Homework 3
Collin Collins
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Problem 1

Suppose $\vec{J}(\vec{r})$ is constant in time but $\rho(\vec{r},t)$ is not-conditions that might prevail, for instance, during the charging of a capacitor.

Part (a)

(a) Show that the charge density at any particular point is a linear function of time:

$$\rho(\vec{r},t) = \rho(\vec{r},0) + \dot{\rho}(\vec{r},0)t$$

where $\dot{\rho}(\vec{r},0)$ is the time derivative of ρ at $t = 0$.

Solution

Seeing $\vec{J}(\vec{r})$ and $\rho(\vec{r},t)$ makes me think to use the continuity equation:

$$\frac{\partial}{\partial t}\Big[\rho(\vec{r},t)\Big] + \nabla \cdot \vec{J}(\vec{r},t) = 0 \tag{1}$$

Here, we can use the information given, namely that $\vec{J}(\vec{r})$ is independent of time:

$$\frac{\partial}{\partial t}\Big[\rho(\vec{r},t)\Big] + \nabla \cdot \vec{J}(\vec{r},t) = 0 \quad \rightarrow \quad \frac{\partial}{\partial t}\Big[\rho(\vec{r},t)\Big] + \nabla \cdot \vec{J}(\vec{r}) = 0$$

Rearranging things in a typical way for dealing with differential equations:

$$\frac{\partial}{\partial t}\Big[\rho(\vec{r},t)\Big] + \nabla \cdot \vec{J}(\vec{r}) = 0 \quad \rightarrow \quad \frac{\partial}{\partial t}\Big[\rho(\vec{r},t)\Big] = -\nabla \cdot \vec{J}(\vec{r})$$

At this point, I see a partial derivative in time that I'd like to remove (to obtain the form of the final result we are asked to prove). Let's do that by integrating both sides with respect to t':

$$\int_{t'=0}^{t'=t} \frac{\partial}{\partial t'}\Big[\rho(\vec{r},t')\Big]dt' = -\int \nabla \cdot \vec{J}(\vec{r})dt'$$

Immediately, my left-hand-side can be simplified with the fundamental theorem of calculus. On the right-hand-side, remember that the integrand is independent of time, and therefore we can treat it as a constant.

$$\rho(\vec{r},t) - \rho(\vec{r},0) = -\nabla \cdot \vec{J}(\vec{r})t \quad \rightarrow \quad \rho(\vec{r},t) = \rho(\vec{r},0) - \nabla \cdot \vec{J}(\vec{r})t$$

We are almost there. The form on the right seems contrived because it is. I want to make it look like the form we were asked to prove: $\rho(\vec{r},t) = \rho(\vec{r},0) + \dot{\rho}(\vec{r},0)t$. Now, we just have to clarify how $\dot{\rho}(\vec{r},0) = -\nabla \cdot \vec{J}(\vec{r})$. For that let's remember our earlier result:

$$\frac{\partial}{\partial t}\Big[\rho(\vec{r},t)\Big] = -\nabla \cdot \vec{J}(\vec{r})$$

If this is true for all t, then it is certainly true for $t = 0$:

$$\frac{\partial}{\partial t}\Big[\rho(\vec{r},0)\Big] = -\nabla \cdot \vec{J}(\vec{r}) \quad \rightarrow \quad \dot{\rho}(\vec{r},0) = -\nabla \cdot \vec{J}(\vec{r})$$

Let's make this substitution in $\rho(\vec{r},t) = \rho(\vec{r},0) - \nabla \cdot \vec{J}(\vec{r})t$.

$$\boxed{\rho(\vec{r},t) = \rho(\vec{r},0) - \nabla \cdot \vec{J}(\vec{r})t \quad \rightarrow \quad \rho(\vec{r},t) = \rho(\vec{r},0) + \dot{\rho}(\vec{r},0)t}$$

Part (b)

This is not an electrostatic or magnetostatic configuration; nevertheless-rather surprisingly-both Coulomb's law (in the form of Eq. 2.8) and the Biot-Savart law (Eq. 5.39) hold, as you can confirm by showing that they satisfy Maxwell's equations. In particular:

(b) Show that

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}\,(\vec{r}\,')\times\hat{R}}{R^2}d\tau'$$

obeys Ampere's law with Maxwell's displacement current term. Part b will be considered as extra credit in case you want to turn-in this corrected version.

Solution

Note that since I don't want to install additional LaTeXpackages and switch to manual typesetting, I will use R as the separation vector $|R| =: |\vec{r} - \vec{r}\,'|$

As a reminder, the full Maxwell-Ampère law is

$$\nabla \times \vec{B}(\vec{r}) = \mu_0\vec{J}(\vec{r}) + \mu_0\epsilon_0\frac{\partial}{\partial t}\Big[\vec{E}(\vec{r},t)\Big] \tag{2}$$
