χ^2 Tests

Grinnell College

November 20, 2024

Review

What we did before, we will do today:

1. Construct a null hypothesis, H_0
2. Collect data and compute our statistic (i.e., $\overline{x}$)
3. Evaluate that statistic in the context of a null distribution, i.e.,

$$t = \frac{\overline{X} - \mu_0}{\hat{\sigma}/\sqrt{n}}$$

4. Reject or fail to reject hypothesis
    - Type I errors
    - Type II errors

Problem

Suppose I am interested in writing an exam with 400 questions, with each question having as possible answers the letters A-E

Rather than choose the solution one-by-one, I randomly assign them for each question with equal probability

Was my random assignment effective for meeting my goals?

	A	B	C	D	E
Expected	80	80	80	80	80
Observed	74	90	76	87	73

Goodness of Fit

The χ^2 (chi squared or "kai" squared) **goodness of fit** test allows us to compared *expected* proportions in k groups against those we *observe*

$$\chi^2 = \sum_{i=1}^{k} \frac{(\text{Expected}_i - \text{Observed}_i)^2}{\text{Expected}_i}$$

Under the null hypothesis, for k groups, the χ^2 goodness of fit test statistic follows a χ^2 distribution with $k-1$ degrees of freedom

$$\chi^2 \sim \chi^2(k-1)$$

	A	B	C	D	E
Expected	80	80	80	80	80
Observed	74	90	76	87	73

$$\chi^2 = \sum_{i=1}^{k} \frac{(\text{Expected}_i - \text{Observed}_i)^2}{\text{Expected}_i}$$

$$= \frac{(74-80)^2}{80} + \frac{(90-80)^2}{80} + \frac{(76-80)^2}{80} + \frac{(87-80)^2}{80} + \frac{(73-80)^2}{80}$$

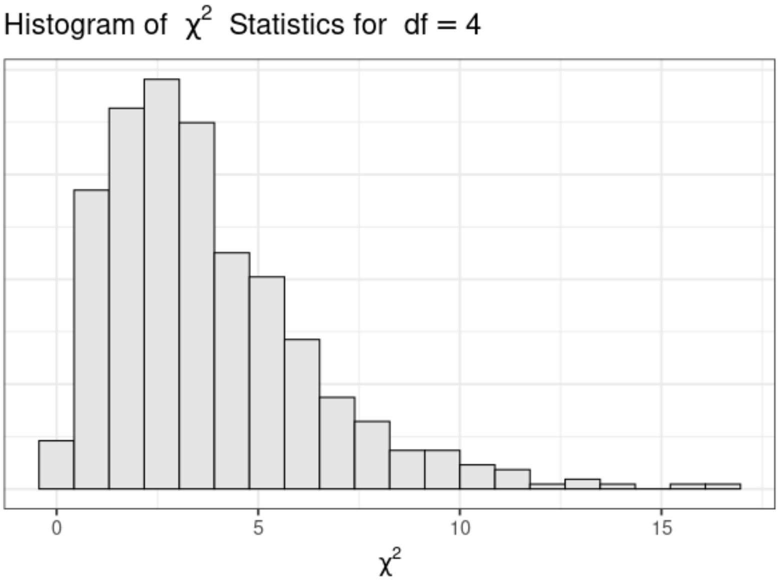
$$= 3.125$$

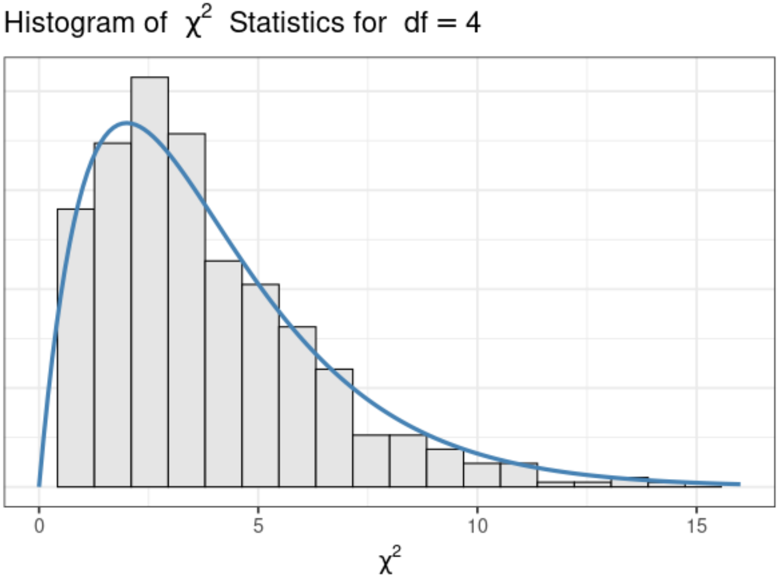
Samples

	A	B	C	D	E
Sample 1	86	68	91	67	88
Sample 2	85	73	81	75	86
Sample 3	79	85	81	73	82
Sample 4	97	87	72	70	74
Sample 5	88	85	73	85	69
Sample 6	85	84	77	83	71
Sample 7	86	69	86	80	79
Sample 8	85	68	72	83	92
Sample 9	76	76	92	75	81
Sample 10	78	83	79	74	86

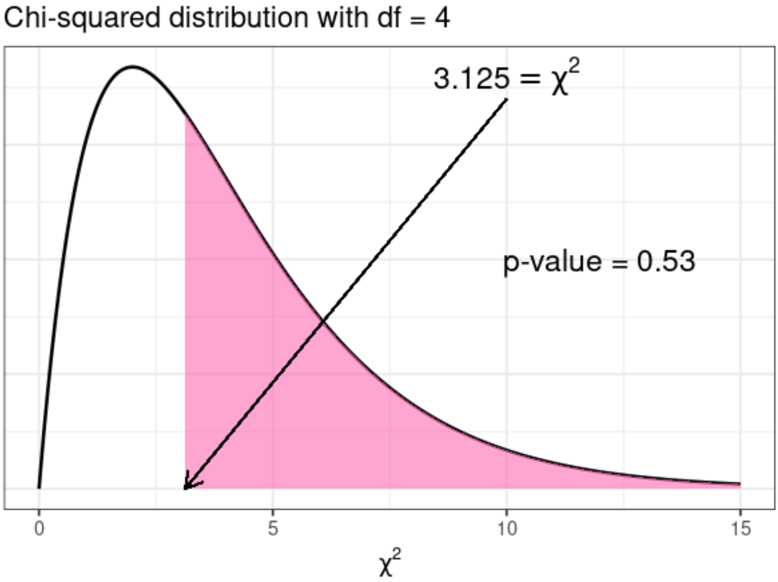
Samples

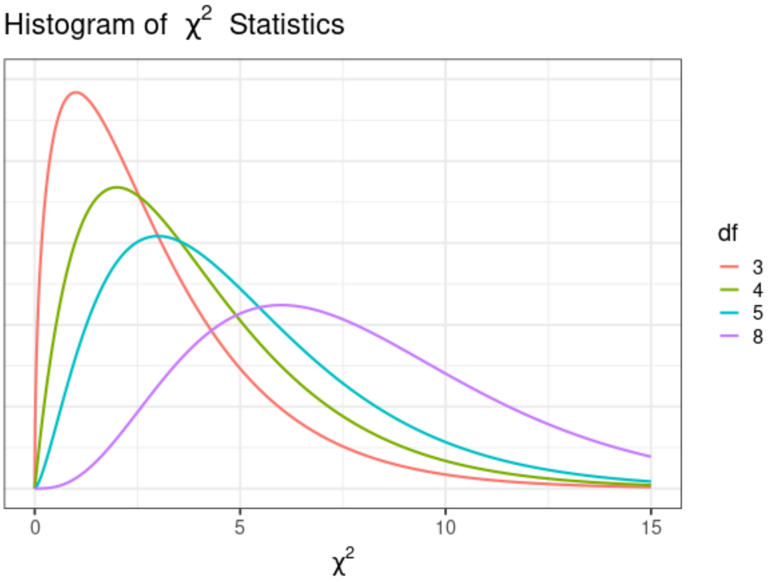
	A	B	C	D	E	χ^2
Sample 1	86	68	91	67	88	6.67
Sample 2	85	73	81	75	86	1.70
Sample 3	79	85	81	73	82	1.00
Sample 4	97	87	72	70	74	6.72
Sample 5	88	85	73	85	69	3.55
Sample 6	85	84	77	83	71	1.75
Sample 7	86	69	86	80	79	2.42
Sample 8	85	68	72	83	92	4.83
Sample 9	76	76	92	75	81	2.52
Sample 10	78	83	79	74	86	1.07


Histogram of χ² Statistics for df = 4
0
5
10
15
χ²


Histogram of χ² Statistics for df = 4
0
5
10
15
χ²

p-value for exam questions




Histogram of χ2 Statistics
0
5
10
15
χ2
df
3
4
5
8

p-value for χ^2

$$\chi^2 = \sum_{i=1}^{k} \frac{(\text{Expected}_i - \text{Observed}_i)^2}{\text{Expected}_i}$$

A few things to note about this statistic:

- It's always positive (or equal to zero)
- The more our observed values deviate from our expected, the larger it gets

From this, we get two facts:

- Our *p*-value is computed as the area *to the right* of our test statistic
- Greater values of χ^2 indicate more evidence against the null hypothesis

Example

Prospective jurors are supposed to be randomly chosen from the eligible adults in a community. The American Civil Liberties Union (ACLU) studied the racial composition of the jury pools in 10 trials in Alameda County, California. Display below is the racial and ethnict composition of the $n = 1,453$ individuals included in the jury pools, along with the distribution of eligivle jurors according to US Census data:

Race Ethnicity	White	Black	Hispanic	Asian	Other	Total
Jury Size	780	117	114	384	58	1453
Census Percentage	54%	18%	12%	15%	1%	100%