PROFESSOR: OK. Now, today we get to move on from integral formulas and methods of integration back to some geometry. And this is more or less going to lead into the kinds of tools you'll be using in multivariable calculus. The first thing that we're going to do today is discuss arc length. Like all of the cumulative sums that we've worked on, this one has a storyline and a picture associated to it, which involves dividing things up. If you have a roadway, if you like, and you have mileage markers along the road, like this, all the way up to, say, s_n here, then the length along the road is described by this parameter, s. Which is arc length. And if we look at a graph of this sort of thing, if this is the last point b, and this is the first point a, then you can think in terms of having points above x_1, x_2, x_3, etc. The same as we did with Riemann sums.

And then the way that we're going to approximate this is by taking the straight lines between each of these points. As things get smaller and smaller, the straight line is going to be fairly close to the curve. And that's the main idea. So let me just depict one little chunk of this. Which is like this. One straight line, and here's the curved surface there. And the distance along the curved surface is what I'm calling delta s, the change in the length between-- so this would be $s_2 - s_1$ if I depicted that one.

So this would be delta s is, say s. $s_i - s_{(i-1)}$, some increment there. And then I can figure out what the length of the orange segment is. Because the horizontal distance is delta x. And the vertical distance is delta y. And so the formula is that the hypotenuse is delta (delta x)^2 + (delta y)^2. Square root. And delta s is approximately that. So what we're saying is that (delta s)^2 is approximately this. So this is the hypotenuse. Squared. And it's very close to the length of the curve.

And the whole idea of calculus is that in the infinitesimal, this is exactly correct. So that's what's going to happen in the limit. And that is the basis for calculating arc length. I'm going to rewrite that formula on the next board. But I'm going to write it in the more customary fashion. We've done this before, a certain amount. But I just want to emphasize it here because this handwriting is a little bit peculiar. This ds is really all one thing. What I really mean is to put the

parenthesis around it. It's one thing. It's not d times s, it's ds. It's one thing. And we square it. But for whatever reason people have gotten into the habit of omitting the parentheses. So you're just going to have to live with that. And realize that this is not d of s^2 or anything like that. And similarly, this is a single number, and this is a single number. Infinitesimal. So that's just the way that this idea here gets written in our notation. And this is the first time we're dealing with squares of infinitesimals. So it's just a little different. But immediately the first thing we're going to do is take the square root.

If I take the square root, that's the square root of dx^2 + dy^2. And this is the form in which I always remember this formula. Let's put it in some brightly decorated form. But there are about four, five, six other forms that you'll derive from this, which all mean the same thing. So this is, as I say, the way I remember it. But there are other ways of thinking of it. And let's just write a couple of them down. The first one is that you can factor out the dx. So that looks like this. 1 + (dy / dx)^2. And then I factored out the dx. So this is a variant. And this is the one which actually we'll be using in practice right now on our examples. So the conclusion is that the arc length, which if you like is this total s_n - s_0, if you like, is going to be equal to the integral from a to b of the square root of 1 + (dy/dx)^2, dx.

In practice, it's also very often written informally as this. The integral ds. So the change in this little variable s, and this is what you'll see notationally in many textbooks. So that's one way of writing it, and of course the second way of writing it which is practically the same thing is square root of 1 + f'(x)^2, dx. Mixing in a little bit of Newton's notation. And this is with y = f(x). So this is the formula for arc length. And as I say, I remember it this way. But you're going to have to derive various variants of it. And you'll have to use some arithmetic to get to various formulas. And there will be more later. Yeah, question.

STUDENT: [INAUDIBLE]

PROFESSOR: OK, the question is, is f'(x)^2 equal to f''(x). And the answer is no. And let's just see what it is. So, for example, if f(x) = x^2, which is an example which will come up in a few minutes, then f'(x) = 2x and f'(x)^2 = (2x)^2, which is 4x^2. Whereas f''(x) is equal to, if I differentiate this another time, it's equal to 2. So they don't mean the same thing. The same thing over here. You can see this dy / dx, this is the rate of change of y with respect to x. The quantity squared. So in other words, this thing is supposed to mean the same as that. Yeah. Another question.

STUDENT: [INAUDIBLE]

PROFESSOR: So the question is, you got a little nervous because I left out these limits. And indeed, I did that on purpose because I didn't want to specify what was going on. Really, if you wrote it in terms of ds, you'd have to write it as starting at s_0 and ending at s_n to be consistent with the variable s. But of course, if you write it in terms of another variable, you put that variable in. So this is what we do when we change variables, right? We have many different choices for these limits. And this is the clue as to which variable we use.

STUDENT: [INAUDIBLE]

PROFESSOR: Correct. s_0 and s_n are not the same thing as a and b. In fact, this is x_n. And this x_0, over here. That's what a and b are. But s_0 and s_n are mileage markers on the road. They're not the same thing as keeping track of what's happening on the x axis. So when we measure arc length, remember it's mileage along the curved path.

So now, I need to give you some examples. And my first example is going to be really basic. But I hope that it helps to give some perspective here. So I'm going to take the example $y = mx$, which is a linear function, a straight line. And then y' would be m, and so ds is going to be the square root of $1 + (y')^2$, dx. Which is the square root of $1 + m^2$, dx. And now, the length, say, if we go from, I don't know, let's say 0 to 10, let's say, of the graph is going to be the integral from 0 to 10 of the square root of $1 + m^2$, dx. Which of course is just 10 square root of $1 + m^2$. Not very surprising. This is a constant. It just factors out and the integral from 0 to 10 of dx is 10.

Let's just draw a picture of this. This is something which has slope m here. And it's going to 10. So this horizontal is 10. And the vertical is 10m. Those are the dimensions of this. And the Pythagorean theorem says that the hypotenuse, not surprisingly, let's draw it in here in orange to remind ourselves that it was the same type of orange that we had over there, this length here is the square root of $10^2 + (10m)^2$. That's the formula for the hypotenuse. And that's exactly the same as this. Maybe you're saying duh, this is obvious. But the point that I'm trying to make is this. If you can figure out these formulas for linear functions, calculus tells you how to do it for every function. The idea of calculus is that this easy calculation here, which you can do without any calculus at all, all of the tools, the notations of differentials and limits and integrals, is going to make you be able to do it for any curve. Because we can break things up into these little infinitesimal bits. This is the whole idea of all of the methods that we had to set up integrals here. This is the main point of these integrals.

Now, so let's do something slightly more interesting. Our next example is going to be the circle, so y = square root of 1-x^2. If you like, that's the graph of a semicircle. And maybe we'll set it up here this way. So that the semicircle goes around like this. And we'll start it here at x = 0. And we'll go over to a. And we'll take this little piece of the circle. So down to here. If you like.

So here's the portion of the circle that I'm going to measure the length of. Now, we know that length. It's called arc length. And I'm going to give it a name, I'm going to call it alpha here. So alpha's the arc length along the circle. Now, let's figure out what it is. First, in order to do this, I have to figure out what y' is. Or, if you like, dy/dx. Now, that's a calculation that we've done a number of times. And I'm going to do it slightly faster. But you remember it gives you a square root in the denominator. And then you have the derivative of what's inside the square root. Which is -2x. But then there's also 1/2, because in disguise it's really (1 - x^2)^(1/2). So we've done this calculation enough times that I'm not going to carry it out completely. I want you to think about what it is. It turns out to -x up here, because the 1/2 and the 2 cancel.

And now the thing that we have to integrate is this arc length element, as it's called, ds. And that's going to be the square root of 1 + (y')^2, dx. And so I'm going to have to carry out the calculation, some messy calculation here. Which is that this is 1 plus the quantity -x over square root of 1 - x^2, squared. So I have to figure out what's under the square root sign over here in order to carry out this calculation. Now let's do that.

This is 1 + x^2 / (1 - x^2). That's what this simplifies to. And then that's equal to, over a common denominator, 1 - x^2. 1 - x^2 + x^2. And there is a little bit of simplification now. Because the two x^2's cancel. And we get 1/(1-x^2). So now I get to finish off the calculation by actually figuring out what the arc length is. And what is it? Well, this alpha is equal to the integral from 0 to a of ds. Well, it's going to be the square root of what I have here. This was a square, this is just what was underneath the square root sign. This is 1 + (y')^2. Have to take the square root of that. So what I get here is dx over the square root of 1 - x^2.

And now, we recognize this. The antiderivative of this is something that we know. This is the inverse sine. Evaluated at 0 and a. Which is just giving us the inverse sine of a, because the inverse sine of 0 is equal to 0. So alpha is equal to the inverse sine of a. That's a very fancy way of saying that sin(alpha) = a. That's the equivalent statement here. And what's going on here is something that's just a little deeper than it looks. Which is this. We've just figured out a geometric interpretation of what's going on here. That is, that we went a distance alpha along

this arc. And now remember that the radius here is 1. And this horizontal distance here is a. This distance here is a. And so the geometric interpretation of this is that this angle is alpha radians. And sin(alpha) = a. So this is consistent with our definition previously, our previous geometric definition of radians. But this is really your first true definition of radians. We never actually-- People told you that radians were the arc length along this curve. This is the first time you're deriving it. This is the first time you're seeing it correctly done. And furthermore, this is the first time you're seeing a correct definition of the sine function.

Remember we had this crazy way, we defined the exponential function, then we had another way of defining the log function as an integral. Then we defined the exponential in terms of it. Well, this is the same sort of thing. What's really happening here is that if you want to know what radians are, you have to calculate this number. If you've calculated this number, then by definition if sine is the thing whose alpha radian amount gives you a, then it must be that this is sine inverse of a. And so the first thing that gets defined is the arcsine. And the next thing that gets defined is the sine, afterwards. This is the way the foundational approach actually works when you start from first principles. This arc length being one of the first principles. So now we have a solid foundation for trig functions. And this is giving that connection. Of course, it's consistent with everything you already knew, so you don't have to make any transitional thinking here. It's just that this is the first time it's being done rigorously. Because you only now have arc length.

So these are examples, as I say, that maybe you already know. And maybe we'll do one that we don't know quite as well. Let's find the length of a parabola. This is Example 3. Now, that was what I was suggesting we were going to do earlier. So this is the function y x^2. $y' = 2x$. And so ds is equal to the square root of $1 + (2x)^2$, dx. And now I can figure out what a piece of a parabola is. So I'll draw the piece of parabola up to a, let's say, starting from 0. So that's the chunk. And then its arc length, between 0 and a of this curve, is the integral from 0 to a of square root of $1 + 4x^2$, dx.

OK, now if you like, this is the answer to the question. But people hate looking at answers when they're integrals if they can be evaluated. So one of the reasons why we went through all this rigmarole of calculating these things is to show you that we can actually evaluate a bunch of these functions here more explicitly. It doesn't help a lot, but there is an explicit calculation of this. So remember how you would do this. So this is just a little bit of review. What we did in techniques of integration. The first step is what? A substitution. It's a trig substitution. And what

is it?

STUDENT: [INAUDIBLE]

PROFESSOR: So x equals something tan(theta). I claim that it's 1/2 tan, and I'm going to call it u. Because I'm going to use theta for something else in a couple of days. OK? So this is the substitution. And then of course dx = 1/2 sec^2 u du, etc. So what happens if you do this? I'll write down the answer, but I'm not going to carry this out. Because every one of these is horrendous. But I think I worked it out. Let's see if I'm lucky. Oh yeah. I think this is what it is. It's a 1/4 ln(2x + square root of (1+4x^2) + 1/2 x square root of (1+4x^2)). Evaluated at a and 0.

So yick. I mean, you know.

STUDENT: [INAUDIBLE]

PROFESSOR: Why I did I make it 1/2? Because it turns out that when you differentiate. So the question is, why is there 1/2 there? If you differentiate it without the 1/2, you get this term and it looks like it's going to be just right. But then if you differentiate this one you get another thing. And it all mixes together. And it turns out that there's more. So it turns out that it's 1/2. Differentiate it and check. So this just an incredibly long calculation. It would take fifteen minutes or something like that. But the point is, you do know in principle how to do these things.

STUDENT: [INAUDIBLE]

PROFESSOR: Oh, he was talking about this 1/2, not this crazy 1/2 here. Sorry.

STUDENT: [INAUDIBLE]

PROFESSOR: Yeah, OK. So sorry about that. Thank you for helping. This factor of 1/2 here comes about because when you square x, you don't get tan^2. When you square 2x, you get (4x)^2 and that matches perfectly with this thing. And that's why you need this factor here. Yeah. Another question, way in the back.

STUDENT: [INAUDIBLE]

PROFESSOR: The question is, when you do this substitution, doesn't the limit from 0 to a change. And the answer is, absolutely yes. The limits in terms of u are not the same as the limits in terms of a. But if I then translate back to the x variables, which I've done here in this bottom formula, of x = 0 and x = a, it goes back to those in the original variables. So if I write things in the original

variables, I have the original limits. If I use the u variables, I would have to change limits. But I'm not carrying out the integration, because I don't want to. So I brought it back to the x formula. Other questions.

OK, so now we're ready to launch into three-space a little bit here. We're going to talk about surface area. You're going to be doing a lot with surface area in multivariable calculus. It's one of the really fun things. And just remember, when it gets complicated, that the simplest things are the most important. And the simple things are, if you can handle things for linear functions, you know all the rest. So there's going to be some complicated stuff but it'll really only involve what's happening on planes. So let's start with surface area. And the example that I'd like to give - this is the only type of example that we'll have - is the surface of rotation. And as long as we have our parabola there, we'll use that one. So we have y = x^2, rotated around the x-axis.

So let's take a look at what this looks like. It's the parabola, which is going like that. And then it's being spun around the x-axis. So some kind of shape like this with little circles. It's some kind of trumpet shape, right? And that's the shape that we're-- Now, again, it's the surface. It's just the metal of the trumpet, not the insides. Now, the principle for figuring out what the formula for area is, is not that different from what we did for surfaces of revolution. But it just requires a little bit of thought and imagination. We have a little chunk of arc length along here. And we're going to spin that around this axis. Now, if this were a horizontal piece of arc length, then it would spin around just like a shell. It would just be a surface. But if it's tilted, if it's tilted, then there's more surface area proportional to the amount that it's tilted. So it's proportional to the length of the segment that you spin around.

So the total is going to be ds, that's one of the factors here. Maybe I'll write that second. That's one of the dimensions. And then the other dimension is the circumference. Which is 2 pi, in this case, y. So that's the end of the calculation. This is the area element of surface area. Now, when you get to 18.02, and maybe even before that, you'll also see some people referring to this area element when it's a curvy surface like this with a notation dS. That's a little confusing because we have a lower case s here. We're not going to use it right now. But the lower case s is usually arc length. The upper case S is usually surface area. So. Also used for dA. The area element. Because this is a curved area element. So let's figure out this example.

So in the example-- ...is equal to x^2 then the situation is, we have the surface area is equal to the integral from, I don't know, 0 to a if those are the limits that we wanted to choose. Of 2 pi x^2, right? Because y = x^2. Times the square root of 1 + 4x^2, dx. Remember we had this

from our previous example. This was ds from previous. And this, of course, is 2 pi y. Now again, the calculation of this integral is kind of long. And I'm going to omit it. But let me just point out that it follows from the same substitution. Namely, $x = 1/2 \tan u$ is going to work for this integral. It's kind of a mess. There's a tan squared here and the secant squared. There's another secant and so on. So it's one of these trig integrals that then takes a while to do.

So that just is going to trail off into nothing. And the reason is that what's important here is more the method. And the setup of the integrals. The actual computation, in fact, you could go to a program and you could plug in something like this and you would spit out an answer immediately. So really what we just want is for you to have enough control to see that it's an integral that's a manageable one. And also to know that if you plugged it in, you would get an answer. When I actually do carry out a calculation, though, what I want to do is to do something that has an answer that you can remember. And that's a nice answer. So that turns out to be the example of the surface area of a sphere. So it's analogous to this 2 here. And maybe I should remember this result here. Which was that the arc length element was given by this. So we'll save that for a second.

So we're going to do this surface area now. So if you like, this is another example. The surface area of a sphere. This is a good example, and one, as I say, that has a really nice answer. So it's worth doing. So first of all, I'm not going to set it up quite the way I did in Example 2. Instead, I'm going to take the general sphere, because I'd like to watch the dependence on the radius. So here this is going to be the radius. It's going to be radius a. And now, if I carry out the same calculations as before, if you think about it for a second, you're going to get this result. And then, the rest of the arithmetic, which is sitting up there in the case, $a = 1$, will give us that ds is equal to what? Well, maybe I'll just carry it out. Because that's always nice. So we have $1 + x^2 / (a^2 - x^2)$. That's $1 + (y')^2$. And now I put this over a common denominator. And I get $a^2 - x^2$. And I have in the numerator $a^2 - x^2 + x^2$. So the same cancellation occurs. But now we get an a^2 in the numerator.

So now I can set up the ds. And so here's what happens. The area of a section of the sphere, so let's see. We're going to start at some starting place x_1, and end at some place x_2. So what does that look like? Here's the sphere. And we're starting at a place x_1. And we're ending at a place x_2. And we're taking more or less the slice here, if you like. The section of this sphere. So the area's going to equal this. And what is it going to be? Well, so I have here 2 pi y. I'll write it out, just leave it as y for now. And then I have ds. So that's always what the

formula is when you're revolving around the x-axis. And then I'll plug in for those things. So 2 pi, the formula for y is square root $a^2 - x^2$. And the formula for ds, well, it's the square root of this times dx. So it's the square root of $a^2 / (a^2 - x^2)$, dx. So this part is ds. And this part is y.

And now, I claim we have a nice cancellation that takes place. Square root of a^2 is a. And then there's another good cancellation. As you can see. Now, what we get here is the integral from x_1 to x_2, of 2 pi a dx, which is about the easiest integral you can imagine. It's just the integral of a constant. So it's 2 pi a $(x_2 - x_1)$. Let's check this in a couple of examples. And then see what it's saying geometrically, a little bit.

So what this is saying-- So special cases that you should always check, when you have a nice formula like this, at least. But really with anything in order to make sure that you've got the right answer. If you take, for example, the hemisphere. So you take 1/2 of this sphere. So that would be starting at 0, sorry. And ending at a. So that's the integral from 0 to a. So this is the case $x_1 = 0$. $x_2 = a$. And what you're going to get is a hemisphere. And the area is 2 pi a times a. Or in other words, 2 pi a^2. And if you take the whole sphere, that's starting at $x_1 = -a$, and $x_2 = a$, you're getting 2 pi a times $(a - (-a))$. Which is 4 pi a^2. That's the whole thing. Yeah, question.

STUDENT: [INAUDIBLE]

PROFESSOR: The question is, would it be possible to rotate around the y-axis? And the answer is yes. It's legal to rotate around the y-axis. And there is-- If you use vertical slices as we did here, that is, well they're sort of tips of slices, it's a different idea. But anyway, it's using dx as the integral of the variable of integration. So we're checking each little piece, each little strip of that type. If we use dx here, we get this. If you did the same thing rotated the other way, and use dy as the variable, you get exactly the same answer. And it would be the same calculation. Because they're parallel. So you're, yep.

STUDENT: [INAUDIBLE]

PROFESSOR: Can you do surface area with shells? Well, the shell shape-- The short answer is not quite. The shell shape is a vertical shell which is itself already three-dimensional, and it has a thickness. So this is just a matter of terminology, though. This thickness is this dx, when we do this rotation here. And then there are two other dimensions. If we have a curved surface, there's no other dimension left to form a shell. But basically, you can chop things up into any

bits that you can actually measure. That you can figure out what the area is. That's the main point.

Now, I said we were going to, we've just launched into three-dimensional space. And I want to now move on to other space-like phenomena. But we're going to do this. So this is also a preparation for 18.02, where you'll be doing this a tremendous amount. We're going to talk now about parametric equations. Really just parametric curves. So you're going to see this now and we're going to interpret it a couple of times, and we're going to think about polar coordinates. These are all preparation for thinking in more variables, and thinking in a different way than you've been thinking before. So I want you to prepare your brain to make a transition here. This is the beginning of the transition to multivariable thinking.

We're going to consider curves like this. Which are described with x being a function of t and y being a function of t. And this letter t is called the parameter. In this case you should think of it-- the easiest way to think of it is as time. And what you have is what's called a trajectory. So this is also called a trajectory. And its location, let's say, at time 0, is this location here. $(x(0), y(0))$, that's a point in the plane. And then over here, for instance, maybe it's $(x(1), y(1))$. And I drew arrows along here to indicate that we're going from this place over to that place. These are later times. $t = 1$ is a later time than $t = 0$. So that's just a very casual, it's just the way we use these notations.

Now let me give you the first example, which is $x = a \cos t$, $y = a \sin t$. And the first thing to figure out is what kind of curve this is. And to do that, we want to figure out what equation it satisfies in rectangular coordinates. So to figure out what curve this is, we recognize that if we square and add-- So we add x^2 to y^2, we're going to get something very nice and clean. We're going to get $a^2 \cos^2 t + a^2 \sin^2 t$. Yeah that's right, OK. Which is just a ^{2} a^2 $(\cos^2 + \sin^2)$, or in other words a^2. So lo and behold, what we have is a circle. And then we know what shape this is now. And the other thing I'd like to keep track of is which direction we're going on the circle. Because there's more to this parameter then just the shape. There's also where we are at what time. This would be, think of it like the trajectory of a planet. So here, I have to do this by plotting the picture and figuring out what happens. So at $t = 0$, we have (x, y) is equal to, plug in here $(a \cos 0, a \sin 0)$. Which is just $a * 1 + a * 0$, so $(a, 0)$. And that's here. That's the point $(a, 0)$. We know that it's the circle of radius a. So we know that the curve is going to go around like this somehow. So let's see what happens at $t = \pi / 2$. So at that point, we have $(x, y) = (a \cos(\pi/2), a \sin(\pi/2))$. Which is $(0, a)$, because sine of $\pi / 2$ is 1.

So that's up here. So this is what happens at t = 0. This is what happens at t = pi / 2. And the trajectory clearly goes this way. In fact, this turns out to be t = pi, etc. And it repeats at t = 2pi. So the other feature that we have, which is qualitative feature, is that it's counterclockwise. Now the last little bit is going to be the arc length. Keeping track of the arc length. And we'll do that next time.