MAE/ECEN 5463
Nonlinear Systems Analysis and Control
Exam 3

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5 point bonus for assignments typeset using LaTeX

Question 1 (50 points)**.** Consider the following model of a two-link planar direct-drive robot manipulator **with torsional springs attached to each of the joints**

$$M(q)\ddot{q} + V_m(q,\dot{q})\dot{q} + F_s(\dot{q}) + F_d\dot{q} + Sq = \tau + d(t,q,\dot{q}),$$

where $q = [q_1\ \ q_2]^T$ and $\dot{q} = [\dot{q}_1\ \ \dot{q}_2]^T$ are the angular positions (rad) and angular velocities (rad/s) of the two links, respectively, $\tau = [\tau_1\ \ \tau_2]^T$ is the torque (N m) produced by the motors that drive the joints, $M(q)$ is the inertia matrix, and $V_m(q,\dot{q})$ is the centripetal-Coriolis matrix, defined as

$$M\left(q\right) := \begin{bmatrix} p_1 + 2p_3c_2\left(q\right) & p_2 + p_3c_2\left(q\right) \\ p_2 + p_3c_2\left(q\right) & p_2 \end{bmatrix}, \qquad V_m\left(q,\dot{q}\right) = \begin{bmatrix} p_3s_2\left(q\right)\dot{q}_2 & -p_3s_2\left(q\right)\left(\dot{q}_1+\dot{q}_2\right) \\ p_3s_2\left(q\right)\dot{q}_1 & 0 \end{bmatrix},$$

where $p_1 = 3.473\,\mathrm{kg\,m^2}$, $p_2 = 0.196\,\mathrm{kg\,m^2}$, $p_3 = 0.242\,\mathrm{kg\,m^2}$, $c_2\left(q\right) = \cos(q_2)$, $s_2\left(q\right) = \sin(q_2)$, and $F_d\dot{q} = \begin{bmatrix} f_{d1} & 0 \\ 0 & f_{d2} \end{bmatrix}\dot{q}$ N m and $F_s(\dot{q}) = [f_{s1}\tanh(\dot{q}_1),\ f_{s2}\tanh(\dot{q}_2)]^T$ N m are the models for dynamic and static friction, respectively, where $f_{d1} = 5.3\,\mathrm{kg\,m/s}$, $f_{d2} = 1.1\,\mathrm{kg\,m/s}$, $f_{s1} = 8.45\,\mathrm{kg\,m/s}$, and $f_{s2} = 2.35\,\mathrm{kg\,m/s}$. The matrix $S = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}$ contains the torsional spring coefficients $s_1 = 0.5\,\mathrm{N\,m/rad}$ and $s_2 = 0.25\,\mathrm{N\,m/rad}$.

Assume that you have sensors on the robot to measure angular positions and angular velocities of both links, i.e., q and $\dot{q}$ are known at each time instance. The quantity $d\left(t,q,\dot{q}\right)$ models a non-vanishing disturbance that is bounded such that $\left\|d\left(t,q,\dot{q}\right)\right\| \leq \overline{d}$ for all $(t,q,\dot{q})$. Let the tracking error, in response to the controller $t \mapsto \tau\left(t\right)$, be defined as

$$e\left(t,0,\begin{bmatrix} q^o \\ \dot{q}^o \end{bmatrix},\tau\left(\cdot\right)\right) = q_d\left(t\right) - \phi_q\left(t,0,\begin{bmatrix} q^o \\ \dot{q}^o \end{bmatrix},\tau\left(\cdot\right)\right),$$

where $t \mapsto q_d\left(t\right) \in \mathcal{C}^2\left(\mathbb{R}_{\geq 0},\mathbb{R}^2\right)$ is a given desired trajectory, and $\phi_q\left(t,0,\begin{bmatrix} q^o \\ \dot{q}^o \end{bmatrix},\tau\left(\cdot\right)\right)$ denotes the $q-$component of the trajectory of the robot, in response to the controller $t \mapsto \tau\left(t\right)$ starting at $t = 0$ from the initial condition $\begin{bmatrix} q^o \\ \dot{q}^o \end{bmatrix}$.

1. (30 points) Assuming that p_1, p_2, p_3, f_{d1}, f_{d2}, f_{s1}, f_{s2}, s_1, and s_2 are **unknown**, design a controller such that the following objectives are met for all initial conditions $\begin{bmatrix} q^o \\ \dot{q}^o \end{bmatrix} \in \mathbb{R}^2$:

   (a) (10 points) For all time, the parameter estimation error, $\tilde{\theta}$, and the tracking error, e are bounded.

   (b) (10 points)The tracking error, e, enters the ball $\overline{\mathrm{B}}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \epsilon\right)$ in finite time, where $\epsilon > 0$ is some positive constant.

   (c) (10 points) The constant ϵ can be made arbitrarily small by tuning control gains.

   Useful property: Letting $r = \dot{e} + \alpha e$, since q_d is now time-varying, we can interpret the function $\frac{1}{2}e^T e + \frac{1}{2}r^T M(q) r$ as a function of e, r, and t, where the explicit time-dependence comes in through q. That is, $V\left(t, \begin{bmatrix} e \\ r \end{bmatrix}\right) = \frac{1}{2}e^T e + \frac{1}{2}r^T M(q(t)) r$. Then, we have

$$
\begin{aligned}
\frac{\partial V\left(t, \begin{bmatrix} e \\ r \end{bmatrix}\right)}{\partial e} &= e^T, \\
\frac{\partial V\left(t, \begin{bmatrix} e \\ r \end{bmatrix}\right)}{\partial r} &= r^T M(q(t)), \text{ and} \\
\frac{\partial V\left(t, \begin{bmatrix} e \\ r \end{bmatrix}\right)}{\partial t} &= \frac{1}{2} r^T \frac{\partial M(q(t))\dot{q}(t)}{\partial (q(t))} r \quad \text{(or, with abuse of notation, } \frac{1}{2} r^T \dot{M}(q(t), \dot{q}(t)) r).
\end{aligned}
$$

   This will allows you to use the skew symmetry property $x^T \left(\frac{1}{2}\frac{\partial (M(q)\dot{q})}{\partial q} - V_m(q, \dot{q})\right) x = 0$ for all $x \in \mathbb{R}^4$ to simplify the design.

2. (20 points) MATLAB experiment to track the desired trajectory

$$
q_d(t) = \begin{bmatrix} (1 - \mathrm{e}^{-2t}) \cos(2t) \\ (1 - \mathrm{e}^{-2t}) \sin(t) \end{bmatrix}.
$$

   (a) Implement your controller in MATLAB using the robot dynamics file available in the Exam 3 folder on Canvas (see the included example for usage). The model in this file has parameter values that are considerably different from those given here, and includes an added nonvanishing disturbance.

   (b) Write a short paragraph comparing the performance of this controller with a PD controller. Use any plots you need to illustrate your comparative analysis. **Quantitative comparisons will be graded higher than qualitative comparisons**.

   (c) Submit your code as a single MATLAB m-file along with your answers.

Question 2 (30 points)**.** Consider the dynamical system

$$
\dot{x}_1 = a x_1 + b x_2, \qquad \dot{x}_2 = u
$$

where $x_1, x_2 \in \mathbb{R}$, a and b are **unknown** constants, and $b > 0$. Design a controller so that x_1 and x_2 decay to zero as time goes to infinity.
Hint: Combine adaptive control and backstepping.

Question 3 (20 points)**.** Consider a dynamical system $\dot{x} = f(t, x)$ where $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the basic conditions. Furthermore, we know that there exists a function $V \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$, a constant $a > 0$ and functions $\alpha_1 \in \mathcal{K}([0, a), \mathbb{R})$, $\alpha_2 \in \mathcal{K}([0, a), \mathbb{R})$, and $\alpha_3 \in \mathcal{K}([0, a), \mathbb{R})$ such that for all $(t, x) \in \mathbb{R}_{\geq t_0} \times \mathrm{B}(0, a)$, we have

$$\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|) \tag{1}$$

$$\frac{\partial V}{\partial x}(t, x) f(t, x) + \frac{\partial V}{\partial t}(t, x) \leq -\alpha_3(\|x\|) \tag{2}$$

1. (10 points) Show that the origin is a uniformly asymptotically stable equilibrium point.

2. (10 points) Find an estimate of the region of attraction of the origin in terms of the functions α_i and the constant a.

Bonus: *(10 points)* Find a class $\mathcal{KL}$ function β in terms of the functions α_i and the constant a such that for all x^o sufficiently close to the origin, $\|\phi(t, 0, x^o)\| \leq \beta(\|x^o\|, t)$

Clearly state any additional assumptions you have to make on the data of the problem for the solution to make sense. The following might help **with the bonus**.

Hint i. **Comparison Lemma (simplified, this says solutions of differential inequalities are bounded by solutions of corresponding differential equations):** Consider a set $\chi \subset \mathbb{R}$ and a scalar differential equation

$$\dot{x} = f(t, x),$$

where $f : \mathbb{R}_{\geq 0} \times \chi \to \mathbb{R}$ is continuous in t for all $x \in \chi$ and locally Lipschitz in x for all $t \in \mathbb{R}_{\geq 0}$. For some initial condition $(t_0, x^o) \in \mathbb{R}_{\geq 0} \times D$, let $[t_0, T)$, be the maximal interval of existence of the solution $t \mapsto \phi(t, t_0, x^o)$, where $0 < T \leq \infty$.

Let $t \mapsto y(t)$ be a differentiable function defined on $[t_0, T)$ such that

$$\frac{\mathrm{d}y}{\mathrm{d}t}(t) \leq f(t, y(t)), \qquad y(t_0) \leq x^o.$$

If, for all $t \in [t_0, T)$, we know that $\phi(t, t_0, x^o) \in D$ and $y(t) \in D$, then

$$y(t) \leq \phi(t, t_0, x^o), \forall t \in [t_0, T).$$

Hint ii. **Differential equations involving class $\mathcal{K}$ functions:** Let $\alpha \in \mathcal{K}([0, a), \mathbb{R})$. If α is locally Lipschitz continuous on $[0, a)$, then for each $x^o \in [0, a)$ and $t_0 \geq 0$, the differential equation

$$\dot{x} = -\alpha(x),$$

with the initial condition $x(t_0) = x^o$, has a unique solution that exists for all $t \geq t_0$. Furthermore,

$$\phi(t, t_0, x^o) = \beta(x^o, t - t_0),$$

for some $\beta \in \mathcal{KL}([0, a) \times [0, \infty), \mathbb{R})$.