuits that include an inductor, capacitor, and resistor connected in series or in parallel are second-order circuits. Figure 14-1 shows second-order circuits driven by an input source, or forcing function.

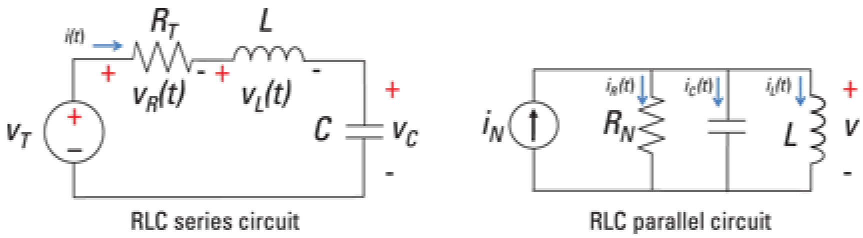

Illustration by Wiley, Composition Services Graphics

Figure 14-1: Examples of second-order circuits.

Getting a unique solution to a second-order differential equation requires knowing the initial states of the circuit. For a second-order circuit, you need to know the initial capacitor voltage and the initial inductor current. Knowing these states at time $t = 0$ provides you with a unique solution for all time after time $t = 0$.

Use these steps when solving a second-order differential equation for a second-order circuit:

1. **Find the zero-input response by setting the input source to 0, such that the output is due only to initial conditions.**
2. **Find the zero-state response by setting the initial conditions equal to 0, such that the output is due only to the input signal.**

   Zero initial conditions means you have 0 initial capacitor voltage and 0 initial inductor current.

   The zero-state response requires you to find the homogeneous and particular solutions:

- **Homogeneous solution:** When there's no input signal or forcing function — that is, when $v_T(t) = 0$ or $i_N(t) = 0$ — you have the *homogeneous solution.*
- **Particular solution:** When you have a nonzero input, the solution follows the form of the input signal, giving you the *particular solution.* For example, if your input is a constant, then your particular solution is also a constant. Likewise, if you have a sine or cosine function as an input, then the output is a combination of sine and cosine functions.

3. **Add up the zero-input and zero-state responses to get the total response.**

   Because you're dealing with linear circuits, you want to use superposition to find the total response. I show you the superposition technique in Chapter 7.

In the following sections, I show you how to find the total response for a second-order differential equation with constant coefficients. I first find the homogeneous solution by using an algebraic characteristic equation and assuming the solutions are exponential functions. The roots to the characteristic equation give you the constants found in the exponent of the exponential function.

Later in this chapter, I analyze an RLC series circuit by applying the preceding steps to get the total response. I set up the appropriate equations using Kirchhoff's voltage law (KVL) and device equations for a capacitor and inductor. Then I determine the zero-input response followed by the calculation of the zero-state response. Finally, I analyze an RLC parallel circuit using the concept of duality, which replaces quantities with their dual quantities. The resulting equations for an RLC parallel circuit are similar to the equations for an RLC series circuit.

Guessing at the elementary solutions: The natural exponential function

I'm giving you just one approach to solving second-order circuits. The good news is that it converts a problem involving a differential equation to one that uses only algebra.

Consider the following differential equation as a numerical example with zero forcing function $v_T(t) = 0$:

$$\frac{d^2v}{dt^2} + 5\frac{dv}{dt} + 6v = 0$$

The solution to this differential equation is called the *homogeneous solution* $v(t)$. One classic approach entails giving your best shot at guessing the solution. Try $v(t) = e^{kt}$. The exponential function works for a first-order equation, so it should work for a second-order equation, too. When you take the derivative of the natural exponential e^{kt}, you get the same thing multiplied by some constant k. You see how the exponential function is your true amigo in solving differential equations like this.

From calculus to algebra: Using the characteristic equation

To solve a homogeneous differential equation, you can convert the differential equation into a characteristic equation, which you solve using algebra. You do this by substituting your guess $v(t) = e^{kt}$ (from the preceding section) into the homogeneous differential equation:

$$\frac{d^2v}{dt^2}+5\frac{dv}{dt}+6v=0$$

$$\frac{d^2}{dt^2}\left(e^{kt}\right)+5\frac{d}{dt}\left(e^{kt}\right)+6e^{kt}=0$$

Factoring out e^{kt} leads you to a characteristic equation:

$$\left(k^2+5k+6\right)e^{kt}=0$$

The coefficient of e^{kt} must be 0, so you can solve for k as follows:

$$k^2+5k+6=0$$
$$k=-2,-3$$

Setting the algebraic equation to 0 gives you a *characteristic equation.* The constant roots –2 and –3 determine the features of the solution $v(t)$.

From these roots, you get a homogeneous solution that's a combination of the solutions e^{-2t} and e^{-3t}:

$$v(t)=c_1e^{-2t}+c_2e^{-3t}$$

The constants c_1 and c_2 are determined by the initial conditions when $t = 0$.

Analyzing an RLC Series Circuit

One second-order circuit consists of a resistor (or network of resistors) hooked up in series with both a capacitor and an inductor. The left diagram in Figure 14-2 shows you what such an RLC series circuit looks like. The rest of Figure 14-2 shows you the RLC series circuit broken into two circuits: One deals with the initial condition, and the other deals with the input source. The top-right diagram shows the zero-input response, setting the input to 0, and the bottom-right diagram deals with the zero-state response, setting the initial conditions to 0.

The following sections walk you through the analysis process for an RLC series circuit.

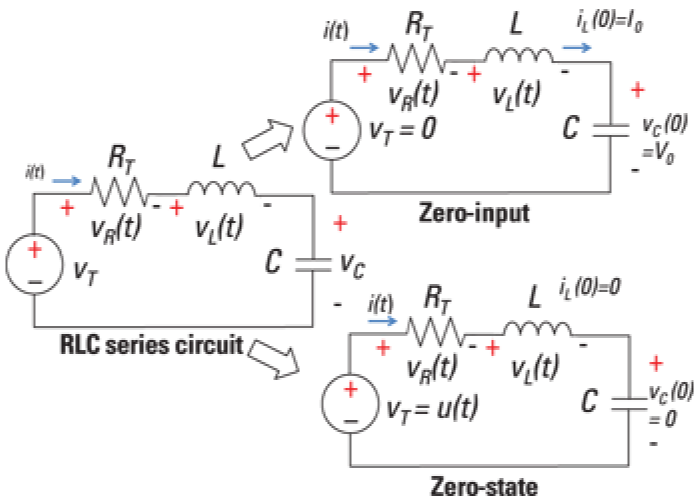

Illustration by Wiley, Composition Services Graphics

Figure 14-2: A second-order RLC series circuit broken into circuits to help you find the zero-input response and zero-state response.

Setting up a typical RLC series circuit

The simple RLC series circuit in Figure 14-2 is driven by a voltage source. Kirchhoff's voltage law (KVL) says the sum of the voltage drops and rises around a loop of a circuit is equal to 0. Using KVL for this circuit gives you the following:

$$v_T(t) = v_R(t) + v_L(t) + v_C(t)$$

The subscript letter *R* is for the resistor, *L* is for the inductor, and *C* is for the capacitor — pretty straightforward, huh? Because these devices are connected in series, they have the same current $i(t)$.

Next, you want to put the resistor voltage and the inductor voltage in terms of the capacitor voltage or its derivative. The voltage across the resistor $v_R(t)$ uses Ohm's law:

$$v_R(t) = Ri(t)$$

The element constraint for an inductor voltage $v_L(t)$ is

$$v_L(t) = L\frac{di(t)}{dt}$$

And for a capacitor current $i(t)$, the device constraint is

$$i(t) = C\frac{dv_C(t)}{dt}$$

Because the series current $i(t)$ flows through each device, you can substitute the capacitor current into the equations for the resistor voltage and inductor voltage. First substitute the capacitor current into the inductor voltage equation:

$$v_L(t) = L\frac{di(t)}{dt}$$
$$v_L(t) = LC\frac{d^2v_C(t)}{dt^2}$$

Next, plug the capacitor current $i(t) = Cdv_C(t)/dt$ into Ohm's law to get the resistor voltage $v_R(t)$:

$$v_R(t) = RC\frac{dv_C(t)}{dt}$$

Now you can plug $v_R(t)$ and $v_L(t)$ into the KVL equation, giving you all the device voltages in terms of the capacitor voltage (or its derivatives):

$$v_T(t) = LC\frac{d^2v_C}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t)$$

You now have a second-order differential equation where the unknown function is the capacitor voltage $v_C(t)$. Knowing $v_C(t)$ gives you the electrical energy stored in the capacitor or the capacitor's state of charge.

In general, the capacitor voltage and inductor current are referred to as *state variables* because these quantities describe the behavior of the circuit at any time.

The RLC series circuit is a second-order circuit because it has two energy-storage devices. It can be described by a second-order differential equation having an applied input voltage $v_T(t)$.

Determining the zero-input response

The top-right diagram of Figure 14-2 shows the input signal $v_T(t) = 0$, which gives you the zero-input response. With the zero-input response, you have no input voltage for all time. This response comes from initial capacitor voltage V_0 and initial inductor current I_0 at time $t = 0$. With zero input, the second-order differential equation reduces to

$$v_T(t) = 0 = LC\frac{d^2 v_{ZI}(t)}{dt^2} + RC\frac{dv_{ZI}(t)}{dt} + v_{ZI}(t)$$

The capacitor voltage is called a *zero-input response,* $v_{ZI}(t)$. The top-right diagram of Figure 14-2 shows the input source set to 0 volts. No external forces (a battery, for example) are acting on the circuit except for its initial states, expressed by the capacitor voltage and inductor current.

You make a reasonable guess at the solution to $v_{ZI}(t)$: the exponential function (see the earlier section "Guessing at the elementary solutions: The natural exponential function"). Substitute the guess into the RLC second-order circuit equation. You can check and verify the solution afterward.

Assume the capacitor voltage and solution to be

$$v_{ZI}(t) = Ae^{kt}$$

The *A* and *k* are arbitrary constants of the zero-input response. You try an exponential function because the time derivative of an exponential is also an exponential.

Substitute the solution $v_{ZI}(t) = Ae^{kt}$ into the differential equation and simplify:

$$LC\frac{d^2v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = 0$$

$$LC\frac{d^2}{dt^2}(Ae^{kt}) + RC\frac{d}{dt}(Ae^{kt}) + Ae^{kt} = 0$$

$$LCk^2Ae^{kt} + RCkAe^{kt} + Ae^{kt} = 0$$

Factoring out Ae^{kt} gives you the algebraic characteristic equation (I also factor out *LC* so that the leading coefficient on k^2 is 1):

$$Ae^{kt}\left(LCk^2 + RCk + 1\right) = 0$$

$$LC\left(Ae^{kt}\right)\left(k^2 + \frac{R}{L}k + \frac{1}{LC}\right) = 0$$

You've transformed the differential equation into an algebraic one. The coefficient of e^{kt} has to equal 0, so use that info to solve for the constant *k:*

$$k^2 + \frac{R}{L}k + \frac{1}{LC} = 0$$

$$k_1,\ k_2 = -\frac{R}{2L} \pm \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}$$

You now have three possible cases and roots under the radical:

$$\text{Case 1: } \left[\left(\frac{R}{L}\right)^2 - \frac{4}{LC}\right] > 0 \;\rightarrow\; \text{two different real roots } (\alpha_1 \text{ and } \alpha_2)$$

$$\text{Case 2: } \left[\left(\frac{R}{L}\right)^2 - \frac{4}{LC}\right] = 0 \;\rightarrow\; \text{two equal real roots } (\alpha_1 = \alpha_2)$$

$$\text{Case 3: } \left[\left(\frac{R}{L}\right)^2 - \frac{4}{LC}\right] < 0 \;\rightarrow\; \text{two complex conjugate roots } (\alpha \pm j\beta)$$

Loosely speaking, Case 1 implies a large resistor R losing lots of energy as heat, with the initial states eventually dying out. Case 3 implies that the resistor is small, where stored energy is being exchanged between the capacitor and inductor. With a sinusoidal input, the stored energy switches between the electrical energy in the capacitor and the magnetic energy in the inductor. This back-and-forth sloshing of energy causes oscillations at the output. Case 2, with two equal and real roots, falls between these two behaviors. Case 2 achieves a faster response than Case 1 but doesn't suffer from the oscillations found in Case 3.

Later in this chapter, Figure 14-3 illustrates the effects of decreasing resistance when you have zero-input response, and Figure 14-4 illustrates the effects of decreasing resistance when you have a step response. Without getting into the analytical detail of how you arrive at these curves, you should observe that for decreasing resistance, you have increasing amplitude of oscillations.

When you know k_1 and k_2, you have the zero-input response $v_{ZI}(t)$. The response $v_{ZI}(t)$ comes from a combination of the two solutions:

$$v_{ZI}(t) = c_1 e^{k_1 t} + c_2 e^{k_2 t}$$

You find the constants c_1 and c_2 by applying two initial conditions: inductor current $i_L(0) = I_0$ and capacitor voltage $v_C(0) = V_0$. At time $t = 0$, the initial capacitor voltage is V_0, and you have

$$v_{ZI}(t) = v_C(t) = c_1 + c_2 = V_0$$

The same current flows through the inductor and the capacitor. You find the inductor's initial current based on the initial condition of the first derivative of capacitor voltage:

$$\left.\frac{dv_C(t)}{dt}\right|_{t=0} = \frac{dv_C(0)}{dt} = \frac{i(0)}{C} = \frac{I_0}{C}$$

Taking the derivative of $v_C(t) = v_{ZI}(t)$ gives you the following:

$$\left.\frac{dv_{ZI}(0)}{dt}\right|_{t=0} = k_1c_1 + k_2c_2 = \frac{I_0}{C}$$

Apply the two initial conditions to give you two equations having two unknowns, c_1 and c_2:

$$v_C(0) = V_0 \quad \rightarrow \quad c_1 + c_2 = V_0$$
$$\frac{dv_C(0)}{dt} = \frac{I_0}{C} \quad \rightarrow \quad k_1c_1 + k_2c_2 = \frac{I_0}{C}$$

Then solve for c_1 and c_2:

$$c_1 = \frac{k_2V_0 - I_0 / C}{k_2 - k_1}$$
$$c_2 = \frac{-k_1V_0 + I_0 / C}{k_2 - k_1}$$

The roots k_1 and k_2 of the characteristic equation reveal the form of the zero-input response $v_C(t) = v_{ZI}(t)$. Based on these roots, you have different solutions for the capacitor voltage $v_C(t)$, which is the *zero-input response* $v_{ZI}(t)$:

$$\text{Case 1: } v_C(t) = c_1e^{\alpha_1 t} + c_2e^{\alpha_2 t} \rightarrow \text{ two different real roots } (\alpha_1 \text{ and } \alpha_2)$$
$$\text{Case 2: } v_C(t) = c_1e^{\alpha t} + c_2te^{\alpha t} \rightarrow \text{ two equal real roots } (\alpha \text{ and } \alpha)$$
$$\text{Case 3: } v_C(t) = c_1e^{\alpha t}\cos\beta t + c_2e^{\alpha t}\sin\beta\text{t} \rightarrow \text{ complex roots } (\alpha \pm j\beta)$$

Figure 14-3 shows three zero-input responses for various values of resistors. Note that real roots mean damping exponentials, and complex roots indicate oscillations. Case 1 (overdamped) doesn't give you oscillations, but the initial conditions die out the most slowly. For Case 3 (underdamped), you get to the desired state faster, but you have oscillations. Case 2 (critically damped) falls between Cases 1 and 3, with faster response than Case 1 and little or none of the oscillations found in Case 3.

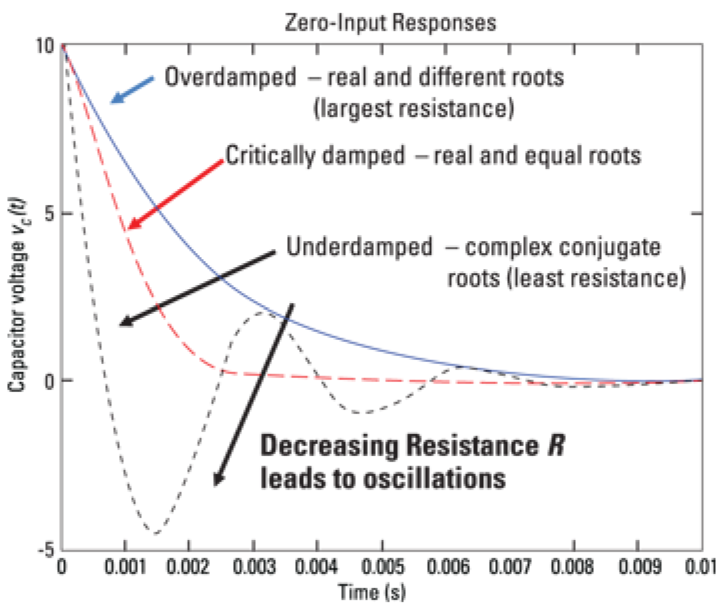

Illustration by Wiley, Composition Services Graphics

Figure 14-3: Zero-input response for varying resistance.

Calculating the zero-state response

Zero-state response means the response of a system under zero initial conditions, implying 0 capacitor voltage and 0 inductor current. When there's an input source $v_T(t)$, you need to find the solution to the homogeneous differential equation and the solution to the differential equation for a particular input.

The circuit in the bottom-right diagram of Figure 14-2 has zero initial conditions and an input voltage of $v_T(t) =$

$u(t)$, where $u(t)$ is a unit step input (I introduce unit step functions in Chapter 11). Mathematically, you can describe a step function $u(t)$ as

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

The input signal is divided into two time intervals. When $t < 0$, $u(t) = 0$. In terms of the capacitor voltage $v(t)$, the second-order differential equation becomes

$$LC\frac{d^2v(t)}{dt^2} + RC\frac{dv(t)}{dt} + v(t) = 0 \qquad t < 0$$

Where $u(t) = 0$ for before time $t = 0$, you have the homogeneous solution $v_h(t)$ when the input is 0:

$$v_h(t) = C_1 e^{k_1 t} + C_2 e^{k_2 t}$$

You determine the arbitrary constants C_1 and C_2 after finding the particular solution and applying the initial condition V_0 of 0 volts. You find the particular solution $v_p(t)$ when $u(t) = 1$ after $t = 0$.

For a step input $v_T(t) = u(t)$, you have the following second-order differential equation:

$$u(t) = LC\frac{d^2v(t)}{dt^2} + RC\frac{dv(t)}{dt} + v(t) \qquad t \geq 0$$

After $t = 0$, the value of the step input $u(t)$ is equal to 1. Substitute $u(t) = 1$ into the preceding equation:

$$1 = LC\frac{d^2v_p(t)}{dt^2} + RC\frac{dv_p(t)}{dt} + v_p(t) \qquad t \geq 0$$

Solve for the capacitor voltage $v_p(t)$ to get the *particular solution,* or *forced response.* The particular solution always depends on the actual input signal.

Because the input is constant after $t = 0$, the particular solution $v_p(t)$ is assumed to be a constant, V_A, as well.

The derivative of a constant is 0:

$$\frac{d}{dt}(V_A) = 0$$

Substitute $v_p(t) = V_A$ and its derivative into the second-order differential equation:

$$1 = LC\underbrace{\frac{d^2 v_p(t)}{dt^2}}_{=0} + RC\underbrace{\frac{dv_p(t)}{dt}}_{0} + v_p(t) \qquad \rightarrow \qquad v_p(t) = 1 = V_A \qquad t \geq 0$$

The particular solution eventually follows the step input after a relatively long period of time. In general, a step input with strength V_A or $V_A u(t)$ leads to a capacitor voltage of V_A.

After finding the solution due to the homogeneous differential equation and the solution for a particular input, you add up the two solutions to get the zero-state response $v_{ZS}(t)$. You find C_1 by applying the zero initial condition.

Adding up the two solutions gives you the zero-state response $v_{ZS}(t)$:

$$v_{ZS}(t) = v_h(t) + v_p(t)$$

Substituting the two solutions into this equation gives you the following:

$$v_{ZS}(t) = C_1 e^{k_1 t} + C_2 e^{k_2 t} + V_A$$

$$\frac{dv_{ZS}(t)}{dt} = C_1 k_1 e^{k_1 t} + C_2 k_2 e^{k_2 t}$$

By definition, at $t = 0$, the initial conditions for a circuit in a zero state is $v_C(0) = i_L(0) = 0$. The zero-state response $v_{ZS}(0)$ is

$$v_{ZS}(0) = 0 = C_1 + C_2 + V_A \quad \rightarrow \quad -V_A = C_1 + C_2$$
$$\frac{dv_{ZS}(0)}{dt} = 0 = C_1k_1 + C_2k_2$$

Solve for C_1 and C_2:

$$C_1 = \frac{k_2V_A}{k_1 - k_2} \text{ and } C_2 = \frac{k_1V_A}{k_2 - k_1}$$

Finishing up with the total response

Add up the zero-input response $v_{ZI}(t)$ and the zero-state response $v_{ZS}(t)$ to get the total response $v(t)$:

$$v(t) = v_{ZI}(t) + v_{ZS}(t)$$
$$v(t) = c_1e^{k_1t} + c_2e^{k_2t} + C_1e^{k_1t} + C_2e^{k_2t} + V_A$$

Do you get good vibes from this solution? If not, you need to verify that the solution is reasonable. When t = 0, the initial voltage across the capacitor is

$$v(0) = c_1e^{-k_1 \cdot 0} + c_2e^{-k_2 \cdot 0} + C_1e^{-k_1 \cdot 0} + C_2e^{-k_2 \cdot 0} + V_A$$
$$= V_0$$

You can substitute c_1, c_2, C_1, and C_2 into this equation (based on the analysis in the previous sections) to confirm that this statement is true.

Next, check out the initial inductor current when you take the derivative of $v(t)$ and evaluate the derivative at t = 0:

$$\left.\frac{dv(0)}{dt}\right|_{t=0} = \frac{dv(0)}{dt} = c_1k_1e^{-k_1 \cdot 0} + c_2k_2e^{-k_2 \cdot 0} + C_1k_1e^{-k_1 \cdot 0} + C_2k_2e^{-k_2 \cdot 0}$$
$$= \frac{I_0}{C}$$

That’s another true statement. If you’re still not feeling good about your solution, look at when the initial conditions die out after a long period of time. The output should just be the step voltage. After a long period of time (or after 5 time constants), you get the following:

$$v(\infty) = c_1 e^{-k_1 \cdot \infty} + c_2 e^{-k_2 \cdot \infty} + C_1 e^{-k_1 \cdot \infty} + C_2 e^{-k_2 \cdot \infty} + V_A$$
$$= V_A$$

Another true statement! The output voltage follows the step input with strength V_A after an extended time. In other words, the capacitor voltage is equal to the strength V_A of the step input after the initial conditions die out.

Figure 14-4 shows several step responses for zero initial conditions for various values of decreasing resistance R. In this example, the step input has a strength of 10 volts. See how all the step responses end up at 10 volts after the time-varying output dies out. Although you initially reach the final value faster with decreasing resistance, you may end up with undesirable wavy behavior.

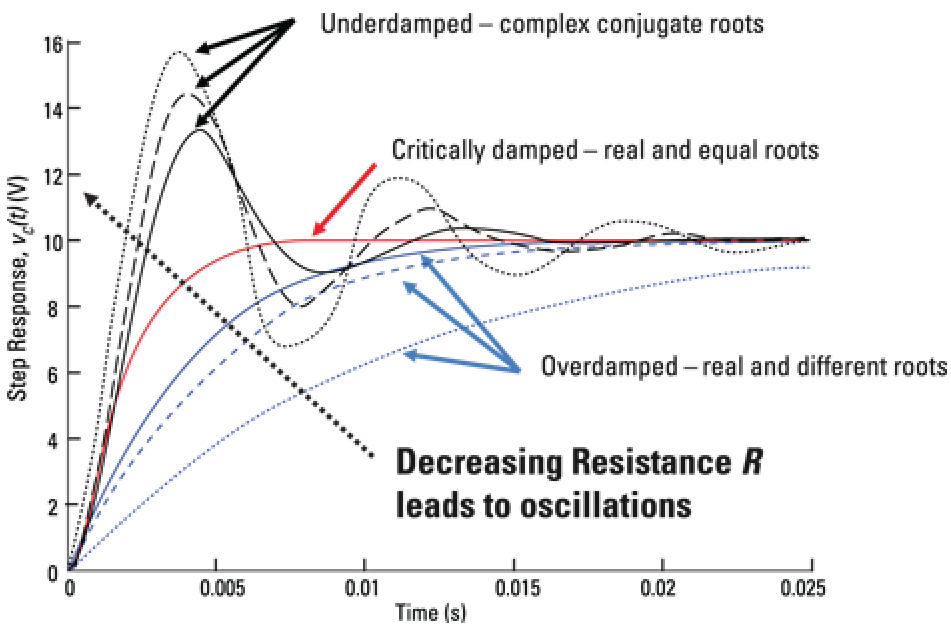

Illustration by Wiley, Composition Services Graphics

Figure 14-4: Step responses for increasing values of resistances.

Analyzing an RLC Parallel Circuit Using Duality

One type of second-order circuit has a resistor, inductor, and capacitor connected in parallel. Check out the example RLC parallel circuit in Figure 14-5. To analyze this second-order circuit, you use basically the same process as for analyzing an RLC series circuit (see the preceding sections).

The left diagram of Figure 14-5 shows an input i_N with initial inductor current I_0 and capacitor voltage V_0. The top-right diagram shows the input current source i_N set equal to zero, which lets you solve for the zero-input response. The bottom-right diagram shows the initial conditions (I_0 and V_0) set equal to zero, which lets you obtain the zero-state response.

In the following sections, I show you how you can use the concept of duality to obtain results similar to the ones you find in an RLC series circuit. With *duality,* you substitute every electrical term in an equation with its dual, or counterpart, and get another correct equation. For example, voltage and current are dual variables.

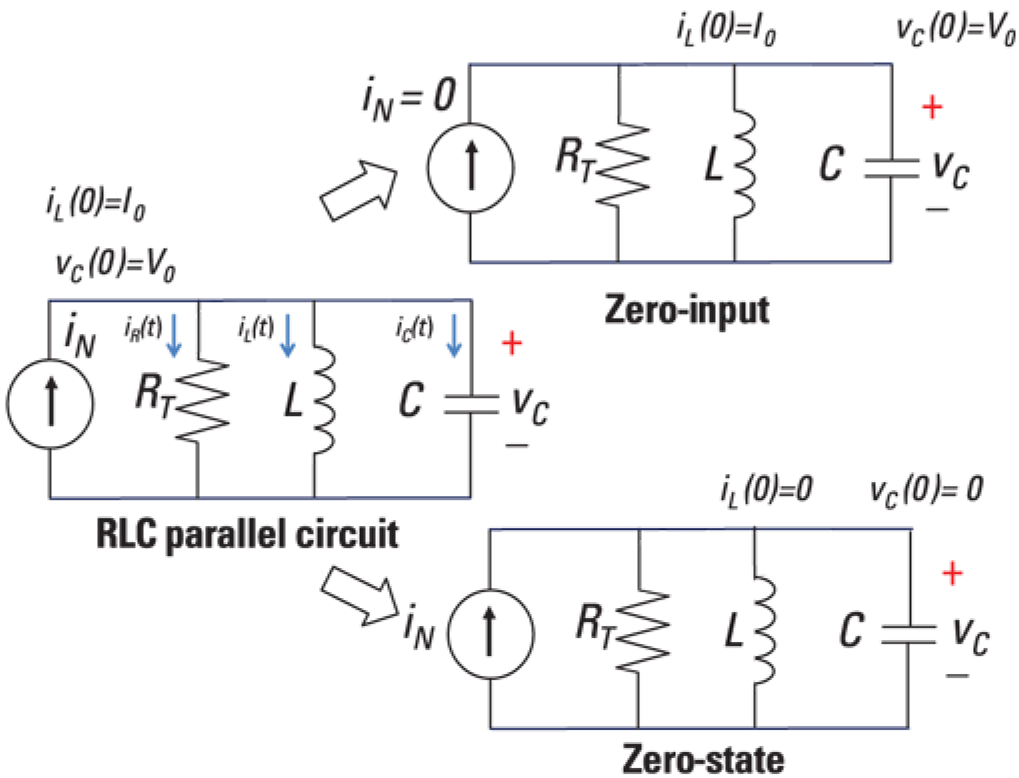

Illustration by Wiley, Composition Services Graphics

Figure 14-5: A second-order RLC parallel circuit broken into circuits to help you find the zero-input response and zero-state response.

Setting up a typical RLC parallel circuit

Because the components of the circuit in Figure 14-5 are connected in parallel, you set up the second-order differential equation by using Kirchhoff's current law (KCL). KCL says the sum of the incoming currents equals the sum of the outgoing currents at a node. Using KCL at Node A of Figure 14-5 gives you

$$i_N(t) = i_R(t) + i_C(t) + i_L(t)$$

Next, put the resistor current and capacitor current in terms of the inductor current. The resistor current $i_R(t)$ is based on the old, reliable Ohm's law:

$$i_R(t) = \frac{v(t)}{R}$$

The element constraint for an inductor is given as

$$v(t) = L\frac{di(t)}{dt}$$

The current $i_L(t)$ is the inductor current, and L is the inductance. This constraint means a changing current generates an inductor voltage. If the inductor current doesn't change, there's no inductor voltage, implying a short circuit.

Parallel devices have the same voltage $v(t)$. You use the inductor voltage $v(t)$ that's equal to the capacitor voltage to get the capacitor current $i_C(t)$:

$$i_C(t) = C\frac{dv(t)}{dt} = LC\frac{d^2 i_L(t)}{dt^2}$$

Now substitute $v(t) = Ldi_L(t)/dt$ into Ohm's law, because you also have the same voltage across the resistor and inductor:

$$i_R(t) = \left(\frac{L}{R}\right)\frac{di_L(t)}{dt}$$

Substitute the values of $i_R(t)$ and $i_C(t)$ into the KCL equation to give you the device currents in terms of the inductor current:

$$i_N(t) = LC\frac{d^2 i_L(t)}{dt^2} + \left(\frac{L}{R}\right)\frac{di_L(t)}{dt} + i_L(t)$$

The RLC parallel circuit is described by a second-order differential equation, so the circuit is a second-order circuit. The unknown is the inductor current $i_L(t)$.

The analysis of the RLC parallel circuit follows along the same lines as the RLC series circuit. Compare the preceding equation with the second-order equation derived from the RLC series circuit (see the earlier section "Calculating the zero-state response" for details):

$$v_T(t) = LC\frac{d^2v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t)$$

The two differential equations have the same form. The unknown solution for the parallel RLC circuit is the inductor current, and the unknown for the series RLC circuit is the capacitor voltage. These unknowns are dual variables.

> With *duality*, you can replace every electrical term in an equation with its dual and get another correct equation. If you use the following substitution of variables in the differential equation for the RLC series circuit, you get the differential equation for the RLC parallel circuit.

$$v_C(t) \leftrightarrow i_L(t)$$
$$v_T(t) \leftrightarrow i_N(t)$$
$$L \leftrightarrow \mathrm{C}$$
$$\mathrm{C} \leftrightarrow L$$
$$R \leftrightarrow G = 1/R$$
$$\text{series} \leftrightarrow \text{parallel}$$
$$\text{KVL} \leftrightarrow \text{KCL}$$

Duality allows you to simplify your analysis when you know prior results. Yippee!

Finding the zero-input response

The results you obtain for an RLC parallel circuit are similar to the ones you get for the RLC series circuit (I cover that series circuit earlier in “Analyzing an RLC Series Circuit”).

As shown in the earlier section “Guessing at the elementary solutions: The natural exponential function,” you have a characteristic equation to the homogeneous equation. For a parallel circuit, you have a second-order and homogeneous differential equation given in terms of the inductor current:

$$0 = \frac{d^2 i_L}{dt^2} + \left(\frac{1}{RC}\right)\frac{di_L(t)}{dt} + \left(\frac{1}{LC}\right)i_L(t)$$

$$k^2 + \frac{1}{RC}k + \frac{1}{LC} = 0 \qquad \rightarrow \qquad k_1,\, k_2 = -\frac{1}{2RC} \pm \frac{1}{2}\sqrt{\left(\frac{1}{RC}\right)^2 - \frac{4}{LC}}$$

The preceding equation gives you three possible cases under the radical:

$$\text{Case 1:}\quad \left[\left(\frac{1}{RC}\right)^2 - \frac{4}{LC}\right] > 0 \;\rightarrow\; \text{two different real roots } (\alpha_1 \text{ and } \alpha_2)$$

$$\text{Case 2:}\quad \left[\left(\frac{1}{RC}\right)^2 - \frac{4}{LC}\right] = 0 \;\rightarrow\; \text{two equal real roots } (\alpha_1 = \alpha_2)$$

$$\text{Case 3:}\quad \left[\left(\frac{1}{RC}\right)^2 - \frac{4}{LC}\right] < 0 \;\rightarrow\; \text{two complex conjugate roots } (\alpha \pm j\beta)$$

The zero-input responses of the inductor responses resemble the form in Figure 14-3, which describes the capacitor voltage.

When you have k_1 and k_2, you have the zero-input response $i_{ZI}(t)$. The solution gives you

$$i_{ZI}(t) = c_1 e^{k_1 t} + c_2 e^{k_2 t}$$

You can find the constants c_1 and c_2 by using the results found in the RLC series circuit, which are given as

$$c_1 = \frac{k_2 V_0 - I_0 / C}{k_2 - k_1}$$

$$c_2 = \frac{-k_1 V_0 + I_0 / C}{k_2 - k_1}$$

Apply duality to the preceding equation by replacing the voltage, current, and inductance with their duals (current, voltage, and capacitance) to get c_1 and c_2 for the RLC parallel circuit:

$$c_1 = \frac{k_2 I_0 - V_0 / L}{k_2 - k_1}$$

$$c_2 = \frac{-k_1 I_0 + V_0 / L}{k_2 - k_1}$$

After you plug in the dual variables, finding the constants c_1 and c_2 is easy.

Arriving at the zero-state response

Zero-state response means zero initial conditions. You need to find the homogeneous and particular solutions of the inductor current when there's an input source $i_N(t)$. Zero initial conditions means looking at the circuit when there's 0 inductor current and 0 capacitor voltage.

When $t < 0$, $u(t) = 0$. The second-order differential equation becomes the following, where $i_L(t)$ is the inductor current:

$$i_N(t) = LC\frac{d^2 i_L(t)}{dt^2} + \left(\frac{L}{R}\right)\frac{di_L(t)}{dt} + i_L(t) \qquad t < 0$$

For a step input where $u(t) = 0$ before time $t = 0$, the homogeneous solution $i_h(t)$ is

$$i_h(t) = C_1 e^{k_1 t} + C_2 e^{k_2 t}$$

Adding the homogeneous solution to the particular solution for a step input $I_A u(t)$ gives you the zero-state response $i_{ZS}(t)$:

$$i_{ZS}(t) = i_h(t) + \underbrace{i_p(t)}_{I_A}$$

Now plug in the values of $i_h(t)$ and $i_p(t)$:

$$i_{ZS}(t) = C_1 e^{k_1 t} + C_2 e^{k_2 t} + I_A$$

Here are the results of C_1 and C_2 for the RLC series circuit:

$$C_1 = \frac{k_2 V_A}{k_1 - k_2} \text{ and } C_2 = \frac{k_1 V_A}{k_2 - k_1}$$

You now apply duality through a simple substitution of terms in order to get C_1 and C_2 for the RLC parallel circuit:

$$C_1 = \frac{k_2 I_A}{k_1 - k_2} \text{ and } C_2 = \frac{k_1 I_A}{k_2 - k_1}$$

Getting the total response

You finally add up the zero-input response $i_{ZI}(t)$ and the zero-state response $i_{ZS}(t)$ to get the total response $i_L(t)$:

$$i_L(t) = i_{ZI}(t) + i_{ZS}(t)$$
$$i_L(t) = c_1 e^{k_1 t} + c_2 e^{k_2 t} + C_1 e^{k_1 t} + C_2 e^{k_2 t} + I_A$$

The solution resembles the results for the RLC series circuit. Also, the step responses of the inductor current follow the same form as the ones shown in the step responses found in Figure 14-4 for the capacitor voltage.