Collin Collins
MATH 3400
26 February 2024

Theorem: Consider the linear, homogeneous, second-order differential equation:

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0 \quad ; \quad \forall x \in \mathbb{R}$$

If $y_1(x)$ is a nontrivial solution, and $y_2(x)$ is defined as:

$$y_2(x) \triangleq y_1(x) \int \frac{e^{-\int p(x)dx}}{y_1^2(x)} dx$$

Then $y_1(x)$ and $y_2(x)$ are linearly independent; thus, the general solution can be expressed as:

$$y_g(x) = C_1 y_1(x) + C_2 y_2(x)$$

Proof: The Wronskian of y_1 and y_2 is:

$$\mathrm{W}\left[y_1, y_2\right] = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_2(x)y_1'(x)$$

Substituting the definition of $y_2(x)$ and differentiating:

$$\mathrm{W}[y_1, y_2] = y_1(x)\left[y_1'(x) \int \frac{e^{-\int p(x)dx}}{y_1^2(x)} dx + y_1(x) \frac{e^{-\int p(x)dx}}{y_1^2(x)}\right] - \left[y_1(x) \int \frac{e^{-\int p(x)dx}}{y_1^2(x)} dx\right] y_1'(x)$$

Simplifying, we obtain:

$$\mathrm{W}[y_1, y_2] = \frac{e^{-\int p(x)dx}}{y_1(x)}$$

For y_1 and y_2 to be linearly independent, their Wronskian must be nonzero. Hence,

$$\frac{e^{-\int p(x)dx}}{y_1(x)} \neq 0 \quad \Longrightarrow \quad e^{-\int p(x)dx} \neq 0 \quad ; \quad \forall p(x) : \int p(x)dx < +\infty.$$

. **QED**