Exercise 42. Let T be a primitive lattice triangle. If $\mathbf{v}$ and $\mathbf{w}$ are vectors corresponding to adjacent sides of T, show that $\mathbf{v}$ and $\mathbf{w}$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^2$.

Solution. Our approach to this problem involves creating a primitive lattice parallelogram based on the vectors $\mathbf{v}$ and $\mathbf{w}$, and then using exercise 41 to conclude $\mathbf{v}$ and $\mathbf{w}$ form a $\mathbb{Z}$-basis for $\mathbb{Z}^2$. Consider the parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$, whose vertices are $(0,0), (v_1, v_2), (w_1, w_2)$, and $(v_1 + w_1, v_2 + w_2)$. We will now show $P(\mathbf{v}, \mathbf{w})$ is primitive.

The additional area of the parallelogram added on to T is simply a copy of T which has been rotated about the midpoint of the far side of the triangle. We wish to show that the transformations required to relate the two shapes preserve the lattice and actually send points to the correct shape.

Consider the map

$$\begin{aligned} &\varphi : \mathbb{R}^2 \to \mathbb{R}^2 \\ &(x, y) \mapsto (x - \frac{1}{2}(v_1 + w_1), y - \frac{1}{2}(v_2 + w_2)). \end{aligned}$$

Note that this map is invertible, with inverse

$$\begin{aligned} &\varphi^{-1} : \mathbb{R}^2 \to \mathbb{R}^2 \\ &(x, y) \mapsto (x + \frac{1}{2}(v_1 + w_1), y + \frac{1}{2}(v_2 + w_2)). \end{aligned}$$

Consider also the linear map given by the matrix

$$\begin{aligned} R &: \mathbb{R}^2 \to \mathbb{R}^2 \\ R &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

The map which will represent our transformation is $T = \varphi^{-1} R \varphi$. What in the world? Well, basically we want to translate the parallelogram so that the midpoint of the diagonal of the parallelogram is the origin, then we want to use Linear Algebra to rotate the plane by π radians, then we want translate the figure back to its original position. Let the upper triangle be herein referred to as T_2, and the original triangle as T_1. We need to show two things:

1. A point $(x, y) \in \mathbb{Z}^2$ if and only if $T((x, y)) \in \mathbb{Z}^2$. This is not immediately obvious since we are adding non-integers to integers in our maps.

2. A point $(x, y) \in T_2$ if and only if $T((x, y)) \in T_1$. Chad, you might say, why not set up the maps so that you start in T_1 and end in T_2? Well, curious reader, I didn't think of that until now, and I don't really want to rework the problem.

We will begin by proving 1. Let $(x, y) \in \mathbb{Z}^2$. A quick computation shows that $T((x, y)) = (v_1 + w_1 - x, v_2 + w_2 - y)$. Now v_1, w_1, v_2, and w_2 are all integers since they are coordinates of the vertices of our starting lattice triangle. The integers are closed under addition and subtraction, so $T((x, y)) \in \mathbb{Z}^2$.

Conversely, suppose that $T((x, y)) = (v_1 + w_1 - x, v_2 + w_2 - y) \in \mathbb{Z}^2$. Choose $t \in \mathbb{Z}$ such that $v_1 + w_1 - x = t$. So $x = v_1 + w_1 - t \in \mathbb{Z}$. Similarly, $y \in \mathbb{Z}$. Thus $(x, y) \in \mathbb{Z}^2$.

We will now prove 2. Suppose $(x, y) \in T_2$. By linear algebra and calculus, $T_2 = \{a\mathbf{v} + b\mathbf{w} : 0 \leq a, b \leq 1 \text{ and } 1 \leq a+b \leq 2\}$. Also, $T_1 = \{a\mathbf{v}+b\mathbf{w} : 0 \leq a, b \leq 1 \text{ and } a+b \leq 1\}$. Choose $a, b \in \mathbb{R}$ such that $(x, y) = (a\mathbf{v}+b\mathbf{w})$. We will consider solely the first coordinate, since the statements concerning the second follow similarly. Recall, the first coordinate of $T((x, y))$ is $v_1 + w_1 - x$. Combining this with our expression for x,

$$\begin{aligned} T((x, y)) = v_1 + w_1 - x &= v_1 + w_1 - av_1 + w_1 \\ &= (1 - a)v_1 + (1 - b)w_1. \end{aligned}$$

Now

$$\begin{aligned} 0 &\leq a \leq 1 \\ 0 &\geq -a \geq -1 \\ 1 &\geq 1 - a \geq 0. \end{aligned}$$

Similarly, $0 \leq 1 - b \leq 1$. Also,

$$\begin{aligned} 1 &\leq a + b \leq 2 \\ -1 &\geq -a - b \geq -2 \\ 1 &\geq 1 - a + 1 - b \geq 0. \end{aligned}$$

Thus $T((x, y)) \in T_1$.

Conversely, suppose that $T((x, y)) = (v_1 + w_1 - x, v_2 + w_2 - y) \in T_1$. Again, we will focus solely on the first coordinate. Choose $a, b \in \mathbb{R}$ such that $v_1 + w_1 - x = av_1 + bw_1$. So $x = (1-a)v_1 + (1-b)w_1$. Now

$$\begin{aligned} 0 &\leq a \leq 1 \\ 0 &\geq -a \geq -1 \\ 1 &\geq 1 - a \geq 0. \end{aligned}$$

Similarly, $0 \leq 1 - b \leq 1$. Also,

$$\begin{aligned} 0 &\leq a + b \leq 1 \\ 0 &\geq -a - b \geq -1 \\ 2 &\geq 1 - a + 1 - b \geq 1. \end{aligned}$$

Thus $(x, y) \in T_2$. We have proven 1 and 2.

Suppose for the sake of contradiction that $T = T_1$ is a primitive lattice triangle, but that $P(\mathbf{v}, \mathbf{w})$ is not a primitive lattice parallelogram. Since T is primitive, there is a non-vertex point $(x, y) \in T_2$ such that $(x, y) \in \mathbb{Z}^2$. By 1 and 2, there must be a corresponding lattice point in T_1. But T_1 is primitive. Thus T_1 is primitive, and T_1 contains a lattice point other than its vertices. We have a contradiction, and thus $P(\mathbf{v}, \mathbf{w})$ is primitive. Thus, by 41, $\mathbf{v}$ and $\mathbf{w}$ form a $\mathbb{Z}$-basis for $\mathbb{Z}^2$.

$\square$