An Introduction to General Relativistic Ray-tracing

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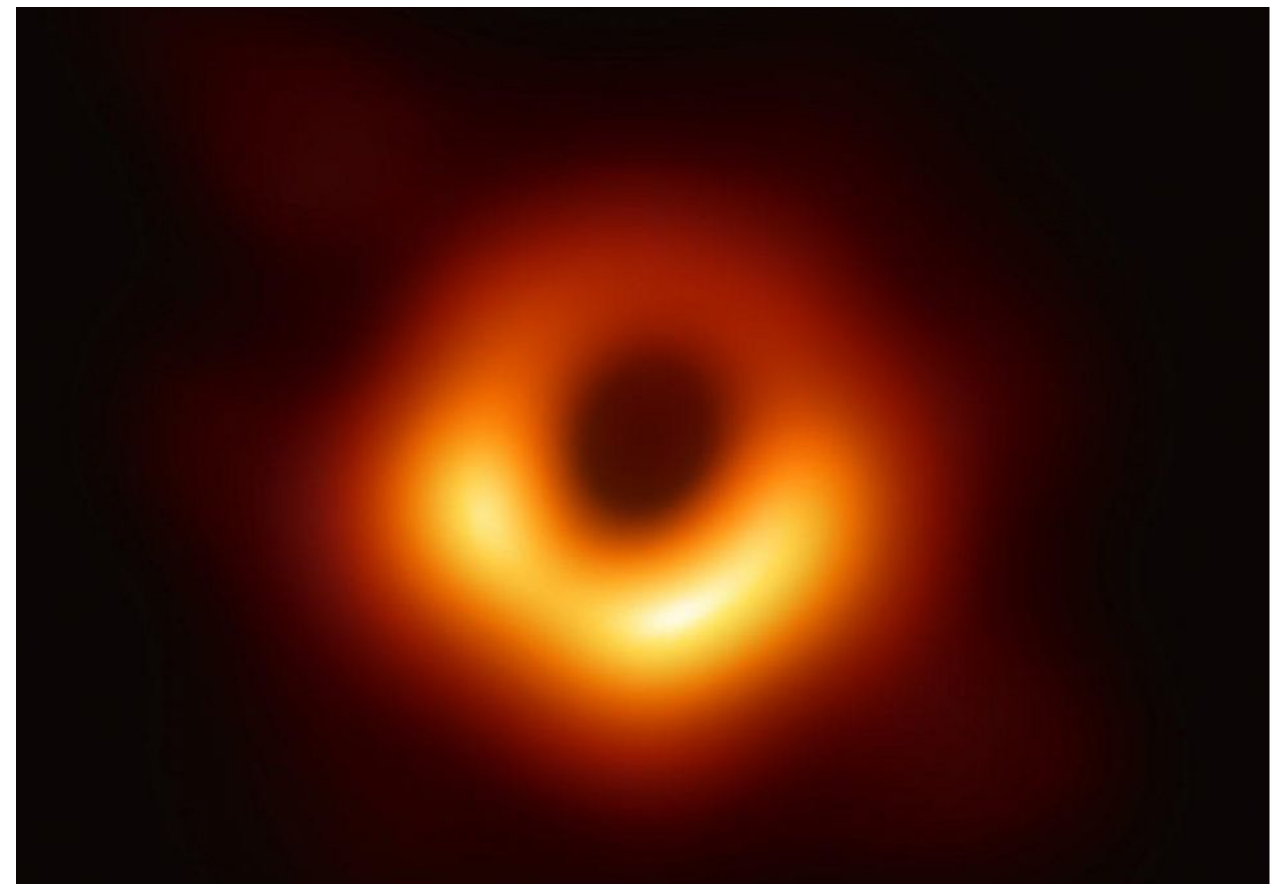
Goal of the class

- **What is general relativistic ray-tracing and why do we need to learn about it?**
- **GR and black hole basics: gravitational unit, event horizon, ISCO, photon sphere, geodesic…**
- **A code on calculating geodesics in the curved space-time around a spinning black hole:**
  - **Exercise 1: a test particle with non-zero mass - apsidal and Lense-Thirring precession**
  - **Exercise 2: a massless test particle — photon orbit**

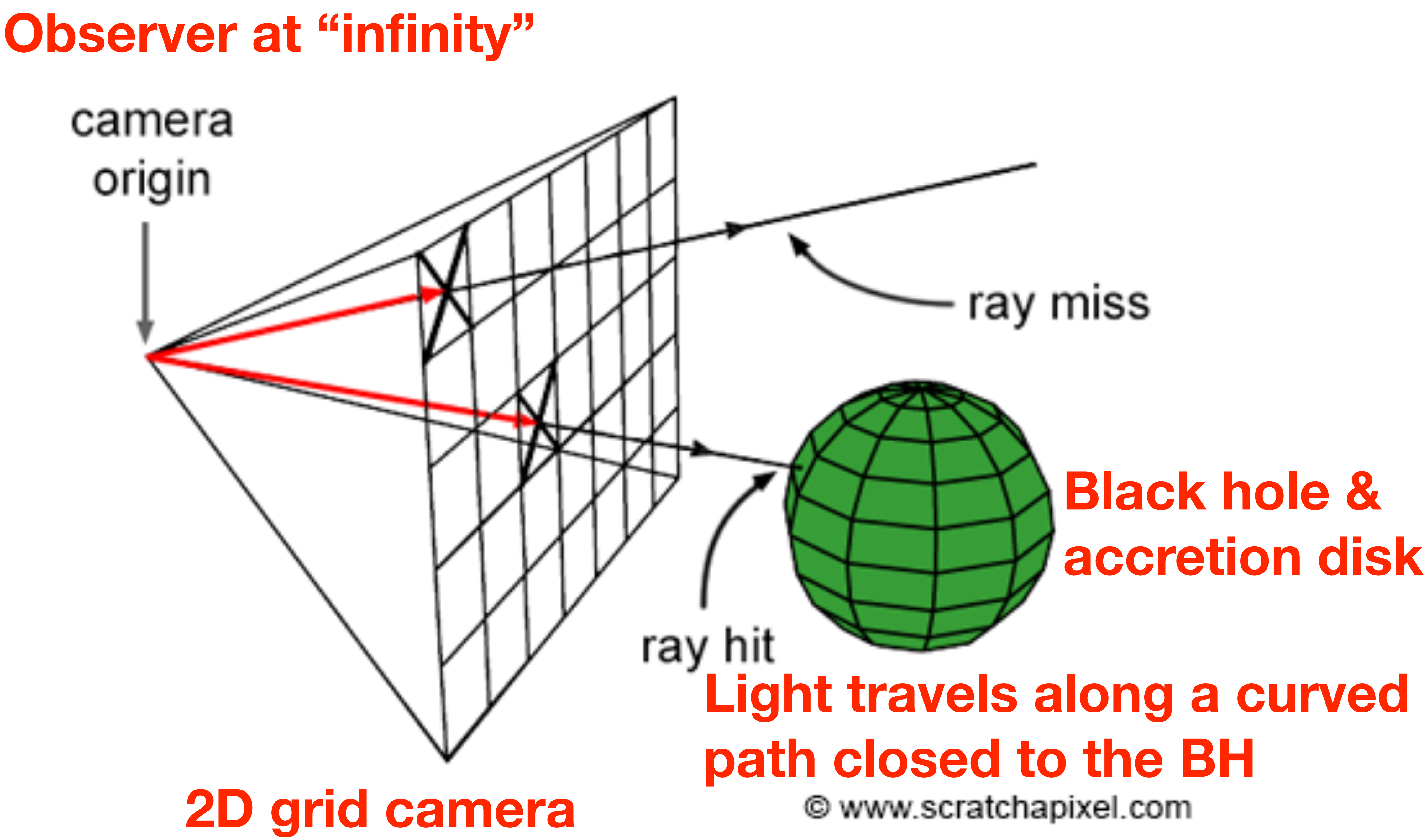
Movie “Interstellar”

Image of M87 by EHT



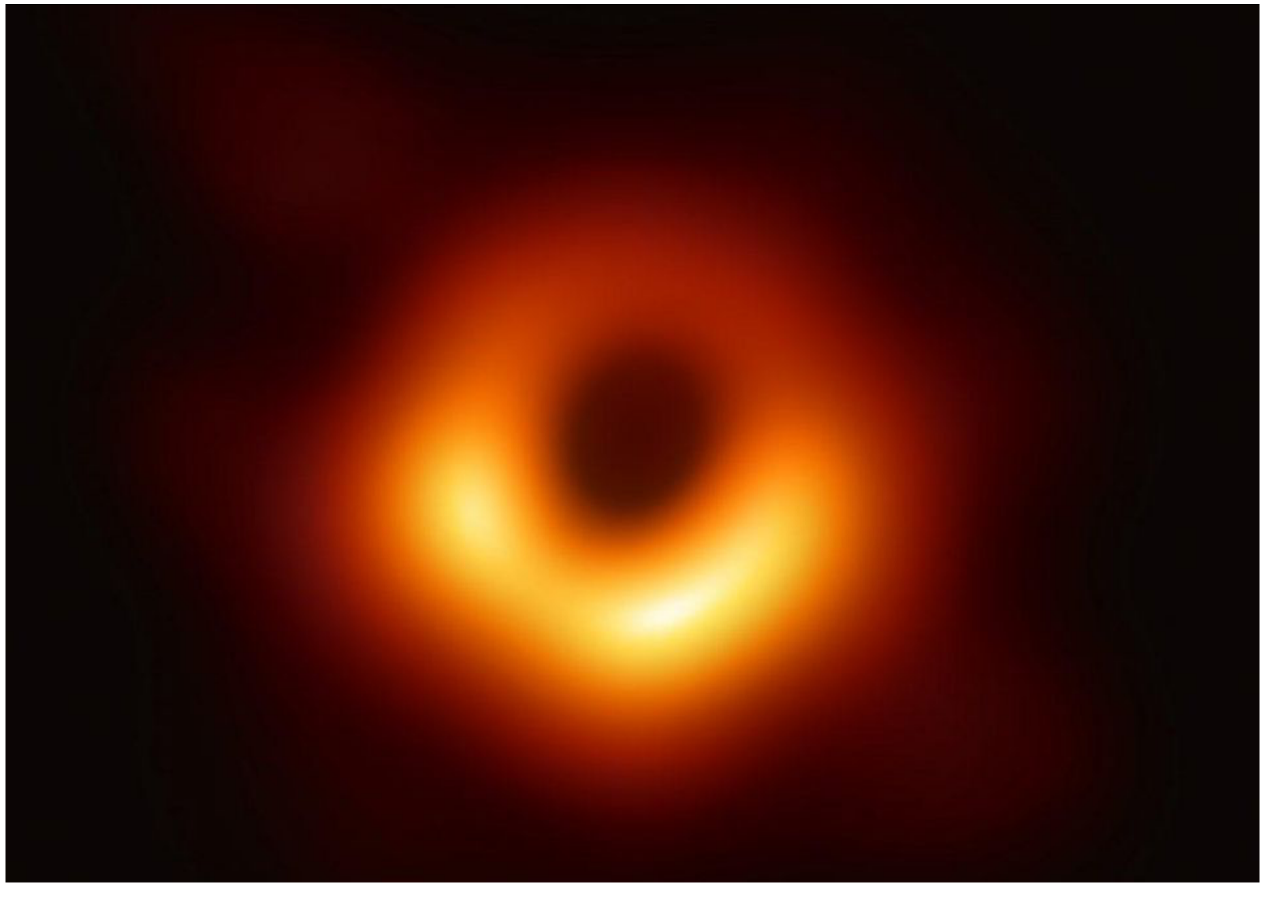
- **What are we really seeing?**
- **Can we already see GR working?**

Standard “Backward” Ray-tracing Method



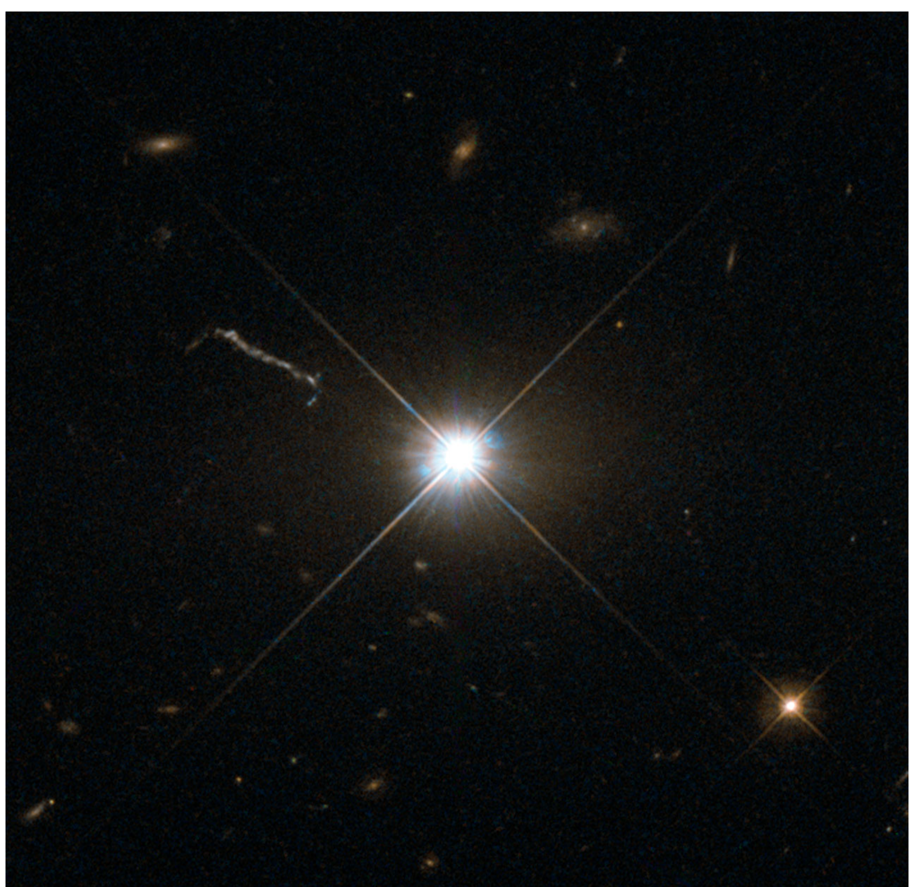
Why studying GR ray-tracing?

- Construct images of accretion disks / black hole shadow



Why studying GR ray-tracing?

- Construct images of accretion disks / black hole shadow
- Calculate accretion disk emission: e.g. Fe Kα emission line at 6.4 keV



The unresolved image of the accretion disk of a faraway black hole

Why studying GR ray-tracing?

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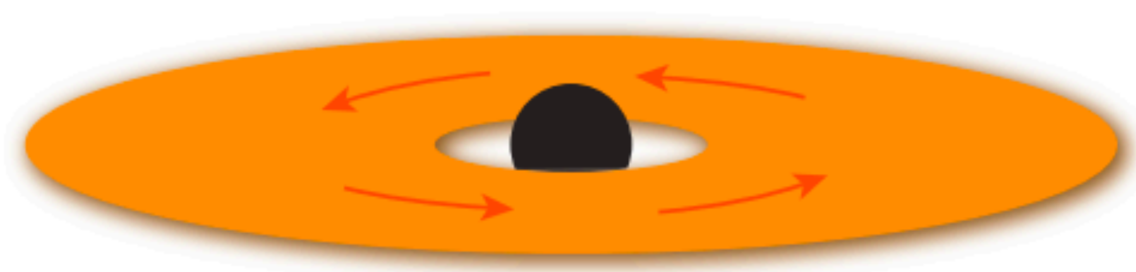
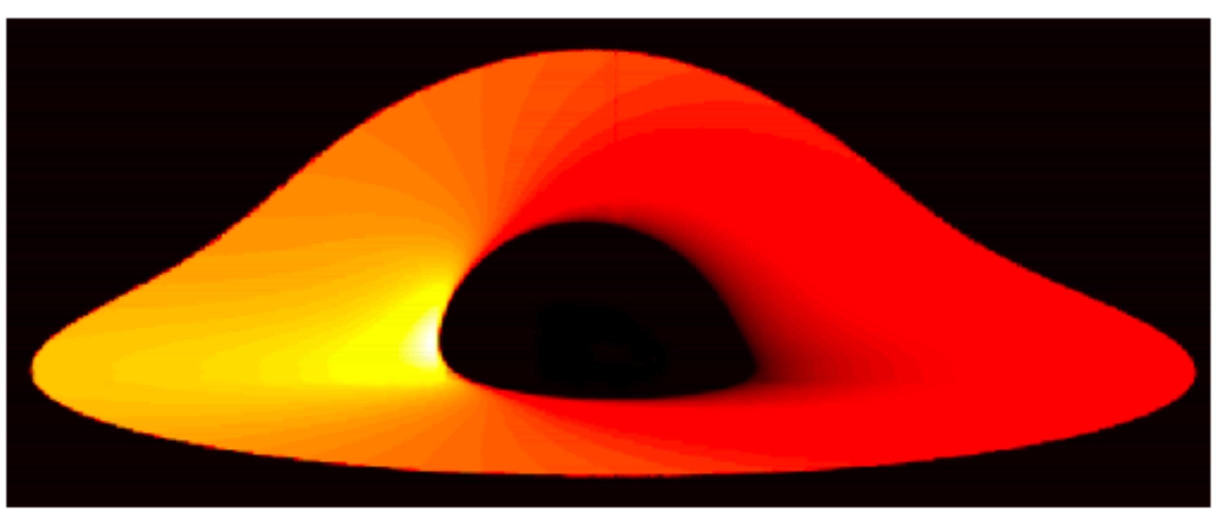
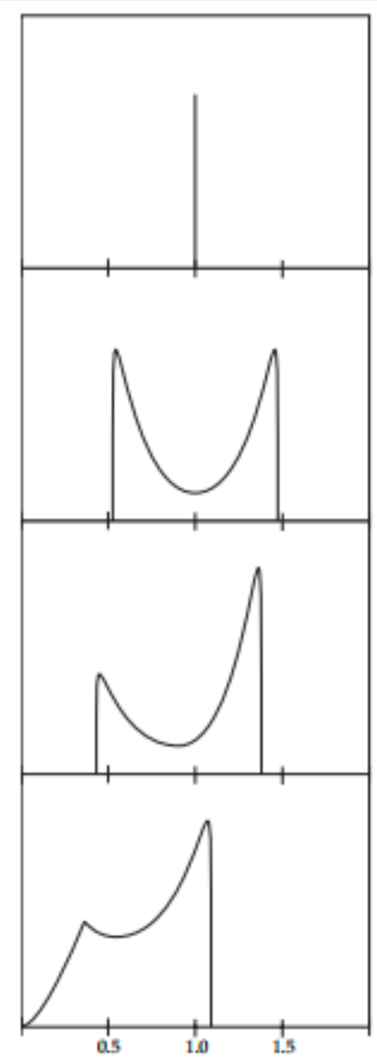


Wilkins 2013





Rest-frame emission line

Newtonian Doppler shift

Special Relativistic Doppler shift

GR Doppler shift + gravitational redshift

Now, some basic concepts… A recap on "gravitational unit"

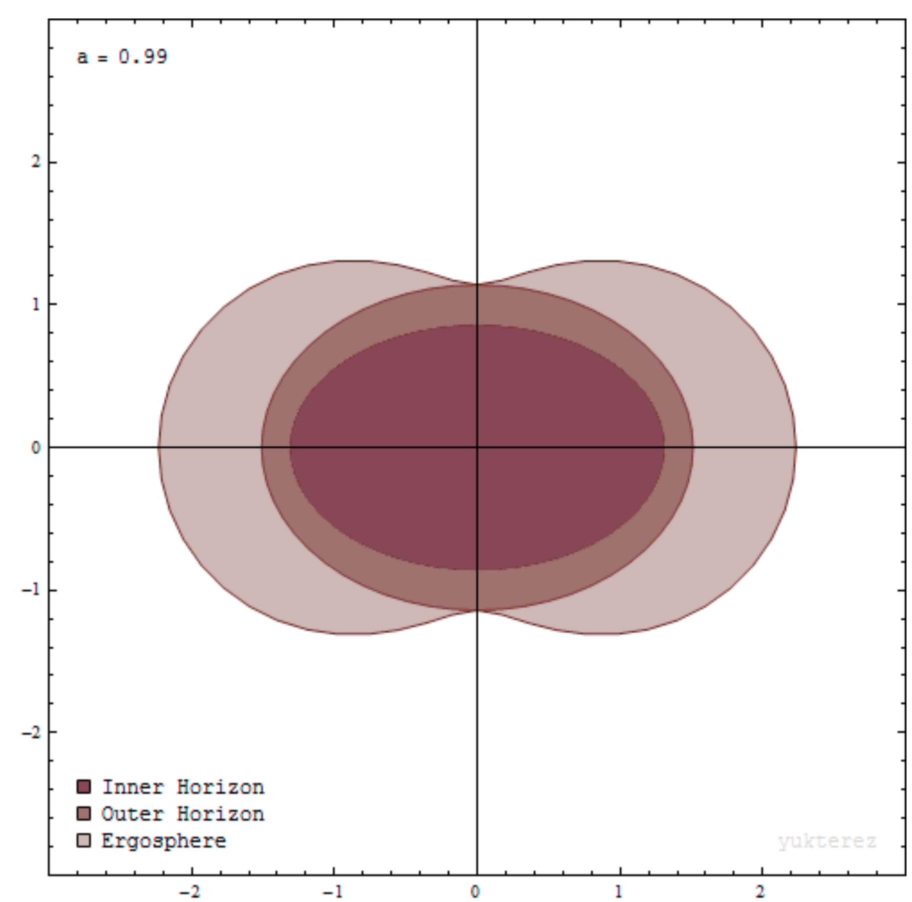
- Constants:
  - gravitational constant **G** [length^3 mass^{-1} time^{-2}]
  - speed of the light **c** [length/time]
  - mass of the black hole **M** [mass]
- Every other unit can be constructed using these constant
- For example, length is in unit: ***$R_g = GM/c^2$ gravitational radius***
- Question: If the orbital time T of an object around a black hole of 1 million solar mass is ~200 in the "code unit" of some GRMHD code, what is the value of T in physical (cgs) unit?

Gravitational Unit

- Question: If the orbital time of an object around a black hole of 1 million solar mass is ~200 in the “code unit”, what is the value of T in physical (cgs) unit?
- A: ~ 10s
- B: ~ 100s
- C: ~ 1000s

Astrophysical black holes

- Two parameters: black hole mass *M* and spin parameter a = *J/Mc*
- a = *0* Schwarzschild black hole; a *>0* Kerr (rotating) black hole; maximum a *=1 (or ~ 0.9998* when the black hole gained mass through accretion (Thorne et al.)
- Event horizon: $R_E = (1 + \sqrt{1 - a^2})R_g$
- Frame-dragging / Lense-Thirring effect
- The ergosphere



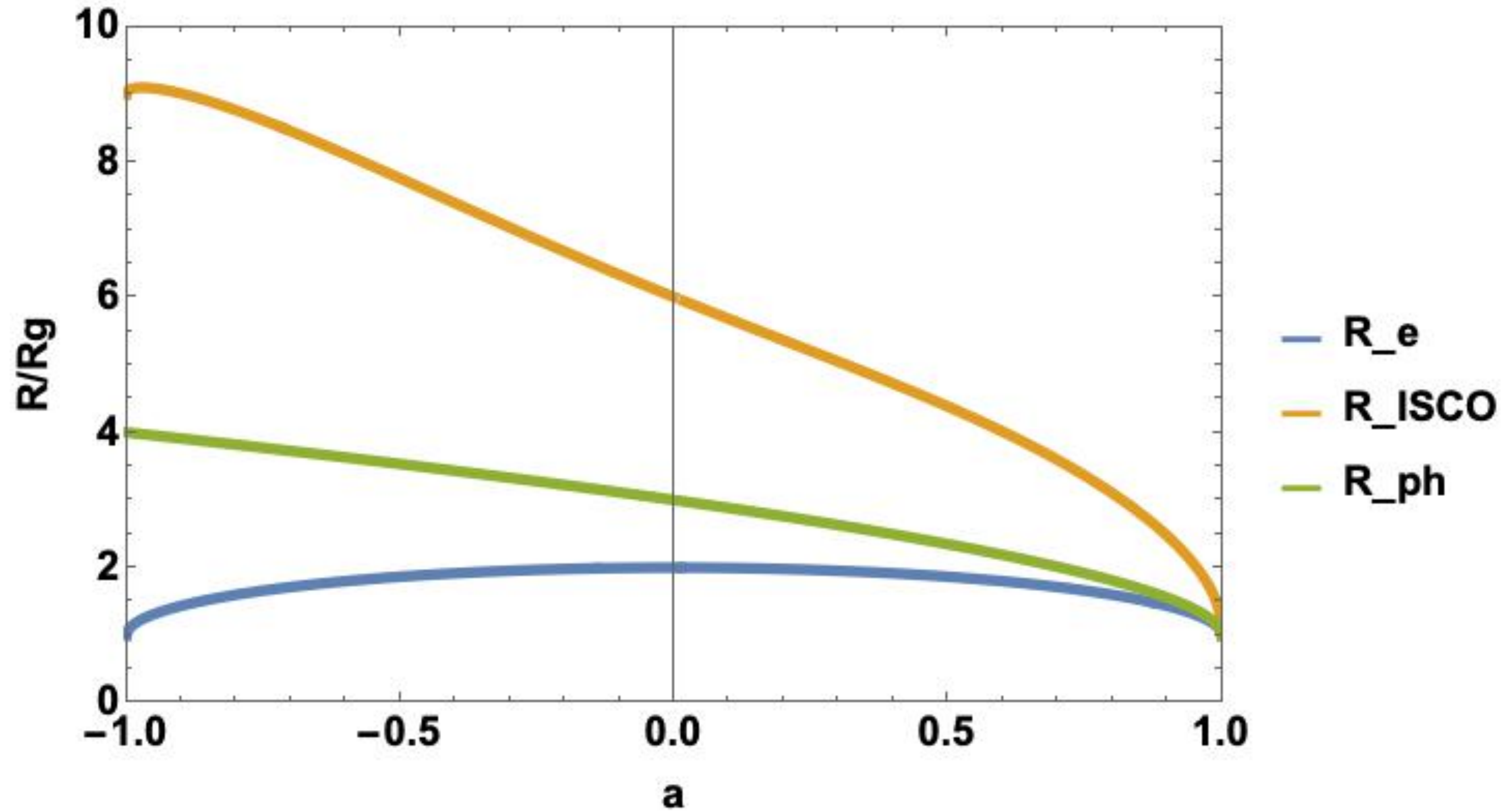
Important radii around black holes

- Gravitational radius:

$$R_g = GM/c^2$$

- Event horizon radius:

$$R_E = (1 + \sqrt{1 - a^2})R_g$$



- Innermost stable circular orbit (ISCO): the smallest circular orbit for a test particle with non-zero mass, often taken as the inner edge of a thin accretion disk

- Photon circular orbit: the only possible radius at which a photon can travel in circles around a black hole

Geodesic

- In general relativity, an object with mass deforms the space-time around it.
- Gravity is no longer an external force, but embedded in the space-time.
- The trajectory of a particle traveling free of other external force follows the geodesic, which is a “straight line” in curved space-time.
- Once we know the space-time metric around a black hole, and the initial position / velocity of a particle, we can calculate its trajectory.

Metric of a Kerr Black hole

$$
\begin{aligned}
d\tau^2 \;=\; & \left(1 - \frac{2Mr}{\Sigma}\right) dt^2 + \frac{4aMr\sin^2\theta}{\Sigma} dt d\phi - \frac{\Sigma}{\Delta} dr^2 \\
& -\Sigma d\theta^2 - \left(r^2 + a^2 + \frac{2a^2 Mr \sin^2\theta}{\Sigma}\right) \sin^2\theta d\phi^2
\end{aligned}
$$

$$\Sigma = r^2 + a^2 \cos^2\theta \text{ and } \Delta = r^2 - 2Mr + a^2.$$

- Boyer-Lindquist (BL) coordinate: a stationary observer at infinity

Properties of the Kerr Metric

$$
\begin{aligned}
d\tau^2 \;&=\; \left(1-\frac{2Mr}{\Sigma}\right)dt^2+\frac{4aMr\sin^2\theta}{\Sigma}dtd\phi-\frac{\Sigma}{\Delta}dr^2\\
&-\Sigma d\theta^2-\left(r^2+a^2+\frac{2a^2Mr\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2
\end{aligned}
$$

$$\Sigma = r^2+a^2\cos^2\theta \text{ and } \Delta = r^2-2Mr+a^2.$$

- Independent of t: conservation of energy E
- Independent of ϕ: conservation of angular momentum L
- There is a mixed term for t and ϕ: Frame-dragging effect
- a=0: reduces to Schwarzschild metric

Metric of a Kerr Black hole

$$
\begin{aligned}
d\tau^2 \;=\;& \left(1-\frac{2Mr}{\Sigma}\right)dt^2+\frac{4aMr\sin^2\theta}{\Sigma}dtd\phi-\frac{\Sigma}{\Delta}dr^2 \\
&-\Sigma d\theta^2-\left(r^2+a^2+\frac{2a^2Mr\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2
\end{aligned}
$$

$$\Sigma = r^2 + a^2\cos^2\theta \text{ and } \Delta = r^2 - 2Mr + a^2.$$

$$
g_{\mu\nu}=\begin{array}{c} \begin{pmatrix} g_{tt} & 0 & 0 & g_{t\phi} \\ 0 & \frac{\Sigma}{\Delta} & 0 & 0 \\ 0 & 0 & \Sigma & 0 \\ g_{t\phi} & 0 & 0 & g_{\phi\phi} \end{pmatrix} \end{array}\begin{array}{c} t \\ r \\ \theta \\ \phi \end{array}
$$

$$
\begin{aligned}
g_{tt} &= -\left(1-\frac{2Mr}{\Sigma}\right) \\
g_{t\phi} &= -\frac{2Mr}{\Sigma}a\sin^2\theta \\
g_{\phi\phi} &= \left[r^2+a^2+\frac{2Mra^2}{\Sigma}\sin^2\theta\right]\sin^2\theta\,.
\end{aligned}
$$

- **Question: what is the metric of flat space-time in this coordinate? (Either have M=a=0, or have r = infinity.)**

- **Question: what is the metric of flat (Minkowskian) space-time in the BL coordinate?**

$$
\begin{array}{cc}
\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} & \begin{matrix} t \\ x \\ y \\ z \end{matrix}
\end{array}
$$

$$
\begin{array}{cc}
g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2(\theta) \end{pmatrix} & \begin{matrix} t \\ r \\ \theta \\ \phi \end{matrix}
\end{array}
$$

(spherical coordinate line element : $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$)

How do we calculate the geodesic?

- **The Lagrangian of a test particle:**

$$\mathcal{L} = \frac{1}{2}\Big[-\left(1-\frac{2Mr}{\Sigma}\right)\dot{t}^2 - \frac{4aMr\sin^2\theta}{\Sigma}\dot{t}\dot{\phi} + \frac{\Sigma}{\Delta}\dot{r}^2 + \Sigma\dot{\theta}^2 + \left(r^2 + a^2 + \frac{2a^2Mr\sin^2\theta}{\Sigma}\right)\sin^2\theta\dot{\phi}^2\Big]$$

$$\Sigma = r^2 + a^2\cos^2\theta \text{ and } \Delta = r^2 - 2Mr + a^2.$$

$\dot{x} = dx/d\lambda$, λ is the affine parameter

How do we calculate the geodesic?

$$\mathcal{L} = \frac{1}{2}\Big[-\left(1-\frac{2Mr}{\Sigma}\right)\dot{t}^2 - \frac{4aMr\sin^2\theta}{\Sigma}\dot{t}\dot{\phi} + \frac{\Sigma}{\Delta}\dot{r}^2$$
$$+\Sigma\dot{\theta}^2 + \left(r^2 + a^2 + \frac{2a^2Mr\sin^2\theta}{\Sigma}\right)\sin^2\theta\dot{\phi}^2\Big]$$

$$\Sigma = r^2 + a^2\cos^2\theta \text{ and } \Delta = r^2 - 2Mr + a^2.$$

Generalized momenta

$$p_{\rm t} = \partial\mathcal{L}/\partial\dot{t} = -E\ ,$$
$$p_{\rm r} = \partial\mathcal{L}/\partial\dot{r} = \frac{\Sigma}{\Delta}\dot{r}\ ,$$
$$p_\theta = \partial\mathcal{L}/\partial\dot{\theta} = \Sigma\dot{\theta}\ ,$$
$$p_\phi = \partial\mathcal{L}/\partial\dot{\phi} = L\ .$$

Euler-Lagrange Equation:

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \text{ or } \dot{p}_x = \frac{\partial L}{\partial x}$$

$$\frac{\partial L}{\partial t} = \frac{\partial L}{\partial \phi} = 0 \Rightarrow p_t, p_\phi = \text{constant}$$

How do we calculate the geodesic?

$$\mathcal{L} = \frac{1}{2}\Big[-\left(1-\frac{2Mr}{\Sigma}\right)\dot{t}^2 - \frac{4aMr\sin^2\theta}{\Sigma}\dot{t}\dot{\phi} + \frac{\Sigma}{\Delta}\dot{r}^2 + \Sigma\dot{\theta}^2 + \left(r^2+a^2+\frac{2a^2Mr\sin^2\theta}{\Sigma}\right)\sin^2\theta\dot{\phi}^2\Big]$$

$$\Sigma = r^2 + a^2\cos^2\theta \text{ and } \Delta = r^2 - 2Mr + a^2.$$

Generalized momenta

$$p_{\mathrm{t}} = \partial\mathcal{L}/\partial\dot{t} = -E\,,$$
$$p_{\mathrm{r}} = \partial\mathcal{L}/\partial\dot{r} = \frac{\Sigma}{\Delta}\dot{r}\,,$$
$$p_{\theta} = \partial\mathcal{L}/\partial\dot{\theta} = \Sigma\dot{\theta}\,,$$
$$p_{\phi} = \partial\mathcal{L}/\partial\dot{\phi} = L\,.$$

Exercise:

Can you calculate this angular momentum L for a non-spinning black hole and assume the particle moves on the equatorial plane ($\theta=\pi/2$)?

Variables & Equations in the code

6 variables: $t, r, \theta, \phi, p_r, p_\theta$

6 ODEs:

$$\dot{t} = E + \frac{2r(r^2+a^2)E - 2aL}{\Sigma\Delta},$$

$$\dot{r} = \frac{\Delta}{\Sigma} p_r \qquad \dot{\theta} = \frac{1}{\Sigma} p_\theta$$

$$\dot{\phi} = \frac{2arE + (\Sigma - 2r)L/\sin^2\theta}{\Sigma\Delta},$$

$$\dot{p}_{\mathrm{r}} = \frac{1}{\Sigma\Delta}\left[(r-1)\left((r^2+a^2)H - \kappa\right) + r\Delta H \right. \\ \left. + 2r(r^2+a^2)E^2 - 2aEL\right] - \frac{2{p_r}^2(r-1)}{\Sigma}$$

$$\dot{p}_\theta = \frac{\sin\theta\cos\theta}{\Sigma}\left[\frac{L^2}{\sin^4\theta} - a^2(E^2+H)\right],$$

Constants of motion:

E: energy

L: angular momentum

Q: Carter's constant

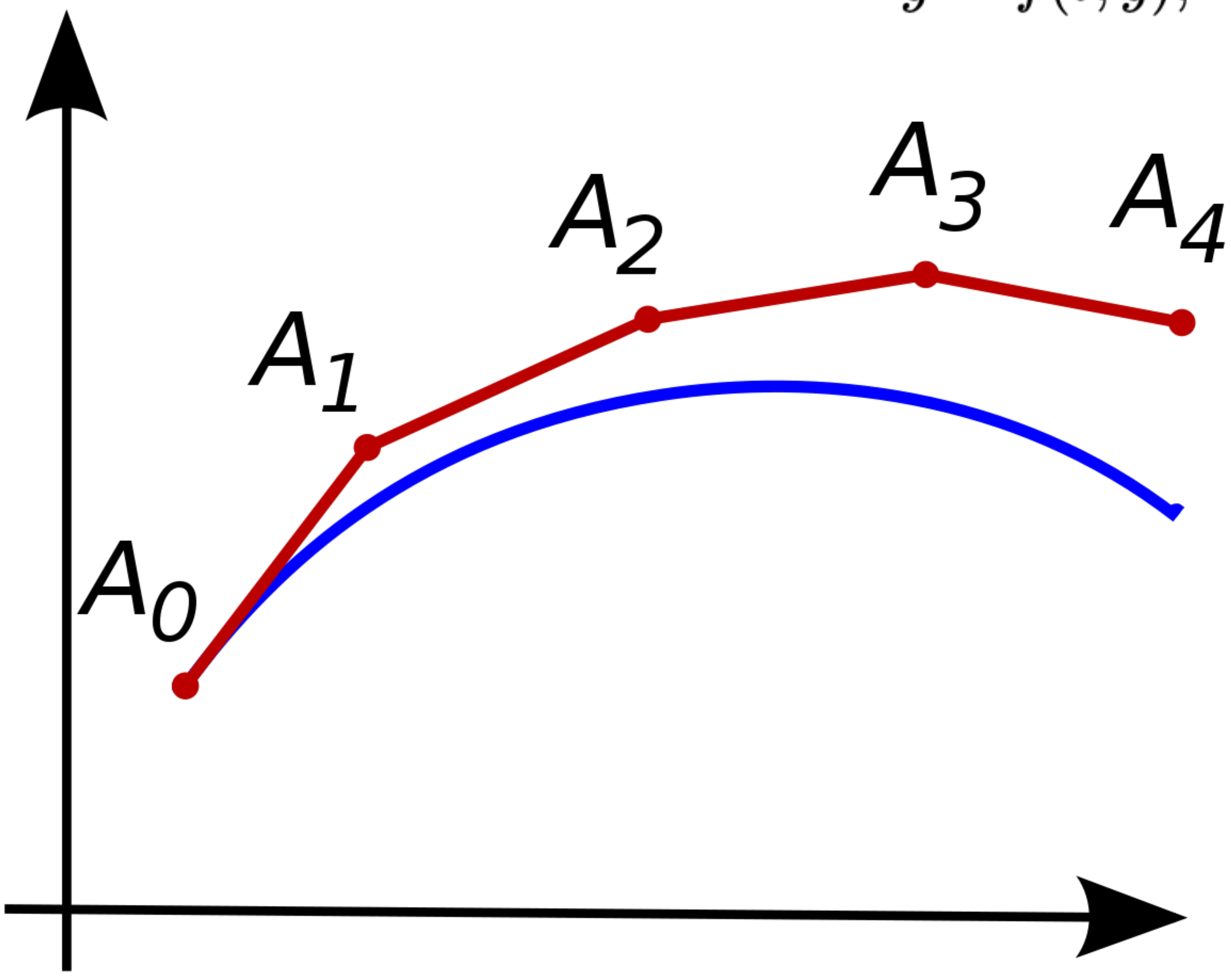
H: Hamiltonian
= 0 for photon;
= -1 for a particle with non-zero mass

$$\kappa = Q^2 + L^2 + a^2(E^2 + H)$$

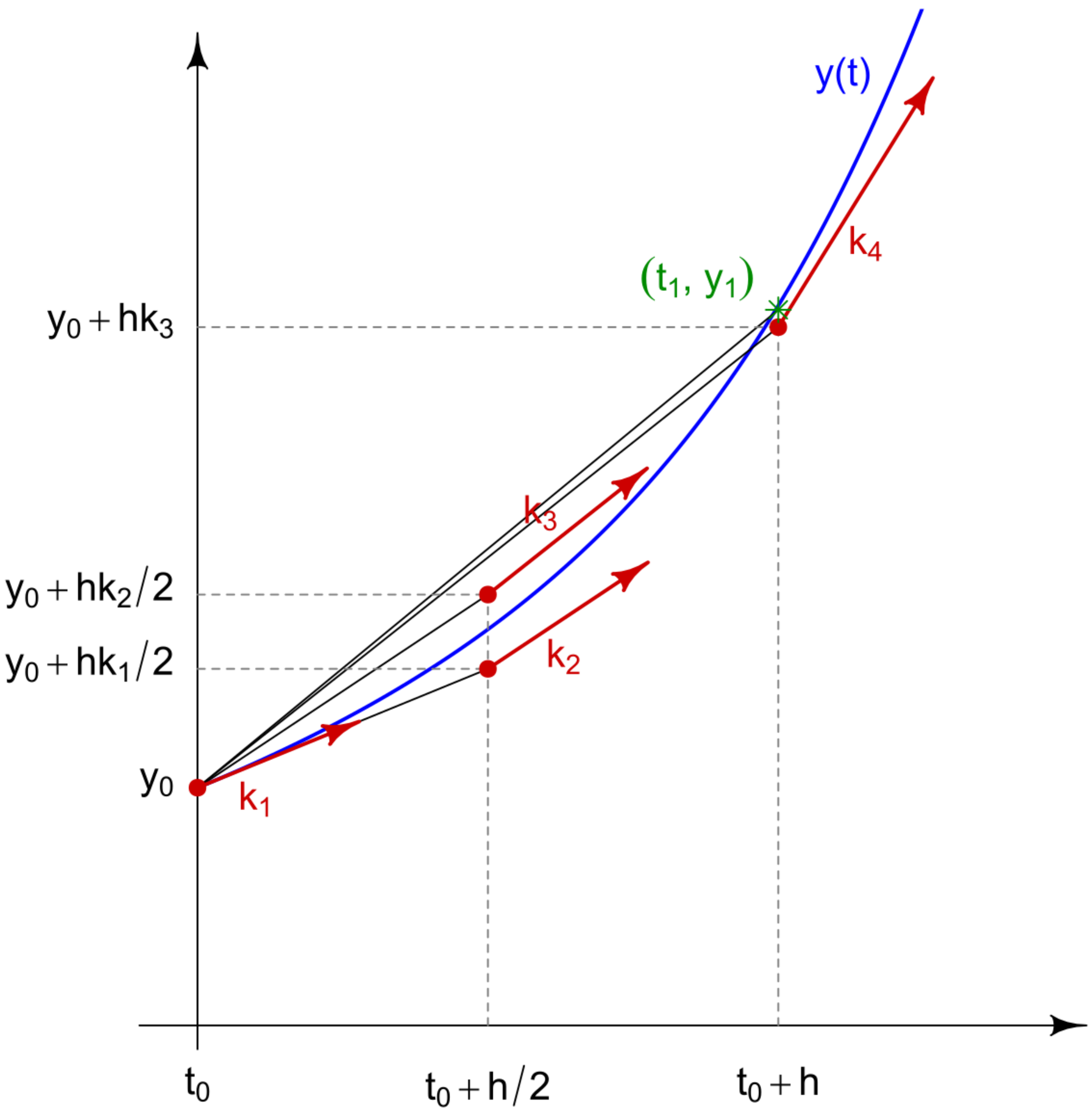
von Fuerst & Wu 2004
and references therein

Solving differential equations: Euler Method

$$\dot{y} = f(t, y), \quad y(t_0) = y_0.$$



Solving differential equations: Runge-Kutta Method



Example: 4th-order RK

$$\dot{y} = f(t, y), \quad y(t_0) = y_0.$$

$$k_1 = h\, f(t_n, y_n),$$

$$k_2 = h\, f\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right),$$

$$k_3 = h\, f\left(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right),$$

$$k_4 = h\, f\left(t_n + h, y_n + k_3\right).$$

$$y_{n+1} = y_n + \tfrac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

Code: 5th order RK, adaptive step size

Exercise 1:
Apsidal precession & Lense-thirring precession

- Connect to your VM machine using: ssh -Y ataschool@…
- Copy this folder from Cartesius to your local VM machine:

  rsync -Pav ccurs071@cartesius.surfsara.nl:/scratch-shared/ccurs071/GRRay /data/.

  Password: ATAschool2019

- Go to the directory of the first exercise: cd /data/GRRay/exercise1
- Find the C code "orbit_mass.cpp" (which evolves the geodesic for a test particle with mass)
- Compile the code: g++ orbit_mass.cpp
- Run the executable: ./a.out
- You should see a file called "trajectory.dat" generated

Exercise 1:
Apsidal precession & Lense-thirring precession

- Open another terminal/window and connect to VM machine and go to the same directory: cd /data/GRRay/exercise1

- Type “gnuplot” to open this software

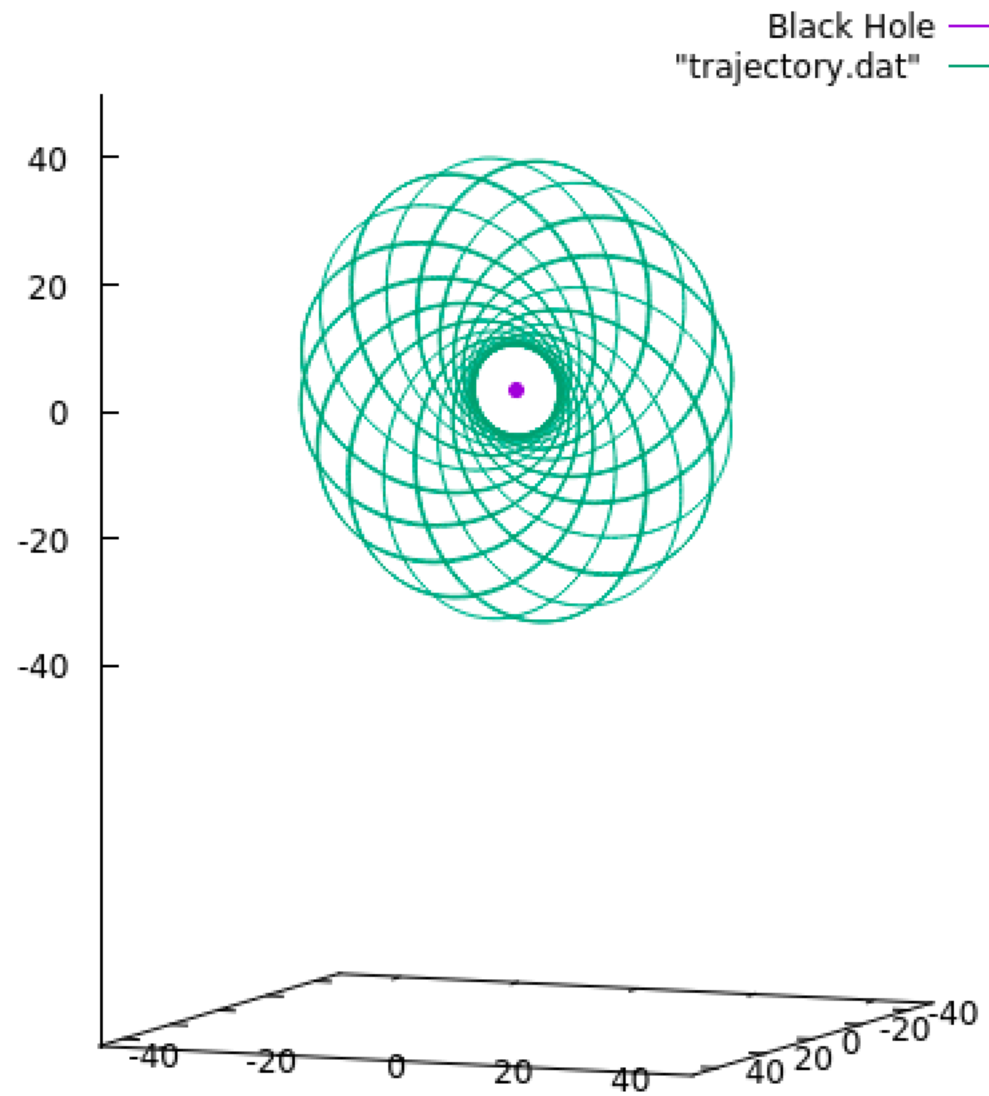
- In gnuplot, type the following lines to see the trajectory of the particle:

```
set parametric
set view equal xyz
set zrange [-50:50]
set xrange [-50:50]
set yrange [-50:50]

splot [0:6.284] [0:6.284] cos(u)*cos(v),cos(u)*sin(v),sin(u) t "Black
Hole", "trajectory.dat" with lines
```

- In case gnuplot doesn’t work on your VM, you can also use the Jupyter ipython notebook in /data/GRray

Exercise 1:
Apsidal precession & Lense-thirring precession



a=0

Exercise 1:
Apsidal precession & Lense-thirring precession

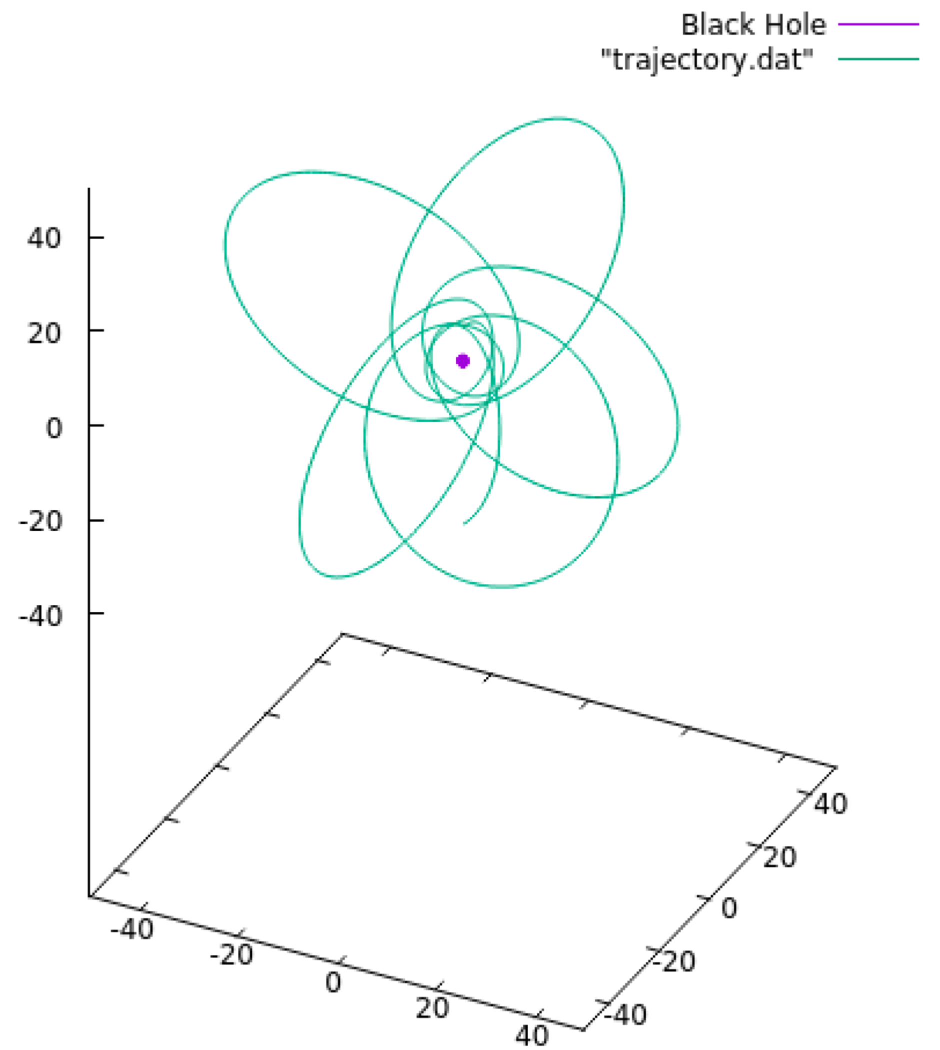
- The original parameters gave us an elliptical orbit around a non-spinning black hole. Do you see the apsidal precession?

- Now change black hole spin parameter a from 0 to 0.9. Compile/run the code again to get another trajectory. Do you see the Lense-Thirring precession? (You can rotate the 3D plot in gnuplot to view it from another angle.)

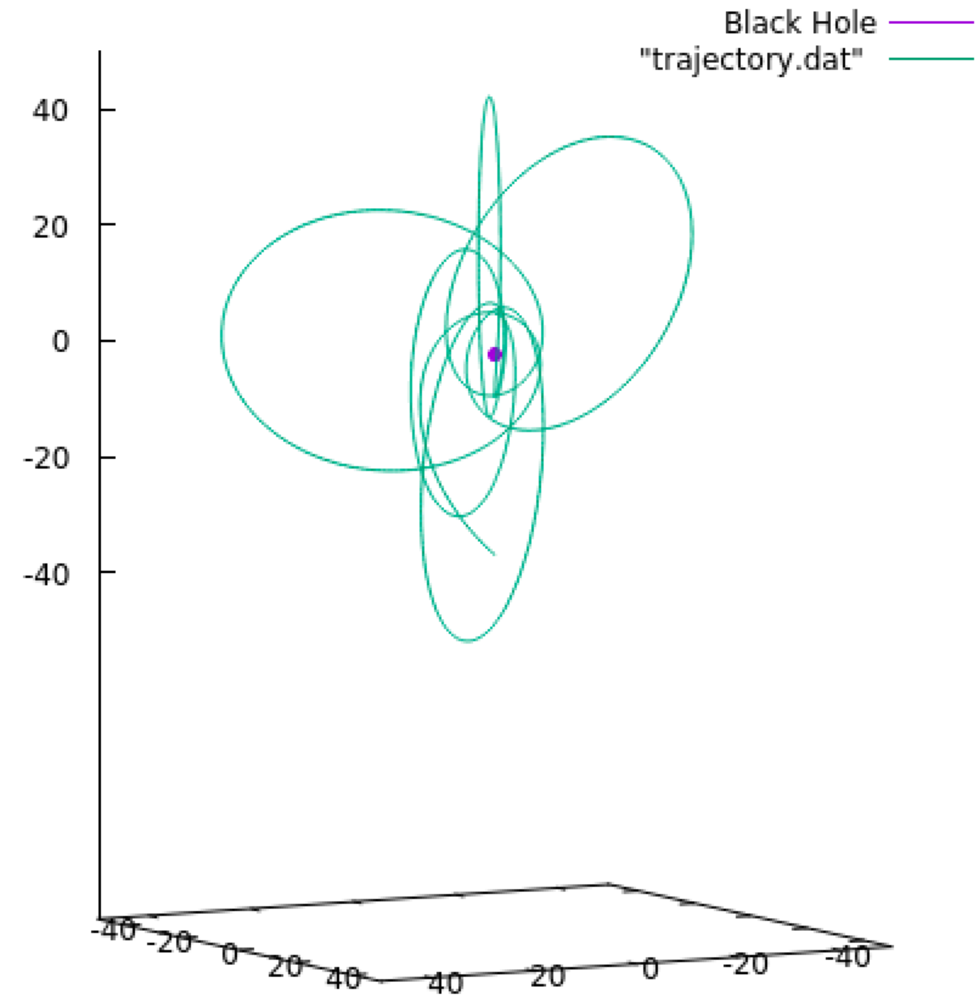
- After class: can you change the test particle initial position and velocity (look for "Edit here") so the particle travels on a Keplerian circular orbit with R=10 R_g on the equatorial plane ? The angular velocity of a circular orbit around a Kerr black hole is:

$$\omega_k = \frac{1}{r^{3/2} + a}$$

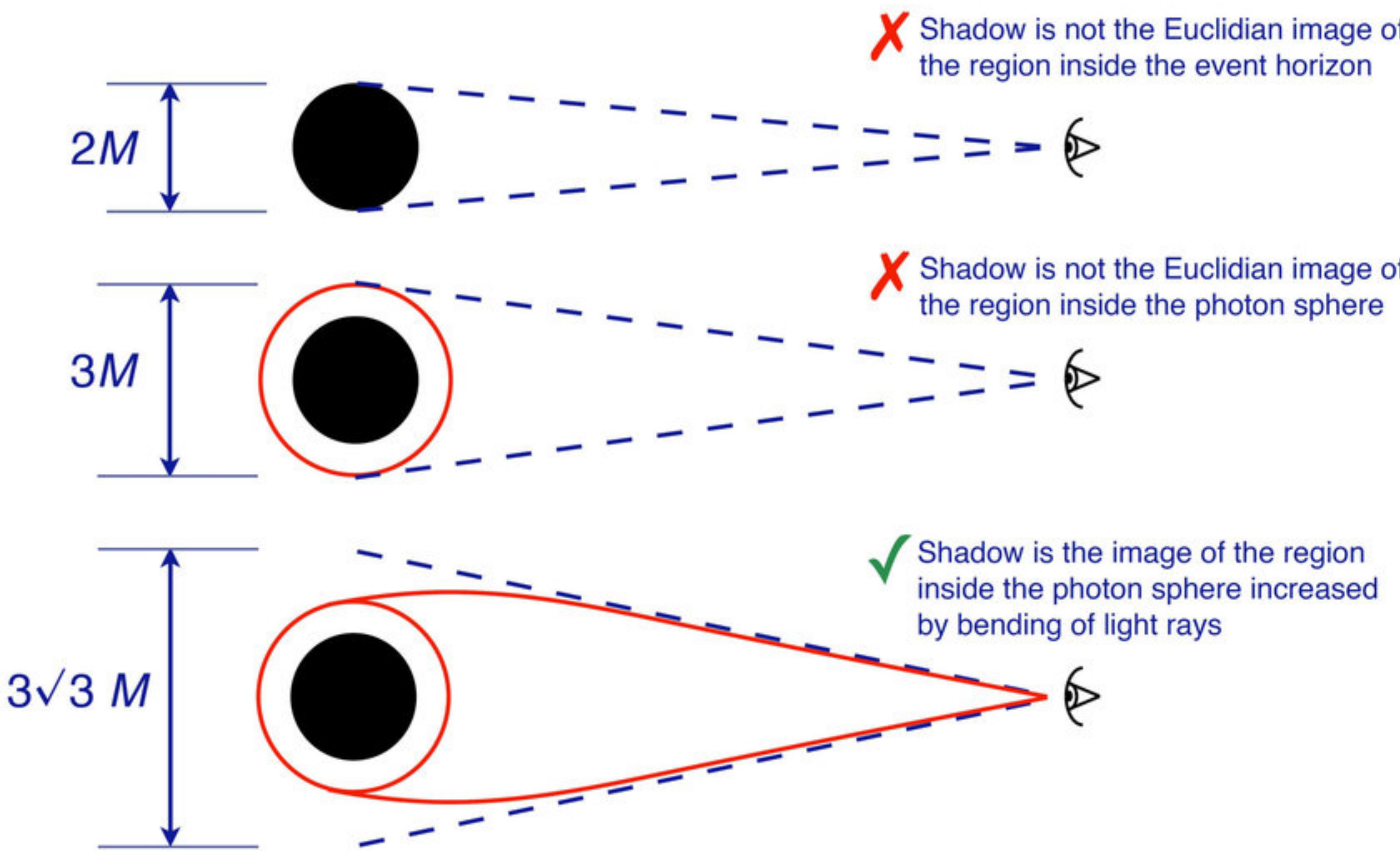
Exercise 1:
Apsidal precession & Lense-thirring precession



a=0.9



Exercise 2:
Black hole shadow / Photon sphere



Exercise 2:
Black hole shadow / Photon sphere

- On your VM machine, go to the directory: /data/GRRay/exercise2
- Find the C code “orbit_photon.cpp” (which evolves the geodesic for a massless test particle)
- Compile the code: g++ orbit_photon.cpp
- Run the executable: ./a.out
- You will be asked “what is the angle alpha deviating from radial infall?” Type 0 for now.
- The code will tell you the photon has plunged into the black hole or escaped

Exercise 2

- Run the code and increase the value of angle alpha (by ~0.001 every time) and see the fate of the photon
- alpha~ 0.0051: the photon orbits around the black hole and back to the observer
- Plot the photon trajectory in gnuplot. Zoom in to check if the photon orbit has r~3 Rg when alpha~ 0.0051

```
set xrange [-10:10]
set yrange [-10:10]
set zrange [-10:10]
(set these ranges to be 1000 if you want to see the
trajectory all the way from the source to black hole,
or 10 if you want check the photon sphere)

splot [0:6.284] [0:6.284]
cos(u)*cos(v),cos(u)*sin(v),sin(u) t "Black Hole",
"trajectory.dat" with lines
```