Lecture 06

Thursday, January 27, 2022 08:57

Nonlinear deterministic dynamical system in continuous tim

$$\dot{x} = f(t, x) \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad f(t, x) = \begin{bmatrix} f_1(t, x) \\ \vdots \\ f_n(t, x) \end{bmatrix} \quad ($$

A **classical solution** to (1) starting from $x^0 \in \mathbb{R}^n$ and t_0 the time interval $[t_0, t_0+T)$ is a differentiable
$t \mapsto \phi(t, t_0, x^0)$ such that

① $\phi(t_0, t_0, x^0) = x^0$

② $\frac{d}{dt} \phi(t, t_0, x^0) = f(t, \phi(t, t_0, x^0))$ for all $t \in [t_0, t$

$x \mapsto f($
$f(\cdot,$

Theorem: Peano existence theorem (1890)
Let
(H1) D be an open and connected subset of
(H2) f is continuous on $I \times D$ for some op
then
(C1) for all $t_0 \in I$ and $x^0 \in D$, there exist interval $I_0 \subseteq I$ such that $t_0 \in I_0$ and a solution to (1) exists on I_0

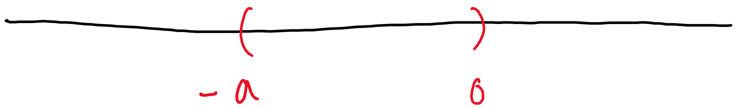
$I \in \mathbb{R}$
$D \in \mathbb{R}^n$

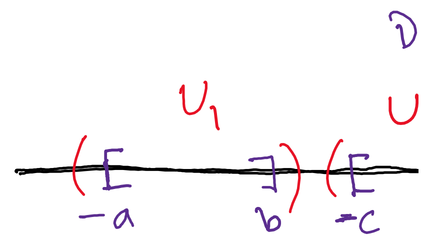
$$I \times D : \{ (t, x) \mid t \in I, x \in D \}$$

such that

Def: $D \subseteq \mathbb{R}^n$ is called disconnected iff there exist open (in the relative topology induced on D), non disjoint ($A \cap B = \emptyset$) such that $D = A \cup B$

disjoint ($A \cap B = \emptyset$), such that $D = A \cup B$





Def: A set $C \subseteq D \subseteq \mathbb{R}^n$ is open in the relative t
D if there exists an open set $U \subseteq \mathbb{R}^n$ such that

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Def: $f : D \to \mathbb{R}^n$ is called continuous at $x \in D$:

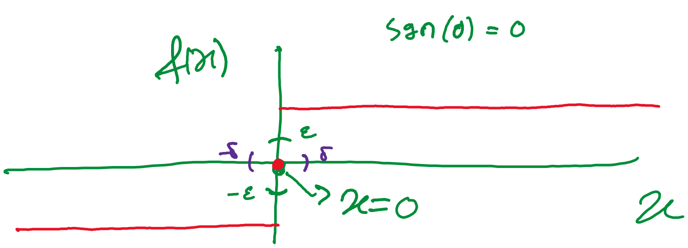
$$\forall \varepsilon > 0, \ \exists \delta > 0 \ | \ y \in B(x,\delta) \cap D \Rightarrow f(y)$$

"in" for all there exists such that

$$B(x,\delta) = \{ y \in \mathbb{R}^n \mid \|x-y\| < \delta \}$$

f is called continuous on D iff it is continuous at $x \in D$ for all $x \in D$

Ex: $f(x) = \mathrm{sgn}(x)$ $\quad x \in \mathbb{R}$



$$\dot{x} = f(t,x) = g(x) + u(t) \qquad u(t) = \begin{cases} 0 & t \\ 1 & t \end{cases}$$

$$\dot{x} = f(t,x) = \begin{cases} g(x) & t < 1 \\ g(x)+1 & t \geq 1 \end{cases}$$

is this contin

Def: (Carathéodory Solutions): A Carathéodory solu

Starting from $x^0 \in \mathbb{R}^n$ at $t_0 \in \mathbb{R}_{\geq 0}$ over $[t_0, t_0$

a locally absolutely continuous function $t \to \phi(t,$

that for all $t \in [t_0, t_0 + T)$

$$\phi(t, t_0, x^0) = x^0 + \int_{t_0}^{t} f(\tau, \phi(\tau, t_0, x^0)$$

Def: A function $\psi : I \to \mathbb{R}^n$ is called locally absolutely

for every interval $[t_1, t_2] \subseteq I$ such that $-\infty$

there exists a Lebesgue integrable function $g : [t$

such that for all $t \in [t_1, t_2]$

$$\psi(t) = \psi(t_1) + \int_{t_1}^{t} g(\tau)\, d\tau$$

Fact: if $f : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is locally absolutely continuous

differentiable almost everywhere.

Also, if $\phi(\cdot, t_0, x^0)$ is a Carathéodory solution to $\dot{x} = f(t,$

then $t \mapsto \phi(t, t_0, x^0)$ is differentiable almost e

for any t where ϕ is "

$$\frac{d}{dt} \phi(t, t_0, x^0) = f\big(t, \phi(t, t_0, x^0$$

Differentiable on a set $A \subseteq \mathbb{R}_{\geq 0}$ such that

$$A \setminus B = A \cap B^c$$

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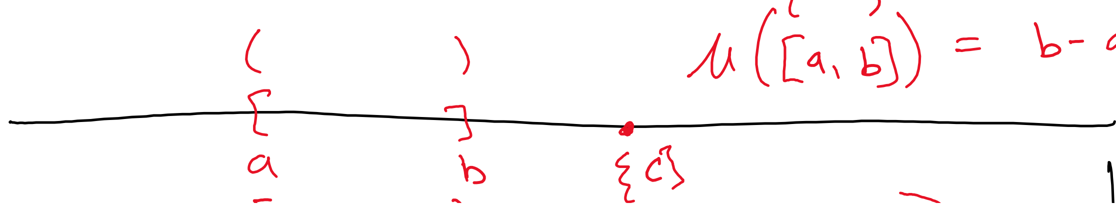
$\mu([a, b]) = b - a$



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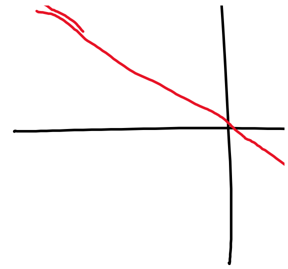
↳ $u=0$



When does a Caratheodory solution exist? When

Example: $\underline{\dot{x} = x^{1/3}}$, $x^0 = 0$, $x \in \mathbb{R}$, $t_0 = 0$

$$\int x^{-1/3}\,dx = \int dt \Rightarrow \int \frac{d}{dx}\left(\frac{3}{2}x^{2/3}\right)dx = \int d$$

$$\frac{3}{2}x^{2/3} = t,$$

$$\phi(t, t_0, x^0) = \left(\frac{2}{3}t\right)^{3/2}$$

$$\phi(t, t_0, x^0) = 0$$

What restrictions can we put on f so that $\dot{x} =$
unique solutions

Picard – Lindlöf theorem.