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MATH 3400
SI Session 5 Practice Problems and Solutions
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Problem 1.

Solve the IVP:

$$2y'' - 3y' + y = 0 \quad ; \quad y(0) = 1 \quad ; \quad y'(0) = 0.$$

Try it out before looking at the solution.

Solution to Problem 1:

To profile this differential equation, let's look at it's order. It's a second order differential equation. Once we see it's order, let's examine the coefficients of each term. They're constants and are not functions of x. Lastly, it is set equal to zero, meaning it is homogenous.

From this profiling, we know that we are working with a 2nd order, linear, constant coefficients, homogenous, differential equation. Let's say that five more times to lock it in $_{\text{just kidding}}$.

This type of differential equation is relatively simple to solve (seriously) if we remember a few things.

1. Find the discriminant
2. Remember which discriminant case corresponds to which solution form
3. Match the discriminant case to the general solution form
4. Use the quadratic formula to find the root[s].
5. Use the initial conditions to find a specific solution.

Let's be reminded of the discriminant trichotomy:

- $D > 0$: two real distinct roots λ_1 and λ_2, The general solution is
$$y_g(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}.$$
- $D = 0$: one real repeated root λ. The general solution is
$$y_g(x) = (C_1 + C_2 x)\, e^{\lambda x}.$$
- $D < 0$: A pair of complex conjugate roots $\lambda = \alpha \pm i\beta$. The general solution is
$$y_g(x) = e^{\alpha x}\Big[C_1 \cos(\beta x) + C_2 \sin(\beta x)\Big].$$

Let's find our discriminant:

$$D = b^2 - 4ac \quad \Longrightarrow \quad D = (-3)^2 - 4(2)(1) \quad \Longrightarrow \quad D = 1 \quad \Longrightarrow \quad \ldots$$

$$\ldots \Longrightarrow \quad \boxed{D > 0 \quad \therefore \quad y_g(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}}$$

Just like that we have the form of our general solution. To complete our answer, we simply need to find the roots of the quadratic in λ.

$$\lambda_{1,2} = \frac{-b \pm \sqrt{D}}{2a}$$

Let's plug in our values:

$$\lambda_{1,2} = \frac{3 \pm 1}{4} \quad \Longrightarrow \quad \boxed{\lambda_1 = 1 \quad ; \quad \lambda_2 = \frac{1}{2}}$$

With our roots, we will plug them into the form of the general solution to get the general solution to this particular problem:

$$\boxed{y_g(x) = C_1 e^x + C_2 e^{\frac{x}{2}}}$$

Since we are given initial conditions, we should use them to determine our constants, C_1 and C_2.

$$y(0) = 1 \quad : \quad 1 = C_1(1) + C_2(1) \quad \Longrightarrow \quad \underline{C_1 + C_2 = 1.}$$

Next we will take the derivative of the general solution to plug in the second initial condition, $y'(0) = 0$.

$$y'(0) = 0 \quad : \quad \frac{d}{dx}\left[y_g(x) = C_1 e^x + C_2 e^{\frac{x}{2}}\right] \quad \Longrightarrow \quad y'(x) = C_1 e^x + \frac{1}{2} C_2 e^{\frac{x}{2}} \quad \Longrightarrow \dots$$

$$\dots \Longrightarrow \quad 0 = C_1(1) + \frac{1}{2} C_2(1) \quad \Longrightarrow \quad \underline{C_1 + \frac{1}{2} C_2 = 0.}$$

Let's use an augmented matrix to solve this system:

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & \frac{1}{2} & 0 \end{array}\right] \quad -R_1{+}R_2 \Longrightarrow R_2 \quad \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -\frac{1}{2} & -1 \end{array}\right] \quad 2R_2{+}R_1 \Longrightarrow R_1 \quad \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & -1 & -2 \end{array}\right] \dots$$

$$\dots \Longrightarrow \quad -R_2 \Longrightarrow R_2 \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right] \quad \therefore \quad \boxed{C_1 = -1 \text{ and } C_2 = 2}.$$

Plugging these into our general solution for this problem, $y_g(x) = C_1 e^x + C_2 e^{\frac{x}{2}}$,

$$\underline{\boxed{y_s(x) = 2e^{\frac{x}{2}} - e^x}}$$

Problem 2.

Solve the differential equation:

$$x^2y'' - 4xy' + 6y = 0 \quad ; \quad x \neq 0 \quad \text{where} \quad y_1(x) = x^2.$$

Try it out before looking at the solution.

Solution to Problem 2:

I should start off by saying that the solution to this problem will seem long. That is mainly because I am showing all steps and giving explanations. I strongly recommend working through a few of these problems to familiarize yourselves with the steps of this technique.

The differential equation comes with a given solution: $y_1(x)$. Because of this, we will use the reduction of order technique to solve the differential equation.

Feel free to use the formula that Dr. B gave us in Equation [1] and see if you get the same answer. It's probably much quicker than the way I do it.

By the way, if you use the formula, **DO NOT FORGET THE NEGATIVE SIGN IN THE EXPONENTIAL'S INTEGRAL**

The Formula

$$y_2 = y_1 \int \frac{e^{-\int p dx}}{y_1^2} dx.$$

Your general solution will be the superposition of y_1 and y_2:

$$y_g(x) = C_1 y_1 + C_2 \left[y_1 \int \frac{e^{-\int p dx}}{y_1^2} dx \right]. \tag{1}$$

End: The Formula

The key to solving a differential equation with reduction of order is:

$$y_2(x) = \mu(x) y_1(x).$$

To find what $\mu(x)$ and then $y_2(x)$ is, we will implicitly differentiate $y_2(x)$ and substitute it into our original differential equation. In most cases, it helps to first insert the value of $y_1(x)$ into $y_2(x)$. Doing this,

$$y_2(x) = x^2 \mu(x).$$

I'm leaving the (x) part in μ and y_2 because it helps to see that both of these are functions of x and not independent variables, themselves. Doing so will ensure that we take our derivatives properly.

After we differentiate, I'll change the notation, so that things are more compact. Differentiating both sides of $y_2(x)$ with respect to x,

$$\frac{d}{dx}\left[y_2(x) = x^2 \mu(x)\right] \xRightarrow{\text{using product rule}} \underline{y_2'(x) = 2x\mu(x) + x^2 \mu'(x)}.$$

Now we will do the same thing again to find the second derivative of $y_2(x)$. Note that we are using product rule.

$$y_2''(x) = \frac{d}{dx}[y_2'(x)] \implies y_2''(x) = 2\mu(x) + 2x\mu'(x) + 2x\mu'(x) + x^2\mu''(x) \implies \ldots$$

$$\ldots \implies \underline{y_2''(x) = 2\mu(x) + 4x\mu'(x) + x^2\mu''(x)}.$$

From now on, for compactness sake, I'll be referring to $\mu(x)$ as μ. $y_2'(x)$ and $y_2''(x)$ will likewise be called y_2' and y_2'', respectively.

With this renaming, we have:

$$\underline{y_2 = x^2\mu}$$

$$\underline{y_2' = 2x\mu + x^2\mu'}$$

$$\underline{y_2'' = 2\mu + 4x\mu' + x^2\mu''}.$$

With these forms of our second solution, we will plug them into our original differential equation: $x^2y'' - 4xy' + 6y = 0$. Making the substitution, we have:

$$x^2[2\mu + 4x\mu' + x^2\mu''] - 4x[2x\mu + x^2\mu'] + 6[x^2\mu] = 0 \qquad \implies \ldots$$

Let's distribute terms so that we can combine like terms.

$$\ldots \implies \quad 2x^2\mu + 4x^3\mu' + x^4\mu'' - 8x^2\mu - 4x^3\mu' + 6x^2\mu = 0.$$

When doing this by hand, it helps to underline like terms in different styles.

$$\underline{2x^2\mu} + \underline{\underline{4x^3\mu'}} + \underline{\underline{\underline{x^4\mu''}}} - \underline{8x^2\mu} - \underline{\underline{4x^3\mu'}} + \underline{6x^2\mu} = 0$$

Let's combine our like terms now that it is clear. It should be noted that this is a place where a lot of mistakes are made. What we should be looking out for as confirmation that we've done things correctly is the vanishing of our μ term entirely. If we don't see that, we have a problem in our arithmetic.

$$x^4\mu'' = 0.$$

The μ term vanished, which is a good sign. The μ' term also vanished which is okay. Let's call $w' = \mu''$ and $w = \mu'$:

$$x^4w' = 0 \qquad \implies \qquad \frac{dw}{dx} = 0.$$

This is separable differential equation. Let's think of what becomes zero when we take the derivative of it. It's a constant, C_1.

$$w = C_1.$$

Now we remember that w was renamed from μ'. Let's "unsubstitue":

$$\mu' = C_1 \qquad \implies \qquad \frac{d\mu}{dx} = C_1 \qquad \implies \qquad d\mu = C_1 dx \qquad \overset{\int}{\implies} \qquad \underline{\mu = C_1x + K}.$$

We will let K be 0. The explanation for doing this is because any constant times y_1 is already part of the general solution, making K redundant. (It's the same reason for letting the constant of integration of $\int p(x)dx$ be 0 for the integrating factor in F.O.L equations).

We now have:

$$\mu = C_1x.$$

To find y_2, we multiply μ by y_1 using:

$$y_2(x) = \mu(x)y_1(x).$$

We then have:

$$\underline{y_2(x) = C_1x^3.}$$

We are given:

$$y_1(x) = x^2$$

Let's put these two together to find the general solution (adding a constant C_2 to y_1)

$$y_g(x) = y_1(x) + y_2(x) \quad \implies \quad \underline{\boxed{y_g(x) = C_1x^3 + C_2x^2}}$$

We could've been more careful in what we called our constant for y_1–for instance, making it C_1–with some forethought, but it really doesn't matter what our constants are called. We could just as correctly ended things with:

$$\underline{\boxed{y_g(x) = \text{🎃}x^3 + \text{👻}x^2}}$$