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MATH 3400
SI Session 11 Final Exam Review
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Problem 1

Suppose a 100 L, well-stirred tank contains 10 L of pure water. At some point, a feed of 2 L/min is started which contains 1 kg/L of salt. The tank begins to drain at a rate of 1 L/min. How much salt is in the tank when the tank is full?

Try it out before looking at the solution.

Solution to Problem 1:
Let's make a list of our knowns:

$$\left\{\begin{array}{ll} V_o = 10 & Q_o = 0 \\ V_{\max} = 100 & V(t) = t + 10 \\ r_{\text{in}} = 2 & r_{\text{out}} = 1 \\ c_{\text{in}} = 1 & c_{\text{out}} = \frac{Q(t)}{t+10} \end{array}\right\}$$

Using the following definition:

$$\frac{dQ}{dt} \triangleq r_{\text{in}}c_{\text{in}} - r_{\text{out}}c_{\text{out}},$$

we have:

$$\frac{dQ}{dt} = (2)(1) - (1)\left(\frac{Q(t)}{t+10}\right).$$

Rearranging:

$$Q' + \underbrace{\frac{1}{t+10}}_{p(t)} Q = \underbrace{2}_{f(t)}.$$

This is a F.O.L. differential equation, so we use the following key to its solution:

$$\mu Q = \int \mu f \, dt.$$

We first find μ, using the following definition:

$$\mu \triangleq e^{\int p dt}.$$

With our value for $p(t)$:

$$\mu = e^{\int \frac{1}{t+10} dt} \quad \Longrightarrow \quad \mu = e^{\ln|t+10|} \quad \Longrightarrow \quad \mu = (t+10).$$

With μ and f, we can find the RHS of the key to F.O.L. by evaluating the integral:

$$\int \mu f \, dt = \int (t+10)(2) dt \quad \Longrightarrow \quad \text{RHS} = 2\left[\frac{t^2}{2} + 10t + C\right] \quad \Longrightarrow \quad \text{RHS} = t^2 + 20t + C_2.$$

Now, let's put things together:

$$\mu Q = \text{RHS} \quad \Longrightarrow \quad \mu Q = t^2 + 20t + C_2 \quad \Longrightarrow \quad Q_g = \frac{t^2 + 20t + C_2}{t+10}.$$

Since we have the initial condition that $Q(0) = 0$, we can find the specific solution for the amount of salt as a function of time:

$$Q(0) = 0 \quad \therefore \quad 0 = \frac{0^2 + 20(0) + C_2}{10} \quad \Longrightarrow \quad C_2 = 0 - 0 - 0 \quad \Longrightarrow \quad C_2 = 0.$$

Our specific solution is:

$$\boxed{Q_s(t) = \frac{t^2 + 20t}{t+10}}$$

We aren't done quite yet. Let's find the amount of time it takes for the tank to fill with our solution:

$$V(t^\star) = V_{\max} \quad \therefore \quad V_{\max} = t^\star + 10 \quad \Longrightarrow \quad 100 = t^\star + 10 \quad \Longrightarrow \quad t^\star = 90 \text{ min.}$$

To find the amount of salt at this time, let's plug this value into our specific solution:

$$Q_s(90) = \frac{(90)^2 + 20(90)}{90 + 10} \quad \Longrightarrow \quad \boxed{Q_s(90) = \frac{90^2 + 1800}{100} \text{ kg}}$$

Problem 2:

Solve the following IVP:

$$yy' = 2y^2 + x \quad ; \quad y(0) = -1.$$

Try it out before looking at the solution.

Solution to Problem 2:
With this problem, we start off by identifying the differential equation type. Since this is first order, let's see if it is first order linear:

$$y' = 2y + \frac{x}{y} \quad \Longrightarrow \quad y' - 2y = xy^{-1}.$$

It is almost F.O.L. save for the power of y being multiplied to our function of x on the RHS. This appears to be a Bernoulli's equation where $n = -1$.

The substitution factor in this case is $v = y^{1-n}$, so for our problem, we have:

$$v = y^{1-(-1)} \quad \Longrightarrow \quad \boxed{v = y^2}$$

Taking the implicit derivative of v:

$$v' = 2yy' \quad \Longrightarrow \quad \boxed{y' = \frac{v'}{2y}}$$

Let's make this substitution:

$$\frac{v'}{2y} - 2y = \frac{x}{y}.$$

Multiplying everything by $2y$ to make our leading derivative monic, we have:

$$v' - 4y^2 = 2x.$$

Noticing that $y^2 = v$ and making the substitution, we have:

$$v' \underbrace{-4}_{p(x)} v = \underbrace{2x}_{f(x)}.$$

This is a F.O.L. differential equation. It's solution is found with:

$$\mu v = \int \mu f dx.$$

Finding μ:

$$\mu \triangleq e^{\int p dx} \quad \Longrightarrow \quad \mu = e^{-4\int dx} \quad \Longrightarrow \quad \mu = e^{-4x}.$$

Evaluating the RHS of the solution key:

$$\text{RHS} = \int e^{-4x}(2x)dx \quad \Longrightarrow \quad \text{RHS} = 2\int xe^{-4x}dx \quad \begin{Bmatrix} u = x & dv = e^{-4x}dx \\ du = dx & v = -\frac{1}{4}e^{-4x} \end{Bmatrix} \quad \Longrightarrow \quad \dots$$

$$\dots \Longrightarrow \quad \text{RHS} = 2\left[-\frac{xe^{-4x}}{4} + \frac{1}{4}\int e^{-4x}dx\right] \quad \Longrightarrow \quad \text{RHS} = 2\left[-\frac{xe^{-4}}{4} + \frac{1}{4}\left(-\frac{1}{4}e^{-4x} + C\right)\right].$$

Distributing the factors of 2 and $\frac{1}{4}$:

$$\mu v = -\frac{xe^{-4x}}{2} - \frac{e^{-4x}}{8} + C_2 \quad \xRightarrow{\text{divide by } \mu} \quad v = -\frac{x}{2} - \frac{1}{8} + C_2e^{4x}.$$

Replacing v with y^2:

$$y^2 = -\frac{x}{2} - \frac{1}{8} + C_2e^{4x} \quad \Longrightarrow \quad \boxed{y_g = \sqrt{-\frac{x}{2} - \frac{1}{8} + C_2e^{4x}}}$$

Let's use our initial condition to find the specific solution:

$$y(0) = -1 \quad \therefore \quad -1 = \sqrt{-\frac{(0)}{2} - \frac{1}{8} + C_2 e^{4(0)}} \quad \implies \quad 1 = -\frac{1}{8} + C_2 \quad \implies \quad \boxed{C_2 = \frac{9}{8}}$$

Our specific solution will have a square root, leading us to think that there are positive and negative values of y for any given value of x. This is not the case, our initial condition tells us that one of our values of y is negative, meaning our solution is concerned with the negative half of the square root:

$$\boxed{y_s = -\left|\sqrt{-\frac{x}{2} - \frac{1}{8} + \frac{9}{8}e^{4x}}\right|}$$

Problem 3

Solve the following IVP using the method of undetermined coefficients:

$$y'' - 2y' + y = \sin(x) \quad ; \quad y(0) = 1 \quad ; \quad y'(0) = 1.$$

Try it out before looking at the solution.

Solution to Problem 3:
With this solution technique, remember that our final solution will be the superposition of our complementary and particular solutions:

$$y_g = y_c + y_p.$$

We start off by finding the solution for the homogenous case (the complementary solution):

$$y'' - 2y' + y = 0.$$

The discriminant is:

$$D \triangleq b^2 - 4ac \quad \Longrightarrow \quad D = (-2)^2 - 4(1)(1) \quad \Longrightarrow \quad D = 4 - 4 \quad \Longrightarrow \quad D = 0.$$

This tells us that our complementary solution will have the form:

$$y_c = C_1 e^{\lambda x} + C_2 x e^{\lambda x}.$$

Let's find our root:

$$\lambda \triangleq \frac{-b \pm \sqrt{D}}{2a} \quad \Longrightarrow \quad \lambda = \frac{2 \pm 0}{2} \quad \Longrightarrow \quad \lambda = 1.$$

Our complementary solution is then:

$$\boxed{y_c = C_1 e^x + C_2 x e^x}$$

To find our particular solution, we examine the form of the inhomogeneity, $\sin(x)$. We make the ansatz that our particular solution will be of the form:

$$y_{p_{\text{guess}}} = A \sin(x) + B \cos(x).$$

Taking the derivatives of our guess:

$$y'_{p_{\text{guess}}} = A \cos(x) - B \sin(x).$$

$$y''_{p_{\text{guess}}} = -A \sin(x) - B \cos(x).$$

We plug these into our differential equation and determine the values of A and B that force a true statement:

$$\big[-A \sin(x) - B \cos(x)\big] - 2\big[A \cos(x) - B \sin(x)\big] + \big[A \sin(x) + B \cos(x)\big] = \sin(x).$$

Let's group terms by their function, making sure to distribute the 2:

$$\big[-A + 2B + A\big] \sin(x) + \big[-B - 2A + B\big] \cos(x) = [1] \sin(x) + [0] \cos(x).$$

The coefficients of the $\sin(x)$ on the LHS need to be equal to 1. The coefficients of the $\cos(x)$ on the LHS need to be equal to 0.

$$-A + 2B + A = 1 \quad \Longrightarrow \quad 2B = 1 \quad \Longrightarrow \quad \boxed{B = \frac{1}{2}}$$

$$-B - 2A + B = 0 \quad \Longrightarrow \quad -2A = 0 \quad \Longrightarrow \quad \boxed{A = 0}$$

We have determined the coefficients, let's plug them into $y_{p_{\text{guess}}}$ to get our particular solution:

$$\boxed{y_p = \frac{1}{2}\cos(x)}$$

Our general solution is then:

$$\boxed{y_g = C_1 e^x + C_2 x e^x + \frac{1}{2}\cos(x)}$$

Using our first initial condition, $y(0) = 1$:

$$1 = C_1 + 0 + \frac{1}{2} \quad \Longrightarrow \quad C_1 = \frac{1}{2}.$$

Taking the derivative of our general solution so that we can use the second initial condition:

$$y_g' = C_1 e^x + C_2 e^x + C_2 x e^x - \frac{1}{2}\sin(x).$$

Now, using the second initial condition, y'(0)=1:

$$1 = C_1 + C_2 + 0 - 0 \quad \Longrightarrow \quad C_1 + C_2 = 1 \quad \Longrightarrow \quad \frac{1}{2} + C_2 = 1 \quad \Longrightarrow \quad C_2 = \frac{1}{2}.$$

Our final, specific solution is:

$$\underline{\underline{\boxed{y_s(x) = \frac{1}{2}\left[e^x + xe^x + \cos(x)\right]}}}$$

Problem 4

Find the general solution using variation of parameters:

$$y'' - 2y' + y = 2e^{3t}.$$

Try it out before looking at the solution.

Solution to Problem 4:
Using the variation of parameters technique will give us a general solution that has this form:

$$y_g = \underbrace{y_c}_{C_1y_1+C_2y_2} + \underbrace{y_p}_{\mu_1y_1+\mu_2y_2}$$

Here, the complementary solution is found by solving the homogenous case of this differential equation. The particular solution is expressed above as: $\mu_1y_1+\mu_2y_2$. Naturally, we want to find out what μ_1 and μ_2 are (y_1 and y_2 come from the complementary solution).

$$\mu_1 \triangleq -\int \frac{y_2f}{W[y_1,y_2]}dt \quad \text{and} \quad \mu_2 \triangleq \int \frac{y_1f}{W[y_1,y_2]}dt.$$

Let's start off by finding the complementary solution.

Oop, it's just the same as the previous problem, so let's use that:

$$y_c = C_1\underbrace{e^t}_{y_1}+C_2\underbrace{te^t}_{y_2}.$$

With y_1 and y_2, we can take the Wronskian:

$$W[y_1,y_2] \triangleq \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

For our values of y_1 and y_2:

$$W[e^t,te^t] = \begin{vmatrix} e^t & te^t \\ e^t & (e^t+te^t) \end{vmatrix} \implies W = e^t(e^t+te^t)-e^t(te^t) \implies W = e^{2t}+te^{2t}-te^{2t} \quad ...$$

$$\boxed{W = e^{2t}}$$

With the Wronskian, we can begin finding μ_1 and μ_2. Let's start with μ_1.

$$\mu_1 = -\int \frac{te^t2e^{3t}}{e^{2t}}dt \implies \mu_1 = -2\int \frac{te^{4t}}{e^{2t}}dt \implies \mu_1 = -2\int te^{2t}dt \quad ...$$

$$\begin{Bmatrix} u = t & dv = e^{2t}dt \\ du = dt & v = \frac{1}{2}e^{2t} \end{Bmatrix} \implies \mu_1 = -2\left[\frac{te^{2t}}{2} - \int \frac{1}{2}e^{2t}dt\right] \implies \boxed{\mu_1 = -te^{2t}+\frac{1}{2}e^{2t}}$$

Now, we find μ_2:

$$\mu_2 = \int \frac{e^t2e^{3t}}{e^{2t}}dt \implies \mu_2 = 2\int \frac{e^{4t}}{e^{2t}}dt \implies \mu_2 = 2\int e^{2t}dt \implies \boxed{\mu_2 = e^{2t}}$$

Our particular solution is then:

$$y_p = \mu_1y_1+\mu_2y_2 \implies y_p = \left(\frac{1}{2}e^{2t} - te^{2t}\right)e^t+(e^{2t})te^t \implies y_p = \frac{1}{2}e^{3t}-te^{3t}+te^{3t} \quad ...$$

$$\boxed{y_p = \frac{1}{2}e^{3t}}$$

Putting everything together to get our general solution:

$$\boxed{y_g(t) = C_1e^t + C_2te^t + \frac{1}{2}e^{3t}}$$

Problem 5

Solve the following IVP using the Laplace method:

$$y'' + 2y' + y = \mu_0(t) \quad ; \quad y(0) = 0 \quad ; \quad y'(0) = 0.$$

Try it out before looking at the solution.

Solution to Problem 5:
I'll start off by giving you the Laplace transforms that were given on the first page of the exam that this problem was inspired by:

t^n	$\frac{n!}{s^{n+1}}$
$e^{-at}f(t)$	$F(s+a)$
$\mu(t-a)f(t-a)$	$e^{-as}F(s)$
$\delta(t-a)$	e^{-as}

Let's begin by taking the Laplace transform of both sides:

$$\mathcal{L}\left\{y''+2y'+y\right\}=\mathcal{L}\left\{\mu_0(t)\right\}.$$

Exploiting the linearity of the Laplace transform operation:

$$\mathcal{L}\left\{y''\right\}+2\mathcal{L}\left\{y'\right\}+\mathcal{L}\left\{y\right\}=\mathcal{L}\left\{\mu_0(t)\right\}.$$

Using that $\big[\mathcal{L}\left\{y''\right\}=s^2Y-sy(0)-y'(0)\big]$, $\big[\mathcal{L}\left\{y'\right\}=sY-y(0)\big]$, $\big[\mathcal{L}\left\{y\right\}=Y\big]$, and $\big[\mathcal{L}\left\{\mu_a(t)\right\}=\frac{e^{-as}}{s}\big]$:

$$\big[s^2Y-sy(0)-y'(0)\big]+2\big[sY-y(0)\big]+\big[Y\big]=\frac{e^{(0)s}}{s}.$$

Factoring out a Y from the terms on the LHS and simplifying the RHS:

$$Y\big[s^2+2s+1\big]\underbrace{-sy(0)-y'(0)-2y(0)}_{0}=\frac{1}{s}\qquad\Longrightarrow\qquad Y[s^2+2s+1]=\frac{1}{s}.$$

Solving for our solution in the s domain:

$$Y=\frac{1}{s[s^2+2s+1]}.$$

Partial fraction decomposition and convolution are two methods used to solve for the solution in the s domain when dealing with Laplace transforms. Here's an explanation of when to use each method:

1. Partial Fraction Decomposition:

    - Use partial fraction decomposition when the denominator of the expression in the s domain is a product of factors (linear or quadratic) and the numerator is a polynomial.
    - The goal of partial fraction decomposition is to split the original fraction into a sum of simpler fractions, each with a denominator that is a single factor from the original denominator.
    - After decomposing the fraction, you can apply the inverse Laplace transform to each individual fraction and then add the results to obtain the solution in the time domain.

2. Convolution:

   - Use convolution when the expression in the s domain has a more complicated numerator, such as a rational function or a non-polynomial function.
   - Convolution is also used when you can easily recognize the inverse Laplace transforms of both the numerator and the denominator separately.
   - In this method, you split the expression into the product of two functions in the s domain, find their individual inverse Laplace transforms, and then perform a convolution integral to obtain the solution in the time domain.

What we will see now, is if the polynomial in the denominator can be factored:

$$Y = \frac{1}{s(s+1)^2}.$$

Since we have the product of linear factors in the denominator and a polynomial in the numerator, partial fraction decomposition would likely be the better way to find our solution in the time domain. Let's go ahead and get started:

$$\frac{1}{s(s+1)^2} = \frac{A}{s} + \frac{B}{(s+1)} + \frac{C}{(s+1)^2} \quad \ldots$$

$$A(s+1)^2 + B(s)(s+1) + C(s) = 1 \quad \ldots$$

Let's expand things just as we would solve an undetermined coefficients problem:

$$A(s^2+2s+1)+B(s)(s+1)+C(s) = 1 \quad \Longrightarrow \quad \underline{\underline{As^2}}+\underline{2As}+A+\underline{\underline{Bs^2}}+\underline{Bs}+\underline{Cs} = 1 \quad \ldots$$

Grouping by the common factors of s:

$$[A+B]s^2 + [2A+B+C]s + [A] = [0]s^2 + [0]s + [1].$$

This gives us three equations for our unknown coefficients:

$$\begin{array}{rcl} (\text{Eq. } 1): & A+B & = 0 \\ (\text{Eq. } 2): & 2A+B+C & = 0 \\ (\text{Eq. } 3): & A & = 1 \end{array}$$

With $A = 1$, we can solve Eq. 1 for B:

$$\boxed{A = 1}$$

$$(1) + B = 0 \quad \Longrightarrow \quad \boxed{B = -1}$$

With A and B known, we can find C in Eq. 2:

$$2(1) + (-1) + C = 0 \quad \Longrightarrow \quad 2 - 1 + C = 0 \quad \Longrightarrow \quad \boxed{C = -1}$$

With the numerators of our partial fractions determined, we can rewrite Y to be the sum of fractions:

$$Y = \frac{1}{s} - \frac{1}{(s+1)} - \frac{1}{(s+1)^2}.$$

Now, when we take the inverse Laplace transform of both sides, we have:

$$\mathcal{L}^{-1}\left\{Y\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}.$$

Let's start off by evaluating the easy inverse Laplace transforms:

$$y(t) = 1 - e^{-t} - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}.$$

Let's give the hard one a name, call it $F(s)$:

$$F(s) = \frac{1}{(s+1)^2}.$$

If we shift our function by taking $F(s+1)$, we have:

$$F(s+1) = \frac{1}{s^2}.$$

Our table above tells us that:

$$\mathcal{L}\left\{e^{-at}f(t)\right\} = F(s+a) \quad \therefore \quad e^{-at}f(t) = \mathcal{L}^{-1}\left\{F(s+a)\right\}.$$

This tells us that to take the inverse Laplace transform of a shifted function in the s domain, we will have a factor of e^{-as} to account for the shift, and it will be multiplied by the inverse Laplace transform of the simplified function.

For us, this means:

$$\mathcal{L}^{-1}\left\{F(s+1)\right\} = e^{-t} \times \mathcal{L}\left\{\frac{1}{s^2}\right\} \quad \Longrightarrow \quad \mathcal{L}^{-1}\left\{F(s+1)\right\} = te^{-t}.$$

Putting everything together, we have:

$$y(t) = 1 - e^{-t} - te^{-t} \quad \Longrightarrow \quad \underline{\underline{\boxed{y(t) = 1 - e^{-t}(1+t)}}}$$

Problem 6

Find the general solution of the following:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ -3 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

In another form:

$$\vec{x'} = \begin{pmatrix} 4 & -2 \\ -3 & -1 \end{pmatrix} \vec{x}$$

Try it out before looking at the solution.

Solution to Problem 6:
To solve a system of differential equations, we first find the eigenvalues of the system.

Two Real, Distinct Eigenvalues:
If we have two real, distinct eigenvalues, λ_1 and λ_2, for the coefficient matrix, we have a general solution of the form:

$$\vec{x}_g = C_1 e^{\lambda_1 t}\vec{v}_1 + C_2 e^{\lambda_2 t}\vec{v}_2.$$

Here, $\vec{v}_1$ and $\vec{v}_2$ are the eigenvectors for the first and second eigenvalues, respectively.

One Degenerate Eigenvalue:
In the case where we have a degenerate eigenvalue, our general solution will have the form of:

$$\vec{x}_g = (C_1 + C_2)e^{\lambda t}\vec{v}_1 + C_2 t e^{\lambda t}\vec{v}_2$$

Where λ is the degenerate eigenvalue, $\vec{v}_1$ is the eigenvector corresponding to the degenerate eigenvalue, and $\vec{v_2}$ is a generalized eigenvector satisfying $(\mathbf{A} - \lambda\mathbf{I})\vec{v}_2 = \vec{v}_1$, where $\mathbf{A}$ is the coefficient matrix and $\mathbf{I}$ is the identity matrix.

Two Complex, Distinct Eigenvalues:
In the case where there are two complex, distinct eigenvalues of the form $\alpha \pm \beta$, the general solution is of the following form:

$$\vec{x}_g = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ where } \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} e^{\alpha t}[\vec{v}_1 \cos(\beta t) - \vec{v}_2 \sin(\beta t)] \\ e^{\alpha t}[\vec{v}_1 \sin(\beta t) + \vec{v}_2 \cos(\beta t)] \end{pmatrix}$$

To solve our problem, let's find the eigenvalues of the coefficient matrix. Below, $\mathbf{A}$ is the coefficient matrix and $\mathbf{I}$ is the identity matrix.

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \implies \begin{vmatrix} 4-\lambda & -2 \\ -3 & -1-\lambda \end{vmatrix} = 0 \implies (4-\lambda)(-1-\lambda) - \big((-3)(-2)\big) = 0 \quad \ldots$$

$$(4-\lambda)(-1-\lambda) - 6 = 0 \implies -4 - 4\lambda + \lambda + \lambda^2 - 6 = 0 \implies \underbrace{\lambda^2 - 3\lambda - 10 = 0}_{\text{characteristic polynomial}} .$$

Solving the characteristic polynomial for λ yields:

$$(\lambda - 5)(\lambda + 2) = 0 \quad \therefore \quad \boxed{\begin{Bmatrix} \lambda_1 = 5 \\ \lambda_2 = -2 \end{Bmatrix}}$$

Here, we have two distinct, real eigenvalues, so our general solution will have the form of:

$$\boxed{\vec{x}_g = C_1 e^{\lambda_1}\vec{v}_1 + C_2 e^{\lambda_2}\vec{v}_2.}$$

The next step is to find the eigenvectors that correspond to each of our eigenvalues. Let's start with the first eigenvalue:

$$(\mathbf{A} - \mathbf{I}\lambda_1)\vec{v}_1 = \vec{0}.$$

For our particular eigenvalue, $\lambda_1 = 5$:

$$\begin{pmatrix} 4-(5) & -2 \\ -3 & -1-(5) \end{pmatrix} \begin{pmatrix} v_a \\ v_b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} -1 & -2 \\ -3 & -6 \end{pmatrix} \begin{pmatrix} v_a \\ v_b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \ldots$$

$$\begin{pmatrix} -1 & -2 & | & 0 \\ -3 & -6 & | & 0 \end{pmatrix} \implies \begin{pmatrix} 1 & 2 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix} \implies v_a + 2v_b = 0 \implies v_a = -2v_b.$$

Now, we pick an arbitrary value for v_b that gives us nice-looking numbers. Let's let $v_b = 1$:

$$\text{If } v_b = 1\text{:} \quad \boxed{\vec{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}}$$

Now, let's find the eigenvector that corresponds to our second eigenvalue.

$$(\mathbf{A} - \mathbf{I}\lambda_2)\vec{v}_2 = \vec{0}.$$

For our particular eigenvalue, $\lambda_2 = -2$:

$$\begin{pmatrix} 4-(-2) & -2 \\ -3 & -1-(-2) \end{pmatrix} \begin{pmatrix} v_c \\ v_d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} 6 & -2 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} v_c \\ v_d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \dots$$

$$\begin{pmatrix} 6 & -2 & | & 0 \\ -3 & 1 & | & 0 \end{pmatrix} \implies \begin{pmatrix} 3 & -1 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix} \implies 3v_c - v_d = 0 \implies v_c = \frac{v_d}{3}.$$

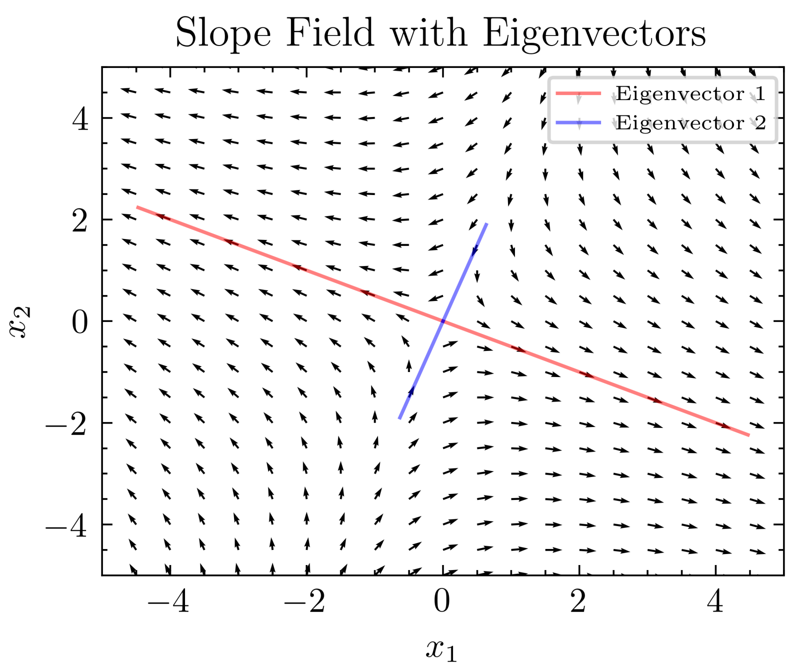
Picking a simple value for v_d to be 3, we have:

$$\text{If } v_d = 3\text{:} \quad \boxed{\vec{v}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}}$$

With our eigenvalues and their corresponding eigenvectors, we can write the general solution for this system of differential equations:

$$\boxed{\vec{x}_g = C_1 e^{5t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}}$$

Below is the slope field and eigenvectors scaled by their eigenvalues for this problem:



Problem 7

Find the power series solution centered at the ordinary point, $x_0 = 0$, for the following differential equation:

$$y'' - xy' = 0.$$

Try it out before looking at the solution.

Solution to Problem 7:
If all of the coefficients of our derivatives are analytic functions, then we can say that our solution will also be an analytic function. As such, it has a power series representation. We make the guess that that solution is of the following form:

$$y_{\text{guess}} = \sum_{n=0}^{\infty} c_n x^n.$$

We take the derivatives of our guess to substitute into our differential equation and find a true statement.

$$y'_{\text{guess}} = \sum_{n=1}^{\infty} (n) c_n x^{n-1}.$$

$$y''_{\text{guess}} = \sum_{n=2}^{\infty} (n-1)(n) c_n x^{n-2}.$$

Before any reindexing, we should be sure to make our substitution and distribute all factors:

$$\left[\sum_{n=2}^{\infty}(n-1)(n)c_n x^{n-2}\right] - \underbrace{x}_{\text{distribute}} \left[\sum_{n=1}^{\infty}(n)c_n x^{n-1}\right] = 0 \quad \ldots$$

$$\left[\sum_{n=2}^{\infty}(n-1)(n)c_n x^{n-2}\right] - \left[\sum_{n=1}^{\infty}(n)c_n x^{n}\right] = 0.$$

Now, we ask ourselves what is stopping us from combining these into one big sum and factoring out x^k? It's typically good convention to start off by ensuring all terms have a common factor of x^k. Let's do that with our first series.

Let $k = n - 2$, implying that $n = k + 2$:

$$\left[\sum_{k=0}^{\infty}(k+1)(k+2)c_{k+2} x^{k}\right] - \underbrace{\left[\sum_{k=1}^{\infty}(k)c_k x^{k}\right]}_{n=k} = 0.$$

Now that all series have a common factor of x^k, we can combine them, right? First, we need to make sure they start at the same index. They don't, so let's fix that:

$$2c_2 + \underbrace{\left[\sum_{k=1}^{\infty}(k+1)(k+2)c_{k+2} x^{k}\right]}_{\text{pulled out the 0th term}} - \left[\sum_{k=1}^{\infty}(k)c_k x^{k}\right] = 0.$$

Now, let's combine the three series into one big series, factoring out an x^k in the process:

$$2c_2 + \sum_{k=1}^{\infty}\left[(k+1)(k+2)c_{k+2} - kc_k\right] x^k = 0.$$

The only way for this statement to be true is if:

$$2c_2 = 0$$

and

$$(k+1)(k+2)c_{k+2} - kc_k = 0$$

These two conditions are where we get our recurrence relations. The conventional form of a recurrence relation is to have the next term in the sequence to be equal to some combination of the previous terms. When we do this we get:

$$c_2 = 0 \quad \text{and} \quad c_{k+2} = \frac{kc_k}{(k+1)(k+2)}.$$

With our recurrence relations, we can begin writing a few of the first coefficients in order to generalize a rule for the coefficients of our power series solution:

$$\begin{aligned}
k = 1: &\quad c_3 = \tfrac{1}{2\times 3}c_1 \\
k = 2: &\quad c_4 = \tfrac{2}{3\times 4}c_2 \implies c_4 = 0 \\
k = 3: &\quad c_5 = \tfrac{3}{4\times 5}c_3 \implies c_5 = \tfrac{1\times 3}{2\times 3\times 4\times 5} \\
k = 4: &\quad c_6 = \tfrac{4}{5\times 6}c_4 \implies c_6 = 0 \\
k = 5: &\quad c_7 = \tfrac{5}{6\times 7}c_5 \implies c_7 = \tfrac{1\times 3\times 5}{2\times 3\times 4\times 5\times 6\times 7}
\end{aligned}$$

Now, if we look at the even terms:

$$\begin{aligned}
p = 1: &\quad c_2 = 0 \\
p = 2: &\quad c_4 = 0 \\
p = 3: &\quad c_6 = 0
\end{aligned}$$

Our rule for the even terms is simple:

$$c_{2p} = 0.$$

Looking at the odd terms:

$$\begin{aligned}
p = 1: &\quad c_3 = \tfrac{1}{2\times 3}c_1 \\
p = 2: &\quad c_5 = \tfrac{1\times 3}{2\times 3\times 4\times 5}c_1 \\
p = 3: &\quad c_7 = \tfrac{1\times 3\times 5}{2\times 3\times 4\times 5\times 6\times 7}c_1
\end{aligned}$$

We see an odd double factorial in the numerator, and a single factorial in the denominator. Using the dummy index, p, to describe this behavior, we have:

$$c_{2p+1} = \frac{(2p-1)!!}{(2p+1)!}c_1.$$

With the coefficients for the even and odd terms of our solution, we have:

$$y_e(x) = 0$$

$$y_o(x) = c_1 \sum_{p=0}^{\infty} \frac{(2p-1)!!}{(2p+1)!} x^{2p+1}.$$

The full general solution is the superposition of the linearly independent solutions:

$$y_g(x) = y_o + y_e$$

$$y_g(x) = c_1 \sum_{p=0}^{\infty} \frac{(2p-1)!!}{(2p+1)!} x^{2p+1}.$$

There is a problem here, however. Notice that we started off with a second order differential equation, which requires two arbitrary constants. y_e being 0 messed that up for us, so we will add a constant to our general solution to ensure that it is in agreement with our expectations:

$$\boxed{y_g(x) = A + c_1 \sum_{p=0}^{\infty} \frac{(2p-1)!!}{(2p+1)!} x^{2p+1}}$$

If we write out a few terms:

$$\boxed{y_g(x) = A + \underbrace{c_1 x}_{(-1)!!=1} + \frac{1}{6} c_1 x^3 + \frac{3}{120} c_1 x^5 + \frac{15}{5040} c_1 x^7 + \frac{105}{362880} c_1 x^9 + \dots}$$

Wolfram Mathematica gives the solution:

$$y(x) = \sqrt{\frac{\pi}{2}} c_1 \operatorname{erfi}\left(\frac{x}{\sqrt{2}}\right) + A.$$

How cool is it that we can find the solution to such a tricky differential equation. If you are curious to see what a plot of the first 100 terms of a specific solution looks like, I've included that below.