Strength of Evidence

Grinnell College

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Warm-up

The culmen (upper bill) of 20 male sharp-shinned hawks were sampled and measured, generating the following statistics:

$$\overline{x} = 10.81, \quad \hat{\sigma} = 3.93$$

With this information, respond to the following:

1. Construct a 90% confidence interval for the average culmen length of male sharp-shinned hawks
2. Construct a t-statistic for the culmen under the null hypothesis $H_0 : \mu = 12.5$
3. Is the value of this hypothesis contained within your 90% CI?
4. Compare your *t*-statistic to the critical value found in (1). Based on this, what conclusion would you come to with your hypothesis?

Review

Monday we introduced the idea of the **null distribution**

If we were to collect many samples of $\overline{X}$, the null distribution refers to the distribution of statistics

$$t = \frac{\overline{X} - \mu_0}{\hat{\sigma}/\sqrt{n}}$$

when $H_0 : \mu = \mu_0$ is true.

Consider the t

Consider the pieces of a t-statistic

$$t = \frac{\overline{x} - \mu_0}{\hat{\sigma}/\sqrt{n}}$$

1. $\overline{x} - \mu_0$ indicates the distance between my observed data and my null hypothesis, though we cannot use this alone as it does not include a degree of "certainty" associated with $\overline{x}$
2. $\hat{\sigma}$ is my estimate of the population's standard deviation. When this is large, there will be more uncertainty in my estimate of $\overline{x}$
3. n represents the number of observations in my sample – the more observations I have, the more confidence I will have in my estimate

We should have a sense of how each of these components impact my t-statistic

Consider the t

$$t = \frac{\overline{x} - \mu_0}{\hat{\sigma}/\sqrt{n}}$$

We can think, then, of the t-statistic as being a measure of evidence *against* the null hypothesis

If:

1. $\overline{x}$ is far from μ and
2. Our certainty in $\overline{x}$ is high (i.e., low $\hat{\sigma}$ or large n)

then our statistic t will be larger. A larger t statistic is less likely than a smaller one

What we consider “large” will depend on the t-distribution

t-distribution

When the null hypothesis is true,

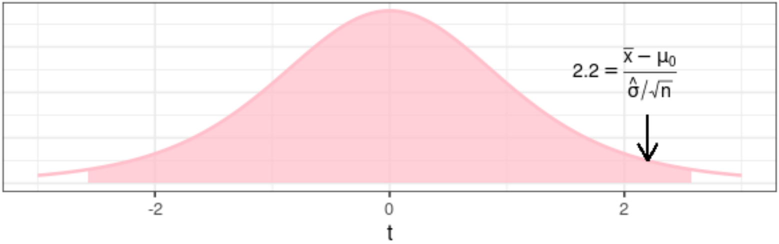
$$t = \frac{\overline{x} - \mu_0}{\hat{\sigma}/\sqrt{n}}$$

follows a t-distribution with $n - 1$ degrees of freedom

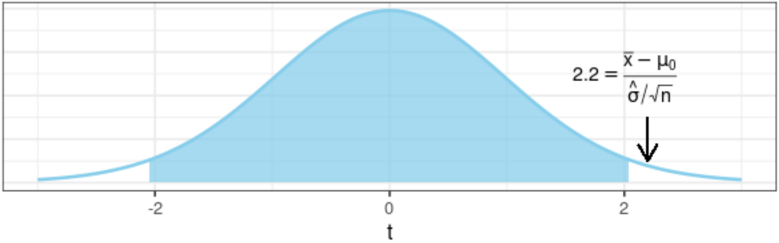
The degrees of freedom tells us, relatively speaking, what values are considered “large”

$t = 2.2$ may be considered “large” when $df = 30$ but not when $df = 5$

95% of a t-distribution with df = 5



95% of a t-distribution with df = 30



In summary

The process goes like this:

1. Assume our null hypothesis $H_0 : \mu = \mu_0$ is true
2. Compute a *t*-statistic with our observed data
3. Ask: is this *t*-statistic “large”?
   - This will depend on the degrees of freedom
   - It will also depend on what range we consider acceptable, i.e., 80%, 95%, 99%, etc.,
4. Either reject or fail to reject depending on how our *t* statistics compares to the relevant critical value

What we would like now is a way to *quantify* the strength of our evidence without comparing our statistic to a particular critical value

p-values

Where *critical values* allow us to move from percentiles to a value, a **p-value** takes us from a value to a percentile

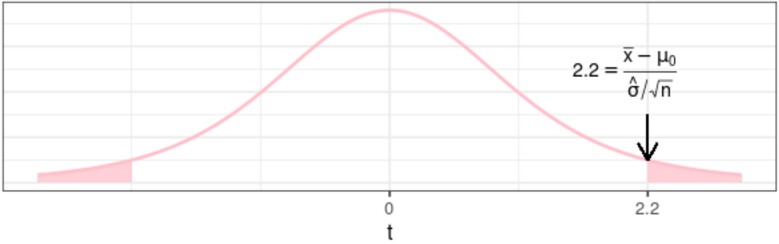
More specifically, a p-values asks: "If the null hypothesis ($H_0 : \mu = \mu_0$) is true, what is the probability that we have observed our data *or something at least as large*"

The proportion of our null distribution at least as large as our observed data is expressed as a probability which we call the p-value:

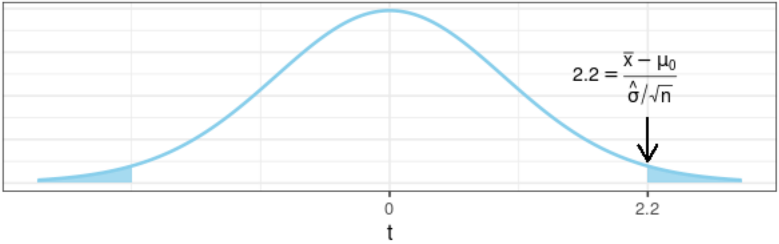
$$\text{p-value} = P(\text{observed data} \mid H_0 \text{ is true})$$

pvalue stuff

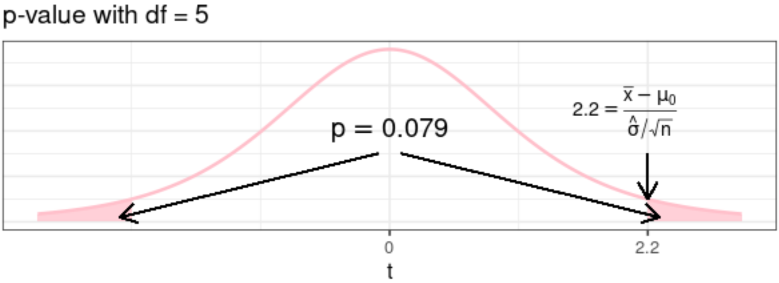
p-value with df = 5

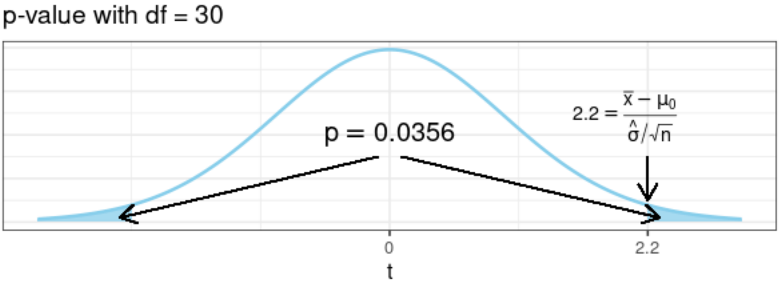


p-value with df = 30



pvalue stuff





pvalue

Just as our *t*-statistic is based on the null hypothesis, so, too, is our p-value

The sampling distribution under the null hypothesis is called our **null distribution**

The p-value, then, is an indication of how likely (or unlikely) our observed data is *under the assumption that the null hypothesis is true*

If we were to have a different null hypothesis, our *p*-value would be different as well

P-values and decision making

Last week we saw that making a decision by comparing our t-statistic with, say, a 90% critical value was associated with a 10% **error rate**

Likewise, comparing our t-statistic with a 95% confidence interval provides us with a 5% error rate

This error rate, called the **Type I Error Rate**, expresses the probability of incorrectly rejecting the null hypothesis when H_0 is true

We represent this error rate with the Greek letter α. So, for a 95% confidence interval, we are conducting a hypothesis test with $\alpha = 0.05$

p-values and decision making

For a given error rate α, there is an equivalence between checking p-values and comparing test statistics with the associated critical values:

1. $$p < \alpha \quad \Leftrightarrow \quad C < |t|$$
2. $$p > \alpha \quad \Leftrightarrow \quad |t| < C$$

Whereas checking $|t| < C$ gives us a binary decision, assessing the p-value allows us to quantify the strength of our evidence

Example

Suppose we are interested in testing the hypothesis that the true average hallux length for male sharp-shinned hawks is 11mm. To this end, we collected two samples:

- **Sample 1:** $\overline{x}_1 = 15.61$, $\hat{\sigma} = 6.72$, $n = 10$
- **Sample 2:** $\overline{x}_2 = 13.61$, $\hat{\sigma} = 6.12$, $n = 25$

We might notice in passing that:

1. Sample 1 has an observed sample mean that is further away from μ_0
2. Sample 2 has more than double the observations as Sample 1
3. The observed variability in both samples is about the same

t-statistics and p-values

We can start by constructing t-statistics for each of our samples

Sample 1:

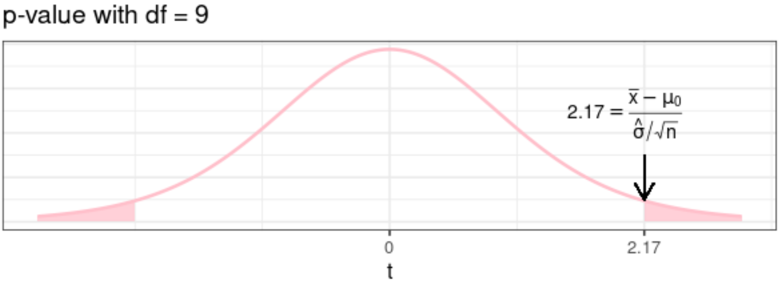
$$t = \frac{15.61 - 11}{6.72/\sqrt{10}} = 2.17$$

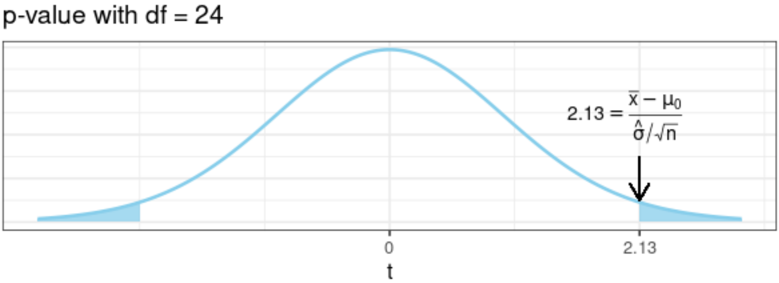
Sample 2:

$$t = \frac{13.61 - 11}{6.12/\sqrt{25}} = 2.13$$

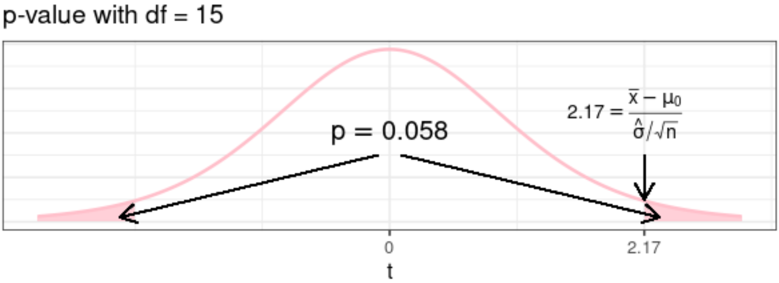
We might conclude that, having the larger t-statistic that Sample 1 provides more evidence against the null; however, the *null distribution* for each statistic is different according to its degrees of freedom

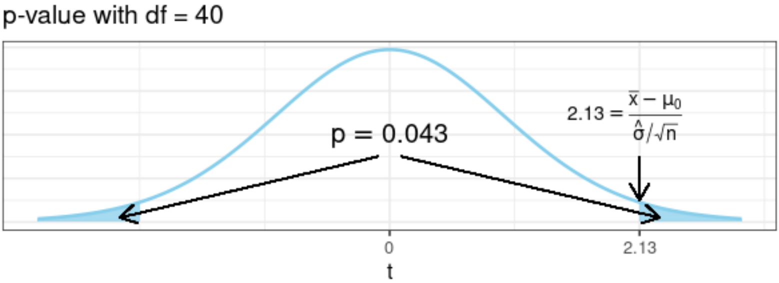
p-values


p-value with df = 9
$2.17 = \frac{\bar{x} - \mu_0}{\hat{\sigma}/\sqrt{n}}$
0
2.17
t


p-value with df = 24
$2.13 = \frac{\bar{x} - \mu_0}{\hat{\sigma}/\sqrt{n}}$
0
2.13
t

p-values


p-value with df = 15
$2.17 = \frac{\overline{x} - \mu_0}{\hat{\sigma}/\sqrt{n}}$
p = 0.058
0
2.17
t


p-value with df = 40
$2.13 = \frac{\overline{x} - \mu_0}{\hat{\sigma}/\sqrt{n}}$
p = 0.043
0
2.13
t

Drawing Conclusions

Suppose we were wishing to test our hypothesis $H_0 : \mu = 11$ with each of our two samples with a Type I error rate of $\alpha = 0.05$

Sample 1:
With a t statistic of $t = 2.17$ following a null distribution with $df = 9$, we find a p-value of $p = 0.058$. Since $p > \alpha$, we fail to reject our null hypothesis

Sample 2:
With a t statistic of $t = 2.13$ following a null distribution with $df = 24$, we find a p-value of $p = 0.043$. Since $p < \alpha$, we reject our null hypothesis

Relationship Between Critical Values and α

As we noted before, we can equivalently check the test statistics for each sample against the appropriate critical values for each null distribution's degrees of freedom:

$$C_9 = 2.262 \qquad C_{24} = 2.063$$

Immediately, we see that for our first sample, $t = 2.17 < C_9$, telling us that our observed data is within the middle 95% and we would fail to reject

Likewise for our second sample, $t = 2.13 > C_{24}$, indicating that we *would reject*

Relationship between CI and α

Finally consider the confidence intervals themselves:

Sample 1:

$$15.61 \pm 2.262 \times \left(6.72/\sqrt{10}\right) = (10.8, 20.4)$$

Sample 2:

$$13.61 \pm 2.063 \times \left(6.12/\sqrt{25}\right) = (11.1, 16.2)$$

Here, we see that the null hypothesis $H_0 : \mu = 11$ is contained within the 95% confidence interval of Sample 1, indicating that we fail to reject, while it is *not* within the interval for Sample 2, indicating rejection

A Note

p-values are useful in that they allow us to compare the size of different t-statistics, relative to the null distribution

In the special case where two samples have the same sample size, we can simply compare the t-statistics directly. Why?

p-value facts

p-values are notorious for how easily they are misinterpreted. Here are a few key facts:

- A *p*-value *is not* the probability that the null hypothesis is false
- A *p*-value *is not* the probability of having observed our data by random chance
- A *p*-value *does not* tell us the magnitude of difference $(\overline{x} - \mu_0)$ or the size of the effect
- A *p*-value *must* be taken in the context of the study; a *p*-value of 0.05 is completely arbitrary
- A *p*-value *is* a probabilistic statement relating observed data to a hypothesis

Review

Hypothesis testing involves formulating statements about our population and then checking the consistency of our hypothesis with observed data

Rather than getting a binary yes/no answer by checking t-statistics with a specific critical value, a **p-value** allows us to *quantify* to what extent our observed data is consistent with a null hypothesis

There is a one-to-one relationship between critical values and our Type I error rate, α

Checking that $t < C$ is equivalent to checking if $p < \alpha$