Easy as π

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The following thesis will discuss the number pi, denoted as π, its history and origin, and several different approximations involving calculus. The number itself will be investigated in a more theoretical sense as well. The irrationality of π will lead to several astounding mathematical conclusions in number theory. A surprising theme of calculus appears throughout the thesis, considering that π has been known to mankind much longer than the concept of calculus. Many other surprises concerning π await the reader.

Chapter 1: What Is π?

As every student learns at least once in his or her mathematics classes, pi is the ratio of the circumference, *C*, of a circle to its diameter, *d*, represented as $\pi = \frac{C}{d}$, a rearrangement of the formula for the circumference of a circle. Pi is also the sixteenth letter of the Greek alphabet (Posamentier, 2004). The symbol π was not used in its modern-day context until 1706 by a self-taught mathematics teacher known as William Jones. Before Jones, π was being used to represent the circumference because the term "circumference" was known as "periphery" at the time, and π is the Greek equivalent of the letter "p" (Rothman, 2009). Unfortunately for Jones, the mathematician who gave π's current definition its popularity is the renowned Leonhard Euler in his famous book *Introductio in analysin infinitorum* in 1748 (Posamentier, 2004).

Pi has been approximated in many different ways. One of the ancient methods was derived by the Greek mathematician Archimedes around 250 B.C. Although pi had been a topic of interest among the Greeks for many years, Archimedes was the first to devise a theoretical approach in calculating this ratio. His method involved regular polygons inscribed in a particular circle of radius r and circumscribed around the same circle. By comparing the perimeters of the polygons to the circumference of the circle, Archimedes was able to find an upper and lower bound for the value of pi. More specifically, he concluded that pi was less than $3\frac{1}{7}$ and greater than $3\frac{10}{71}$ (Groleau, 2003).

Archimedes' approximation only required basic concepts of geometry, but it is considered a sufficient value for most people even today. However, with the development of more advanced mathematics such as trigonometry and calculus, mathematicians are constantly finding new ways of approximating this fine ratio.

Calculus was not explicitly used until the late 17th century. Sir Isaac Newton and Gottfried Wilhelm Leibniz independently developed theories of differential and integral calculus, but Leibniz published his work a decade before Newton. Nevertheless, calculus-based approximations of pi were not even considered until after this time (Mastin, 2010).

Approximations of pi have been developed using infinite series expansions of functions. By definition, the sum of an infinite series is the limit $S = \lim_{N\to\infty} S_N$, where S_N is the N^{th} partial sum. If the limit exists, then the infinite series converges to the sum S. One particular example involves the power series for $\tan^{-1} x$. A power series with center c is an infinite series

$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots,$$

where x is a variable. The power series for arctangent is as follows:

Theorem 1: Power Series for Arctangent

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

for -1< x< 1.

Proof: Using the geometric series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

for -1 < x < 1 and substituting $-x^2$ for x, the following alternating series is produced:

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

for -1 < x < 1. By integrating term-by-term, it can be seen that

$$\tan^{-1} x = \int \frac{1}{1+x^2}\,dx = \int (1 - x^2 + x^4 - x^6 + \cdots)dx$$
$$= c + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

for -1 < x < 1. By setting x = 0, it can be seen that c = 0, thus proving

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

for -1 < x < 1. ■
(Rogawski, 2012).

Evaluating this series at special values produces pi-related solutions. An expected value to try first is 1; however, the power series needs to be tested for convergence at 1 using the alternating series test for $\sum_{n=0}^{\infty}(-1)^n a_n$. Evaluated at 1, the terms $a_n = \frac{1}{2n+1}$ are positive and decrease to 0, so the series converges (Rogawski, 2012). It is now acceptable to try 1. The power series for arctangent evaluated at 1 equals $\frac{\pi}{4}$. This results in the fascinating identity

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

known as the Gregory-Leibniz series. Simply multiplying by 4 on both sides of the equation yields an infinite series that converges to pi. How useful is this approximation? The followed table represents the approximation:

Number of terms	Partial Sum	Approximate Value
1	4*1	4
2	$4*(1-\frac{1}{3})$	2.666666666
3	$4*(1-\frac{1}{3}+\frac{1}{5})$	**3**.466666666
4	$4*(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7})$	2.895238095
5	$4*(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9})$	**3**.339682539
6	$4*(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11})$	2.976046176
7	$4*(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13})$	**3**.283738483
8	$4*(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15})$	**3**.017071817
9	$4*(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\frac{1}{17})$	**3**.252365934
10	$4*(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\frac{1}{17}-\frac{1}{19})$	**3**.041839618
100	$4*(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots-\frac{1}{199})$	**3.1**31592903
1000	$4*(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots-\frac{1}{1999})$	**3.1**31592902
10,000	$4*(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots-\frac{1}{19{,}999})$	**3.141**492653

100,000	$4*(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots-\frac{1}{199{,}999})$	**3.14158**2653
1,000,000	$4*(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots-\frac{1}{1{,}999{,}999})$	**3.141591**653
10,000,000	$4*(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots-\frac{1}{19{,}999{,}999})$	**3.141592**553
100,000,000	$4*(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots-\frac{1}{199{,}999{,}999})$	**3.1415926**43

(Posamentier, 2004).

Clearly, this series is approximating pi; however, it is converging exceptionally slowly. The bold digits represent the accurate decimal places of pi. Notice that after 100,000,000 terms, the approximation is only accurate to seven decimal places. This speed is due to the fact that the chosen value of 1 is at the right endpoint of the interval of convergence for the series. Choosing a value closer to 0 and, therefore, farther from this endpoint should result in a faster convergence. This value must be chosen strategically- a value whose arctangent involves pi. Arctangent evaluated at $\frac{\sqrt{3}}{3}$ is $\frac{\pi}{6}$. This results in another remarkable identity based on Gregory's series,

$$\frac{\pi}{6}=\frac{\sqrt{3}}{3}-\frac{\sqrt{3}}{3^2}\left(\frac{1}{3}\right)+\frac{\sqrt{3}}{3^3}\left(\frac{1}{5}\right)-\frac{\sqrt{3}}{3^4}\left(\frac{1}{7}\right)+\cdots.$$

Perhaps this series will be more effective. The next table approximates pi using this value:

Number of terms	Partial Sum	Approximate Value
1	$6*\frac{\sqrt{3}}{3}$	**3**.464101615
2	$6*(\frac{\sqrt{3}}{3}-\frac{\sqrt{3}}{3^2}(\frac{1}{3}))$	**3**.079201436
3	$6*(\frac{\sqrt{3}}{3}-\frac{\sqrt{3}}{3^2}\left(\frac{1}{3}\right)+\frac{\sqrt{3}}{3^3}\left(\frac{1}{5}\right))$	**3.1**56181472
4	$6*(\frac{\sqrt{3}}{3}-\frac{\sqrt{3}}{3^2}\left(\frac{1}{3}\right)+\frac{\sqrt{3}}{3^3}\left(\frac{1}{5}\right)-\frac{\sqrt{3}}{3^4}\left(\frac{1}{7}\right))$	**3.1**37852892
5	$6*(\frac{\sqrt{3}}{3}-\frac{\sqrt{3}}{3^2}\left(\frac{1}{3}\right)+\frac{\sqrt{3}}{3^3}\left(\frac{1}{5}\right)-\frac{\sqrt{3}}{3^4}\left(\frac{1}{7}\right)+\frac{\sqrt{3}}{3^5}\left(\frac{1}{9}\right))$	**3.14**2604746
6	$6*(\frac{\sqrt{3}}{3}-\frac{\sqrt{3}}{3^2}\left(\frac{1}{3}\right)+\frac{\sqrt{3}}{3^3}\left(\frac{1}{5}\right)-\frac{\sqrt{3}}{3^4}\left(\frac{1}{7}\right)+\frac{\sqrt{3}}{3^5}\left(\frac{1}{9}\right)-\frac{\sqrt{3}}{3^6}\left(\frac{1}{11}\right))$	**3.141**308786
7	$6*(\frac{\sqrt{3}}{3}-\frac{\sqrt{3}}{3^2}\left(\frac{1}{3}\right)+\frac{\sqrt{3}}{3^3}\left(\frac{1}{5}\right)-\frac{\sqrt{3}}{3^4}\left(\frac{1}{7}\right)+\frac{\sqrt{3}}{3^5}\left(\frac{1}{9}\right)-\frac{\sqrt{3}}{3^6}\left(\frac{1}{11}\right)+\frac{\sqrt{3}}{3^7}\left(\frac{1}{13}\right))$	**3.141**674313
8	$6*(\frac{\sqrt{3}}{3}-\frac{\sqrt{3}}{3^2}\left(\frac{1}{3}\right)+\frac{\sqrt{3}}{3^3}\left(\frac{1}{5}\right)-\frac{\sqrt{3}}{3^4}\left(\frac{1}{7}\right)+\frac{\sqrt{3}}{3^5}\left(\frac{1}{9}\right)-\frac{\sqrt{3}}{3^6}\left(\frac{1}{11}\right)+\frac{\sqrt{3}}{3^7}\left(\frac{1}{13}\right)-\frac{\sqrt{3}}{3^8}\left(\frac{1}{15}\right))$	**3.1415**68716
9	$6*(\frac{\sqrt{3}}{3}-\frac{\sqrt{3}}{3^2}\left(\frac{1}{3}\right)+\frac{\sqrt{3}}{3^3}\left(\frac{1}{5}\right)-\frac{\sqrt{3}}{3^4}\left(\frac{1}{7}\right)+\frac{\sqrt{3}}{3^5}\left(\frac{1}{9}\right)-\frac{\sqrt{3}}{3^6}\left(\frac{1}{11}\right)+\frac{\sqrt{3}}{3^7}\left(\frac{1}{13}\right)-\frac{\sqrt{3}}{3^8}\left(\frac{1}{15}\right)+\frac{\sqrt{3}}{3^9}\left(\frac{1}{17}\right))$	**3.14159**9774
10	$6*(\frac{\sqrt{3}}{3}-\frac{\sqrt{3}}{3^2}\left(\frac{1}{3}\right)+\frac{\sqrt{3}}{3^3}\left(\frac{1}{5}\right)-\frac{\sqrt{3}}{3^4}\left(\frac{1}{7}\right)+\frac{\sqrt{3}}{3^5}\left(\frac{1}{9}\right)-\frac{\sqrt{3}}{3^6}\left(\frac{1}{11}\right)+\frac{\sqrt{3}}{3^7}\left(\frac{1}{13}\right)-\frac{\sqrt{3}}{3^8}\left(\frac{1}{15}\right)+\frac{\sqrt{3}}{3^9}\left(\frac{1}{17}\right)-\frac{\sqrt{3}}{3^{10}}\left(\frac{1}{19}\right))$	**3.14159**0511
11	$6*(\frac{\sqrt{3}}{3}-\frac{\sqrt{3}}{3^2}\left(\frac{1}{3}\right)+\frac{\sqrt{3}}{3^3}\left(\frac{1}{5}\right)-\frac{\sqrt{3}}{3^4}\left(\frac{1}{7}\right)+\frac{\sqrt{3}}{3^5}\left(\frac{1}{9}\right)-\frac{\sqrt{3}}{3^6}\left(\frac{1}{11}\right)+\frac{\sqrt{3}}{3^7}\left(\frac{1}{13}\right)-\frac{\sqrt{3}}{3^8}\left(\frac{1}{15}\right)+\frac{\sqrt{3}}{3^9}\left(\frac{1}{17}\right)-\frac{\sqrt{3}}{3^{10}}\left(\frac{1}{19}\right)+\frac{\sqrt{3}}{3^{11}}\left(\frac{1}{21}\right))$	**3.14159**3305
12	$6*(\frac{\sqrt{3}}{3}-\frac{\sqrt{3}}{3^2}\left(\frac{1}{3}\right)+\frac{\sqrt{3}}{3^3}\left(\frac{1}{5}\right)-\frac{\sqrt{3}}{3^4}\left(\frac{1}{7}\right)+\frac{\sqrt{3}}{3^5}\left(\frac{1}{9}\right)-\frac{\sqrt{3}}{3^6}\left(\frac{1}{11}\right)+\frac{\sqrt{3}}{3^7}\left(\frac{1}{13}\right)-\frac{\sqrt{3}}{3^8}\left(\frac{1}{15}\right)+\frac{\sqrt{3}}{3^9}\left(\frac{1}{17}\right)-\frac{\sqrt{3}}{3^{10}}\left(\frac{1}{19}\right)+\frac{\sqrt{3}}{3^{11}}\left(\frac{1}{21}\right)-\frac{\sqrt{3}}{3^{12}}\left(\frac{1}{23}\right))$	**3.141592**455
13	$6*(\frac{\sqrt{3}}{3}-\frac{\sqrt{3}}{3^2}\left(\frac{1}{3}\right)+\frac{\sqrt{3}}{3^3}\left(\frac{1}{5}\right)-\frac{\sqrt{3}}{3^4}\left(\frac{1}{7}\right)+\frac{\sqrt{3}}{3^5}\left(\frac{1}{9}\right)-\frac{\sqrt{3}}{3^6}\left(\frac{1}{11}\right)+\frac{\sqrt{3}}{3^7}\left(\frac{1}{13}\right)-\frac{\sqrt{3}}{3^8}\left(\frac{1}{15}\right)+\frac{\sqrt{3}}{3^9}\left(\frac{1}{17}\right)-\frac{\sqrt{3}}{3^{10}}\left(\frac{1}{19}\right)+\frac{\sqrt{3}}{3^{11}}\left(\frac{1}{21}\right)-\frac{\sqrt{3}}{3^{12}}\left(\frac{1}{23}\right)+\frac{\sqrt{3}}{3^{13}}\left(\frac{1}{25}\right))$	**3.1415927**16
14	$6*(\frac{\sqrt{3}}{3}-\frac{\sqrt{3}}{3^2}\left(\frac{1}{3}\right)+\frac{\sqrt{3}}{3^3}\left(\frac{1}{5}\right)-\frac{\sqrt{3}}{3^4}\left(\frac{1}{7}\right)+\frac{\sqrt{3}}{3^5}\left(\frac{1}{9}\right)-\frac{\sqrt{3}}{3^6}\left(\frac{1}{11}\right)+\frac{\sqrt{3}}{3^7}\left(\frac{1}{13}\right)-\frac{\sqrt{3}}{3^8}\left(\frac{1}{15}\right)+\frac{\sqrt{3}}{3^9}\left(\frac{1}{17}\right)-\frac{\sqrt{3}}{3^{10}}\left(\frac{1}{19}\right)+\frac{\sqrt{3}}{3^{11}}\left(\frac{1}{21}\right)-\frac{\sqrt{3}}{3^{12}}\left(\frac{1}{23}\right)+\frac{\sqrt{3}}{3^{13}}\left(\frac{1}{25}\right)-\frac{\sqrt{3}}{3^{14}}\left(\frac{1}{27}\right))$	**3.1415926**36

15	$6*(\frac{\sqrt{3}}{3}-\frac{\sqrt{3}}{3^2}\left(\frac{1}{3}\right)+\frac{\sqrt{3}}{3^3}\left(\frac{1}{5}\right)-\frac{\sqrt{3}}{3^4}\left(\frac{1}{7}\right)+\frac{\sqrt{3}}{3^5}\left(\frac{1}{9}\right)-\frac{\sqrt{3}}{3^6}\left(\frac{1}{11}\right)+\frac{\sqrt{3}}{3^7}\left(\frac{1}{13}\right)-\frac{\sqrt{3}}{3^8}\left(\frac{1}{15}\right)+\frac{\sqrt{3}}{3^9}\left(\frac{1}{17}\right)-\frac{\sqrt{3}}{3^{10}}\left(\frac{1}{19}\right)+\frac{\sqrt{3}}{3^{11}}\left(\frac{1}{21}\right)-\frac{\sqrt{3}}{3^{12}}\left(\frac{1}{23}\right)+\frac{\sqrt{3}}{3^{13}}\left(\frac{1}{25}\right)-\frac{\sqrt{3}}{3^{14}}\left(\frac{1}{27}\right)+\frac{\sqrt{3}}{3^{15}}\left(\frac{1}{29}\right))$	**3.141592661**
16	$6*(\frac{\sqrt{3}}{3}-\frac{\sqrt{3}}{3^2}\left(\frac{1}{3}\right)+\frac{\sqrt{3}}{3^3}\left(\frac{1}{5}\right)-\frac{\sqrt{3}}{3^4}\left(\frac{1}{7}\right)+\frac{\sqrt{3}}{3^5}\left(\frac{1}{9}\right)-\frac{\sqrt{3}}{3^6}\left(\frac{1}{11}\right)+\frac{\sqrt{3}}{3^7}\left(\frac{1}{13}\right)-\frac{\sqrt{3}}{3^8}\left(\frac{1}{15}\right)+\frac{\sqrt{3}}{3^9}\left(\frac{1}{17}\right)-\frac{\sqrt{3}}{3^{10}}\left(\frac{1}{19}\right)+\frac{\sqrt{3}}{3^{11}}\left(\frac{1}{21}\right)-\frac{\sqrt{3}}{3^{12}}\left(\frac{1}{23}\right)+\frac{\sqrt{3}}{3^{13}}\left(\frac{1}{25}\right)-\frac{\sqrt{3}}{3^{14}}\left(\frac{1}{27}\right)+\frac{\sqrt{3}}{3^{15}}\left(\frac{1}{29}\right)-\frac{\sqrt{3}}{3^{16}}\left(\frac{1}{31}\right))$	**3.141592653**

After the sixteenth term, the approximation has nine decimal digits of accuracy. This accuracy could not be achieved after 100,000,000 terms for arctangent evaluated at 1! In theory, one can use half angle trigonometric formulae in order to find smaller and smaller pi-related values and, in turn, faster approximations.

Another method for speeding up convergence involves the sum of two series. Arctangent evaluated at 1 equals the sum of arctangent evaluated at $\frac{1}{2}$ and arctangent evaluated at $\frac{1}{3}$. Since both $\frac{1}{2}$ and $\frac{1}{3}$ are smaller than 1, the two power series will converge much more quickly. These specific values are important because the sum of the two arctangents is 1, so the sum of the power series must converge to $\frac{\pi}{4}$ as well. The following trigonometric identity is used to evaluate this sum:

Theorem 2: Sum of Arctangents Identity

$$\tan^{-1} x+\tan^{-1} y=\tan^{-1}\frac{x+y}{1-xy}$$

for x,y $\in \mathbb{R}$, xy$\neq$ 1.

Proof Part 1:

Prove that $\tan(A+B)=\frac{\text{tanA+tanB}}{\text{1−tanAtanB}}$. Note that $\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}$.

By using angle sum trigonometric identities, $\frac{\sin(A+B)}{\cos(A+B)}=\frac{\text{sinAcosB+cosAsinB}}{\text{cosAcosB−sinAsinB}}$.

Multiply the numerator and denominator by $\frac{1}{\text{cosAcosB}}$ in order to eliminate the extra cosines as follows:

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\frac{1}{\text{cosAcosB}}(\text{sinAcosB}+\text{cosAsinB})}{\frac{1}{\text{cosAcosB}}(\text{cosAcosB}-\text{sinAsinB})} = \frac{\frac{\text{sinA}}{\text{cosA}}+\frac{\text{sinB}}{\text{cosB}}}{1-\frac{\text{sinAsinB}}{\text{cosAcosB}}}.$$

Rewrite the equation to obtain:

$$\tan(A+B) = \frac{\text{tanA} + \text{tanB}}{1 - \text{tanAtanB}}.$$

(Terr, 2009).

Part 2:

Let $\tan^{-1} x = \text{A}, \tan^{-1} \text{y} = \text{B},$ so $\text{tanA} = \text{x}, \text{tanB} = \text{y}$. By substitution on the right hand side of the above identity, it can be seen that

$$\tan(A+B) = \frac{\text{x}+\text{y}}{1-\text{xy}}.$$

Take the arctangent on both sides to obtain

$$\text{A} + \text{B} = \tan^{-1}(\frac{\text{x}+\text{y}}{1-\text{xy}}).$$

By substitution on the left hand side,

$$\tan^{-1}\text{x} + \tan^{-1}\text{y} = \tan^{-1}\frac{\text{x}+\text{y}}{1-\text{xy}},$$

as desired. ■

Using this identity, it is clear that $\tan^{-1}(\frac{1}{2}) + \tan^{-1}(\frac{1}{3}) = \tan^{-1}\frac{\frac{1}{2}+\frac{1}{3}}{1-(\frac{1}{2})(\frac{1}{3})} =$ $\tan^{-1} 1$. How much more quickly will the approximation for arctangent evaluated at 1 converge when taking the sum of the arctangents evaluated at the two fractional values? The following table will compare approximate values of the original and the sum.

Number of terms	Approx. Value using $4(\tan^{-1} 1)$	Approx. Value using $4(\tan^{-1}(\frac{1}{2}) + \tan^{-1}(\frac{1}{3}))$
1	4	**3**.333333333
2	2.666666666	**3.1**17283951
3	**3**.466666666	**3.14**5576132
4	2.895238095	**3.14**0850562
5	**3**.339682539	**3.141**741197
6	2.976046176	**3.1415**61588

7	3.283738483	**3.141599341**
8	3.017071817	**3.141591184**
9	3.252365934	**3.141592981**
10	3.041839618	**3.14159258**
11	3.232315809	**3.14159267**
12	3.058402766	**3.14159265**
13	3.218402766	**3.141592654**
14	3.070254618	**3.141592653**

Again, the power series is converging much more swiftly. This sum actually converges slightly more quickly than the arctangent evaluated at $\frac{\sqrt{3}}{3}$. After the fourteenth term, the approximation is accurate to nine decimal places, which is two terms sooner than the power series of $\tan^{-1}\frac{\sqrt{3}}{3}$.

Pi has many other features besides approximation. Some of these features have led to astounding conclusions about certain mathematical phenomena. In recent years, calculus has been a major tool in arriving to these conclusions.

Chapter 2: What Kind of Number Is π?

Pi has been described as an irrational and transcendental number, and it has been found in many mathematical subdivisions including geometry, trigonometry, calculus, number theory, and even probability! It touches infinity through several infinite series as mentioned earlier. Pi is also connected to waves such as pulse monitors and rhythms of sleep through trigonometric functions (Strogatz, 2015). This concludes that even ones own breathing patterns can be calculated in terms of pi. The reader can imagine that he or she is indirectly inhaling and exhaling pi! Clearly, this is a number worth the enthusiasm!

The next chapters will cover several theoretical proofs about the nature of pi, which result in a highly fascinating conclusion about prime numbers and surprisingly, probability. First, pi's irrationality must be proven.

Pi was first proven irrational in 1761 by Johann Heinrich Lambert. Lambert's proof involved continued fractions, which was considered too much of a digression from the main focus of proving pi's irrationality (Laczkovich, 1997). In 1794, Adrien-Marie Legendre was the first to prove pi's irrationality by proving the square of pi's irrationality (Beckmann, 1971). Ivan Niven developed his own proof almost 200 years later in 1946. Niven's proof simply involves algebra and basic calculus, making it a more commonly used proof in number theory textbooks. Niven also proves the irrationality of π^2, which, in turn, proves the irrationality of π.

Theorem 3: π^2 is irrational.
Proof: Consider the function

$$f(x) = \frac{x^n(1-x)^n}{n!},$$

where n is any positive integer. Motivation for this function is given by the binomial theorem, which describes an expansion of powers of binomials. The expansion of $(1-x)^n$ is as follows:

$$(1-x)^n = 1 - nx + \binom{n}{2}x^2 - \cdots + (-1)^n\binom{n}{n}x^n.$$

Multiply by x^n: $$x^n(1-x)^n = x^n - nx^{n+1} + \binom{n}{2}x^{n+2} - \cdots + (-1)^n\binom{n}{n}x^{2n}.$$

Note, for any nonnegative integer $k \le n$, the binomial coefficient $\frac{n(n-1)(n-2)\ldots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$ is always an integer.

Some observations will be made regarding f(x). For any integer $n \ge 1$, the following hold.

a) f(x) is a polynomial of the form $f(x) = \frac{1}{n!}\sum_{i=n}^{2n} c_i x^i$, and all $c_i \in \mathbb{Z}$.
b) For $0<x<1$, $0< f(x)<\frac{1}{n!}$.
c) The derivatives $f^{(k)}(0)$ and $f^{(k)}(1)$ are integers for all $k \ge 0$.

Clearly a) holds, since it represents the function in summation notation, and the binomial coefficient, represented by c_i, is always an integer.

For b), observe that for any x such that $0<x<1$, then $0<x^n<1$ and $0<(1-x)^n<1$, which implies that $0<x^n(1-x)^n<1$. By multiplying across by $\frac{1}{n!}$, the inequality is obtained, $0<\frac{x^n(1-x)^n}{n!}<\frac{1}{n!}$.

For c), $f^{(k)}(0)=0$ if $k<n$ or $k>2n$ since those k's do not appear in index of the summation; therefore those coefficients in the polynomial are 0. For $n\le k\le 2n$, $f^{(k)}(0)=\frac{k!}{n!}c_k\in\mathbb{Z}$ because the $\frac{k!}{n!}\in\mathbb{Z}$ since $k\ge n$ and $c_k\in\mathbb{Z}$ by a). $f^{(k)}(1)$ is equivalent to $f^{(k)}(0)$ due to the symmetry of f(x). Specifically, $f(x)=f(1-x)$ implies that $f^{(k)}(x)=(-1)^k f^{(k)}(1-x)$ and so $f^{(k)}(1)=(-1)^k f^{(k)}(0)\in\mathbb{Z}$.

By way of contradiction, assume π^2 is rational, so $\pi^2=\frac{a}{b}$, where a and b are positive integers. Let $\mathcal{F}(x)=b^n(\pi^{2n}f(x)-\pi^{2(n-1)}f^{(2)}(x)+\pi^{2(n-2)}f^{(4)}(x)-\cdots+(-1)^n f^{(2n)}(x))$. Observe for all $0\le k\le n$, $b^n\pi^{2(n-k)}=b^n(\frac{a}{b})^{n-k}$ by assumption. After some algebraic manipulations,

$$b^n(\frac{a}{b})^{n-k}=a^{n-k}b^k\in\mathbb{Z}.$$

Hence, $\mathcal{F}(x)=a^n f(x)-a^{n-1}bf^{(2)}(x)+a^{n-2}b^2f^{(4)}(x)-\cdots+(-1)^n b^n f^{(2n)}(x)$.

Since $f^{(k)}(0)$ and $f^{(k)}(1)$ are integers by c), $\mathcal{F}(0)$ and $\mathcal{F}(1)$ are also integers. By differentiating $\mathcal{F}$ twice,
$\mathcal{F}''(x)=b^n(\pi^{2n}f^{(2)}(x)-\pi^{2(n-1)}f^{(4)}(x)+\pi^{2(n-2)}f^{(6)}(x)-\cdots+(-1)^n f^{(2n+2)}(x))$.

Note that everything is a constant except for $f^{(k)}(x)$. Also, by a), $f^{(2n+2)}(x)=0$ since $2n+2>2n$ and so the last term disappears.

Next observe that

$$\mathcal{F}''(x)+\pi^2\mathcal{F}(x)=b^n\pi^{2n+2}f(x)$$

because every term except the first cancels since the series alternates. After some algebraic manipulations,

$$b^n\pi^{2n+2}f(x)=b^n\left(\frac{a^{n+1}}{b^{n+1}}\right)f(x)=\left(\frac{a}{b}\right)a^n f(x)=\pi^2a^n f(x)$$

by assumption. By differentiating, it can be seen that

$$\begin{aligned}&\frac{d}{dx}(\mathcal{F}'(x)\sin\pi x-\pi\mathcal{F}(x)\cos\pi x)\\&=\pi\cos\pi x\mathcal{F}'(x)+\mathcal{F}''(x)\sin\pi x+\pi^2\sin\pi x\mathcal{F}(x)-\pi\mathcal{F}'(x)\cos\pi x\\&=\sin\pi x\big(\mathcal{F}''(x)+\pi^2\mathcal{F}(x)\big).\end{aligned}$$

By substitution,

$$\sin\pi x\big(\mathcal{F}''(x) + \pi^2\mathcal{F}(x)\big) = \sin\pi x\big(b^n\pi^{2n+2}f(x)\big) = \sin\pi x\big(\pi^2a^nf(x)\big).$$

By the fundamental theorem of calculus,

$$\pi^2a^n\int_0^1 f(x)\sin\pi x dx = (\mathcal{F}'(x)\sin\pi x - \pi\mathcal{F}(x)\cos\pi x)\,|_0^1 =$$

$$\mathcal{F}'(1)\sin\pi - \pi\mathcal{F}(1)\cos\pi - (\mathcal{F}'(0)\sin 0 - \pi\mathcal{F}(0)\cos 0) = \pi\big(\mathcal{F}(1) + \mathcal{F}(0)\big).$$

By canceling like π terms, $\pi a^n \int_0^1 f(x)\sin\pi x dx = \mathcal{F}(1) + \mathcal{F}(0) \in \mathbb{Z}$.

Since $0< f(x)<\frac{1}{n!}$ for $0<x<1$ by b) and $0<\sin\pi x<1$ for $0<x<1$, then $0<f(x)\sin\pi x<\frac{1}{n!}$ for $0<x<1$. For any integer $n\geq1$, this inequality implies that

$$0 < \pi a^n\int_0^1 f(x)\sin\pi x dx < \frac{\pi a^n}{n!}.$$

But, for any number a, $\lim_{n\to\infty}\frac{a^n}{n!} = 0$. By choosing n large enough so $\frac{\pi a^n}{n!} < 1$, $0< \pi a^n \int_0^1 f(x)\sin\pi x dx < 1$. This is a contradiction since $\pi a^n \int_0^1 f(x)\sin\pi x dx = \mathcal{F}(1) + \mathcal{F}(0)$ is an integer. Hence π^2 is irrational. ■
(Hardy, 1960).

Since π^2 is irrational, π can easily be proven irrational as well.

Theorem 4: π is irrational.
Proof: By way of contradiction, assume π is rational. Rational numbers are closed under multiplication, which means the product of two rational numbers is also a rational number. So,

$$\pi * \pi = \pi^2 \in \mathbb{Q}.$$

This is a contradiction since π^2 is irrational; therefore, π cannot be rational. ■

The transcendence of pi was proven in 1882 by Ferdinand von Lindemann (Berggren, 1997). This more advanced proof will not be provided in this thesis; however, the discovery is worth mentioning. By definition, a number is transcendental if it is not algebraic, i.e., it does not satisfy any non-zero polynomial equation with integer coefficients. This conclusion settled the long-debated question of the possibility of squaring a circle. Since pi is transcendental, and therefore not the solution to any polynomial equation, it is certainly not the solution to a quadratic equation. Lindemann consequently proved that it is impossible to square a circle. (Hardy, 1960).

Chapter 3: Pi and Number Theory

A new function will now be introduced that, strangely enough, relates to pi. This function, known as the Riemann zeta function, can be used to prove that there are infinitely many prime numbers. This proof relies on knowing that π^2 is irrational.

What is this Riemann zeta function? It is the simplest of Dirichlet series, defined by $\zeta(s)= \sum_{n=1}^{\infty}\frac{1}{n^s}$. A Dirichlet series is a series of the form $F(s)= \sum_{n=1}^{\infty}\frac{a_n}{n^s}$, where the variable s may be real or complex. In the case of the zeta function, $s \in \mathbb{C}$ such that $\Re\, s > 1$ (Hardy, 1960). This is because, for $s \in \mathbb{R}$, the zeta function reduces to a p-series, which is an infinite series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ that converges if p>1 and diverges otherwise. Proving that a p-series converges if p>1 can be done quickly using the integral test. If $\int_1^{\infty}\frac{1}{x^p}dx$ converges, then $\sum_{n=1}^{\infty}\frac{1}{n^p}$ also converges. For p>1, the following integral results:

$$\int_1^{\infty}\frac{1}{x^p}dx = \lim_{R\to\infty}\int_1^{R} x^{-p}\,dx = \lim_{R\to\infty}\frac{x^{-p+1}}{-p+1}\Big|_1^R = \lim_{R\to\infty}\left(\frac{1}{R^{p-1}(-p+1)} - \frac{1}{-p+1}\right) = \frac{1}{p-1}.$$

Since the limit equals a finite number, the summation converges (Rogawski, 2012).

Certain special values of the zeta function are in terms of pi. This thesis will analyze $\zeta(2)$, which is the sum of the reciprocals of the squares. Evaluating $\zeta(2)$ was known as the Basel Problem, and Euler discovered the exact value of $\frac{\pi^2}{6}$ in 1731 (Sangwin, 2001). The following proof is only twenty-three years old and due to the work of Frits Beukers, Eugenio Calabi, and Johan Kolk.

Theorem 5: $\zeta(2) = \frac{\pi^2}{6}$.

Proof: Given the summation, break it up into the even n's and odd n's:

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{\text{odd } n}^{\infty} \frac{1}{n^2} + \sum_{\text{even } n}^{\infty} \frac{1}{n^2}$$
$$= \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} + \sum_{k=1}^{\infty} \frac{1}{(2k)^2}$$
$$= \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} + \frac{1}{4}\sum_{k=1}^{\infty} \frac{1}{k^2}.$$

This implies that

$$\frac{3}{4}\zeta(2) = \frac{3}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}.$$

Note that the iterated integral

$$\int_0^1\int_0^1 x^{2k}y^{2k}\,dxdy = \int_0^1 \frac{x^{2k+1}}{2k+1}\Big|_0^1 y^{2k}dy = \int_0^1 \frac{1}{2k+1}y^{2k}dy = \frac{1}{2k+1}\frac{y^{2k+1}}{2k+1}\Big|_0^1$$
$$= \frac{1}{(2k+1)^2}.$$

So, by substitution,

$$\frac{3}{4}\zeta(2) = \sum_{k=0}^{\infty}\int_0^1\int_0^1 x^{2k}y^{2k}\,dxdy = \int_0^1\int_0^1\sum_{k=0}^{\infty}(x^2y^2)^k dxdy.$$

The interchange of summation and integration is allowed since a power series may be integrated term-by-term. Using the well-known geometric series formula $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}, |r| < 1$, and substituting x^2y^2 for r, it can be seen that

$$\frac{3}{4}\zeta(2) = \int_0^1\int_0^1 \frac{1}{1-x^2y^2}\,dxdy.$$

This improper integral, which must converge, will now be evaluated.

Let $x=\frac{\sin u}{\cos v}$, $y=\frac{\sin v}{\cos u}$, and by a change of variables,

$$\int_0^1 \int_0^1 \frac{1}{1-x^2y^2}\,dxdy = \iint_{\mathcal{R}} \frac{1}{1-(\frac{\sin u}{\cos v})^2(\frac{\sin v}{\cos u})^2}|J|dudv,$$

where $|J|$ is the absolute value of the Jacobian of the mapping and $\mathcal{R}$ is the new region. The Jacobian is computed as

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\cos u}{\cos v} & \sin u \sec v \tan v \\ \sin v \sec u \tan u & \frac{\cos v}{\cos u} \end{vmatrix} = 1 - \tan^2 u \tan^2 v.$$

Now, since $0 \le x \le 1$ and $0 \le y \le 1$, then $0 \le \frac{\sin u}{\cos v}\frac{\sin v}{\cos u} \le 1$ or $0 \le \tan u \tan v \le 1$. This implies that $0 \le \tan^2 u \tan^2 v \le 1$, hence $|J| = 1 - \tan^2 u \tan^2 v$. By substitution,

$$\iint_{\mathcal{R}} \frac{1}{1-\left(\frac{\sin u}{\cos v}\right)^2(\frac{\sin v}{\cos u})^2}|J|dudv = \iint_{\mathcal{R}} \frac{1}{1-(\tan u)^2(\tan v)^2}(1-\tan^2 u \tan^2 v)dudv =$$

$\iint_{\mathcal{R}} 1dudv.$

Next, the region must be found. There is no loss to restrict u and v so that $0 \le u < \frac{\pi}{2}$ and $0 \le v < \frac{\pi}{2}$. The inequality $0 \le x \le 1$ implies that $0 \le \frac{\sin u}{\cos v} \le 1$. Since $\cos v = \sin(\frac{\pi}{2} - v)$, then $0 \le \sin u \le \sin\left(\frac{\pi}{2} - v\right)$. Taking the inverse of sine throughout results in inequality $0 \le u \le \frac{\pi}{2} - v$. An analogous argument applied to $0 \le y \le 1$ shows that $0 \le v \le \frac{\pi}{2} - u$. So $\mathcal{R}$ is simply the triangular region given by $\left\{(u,v): u \ge 0, v \ge 0, u + v \le \frac{\pi}{2}\right\}$. This triangular region has base and height of $\frac{\pi}{2}$.

Since the integrand is a constant, specifically 1, the double integral can be represented as the area of the region $\mathcal{R}$. Therefore,

$$\iint_{\mathcal{R}} 1dudv = Area(\mathcal{R}) = \frac{1}{2}\left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{8}.$$

Finally,

$$\frac{3}{4}\zeta(2) = \frac{\pi^2}{8}$$

and, therefore,

$$\zeta(2) = \frac{\pi^2}{6}. \blacksquare$$

(Beukers, 1993).

The zeta function is vital in the theory of prime numbers. In 1737, Leonhard Euler discovered the following remarkable identity that represented zeta as a product over primes (Hardy, 1960).

Theorem 6: Zeta as a product over primes

$$\zeta(s) = \prod_{p\ prime} \frac{1}{1-\frac{1}{p^s}},$$

where $\Re\, s > 1$.

Proof: Using a sieving method on $\zeta(s)$, first sieve out $\frac{1}{2^s}$ and all elements with a factor of 2:

$$\zeta(s) - \frac{1}{2^s}\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \cdots$$

so that

$$\left(1-\frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \cdots.$$

Repeat the procedure with the next term, $\frac{1}{3^s}$, to get

$$\left(1-\frac{1}{2^s}\right)\zeta(s) - \frac{1}{3^s}\left(1-\frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \cdots$$

so that

$$\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \cdots.$$

Because of the unique prime factorization of integers, the sieving method can continue indefinitely to obtain

$$\ldots\left(1-\frac{1}{5^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{2^s}\right)\zeta(s) = 1.$$

For $\Re\, s > 1$, it can be shown that this product converges since the sequence of partial products converges to a nonzero value. By dividing the left hand side,

$$\zeta(s) = \frac{1}{\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{5^s}\right)\ldots}$$

$$= \prod_p \frac{1}{1-\frac{1}{p^s}}. \blacksquare$$

By substituting pi-related values into zeta, one can use the acquired knowledge of π^2 to deduce the infinitude of prime numbers. This next proof is a simple, but surprising proof of Euclid's famous result that there exist infinitely many prime numbers.

Theorem 7: There exist infinitely many primes.

Proof: Assume there are finitely many primes $p_1, p_2, p_3, \dots, p_r$. Based on the Euler's Product Formula for $\zeta(s)$,

$$\zeta(s) = \prod_{j=1}^{r} \frac{1}{1 - \frac{1}{p_j^{s}}}$$

for any s such that $\Re$ s > 1. By letting s=2, the following identity is formed:

$$\zeta(2) = \prod_{j=1}^{r} \frac{1}{1 - \frac{1}{p_j^{2}}},$$

which implies that

$$\zeta(2) = \frac{1}{\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\dots\left(1-\frac{1}{p_r^2}\right)}.$$

Hence,

$$\frac{\pi^2}{6} = \frac{1}{\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\dots\left(1-\frac{1}{p_r^2}\right)} = \left(\frac{4}{3}\right)\left(\frac{9}{8}\right)\left(\frac{25}{24}\right)\dots\left(\frac{p_r^{\,2}}{p_r^{\,2}-1}\right)$$

and so,

$$\pi^2 = 6\left(\frac{4}{3}\right)\left(\frac{9}{8}\right)\left(\frac{25}{24}\right)\dots\left(\frac{p_r^{\,2}}{p_r^{\,2}-1}\right).$$

Now the right-hand side is rational; however π^2 is irrational. This is a contradiction. ■

Euclid also proved the fact that there are infinitely many primes around 300 B.C. in his famous book Elements. This proof does not require π, nor does it involve calculus. Euclid begins by proving that every natural number greater than one has a prime divisor using the Well-Ordering Property. By way of contradiction, he assumes that there exists a natural number greater than one with no prime divisors.

Based on the Well-Ordering Property, there must be a smallest element, say n, in this nonempty subset of numbers. Since n divides n implies that n is not a prime, then a number m divides n for some m such that 1<m<n. This m must have a prime divisor p such that 1<p<m. By the transitive property, p must divide n as well. This is a contradiction since it was assumed that n had no prime divisors.

Euclid uses the fact that every natural number greater than one has a prime divisor in order to prove that there exist infinitely many primes. By way of contradiction, he assumes that there are only finitely many primes, denoted as $p_1, p_2, \ldots p_r$. He considered a number N = $p_1 p_2 \ldots p_r + 1$. Since N>1, N must have a prime divisor p. Since there are finitely many primes, p must be one of the primes $p_1, p_2, \ldots p_r$. Hence, p divides N and p divides $p_1 p_2 \ldots p_r$. This implies that p divides any linear combination of the two, so p divides N-$p_1 p_2 \ldots p_r$; therefore, p divides 1. If p divides 1, then $p \leq 1$. This is a contradiction, so there must exist infinitely many prime numbers (Rosen, 2011).

Euclid is commended for such a profound discovery at his time. Arriving at this conclusion using π^2, however, is much more fascinating. π^2 is neither natural nor prime, but after much calculus and number theory, this number does, in fact, prove that there exist infinitely many primes as well.

Chapter 4: Pi and Probability

If proving that there are infinitely many primes using π was not farfetched enough, this next connection to π should suffice! In 1849, Peter Gustav Lejeune Dirichlet showed that the probability that two randomly selected natural numbers are relatively prime is $\frac{6}{\pi^2}$ (Posamentier, 2004). Notice that the stated probability is the reciprocal of the value of ζ(2). This is not a coincidence! A plausibility argument, rather than a rigorous proof, will be provided. First, it must be shown that the probability that two randomly selected natural numbers do not share a prime divisor p is $1-\frac{1}{p^2}$.

The probability that one natural number less than or equal to n is divisible by p is $\frac{1}{n}\left[\frac{n}{p}\right]$, where $\left[\frac{n}{p}\right]$ is the greatest integer less than or equal to $\frac{n}{p}$. If $\left[\frac{n}{p}\right]=\frac{n}{p}$, then the probability is simply $\frac{1}{p}$. The probability that two randomly chosen natural numbers less than or equal to n are both divisible by p is $\frac{1}{n}\left[\frac{n}{p}\right]*\frac{1}{n}\left[\frac{n}{p}\right]$ or $\frac{1}{n^2}\left[\frac{n}{p}\right]^2$. The product is taken since each choice is considered an independent event. That is, the outcome that the first randomly chosen natural number is divisible by p is independent of the outcome that the second randomly chosen natural number is divisible by p. Again, if $\left[\frac{n}{p}\right]=\frac{n}{p}$, then the probability is simply $\frac{1}{p^2}$. By the law of complementary events, the probability that two randomly selected natural numbers are not both divisible by p is $1-\frac{1}{n^2}\left[\frac{n}{p}\right]^2$. The limit as $n\to\infty$ of this probability must be taken. By the properties of limits,

$$\lim_{n\to\infty}\left(1-\frac{1}{n^2}\left[\frac{n}{p}\right]^2\right)=\lim_{n\to\infty}1-\lim_{n\to\infty}\frac{1}{n^2}\left[\frac{n}{p}\right]^2=1-\left(\lim_{n\to\infty}\frac{1}{n}\left[\frac{n}{p}\right]\right)^2.$$

The greatest integer function [x] satisfies the inequality $x-1<[x]\leq x$ for all $x\in\mathbb{R}$ (Rosen, 2011). So, $\frac{n}{p}-1<\left[\frac{n}{p}\right]\leq\frac{n}{p}$. By multiplying the inequality by $\frac{1}{n}$, it can be seen that

$$\frac{1}{p} - \frac{1}{n} < \frac{\left[\frac{n}{p}\right]}{n} \leq \frac{1}{p}.$$

Then, take the limit of each bound, so $\lim_{n\to\infty}(\frac{1}{p} - \frac{1}{n}) = \frac{1}{p}$ and $\lim_{n\to\infty} \frac{1}{p} = \frac{1}{p}$. By the Squeeze Theorem, $\lim\limits_{n\to\infty} \frac{\left[\frac{n}{p}\right]}{n} = \frac{1}{p}$. Therefore, $\lim_{n\to\infty}(1 - \frac{1}{n^2}\left[\frac{n}{p}\right]^2) = 1 - \frac{1}{p^2}$ as desired (Rogawski, 2012).

The probability that two randomly chosen natural numbers less than or equal to n are relatively prime, meaning they do not share any primes, can be expressed by a complicated formula using a method known as inclusion and exclusion. The limit of this probability as n approaches infinity is $\prod_p(1 - \frac{1}{p^2})$, which is the reciprocal of $\zeta(2)$. Since $\zeta(2) = \frac{\pi^2}{6}$, $\prod_p(1 - \frac{1}{p^2}) = \frac{6}{\pi^2}$.

The value $\frac{6}{\pi^2}$, approximately 61%, is also the probability that a randomly chosen natural number is square-free, which means that the number is not divisible by the square of any prime. That is, it is the product of distinct primes. The argument is similar, and the mathematical value is equivalent, but the result pertains to a completely different event. The first event involves two numbers being relatively prime, while the second event only involves one number being square-free. The results are quite astounding indeed!

Conclusion

Pi is a strange, fascinating, irrational, and transcendental number that is geometrically based, yet involved in trigonometry, calculus, number theory, and probability. It can be approximated using infinite series, some converging more quickly than others. It has been proven irrational time and time again using numerous methods. It comes up in number theory through the Riemann zeta function and the Euler Product Formula. Surprisingly enough, it can be used to prove that infinitely many primes exist. The peculiar number is even found in probability!

This thesis covered a wide range of topics involving pi, including the major theme of calculus throughout the paper. There is still much more information concerning pi that has already been discovered, and even further curiosities that remain a mystery today. Pi is simply a number, but complexly, it is so much more!

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