Homework 3
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Problem 1

Suppose $\mathbf{J}(\mathbf{r})$ is constant in time but $\rho(\mathbf{r}, t)$ is not-conditions that might prevail, for instance, during the charging of a capacitor.

Part (a)

(a) Show that the charge density at any particular point is a linear function of time:

$$\rho(\mathbf{r}, t) = \rho(\mathbf{r}, 0) + \dot{\rho}(\mathbf{r}, 0)t$$

where $\dot{\rho}(\mathbf{r}, 0)$ is the time derivative of ρ at $t = 0$.

Solution

Seeing $\mathbf{J}(\mathbf{r})$ and $\rho(\mathbf{r}, t)$ makes me think to use the continuity equation:

$$\frac{\partial}{\partial t}\Big[\rho(\mathbf{r}, t)\Big] + \nabla \cdot \mathbf{J}(\mathbf{r}, t) = 0 \tag{1}$$

Here, we can use the information given, namely that $\mathbf{J}(\mathbf{r})$ is independent of time:

$$\frac{\partial}{\partial t}\Big[\rho(\mathbf{r}, t)\Big] + \nabla \cdot \mathbf{J}(\mathbf{r}, t) = 0 \quad \rightarrow \quad \frac{\partial}{\partial t}\Big[\rho(\mathbf{r}, t)\Big] + \nabla \cdot \mathbf{J}(\mathbf{r}) = 0$$

Rearranging things in a typical way for dealing with differential equations:

$$\frac{\partial}{\partial t}\Big[\rho(\mathbf{r}, t)\Big] + \nabla \cdot \mathbf{J}(\mathbf{r}) = 0 \quad \rightarrow \quad \frac{\partial}{\partial t}\Big[\rho(\mathbf{r}, t)\Big] = -\nabla \cdot \mathbf{J}(\mathbf{r})$$

At this point, I see a partial derivative in time that I'd like to remove (to obtain the form of the final result we are asked to prove). Let's do that by integrating both sides with respect to t':

$$\int_{t'=0}^{t'=t} \frac{\partial}{\partial t'}\Big[\rho(\mathbf{r}, t')\Big] dt' = -\int \nabla \cdot \mathbf{J}(\mathbf{r}) dt'$$

Immediately, my left-hand-side can be simplified with the fundamental theorem of calculus. On the right-hand-side, remember that the integrand is independent of time, and therefore we can treat it as a constant.

$$\rho(\mathbf{r}, t) - \rho(\mathbf{r}, 0) = -\nabla \cdot \mathbf{J}(\mathbf{r})t \quad \rightarrow \quad \rho(\mathbf{r}, t) = \rho(\mathbf{r}, 0) - \nabla \cdot \mathbf{J}(\mathbf{r})t$$

We are almost there. The form on the right seems contrived because it is (you've given me the answer and I want whatever I'm doing to be guided by that form). So I've rearranged things to resemble the form we were asked to prove: $\rho(\mathbf{r},t) = \rho(\mathbf{r},0) + \dot{\rho}(\mathbf{r},0)t$. Now, we just have to clarify how $\dot{\rho}(\mathbf{r},0) = -\nabla \cdot \mathbf{J}(\mathbf{r})$. For that let's remember our earlier result:

$$\frac{\partial}{\partial t}\Big[\rho(\mathbf{r},t)\Big] = -\nabla \cdot \mathbf{J}(\mathbf{r})$$

If this is true for all t, then it is certainly true for $t = 0$:

$$\frac{\partial}{\partial t}\Big[\rho(\mathbf{r},0)\Big] = -\nabla \cdot \mathbf{J}(\mathbf{r}) \quad \rightarrow \quad \dot{\rho}(\mathbf{r},0) = -\nabla \cdot \mathbf{J}(\mathbf{r})$$

Let's make this substitution in $\rho(\mathbf{r},t) = \rho(\mathbf{r},0) - \nabla \cdot \mathbf{J}(\mathbf{r})t$.

$$\boxed{\rho(\mathbf{r},t) = \rho(\mathbf{r},0) - \nabla \cdot \mathbf{J}(\mathbf{r})t \quad \rightarrow \quad \rho(\mathbf{r},t) = \rho(\mathbf{r},0) + \dot{\rho}(\mathbf{r},0)t}$$

Part (b)

This is not an electrostatic or magnetostatic configuration; nevertheless-rather surprisingly-both Coulomb's law (in the form of Eq. 2.8) and the Biot-Savart law (Eq. 5.39) hold, as you can confirm by showing that they satisfy Maxwell's equations. In particular:

(b) Show that

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}') \times \hat{R}}{R^2} d\tau'$$

obeys Ampere's law with Maxwell's displacement current term. Part b will be considered as extra credit in case you want to turn-in this corrected version.

Solution

Note that since I don't want to install additional LaTeXpackages and switch to manual typesetting, I will use R as the separation vector $|R| =: |\mathbf{r} - \mathbf{r}'|$

As a reminder, the full Maxwell-Ampère law is

$$\nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r}) + \mu_0\epsilon_0 \frac{\partial}{\partial t}\Big[\mathbf{E}(\mathbf{r},t)\Big] \tag{2}$$

For this problem, we define the electric field with Coulomb's law:

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}',t)\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} d\tau' \tag{3}$$

As we showed in the first part, the charge density varies linearly in time and the current density is independent of time. Because ρ is changing, however, we have:

$$\nabla \cdot \mathbf{J} = -\frac{\partial}{\partial t}\rho$$

To account for this current, we have see the need for the additional term in Eq [2]: $\mu_0\epsilon_0 \frac{\partial}{\partial t}\Big[\mathbf{E}(\mathbf{r},t)\Big]$, which provides the missing contribution of current from the static version of Ampère's law.

Let's simplify the notation here with

$$\hat{R} =: \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \quad \text{and} \quad R =: |\mathbf{r} - \mathbf{r}'|$$

The magnetic field is

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') \times \frac{\hat{R}}{R^2} d\tau'$$

Taking the curl, we have

$$\nabla \times \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \nabla \times \left[\mathbf{J}(\mathbf{r}') \times \frac{\hat{R}}{R^2} \right] d\tau'$$

Let's let (hehe)

$$\mathbf{a} =: \mathbf{J}(\mathbf{r}') \quad \text{and} \quad \mathbf{A}(\mathbf{r}) =: \frac{\hat{R}}{R^2}$$

We remember the vector identity

$$\nabla \times (\mathbf{a} \times \mathbf{A}(\mathbf{r})) = \mathbf{a} \left(\nabla \cdot \mathbf{A}(\mathbf{r}) \right) - \left(\mathbf{a} \cdot \nabla \right) \mathbf{A}(\mathbf{r})$$

Since $\mathbf{a}$ is independent of $\mathbf{r}$, the second term of our vector identity vanishes and we are left with

$$\nabla \times (\mathbf{a} \times \mathbf{A}(\mathbf{r})) = \mathbf{a} \left(\nabla \cdot \mathbf{A}(\mathbf{r}) \right) \quad \rightarrow \quad \mathbf{J}(\mathbf{r}') \nabla \cdot \frac{\hat{R}}{R^2}$$

From PHYS 4031, we learned that

$$\nabla \cdot \frac{\hat{R}}{R^2} \quad \rightarrow \quad \nabla \cdot \left[\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right] = 4\pi \delta^3(\mathbf{r} - \mathbf{r}')$$

Therefore

$$\mathbf{J}(\mathbf{r}') \nabla \cdot \left[\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right] \quad \rightarrow \quad 4\pi \mathbf{J}(\mathbf{r}') \delta^3(\mathbf{r} - \mathbf{r}')$$

Making this substitution into integral given in the problem statement,

$$\boxed{\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int 4\pi \mathbf{J}(\mathbf{r}') \delta^3(\mathbf{r} - \mathbf{r}') d\tau' = \mu_0 \mathbf{J}(\mathbf{r})}$$

This is not the end of the problem, as we are not working with a time-independent configuration.

Continuing, we consider the time-dependent electric field term in Eq. (2):

$$\mu_0 \epsilon_0 \frac{\partial}{\partial t} \Big[\mathbf{E}(\mathbf{r}, t) \Big] = \mu_0 \epsilon_0 \frac{1}{4\pi\epsilon_0} \int \underbrace{\frac{\partial}{\partial t} \Big[\rho(\mathbf{r}', t) \Big]}_{\text{cont. eq.}} \frac{\hat{R}}{R^2} d\tau'. \tag{4}$$

Since the charge density ρ changes in time but $\mathbf{J}$ and ρ obey the continuity equation, we have

$$\frac{\partial \rho(\mathbf{r}', t)}{\partial t} = -\nabla' \cdot \mathbf{J}(\mathbf{r}').$$

Making this substitution,

$$\mu_0\epsilon_0\frac{1}{4\pi\epsilon_0}\int\big[\nabla'\cdot\mathbf{J}(\mathbf{r}')\big]\frac{\hat{R}}{R^2}d\tau' \quad\longrightarrow\quad -\mu_0\epsilon_0\frac{1}{4\pi\epsilon_0}\int\big[\nabla'\cdot\mathbf{J}(\mathbf{r}')\big]\frac{\hat{R}}{R^2}d\tau'.$$

Let's define

$$f(\mathbf{r}') := \frac{\hat{R}}{R^2} \quad \text{and} \quad \mathbf{J}(\mathbf{r}') \quad \text{as our vector field.}$$

Then, using integration by parts (and assuming that any surface term vanishes at infinity):

$$\int_V\big[\nabla'\cdot\mathbf{J}(\mathbf{r}')\big]f(\mathbf{r}')d\tau' = \underbrace{\oint_S f\mathbf{J}\cdot d\mathbf{a}'}_{\text{vanishes}} - \int_V \mathbf{J}(\mathbf{r}')\cdot\big[\nabla' f(\mathbf{r}')\big]d\tau'.$$

So,

$$-\mu_0\epsilon_0\frac{1}{4\pi\epsilon_0}\int\nabla'\cdot\mathbf{J}(\mathbf{r}')\frac{\hat{R}}{R^2}d\tau' = \mu_0\epsilon_0\frac{1}{4\pi\epsilon_0}\int\mathbf{J}(\mathbf{r}')\cdot\nabla'\Big[\frac{\hat{R}}{R^2}\Big]d\tau'.$$

Putting it all together, we see that the total curl of $\mathbf{B}$ is:

$$\nabla\times\mathbf{B}(\mathbf{r}) = \mu_0\mathbf{J}(\mathbf{r}) + \mu_0\epsilon_0\frac{\partial}{\partial t}\Big[\mathbf{E}(\mathbf{r},t)\Big].$$

When we include the extra term arising from $\partial\rho/\partial t$ (and thus from $\partial\mathbf{E}/\partial t$), the "missing piece" is exactly the displacement current term $\mu_0\epsilon_0\partial\mathbf{E}/\partial t$. This confirms that a Biot–Savart style **B** field *does* satisfy Ampère's law *including* Maxwell's correction, provided ρ and **J** obey the continuity equation.

$$\boxed{\nabla\times\mathbf{B}(\mathbf{r}) = \mu_0\mathbf{J}(\mathbf{r}) + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}(\mathbf{r},t)}$$
