Part III

Turning to Angular Momentum and Spin

"Great–differential equations brought us Newton's Law of Universal Gravitation, Maxwell's field equations, and now Stuart's Rate of Hair Loss."

In this part . . .

Things that spin and rotate — that's the topic of this part. Quantum physics has all kinds of things to say about how angular momentum and spin are quantized, and you see it all in this part.

Chapter 5

Working with Angular Momentum on the Quantum Level

In This Chapter

- Angular momentum
- Angular momentum and the Hamiltonian
- Matrix representation of angular momentum
- Eigenfunctions of angular momentum

In classical mechanics, you may measure angular momentum by attaching a golf ball to a string and whirling it over your head. In quantum mechanics, think in terms of a single molecule made up of two bound atoms rotating around each other. That's the level at which quantum mechanical effects become noticeable. And at that level, it turns out that angular momentum is quantized. And since that has tangible results in many cases, such as the spectrum of excited atoms, it's an important topic.

Besides having kinetic and potential energy, particles can also have *rotational energy*. Here's what the Hamiltonian (total energy — see Chapter 4) looks like:

$$H = \frac{L^2}{2I}$$

Here, L is the angular momentum operator and I is the rotation moment of inertia. What are the eigenstates of angular momentum? If L is the angular momentum operator, and l is an eigenvalue of L, then you could write the following:

$$H|l> = \frac{L^2}{2I}|l> \quad \text{Incomplete!}$$

But that turns out to be incomplete because angular momentum is a vector in three-dimensional space — and it can be pointing any direction. Angular momentum is typically given by a magnitude and a component in one

direction, which is usually the Z direction. So in addition to the magnitude *l,* you also specify the component of L in the Z direction, L_z (the choice of Z is arbitrary — you can just as easily use the X or Y direction).

If the quantum number of the Z component of the angular momentum is designated by *m,* then the complete eigenstate is given by $|l, m>$, so the equation becomes the following:

$$H|l,m>=\frac{L^2}{2I}|l,m>$$

That's the kind of discussion about eigenstates that I cover in this chapter, and I begin with a discussion of angular momentum.

Ringing the Operators: Round and Round with Angular Momentum

Take a look at Figure 5-1, which depicts a disk rotating in 3D space. Because you're working in 3D, you have to go with vectors to represent both magnitude and direction.

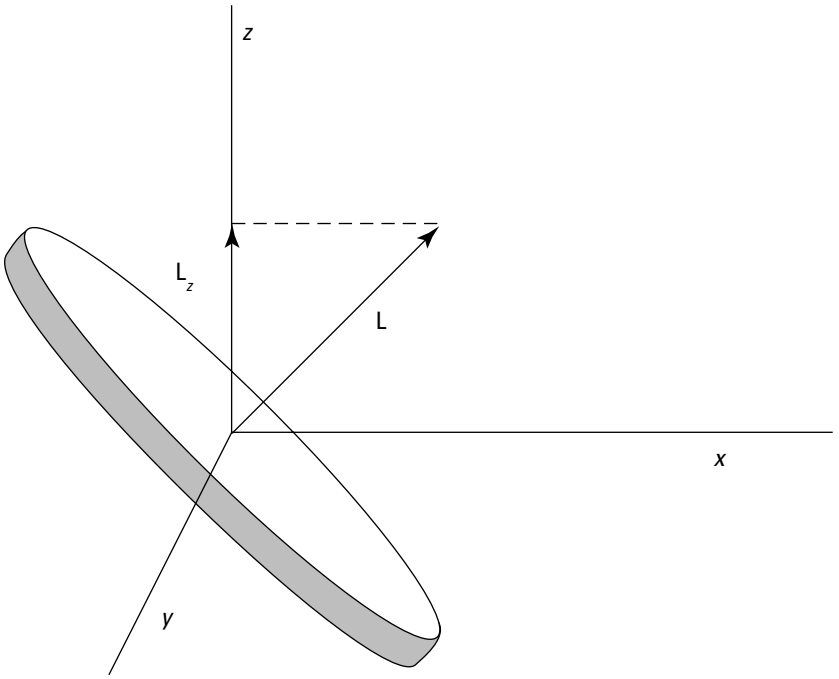

Figure 5-1: A rotating disk with angular momentum vector L.

As you can see, the disk's angular momentum vector, **L,** points perpendicular to the plane of rotation. Here, you can apply the right-hand rule: If you wrap your right hand in the direction something is rotating, your thumb points in the direction of the **L** vector.

Having the **L** vector point out of the plane of rotation has some advantages. For example, if something is rotating at a constant angular speed, the **L** vector will be constant in magnitude and direction — which makes more sense than having the **L** vector rotating in the plane of the disk's rotation and constantly changing direction.

Because **L** is a 3D vector, it can point in any direction, which means that it has *x, y,* and *z* components, L_x, L_y, and L_z (which aren't vectors, just magnitudes). You can see L_z in Figure 5-1.

L is the vector product of position **R** and linear momentum **P**, so ($\mathbf{L} = \mathbf{R} \times \mathbf{P}$). You can also write L_x, L_y, and L_z at any given moment in terms of operators like this, where P_x, P_y, and P_z are the *momentum operators* (which return the momentum in the *x, y,* and *z* directions) and X, Y, and Z are the *position operators* (which return the position in the *x, y,* and *z* directions):

- $L_x = YP_z - ZP_y$
- $L_y = ZP_x - XP_z$
- $L_z = XP_y - YP_x$

You can write the momentum operators P_x, P_y, and P_z as

$$P_x = -i\hbar\frac{\partial}{\partial x}$$

$$P_y = -i\hbar\frac{\partial}{\partial y}$$

$$P_z = -i\hbar\frac{\partial}{\partial z}$$

In the same way you can represent the position operators by their equivalent coordinates, i.e.

- $X = x$
- $Y = y$
- $Z = z$

Then if we substitute these operator representations into the equations for L_x, L_y, and L_z, you get,

$$L_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right)$$

$$L_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right)$$

$$L_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right)$$

Finding Commutators of L_x, L_y, and L_z

First examine L_x, L_y, and L_z by taking a look at how they commute; if they commute (for example, if $[L_x, L_y] = 0$), then you can measure any two of them (L_x and L_y, for example) exactly. If not, then they're subject to the *uncertainty relation,* and you can't measure them simultaneously exactly.

Okay, so what's the commutator of L_x and L_y? Using $L_x = YP_z - ZP_y$ and $L_y = ZP_x - XP_z$, you can write the following:

$$[L_x, L_y] = [YP_z - ZP_y, ZP_x - XP_z]$$

You can write this equation as

$$\begin{aligned}[L_x, L_y] &= [YP_z, ZP_x] - [YP_z, XP_z] - [ZP_y, ZP_x] + [ZP_y, XP_z] \\ &= Y\left[P_z, Z\right]P_x + X\left[Z, P_z\right]P_y \\ &= i\hbar(XP_y - YP_x)\end{aligned}$$

But $XP_y - YP_x = L_z$, so $[L_x, L_y] = i\hbar L_z$. So L_x and L_y don't commute, which means that you can't measure them both simultaneously with complete precision. You can also show that $[L_y, L_z] = i\hbar L_x$ and $[L_z, L_x] = i\hbar L_y$.

Because none of the components of angular momentum commute with each other, you can't measure any two simultaneously with complete precision. Rats.

That also means that the L_x, L_y, and L_z operators can't share the same eigenstates. So what can you do? How can you find an operator that shares eigenstates with the various components of L so that you can write the eigenstates as $|l, m\rangle$?

The usual trick here is that the square of the angular momentum, L^2, is a scalar, not a vector, so it'll commute with the L_x, L_y, and L_z operators, no problem:

- $[L^2, L_x] = 0$
- $[L^2, L_y] = 0$
- $[L^2, L_z] = 0$

Okay, cool, you're making progress. Because L_x, L_y, and L_z don't commute, you can't create an eigenstate that lists quantum numbers for any two of them. But because L^2 commutes with them, you can construct eigenstates that have eigenvalues for L^2 and any *one* of L_x, L_y, and L_z. By convention, the direction that's usually chosen is L_z.

Creating the Angular Momentum Eigenstates

Now's the time to create the actual eigenstates, $|l, m\rangle$, of angular momentum states in quantum mechanics. When you have the eigenstates, you'll also have the eigenvalues, and when you have the eigenvalues, you can solve the Hamiltonian and get the allowed energy levels of an object with angular momentum.

Don't make the assumption that the eigenstates are $|l, m\rangle$; rather, say they're $|\alpha, \beta\rangle$, where the eigenvalue of L^2 is $L^2|\alpha, \beta\rangle = \hbar^2\alpha|\alpha, \beta\rangle$. So the eigenvalue of L^2 is $\hbar^2\alpha$, where you have yet to solve for α. Similarly, the eigenvalue of L_z is $L_z|\alpha, \beta\rangle = \hbar\beta|\alpha, \beta\rangle$.

To proceed further, you have to introduce *raising* and *lowering* operators (as you do with the harmonic oscillator in Chapter 4). That way, you can solve for the ground state by, for example, applying the lowering operator to the ground state and setting the result equal to zero — and then solving for the ground state itself.

In this case, the raising operator is L_+ and the lowering operator is L_-. These operators raise and lower the L_z quantum number. In a way analogous to the raising and lowering operators in Chapter 4, you can define the raising and lowering operators this way:

- **Raising:** $L_+ = L_x + iL_y$
- **Lowering:** $L_- = L_x - iL_y$

These two equations mean that

$$L_x = \frac{1}{2}\left(L_+ + L_-\right)$$

$$L_y = \frac{1}{2i}\left(L_+ - L_-\right)$$

You can also see that

$$L_+L_- = L_x^2 + L_y^2 + \hbar L_z = L^2 - L_z^2 + \hbar L_z$$

That means the following are all equal to L^2:

- $L^2 = L_+L_- + L_z^2 - \hbar L_-$
- $L^2 = L_-L_+ + L_z^2 + \hbar L_z$
- $L^2 = \frac{1}{2}(L_+L_- + L_-L_+) + L_z^2$

You can also see that these equations are true:

- $[L^2, L_\pm] = 0$
- $[L_+, L_-] = 2\hbar L_z$
- $[L_z, L_\pm] = \pm\hbar L_\pm$

Okay, now you can put all this to work. You're getting to the good stuff.

Take a look at the operation of L_+ on $|\alpha, \beta>$:

$$L_+|\alpha, \beta> = ?$$

To see what $L_+|\alpha, \beta>$ is, start by applying the L_z operator on it like this:

$$L_zL_+|\alpha, \beta> = ?$$

From $[L_z, L_\pm] = \pm\hbar L_\pm$, you can see that $L_zL_+ - L_+L_z = \hbar L_+$, so

$$L_zL_+|\alpha, \beta> = L_+L_z|\alpha, \beta> + \hbar L_+|\alpha, \beta>$$

And because $L_z|\alpha, \beta> = \hbar\beta|\alpha, \beta>$, you have the following:

$$L_zL_+|\alpha, \beta> = \hbar(\beta + 1)L_+|\alpha, \beta>$$

This equation means that the eigenstate $L_+|\alpha, \beta>$ is also an eigenstate of the L_z operator, with an eigenvalue of $(\beta + 1)$. Or in a more comprehensible way:

$$L_+|\alpha, \beta> = c|\alpha, \beta + 1>$$

where *c* is a constant you find later in "Finding the Eigenvalues of the Raising and Lowering Operators."

So the L_+ operator has the effect of rasing the β quantum number by 1. Similarly, the lowering operator does this:

$$L_-|\alpha, \beta> = d|\alpha, \beta - 1>$$

Now take a look at what $L^2L_+|\alpha, \beta\rangle$ equals:

$$L^2 L_+ |\alpha, \beta\rangle = ?$$

Because L^2 is a scalar, it commutes with everything. $L^2 L_+ - L_+ L^2 = 0$, so this is true:

$$L^2 L_+ |\alpha, \beta\rangle = L_+ L^2 |\alpha, \beta\rangle$$

And because $L^2 |\alpha, \beta\rangle = \alpha\hbar^2 |\alpha, \beta\rangle$, you have the following equation:

$$L^2 L_+ |\alpha, \beta\rangle = \alpha\hbar^2 L_+ |\alpha, \beta\rangle$$

Similarly, the lowering operator, L_-, gives you this:

$$L^2 L_- |\alpha, \beta\rangle = \alpha\hbar^2 L_- |\alpha, \beta\rangle$$

So the results of these equations mean that the $L_\pm$ operators don't change the α eigenvalue of $|\alpha, \beta\rangle$ at all.

Okay, so just what *are* α and β? Read on.

Finding the Angular Momentum Eigenvalues

The eigenvalues of the angular momentum are the possible values the angular momentum can take, so they're worth finding. Let's take a look at how to do just that.

Deriving eigenstate equations with β_{max} and β_{min}

Note that $L^2 - L_z^2 = L_x^2 + L_y^2$, which is a positive number, so $L^2 - L_z^2 \geq 0$. That means that

$$\langle \alpha, \beta | \left(L^2 - L_z^2 \right) | \alpha, \beta \rangle \geq 0$$

And substituting in $L^2|\alpha, \beta> = \alpha\hbar^2|\alpha, \beta>$ and $L_z^2|\alpha, \beta> = \beta\hbar|\alpha, \beta>$, and using the fact that the eigenstates are normalized, gives you this:

$$<\alpha, \beta\left|\left(L^2 - L_z^2\right)\right|\alpha, \beta> = \hbar^2\left(\alpha - \beta^2\right) \geq 0$$

Therefore, $\alpha \geq \beta^2$. So there's a maximum possible value of β, which you can call β_{max}.

You can be clever now, because there has to be a state $|\alpha, \beta_{max}>$ such that you can't raise β any more. Thus, if you apply the raising operator, you get zero:

$$L_+|\alpha, \beta_{max}> = 0$$

Applying the lowering operator to this also gives you zero:

$$L_-L_+|\alpha, \beta_{max}> = 0$$

And because $L_-L_+ = L^2 - L_z^2 - \hbar L_z$, that means the following is true:

$$(L^2 - L_z^2 - \hbar L_z)|\alpha, \beta_{max}> = 0$$

Putting in $L^2|\alpha, \beta_{max}> = \alpha\hbar^2$ and $L_z|\alpha, \beta_{max}> = \beta_{max}\hbar|\alpha, \beta_{max}>$ gives you this:

$$(\alpha - \beta_{max}^2 - \beta_{max})\hbar^2 = 0$$
$$\alpha = \beta_{max}(\beta_{max} + 1) = 0$$

Cool, now you know what α is. At this point, it's usual to rename β_{max} as *l* and β as *m*, so $|\alpha, \beta>$ becomes $|l, m>$ and

- $L^2|l, m> = l(l + 1)\hbar^2|l, m>$
- $L_z|l, m> = m\hbar|l, m>$

You can say even more. In addition to a β_{max}, there must also be a β_{min} such that when you apply the lowering operator, L_-, you get zero, because you can't go any lower than β_{min}:

$$L_-|l, \beta_{min}> = 0$$

And you can apply L_+ on this as well:

$$L_+L_-|l, \beta_{min}> = 0$$

From $L_-L_+ = L^2 - L_z^2 + \hbar L_z$, you know that

$$(L^2 - L_z^2 + \hbar L_z)|\alpha, \beta_{min}\rangle = 0$$

which gives you the following:

$$(\alpha - \beta_{min}^2 + \beta_{min})\hbar^2 = 0$$
$$\alpha - \beta_{min}^2 + \beta_{min} = 0$$
$$\alpha = \beta_{min}^2 - \beta_{min}$$
$$\alpha = \beta_{min}(\beta_{min} - 1)$$

And comparing this equation to $\alpha = \beta_{max}(\beta_{max} + 1) = 0$ gives you

$$\beta_{max} = -\beta_{min}$$

Note that because you reach $|\alpha, \beta_{min}\rangle$ by n successive applications of L_- on $|\alpha, \beta_{max}\rangle$, you get the following:

$$\beta_{max} = \beta_{min} + n$$

Coupling these two equations gives you

$$\beta_{max} = {}^{n}/_{2}$$

Therefore, β_{max} can be either an integer or half an integer (depending on whether n is even or odd).

Because $l = \beta_{max}$, $m = \beta$, and n is a positive number, you can find that $-l \le m \le l$. So now you have it:

- The eigenstates are $|l, m\rangle$.
- The quantum number of the total angular momentum is l.
- The quantum number of the angular momentum along the z axis is m.
- $L^2|l, m\rangle = \hbar^2 l(l + 1)|l, m\rangle$, where $l = 0, {}^{1}/_{2}, 1, {}^{3}/_{2}, \ldots$
- $L_z|l, m\rangle = \hbar m|l, m\rangle$, where $m = -l, -(l - 1), \ldots, l - 1, l$.
- $-l \le m \le l$.

For each l, there are $2l + 1$ values of m. For example, if $l = 2$, then m can equal –2, –1, 0, 1, or 2. If $l = {}^{5}/_{2}$, then m can equal $-{}^{5}/_{2}$, $-{}^{3}/_{2}$, $-{}^{1}/_{2}$, ${}^{1}/_{2}$, ${}^{3}/_{2}$, and ${}^{5}/_{2}$.

You can see a representative L and L_z in Figure 5-2. L is the total angular momentum and L_z is the projection of that total angular momentum on the z axis.

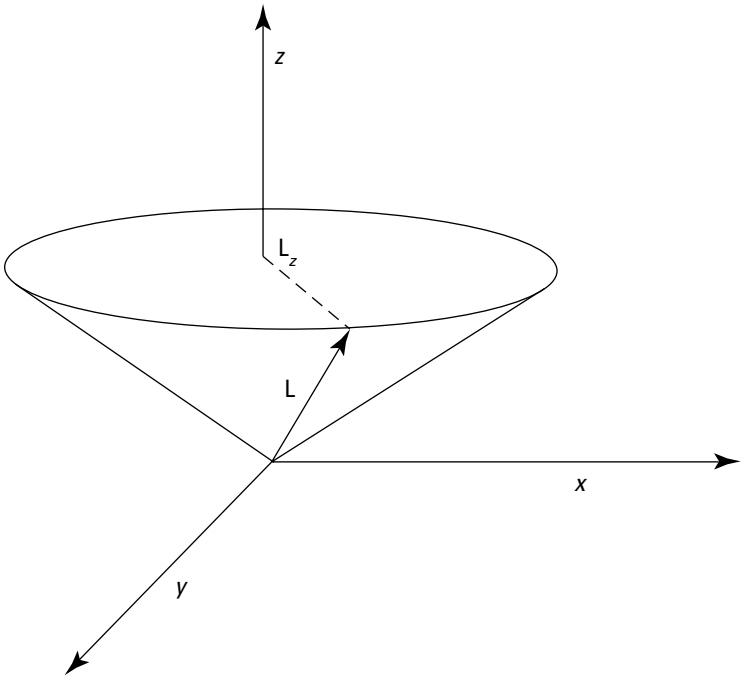

Figure 5-2: L and L_z.

Getting rotational energy of a diatomic molecule

Here's an example that involves finding the rotational energy spectrum of a diatomic molecule. Figure 5-3 shows the setup: A rotating diatomic molecule is composed of two atoms with masses m_1 and m_2. The first atom rotates at $r = r_1$, and the second atom rotates at $r = r_2$. What's the molecule's rotational energy?

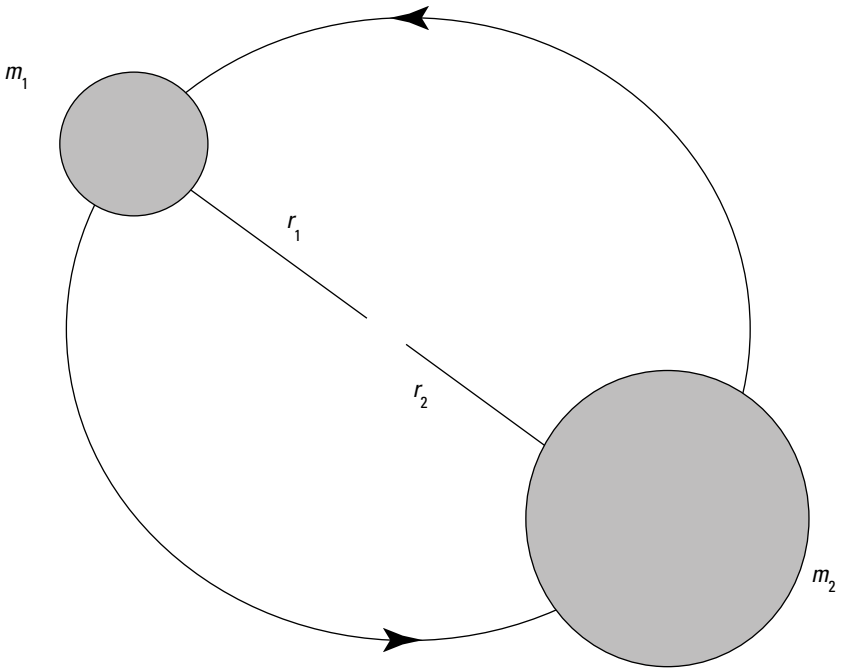

Figure 5-3: A rotating diatomic molecule.

The Hamiltonian (as you can see at the chapter intro) is

$$H = \frac{L^2}{2I}$$

I is the rotational moment of inertia, which is

$$I = m_1 r_1^2 + m_2 r_2^2 = \mu r^2$$

where $r = |r_1 - r_2|$ and $\mu = \frac{m_1 m_2}{m_1 + m_2}$.

Because $L = I\omega$, $L = \mu r^2 \omega$. Therefore, the Hamiltonian becomes

$$H = \frac{L^2}{2I} = \frac{L^2}{2\mu r^2}$$

So applying the Hamiltonian to the eigenstates, $|l, m>$, gives you the following:

$$H|l,m> = \frac{L^2}{2\mu r^2}|l,m>$$

And as you know, $L^2|l, m> = l(l + 1)\hbar^2|l, m>$, so this equation becomes

$$H|l,m> = \frac{L^2}{2\mu r^2}|l,m> = \frac{l(l+1)\hbar^2}{2\mu r^2}|l,m>$$

And because $H|l, m> = E|l, m>$, you can see that

$$E = \frac{l(l+1)\hbar^2}{2\mu r^2}$$

And that's the energy as a function of *l,* the angular momentum quantum number.

Finding the Eigenvalues of the Raising and Lowering Operators

This section looks at finding the eigenvalues of the raising and lowering angular momentum operators, which raise and lower a state's *z* component of angular momentum.

Start by taking a look at L_+, and plan to solve for *c:*

$$L_+|l, m> = c|l, m + 1>$$

So $L_+|l, m>$ gives you a new state, and multiplying that new state by its transpose should give you c^2:

$$(L_+|l, m>)^\dagger L_+|l, m> = c^2$$

To see this equation, note that $(L_+|l, m>)^\dagger L_+|l, m> = c^2<l, m + 1|l, m + 1> = c^2$. On the other hand, also note that $(L_+|l, m>)^\dagger L_+|l, m> = <l, m|L_+L_-|l, m>$, so you have

$$<l, m|L_+L_-|l, m> = c^2$$

What do you do about $L_+ L_-$? Well, you see earlier in the chapter, in "Creating the Angular Momentum Eigenstates," that this is true: $L_+L_- = L^2 - L_z^2 + \hbar L_z$. So your equation becomes the following:

$$< l, m \left| \left(L^2 - L_z{}^2 + \hbar L_z \right) \right| l, m > = c^2$$

Great! That means that *c* is equal to

$$c = \left(< l, m \left| \left(L^2 - L_z{}^2 + \hbar L_z \right) \right| l, m > \right)^{1/2}$$

So what is $\left(< l, m \left| \left(L^2 - L_z{}^2 + \hbar L_z \right) \right| l, m > \right)^{1/2}$? Applying the L^2 and L_z operators gives you this value for *c:*

$$c = \hbar [l(l + 1) - m(m + 1)]^{1/2}$$

And that's the eigenvalue of L_+, which means you have this relation:

$$L_+ | l, m> = \hbar [l(l + 1) - m(m + 1)]^{1/2} | l, m + 1>$$

Similarly, you can show that L_- gives you the following:

$$L_- | l, m> = \hbar [l(l + 1) - m(m - 1)]^{1/2} | l, m - 1>$$

Interpreting Angular Momentum with Matrices

Chapter 4 covers a matrix interpretation of harmonic oscillator states and operators, and you can handle angular momentum the same way (which often makes understanding what's going on with angular momentum easier). You get to take a look at the matrix representation of angular momentum on a quantum level now.

Consider a system with angular momentum, with the total angular momentum quantum number $l = 1$. That means that *m* can take the values –1, 0, and 1. So you can represent the three possible angular momentum states like this:

$$|1,-1\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$|1,0\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$|1,1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Okay, so what are the operators you've seen in this chapter in matrix representation? For example, what is L^2? You can write L^2 this way in matrix form:

$$L^2 = \begin{bmatrix} \langle 1,1|L^2|1,1\rangle & \langle 1,1|L^2|1,0\rangle & \langle 1,1|L^2|1,-1\rangle \\ \langle 1,0|L^2|1,1\rangle & \langle 1,0|L^2|1,0\rangle & \langle 1,0|L^2|1,-1\rangle \\ \langle 1,-1|L^2|1,1\rangle & \langle 1,-1|L^2|1,0\rangle & \langle 1,-1|L^2|1,-1\rangle \end{bmatrix}$$

Okay, $\langle 1, 1 | L^2 | 1, 1\rangle = l(l+1)\hbar^2 = 2\hbar^2$; $\langle 1, 1 | L^2 | 1, 0\rangle = 0$; $\langle 1, 0 | L^2 | 1, 0\rangle = 2\hbar^2$; and so on; Therefore, the preceding matrix becomes the following:

$$L^2 = \begin{bmatrix} 2\hbar^2 & 0 & 0 \\ 0 & 2\hbar^2 & 0 \\ 0 & 0 & 2\hbar^2 \end{bmatrix}$$

And you can also write this as

$$L^2 = 2\hbar^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So in matrix form, the equation $L^2 | 1, 1\rangle = 2\hbar^2 | 1, 1\rangle$ becomes

$$2\hbar^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 2\hbar^2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

How about the L_+ operator? As you probably know (from the preceding section), $L_+ | l, m\rangle = \hbar[l(l+1) - m(m+1)]^{1/2} | l, m+1\rangle$. In this example, $l = 1$ and $m = 1, 0$, and -1. So you have the following:

- $L_+ | 1, 1\rangle = 0$
- $L_+ |1,0\rangle = \sqrt{2}\hbar |1,1\rangle$
- $L_+ |1,-1\rangle = \sqrt{2}\hbar |1,0\rangle$

So the L_+ operator looks like this in matrix form:

$$L_+ = \sqrt{2}\hbar \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, $L_+ | 1, 0\rangle$ would be

$$L_+ |1,0\rangle = \sqrt{2}\hbar \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

And this equals

$$L_+ |1,0\rangle = \sqrt{2}\hbar \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \sqrt{2}\hbar \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

In other words, $\sqrt{2}\hbar |1,1\rangle = L_+ |1,0\rangle$.

Okay, what about L_-? You know that

$$L_-|l, m\rangle = \hbar[l(l+1) - m(m-1)]^{1/2}|l, m-1\rangle.$$

In this example, l = 1 and m = 1, 0, and –1.

So that means the following:

- $L_-|1,1\rangle = \sqrt{2}\hbar|1,0\rangle$
- $L_-|1,0\rangle = \sqrt{2}\hbar|1,-1\rangle$
- $L_-|1,-1\rangle = 0$

So the L_- operator looks like this in matrix form:

$$L_- = \sqrt{2}\hbar\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

That means that $L_-|1, 1\rangle$ would be

$$L_-|1,1\rangle = \sqrt{2}\hbar\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

This equals

$$L_-|1,1\rangle = \sqrt{2}\hbar\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \sqrt{2}\hbar\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Which tells you that

$$L_-|1,1\rangle = \sqrt{2}\hbar|1,0\rangle$$

Just as you'd expect.

Okay, you've found L^2, L_+, and L_-. Finding the matrix representation of L_z is simple because

- $\hbar |1, 1\rangle = L_z |1, 1\rangle$
- $0 = L_z |1, 0\rangle$
- $-\hbar |1, 1\rangle = L_z |1, -1\rangle$

So you have that

$$L_z = \hbar \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Thus $L_z |1, -1\rangle$ equals

$$L_z |1,-1\rangle = \hbar \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

And this equals

$$L_z |1,-1\rangle = \hbar \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = -\hbar \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

So $L_z |1, -1\rangle = -\hbar |1, -1\rangle$.

Now what about finding the L_x and L_y operators? That's not as hard as you may think, because

$$L_x = \frac{1}{2}\left(L_+ + L_-\right)$$

and

$$L_y = \frac{i}{2}\left(L_- - L_+\right)$$

Take a look at L_x first. L_+ equals

$$L_+ = \sqrt{2}\hbar \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

And L_- equals

$$L_- = \sqrt{2}\hbar \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

So L_x equals:

$$L_x = \frac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Okay, now what about L_y? $L_y = \frac{i}{2}(L_- - L_+)$, so:

$$L_y = \frac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}$$

Cool. This is going pretty well — how about calculating $[L_x, L_y]$? To do that, you need to calculate $[L_x, L_y] = L_xL_y - L_yL_x$. First find L_xL_y:

$$L_xL_y = \frac{\hbar^2}{2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}$$

This equals

$$L_x L_y = \frac{\hbar^2}{2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} = \frac{\hbar^2}{2}\begin{bmatrix} i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & -i \end{bmatrix}$$

And similarly, $L_y L_x$ equals

$$L_y L_x = \frac{\hbar^2}{2}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

And this equals

$$L_y L_x = \frac{\hbar^2}{2}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \frac{\hbar^2}{2}\begin{bmatrix} -i & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & i \end{bmatrix}$$

So

$$\left[L_x, L_y\right] = L_x L_y - L_y L_x = \frac{\hbar^2}{2}\begin{bmatrix} 2i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2i \end{bmatrix}$$

And this equals

$$\left[L_x, L_y\right] = L_x L_y - L_y L_x = i\hbar^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

But because

$$L_z = \hbar \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

You can write the commutator, $[L_x, L_y]$ like this:

$$\left[L_x, L_y \right] = i\hbar^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = i\hbar \left(\hbar \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right) = i\hbar L_z$$

This is just the old result that we know and love, so it all checks out!

Rounding It Out: Switching to the Spherical Coordinate System

So far, this chapter has been dealing with angular momentum in terms of bras and kets, such as:

$$\sqrt{2}\hbar \left| 1,0 \right> = L_- \left| 1,1 \right>$$

The charm of bras and kets is that they don't limit you to any specific system of representation (see Chapter 2). So you have the general eigenstates, but what are the actual *eigenfunctions* of L_z and L^2? That is, you're going to try to find the actual functions that you can use with the angular momentum operators like L^2 and L_z.

To find the actual eigenfunctions (not just the eigenstates), you turn from rectangular coordinates, *x, y,* and *z*, to spherical coordinates because it'll make the math much simpler (after all, angular momentum is about things going around in circles). Figure 5-4 shows the spherical coordinate system.

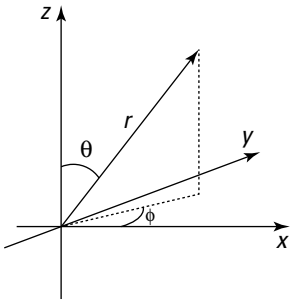

Figure 5-4: The spherical coordinate system.

In the rectangular (Cartesian) coordinate system, you use *x, y,* and *z* to orient yourself. In the spherical coordinate system, you also use three quantities: *r,* θ, and ϕ, as Figure 5-4 shows. You can translate between the spherical coordinate system and the rectangular one this way: The *r* vector is the vector to the particle that has angular momentum, θ is the angle of *r* from the z axis, and ϕ is the angle of *r* from the x axis.

- $x = r\sin\theta\cos\phi$
- $y = r\sin\theta\sin\phi$
- $z = r\cos\theta$

Consider the equations for angular momentum:

$$L_x = YP_z - ZP_y = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$L_y = ZP_x - XP_z = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$L_z = XP_y - YP_x = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

When you take the angular momentum equations with the spherical-coordinate-system conversion equations, you can derive the following:

- $L_z = -i\hbar\frac{\partial}{\partial\phi}$
- $L^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$

- $L_+ = L_x + iL_y = \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + \frac{i\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi}\right)$
- $L_- = L_x - iL_y = \hbar e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + \frac{i\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi}\right)$

Okay, these equations look pretty involved. But there's one thing to notice: They depend only on θ and ϕ, which means their eigenstates depend only on θ and ϕ, not on *r*. So the eigenfunctions of the operators in the preceding list can be denoted like this:

$$<\theta, \phi | l, m>$$

Traditionally, you give the name $Y_{lm}(\theta, \phi)$ to the eigenfunctions of angular momentum in spherical coordinates, so you have the following:

$$Y_{lm}(\theta, \phi) = <\theta, \phi | l, m>$$

All right, time to work on finding the actual form of $Y_{lm}(\theta, \phi)$. You know that when you use the L^2 and L_z operators on angular momentum eigenstates, you get this:

$$L^2 | l, m> = l(l + 1)\hbar^2 | l, m>$$

$$L_z | l, m> = m\hbar | l, m>$$

So the following must be true:

- $L^2 Y_{lm}(\theta, \phi) = l(l + 1)\hbar^2 Y_{lm}(\theta, \phi)$
- $L_z Y_{lm}(\theta, \phi) = m\hbar Y_{lm}(\theta,\phi)$

In fact, you can go further. Note that L_z depends only on θ, which suggests that you can split $Y_{lm}(\theta,\phi)$ up into a part that depends on θ and a part that depends on ϕ. Splitting $Y_{lm}(\theta,\phi)$ up into parts looks like this:

$$Y_{lm}(\theta,\phi) = \Theta_{lm}(\theta)\Phi_m(\phi)$$

That's what makes working with spherical coordinates so helpful — you can split the eigenfunctions up into two parts, one that depends only on θ and one part that depends only on ϕ.

The eigenfunctions of L_z in spherical coordinates

Start by finding the eigenfunctions of L_z in spherical coordinates. In spherical coordinates, the L_z operator looks like this:

$$L_z = -i\hbar\frac{\partial}{\partial\phi}$$

So $L_z Y_{lm}(\theta, \phi) = L_z \Theta_{lm}(\theta)\Phi_m(\phi)$ is

$$L_z\Theta_{lm}(\theta)\Phi_m(\phi) = -i\hbar\frac{\partial}{\partial\phi}\Theta_{lm}(\theta)\Phi_m(\phi)$$

which is the following:

$$L_z\Theta_{lm}(\theta)\Phi_m(\phi) = -i\hbar\Theta_{lm}(\theta)\frac{\partial\Phi_m}{\partial\phi}(\phi)$$

And because $L_z Y_{lm}(\theta,\phi) = m\hbar Y_{lm}(\theta,\phi)$, this equation can be written in this version:

$$-i\hbar\Theta_{lm}(\theta)\frac{\partial\Phi_m}{\partial\phi}(\phi) = m\hbar\Theta_{lm}(\theta)\Phi_m(\phi)$$

Cancelling out terms from the two sides of this equation gives you this differential equation:

$$-i\frac{\partial\Phi_m}{\partial\phi}(\phi) = m\Phi_m(\phi)$$

This looks easy to solve, and the solution is just

$$\Phi_m(\phi) = Ce^{im\phi}$$

where C is a constant of integration.

You can determine C by insisting that $\Phi_m(\phi)$ be normalized — that is, that the following hold true:

$$\int_0^{2\pi}\Phi^*_m(\phi)\Phi_m(\phi)d\phi = 1$$

which gives you

$$C = \frac{1}{(2\pi)^{1/2}}$$

So $\Phi_m(\phi)$ is equal to this:

$$\Phi_m(\phi) = \frac{e^{im\phi}}{(2\pi)^{1/2}}$$

You're making progress — you've been able to determine the form of $\Phi_m(\phi)$, so $Y_{lm}(\theta,\phi) = \Theta_{lm}(\theta)\,\Phi_m(\phi)$, which equals

$$Y_{lm}(\theta,\phi) = \Theta_{lm}(\theta)\Phi_m(\phi) = \Theta_{lm}(\theta)\frac{e^{im\phi}}{(2\pi)^{1/2}}$$

That's great — you're halfway there, but you still have to determine the form of $\Theta_{lm}(\theta)$, the eigenfunction of L^2. That's coming up next.

The eigenfunctions of L^2 in spherical coordinates

Now you're going to tackle the eigenfunction of L^2, $\Theta_{lm}(\theta)$. You already know that in spherical coordinates, the L^2 operator looks like this:

$$L^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

That's quite an operator. And you know that

$$Y_{lm}(\theta,\phi) = \Theta_{lm}(\theta)\frac{e^{im\phi}}{(2\pi)^{1/2}}$$

So applying the L^2 operator to $Y_{lm}(\theta,\phi)$ gives you the following:

$$L^2Y_{lm}(\theta,\phi) = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\left(\Theta_{lm}(\theta)\frac{e^{im\phi}}{(2\pi)^{1/2}}\right)$$

And because $L^2Y_{lm}(\theta, \phi) = l(l + 1)\hbar^2Y_{lm}(\theta, \phi) = l(l + 1)\hbar^2\Theta lm(\theta)\Phi_m(\phi)$, this equation becomes

$$-\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\left(\Theta_{lm}(\theta)\frac{e^{im\phi}}{(2\pi)^{1/2}}\right)$$

$$=l(l+1)\hbar^2\Theta_{lm}(\theta)\frac{e^{im\phi}}{(2\pi)^{1/2}}$$

Wow, what have you gotten in to? Cancelling terms and subtracting the right-hand side from the left finally gives you this differential equation:

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\left(\Theta_{lm}(\theta)e^{im\phi}\right)+l(l+l)\Theta_{lm}(\theta)e^{im\phi}=0$$

Combining terms and dividing by $e^{im\phi}$ gives you the following:

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)-\frac{m^2}{\sin^2\theta}+l(l+1)\right]\Theta_{lm}(\theta)=0$$

Holy cow! Isn't there someone who's tried to solve this kind of differential equation before? Yes, there is. This equation is a *Legendre differential equation*, and the solutions are well-known. (Whew!) In general, the solutions take this form:

$$\Theta_{lm}(\theta) = C_{lm}P_{lm}(\cos\theta)$$

where $P_{lm}(\cos\theta)$ is the *Legendre function*.

So what are the Legendre functions? You can start by separating out the *m* dependence, which works this way with the Legendre functions:

$$P_{lm}(x)=\left(1-x^2\right)^{m/2}\frac{d^m}{dx^m}P_l(x)$$

where $P_l(x)$ is called a *Legendre polynomial* and is given by the Rodrigues formula:

$$P_l(x)=\frac{(-1)^l}{2^l l!}\frac{d^l}{dx^l}\left(1-x^2\right)^l$$

You can use this equation to derive the first few Legendre polynomials like this:

- $P_0(x) = 1$
- $P_1(x) = x$
- $P_2(x) = {}^1/_2\,(3x^2 - 1)$
- $P_3(x) = {}^1/_2\,(5x^3 - 3x)$
- $P_4(x) = {}^1/_8\,(35x^4 - 30x^2 + 3)$
- $P_5(x) = {}^1/_8\,(63x^5 - 70x^3 + 15x)$

and so on. That's what the first few $P_l(x)$ polynomials look like. So what do the associated Legendre functions, $P_{lm}(x)$ look like? You can also calculate them. You can start off with $P_{l0}(x)$, where $m = 0$. Those are easy because $P_{l0}(x) = P_l(x)$, so

- $P_{10}(x) = x$
- $P_{20}(x) = {}^1/_2\,(3x^2 - 1)$
- $P_{30}(x) = {}^1/_2\,(5x^3 - 3x)$

Also, you can find that

- $P_{11}(x) = (1 - x^2)^{1/2}$
- $P_{21}(x) = 3x(1 - x^2)^{1/2}$
- $P_{22}(x) = 3(1 - x^2)$
- $P_{31}(x) = \frac{3}{2}\left(5x^2 - 1\right)\left(1 - x^2\right)^{1/2}$
- $P_{32}(x) = 15x(1 - x^2)$
- $P_{33}(x) = 15x(1 - x^2)^{3/2}$

These equations give you an overview of what the P_{lm} functions look like, which means you're almost done. As you may recall, $\Theta_{lm}(\theta)$, the θ part of $Y_{lm}(\theta, \phi)$, is related to the P_{lm} functions like this:

$$\Theta_{lm}(\theta) = C_{lm}P_{lm}(\cos\theta)$$

And now you know what the P_{lm} functions look like, but what do C_{lm}, the constants, look like? As soon as you have those, you'll have the complete angular momentum eigenfunctions, $Y_{lm}(\theta, \phi)$, because $Y_{lm}(\theta, \phi) = \Theta_{lm}(\theta)\Phi_m(\phi)$.

You can go about calculating the constants C_{lm} the way you always calculate such constants of integration in quantum physics — you normalize the eigenfunctions to 1. For $Y_{lm}(\theta,\phi) = \Theta_{lm}(\theta)\Phi_m(\phi)$, that looks like this:

$$\int_0^{2\pi}\int_0^{\pi} Y_{lm}^{*}(\theta,\phi)Y_{lm}(\theta,\phi)\sin\theta\, d\theta d\phi = 1$$

Substitute the following three quantities in this equation:

- $Y_{lm}(\theta, \phi) = \Theta_{lm}(\theta)\,\Phi_m(\phi)$
- $\Phi_m(\phi) = \dfrac{e^{im\phi}}{(2\pi)^{1/2}}$
- $\Theta_{lm}(\theta) = C_{lm}P_{lm}(\cos\theta)$

You get the following:

$$\frac{|C_{lm}|^2}{2\pi}\int_0^{2\pi} d\phi\int_0^{\pi}|P_{lm}(\cos\theta)|^2\sin\theta\, d\theta = 1$$

The integral over ϕ gives 2π, so this becomes

$$|C_{lm}|^2\int_0^{\pi}|P_{lm}(\cos\theta)|^2\sin\theta\, d\theta = 1$$

You can evaluate the integral to this:

$$|C_{lm}|^2\frac{2}{2l+1}\frac{(l+|m|)!}{(l-|m|)!} = 1$$

So in other words:

$$C_{lm} = (-1)^{|m|}\left[\frac{(2l+1)(l-|m|)!}{2(l+|m|)!}\right]^{1/2}$$

Which means that

$$\Theta_{lm}(\theta) = (-1)^{|m|}\left[\frac{(2l+1)(l-|m|)!}{2(l+|m|)!}\right]^{1/2}P_{lm}(\cos\theta)$$

So $Y_{lm}(\theta, \phi) = \Theta_{lm}(\theta)\Phi_m(\phi)$, which is the angular momentum eigenfunction in spherical coordinates, is

$$Y_{lm}(\theta,\phi) = (-1)^{|m|}\left[\frac{(21+1)(1-|m|)!}{4\pi(1+|m|)!}\right]^{1/2} P_{lm}(\cos\theta)e^{im\phi}$$

The functions given by this equation are called the *normalized spherical harmonics*. Here are what the first few normalized spherical harmonics look like:

- $Y_{00}(\theta,\phi) = \frac{1}{(4\pi)^{1/2}}$
- $Y_{10}(\theta,\phi) = \frac{3}{4\pi}^{1/2}\cos\theta$
- $Y_{1\pm1}(\theta,\phi) = \mp\left(\frac{3}{8\pi}\right)^{1/2} e^{\pm i\phi}\sin\theta$
- $Y_{20}(\theta,\phi) = \left(\frac{5}{16\pi}\right)^{1/2}\left(3\cos^2\theta - 1\right)$
- $Y_{2\pm1}(\theta,\phi) = \mp\left(\frac{15}{8\pi}\right)^{1/2} e^{\pm i\phi}\sin\theta\cos\theta$
- $Y_{2\pm2}(\theta,\phi) = \left(\frac{15}{32\pi}\right)^{1/2} e^{\pm 2/\phi}\sin^2\theta$

In fact, you can use these relations to convert the spherical harmonics to rectangular coordinates:

- $\sin\theta\cos\phi = \frac{x}{r}$
- $\sin\theta\sin\phi = \frac{y}{r}$
- $\cos\theta = \frac{z}{r}$

Substituting these equations into

$Y_{lm}(\theta,\phi) = (-1)^{|m|}\left[\frac{(21+1)(1-|m|)!}{4\pi(1+|m|)!}\right]^{1/2} P_{lm}(\cos\theta)e^{im\phi}$ gives you the

spherical harmonics in rectangular coordinates:

- $Y_{00}(x,y,z) = \frac{1}{(4\pi)^{1/2}}$
- $Y_{10}(x,y,z) = \left(3/4\pi\right)^{1/2} z/r$

- $Y_{1\pm1}(x,y,z) = \mp\left(3/8\pi\right)^{1/2}(x \pm iy)/r$
- $Y_{20}(x,y,z) = \left(5/16\pi\right)^{1/2}(3z^2 - r^2)/r^2$
- $Y_{2\pm1}(x,y,z) = \mp\left(15/8\pi\right)^{1/2} z(x \pm iy)/r^2$
- $Y_{2\pm2}(x,y,z) = \mp\left(15/32\pi\right)^{1/2}(x^2 - y^2 \pm 2ixy)/r^2$

Chapter 6

Getting Dizzy with Spin

In This Chapter

- Discovering spin with the Stern-Gerlach experiment
- Looking at eigenstates and spin notation
- Understanding fermions and bosons
- Comparing the spin operators with angular momentum operators
- Working with spin $^1/_2$ and Pauli matrices

Physicists have suggested that orbital angular momentum is not the only kind of angular momentum present in an atom — electrons could also have *intrinsic* built-in angular momentum. This kind of built-in angular momentum is called *spin*. Whether or not electrons actually spin will never be known — they're as close to point-like particles as you can come, without any apparent internal structure. Yet the fact remains that they have intrinsic angular momentum. And that's what this chapter is about — the intrinsic, built-in quantum mechanical spin of subatomic particles.

The Stern-Gerlach Experiment and the Case of the Missing Spot

The Stern-Gerlach experiment unexpectedly revealed the existence of spin back in 1922. Physicists Otto Stern and Walther Gerlach sent a beam of silver atoms through the poles of a magnet — whose magnetic field was in the z direction — as you can see in Figure 6-1.

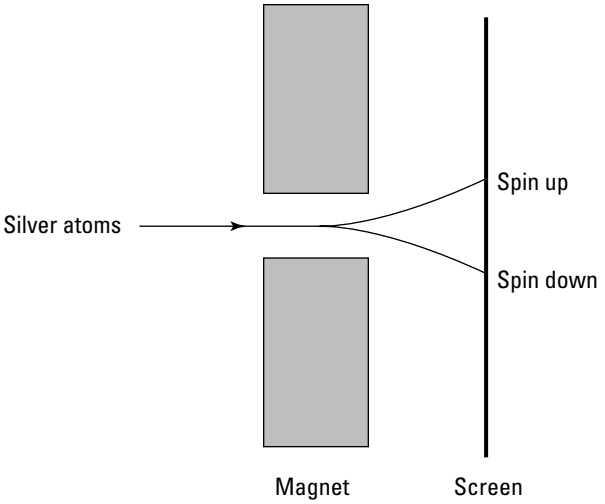

Figure 6-1: The Stern-Gerlach experiment.

Because 46 of silver's 47 electrons are arranged in a symmetrical cloud, they contribute nothing to the orbital angular momentum of the atom. The 47th electron can be in

- The 5*s* state, in which case its angular momentum is $l = 0$ and the z component of that angular momentum is 0
- The 5*p* state, in which case its angular momentum is $l = 1$, which means that the z component of its angular momentum can be –1, 0, or 1

That means that Stern and Gerlach expected to see one or three spots on the screen you see at right in Figure 6-1, corresponding to the different states of the z component of angular momentum.

But famously, they saw only two spots. This puzzled the physics community for about three years. Then, in 1925, physicists Samuel A. Goudsmit and George E. Uhlenbeck suggested that electrons contained intrinsic angular momentum — and that intrinsic angular momentum is what gave them a magnetic moment that interacted with the magnetic field. After all, it was apparent that some angular momentum other than orbital angular momentum was at work here. And that built-in angular momentum came to be called *spin*.

The beam of silver atoms divides in two, depending on the spin of the 47th electron in the atom, so there are two possible states of spin, which came to be known as *up* and *down*.

Spin is a purely quantum mechanical effect, and there's no real classical analog. The closest you can come is to liken spin to the spin of the Earth as it goes around the sun — that is, the Earth has both spin (because it's rotating on its axis) and orbital angular momentum (because it's revolving around the sun). But even this picture doesn't wholly explain spin in classical terms, because it's conceivable that you could stop the Earth from spinning. But you can't stop electrons from possessing spin, and that also goes for other subatomic particles that possess spin, such as protons.

Spin doesn't depend on spatial degrees of freedom; even if you were to have an electron at rest (which violates the uncertainty principle), it would still possess spin.

Getting Down and Dirty with Spin and Eigenstates

Spin throws a bit of a curve at you. When dealing with orbital angular momentum (see Chapter 5), you can build angular momentum operators because orbital angular momentum is the product of momentum and radius. But spin is built in; there's no momentum operator involved. So here's the crux: You cannot describe spin with a differential operator, as you can for orbital angular momentum.

In Chapter 5, I show how orbital angular momentum can be reduced to these differential operators:

- $\mathrm{L}_x = \mathrm{YP}_z - \mathrm{ZP}_y = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right)$
- $\mathrm{L}_y = \mathrm{ZP}_x - \mathrm{XP}_z = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right)$
- $\mathrm{L}_z = \mathrm{XP}_y - \mathrm{YP}_x = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right)$

And you can find eigenfunctions for angular momentum, such as Y_{20}:

$$\mathrm{Y}_{20}\left(x,y,z\right) = \left(\frac{5}{16\pi}\right)^{1/2}\left(\frac{3z^2 - r^2}{r^2}\right)$$

But because you can't express spin using differential operators, you can't find eigenfunctions for spin as you do for angular momentum. So that means that you're left with the bra and ket way of looking at things (bras and kets aren't tied to any specific representation in spatial terms).

In Chapter 5, you also take a look at things in angular momentum terms, introducing the eigenstates of orbital angular momentum like this: $|l, m\rangle$ (where l is the angular momentum quantum number and m is the quantum number of the z component of angular momentum).

You can use the same notation for spin eigenstates. As with orbital angular momentum, you can use a total spin quantum number and a quantum number that indicates the spin along the z axis (***Note:*** There's no true z axis built in when it comes to spin — you introduce a z axis when you apply a magnetic field; by convention, the z axis is taken to be in the direction of the applied magnetic field).

The letters given to the total spin quantum number and the z-axis component of the spin are s and m (you sometimes see them written as s and m_s). In other words, the eigenstates of spin are written as $|s, m\rangle$.

So what possible values can s and m take? That's coming up next.

Halves and Integers: Saying Hello to Fermions and Bosons

In analogy with orbital angular momentum, you can assume that m (the z-axis component of the spin) can take the values $-s, -s + 1, \ldots, s - 1$, and s, where s is the total spin quantum number. For electrons, Stern and Gerlach observed two spots, so you have $2s + 1 = 2$, which means that $s = {}^1/_2$. And therefore, m can be $+{}^1/_2$ or $-{}^1/_2$. So here are the possible eigenstates for electrons in terms of spin:

$|{}^1/_2, {}^1/_2\rangle$

$|{}^1/_2, -{}^1/_2\rangle$

So do all subatomic particles have $s = {}^1/_2$? Nope. Here are their options:

- **Fermions:** In physics, particles with half-integer spin are called *fermions.* They include electrons, protons, neutrons, and so on, even quarks. For example, electrons, protons, and neutrons have spin $s = {}^1/_2$, and delta particles have $s = {}^3/_2$.

- **Bosons:** Particles with integer spin are called *bosons.* They include photons, pi mesons, and so on; even the postulated particles involved with the force of gravity, *gravitons,* are supposed to have integer spin. For example, pi mesons have spin $s = 0$, photons have $s = 1$, and so forth.

So for electrons, the spin eigenstates are $|{}^1/_2, {}^1/_2\rangle$ and $|{}^1/_2, -{}^1/_2\rangle$. For photons, the eigenstates are $|1, 1\rangle$, $|1, 0\rangle$, and $|1, -1\rangle$. Therefore, the possible eigenstates depend on the particle you're working with.

Spin Operators: Running Around with Angular Momentum

Because spin is a type of built-in angular momentum, the spin operators have a lot in common with the orbital angular momentum operators. In Chapter 5, I discuss the orbital angular momentum operators L^2 and L_z, and as you may expect, there are analogous spin operators, S^2 and S_z. However, these operators are just operators; they don't have a differential form like the orbital angular momentum operators do.

In fact, all the orbital angular momentum operators, such as L_x, L_y, and L_z, have analogs here: S_x, S_y, and S_z. The commutation relations among L_x, L_y, and L_z are the following:

- $[L_x, L_y] = i\hbar L_z$
- $[L_y, L_z] = i\hbar L_x$
- $[L_z, L_x] = i\hbar L_y$

And they work the same way for spin:

- $[S_x, S_y] = i\hbar S_z$
- $[S_y, S_z] = i\hbar S_x$
- $[S_z, S_x] = i\hbar S_y$

The L^2 operator gives you the following result when you apply it to an orbital angular momentum eigenstate:

$$L^2|l, m\rangle = l(l + 1)\hbar^2|l, m\rangle$$

And just as you'd expect, the S^2 operator works in an analogous fashion:

$$S^2|s, m\rangle = s(s+1)\hbar^2|s, m\rangle$$

The L_z operator gives you this result when you apply it to an orbital angular momentum eigenstate (see Chapter 5):

$$L_z|l, m\rangle = m\hbar|l, m\rangle$$

And by analogy, the S_z operator works this way:

$$S_z|s, m\rangle = m\hbar|s, m\rangle$$

What about the raising and lowering operators, L_+ and L_-? Are there analogs for spin? In angular momentum terms, L_+ and L_- work like this:

$$L_+|l,m\rangle = \hbar\left[l(l+1) - m(m+1)\right]^{1/2}|l,m+1\rangle$$

$$L_-|l,m\rangle = \hbar\left[l(l+1) - m(m-1)\right]^{1/2}|l,m-1\rangle$$

There are spin raising and lowering operators as well, S_+ and S_-, and they work like this:

$$S_+|s,m\rangle = \hbar\left[s(s+1) - m(m+1)\right]^{1/2}|s,m+1\rangle$$

$$S_-|s,m\rangle = \hbar\left[s(s+1) - m(m-1)\right]^{1/2}|s,m-1\rangle$$

In the next section, I take a special look at particles with spin 1/2.

Working with Spin 1/2 and Pauli Matrices

Spin 1/2 particles (fermions) need a little extra attention. The eigenvalues of the S^2 operator here are

$$S^2|s,m\rangle = s(s+1)\hbar^2|s,m\rangle = \frac{3}{4}\hbar^2|s,m\rangle$$

And the eigenvalues of the S_z operator are

$$S_z \left| s,m > = m\hbar \right| s,m > = \pm \frac{\hbar}{2} \left| s,m > \right.$$

You can represent these two equations graphically as shown in Figure 6-2, where the two spin states have different projections along the *z* axis.

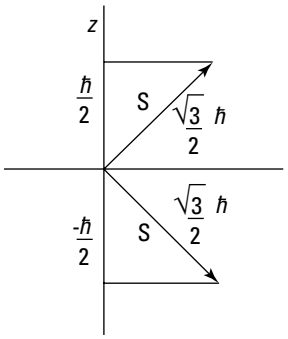

Figure 6-2: Spin magnitude and *z* projection.

Spin ½ matrices

Time to take a look at the spin eigenstates and operators for particles of spin ½ in terms of matrices. There are only two possible states, spin up and spin down, so this is easy. First, you can represent the eigenstate $|{}^1/_2, {}^1/_2>$ like this:

$$\left| 1/2, 1/2 \right\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

And the eigenstate $|{}^1/_2, -{}^1/_2>$ looks like this:

$$\left| 1/2, -1/2 \right\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Now what about spin operators like S^2? The S^2 operator looks like this in matrix terms:

$$S^2 = \begin{bmatrix} \langle 1/2, 1/2 | S^2 | 1/2, 1/2 \rangle & \langle 1/2, 1/2 | S^2 | 1/2, -1/2 \rangle \\ \langle 1/2, -1/2 | S^2 | 1/2, 1/2 \rangle & \langle 1/2, -1/2 | S^2 | 1/2, -1/2 \rangle \end{bmatrix}$$

And this works out to be the following:

$$S^2 = \frac{3}{4}\hbar^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Similarly, you can represent the S_z operator this way:

$$S_z = \begin{bmatrix} \langle 1/2, 1/2 | S_z | 1/2, 1/2 \rangle & \langle 1/2, 1/2 | S_z | 1/2, -1/2 \rangle \\ \langle 1/2, -1/2 | S_z | 1/2, 1/2 \rangle & \langle 1/2, -1/2 | S_z | 1/2, -1/2 \rangle \end{bmatrix}$$

This works out to

$$S_z = \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Using the matrix version of S_z, for example, you can find the z component of the spin of, say, the eigenstate $|{}^1/_2, -{}^1/_2\rangle$. Finding the z component looks like this:

$$S_z \, |{}^1/_2, -{}^1/_2\rangle$$

Putting this in matrix terms gives you this matrix product:

$$\frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Here's what you get by performing the matrix multiplication:

$$\frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\frac{\hbar}{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

And putting this back into ket notation, you get the following:

$$S_z \left| \tfrac{1}{2}, -\tfrac{1}{2} \right> = \frac{-\hbar}{2} \left| \tfrac{1}{2}, -\tfrac{1}{2} \right>$$

How about the raising and lowering operators S_+ and S_-? The S_+ operator looks like this:

$$S_+ = \hbar \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

And the lowering operator looks like this:

$$S_- = \hbar \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

So, for example, you can figure out what $S_+ | {}^1/_2, -{}^1/_2 >$ is. Here it is in matrix terms:

$$\hbar \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Performing the multiplication gives you this:

$$\hbar \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \hbar \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Or in ket form, it's $S_+ | {}^1/_2, -{}^1/_2 > = \hbar | {}^1/_2, {}^1/_2 >$. Cool.

Pauli matrices

Sometimes, you see the operators S_x, S_y, and S_z written in terms of *Pauli matrices*, σ_x, σ_y, and σ_z. Here's what the Pauli matrices look like:

$$\sigma_x = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$\sigma_y = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix}$$

$$\sigma_z = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$$

Now you can write S_x, S_y, and S_z in terms of the Pauli matrices like this:

$$S_x = \frac{\hbar}{2}\sigma_x$$

$$S_y = \frac{\hbar}{2}\sigma_y$$

$$S_z = \frac{\hbar}{2}\sigma_z$$

Whoo! And that concludes your look at spin.