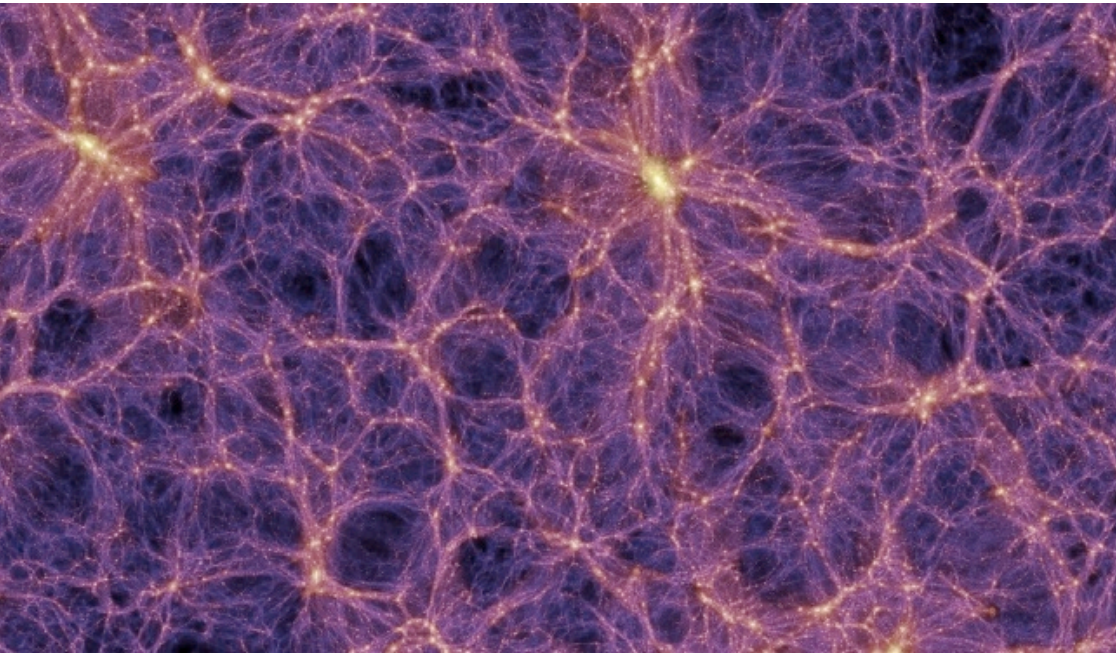
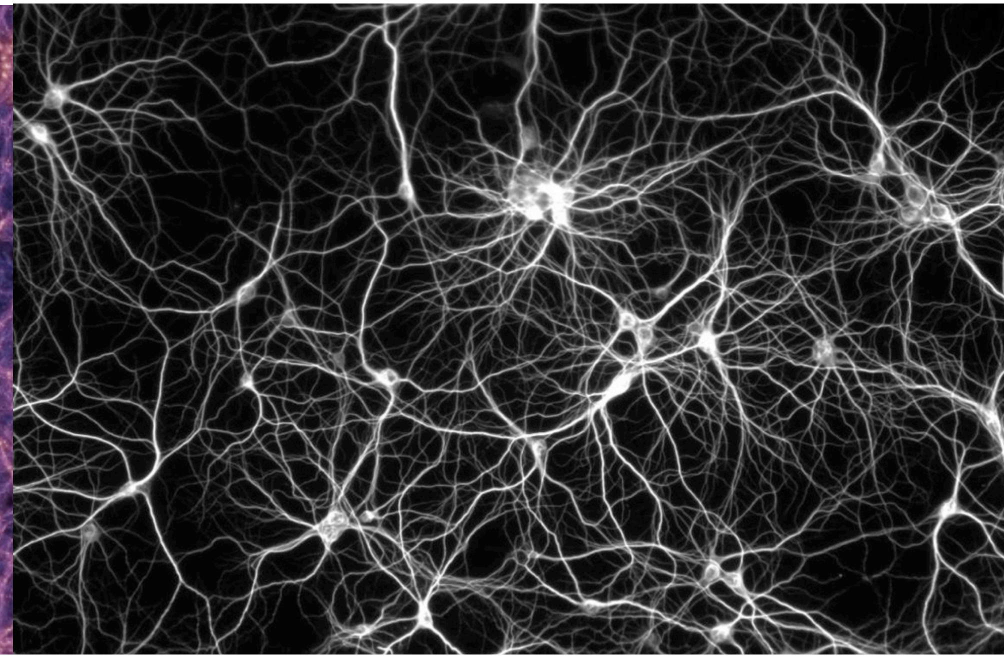
Energy Scales of Physical Phenomena





energy scales: rest-mass

We can associate the rest mass energy mc^2 with each particle of mass m.

$$mc^2 = \begin{cases} m_e c^2 \approx 0.5\mathrm{MeV} \\ m_p c^2 \approx 1\mathrm{GeV} \end{cases}$$

If the particles of the system have internal structure (molecular, atomic, nuclear, etc) then we get further energy scales that are characteristic of the interactions. The simplest is the atomic binding energy of atoms and molecules, which arises from the electromagnetic coupling between particles.

energy scales: atomic

Size and energy of the ground state of a hydrogen atom:

$$a_0 = \frac{\hbar^2}{m_e q^2} \approx 5.2 \times 10^{-9} \text{ cm}$$

$$\epsilon_a = \frac{m_e q^4}{2\hbar^2} = \frac{1}{2}\alpha^2 m_e c^2 \approx 13.6 \text{ eV}$$

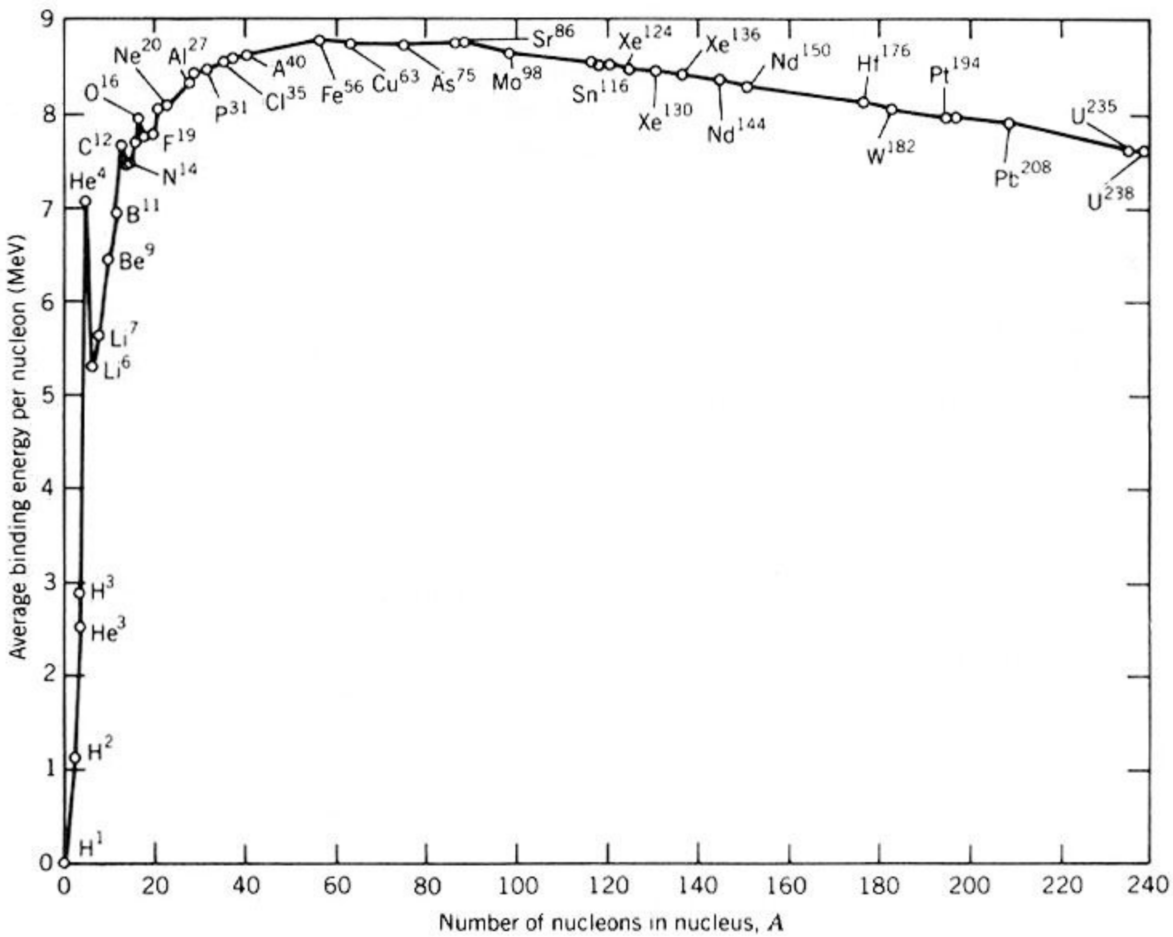
$\alpha = q^2/(\hbar c)$ is the fine structure constant.

When the atoms of size a_0 are closely packed:

$$n_{\text{solid}} \approx (2a_0)^{-3} \approx 10^{24} \text{ cm}^{-3}.$$

energy scales: nuclear

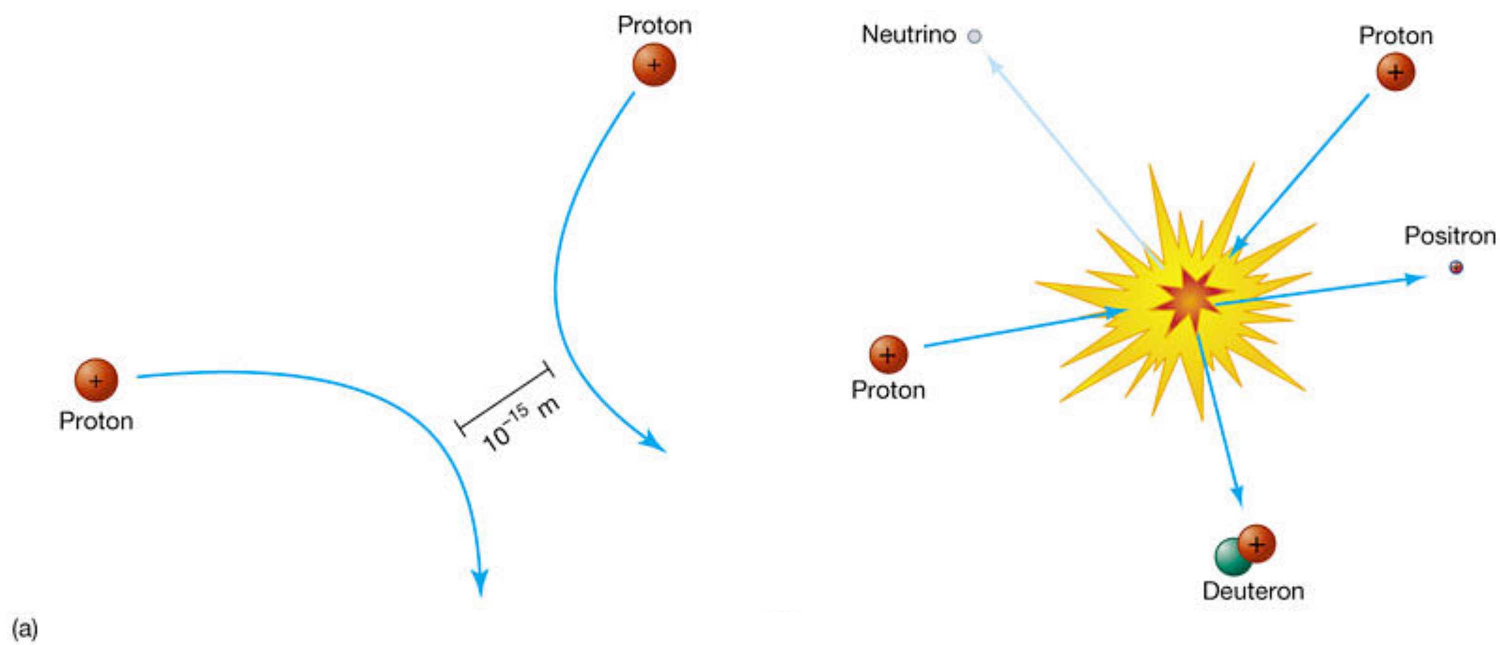
Atomic nuclei are bound by the strong interaction force that produces a binding energy per particle about 8 MeV, which is the characteristic scale for nuclear energy levels



energy scales: nuclear

In the astrophysical context, a more relevant energy scale is the one at which nuclear reactions can be triggered. Nuclear reactions to occur through quantum-mechanical tunneling when the de Broglie wavelength $\lambda_{\rm deB} = h/(m_p v)$ of the two protons overlap. This occurs when the energy is approximately

$$\epsilon_{\rm nucl} \approx \frac{\alpha^2}{2\pi^2}\, m_p c^2 \approx 1 \text{ keV}.$$



energy scales: gravitational

In the non-relativistic Newtonian theory of gravity, the gravitational binding energy:

$$E_{\text{grav}} \approx \frac{GM^2}{R} \approx \frac{Gm_p^2}{R} N^2.$$

The potential energy per particle:

$$\epsilon_g = \frac{E_{\text{grav}}}{N} = \frac{Gm_p^2}{R} N = \frac{4\pi}{3}^{1/3} Gm_p^2 N^{2/3} n^{1/3}.$$

energy scales: thermal

The behavior of the system depends on the origin of the momentum distribution of the particles. The familiar situation is the one in which short-range interactions between particles effectively exchange the energy so as to randomize the momentum distribution. When such a system is in steady state, we can assume that the local thermodynamical equilibrium, characterized by a temperature T, exists in the system:

$$\epsilon \approx k_B T$$

In this case, the momentum and the kinetic energy of the particle vanishes when

$$T \to 0.$$

energy scales: degeneracy

The mean energy of a system of electrons will not vanish at zero temperature because electrons obey the Pauli exclusion principle, which requires that the number of electrons that can occupy any quantum state be two.

The number of quantum states with a phase space volume $p_F = \hbar(3\pi^2 n)^{1/3}$.
sets the quantum mechanical scale of the energy

$$\epsilon_F = \sqrt{p_F^2 c^2 + m^2 c^4} - mc^2 \approx \begin{cases} \frac{p_F^2}{2m} = \left(\frac{\hbar^2}{2m}\right)(3\pi^2 n)^{2/3} & \text{if } mc^2 \gg \epsilon_F \\ p_F c = (\hbar c)(3\pi^2 n)^{1/3} & \text{if } mc^2 \ll \epsilon_F \end{cases}$$

energy scales: pressure

Pressure exerted by the a system of particles as the rate of momentum transferred (per unit area) from particles of energy ϵ:

$$\begin{cases} \frac{2}{3} < n\epsilon > & \text{if } mc^2 \gg \epsilon \\ \frac{1}{3} < n\epsilon > & \text{if } mc^2 \ll \epsilon \end{cases}$$

Pressure ~ (Energy per particle) x (number density)

Energy per particle

1 GeV — 0.5 MeV — 1 keV — 13.6 eV

$\varepsilon_p \approx m_p c^2$

$\varepsilon_e \approx m_e c^2$

$\varepsilon_{\text{nucl}} \approx \dfrac{\alpha^2}{2\pi^2}\, m_p c^2$

$\varepsilon_a = \dfrac{m_e q^4}{2\hbar^2} = \dfrac{1}{2}\alpha^2 m_e c^2$

density

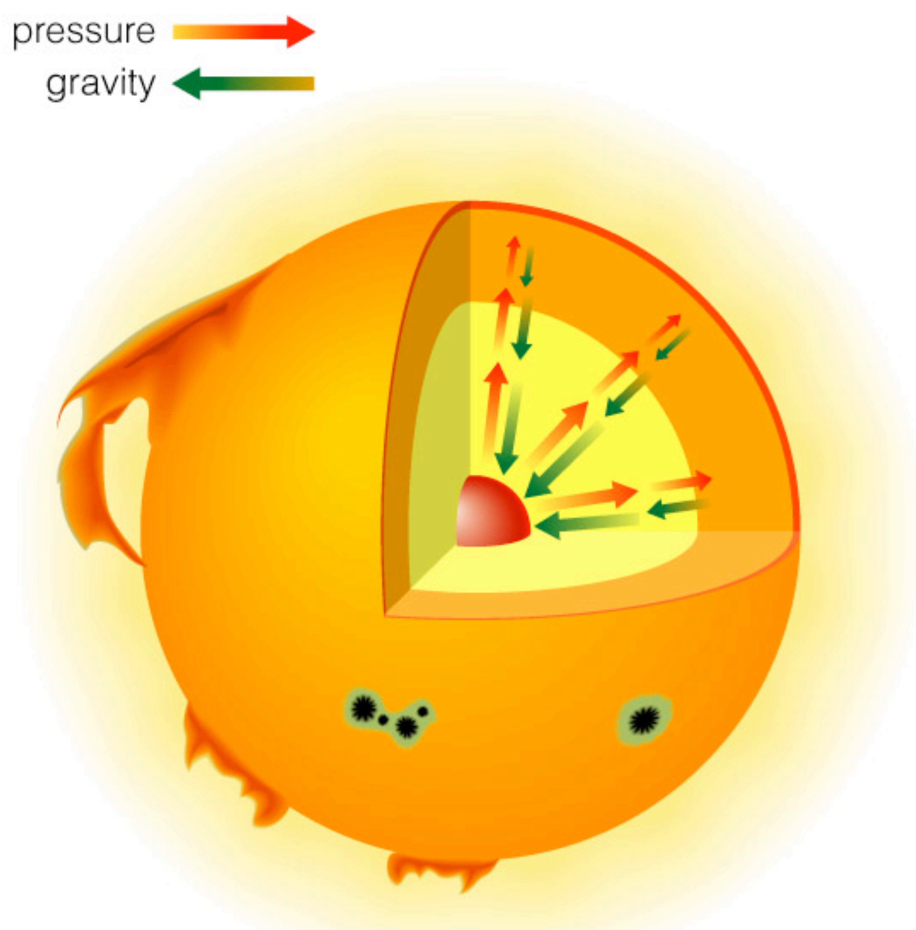
thermal energy

gravity

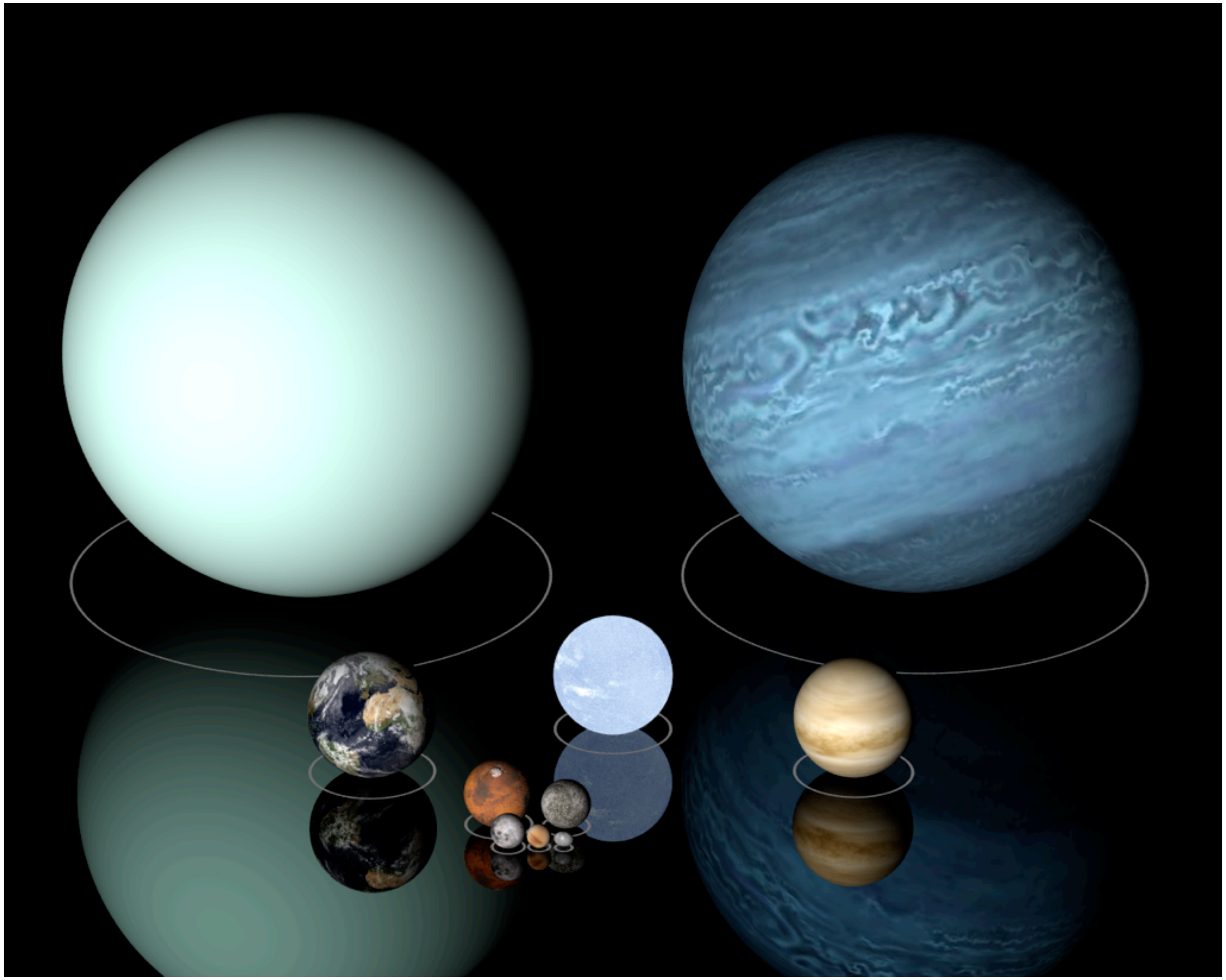
pressure

Varieties of Astrophysical Structures

We now turn to the question of trying to determine broad features of astronomical systems from the description of the key physical energy scales. It is clear that we are interested in systems that are massive enough so that gravity plays a role in their dynamics.



structures: giant planets



structures: giant planets

The laboratory systems have negligible gravitational potential energy. The atomic binding energy (per particle) of a system is approximately

$$\epsilon_a \approx \alpha^2 m_e c^2 \approx \frac{q^2}{a_0} \approx q^2 n^{1/3}$$

If the atoms are closely packed

$$n a_0^3 = 1$$

The gravitational energy per particle is

$$\epsilon_g = (\frac{4\pi}{3})^{1/3} G m_p^2 N^{2/3} n^{1/3}$$

Their ratio is given by

$$\Re_{ga} = \frac{\epsilon_a}{\epsilon_g} = \left(\frac{\alpha}{\alpha_G}\right)\left(\frac{1}{N^{2/3}}\right) = \left(\frac{N_G}{N}\right)^{2/3} \approx \left(\frac{10^{54}}{N}\right)^{2/3}$$

$$\alpha_G = (G m_p^2/\hbar c) \approx 6 \times 10^{-39}$$

structures: giant planets

Clearly, the number $N_G = \alpha^{3/2}\alpha_G^{-3/2} \approx 10^{54}$, arising out of fundamental constants, sets the smallest scale in astrophysics, in which the gravitational binding energy becomes as important as the electromagnetic binding energy of matter.

The corresponding mass

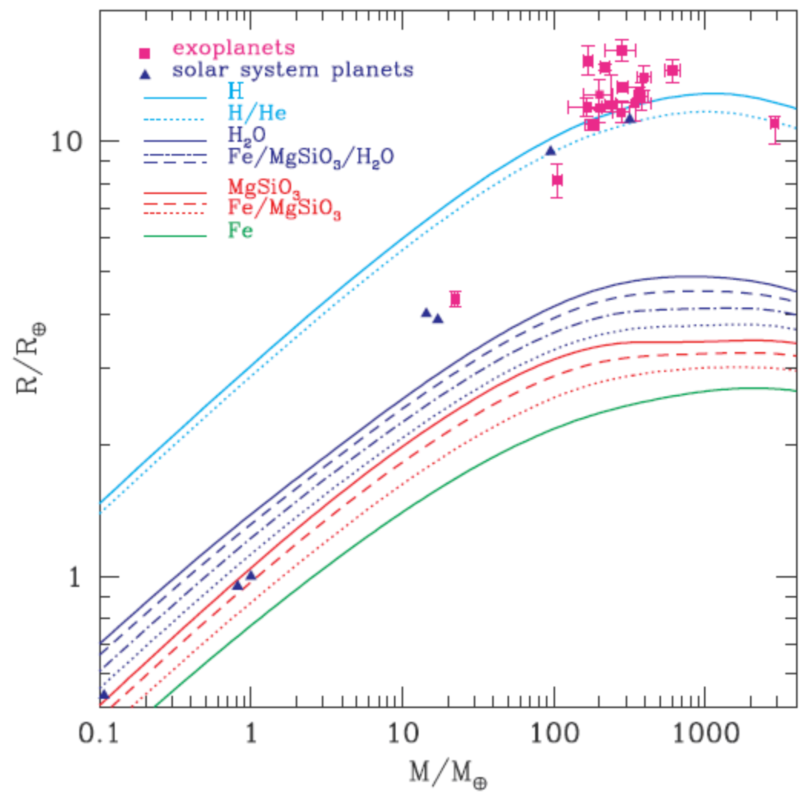
$$M_{\rm planet} \approx N_G m_p \approx 10^{30}\ {\rm g}$$

and length scale

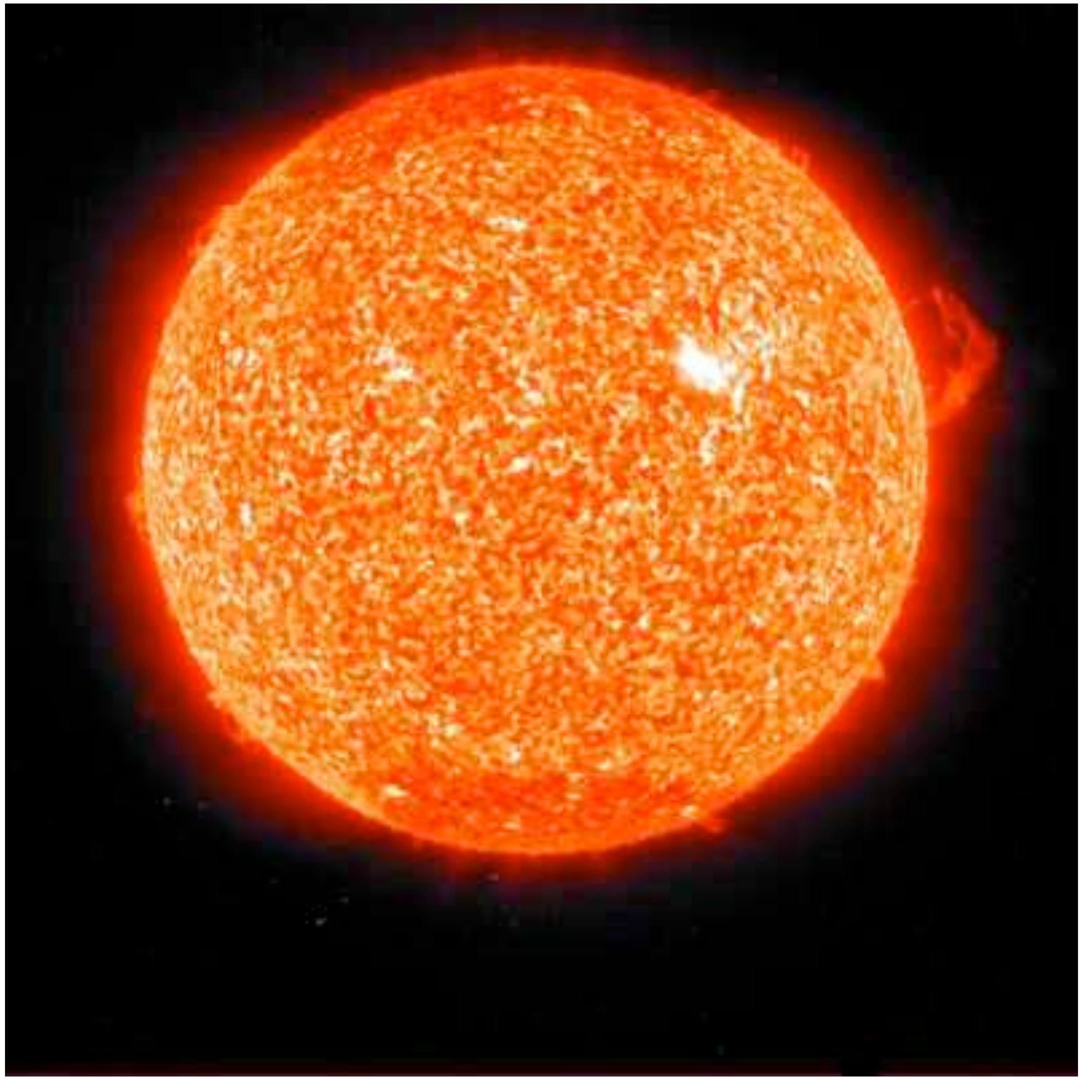
$$R_{\rm planet} \approx N_G^{1/3} a_0 \approx 10^{10}\ {\rm cm}$$

For a system with $na_0^3 = 1$:

$\epsilon_a \sim \epsilon_F$ provided $k_B T \leq \epsilon_a \approx \epsilon_F$



structures: stars



structures: existence stars

When the mass of the system is increased, the gravitational pressure increases and - to balance it - both the Fermi pressure and the thermal pressure will increase:

$$\epsilon = \epsilon_f(n) + k_B T$$

$$P \approx n k_B T + n\epsilon_F$$

This pressure can balance gravitational pressure if $(k_B T + \epsilon_F) \approx \epsilon_g$. For a classical system we get

$$k_B T \approx G m_p^2 N^{2/3} n^{1/3} - \frac{(3\pi^2)^{2/3}}{2}\frac{\hbar^2}{m_e} n^{2/3}.$$

The maximum temperature of the system is reached when $n = n_c$, with

$$n_c \approx \frac{\alpha_G}{(3\pi^2)^{2/3}}\left(\frac{N^{2/3}}{\lambda\!\!\!^{-}_e}\right); \quad k_B T \approx \frac{\alpha_G^2}{2(3\pi^2)^{2/3}}(N^{4/3} m_e c^2),$$

$$\lambda\!\!\!^{-}_e = (\hbar/m_e c)$$

structures: existence stars

An interesting phenomena arises when the maximum temperature is sufficiently high to trigger nuclear fusion $\epsilon_{\rm nucl} = \eta\alpha^2 m_p c^2$, with $\eta \approx 0.1$. The energy corresponding to the maximum temperature $k_B T$ will be larger than $\epsilon_{\rm nucl}$ when

$$N > (2\eta)^{3/4}(3\pi^2)^{1/2}\left(\frac{m_P}{m_e}\right)^{3/4}\left(\frac{\alpha}{\alpha_G}\right)^{3/2} \approx 4\times 10^{56}$$

for $\eta \approx 0.1$. The corresponding condition on mass is

$$M > M_* \approx (2\eta)^{3/4}(3\pi^2)^{1/2}\left(\frac{m_P}{m_e}\right)^{3/4}\left(\frac{\alpha}{\alpha_G}\right)^{3/2} m_p \approx 4\times 10^{32} \ \ \mathrm{g}$$

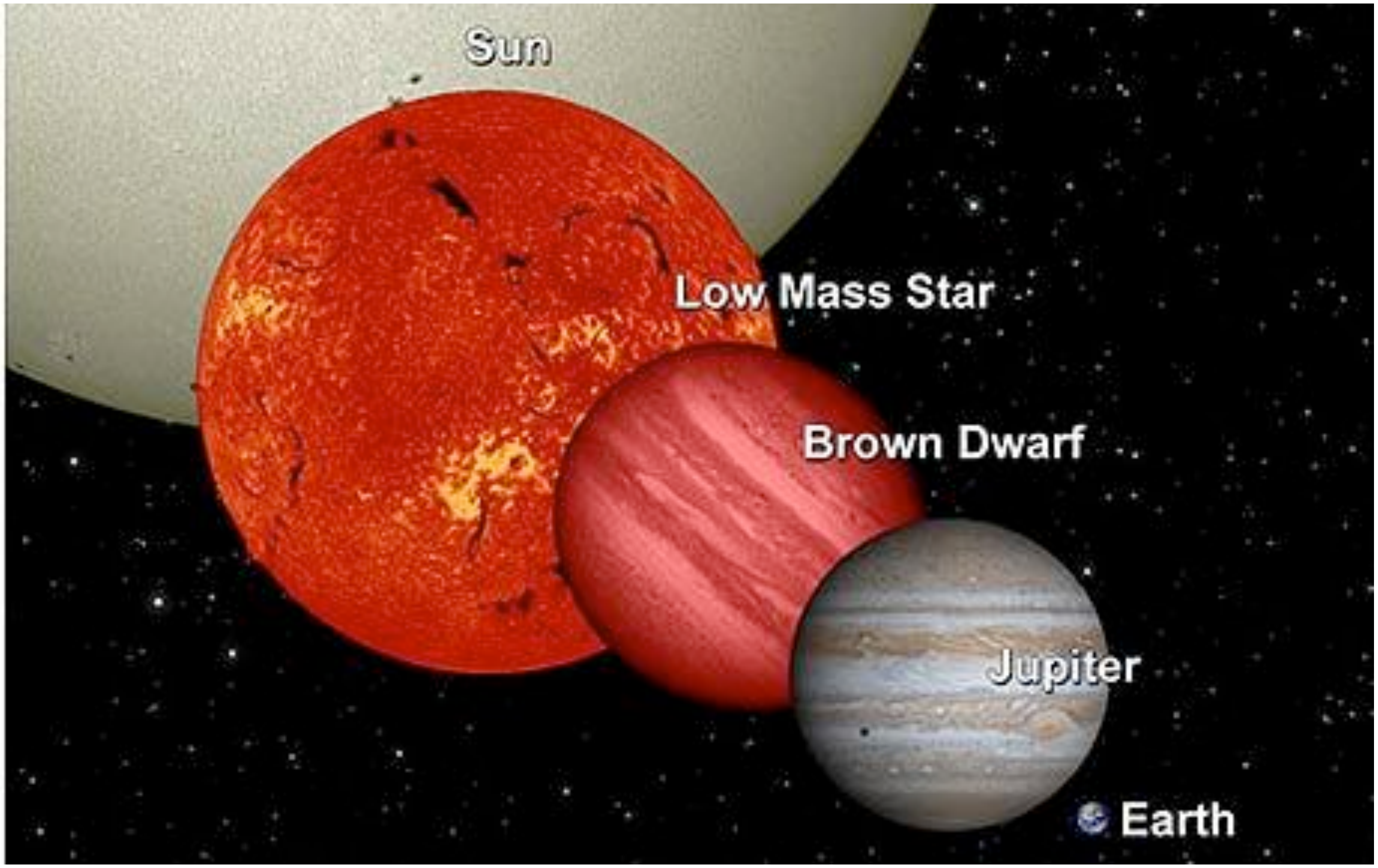
which is comparable with the mass of the smallest stars observed in our Universe.

The mass of the Sun, for example, is

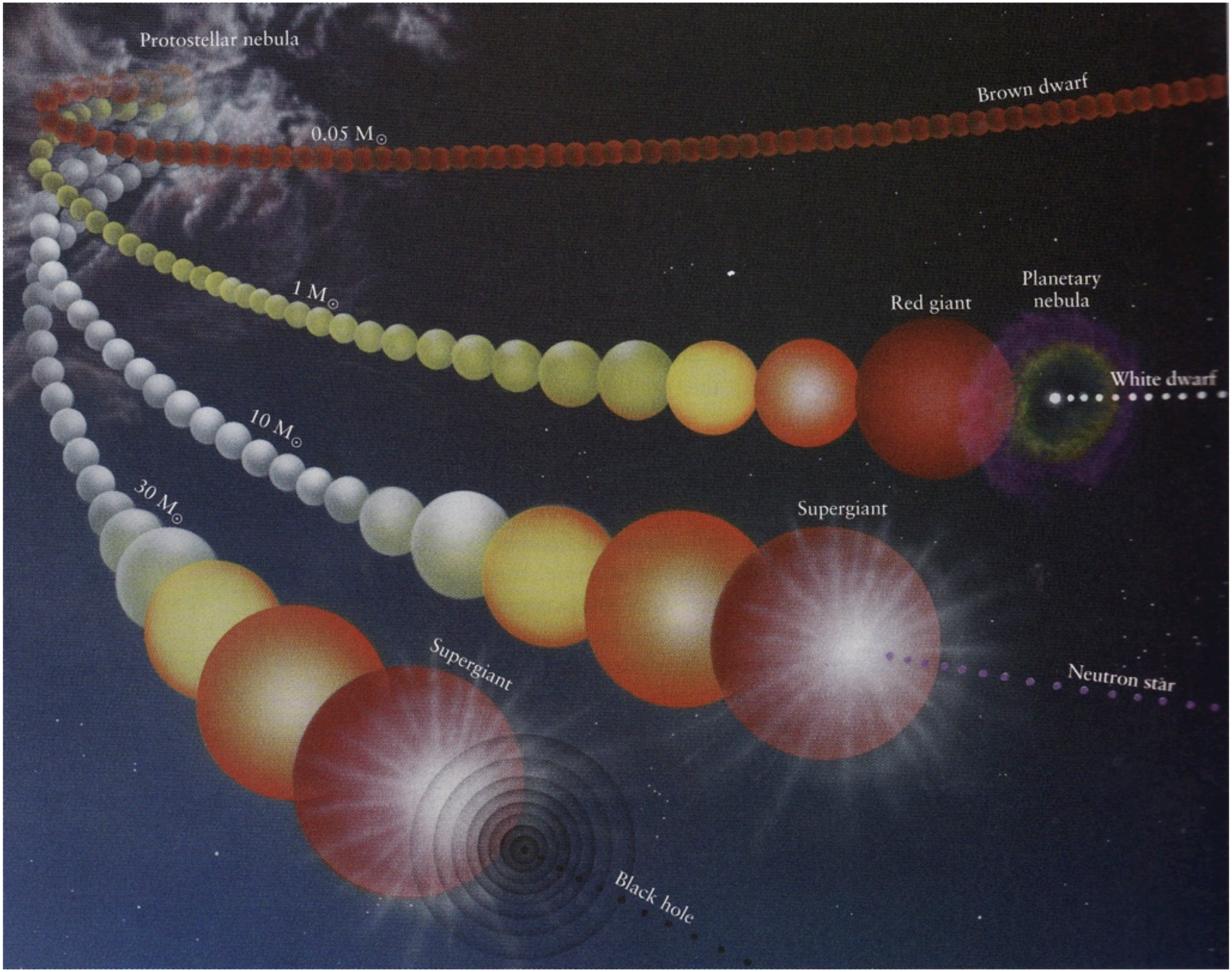
$$M_\odot = 2\times 10^{33}\ \mathrm{g}$$

structures: existence stars

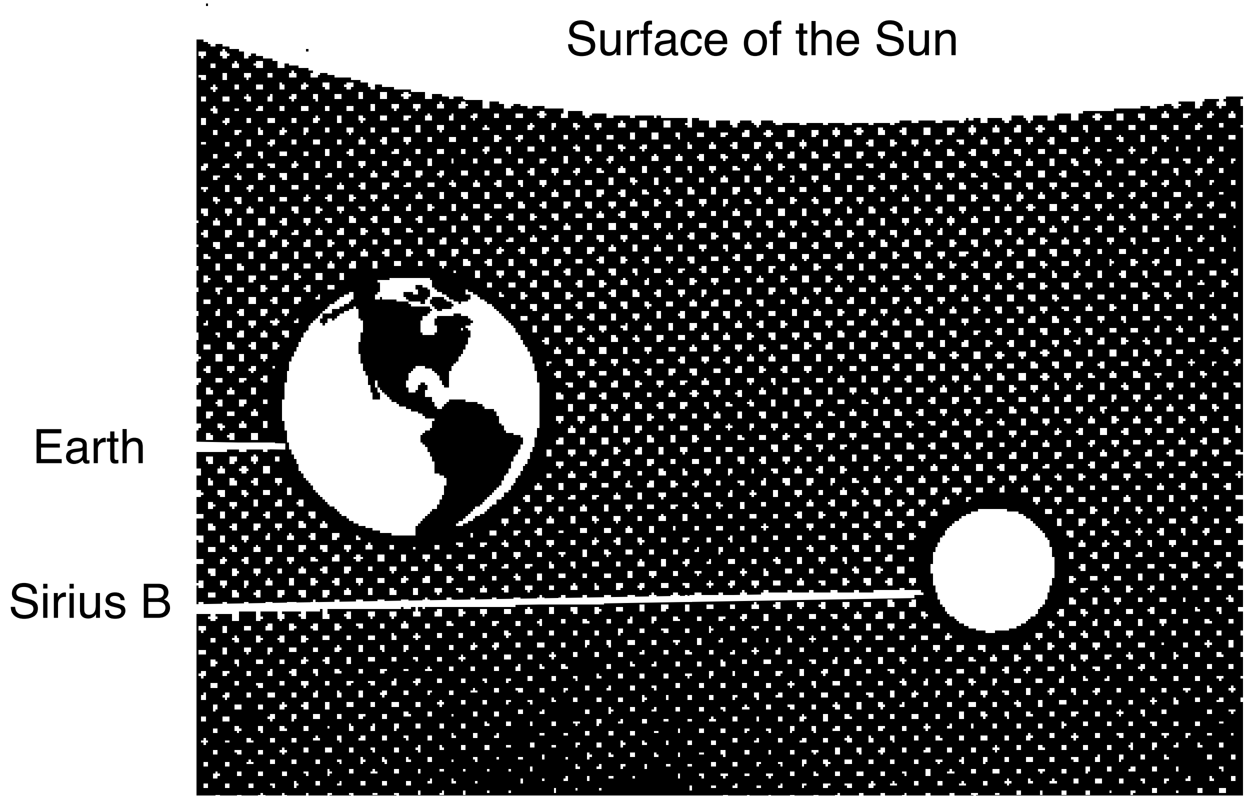
The lowest-mass star that can fuse H into He in its core has a mass of about 0.07 solar masses



stellar evolution



stellar corpses: white dwarfs



structures: white dwarfs

When the nuclear fuel in the star is exhausted, the gravitational force will start contracting the matter again and the density will be sufficiently high so that the quantum degeneracy pressure will dominate over thermal pressure.

$$\epsilon_g = \frac{E_{\text{grav}}}{N} = \frac{Gm_p^2}{R} N = \frac{4\pi}{3}^{1/3} G m_p^2 N^{2/3} n^{1/3}$$

$$\epsilon_F = \sqrt{p_F^2 c^2 + m^2 c^4} - mc^2 \approx \begin{cases} \frac{p_F^2}{2m} = \left(\frac{\hbar^2}{2m}\right) (3\pi^2 n)^{2/3} & \text{if } mc^2 \gg \epsilon_F \\ p_F c = (\hbar c)(3\pi^2 n)^{1/3} & \text{if } mc^2 \ll \epsilon_F \end{cases}$$

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When particles are not relativistic, the condition can be satisfied at equality if

$$n^{1/3} = \frac{2}{(3\pi^2)^{2/3}}\left(\frac{Gm_p^2 m_e}{\hbar^2}\right)N^{2/3}$$

structures: white dwarfs

With $n = (3N/4\pi R^3)$ and $N = (M/m_p)$, this reduces to the following mass-radius relation:

$$RM^{1/3} \approx \alpha_G^{-1} \lambdabar_e m_p^{1/3} \approx 8.7 \times 10^{-3} R_\odot m_\odot^{1/3}$$

structures: white dwarfs

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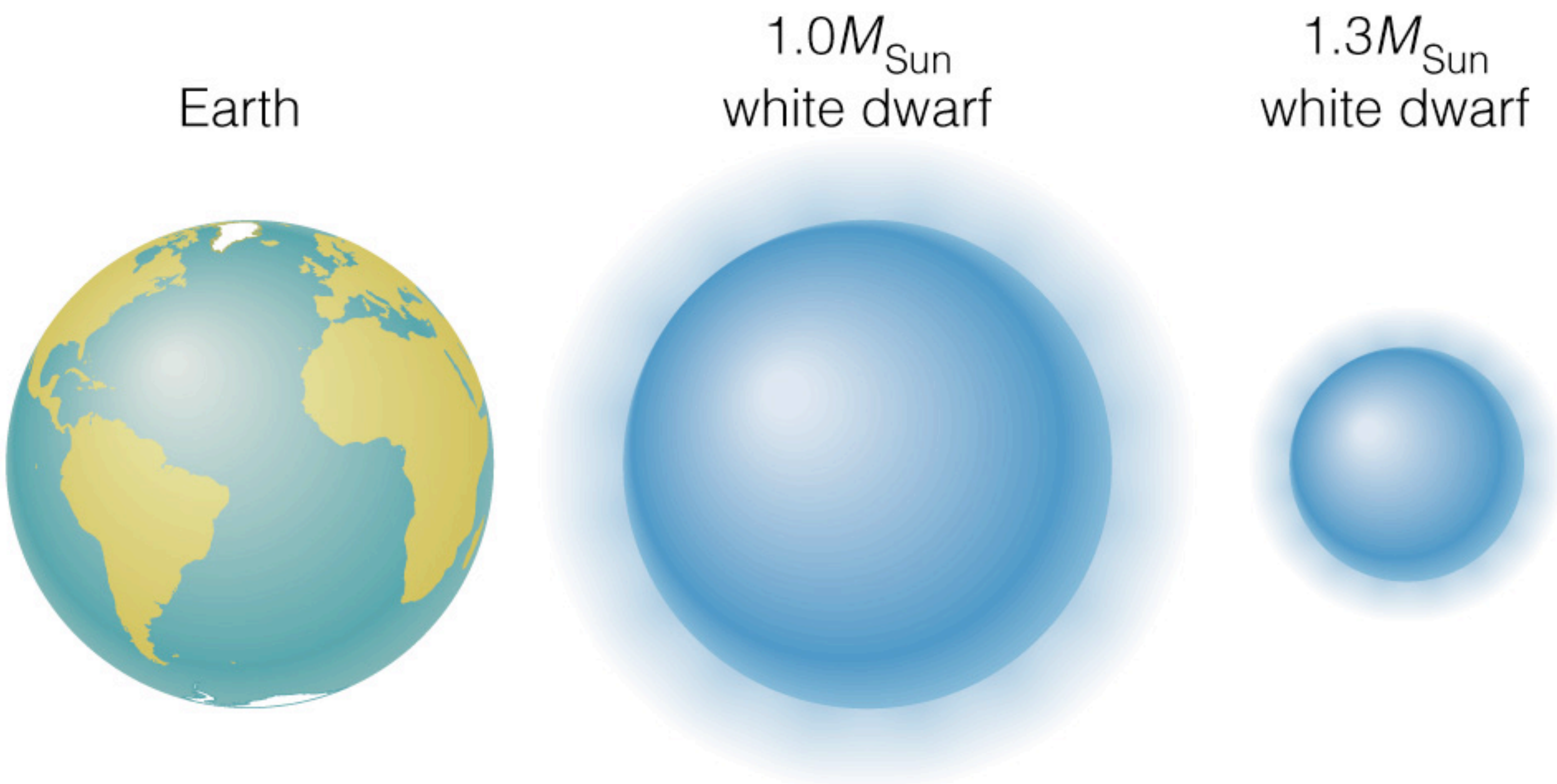


chandrasekhar mass limit

When the density is still higher, The Fermi energy has to be supplied by relativistic particles: $\epsilon_F \approx \hbar c n^{1/3}$

Thus

$$\hbar c \geq G m_p^2 N^{2/3}$$

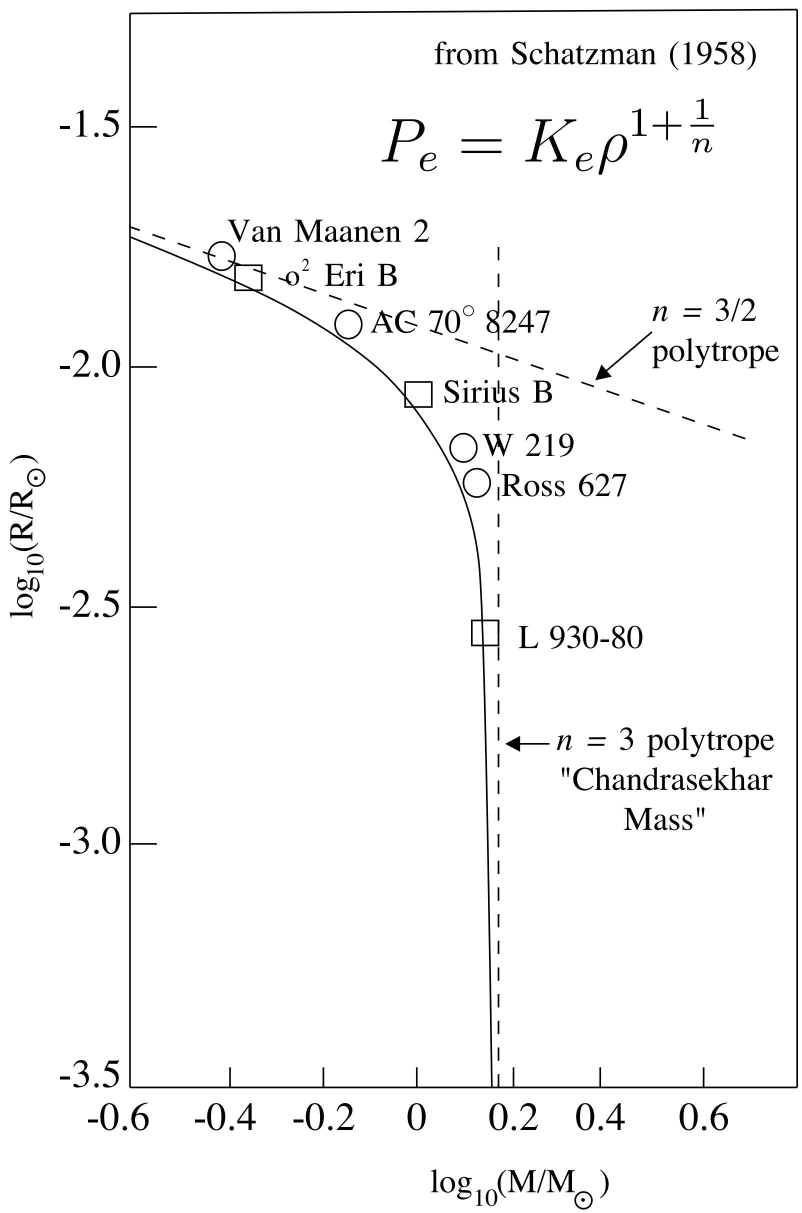
or

$$N \leq \alpha_G^{-3/2}$$

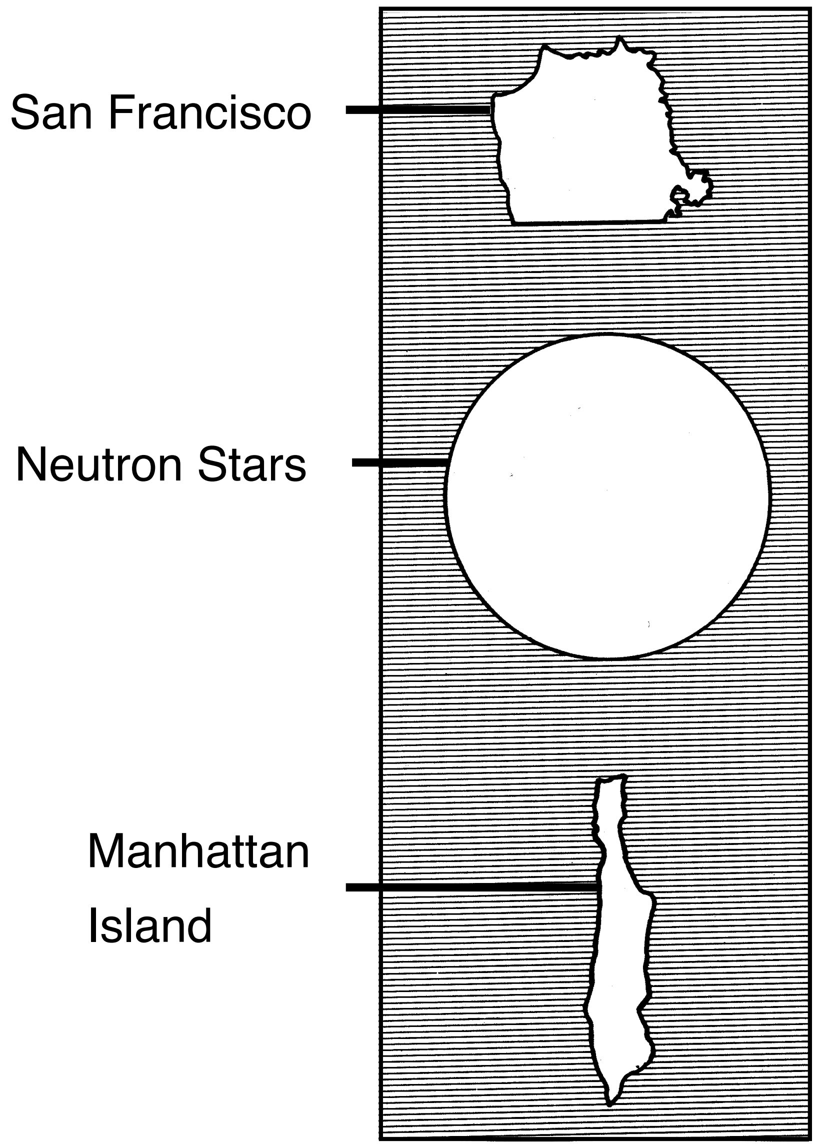
The corresponding mass bound (called Chandrasekhar limit) is

$$M \leq m_p \alpha_G^{-3/2} \approx 1 M_\odot$$

white dwarf structure



neutron stars



structures: neutron stars

As the density increases, electrons combine with protons through inverse beta decay to form neutrons, which can provide the degeneracy pressure.

$$n^{1/3} = \frac{2}{(3\pi^2)^{2/3}} \left(\frac{G m_p^2 m_e}{\hbar^2} \right) N^{2/3}$$

is still applicable with m_e replaced by m_n :

$$R M^{1/3} \approx \alpha_G^{-1} \lambda\!\!\!^{-}_n m_p^{1/3} \approx 8.7 \times 10^{-5} R_\odot m_\odot^{1/3}$$

$$(\lambda\!\!\!^{-}_n / \lambda\!\!\!^{-}_e) \approx 10^{-3}$$

maximum density

Neutrons are relativistic when $p_n \leq m_n c = \frac{h}{\lambda_{cn}}$

$$\Rightarrow \lambda_{cn} \approx \frac{h}{m_{cn}c} \approx 1.3 \times 10^{-13} \text{ cm} = 1.3 \text{ fm}$$

$$\rho_{n,rel} \approx m_n/\lambda_{cn}^3 \approx 7 \times 10^{14} \text{ g cm}^{-3}.$$

The mass density corresponding to a Compton wavelength:

$$\rho_{max} \approx \text{few} \times 10^{15} \text{ g cm}^{-3}.$$

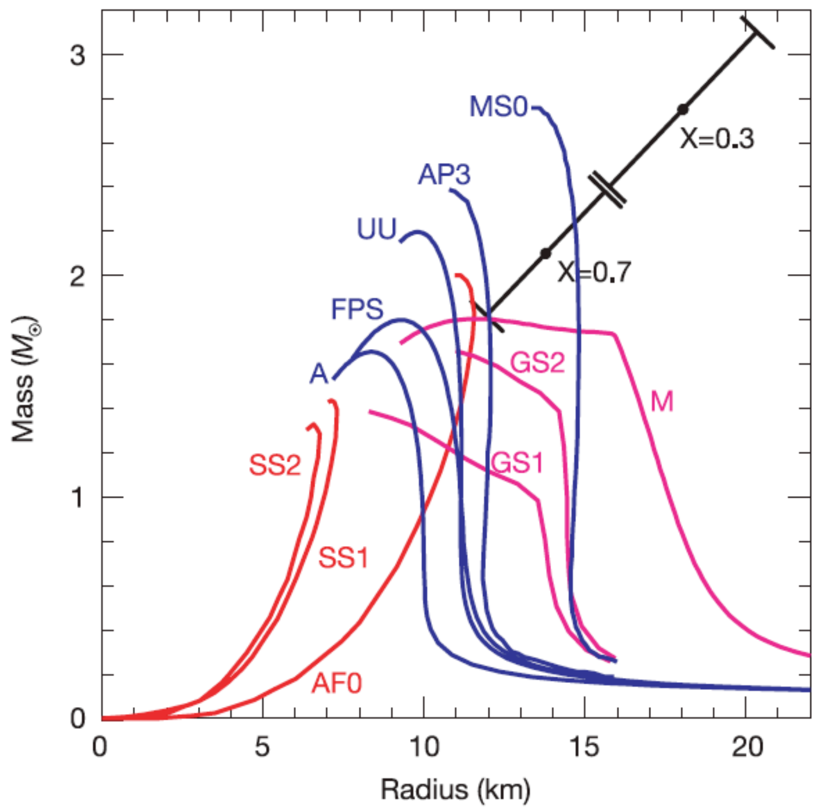
At these densities nuclear reactions are strong and they play a role in determining the structure of the neutron star.

structure of neutron stars

The characteristic radius can be obtained:

$$\frac{4}{3}\pi \rho_{max} R_c^3 \approx 1\ M_\odot$$
$$\Rightarrow R_c \approx 7 \times 10^5 \text{ cm.}$$

For real stars, with density gradients: $8 < R_{NS} < 20$ km.



structures: black holes

General relativistic effects become important when

$$R_{\rm gm} \equiv E_{\rm grav}/E_{\rm mass} \sim 1,$$

where

$$R_{\rm gm} \approx 0.7(M/10^{33}\ {\rm g})(R/1\ {\rm km})^{-1}.$$

Energy per particle

1 GeV	0.5 MeV	1 keV	13.6 eV
$\varepsilon_p \approx m_p c^2$	$\varepsilon_e \approx m_e c^2$	$\varepsilon_{\mathrm{nucl}} \approx \frac{\alpha^2}{2\pi^2} m_p c^2$	$\varepsilon_a = \frac{m_e q^4}{2\hbar^2} = \frac{1}{2}\alpha^2 m_e c^2$

Black Holes / Neutron Stars	White Dwarfs	Stars	Planets
$\varepsilon_g \approx m_p c^2$	$\varepsilon_g \approx \varepsilon_F$	$\varepsilon_g \approx (k_B T + \varepsilon_F)$ $k_B T \gtrsim \varepsilon_{\mathrm{nucl}}$	$\varepsilon_g \approx \varepsilon_a$

density

$$\varepsilon_F = \sqrt{p_F^2 c^2 + m^2 c^4} - mc^2$$

thermal energy

$$\varepsilon \approx k_B T$$

gravity

$$\varepsilon_g = \frac{E_{\mathrm{grav}}}{N} = \frac{G m_p^2}{R} N = {\frac{4\pi}{3}}^{1/3} G m_p^2 N^{2/3} n^{1/3}$$

pressure

$$\wp_{\mathrm{NR}} \approx (2/3) < n\varepsilon > \qquad \wp_{\mathrm{ER}} \approx (1/3) < n\varepsilon >$$