Homework 2

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Problem 1. Show that convolution is associative, $(f \star g) \star h = f \star (g \star h)$

- $[f(x) * g(x)] * h(x) = \int_{-\infty}^{+\infty} [\int_{-\infty}^{+\infty} f(\beta)g(\alpha - \beta)d\beta]\, h(x - \alpha)d\alpha$
- Interchange the order of integration:

  $[f(x) * g(x)] * h(x) = \int_{-\infty}^{+\infty} f(x)[\int_{-\infty}^{+\infty} g(\alpha - \beta)h(x - \alpha)d\alpha]\, d\beta$
- The inner integral is the convolution $g(x - \beta) * h(x)$,
  let $g(x - \beta) * h(x) = u(x)$, then:

$$\begin{aligned}
[f(x) * g(x)] * h(x) &= \int_{-\infty}^{+\infty} f(\beta)u(x - \beta)d\beta \\
&= f(x) * u(x) \\
&= f(x) * [g(x) * h(x)]
\end{aligned}$$

Problem 2. Show that Correlation is not associative

- Given two functions $f(x)$ and $g(x)$, cross correlation of $f(x)$ with $g(x)$ is:

$$f(x) \star g(x) = \int_{-\infty}^{+\infty} f(\alpha)g(\alpha - x)d\alpha \tag{1}$$

- Although this operation looks similar to convolution, there is a critical difference: the function $g(x)$ is not folded as in convolution. if we make the change of variable $\beta = \alpha - x$, Eq.1 becomes:

$$f(x) \star g(x) = \int_{-\infty}^{+\infty} f(\beta + x)g(\beta)d\beta \tag{2}$$

- Conclusion:

$$f(x) \star g(x) \neq g(x) \star f(x) \tag{3}$$

Problem 3. If we have an image I of dimension HxW and we convolve it with a filter f of size MxN using the spatial domain formula given in class, what will the complexity be? How about if we use the FFT "trick"?

- The complexity of spatial convolution is $O((HWMN)^2)$ for each pixel in the image the convolution is the adjustment of pixel values in the filter
- The complexity of FFT in for this operation is $O((HW)^2) * O(N \log N)$

Problem 4. 2D isotropic Gaussian filter. Show that the filter is the product of two functions $f(x, y) = f_1(x)f_2(y)$

- Formula for discrete 2D convolution:

$$(I * f)(x, y) \equiv \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} I(x-u, y-v)f(u, v) \tag{4}$$

- $$I[x, y] * f[x, y] = \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} I(n_1, n_2) * f(x-n_1, y-n_2) \tag{5}$$

- 2D isotropic Gaussian filter:

$$g(x, y) = \frac{1}{2\pi\sigma^2} e^{\frac{-(x^2+y^2)}{2\sigma^2}} \tag{6}$$

Because the Gaussian filter is separable, first convolve each row with a 1D filter, then convolve each column with a 1D filter. By way of associativity, separate Gaussian:

$$G_\sigma * f = [g_{\sigma_x} * g_{\sigma_y}] * f = g_{\sigma_x} * [g_{\sigma_y} * f] \tag{7}$$

$$g_{\sigma_x} = \frac{1}{2\pi\sigma^2} \exp \frac{-x^2}{2\sigma^2} \tag{8}$$

$$g_{\sigma_y} = \frac{1}{2\pi\sigma^2} \exp \frac{-y^2}{2\sigma^2} \tag{9}$$

$$(I * G)(x, y) \equiv \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} g_{\sigma_x} * [g_{\sigma_y} * I(x, y)] \tag{10}$$

Problem 5. Discrete Fourier transform (DFT)

I constructed a square periodic wave with box function of 5 and 10 sample units, shown in figure 1. I took the discrete Fourier transform using matlab function FFT on the data and using fftshift to center the magnitude around 0.

Figure 1: (a) A plot of the pulse function with unit height 5 and 10.

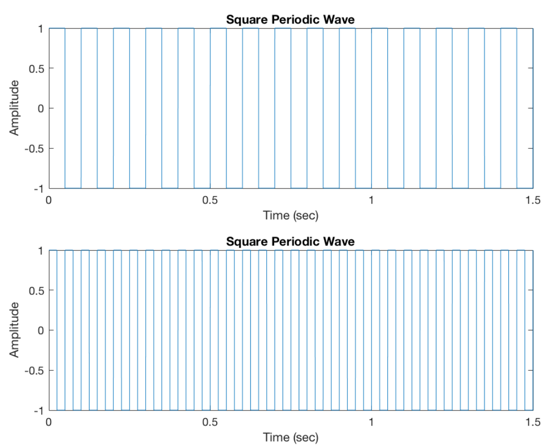
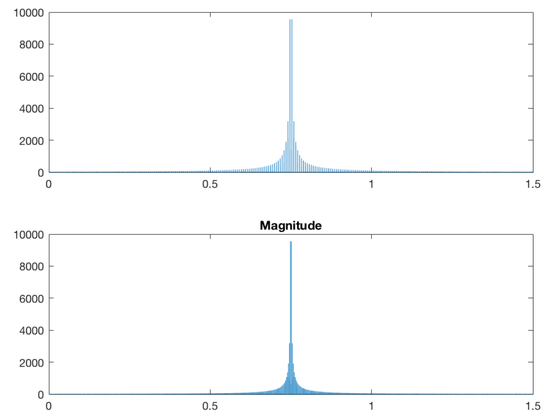


Figure 2: (b) A plot of the magnitudes.



In Figure 3, plots of two Gaussians are shown with $\sigma = 1 and \sigma = 2$. Then performed discrete Fourier transform and zero centered the magnitudes.

Figure 3: (a) A plot of the Gaussians with $\sigma = 1 and \sigma = 2$.

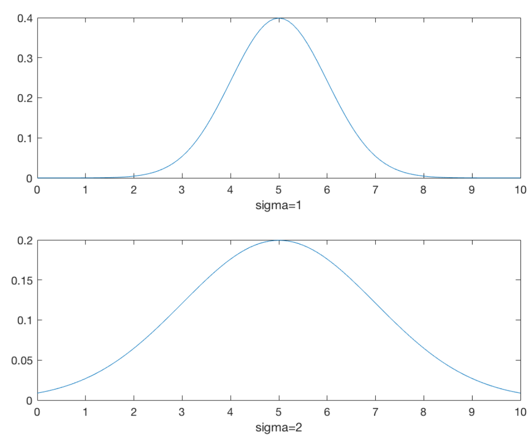


Figure 4: (b) A plot of their magnitudes.

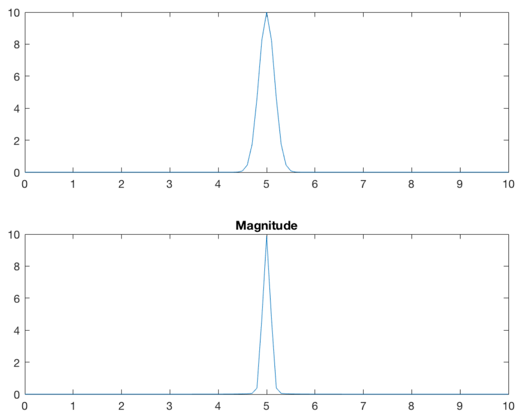
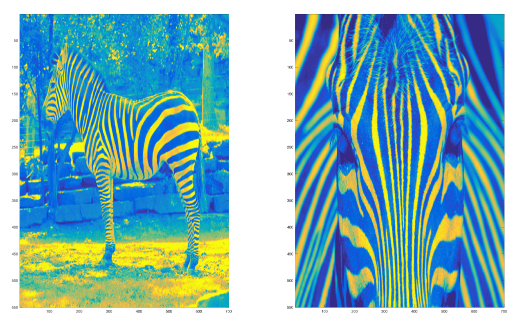


Figure 5 displays the two input images shown in gray scale. Perform 2D DFT on these images and combine their phase and magnitude to create a new image.

Figure 5: (a) Two input images shown in gray scale.



Problem 6. gradient based edge detector

Taking a derivative multiplies the signal spectrum and enhances high frequency noise. To overcome this, we have to suppress high frequency noice by applying a smoothing filter such as the Gaussian filter over the noisy image. We need to define direction in which the derivative is taken, for the two-dimensional case, we have the horizontal and vertical direction.

Starting with an RGB image of the zebra, convert into gray scale and display in Figure 6. Using a Gaussian filter for spatial smoothing, take the 2D derivative and look at the gradient magnitude and gradient direction in Figure 7. For $\sigma = 0.5$ the results are shown in Figure 8.

Figure 6: The gray scale image of a zebra.

Figure 7: The gradient magnitude and orientation.

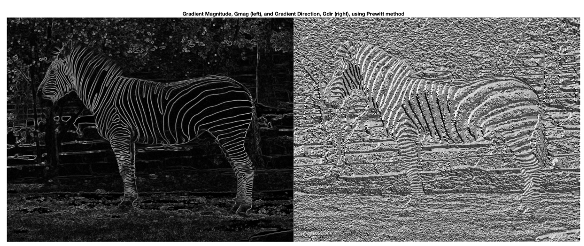
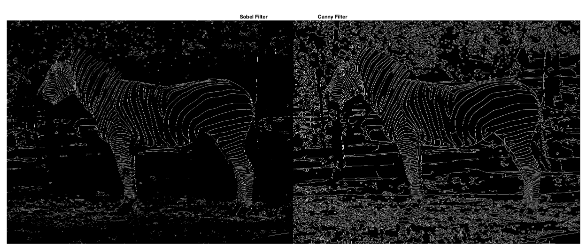


Figure 8: Detected edge.



Problem 7. object detector based on correlation

Using the Dilbert image, I chose a clip of the image containing an object that I would like to detect, as shown in Figure 9.

Figure 9: Object to be detected using correlation

(a) Dilbert image (b) target object

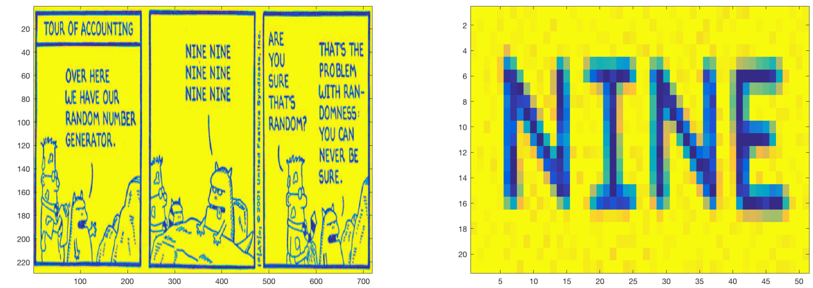


Figure 10: Cross-correlation

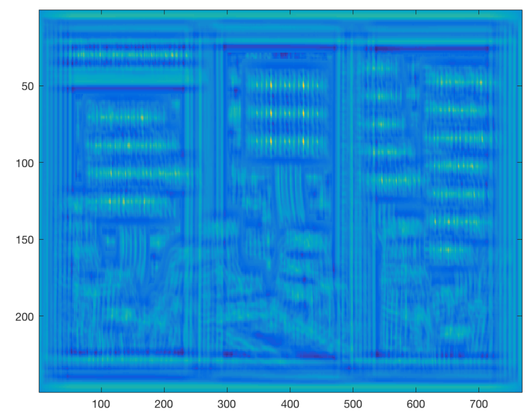
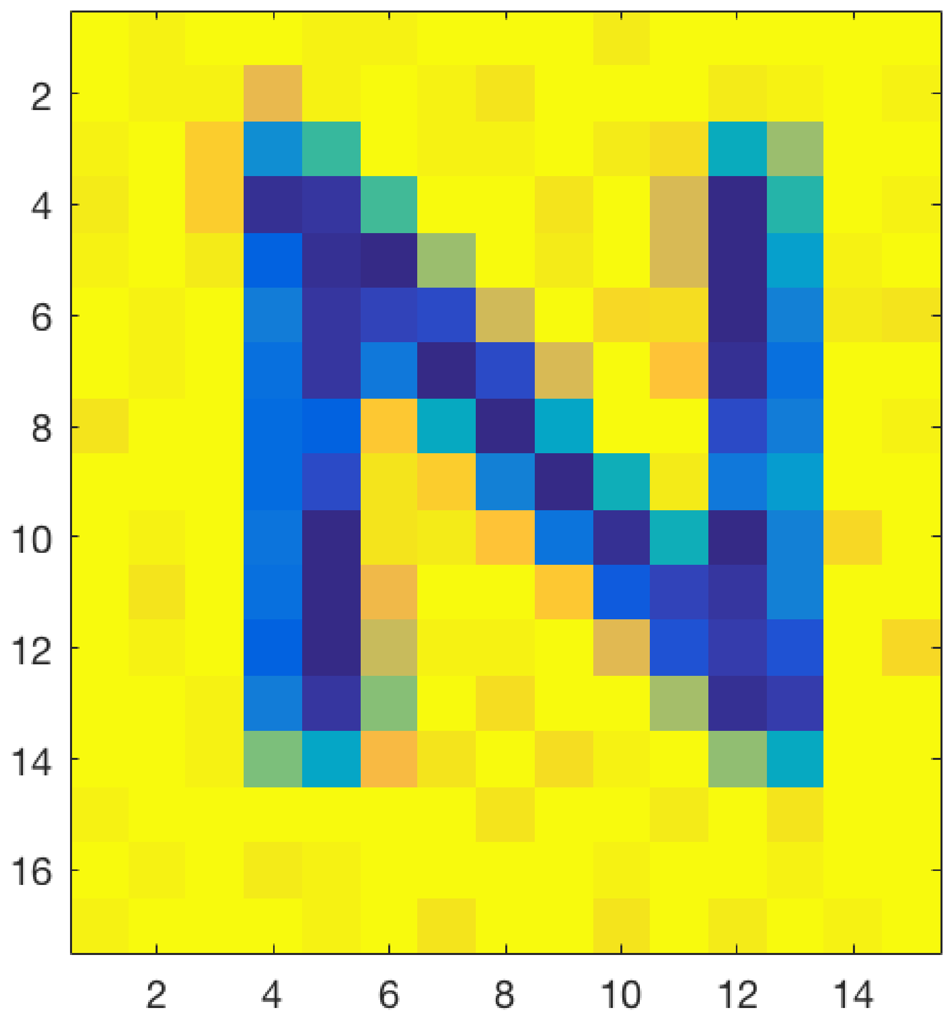


Figure 11: Detected object using greatest correlation is circled

Problem 8. Compute sum of squared differences (SSD)

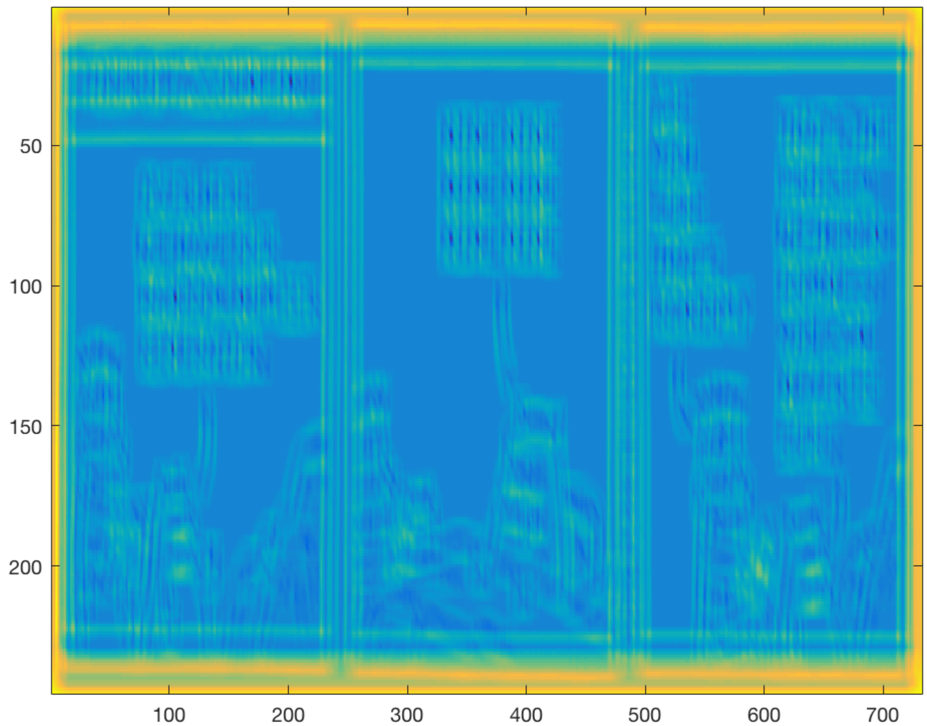
Using the same Dilber timage as in Figure 11, I computed the sum of squared differences (SSD), instead of correlation, between the image and the target shown in Figure 12

Figure 12: Target object



The SSD between the image I and the target T is shown in Figure 13

Figure 13: Sum of squared differences between the Image and the Target



The detected object is identified with a red circle in Figure 14

Figure 14: Target object