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MATH 3400
SI Session 10 Practice Problems and Solutions
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Problem 1

Find $y_s(x)$ using the Laplace transform solution technique for the following IVP.

$$5y'' + 10y = \delta(t-6) \quad ; \quad y(0) = y'(0) = 0.$$

Try it out before looking at the solution.

Solution to Problem 1:

Below are some important Laplace transforms to remember:

$y(t)$	$Y(s)$
$y'(t)$	$sY(s) - y(0)$
$y''(t)$	$s^2Y(s) - sy(0) - y'(0)$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\mu_a(t)$	$\frac{e^{-as}}{s}$
$\delta(t-a)$	e^{-as}
$(f \star g)(t)$	$F(s)G(s)$

Table 1: The most important Laplace transforms. Note the taking the Laplace transform means going from left to right. Taking the inverse Laplace transform means going from right to left.

Here are two definitions that are essential for this solution technique:

$$\mathcal{L}\{f(t)\} \triangleq \int_0^\infty f(t)e^{-st}dt. \tag{1}$$

$$(f \star g)(t) \triangleq \int_0^t f(\tau)g(t-\tau)d\tau. \tag{2}$$

We can start off by taking the Laplace transform of both sides of our differential equation:

$$\mathcal{L}\{5y'' + 10y\} = \mathcal{L}\{\delta(t-6)\}.$$

Since the Laplace transform is a linear operator, we can separate by plus and minus signs and pull out constants:

$$5\mathcal{L}\{y''\} + 10\mathcal{L}\{y\} = \mathcal{L}\{\delta(t-6)\}.$$

Using our known Laplace transforms, we go from the t domain to the s domain by performing the Laplace transform operation of each of our arguments.

$$5[s^2Y - sy(0) - y'(0)] + 10[Y] = e^{-6s}.$$

The Laplace transform turns a differential equation into a simple equation, we use algebra to solve for our function in the s domain, $Y(s)$. The last step is to translate our function into the t domain, $y(t)$.

Let's start off by using our initial conditions and solving for $Y(s)$:

$$5s^2Y + 10Y = e^{-6s} \quad \rightarrow \quad Y(s) = \frac{e^{-6s}}{5s^2+10}.$$

The last step is to take in the inverse Laplace transform of both sides.

$$\mathcal{L}^{-1}\left\{Y(s)\right\} = \mathcal{L}^{-1}\left\{\frac{e^{-6s}}{5s^2+10}\right\} \quad \rightarrow \quad y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-6s}}{5s^2+10}\right\}.$$

At this point, we see if what we are taking the inverse Laplace transform of matches the form of anything that we know. Directly, it doesn't. Just like taking derivatives and integrals of the product of two functions means that we have to use *fancy* multiplication (product rule and integration by parts, respectively), when we want to take the inverse Laplace transform of the product of two functions of s, we use convolution.

We known how to take the inverse Laplace transform of the product of two functions of s. That can be seen in our table above:

$$\mathcal{L}\left\{(f \star g)(t)\right\} = F(s)G(s) \quad \therefore \quad \mathcal{L}^{-1}\left\{F(s)G(s)\right\} = (f \star g)(t),$$

where f convolved with g is given in Definition [2] as:

$$(f \star g)(t) \triangleq \int_0^t f(\tau)g(t-\tau)d\tau.$$

What we need to do now is to find a way to express $\frac{e^{-6s}}{5s^2+10}$ as the product of two functions of s (both of which have the form of a Laplace transform that we are familiar with).

$$\text{Let}: \quad A(s) \triangleq e^{-6s} \quad \text{and} \quad B(s) \triangleq \frac{1}{5s^2+10}.$$

We now have:

$$y(t) = \mathcal{L}^{-1}\left\{A(s)B(s)\right\}.$$

Taking this inverse Laplace transform means that we will be convolving $a(t)$ with $b(t)$. To do this, we first need to know what $a(t)$ and $b(t)$ are.

$$A(s) = e^{-6s} \quad \therefore \quad a(t) = \mathcal{L}^{-1}\left\{e^{-6s}\right\} \quad \rightarrow \quad \boxed{a(t) = \delta(t-6)}$$

$$B(s) = \frac{1}{5s^2+10} \quad \therefore \quad b(t) = \mathcal{L}^{-1}\left\{\frac{1}{5s^2+10}\right\} \quad \cdots$$

Here, we see that $B(s)$ is most similar to the Laplace transform of $\sin(at)$. What we need to do is manipulate $B(s)$ until it is exactly in the form of the Laplace transform of $\sin(at)$. Let's start off by pulling out an extra factor of 5 in the denominator.

$$b(t) = \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s^2+2}\right\}.$$

Notice that the Laplace transform of $\sin(at)$ has an a in the numerator and an a^2 in the denominator. How do we write 2 as the square of a number?

$$b(t) = \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s^2+(\sqrt{2})^2}\right\}.$$

Now, to force a factor of $a = \sqrt{2}$ in the numerator we can multiply by $\frac{\sqrt{2}}{\sqrt{2}}$ and pull the $\frac{1}{\sqrt{2}}$ outside:

$$b(t) = \frac{1}{5}\mathcal{L}^{-1}\left\{\left(\frac{\sqrt{2}}{\sqrt{2}}\right)\frac{1}{s^2+(\sqrt{2})^2}\right\} \quad \rightarrow \quad b(t) = \frac{1}{5\sqrt{(2)}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+(\sqrt{2})^2}\right\}.$$

Making these alterations lets us take this inverse Laplace transform based on our known transformations in Table [1] :

$$\boxed{b(t) = \frac{1}{5\sqrt{2}} \sin\left(t\sqrt{2}\right)}$$

Since we know $a(t)$ and $b(t)$, we can perform the convolution operation:

$$y(t) = (a \star b)(t) = \frac{1}{5\sqrt{2}} \int_0^t \delta(\tau - 6) \sin\left((t - \tau)\sqrt{2}\right) d\tau.$$

This convolution integral will replace all τ's with t's when we evaluate the integral at each bound. Since the integral is definite, there will be no arbitrary constants. This is, therefore, our specific solution:

$$\boxed{\frac{1}{5\sqrt{2}} \int_0^t \delta(\tau - 6) \sin\left((t - \tau)\sqrt{2}\right) d\tau}$$