ECEN5463 | EX 1

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Problem 1

FitzHugh-Nagumo model:

$$\dot{u} = u - \frac{1}{3}u^3 - w + I,$$

$$\dot{w} = c(b_0 + b_1 u - w)$$

Where $c, b_0, b_1 > 0$

```
syms u w I c b0 b1

f = [u-u^3/3-w+I; c*(b0+b1*u-w)];

jac = jacobian(f,[u,w])
```

jac =

$$\begin{pmatrix} 1-u^2 & -1 \\ b_1\, c & -c \end{pmatrix}$$

1-1

Let $c = 0.1$, $b_0 = 2$, and $b_1 = 1.5$. For each of the values $I = 0$ and $I = 2$

```
jac1 = subs(jac, [b1,c], [1.5,0.1]);
f11 = subs(f, [c,b0,b1,I], [0.1,2,1.5,0]);
f12 = subs(f, [c,b0,b1,I], [0.1,2,1.5,2]);
```

1-1-1 Use linearization to find all equiplibrium points and determine their types

```
sol = solve(f11==0, 'real', true, 'ReturnConditions', true);
eq_points11 = vpa([sol.u, sol.w],2)
```

eq_points11 = $\begin{pmatrix} -1.5 & -0.32 \end{pmatrix}$

```
jac11 = cell(size(eq_points11,1),1);
eig11 = cell(size(eq_points11,1),1);

for i = 1:size(eq_points11,1)
```

```
    jac11{i} = subs(jac1, u, eq_points11(i,1));
    eig11{i} = eig(jac11{i});
    disp(vpa(eig11{i},2));
end
```

$$\begin{pmatrix} -0.23 \\ -1.3 \end{pmatrix}$$

```
sol = solve(f12==0, 'real', true, 'ReturnConditions', true);
eq_points12 = vpa([sol.u, sol.w],2)
```

eq_points12 = $\begin{pmatrix} 0 & 2.0 \end{pmatrix}$

```
jac12 = cell(size(eq_points12,1),1);
eig12 = cell(size(eq_points12,1),1);

for i = 1:size(eq_points12,1)
    jac12{i} = subs(jac1, u, eq_points12(i,1));
    eig12{i} = eig(jac12{i});
    disp(vpa(eig12{i},2));
end
```

$$\begin{pmatrix} 0.059 \\ 0.84 \end{pmatrix}$$

Case $I = 0$:

$\lambda_1$is purely negative and real, so $x = [-0.23, -1.3]^T$ is a stable node

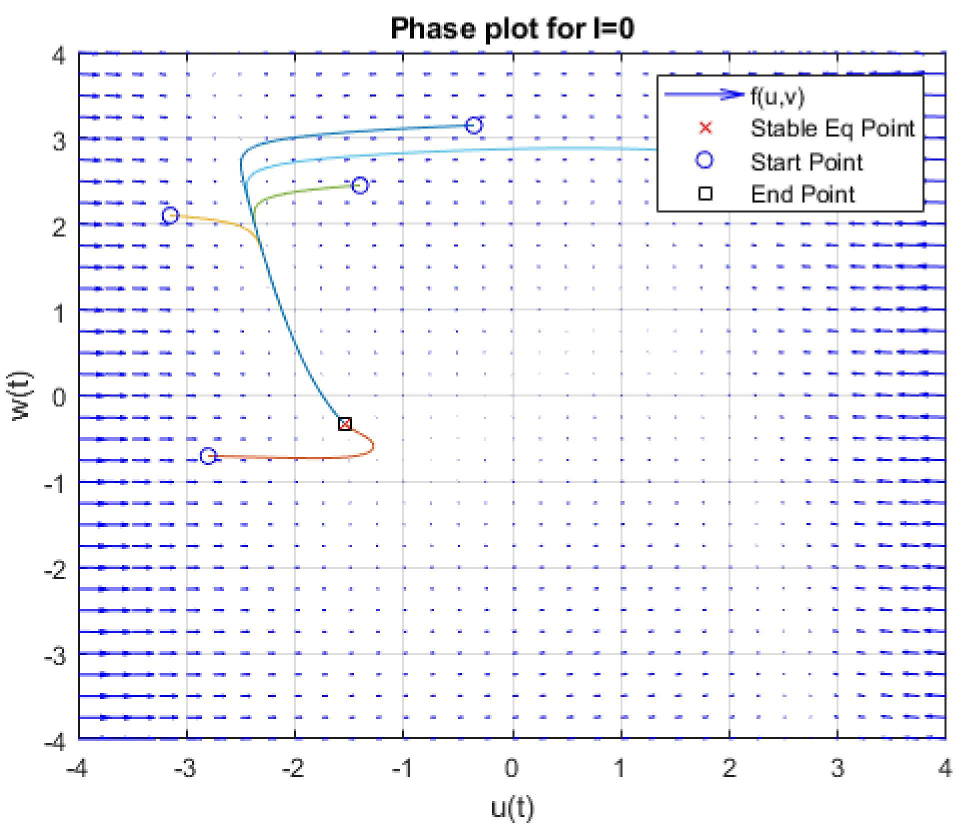
Case $I = 2$:

λ_1 is purely positive and real, so $x = [0, 2]^T$ is an unstable node

1-1-2 Construct the 2D phase portrait on [-4 4] x[-4 4]

```
% P = [I;c;b0;b1]
g = @(t,x,P) [x(1)-x(1).^3./3-x(2)+P(1); P(2)*(P(3)+P(4)*x(1)-x(2))];

figure()
params = [0,0.1,2,1.5];
[ut,wt] = meshgrid(-4:0.25:4, -4:0.25:4);

du = zeros(size(ut));
dw = zeros(size(ut));
for i=1:numel(ut)
    fprime = g(0,[ut(i);wt(i)], params);
    du(i) = fprime(1);
    dw(i) = fprime(2);
end
quiver(ut,wt,du,dw, 'b')

grid on
```

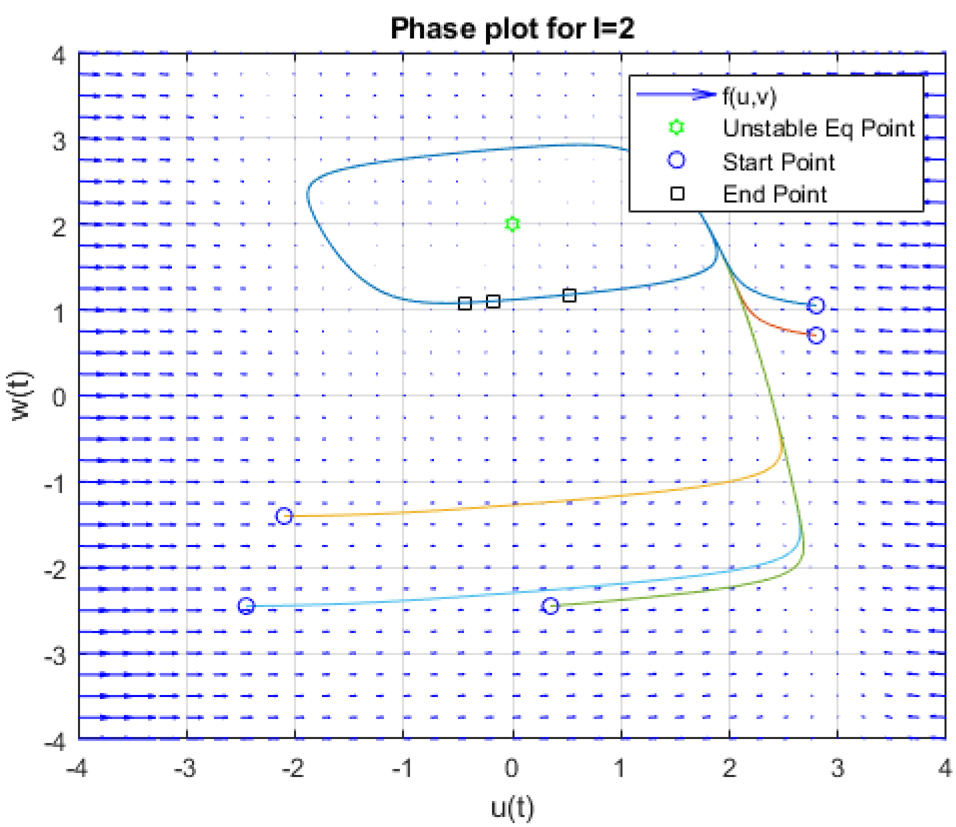
```
ylim([-4,4])
xlim([-4,4])
xlabel("u(t)")
ylabel("w(t)")
title("Phase plot for I=0")
hold on
plot(eq_points11(1,1),eq_points11(1,2), "rx")


for i = 1:5
    y0 = 0.35*(randi(20,2,1) - 10);
    [ts,ys] = ode45(@(t,y) g(t,y,params),[0,50],[y0]);
    plot(ys(:,1),ys(:,2))
    plot(ys(1,1),ys(1,2),'bo') % starting point
    plot(ys(end,1),ys(end,2),'ks') % ending point
end

legend("f(u,v)", "Stable Eq Point", "", "Start Point", "End Point")
```



This equilibrium point is behaving as a stable node. There are no oscillatory components in the trajectories, which indicate there is no imaginary component. All trajectories converge, which indicate the real component of each eigen value is negative.

```
% P = [I;c;b0;b1]
figure()
params = [2,0.1,2,1.5];

du = zeros(size(ut));
```

```
dw = zeros(size(ut));
for i=1:numel(ut)
    fprime = g(0,[ut(i);wt(i)],params);
    du(i) = fprime(1);
    dw(i) = fprime(2);
end
quiver(ut,wt,du,dw, 'b')

grid on
ylim([-4,4])
xlim([-4,4])
xlabel("u(t)")
ylabel("w(t)")
title("Phase plot for I=2")
hold on
plot(eq_points12(1,1),eq_points12(1,2), "gh")


for i = 1:5
    y0 = 0.35*(randi(20,2,1) - 10);
    [ts,ys] = ode45(@(t,y) g(t,y,params),[0,50],[y0]);
    plot(ys(:,1),ys(:,2))
    plot(ys(1,1),ys(1,2),'bo') % starting point
    plot(ys(end,1),ys(end,2),'ks') % ending point
end

legend("f(u,v)", "Unstable Eq Point", "", "Start Point", "End Point")
```



While the linearization shows that there is an unstable node, the phase plot indicates that the equilibrium point behaves more a like a center. The imaginary components are not indicated in the linearization as the oscillations occur too far away from the equilibrium point, which locally behaves like a node.

1-1-3

The phase plot for case ($I = 0$) largely agrees with the linearization analysis, which both show the equilibrium point behaving as a stable node.

There is a disagreement in the second case ($I = 2$). While the linearization coorectly predicted the unstability of the equilibrium point, the eigen values had no imaginary components. The phase plot clearly indicates an oscillatory behavior in the trajectory, which makes the equilibrium point appear to be a center. The discrepancy is due to the linearization breaking down as [u,x] moves away from the equiplibrium point.

1-2

Let $c = 0.1$, $b_0 = 2$, and $b_1 = 0.5$. For each of the values $I = 0$ and $I = 2$

```
jac2 = subs(jac, [b1,c], [0.5,0.1]);
f21 = subs(f, [c,b0,b1,I], [0.1,2,0.5,0]);
f22 = subs(f, [c,b0,b1,I], [0.1,2,0.5,2]);
```

1-2-1 Use linearization to find all equiplibrium points and determine their types

```
sol = solve(f21==0, 'real', true, 'ReturnConditions', true);
eq_points21 = vpa([sol.u, sol.w],2)
```

eq_points21 = $\begin{pmatrix} -2.1 & 0.95 \end{pmatrix}$

```
jac21 = cell(size(eq_points21,1),1);
eig21 = cell(size(eq_points21,1),1);

for i = 1:size(eq_points21,1)
    jac21{i} = subs(jac2, u, eq_points21(i,1));
    eig21{i} = eig(jac21{i});
    disp(vpa(eig21{i},2));
end
```

$$\begin{pmatrix} -0.12 \\ -3.4 \end{pmatrix}$$

```
sol = solve(f22==0, 'real', true, 'ReturnConditions', true);
eq_points22 = vpa([sol.u, sol.w],2)
```

eq_points22 =

$$\begin{pmatrix} 0 & 2.0 \\ -1.2 & 1.4 \\ 1.2 & 2.6 \end{pmatrix}$$

```
jac22 = cell(size(eq_points22,1),1);
eig22 = cell(size(eq_points22,1),1);

for i = 1:size(eq_points22,1)
    jac22{i} = subs(jac2, u, eq_points22(i,1));
    eig22{i} = eig(jac22{i});
    disp(vpa(eig22{i},2));
end
```

$$\begin{pmatrix} -0.052 \\ 0.95 \end{pmatrix}$$

$$\begin{pmatrix} -0.3 + 0.1\,\mathrm{i} \\ -0.3 - 0.1\,\mathrm{i} \end{pmatrix}$$

$$\begin{pmatrix} -0.3 + 0.1\,\mathrm{i} \\ -0.3 - 0.1\,\mathrm{i} \end{pmatrix}$$

Case $I = 0$:

$\lambda_1$is purely negative and real, so $x = [-0.12, -3.4]^T$ is a stable node

Case $I = 2$:

λ_1 is purely real with both positive and negaitve components, so $x = [0, 2]^T$ is a saddle point

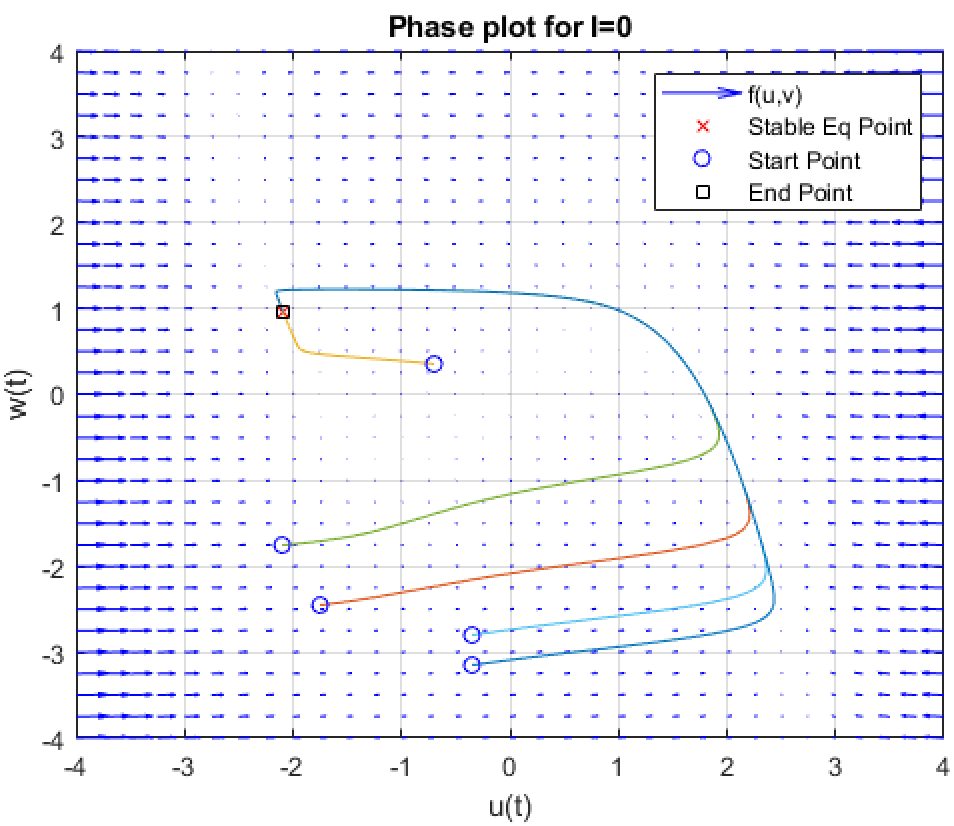
λ_2 is complex with negaitve real components, so $x = [-1.2, 1.4]^T$ is a stable spiral

λ_3 is complex with negative real components, so $x = [1.2,\ 2.6]^T$ is a stable spriral

1-2-2 Construct the 2D phase portrait on [-4 4] x[-4 4]

```
% P = [I;c;b0;b1]
g = @(t,x,P) [x(1)-x(1).^3./3-x(2)+P(1); P(2)*(P(3)+P(4)*x(1)-x(2))];

figure()
params = [0,0.1,2,0.5];

du = zeros(size(ut));
dw = zeros(size(ut));
for i=1:numel(ut)
    fprime = g(0,[ut(i);wt(i)], params);
    du(i) = fprime(1);
    dw(i) = fprime(2);
end
quiver(ut,wt,du,dw, 'b')
```

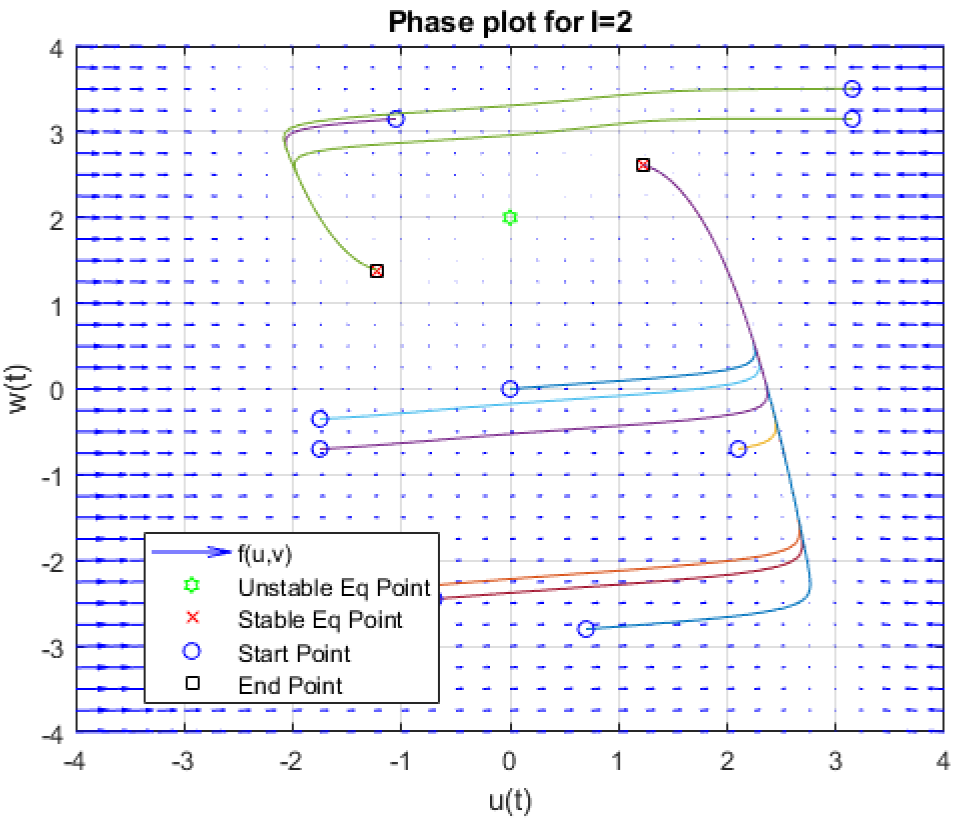
```
grid on
ylim([-4,4])
xlim([-4,4])
xlabel("u(t)")
ylabel("w(t)")
title("Phase plot for I=0")
hold on
plot(eq_points21(1,1),eq_points21(1,2), "rx")


for i = 1:5
    y0 = 0.35*(randi(20,2,1) - 10);
    [ts,ys] = ode45(@(t,y) g(t,y,params),[0,50],[y0]);
    plot(ys(:,1),ys(:,2))
    plot(ys(1,1),ys(1,2),'bo') % starting point
    plot(ys(end,1),ys(end,2),'ks') % ending point
end

legend("f(u,v)", "Stable Eq Point", "", "Start Point", "End Point")
```



This equilibrium point is behaving as a stable node. There are no oscillatory components in the trajectories, which indicate there is no imaginary component. All trajectories converge, which indicate the real component of each eigen value is negative.

```
% P = [I;c;b0;b1]
figure()
params = [2,0.1,2,0.5];
```

```
du = zeros(size(ut));
dw = zeros(size(ut));
for i=1:numel(ut)
    fprime = g(0,[ut(i);wt(i)],params);
    du(i) = fprime(1);
    dw(i) = fprime(2);
end
quiver(ut,wt,du,dw, 'b')

grid on
ylim([-4,4])
xlim([-4,4])
xlabel("u(t)")
ylabel("w(t)")
title("Phase plot for I=2")
hold on

plot(eq_points22(1,1),eq_points22(1,2), "gh")
plot(eq_points22(2,1),eq_points22(2,2), "rx")
plot(eq_points22(3,1),eq_points22(3,2), "rx")

for i = 1:10
    y0 = 0.35*(randi(20,2,1) - 10);
    [ts,ys] = ode45(@(t,y) g(t,y,params),[0,50],[y0]);
    plot(ys(:,1),ys(:,2))
    plot(ys(1,1),ys(1,2),'bo') % starting point
    plot(ys(end,1),ys(end,2),'ks') % ending point
end

legend("f(u,v)", "Unstable Eq Point", "Stable Eq Point","","", "Start Point", "End Point", "Loc
```



The phase plot show three equilibrium points: one saddle point and to stable points. The stable points do not exhibit much oscillation, appearing more similiar to stable nodes than stable spirals.

1-2-3

The phase plot for case $(I = 0)$ largely agrees with the linearization analysis, which both show the equilibrium point behaving as a stable node.

The phase plot for case $(I = 2)$ largely agrees with the linearization analysis, though the stable equilibrium points do not exhibit much oscillation. This indicates there is little to no imaginary component as [u,v] distances from the point. This being said, the linearization showed a weak imaginary component, suggesting that there wouldn't be much oscillation.

Problem 2

Consider the system

$$f(x) = \dot{x} = \begin{bmatrix} -\mathrm{ax}_1 + x_2 \\ \dfrac{x_1^2}{1 + x_1^2} - \mathrm{bx}_2 \end{bmatrix}$$, where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $a, b > 0$

2-1 Show that the nonnegative quadrant is positively invariant for all $a, b > 0$

Positive quadrant: $Q = \{x = [x_1, x_2]^T \in \mathbb{R}^2 | x_1 \geq 0, x_2 \geq 0\}$

Positive invariance: $f^T(x)n(x) \leq 0$, where $n(x)$ is the normal to the boundary at x

2 boundaries to the nonnegative quadrant: $\partial M_1(x_1) = [x_1, 0]^T$ and $\partial M_2 = [0, x_2]^T$.

Normals: $n_1(x_1) = [x_1, -1]^T$ and $n_2(x_2) = [-1, x_2]^T$

```
syms x1 x2 a b

f = [-a*x1 + x2; x1^2/(1+x1^2)-b*x2]
```

f =

$$\begin{pmatrix} x_2 - a\,x_1 \\ \dfrac{x_1{}^2}{x_1{}^2 + 1} - b\,x_2 \end{pmatrix}$$

```
n1 = [x1;-1];
n2 = [-1;x2];

f1n = transpose(f)*n1
```

f1n =

$$b\,x_2 - \frac{x_1{}^2}{x_1{}^2 + 1} + x_1\,(x_2 - a\,x_1)$$

Evaluate the normal at $\partial M_1 = [x_1, 0]^T$

```
f1n = simplify(subs(f1n, x2, 0))
```

f1n =

$$-a\,x_1{}^2 - \frac{x_1{}^2}{x_1{}^2 + 1}$$

```
solve(f1n, a)
```

ans =

$$-\frac{1}{x_1{}^2 + 1}$$

Assuming $a > 0$, then $f^T(x)n_1(x) \leq 0, x \in \partial M_1$

```
f2n = transpose(f)*n2
```

f2n =

$$a\,x_1 - x_2 - x_2\left(b\,x_2 - \frac{x_1{}^2}{x_1{}^2 + 1}\right)$$

Evaluate the normal at $\partial M_2 = [0, x_2]^T$

```
f2n = simplify(subs(f2n,x1,0))
```

f2n = $-b\,x_2{}^2 - x_2$

```
solve(f2n, b)
```

ans =

$$-\frac{1}{x_2}$$

Assuming $b > 0$, then $f^T(x)n_2(x) \le 0, x \in \partial M_2$

Once x enters the nonnegative quadrant ($x \in Q$) and $a, b > 0$, then $f(x) = \dot{x}$ remains positive on the boundaries of the quadrant. This implies that the system will never leave Q, thus suggesting that the nonnegative quadrant is positively invariant with respect to $f(x)$.

2-2 Find a condition on the parameters a and b such that the nonnegative quadrant contains multiple equilibria

```
sol = solve(f==0, [x1,x2]);
eq_points = simplify([sol.x1, sol.x2])
```

eq_points =

$$\begin{pmatrix} 0 & 0 \\ -\frac{\sigma_1 - 1}{2\,a\,b} & -\frac{\sigma_1 - 1}{2\,b} \\ \frac{\sigma_1 + 1}{2\,a\,b} & \frac{\sigma_1 + 1}{2\,b} \end{pmatrix}$$

where

$$\sigma_1 = \sqrt{1 - 4\,a^2\,b^2}$$

The origin is always an equilibrium point

Condition: One other exists if $a^2b^2 = \frac{1}{4}$

Condition: Two others exist if $a^2b^2 < \frac{1}{4}$

Determine their type while setting $a = \sqrt{\frac{0.9}{4b^2}}$:

```
f2 = subs(f, a, sqrt(0.9/(4*b^2)));
sol = solve(f2==0, [x1,x2]);
eq_points = simplify([sol.x1, sol.x2])
```

eq_points =

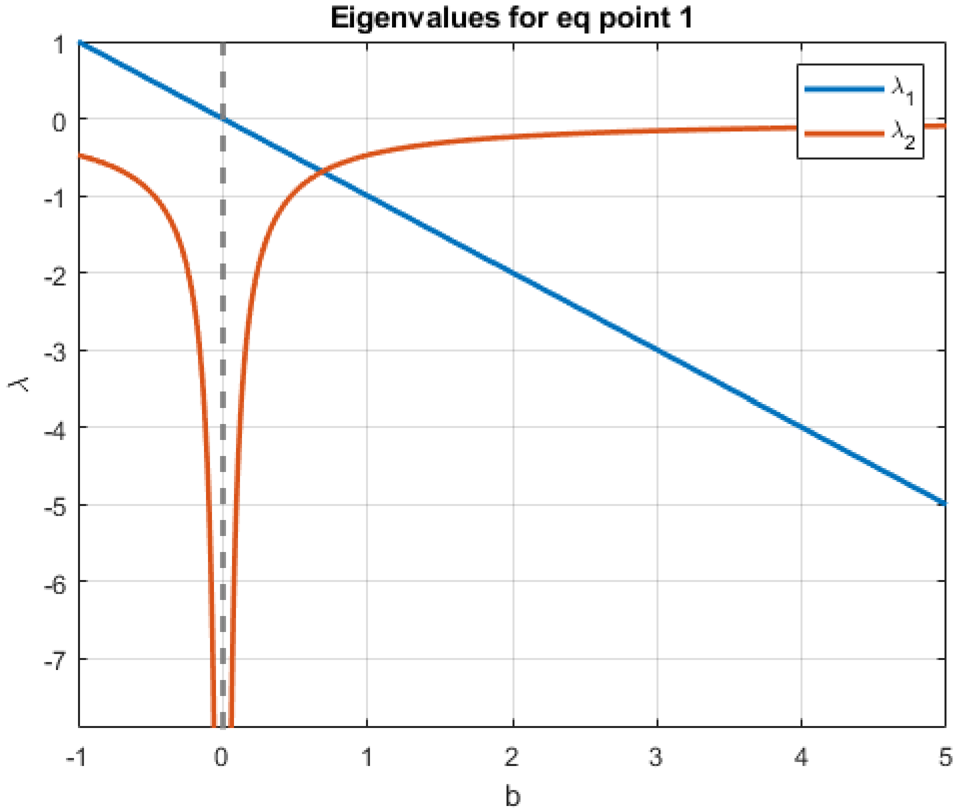
$$\begin{pmatrix} 0 & 0 \\ \dfrac{\sqrt{10}-1}{3\,b\,\sqrt{\dfrac{1}{b^2}}} & -\dfrac{\sqrt{10}-10}{20\,b} \\ \dfrac{\sqrt{10}+1}{3\,b\,\sqrt{\dfrac{1}{b^2}}} & \dfrac{\sqrt{10}+10}{20\,b} \end{pmatrix}$$

```
jac = jacobian(f2, [x1,x2]);
jacs = cell(size(eq_points,1), 1);
eigs = cell(size(eq_points,1), 1);
```

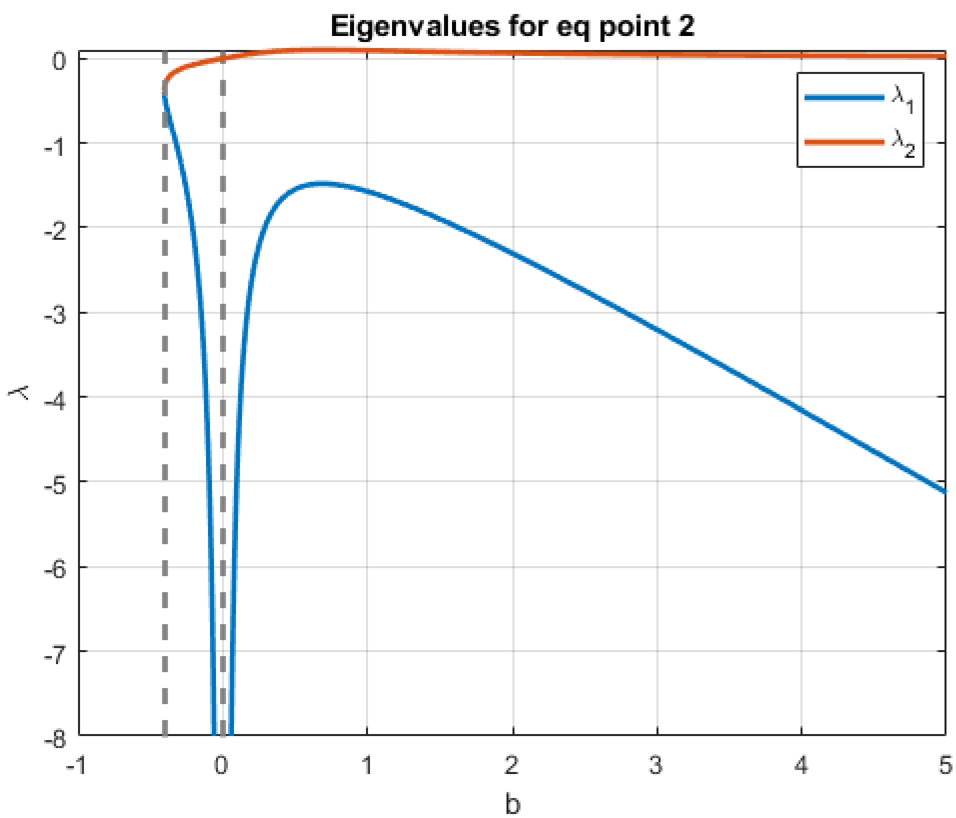
Plot eigen values over b and show $\lim_{b-\infty} \lambda(b)$. Note that as $b \to \infty$, $a \to 0$, so this condition will never be fully met as a must be greater than 0.

```
for i = 1:size(eq_points,1)
    jacs{i} = subs(jac, [x1,x2], [eq_points(i,1), eq_points(i,2)]);
    eigs{i} = vpa(eig(jacs{i}),2);
    disp(limit(eigs{i},b,Inf))
    figure()
    fplot(eigs{i}, [-1,5], "LineWidth", 2);
    grid on
    title("Eigenvalues for eq point " + i)
    legend("\lambda_1", "\lambda_2")
    xlabel("b")
    ylabel("\lambda")
end
```

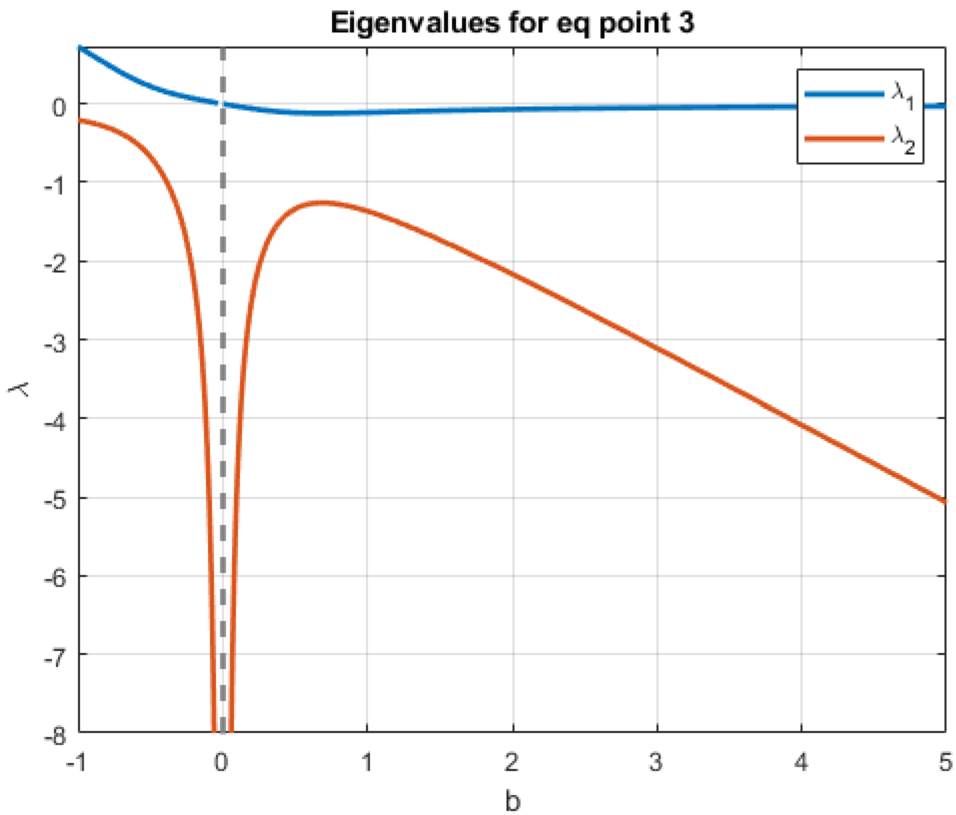
$$\begin{pmatrix} -\infty \\ 0 \end{pmatrix}$$



$$\begin{pmatrix} -\infty \\ 0 \end{pmatrix}$$



$$\begin{pmatrix} -\infty \\ -\infty \end{pmatrix}$$



Thus

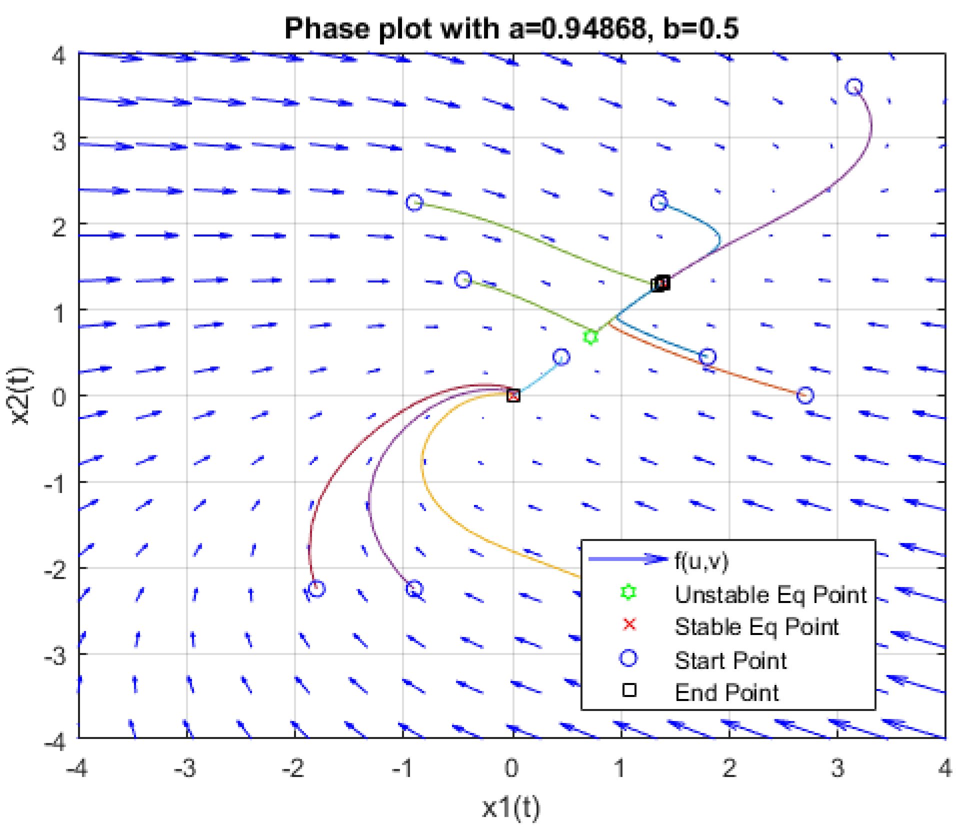
λ_1 is a stable node

λ_2 is a saddle point

λ_3 is a stable node

```
figure()

g = @(t,x,P) [-P(1)*x(1) + x(2); x(1)^2/(1+x(1)^2)-P(2)*x(2)];

bp = 0.5;
ap = sqrt(0.9/(4*bp^2));
params = [ap,bp];

high = 4;
low = -4;
d = 2;
n = d*(high-low);

[x1t, x2t] = meshgrid(linspace(low,high,n), linspace(low,high,n));
dx1 = zeros(size(x1t));
dx2 = zeros(size(x1t));
```

```
for i=1:numel(x1t)
    fprime = g(0,[x1t(i);x2t(i)],params);
    dx1(i) = fprime(1);
    dx2(i) = fprime(2);
end
quiver(x1t,x2t,dx1,dx2, 'b')

grid on
ylim([low,high])
xlim([low,high])
xlabel("x1(t)")
ylabel("x2(t)")
title("Phase plot with a=" + ap + ", b=" + bp)
hold on

plot(subs(eq_points(2,1), b, bp), subs(eq_points(2,2), b, bp), "gh")
plot(eq_points(1,1),eq_points(1,2), "rx")
plot(subs(eq_points(3,1), b, bp), subs(eq_points(3,2), b, bp), "rx")

for i = 1:10
    y0 = 0.9*(randi(n,2,1) - n/2)/d;
    [ts,ys] = ode45(@(t,y) g(t,y,params),[0,50], y0);
    plot(ys(:,1),ys(:,2))
    plot(ys(1,1),ys(1,2),'bo') % starting point
    plot(ys(end,1),ys(end,2),'ks') % ending point
end
legend("f(u,v)", "Unstable Eq Point", "Stable Eq Point","","", "Start Point", "End Point", "Loc
```



The phase plot agrees:

- λ_1 is a stable node
- λ_2 is a saddle point
- λ_3 is a stable node

Problem 3

Using $V(x) = x^T Px$ as a candidate lyapunov function, show that $x = 0$ is a globally asympotitcally stable equilibrium point of the linear system

$\dot{x} = \mathrm{Ax}$

- ***Definition:*** An equilibrium point of $\dot{x} = f(x)$ is globally asymptotically stable if it is stable and its domain of attraction is $\mathbb{R}^n$. When the system is linear, that is, $f(x) = Ax$ for some matrix $A \in \mathbb{R}^{nxn}$, stability and asympotic stability can be verified easiliy using the Eigenvalues of the matrix A.
- ***Definition:*** A function $f(x)$ is called positive definite in U iff:
- $f(0) = 0$
- $\forall\, x \in U, x \neq 0 \mid f(x) > 0$

$V(x) = x^T Px \qquad A^T P + PA = -Q$

Symmetry of P: $\dfrac{\partial V(x)}{\partial x} = x^T(P + P^T) = x^T P + x^T P^T = Px + x^T P$

$\dfrac{\partial V(x)}{\partial x} f(x) = (Px + x^T P)\dot{x} = Px\dot{x} + x^T P\dot{x} = \dot{x}^T Px + x^T P\dot{x}$

$= (Ax)^T Px + x^T P(Ax)$

$= x^T A^T Px + x^T PAx$

$= -x^T Qx$

As $Q \succ 0$, $\forall x \in U \backslash \{0\} \mid -x^T Qx < 0$ and $x = 0 \mid x^T Qx = 0$

In otherwords, $Q \succ 0$ implies that $-x^T Qx \prec 0$, which shows $\dot{x} = Ax$ is locally asympotically stable

For global asympototic stability, we have $\lim\limits_{i\to\infty} V(x_i) = \infty$ whenever $\lim\limits_{i\to\infty} ||x_i|| = \infty$

$V(x) = x^T P x \succ 0$ implies that $\forall\, x \in U \setminus \left\{ \vec{0} \right\} \mid f(x) > 0$

Thus, $\lim_{i\to\infty} V(x_i)$ must go to ∞ as $\lim_{i\to\infty} ||x_i||$goes to ∞.

Problem 4

Show that the equilibrium point at the origin of the linear time-invariant system $\dot{x} = Ax$ is Lyapunov stable, where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

- **<u>Definition:</u>** A function is call Lyapunov stable if $\forall \varepsilon > 0, \exists \delta > 0 \mid ||x(0)|| < \delta \Longrightarrow ||x(t)|| < \varepsilon, \forall t$
- **<u>Theorem 3.1:</u>** The equilibrium point at $x = 0$ of the linear function $\dot{x} = \mathrm{Ax}$ is Lyapunov stable iff all eigenvalues of A satisfy $\mathrm{Re}[\lambda_i] \le 0$ and for every eigenvalue with $\mathrm{Re}[\lambda_i] = 0$ and algebraic multiplicity $q_i \ge 2$, $\mathrm{rank}(A - \lambda_i I) = n - q_i$, where n is the dimension of x.

```
A = [0,1,0,0;-1,0,0,0;0,0,0,1;0,0,-1,0];
lambda = eig(A)
```

```
lambda = 4×1 complex
   0.0000 + 1.0000i
   0.0000 - 1.0000i
   0.0000 + 1.0000i
   0.0000 - 1.0000i
```

$\lambda_1 = 0 + 1i \quad \lambda_2 = 0 - 1i \quad \lambda_3 = 0 + 1i \quad \lambda_4 = 0 - 1i$

$\mathrm{Re}[\lambda_i] = 0$ for all eigenvalues, so further analysis is required to determine stability.

There are two eigenvalues $\lambda = i$ and $\lambda = -i$, each with algebraic multiplicity of 2.Thus, if $\mathrm{rank}(A - \lambda_i I) = n - q_i = 2$ for each eigenvalue, this system is Lyapunov stable.

```
rank(A-lambda(1)*eye(size(A)))
```

```
ans = 2
```

```
rank(A-lambda(2)*eye(size(A)))
```

```
ans = 2
```

Thus, $\dot{x} = \mathrm{Ax}$ is Lyapunov stable

Problem 5

Model of a two-lin planar direct-drive robot manipulator:

$$M(q)\ddot{q} + V_m(q,\dot{q})\dot{q} + F_s(\dot{q}) + F_d(\dot{q}) = \tau$$

- Angular positions: $q = [q_1 \;\; q_2]^T$ (rad)
- Angular velocities: $\dot{q} = [\dot{q}_1 \;\; \dot{q}_2]^T$ (rad/s)
- Torque: $\tau = [\tau_1 \;\; \tau_2]^T$ (N m)

Convert to state-space form $f(x) = \dot{x}$, where $x = [q \;\; \dot{q}]^T$

- $M(q)\ddot{q} = \tau - V_m(q,\dot{q})\dot{q} - F_s(\dot{q}) - F_d(\dot{q})$
- $\ddot{q} = M^{-1}(q)\left(\tau - V_m(q,\dot{q})\dot{q} - F_S(\dot{q}) - F_d(\dot{q})\right)$

$$f(x) = \dot{x} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ M^{-1}(q)\left(\tau - V_m(q,\dot{q})\dot{q} - F_S(\dot{q}) - F_d(\dot{q})\dot{q}\right) \end{bmatrix}$$

Prove that origin of the system the system $\dot{x} = f(x)$ is a globally asympotically stable equilibrium point under the PD controller $\tau(t) = -k_p q(t) - k_D \dot{q}(t)$.

Under control:

$$f(x) = \begin{bmatrix} \dot{q} \\ M^{-1}(q)\left(-k_P q - k_D\dot{q} - V_m(q,\dot{q})\dot{q} - F_S(\dot{q}) - F_d(\dot{q})\dot{q}\right) \end{bmatrix}$$

Given $V(x) = \frac{1}{2}q^T k_P q + \frac{1}{2}\dot{q}^T M(q)\dot{q}$

and $\frac{\partial V(x)}{\partial x} = \begin{bmatrix} \frac{\partial V(x)}{\partial q} & \frac{\partial V(x)}{\partial \dot{q}} \end{bmatrix} = \begin{bmatrix} q^T k_P + \frac{1}{2}\dot{q}^T \frac{\partial (M(q)\dot{q})}{\partial q} & \dot{q}^T M(q) \end{bmatrix}$

where $M(q), k_P, k_D \succ 0$ and k_P, k_D are diagonal (which implies symmetry)

<u>First:</u> Establish $V(x)$ as a valid candidate Lyapunov function

Criterion:

- $V(x)$ must be continuously differentiable
- $V(x)$ must be positive definite
- $f\left(\overrightarrow{0}\right) = 0$

$V(x)$ is constructed in quadratic form of positive definite matrices, so it must be positive definite. Furthermore, there are no discontinuous points due to its quadratic nature. In order to satisfy the requirement that $f\left(\overrightarrow{0}\right) = 0$, we acknowledge $\tanh(0) = 0$ and thus all terms of $-k_P q - k_D \dot{q} - V_m(q,\dot{q})\dot{q} - F_S(\dot{q}) - F_d(\dot{q})\dot{q}$ are 0 when $x = [q\ \dot{q}]^T = 0$.

<u>**Second:**</u> Find $\dot{V}(x) = \frac{\partial V(x)}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial V(x)}{\partial x} f(x)$. The system is Lyapunov stable at the origin if $\forall x \in U \mid \dot{V}(x) \le 0$ and locally asympotically stable if $V\left(\overrightarrow{0}\right) = 0$ and $\forall x \in U \backslash \left\{\overrightarrow{0}\right\} \mid \dot{V}(x) < 0$.

$$\frac{\partial V(x)}{\partial x} f(x) = \left(q^T k_P + \frac{1}{2}\dot{q}^T \frac{\partial (M(q)\dot{q})}{\partial q}\right)\dot{q} + \left(\dot{q}^T M(q)\right)\left(M^{-1}(q)\left(\tau - V_m(q,\dot{q})\dot{q} - F_s(\dot{q}) - F_d\dot{q}\right)\right)$$

$$= q^T k_P \dot{q} + \frac{1}{2}\dot{q}^T \frac{\partial (M(q)\dot{q}}{\partial q}\dot{q} - \dot{q}^T M(q) M^{-1}(q)\left(k_P q + k_D \dot{q} + V_m(q,\dot{q})\dot{q}\right) + F_s(\dot{q}) + F_d\dot{q}\right)$$

$$= q^T k_P \dot{q} + \frac{1}{2}\dot{q}^T \frac{\partial (M(q)\dot{q}}{\partial q}\dot{q} - \dot{q}^T k_P q - \dot{q}^T k_D \dot{q} - \dot{q}^T V_m(q,\dot{q})\dot{q}) - \dot{q}^T F_s(\dot{q}) - \dot{q}^T F_d \dot{q}$$

$$= \dot{q}^T\left(\frac{1}{2}\frac{\partial M(q)\dot{q}}{\partial q} - V_m(q,\dot{q}) - k_D - F_d\right)\dot{q} + q^T k_P \dot{q} - \dot{q}^T k_P q - \dot{q}^T F_s(\dot{q})$$

Exploiting that $\frac{1}{2}\frac{\partial M(q)\dot{q}}{\partial q} - V_m(q,\dot{q})$ is skew symmetric and k_D, k_P are symmetric:

$$= \dot{q}^T(-k_D - F_d)\dot{q} + q^T(k_P - k_P)\dot{q} - \dot{q}^T F_s(\dot{q})$$

$$= -\left(\dot{q}^T k_D \dot{q} + \dot{q}^T F_d \dot{q}\right) - \dot{q}^T F_s(\dot{q})$$

$$= -\dot{q}^T k_D \dot{q} - \dot{q}^T F_d \dot{q} - \dot{q}^T F_s(\dot{q})$$

Thus, $\dot{V}(x) = -\dot{q}^T k_D \dot{q} - \dot{q}^T F_d \dot{q} - \dot{q}^T F_s(\dot{q})$

We know the first term is negative definite as k_D is stated as positive definite in the problem definition. Next, we must check the other two terms.

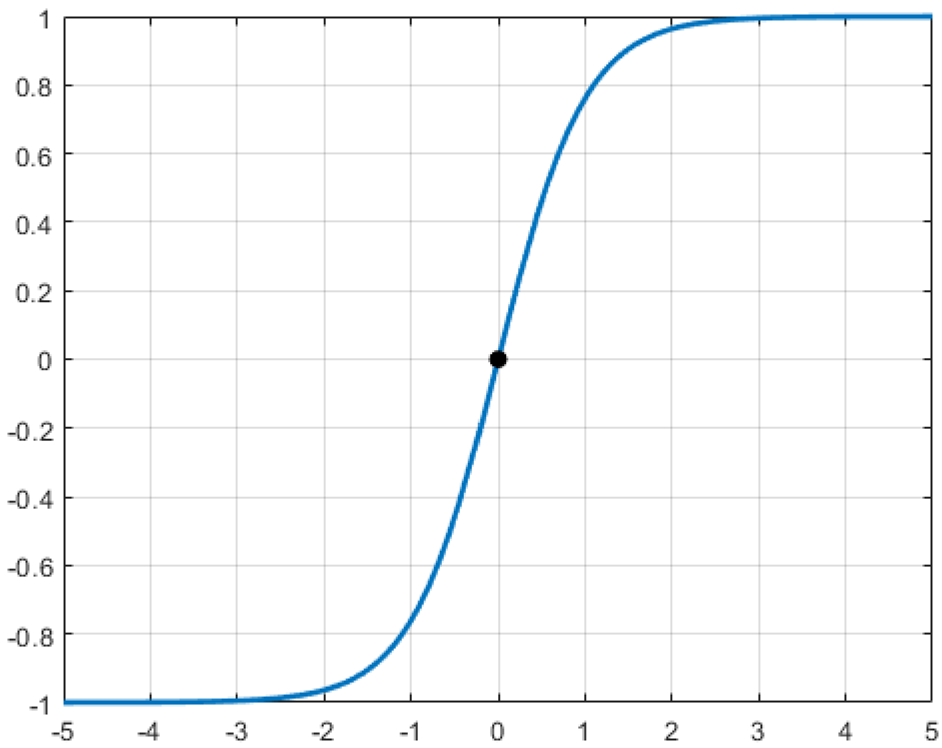
```
Fd = [5.3, 0; 0, 1.1];
eig(Fd)
```

```
ans = 2×1
    1.1000
    5.3000
```

As the eigenvalues of F_d are purely positive, $\dot{q}^T F_d \dot{q}$ is also positive definite.

Finally, $F_s(\dot{q}) = \begin{bmatrix} f_{s,1}\tanh(\dot{q_1}) \\ f_{s,2}\tanh(\dot{q_2}) \end{bmatrix}$, where $f_{s,1} = 8.45$ kg m/s and $f_{s,2} = 2.35$ kg m/s. $\tanh(x)$ is monotonically increasing and crosses the x-axis at $x = 0$. A plot is shown below.

```
syms x

figure()
fplot(tanh(x), "LineWidth", 2);

hold on
grid on
plot(0,0, '-ko', 'MarkerFaceColor', 'k')
```



Thus, the sign of $\tanh(\dot{q})$ is equivalent to the sign of $\dot{q}$, which implies that $\dot{q}^T F_s(\dot{q})$ positive for all $q \neq 0$.

Therefore, all terms of $\dot{V}(x)$ are negative definite. As such, $\dot{V}(x) < 0$ for all $x \neq 0$ and $\dot{V}\left(\vec{0}\right) = 0$. This implies that the origin is a local asympotically stable equilibrium point of the system $\dot{x} = f(x)$.

Finally: Prove that the origin is a globally asymptotically stable equilibrium point.

The system is globally asympotically stable if it is LAS and $||x|| \rightarrow \infty \Rightarrow V(x) \rightarrow \infty$

$$V(x) = \frac{1}{2}q^T k_P q + \frac{1}{2}\dot{q}^T M(q)\dot{q}$$

The first term clearly tends toward ∞ when $x \rightarrow \infty$ as k_P is a constant positive definite diagonal matrix. The second term requires a bit more analysis. As given in the hints, $M(q)$ is uniformly positive definite, which implies the existence of a constant $\bar{m}$ such that the minimum eigenvalue of $M(q)$ is larger than $\bar{m}$ for all $q \in \mathbb{R}^2$.

Additionally, it is given that the following inequality holds for all $q, \dot{q} \in \mathbb{R}^2$:

$\lambda_{\min}(M(q))||\dot{q}||^2 \leq \dot{q}^T M(q)\dot{q} \leq \lambda_{\max}(M(q))||\dot{q}||^2$

This is equivalent to $\bar{m}||\dot{q}||^2 \leq \dot{q}^T M(q)\dot{q} \leq \lambda_{\max}(M(q))||\dot{q}||^2$

Disregarding the third term, we can see that $\dot{q}^T M(q)\dot{q} \geq \bar{m}||\dot{q}||^2$. As $\dot{q} \subset x$, $||x|| \rightarrow \infty \Rightarrow ||\dot{q}|| \rightarrow \infty$ and $\bar{m}||\dot{q}||^2 \rightarrow \infty$ for all $\bar{m} > 0$.

Thus $\dot{q} \rightarrow \infty \Rightarrow \dot{q}^T M(q)\dot{q} \rightarrow \infty$ and $V(x) \rightarrow \infty$ as $||x|| \rightarrow \infty$. The origin is globally asympotically stable.