Journal of the Textile Institute

ISSN: 0040-5000 (Print) 1754-2340 (Online) Journal homepage: http://www.tandfonline.com/loi/tjti20

# 11—AN ENERGY METHOD FOR CALCULATIONS IN FABRIC MECHANICS PART I: PRINCIPLES OF THE METHOD

J. W. S. Hearle & W. J. Shanahan

Vol. 69 No. 4 APRIL, 1978

# JOURNAL
*of the*
# TEXTILE INSTITUTE



## 11—AN ENERGY METHOD FOR CALCULATIONS IN FABRIC MECHANICS
## PART I: PRINCIPLES OF THE METHOD*

By J. W. S. HEARLE and W. J. SHANAHAN

An energy method for calculations in fabric mechanics is described. The aim behind the development of this method was to find an approach that could be applied in a uniform manner to a variety of fabric structures and types of deformation.

In the first part of the paper, the principle of the method is described for the simplest case: that in which the fabric structure acts only as a load-bearing mechanism, whose changes in internal energy during deformation are negligible compared with changes in potential energy due to externally applied loads. In the second part, the method is extended to include the effects of strain energy in the component fibres and yarns.

### 1. INTRODUCTION

A review by Grosberg[1] has described approaches used for calculating fabric geometry and properties: one of the conclusions was that a greater uniformity of approach would be desirable. Apart from reducing confusion, there could be a considerable advantage in a uniform approach, since standard computational facilities might then be applied to a wide variety of fabric studies and properties. It is also desirable to use methods that are compatible with those applied to yarns on the one hand and to complex fabric deformations on the other.

Roughly speaking, approaches to fabric structure and properties may be divided into two classes: the descriptive geometrical and the mechanistic[1]. The descriptive geometrical approach involves the prescription of a geometric form for a particular fabric structure, whereas in the mechanistic approach an attempt is made to derive the geometry of the structure from a knowledge of its topological characteristics and the mechanical properties of the structural components (fibres, yarns, etc.). The dividing line between these approaches is not a sharp one, since most treatments that claim to be mechanistic contain some elements of the descriptive, and the choice of geometric models depends implicitly on mechanical ideas. The advantage of the descriptive geometrical approach is that the structural models are mathematically simpler and require simpler calculations. On the other hand, the information that may be obtained is inherently more limited. A mechanistic model is likely to be capable of supplying more information, provided that its idealizations are sufficiently realistic, but at the expense of greater complexity and usage of computer time. Clearly, the simpler geometrical approach is to be preferred whenever it is adequate for the purpose in hand.

The object of this paper is to describe a uniform approach to the mechanical analysis of descriptive geometrical models of fabrics that are characterized by a repetitive unit cell. The approach is based on an energy method very similar to one that has been applied to yarns[2] and non-woven fabrics[3], though in the authors' knowledge there has been little systematic use of such methods in the mechanics of woven and knitted fabrics.

*Paper submitted by a Fellow of the Textile Institute (Professor J. W. S. Hearle).

This paper deals firstly with the principles of the method in situtations where energy changes in the component yarns can be neglected; the analysis then links the equations defining the geometry directly to the forces applied to the fabric, or more strictly to the ratios of forces, since a system so defined is indeterminate in relation to absolute values of forces. Secondly, an extension of the method enables energy changes in and between the yarns and fibres to be brought into the analysis. The computational techniques are described in an Appendix*, and examples of applications will be discussed in a second paper.

## 2. DESCRIPTION OF THE ENERGY METHOD FOR STRUCTURES WITH NEGLIGIBLE INTERNAL ENERGY

### 2.1 Structures Subject to Two External Geometric Variables

The principle of the method is demonstrated first by a simple case, which can then be further generalized. This simple case is that of a rectangle of fabric dimensions $x \times y$, subject to biaxial tension $F_x$ and $F_y$ as shown in Fig. 1. It is assumed that there is a

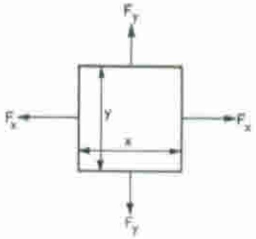


Fig. 1
Fabric rectangle under biaxial tension

geometrical model of the fabric structure, which may be analysed to yield one or more relations equivalent to the form:

$$\mathrm{f}(x, y) = 0. \qquad \ldots\ldots(1)$$

Alternatively, this could be regarded as defining a function

$$y = y(x), \qquad \ldots\ldots(2)$$

$x$ being treated as the independent variable.

Clearly, such a model possesses one degree of freedom. (Note that the range of validity of the functional relations Equations (1) and (2) may be limited by effects such as jamming.) It is tacitly assumed here that the fabric acts purely as a load-bearing mechanism, in which there is no stored elastic energy: an example of such an idealized fabric would be one composed of flexible, inextensible yarns.

The energy $U$ associated with the system will be solely the potential energy of the applied loads, $F_x$ and $F_y$. An arbitrary zero point may be taken at $x = 0$ and $y = 0$ to give:

$$U = -F_x x - F_y y. \qquad \ldots\ldots(3)$$

By the principle of minimum energy, equilibrium will occur when†:

$$\frac{\mathrm{d}U}{\mathrm{d}x} = -F_x - F_y\frac{\mathrm{d}y}{\mathrm{d}x} = 0, \qquad \ldots\ldots(4)$$

*The Appendix is deposited at the British Library, Lending Division, as SUP 12055 under the Supplementary Publications Scheme.

†Note: (i) Since there is only degree of freedom, the same Equation (4) is obtained by putting $\mathrm{d}U/\mathrm{d}y = 0$.
(ii) Equation (4) can also be obtained from the principle of conservation of energy by putting:
external work (for small displacements)
$= \mathrm{d}W = F_x\,\mathrm{d}x + F_y\,\mathrm{d}y$
$=$ change in internal energy
$= \mathrm{d}E = 0$, in this special case.
A special case of this equation has been given by Grosberg[7].

i.e., when:

$$\frac{F_x}{F} = -\frac{dy}{dx}. \quad \ldots\ldots(5)$$

Thus the load–deformation behaviour of the fabric under these conditions is found in terms of the derivative $dy/dx$. The point to be noted is that, no matter what the details of the particular geometry, we always obtain the load–deformation relation in the same manner, i.e., by differentiating the geometrically determined relation between $y$ and $x$. This uniformity of approach to differing structures lends itself to the development of computational facilities that might apply the method in a routine fashion, in contrast to force methods, which involve considering afresh in each new case conditions of internal force balance in a manner that is not likely to be easily mechanized.

## 2.2 Simple Examples

### *2.2.1 Illustration of Principle*

A trivial example will illustrate the principle of the method. Suppose we have a freely pin-jointed lattice, with rigid links of length $l$, as shown in Fig. 2(*a*), subject to forces along the diagonals. The geometry is defined by:

$$x^2+y^2 = 4l^2. \quad \ldots\ldots(6)$$

Differentiation gives:

$$2x \,.\, dx+2y \,.\, dy = 0,$$

i.e.:

$$\frac{dy}{dx} = -\frac{x}{y}. \quad \ldots\ldots(7)$$

Hence:

$$\frac{F_x}{F_y} = \frac{x}{y}. \quad \ldots\ldots(8)$$

This result is, of course, easily obtained from the equilibrium of forces, for example, by the vector diagram of Fig. 2(*b*), since the resultant force must be along the links.

If the rods have a finite width, the lattice will jam when parallel rods come into contact, and this condition defines two limiting states for the range of validity of the solution.

### *2.2.2 Geometry of the Plain Weave*

As another example, consider the Peirce geometry of the plain weave[4]. This geometry leads to the following equations:

$$\left.\begin{aligned} x &= (l_1 - D\theta_1)\cos\theta_1 + D\sin\theta_1, \\ y &= (l_2 - D\theta_2)\cos\theta_2 + D\sin\theta_2, \\ h_1 &= (l_1 - D\theta_1)\sin\theta_1 + D(1-\cos\theta_1), \\ h_2 &= (l_2 - D\theta_2)\sin\theta_2 + D(1-\cos\theta_2), \\ h_1 + h_2 &= D, \end{aligned}\right\} \quad \ldots\ldots(9)$$

where $y$ and $x$ are dimensions of a single weave repeat. The meaning of the remaining parameters is indicated in Fig. 3. The equations do define a function $y$ of $x$, though the definition is implicit. Since there are five equations in six unknowns ($x$, $y$, $h_1$, $h_2$, $\theta_1$, $\theta_2$), $l_1$, $l_2$, and $D$ being assumed to be given constants, it should be sufficient to assign a value of $x$ in order to determine all the remaining unknowns, including $y$.

It can be shown that the derivative of $y$ with respect to $x$ is:

$$\frac{dy}{dx} = -\frac{\tan\theta_2}{\tan\theta_1}, \quad \ldots\ldots(10)$$

so that, from Equation (5):

$$\frac{F_x}{F_y} = \frac{\tan\theta_2}{\tan\theta_1}. \qquad \ldots\ldots(11)$$

This is the same result as that which could be obtained by balancing the interlacing forces that arise from thread tensions.

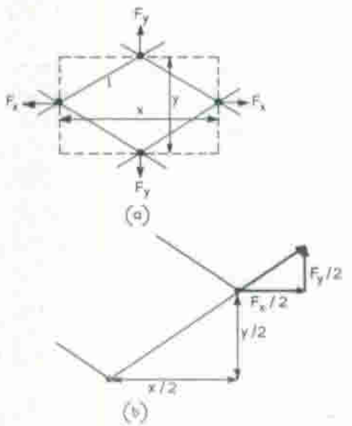


Fig. 2
(*a*) Lattice structure under biaxial load
(*b*) Vector diagram of forces

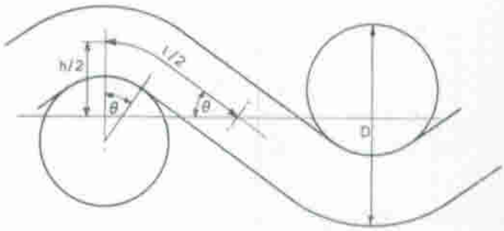


Fig. 3
Peirce geometry of plain weave

Equation (11) is the additional equation needed to solve the geometry of Equations (9) under any applied forces (or to determine the ratio of forces from a given geometry). If $F_x/F_y$, and thus $\tan\theta_2/\tan\theta_1$, are given, the other geometrical quantities, notably $x$ and $y$, can be found by any of the graphical or numerical methods that have been used on the Peirce geometry. At the present time, the best method would be the general interactive computational method, which was the forerunner of QAS (see Appendix) and was described by Hearle, Konopasek, and Newton[5] as applied to the racetrack geometry of Kemp[6]. The Peirce geometry is a special case of this with zero flattening.

### 2.3 Structures Subject to Three External Geometric Variables

The format of the simple example given above is limited to problems with two degrees of freedom, such as biaxial loading with changes of length of two orthogonal axes. However, the method is easily extended to include more modes of deformation. For example, suppose in-plane shear is allowed, with the introduction of the moment $M$ and an angular displacement $\phi$. We then have:

the geometric relation:

$$f(x, y, \phi) = 0; \qquad \ldots\ldots(12)$$

the energy relation:

$$U = -F_x x - F_y y - M\phi; \qquad \ldots\ldots(13)$$

from the minimum-energy criterion:

$$\left(\frac{\partial U}{\partial x}\right)_\phi = -F_x - F_y\left(\frac{\partial y}{\partial x}\right)_\phi = 0,$$

and

$$\left(\frac{\partial U}{\partial \phi}\right)_x = -F_y\left(\frac{\partial y}{\partial \phi}\right)_x - M = 0, \qquad \ldots\ldots(14)$$

leading to:

$$\frac{F_x}{F_y} = -\left(\frac{\partial y}{\partial x}\right)_\phi, \qquad \text{......(15)}$$

$$\frac{M}{F_y} = -\left(\frac{\partial y}{\partial \phi}\right)_x, \qquad \text{......(16)}$$

and

$$\frac{M}{F_x} = \left(\frac{\partial y}{\partial \phi}\right)_x \left(\frac{\partial x}{\partial y}\right)_\phi = -\left(\frac{\partial x}{\partial \phi}\right)_y. \qquad \text{......(17)}$$

These are the relations that apply to any in-plane fabric deformation.

### 2.4 Generalized Forces and Displacements

The bending of a fabric out of plane will bring in three other modes of uniform deformation of a sheet material. More complicated problems may involve other forms of loading, such as transverse compression or loads within a unit cell. The relations should therefore be formulated in a general form applicable to any number of degrees of freedom of the system. Associated with each degree of freedom, characterized by the subscript $i$, there will be a generalized force $F_i$ and a generalized displacement $x_i$ (either combinations of force with distance or of moment with angle, twist, or curvature).

The relations then become:

(*a*) derived from the geometry, and introducing the parameters needed to characterize the geometry, a set of equations that is equivalent to the form:

$$\mathrm{f}(x_1 \ldots x_i \ldots) = 0; \qquad \text{......(18)}$$

(*b*) the energy equation:

$$U = \Sigma(-F_i x_i). \qquad \text{......(19)}$$

Because of the indeterminacy of the absolute values of the forces (which does not arise later with a deformable material), we must write the minimum-energy criterion $(\partial U/\partial x_i) = 0$ with one parameter $x_{i1}$ as the variable for differentiation and a second parameter $x_{i2}$ selected as a dependent variable. This gives:

$$\left(\frac{\partial U}{\partial x_{i1}}\right)_{\text{all except } x_{i1}, x_{i2}} = 0 \qquad \text{......(20)}$$

and

$$-F_{i1} - F_{i2}\left(\frac{\partial x_{i2}}{\partial x_{i1}}\right)_{\text{all except } x_{i1}, x_{i2}} = 0,$$

so that:

$$\frac{F_{i1}}{F_{i2}} = -\left(\frac{\partial x_{i2}}{\partial x_{i1}}\right)_{\text{all except } x_{i1}, x_{i2}}. \qquad \text{......(21)}$$

This form could alternatively have been inferred from the three parameter equations, Equations (15)–(17).

## 3. INCLUSION OF ENERGY OF STRUCTURAL COMPONENTS

### 3.1 Description of the Method for a System with Two External and Two Internal Independent Variables

It is now necessary to extend the method to the much more important problems where the fibres and yarns deform. Once again, it is convenient to start with simple examples, to illustrate the basic principles, and then give a more general formulation. It is also convenient to start with the assumption that the yarns deform elastically with no energy loss and then to consider the question of inelastic effects in a later section of the paper.

We can generalize the simple case with two degrees of freedom described in Section 2.1 to include yarn extension. We consider the modular lengths $l_1$ and $l_2$ of two sets of yarn (e.g., the warp and weft in a woven fabric) in the unit cell as subject to change under load from initial values $L_1$ and $L_2$. The yarn properties will be given by their measured load–elongation curves:

$$f_1 = f_1(l_1); \qquad \ldots\ldots(22a)$$

$$f_2 = f_2(l_2), \qquad \ldots\ldots(22b)$$

where $f_1$ and $f_2$ are the loads.

However, we are initially interested in the stored elastic energies in the yarns, $U_1$ and $U_2$, presumed for the time being to be single-valued functions of elongation. These are given by the areas under the load–elongation curves:

$$U_1 = \int_{L_1}^{l_1} f_1 \, \mathrm{d}l_1 = U_1(l_1); \qquad \ldots\ldots(23a)$$

$$U_2 = \int_{L_2}^{l_2} f_2 \, \mathrm{d}l_2 = U_2(l_2). \qquad \ldots\ldots(23b)$$

The geometric equations of the nodes are now expressed by the symbolic form:

$$\mathrm{f}(x, y, l_1, l_2) = 0, \qquad \ldots\ldots(24)$$

where $x$ and $y$ are the external dimensions associated with the two degrees of freedom.

The total-energy equation now becomes:

$$U = -F_x x - F_y y + U_1(l_1) + U_2(l_2), \qquad \ldots\ldots(25)$$

and the minimum-energy criteria become:

$$\left(\frac{\partial U}{\partial x}\right)_{l_1, l_2} = -F_x - F_y\left(\frac{\partial y}{\partial x}\right)_{l_1, l_2} = 0, \qquad \ldots\ldots(26)$$

$$\left(\frac{\partial U}{\partial l_1}\right)_{x, l_2} = -F_y\left(\frac{\partial y}{\partial l_1}\right)_{x, l_2} + \frac{\mathrm{d}U_1}{\mathrm{d}l_1} = 0, \qquad \ldots\ldots(27)$$

and

$$\left(\frac{\partial U}{\partial l_2}\right)_{x, l_1} = -F_y\left(\frac{\partial y}{\partial l_2}\right)_{x, l_1} + \frac{\mathrm{d}U_2}{\mathrm{d}l_2} = 0. \qquad \ldots\ldots(28)$$

Hence:

$$\frac{F_x}{F_y} = \left(\frac{\partial y}{\partial x}\right)_{l_1, l_2}, \qquad \ldots\ldots(29)$$

$$F_y\left(\frac{\partial y}{\partial l_1}\right)_{x, l_2} = \frac{\mathrm{d}U_1}{\mathrm{d}l_1}, \qquad \ldots\ldots(30)$$

and

$$F_y\left(\frac{\partial y}{\partial l_2}\right)_{x, l_1} = \frac{\mathrm{d}U_2}{\mathrm{d}l_2}. \qquad \ldots\ldots(31)$$

These equations*, together with the geometrical relation, Equation (3), enable the four unknowns $x$, $y$, $l_1$, and $l_2$ to be found if the forces $F_x$ and $F_y$ are given. In contrast to the previous situation, the forces are absolutely determinate.

*It should be noted that one of the external dimensional variables $x$ or $y$ must be taken as dependent in writing the minimum-energy criteria; it is determined by the geometrical relation. The present authors have arbitrarily taken $x$ as independent and $y$ as dependent, but an equivalent set of equations would be obtained by reversing these.

The partial differentials $(\partial y/\partial x)_{l_1, l_2}$, etc., are found from the geometrical relations. The terms $\mathrm{d}U_1/\mathrm{d}l_1$ and $\mathrm{d}U_2/\mathrm{d}l_2$ take us back to the load–elongation relations:

$$\frac{\mathrm{d}U_1}{\mathrm{d}l_1} = f_1(l_1); \qquad \ldots\ldots(32a)$$

$$\frac{\mathrm{d}U_2}{\mathrm{d}l_2} = f_2(l_2). \qquad \ldots\ldots(32b)$$

We note in passing that, in this instance, this also means that the forces in the yarns in the structure have been determined, although this will not always be so in more complicated cases.

### 3.2 A Trivial Example

Suppose that the lattice model shown in Fig. 2 is made up of identical rods obeying Hooke's Law. In this instance, there is a further simplification in that only one type of component and one length $l$ are involved. The equations become:

$$\text{geometric: } x^2+y^2 = 4l^2 \qquad \ldots\ldots(33)$$

and

$$\text{component: } \frac{\mathrm{d}U}{\mathrm{d}l} = 4f = 4k(l-l_0), \qquad \ldots\ldots(34)$$

where $k$ is the spring constant for a single rod and $l_0$ the unstrained length of the rod.

Differentiation gives:

$$2x\,.\,\mathrm{d}x+2y\,.\,\mathrm{d}y = 8l\,.\,\mathrm{d}l, \qquad \ldots\ldots(35)$$

$$\left(\frac{\partial y}{\partial x}\right)_l = -\frac{x}{y}, \qquad \ldots\ldots(36a)$$

$$\left(\frac{\partial y}{\partial l}\right)_x = \frac{4l}{y}. \qquad \ldots\ldots(36b)$$

Equation (29) thus gives as before:

$$\frac{F_x}{F_y} = \frac{x}{y}. \qquad \ldots\ldots(37)$$

Equation (30) gives:

$$-F_y\frac{4l}{y}+4k(l-l_0) = 0, \qquad \ldots\ldots(38)$$

from which:

$$l = \frac{kyl_0}{ky-F_y}, \qquad \ldots\ldots(39)$$

and hence, from Equation (33):

$$x^2+y^2 = 4\left(\frac{kyl_0}{ky-F_y}\right)^2. \qquad \ldots\ldots(40)$$

Equations (37) and (40) together determine the two unknowns from the set $x$, $y$, $F_x$, $F_y$. For example, they will reduce to the quadratic equation:

$$y^2-2\frac{F_y}{k}y+\frac{F_y^2}{k^2}-\frac{4l_0^2}{(1+F_x^2/F_y^2)} = 0. \qquad \ldots\ldots(41)$$

### 3.3 The Peirce Fabric Model

If we consider the simple fabric model described in Section 2.2.2, but with the yarns allowed to extend and following general load–elongation relations of the form given by

Equations (22*a*) and (22*b*), then the Equations (29)–(37) yield (since $\partial y/\partial l_1 = \tan\theta_2/\sin\theta_1$ and $\partial y/\partial l_2 = 1/\cos\theta_2$):

$$\frac{F_x}{F_y} = \frac{\tan\theta_2}{\tan\theta_1}, \qquad \ldots\ldots(42)$$

$$F_y \tan\theta_2/\sin\theta_1 = f_1(l_1), \qquad \ldots\ldots(43)$$

and

$$F_y/\cos\theta_2 = f_2(l_2). \qquad \ldots\ldots(44)$$

Together with the set of geometric relations, this gives a full solution of the problem. As in the first example, the result is the same as could be obtained in a somewhat simpler manner by considering internal forces. We may note that:

$$F_y \tan\theta_2/\sin\theta_1 = F_x/\cos\theta_1 = \text{tension in the weft yarn.}$$

### 3.4 Energy Terms Associated with Dependent Geometrical Parameters

If we consider the previous example further, we can see that it is possible to include energy terms associated with the resistance to bending of the yarns. For yarns that are assumed to be linearly elastic in bending, the bending strain energy can be defined[8] as:

$$U_B = \int \tfrac{1}{2} bK^2 \, ds, \qquad \ldots\ldots(45)$$

where $K$ = curvature, $b$ = bending rigidity, and $s$ = length measured along the yarn.

Applying the definition to the Peirce model gives the bending energy of one of the yarns as:

$$U_B = \frac{2b\theta}{D}. \qquad \ldots\ldots(46)$$

Still regarding $D$ as a constant, we can see that $U_B$ is a function of the parameter $\theta$, the weave angle. Such a parameter does not appear explicitly in the geometric relation, Equation (24).

In the Peirce geometry, the geometric relation, Equation (24) is, in fact, defined by several simultaneous equations (Equations (9)), and the weave angles are parameters eliminated in resolving these equations. Thus the weave angles can be regarded as secondary dependent geometric parameters that are functions of the primary independent geometric parameters $(x, l_1, l_2)$. In principle, $\theta$ could be substituted out of Equation (46) in order to give an energy function dependent only upon the independent geometrical parameters, but, in practice, because $\theta$ is defined explicitly, this would not be possible.

It is desirable, then, to generalize the formulation of Equations (29), (30), and (31) to allow for the existence of a set of dependent geometric parameters, say $\theta_1$, $\theta_2$, such that the energy could be given as:

$$U = -F_x x - F_y y + U_1(l_1) + U_2(l_2) + U_3(\theta_1, \theta_2).$$

The minimum-energy conditions then become:

$$\frac{F_x}{F_y} = \left(\frac{\partial y}{\partial x}\right)_{l_1, l_2} + \left(\frac{\partial U_3}{\partial \theta_1}\right)_{\theta_2} \cdot \left(\frac{\partial \theta_1}{\partial x}\right)_{l_1, l_2} + \left(\frac{\partial U_3}{\partial \theta_2}\right)_{\theta_1} \cdot \left(\frac{\partial \theta_2}{\partial x}\right)_{l_1, l_2}, \qquad \ldots\ldots(47)$$

$$F_y\left(\frac{\partial y}{\partial l_1}\right)_{x, l_2} = \frac{\partial U_1}{\partial l_1} + \left(\frac{\partial U_3}{\partial \theta_1}\right)_{\theta_2} \cdot \left(\frac{\partial \theta_1}{\partial l_1}\right)_{x, l_2} + \left(\frac{\partial U_3}{\partial \theta_2}\right)_{\theta_1} \cdot \left(\frac{\partial \theta_2}{\partial l_1}\right)_{x, l_2}, \qquad \ldots\ldots(48)$$

and

$$F_y\left(\frac{\partial y}{\partial l_2}\right)_{x, l_1} = \frac{\partial U_2}{\partial l_2} + \left(\frac{\partial U_3}{\partial \theta_1}\right)_{\theta_2} \cdot \left(\frac{\partial \theta_1}{\partial l_2}\right)_{x, l_1} + \left(\frac{\partial U_3}{\partial \theta_2}\right)_{\theta_1} \cdot \left(\frac{\partial \theta_2}{\partial l_2}\right)_{x, l_1}. \qquad \ldots\ldots(49)$$

In order to apply this to the Pierce model, the appropriate partial differentiation of the geometric and the energy equations must be undertaken.

## 4. GENERAL TREATMENT

### 4.1 Mathematical Formulation

For a general treatment, we must allow for any number of modes of external deformation with associated forces or moments and for any number of modes of deformation of the material components, with associated energy changes. The latter would certainly include, in addition to yarn elongation, yarn-bending and yarn lateral compression.

It is assumed that the geometry gives one or more relations equivalent to:

$$f(x_1, x_2, \dots x_l, \rho_1, \rho_2, \dots \rho_n) = 0, \qquad \dots\dots(50)$$

where $x_1, x_2, \dots x_l$ are the generalized dimensions or displacements or both associated with external deformation, and $\rho_1, \rho_2, \dots \rho_n$ are a set of independent geometrical parameters (such as the yarn modular lengths in previous examples).

The total energy, $U$, is assumed to be given by:

$$U = \sum_{i=1}^{l} (-F_i x_i) + E(\rho_1, \rho_2, \dots \rho_n, \alpha_1, \alpha_2, \dots \alpha_m), \qquad \dots\dots(51)$$

where, as in Section 2.4, $F_i$ are the generalized external forces associated with the generalized displacements $x_i$, $(\alpha_1, \alpha_2, \dots \alpha_m)$ are the dependent geometric parameters (such as weave angles) discussed above, and $E$ is the strain energy.

The minimum-energy conditions for the general case, with $x_1$ arbitrarily chosen as the dependent mode of deformation (any of $x_1, \dots x_l$ could be chosen to give $l$ different but equivalent formulations) are:

$$\left.\begin{aligned} F_1\frac{\partial x_1}{\partial x_k} + F_k &= \sum_{j=1}^{m} \frac{\partial E}{\partial \alpha_j}\cdot\frac{\partial \alpha_j}{\partial x_k}, \qquad k = 2, 3, \dots l \\ F_1\frac{\partial x_1}{\partial \rho_i} &= \frac{\partial E}{\partial \rho_i} + \sum_{j=1}^{m} \frac{\partial E}{\partial \alpha_j}\cdot\frac{\partial \alpha_j}{\partial \rho_i}. \qquad i = 1, 2, \dots n \end{aligned}\right\} \qquad \dots\dots(52)$$

This gives a total of $(l+n-1)$ equations. If these are taken in conjunction with the geometrical relation, Equation (30), there is a system of $l+n$ equations describing the behaviour of the structure under the specified types of load. If $\alpha_1, \alpha_2 \dots \alpha_m$, which may be eliminated, are ignored, there are $2l+n$ unknowns $(F_1, F_2, \dots F_l, x_1, x_2, \dots x_l, \rho_1, \rho_2, \dots \rho_n)$. Thus, in general, the system will be completely defined if $l$ of these quantities are given, i.e., a total of half of the combined set of forces and displacements would be sufficient, for example, in the situation where all the forces or all the displacements were given: this conforms with the physical expectation.

### 4.2 Other Physical Considerations

A limitation of the method, as so far presented, is that it is restricted to materials in which there is a well-defined strain energy, which is a single-valued function of strain. This leads to single-valued solutions. However, the method need not be so restricted. It can be applied to materials that show hysteresis, provided that the strain-energy terms, which appear as differential coefficients with respect to strain, are related to the previous history of deformation. With care, this may be accommodated within the computing routine. Similarly, time-dependent effects or thermal changes could also be included.

Frictional effects are more difficult for two reasons. Firstly, there is a basic indeterminacy in that, because frictional forces can act in any direction, there will be ranges of possible minimum-energy solutions, in the sense that a change in either direction leads to an increase of energy. Secondly, there is the problem that, in order to determine frictional forces within a structure, normal loads must be known. However, it may be possible to find ways of bringing frictional effects into the method in at least some structural models.

## 5. THE PROBLEM AND ITS SOLUTION

### 5.1 The Problem

By looking at the simple example of Peirce geometry described in Section 2.2.2, it is possible to obtain some idea of the difficulties likely to be encountered in using the methods described. Firstly, it can be seen that the geometrical model involved, simple though it is, does not yield an explicit relation of the form $y = f(x)$. The relation is, in fact, defined implicitly through a set of five non-linear simultaneous equations. Generally, this is the sort of situation that can be expected. The energy method yields one extra equation, so that solving load–deformation problems with this model requires the solution of six non-linear simultaneous equations. With less simple models, the situation becomes worse, with the equations, both those of the geometry itself and those derived from it by the energy method, becoming more numerous and more complex.

A second difficulty arises from the first. Because $y = f(x)$ is defined implicitly, the derivative $dy/dx$ requires some manipulative labour to obtain. Each of the five equations of the geometry must be differentiated with respect to $x$, to produce a set of five linear equations in the five derivatives

$$\frac{dy}{dx}, \quad \frac{d\theta_1}{dx}, \quad \frac{d\theta_2}{dx}, \quad \frac{dh_1}{dx}, \quad \frac{dh_2}{dx},$$

which may then be solved to yield the result:

$$\frac{dy}{dx} = -\frac{\tan \theta_2}{\tan \theta_1}$$

as used in Section 2.2.2. In a more general case, where $y = f(x, p_1, p_2, \ldots p_n)$, $(n+1)$ such differentiations (partial) must be carried out. With models of any complexity, the amount of algebraic manipulation becomes quite large and is not likely to result in such convenient solutions as those found in this example. The tedium of these manipulations not only reduces the attraction of the method but is also likely to be a source of errors.

Thus there are two main areas of difficulty:

(i) the numerical problem of solving non-linear simultaneous equations; and
(ii) the algebraic-manipulative problem involved in obtaining expressions for derivatives.

### 5.2 The Solution

The interactive system called QAS (Question Answering Systems on Algebraic Equations), developed at the University of Manchester Institute of Science and Technology by Konopasek[9], has just the features required to help solve sets of non-linear simultaneous equations: in particular, it allows the user to perform conveniently the kinds of manipulation necessary for experimenting with his sets of equations.

It has now been possible to develop QAS to mechanize the process of symbolic differentiation. A description of this method and the computational procedures are described in the Appendix*. In essence, the user writes down three types of determining equation: (*a*) the geometrical model, (*b*) the material properties, and (*c*) the differential equations of minimum energy as derived in this paper. Provided that these are in forms allowed under QAS rules (and this can usually be achieved without difficulty), then the computation proceeds automatically, includes the differentiation, and, with appropriate numerical input, leads to the required numerical output.

## 6. CONCLUSION

This paper introduces a general energy method of determining the state of a fabric, whose structure can be characterized by a set of geometric equations, under imposed

*SUP 12055.

external forces. The method is particularly applicable to woven, knitted, and other fabrics with a repetitive unit cell, although it is possible that it could be applied to irregular non-woven fabrics, where the complete cell would be the whole specimen, if these could be characterized by equivalent unit cells.

The method is first introduced in an application to idealized fabrics in which there are no energy changes associated with deformation of the fibres or yarns. This more limited approach has been used, partly to clarify the arguments, and partly because there are problems in which the simplifying assumption is acceptable and useful. The method would then predict, for example, the state of a fabric under forces that are large enough to overcome bending resistances (so that the yarns follow the shortest constrained paths, with straight free lengths and close contact with constraints) but small enough to make yarn extension negligible. In these circumstances, the method links any geometric models of fabrics that have been proposed to the ratios of imposed forces and moments.

The method is then extended to the much more important problems where the fibres and yarns deform. Finally, powerful computational methods for handling the problems are introduced and described in more detail in an Appendix*. The combination of these features make it worth while to devote more effort to the formulation of better geometric models and to models of more complicated fabrics. Attention should be paid to the means of introducing such models by direct computational methods.

## ACKNOWLEDGEMENT

The work presented in this paper was carried out while one of the authors (W.J.S.) was the holder of an I.C.I. Fellowship. The authors would also like to thank Dr. M. Konopasek for many helpful discussions on computational methods.

## REFERENCES

1 P. Grosberg in 'Contributions of Science to the Development of the Textile Industry' (edited by M. Cordelier and P. W. Harrison), l'Institut Textile de France and the Textile Institute, Boulogne-sur-Seine and Manchester, 1975, p. 179.
2 J. W. S. Hearle and M. Konopasek. *Appl. Polymer Symp.*, 1975, No. 27, 253.
3 J. W. S. Hearle and A. Newton. *Text. Res. J.*, 1967, **37**, 778.
4 F. T. Peirce. *J. Text. Inst.*, 1937, **28**, T45.
5 J. W. S. Hearle, M. Konopasek, and A. Newton in 'New Ways to Produce Textiles' (edited by P. W. Harrison), the Textile Institute, Manchester, 1972, p. 133.
6 A. Kemp. *J. Text. Inst.*, 1958, **49**, T44.
7 P. Grosberg in 'Structural Mechanics of Fibers, Yarns and Fabrics', Vol. 1, by J. W. S. Hearle, P. Grosberg, and S. Backer, Wiley, New York, 1969, p. 342.
8 A. E. H. Love. 'The Mathematical Theory of Elasticity', Cambridge University Press, 1920.
9 M. Konopasek. 'Question Answering System on Sets of Algebraic Equations' (Description and Provisional User's Manual), University of Manchester Institute of Science and Technology, Manchester, 1974; *Proc. European Computing Congress, London*, 1975.

Department of Textile Technology,
University of Manchester Institute of Science and Technology,
Manchester. 1.

*Received* 6.1.1977
*Accepted for publication* 14.4.1977

*SUP 12055.