Lecture 10

Thursday, February 10, 2022 9:03 AM

$$\left.\begin{aligned}\dot{x}_1 &= \frac{x_1}{3}(x_1^2+x_2^2-2) - \frac{4}{3}x_1x_2^2 \\ \dot{x}_2 &= 4x_1^2x_2 + x_2(x_1^2+x_2^2-2)\end{aligned}\right\} \quad \dot{x} = f(x) \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$$

$$f(x) = \begin{bmatrix} \frac{x_1}{3}(x_1^2+x_2^2-2) - \frac{4}{3}x_1x_2^2 \\ 4x_1^2x_2 + x_2(x_1^2+x_2^2-2) \end{bmatrix} \qquad f(0) = 0$$

$$V(x) = 3x_1^2 + x_2^2 \longrightarrow \|x\|^2 \le V(x) \le 3\|x\|^2$$

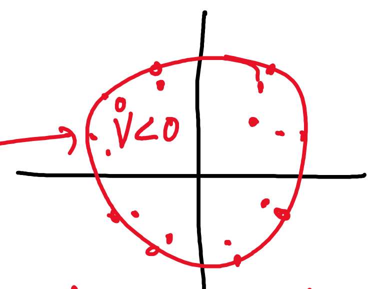
$$\frac{\partial V(x)}{\partial x} = \begin{bmatrix} \frac{\partial V(x)}{\partial x_1} & \frac{\partial V(x)}{\partial x_2} \end{bmatrix}$$

$$\frac{\partial V(x)}{\partial x} f(x) = \begin{bmatrix} 6x_1 & 2x_2 \end{bmatrix} \begin{bmatrix} \frac{x_1}{3}(x_1^2+x_2^2-2) - \frac{4}{3}x_1x_2^2 \\ 4x_1^2x_2 + x_2(x_1^2+x_2^2-2) \end{bmatrix}$$

$$= 2x_1^2(x_1^2+x_2^2-2) - \cancel{8x_1^2x_2^2} + \cancel{8x_1^2x_2^2} + 2x_2^2(x_1^2+x_2^2-2)$$

$$\frac{\partial V(x)}{\partial x} f(x) = 2(x_1^2+x_2^2)(x_1^2+x_2^2-2)$$

If $D = \{ x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 2 \}$



Then for all $x \in D \setminus \{\vec{0}\}$ $\quad \frac{\partial V(x)}{\partial x} f(x) < 0$

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt}$$

$\therefore$ $x = 0$ is a LAS eq. pt. of $\dot{x} = f(x)$

Exercise: Is D a domain of attraction?

$$\dot{x}_1 = -\frac{6x_1}{(1+x_1^2)^2} + 2x_2$$

$$\dot{x}_2 = -\frac{2(x_1+x_2)}{(1+x_1^2)^2}$$

f is loc. Lip.

$$V(x) = \frac{x_1^2}{(1+x_1^2)} + x_2^2 \qquad \text{PD, Cts diff.}$$

$$\frac{\partial V}{\partial x}(x) f(x) = \frac{-12x_1^2}{(x_1^2+1)^4} - \frac{4x_2^2}{(x_1^2+1)^2}$$

$D = \mathbb{R}^2$

$\dot{V} < 0$ for all $x \in \mathbb{R}^2 \setminus \{\vec{0}\}$, $\frac{\partial v}{\partial x}(x) f(x) < 0 \Rightarrow$ LAS of $x = 0$

$V(x$

Q: Is $\mathbb{R}^2$ a domain of attraction?

Theorem: (Barbashin-Krasovskii theorem)

If:

H1: $f(0) = 0$, $f: \mathbb{R}^n \to \mathbb{R}^n$ is loc. Lip. cts.

H2: $V: \mathbb{R}^n \to \mathbb{R}$ is cts. diff. p.D.

H3: Given any sequence $\{x_i\}_{i=1}^{\infty}$ such that $\lim_{i \to \infty} \|x_i\| = \infty$

we have $\lim_{i \to \infty} V(x_i) = \infty$ (V is radially unbounded)

H4: $\frac{\partial V}{\partial x}(x) f(x) < 0$ for all $x \in \mathbb{R}^n \setminus \{\vec{0}\}$

C1: $x = 0$ is a globally asymptotically stable eq. pt. of $\dot{x} = f(x)$

$$\boxed{H_3 \Leftrightarrow \text{ (for cts. diff functions) sublevel sets of } V \text{ are compact}}$$

Def. A c-sublevel set of V is the set $\{x \in \mathbb{R}^n \mid V(x) \leq c\}$

Theorem: If (Lyapunov's second theorem)

H1: $D \subseteq \mathbb{R}^n$ is open, connected, and $\vec{0} \in D$

H2: $f(\vec{0}) = 0$

H3: f is locally Lipschitz continuous on D

H4: $V: D \to \mathbb{R}$ is continuously differentiable and positive definite

then the eq. pt. $x = 0$ of $\dot{x} = f(x)$ is

C1: stable if $\left(\frac{\partial V}{\partial x}(x)\right) f(x) \leq 0 \ \forall x \in D$

$\left(\frac{\partial}{\partial x}\right)$

C2: locally asymptotically stable if

$$\left(\frac{\partial V(x)}{\partial x}\right) f(x) < 0 \quad \text{for all } x \in D \setminus \{\vec{0}\}$$

Objective: Given $\varepsilon > 0$ we need to find $\delta > 0$ so that $\forall x^0 \in B(0,\delta)$

① $\phi(t, x^0)$ exists for all $t \geq 0$

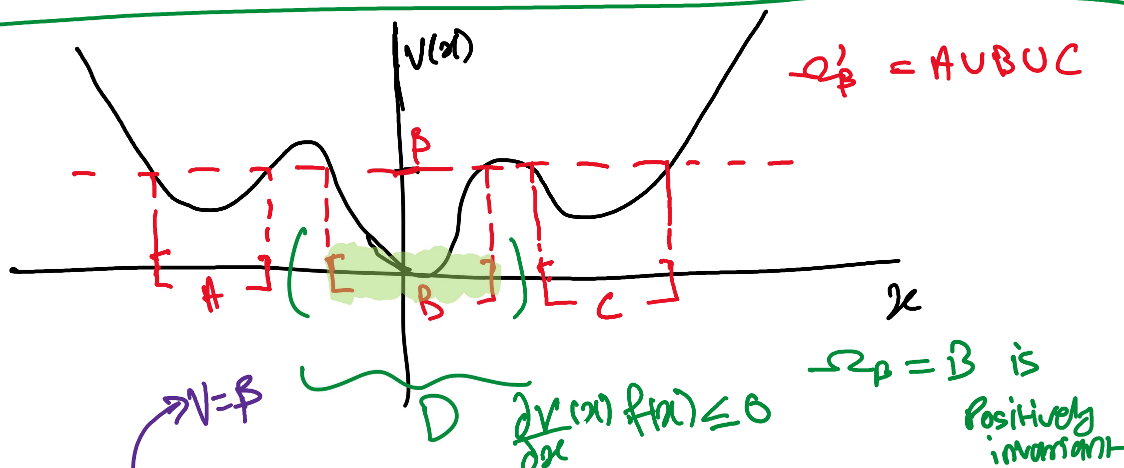
② $\phi(t, x^0) \in B(0, \varepsilon)$ for all $t \geq 0$

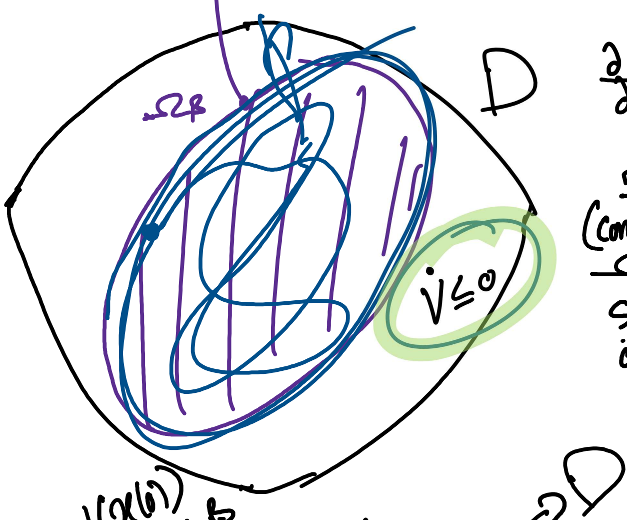
Claim:

For any $\beta \in \mathbb{R}$, let Ω'_β be a β-sublevel set of V. Let Ω_β be any connected component of Ω'_β.

If Ω_β is closed and $\Omega_\beta \subseteq D$ then Ω_β is positively (forward) invariant with respect to $\dot{x} = f(x)$.

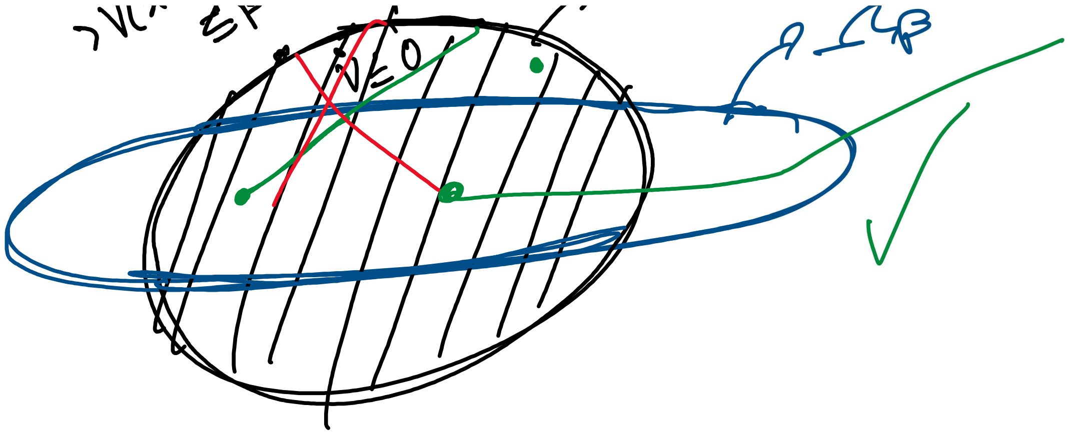
In addition, if Ω_β is compact then $\phi(t, x^0)$ exists for all $t \geq 0$

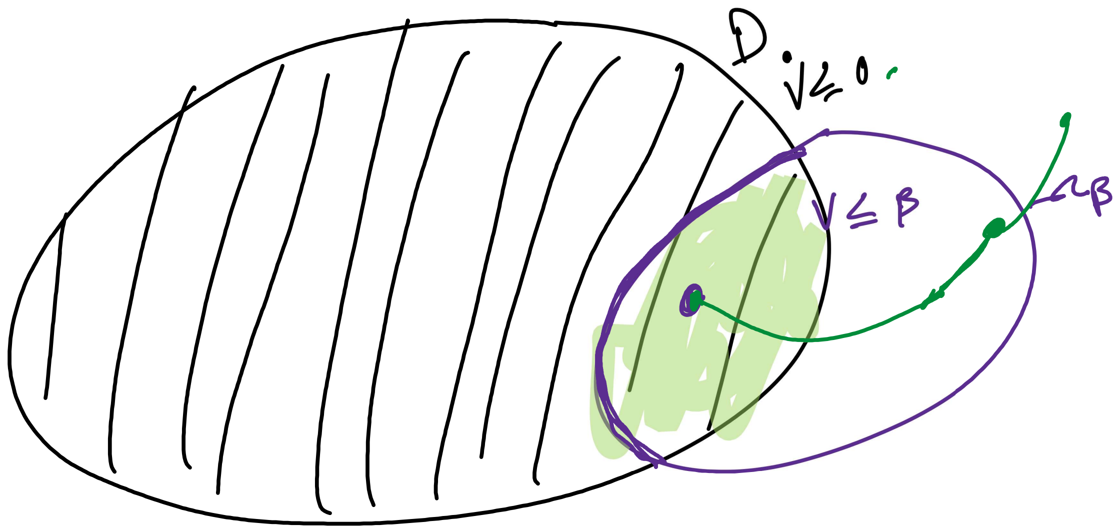




$\frac{\partial V}{\partial x}(x) f(x) \leq 0$ for all $x \in D$

If we can find $\beta > 0$ such that (compact) $\Omega_\beta \subset D$ then all solutions starting in Ω_β exist for all t and stay in Ω_β for all t


V ≤ 0
Ωβ


D
V ≤ 0
V ≤ β
Ωβ