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MATH 3400
SI Session 9 Practice Problems and Solutions
8 April 2024

Problem 1

Compute the power series solution for the differential equation about the ordinary point, $x_0 = 0$:

$$y'' + 2xy' + 2y = 0.$$

Try it out before looking at the solution.

Solution to Problem 1:

To begin, we will write that the final form of the power series of our solution is the sum of two linearly independent solutions:

$$y(x) = y_1(x) + y_2(x).$$

Much like other techniques, we will begin with a guess of our solution. This time the guess is the simple power series:

$$y_{\text{guess}}(x) = \sum_{n=0}^{\infty} c_n x^n.$$

We can differentiate and integrate a power series term-by-term, so let's do that for our guess in order to write our differential equation as a sum of power series.

$$y'_{\text{guess}}(x) = \frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n x^n\right] \quad \rightarrow \quad y'_{\text{guess}}(x) = \sum_{n=1}^{\infty}(n)c_n x^{n-1}.$$

We begin our derivative's series at $n = 1$ because the 0^{th} term is 0:

$$y_{\text{guess}}(x) = \underbrace{c_0}_{n=0} + \underbrace{c_1 x}_{n=1} + \underbrace{c_2 x^2}_{n=2} + \dots$$

$$y'_{\text{guess}}(x) = \underbrace{0}_{n=0} + \underbrace{c_1}_{n=1} + \underbrace{2c_2 x}_{n=2} + \dots$$

Next, we find the second derivative of our solution form (which will begin at $n = 2$).

$$y''_{\text{guess}}(x) = \frac{d}{dx}\left[\sum_{n=1}^{\infty}(n)c_n x^{n-1}\right] \quad \rightarrow \quad y''_{\text{guess}}(x) = \sum_{n=2}^{\infty}(n-1)(n)c_n x^{n-2}.$$

We now have the following representations of our guess and their derivatives:

$$y_{\text{guess}}(x) = \sum_{n=0}^{\infty} c_n x^n.$$

$$y'_{\text{guess}}(x) = \sum_{n=1}^{\infty}(n)c_n x^{n-1}.$$

$$y''_{\text{guess}}(x) = \sum_{n=2}^{\infty}(n-1)(n)c_n x^{n-2}.$$

Let's plug these into our differential equation: $y'' + 2xy' + 2y = 0$.

$$\left[\sum_{n=2}^{\infty}(n-1)(n)c_n x^{n-2}\right] + 2x\left[\sum_{n=1}^{\infty}(n)c_n x^{n-1}\right] + 2\left[\sum_{n=0}^{\infty} c_n x^n\right] = 0.$$

Our goal is to be able to combine these series into one big series and factor out a common term of x^n. To do that, we will need to make sure all series:

1. Have a factor of x^n.
2. Begin at the same index

$\boxed{\text{IMPORTANT NOTE}}$:

Before we reindex any of our power series, we want to plug them into our differential equation. Why? Because we want to distribute terms like the $2x$ into our series before we reindex, or else we'll have to reindex twice (which is not fun).

Let's begin with the left-most term of our differential equation, $y''(x)$.

Since we don't distribute anything to this power series, we can reindex immediately:

$$\text{Let } k = n - 2, \text{ implying } n = k + 2.$$

$$\sum_{n=2}^{\infty}(n-1)(n)c_n x^{n-2} \quad \rightarrow \quad \sum_{k=0}^{\infty}((k+2)-1)(k+2)c_{k+2}x^k \quad \rightarrow \quad \sum_{k=0}^{\infty}(k+1)(k+2)c_{k+2}x^k.$$

To show that I've already reindexed, I prefer to keep the index as k. We now have:

$$y''_{\text{guess}}(x) = \sum_{k=0}^{\infty}(k+1)(k+2)c_{k+2}x^k$$

Now, let's do the same, but with the second term of our differential equation.

We will distribute $2x$ into the power series representation of $y'(x)$:

$$2x\left[\sum_{n=1}^{\infty}(n)c_n x^{n-1}\right] \quad \rightarrow \quad \sum_{n=1}^{\infty}2(n)c_n x^{n-1+1} \quad \rightarrow \quad \sum_{n=1}^{\infty}2(n)c_n x^n.$$

Since this power series now contains a factor of x^n, let's reindex n to be k:

$$2xy'_{\text{guess}}(x) = \sum_{k=1}^{\infty}2(k)c_k x^k$$

Let's work with the last term of our series-expanded differential equation:

$$2\left[\sum_{n=0}^{\infty}c_n x^n\right] \quad \rightarrow \quad \sum_{n=0}^{\infty}2c_n x^n.$$

This term already contains a factor of x^n, so let's let $n = k$ and write our last distributed term out:

$$2y_{\text{guess}}(x) = \sum_{k=0}^{\infty}2c_k x^k$$

At this point we have all of our power series containing a factor of x^k. When we put them together we have:

$$y'' + 2xy' + 2y = \left[\sum_{k=0}^{\infty}(k+1)(k+2)c_{k+2}x^k\right] + \left[\sum_{k=1}^{\infty}2(k)c_k x^k\right] + \left[\sum_{k=0}^{\infty}2c_k x^k\right] = 0.$$

Notice that we are one step away from making one big $\sum$. We need all of our series to start at $k = 1$.

To do this, we will pull the 1st terms out of our first and last series.

Pulling the first term out of $\sum_{k=0}^{\infty}(k+1)(k+2)c_{k+2}x^k$, we have:

$$\left[\sum_{k=0}^{\infty}(k+1)(k+2)c_{k+2}x^k\right] \quad \to \quad \underbrace{2c_2}_{0^{\text{th}}\text{ term}} + \left[\sum_{k=1}^{\infty}(k+1)(k+2)c_{k+2}x^k\right].$$

Pulling the first term out of $\sum_{k=0}^{\infty}2c_k x^k$, we have:

$$\left[\sum_{k=0}^{\infty}2c_k x^k\right] \quad \to \quad \underbrace{2c_0}_{0^{\text{th}}\text{ term}} + \left[\sum_{k=1}^{\infty}2c_k x^k\right].$$

Rewriting our differential equation:

$$2c_2 + \left[\sum_{k=1}^{\infty}(k+1)(k+2)c_{k+2}x^k\right] + \left[\sum_{k=1}^{\infty}2(k)c_k x^k\right] + 2c_0 + \left[\sum_{k=1}^{\infty}2c_k x^k\right] = 0.$$

We will combine our sums since they begin at the same index and we will factor out x^k since all of them have that term in common.

$$\underbrace{2(c_2 + c_0)}_{\text{this part } = 0} + \sum_{k=1}^{\infty}\underbrace{\left[(k+1)(k+2)c_{k+2} + 2c_k(k+1)\right]}_{\text{this part also } = 0} x^k = 0$$

Let's set our coefficients to be equal to zero. Here the coefficients outside of the sum are going to be equal to zero when $k = 0$ and the coefficients inside the sum are going to be equal to zero when $k = 1, 2, 3 \ldots$

$$\begin{aligned} k &= 0 & 2(c_2 + c_0) &= 0 \\ k &= 1, 2, 3, \ldots & (k+1)(k+2)c_{k+2} + 2c_k(k+1) &= 0 \end{aligned}$$

This means that

$$2(c_2 + c_0) = 0 \quad \to \quad \boxed{c_2 = -c_0}$$

and

$$(k+1)(k+2)c_{k+2} + 2c_k(k+1) = 0 \quad \to \quad c_{k+2} = -\frac{2c_k(k+1)}{(k+1)(k+2)} \quad \to \ldots$$

$$\ldots \quad \to \quad c_{k+2} = \boxed{-\frac{2c_k}{(k+2)}}$$

Let's use the boxed relations to write out some of the first few coefficients, starting at $k = 1$.

$$\begin{array}{llcl}
k=1: & c_3 = -\frac{2c_1}{3} & \overset{\text{rewriting}}{\rightarrow} & c_3 = -\frac{2}{3\times1}c_1 \\
k=2: & c_4 = -\frac{2c_2}{4} & \overset{c_2=-c_0}{\rightarrow} & c_4 = \frac{2}{4}c_0 \\
k=3: & c_5 = -\frac{2c_3}{5} & \overset{\text{using } c_3}{\rightarrow} & c_5 = \frac{2\times2}{5\times3\times1}c_1 \\
k=4: & c_6 = -\frac{2c_4}{6} & \overset{\text{using } c_4}{\rightarrow} & c_6 = -\frac{2\times2}{6\times4}c_0 \\
k=5: & c_7 = -\frac{2c_5}{7} & \overset{\text{using } c_5}{\rightarrow} & c_7 = -\frac{2\times2\times2}{7\times5\times3\times1}c_1 \\
k=6: & c_8 = -\frac{2c_6}{8} & \overset{\text{using } c_6}{\rightarrow} & c_8 = \frac{2\times2\times2}{8\times6\times4}c_0 \\
k=7: & c_9 = -\frac{2c_7}{9} & \overset{\text{using } c_7}{\rightarrow} & c_9 = \frac{2\times2\times2\times2}{9\times7\times5\times3\times1}c_1 \\
k=8: & c_{10} = -\frac{2c_8}{10} & \overset{\text{using } c_8}{\rightarrow} & c_{10} = -\frac{2\times2\times2\times2}{10\times8\times6\times4}c_0 \\
k=9: & c_{11} = -\frac{2c_9}{11} & \overset{\text{using } c_9}{\rightarrow} & c_{11} = -\frac{2\times2\times2\times2\times2}{11\times9\times7\times5\times3\times1}c_1
\end{array}$$

It's time for a reality check. If things are going right, we should see that all of our coefficients are in terms of two coefficients. In this case, c_0 and c_1. If you don't end up with something like this, make sure you've simplified completely.

Let's split things up into c_0's and c_1's to make the pattern more clear.

The c_1 terms are below. p is a dummy index that will help us determine the parts of our sequence that define the pattern of our odd-numbered terms.

$$\begin{array}{ll}
p=1: & c_3 = -\frac{2}{3\times1}c_1 \\
p=2: & c_5 = \frac{2\times2}{5\times3\times1}c_1 \\
p=3: & c_7 = -\frac{2\times2\times2}{7\times5\times3\times1}c_1 \\
p=4: & c_9 = \frac{2\times2\times2\times2}{9\times7\times5\times3\times1}c_1 \\
p=5: & c_{11} = -\frac{2\times2\times2\times2\times2}{11\times9\times7\times5\times3\times1}c_1
\end{array}$$

We'll start off with some rules for c_1.

1. Subscript starts at 3 and iterates over odd numbers, so our coefficient is defined in terms of p as c_{2p+1}.

2. Sign starts (-1) for c_3 and oscillates from (-1) to $(+1)$, so that portion of our coefficients' recurrence relation is $(-1)^p$.

3. Numerator has a 2 raised to a power of 1 for c_3 and 5 for c_{11}, so the numerator follows the patter of 2^p.

4. Denominator has an odd number factorial defined by the constant's subscript, this is defined using $(2p + 1)!!$, which will be discussed in a moment.

Let's write the sequence that satisfies these rules and explain each part after:

$$\underline{c_{2p+1} = (-1)^p \left[\frac{2^p}{(2p+1)!!}\right] c_1.}$$

The double factorial notation was invented in 1902 by physicist Arthur Schuster, and we should be glad that someone came up with it because it makes our work much more simple. In short, it's a factorial that increments down by two numbers instead of one.

Feel free to check that this sequence makes sense. Plug in some values of p to see if the results matches our c_{2p+1} sequence.

The sequence that we have written is a formal and compact way of representing all odd-numbered coefficients, c_{2p+1} of the terms of our differential equation solution. The odd-numbered coefficients correspond to the odd-numbered powers of x: x^{2p+1}. This supports the notion that our solution will be the combination of two linearly independent solutions $y_1(x)$ and $y_2(x)$.

It makes no difference which of the two previously mentions halves of the solution the c_{2p+1} terms correspond to, let's arbitrarily pick them to be for the $y_2(x)$ half of our final solution.

Remembering the form of our solution guess:

$$y_{\text{guess}}(x) = \sum_{n=0}^{\infty} c_n x^n,$$

The odd-numbered terms will be of the form:

$$y_2(x) = c_1 \sum_{p=0}^{\infty} c_{2p+1} x^{2p+1}.$$

Which can be fully written as:

$$\boxed{y_2(x) = c_1 \sum_{p=0}^{\infty} (-1)^p \left[\frac{2^p}{(2p+1)!!}\right] x^{2p+1}}$$

Something to note: our solution begins at $p = 0$, which is how we obtain the $x^{2(0)+1} \to x$ term of our solution.

Now, let's examine the c_0 terms.

$$\begin{aligned} p = 1: &\quad c_2 = -\tfrac{2^0}{1} c_0. \\ p = 2: &\quad c_4 = \tfrac{2}{4} c_0, \\ p = 3: &\quad c_6 = -\tfrac{2\times 2}{6\times 4} c_0, \\ p = 4: &\quad c_8 = \tfrac{2\times 2\times 2}{8\times 6\times 4} c_0, \\ p = 5: &\quad c_{10} = -\tfrac{2\times 2\times 2\times 2}{10\times 8\times 6\times 4} c_0. \end{aligned}$$

We'll start off with some rules for c_0.

1. Subscript starts at 2 and iterates over even numbers, so our coefficients follow the rule c_{2p}.

2. Sign starts at (-1) for c_2 and oscillates from (-1) to $(+1)$, so we have a factor of $(-1)^p$.

3. Numerator has a 2 raised to a power of 0 for c_2 and 4 for c_{10}, the rule that governs this behavior is 2^{p-1}.

4. Denominator has an even number factorial defined by the constant's subscript, but it's missing the 2 term, yielding a factor of $\frac{2}{(2p)!!}$.

Let's write the sequence that satisfies these rules:

$$c_{2p} = (-1)^p \left[\frac{2\left(2^{p-1}\right)}{(2p)!!} \right] c_0.$$

Notice that we can distribue the 2 to the 2^{p-1} term, giving us:

$$\underline{c_{2p} = (-1)^p \left[\frac{2^p}{(2p)!!} \right] c_0}$$

We now have the recurrence relation for the even-numbered, c_{2p}, coefficients. These coefficients belong to the even-powered x terms: x^{2p}. This half of our final solution will be called $y_1(x)$. Similar to the $y_2(x)$ half of our solution, it will be of the form:

$$y_1(x) = c_0 \sum_{p=0}^{\infty} c_{2p} x^{2p}.$$

In its full form:

$$\boxed{y_1(x) = c_0 \sum_{p=0}^{\infty} (-1)^p \left[\frac{2^p}{(2p)!!} \right] x^{2p}}$$

Our final solution is then:

$$y(x) = y_1(x) + y_2(x) \quad \rightarrow \quad y(x) = \sum_{p=0}^{\infty} c_0 c_{2p} x^{2p} + c_1 c_{2p+1} x^{2p+1} \quad \rightarrow \ldots$$

$$\underline{\boxed{y(x) = \sum_{p=0}^{\infty} c_0 \left((-1)^p \left[\frac{2^p}{(2p)!!} \right] \right) x^{2p} + c_1 \left((-1)^p \left[\frac{2^p}{(2p+1)!!} \right] \right) x^{2p+1}}}$$

Writing some terms out:

$$y(x) = c_0 + c_1 x - c_0 x^2 - c_1 \frac{2}{3} x^3 + c_0 \frac{1}{2} x^4 + c_1 \frac{4}{15} x^5 - c_0 \frac{1}{6} x^6 - c_1 \frac{8}{105} x^7 + \ldots$$