ECEN5463 | HW 1

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Problem 1

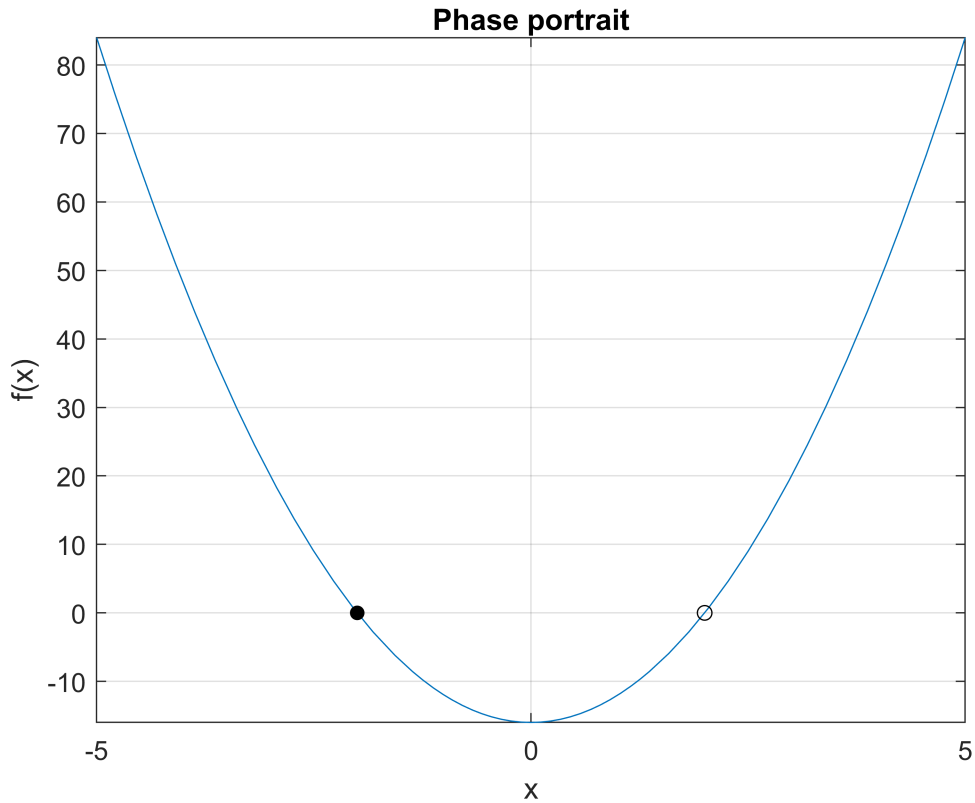
1-1 $\dot{x} = 4x^2 - 16$

```
syms x

f = 4*x^2 - 16;

sol = solve(f==0);
eq_points = sol
```

eq_points =

$$\begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

```

figure()
fplot(f, "DisplayName", "f(x)")
hold on;
grid on;

for i = 1:size(eq_points,1)
    if subs(diff(f), x, eq_points(i)) < 0
        scatter(eq_points(i), subs(f, x, eq_points(i)), 30,'k', ...
            'filled', 'DisplayName', 'Stable')
    else
        scatter(eq_points(i), subs(f, x, eq_points(i)), 30,'k', ...
            'DisplayName', 'Unstable')
    end
end
xlabel("x");
ylabel("f(x)")
title("Phase portrait")
```



1-2 $\dot{x} = x - x^3$

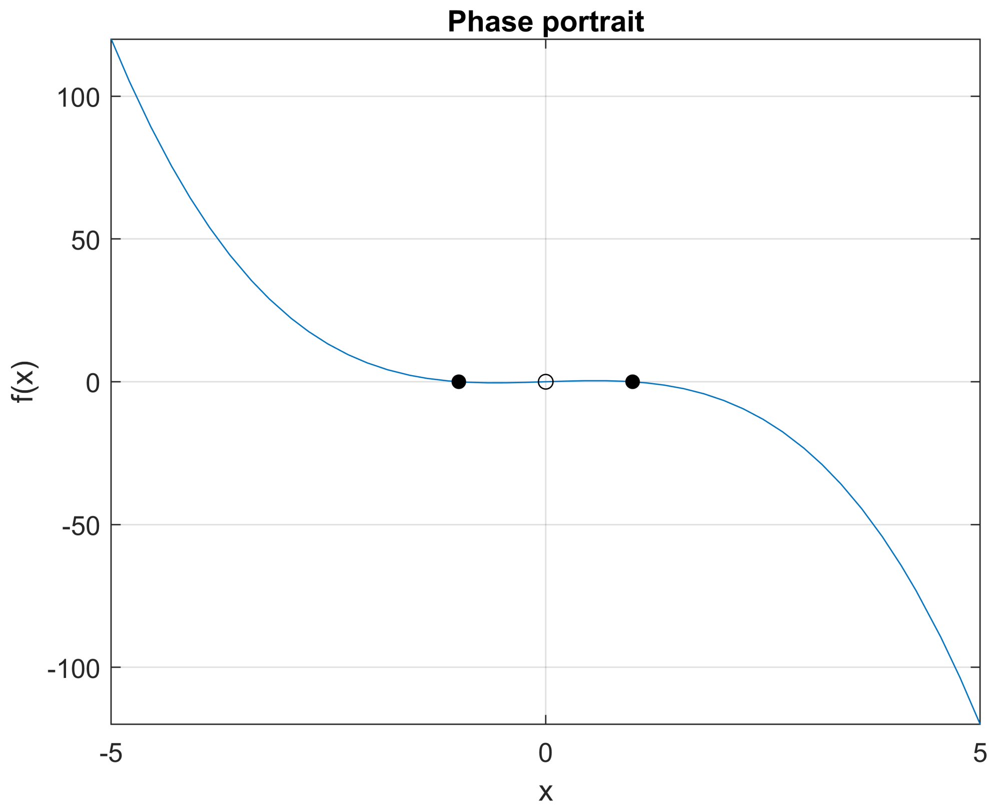
```
syms x

f = x - x^3;

sol = solve(f==0);
eq_points = sol
```

eq_points =

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

```
figure()
fplot(f, "DisplayName", "f(x)")
hold on;
grid on;

for i = 1:size(eq_points,1)
    if subs(diff(f), x, eq_points(i)) < 0
        scatter(eq_points(i), subs(f, x, eq_points(i)), 30,'k', ...
            'filled', 'DisplayName', 'Stable')
    else
        scatter(eq_points(i), subs(f, x, eq_points(i)), 30,'k', ...
            'DisplayName', 'Unstable')
```

```
    end
end
xlabel("x");
ylabel("f(x)")
title("Phase portrait")
```



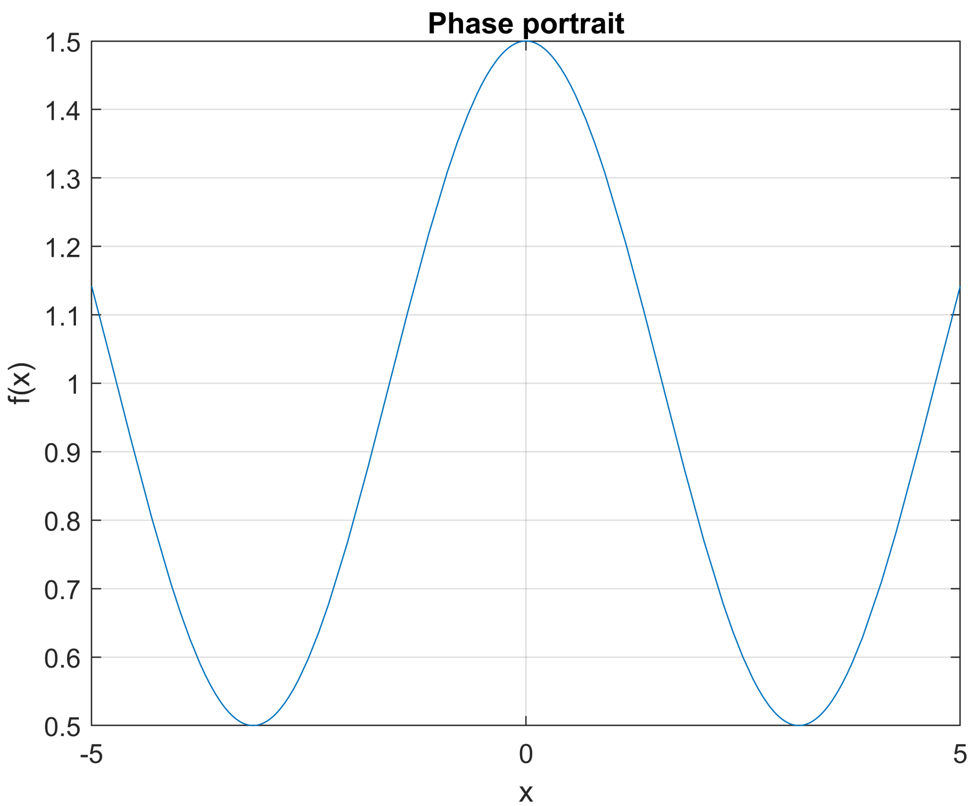
1-3 $\dot{x} = 1 + \frac{1}{2}\cos(x)$

```
syms x

f = 1+0.5*cos(x);

sol = solve(f==0, 'real', true);
eq_points = sol
```

```
eq_points =

Empty sym: 0-by-1
```

```

figure()
fplot(f, "DisplayName", "f(x)")
hold on;
grid on;

for i = 1:size(eq_points,1)
```

```
    if subs(diff(f), x, eq_points(i)) < 0
        scatter(eq_points(i), subs(f, x, eq_points(i)), 30,'k', ...
            'filled', 'DisplayName', 'Stable')
    else
        scatter(eq_points(i), subs(f, x, eq_points(i)), 30,'k', ...
            'DisplayName', 'Unstable')
    end
end
xlabel("x");
ylabel("f(x)")
title("Phase portrait")
```



Note: 1-3 does **NOT** have any stability points as it never crosses $f(x) = 0$

1-4 $\dot{x} = e^{-x}\sin(x)$

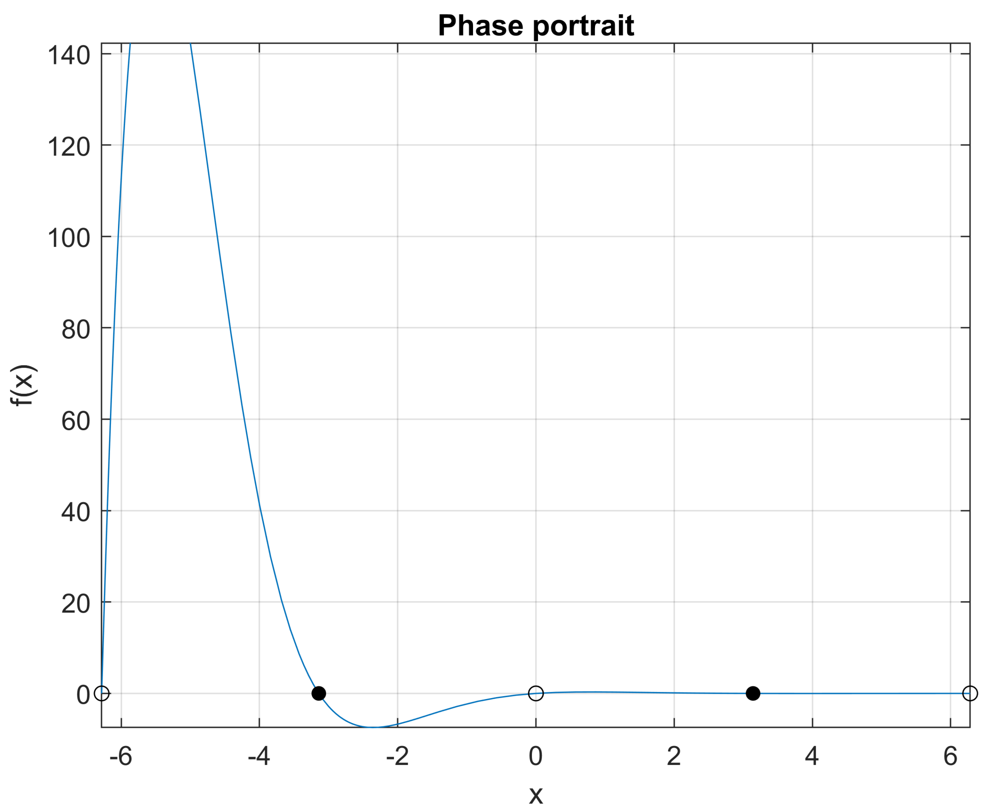
```
syms x

f = exp(-x)*sin(x)
```

f = $e^{-x} \sin(x)$

```

sol = solve(f==0, 'real', true, 'returnconditions', true);
eq_points = sol.x
```

eq_points = πk

```
figure()
fplot(f, "DisplayName", "f(x)")
hold on;
grid on;

for i = -2:2
    point = subs(eq_points, k, i);
    if subs(diff(f), x, point) < 0
        scatter(point, subs(f, x, point), 30,'k', ...
            'filled', 'DisplayName', 'Stable')
    else
        scatter(point, subs(f, x, point), 30,'k', ...
            'DisplayName', 'Unstable')
    end
end
xlabel("x");
ylabel("f(x)")
title("Phase portrait")
```



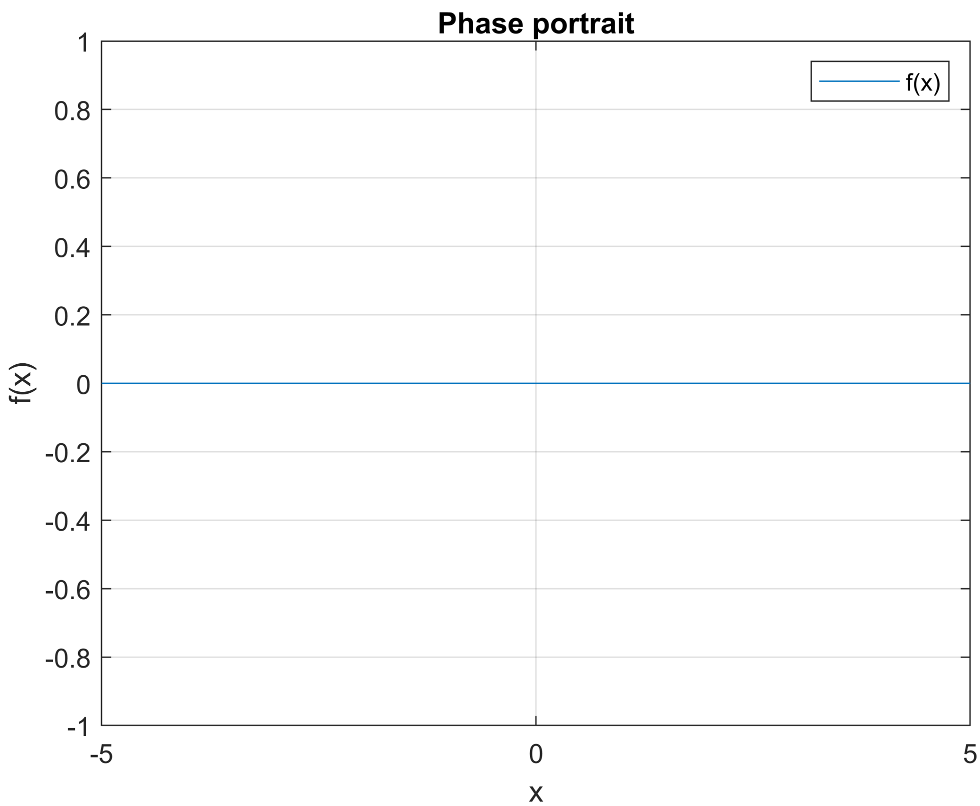
$f(x)$ has an infinite number of eq points at $x = k\pi$, alternating between stable and unstable where $x = 0$ is unstable

Problem 2

2-1 Find $f(x)$ such that every real number is an eq point

$f(x) = 0$

```
syms x
f = 0*x;

figure()
fplot(f, "DisplayName", "f(x)")
hold on;
grid on;
xlabel("x");
ylabel("f(x)")
title("Phase portrait")
legend()
```

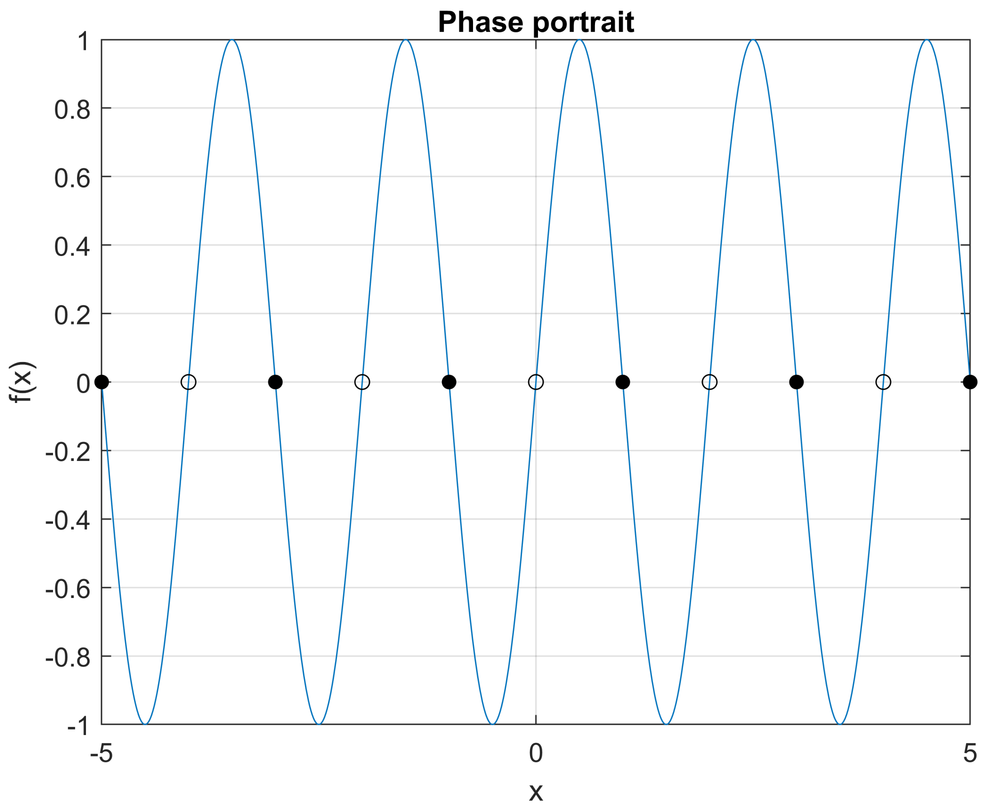


2-2 Every integer is an eq point, and there are no others

$f(x) = \sin(\pi x)$

```
syms x
f = sin(pi*x);

sol = solve(f==0, 'real', true, 'ReturnConditions', true);
eq_points = sol.x;

figure()
fplot(f, "DisplayName", "f(x)")
hold on;
grid on;
```

```
for i = -5:5
    point = subs(eq_points, k, i);
    if subs(diff(f), x, point) < 0
        scatter(point, subs(f, x, point), 30,'k', ...
            'filled', 'DisplayName', 'Stable')
    else
        scatter(point, subs(f, x, point), 30,'k', ...
            'DisplayName', 'Unstable')
    end
end
xlabel("x");
ylabel("f(x)")
title("Phase portrait")
```



2-3 There are precisely 3 eq points, and all are stable

This is not possible in 1 dimension as eq points must alternate between stable and unstable

2-4 Every point on the circle of radius 1 is an eq point and there are no others

- $r = \sqrt{x^2 + y^2}$

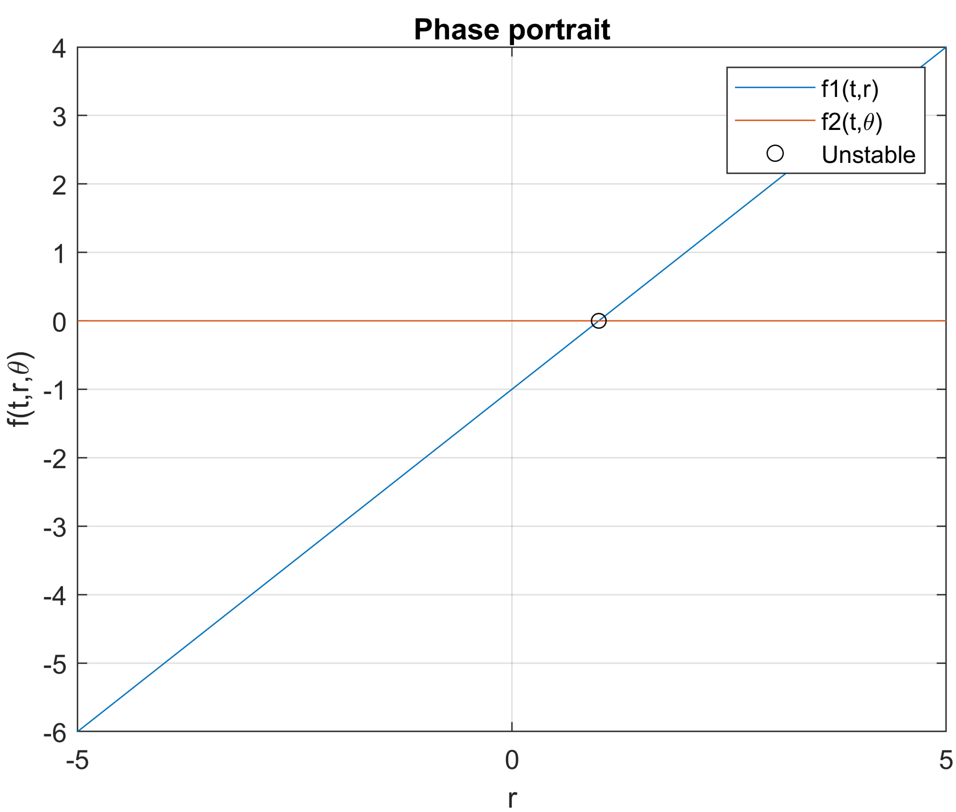
  $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$f(t, r, \theta) = \begin{bmatrix} r - 1 \\ 0 \end{bmatrix}$$

```
syms r theta

f  = [(r-1);0];

sol = solve(f==0, 'real', true, 'ReturnConditions', true);
eq_points = [sol.r]
```

eq_points = 1

```
figure()
fplot(f(1), "DisplayName", "f1(t,r)")
hold on;
fplot(f(2), "DisplayName", "f2(t,\theta)")

grid on;
legend()

scatter(eq_points(1), subs(f(1), r, eq_points(1)), 30,'k', ...
    'DisplayName', 'Unstable')

xlabel("r")
ylabel("f(t,r,\theta)")
title("Phase portrait")
```



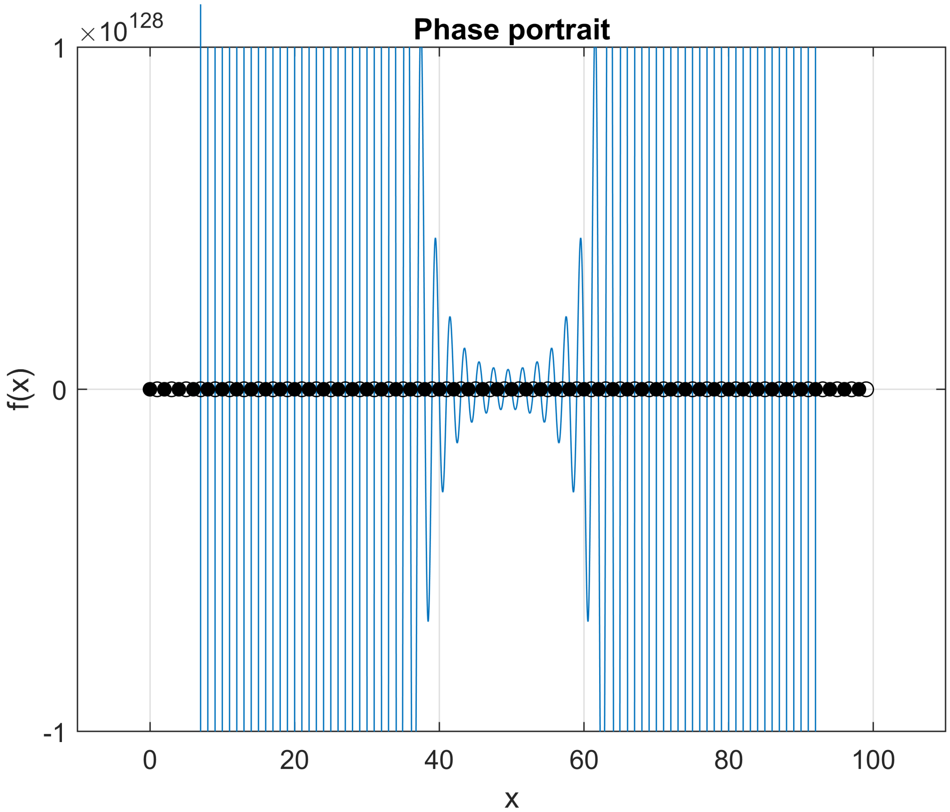
2-5 There are precisely 100 equilibrium points

$$f(x) = x(x-1)(x-2)(x-3)\ldots(x-99) = \prod_{i=0}^{99}(x-i)$$

```
syms x
f = x;
for i=1:99
    f = f*(x-i);
end
sol = solve(f==0, 'real', true, 'ReturnConditions', true);
eq_points = sol.x;
disp(size(eq_points));
```

```
   100     1
```

```

figure()
fplot(f, [-10,110])
ylim([-10^128, 10^128])
hold on
grid on
for i = 1:size(eq_points,1)
    if subs(diff(f), x, eq_points(i)) < 0
        scatter(eq_points(i), subs(f, x, eq_points(i)), 30,'k', ...
            'filled', 'DisplayName', 'Stable')
    else
        scatter(eq_points(i), subs(f, x, eq_points(i)), 30,'k', ...
            'DisplayName', 'Unstable')
    end
end
xlabel("x");
ylabel("f(x)")
title("Phase portrait")
```



Problem 3

$\dot{x} = \sigma\,(y - x) \qquad \dot{y} = \text{rx} - y - \text{xz} \qquad \dot{z} = \text{xy} - \text{bz}$

3-1 Find the equilibrium points

```
syms x y z s r b

X = [x;y;z];
f = [s*(y-x); r*x-y-x*z; x*y-b*z];

sol = solve(f==0, X);
eq_points = [sol.x, sol.y, sol.z]
```

eq_points =

$$\begin{pmatrix} 0 & 0 & 0 \\ -\sqrt{b\ (r-1)} & -\sqrt{b\ (r-1)} & r-1 \\ \sqrt{b\ (r-1)} & \sqrt{b\ (r-1)} & r-1 \end{pmatrix}$$

3-2 Linearize the dynamics about the eq points

```
jac = jacobian(f,X);
```

3-2-1 $r \leq 1$ and $b, \sigma > 0$

```
eigs = cell(3,1);
```

```
for i = 1:3
    eigs{i,1} = eig(subs(jac, X, eq_points(i,:)'));
    eigs{i,1} = subs(eigs{i,1}, [r,b,s], [0,1,1]);
    vpa(eigs{i,1},2)
end
```

```
ans =
```

$$\begin{pmatrix} -1.0 \\ -1.0 \\ -1.0 \end{pmatrix}$$

```
ans =
```

$$\begin{pmatrix} 0.62 + 1.5\text{e-}11\,\mathrm{i} \\ -1.6 - 7.3\text{e-}12\,\mathrm{i} \\ -2.0 - 7.3\text{e-}12\,\mathrm{i} \end{pmatrix}$$

```
ans =
```

$$\begin{pmatrix} 0.62 + 1.5\text{e-}11\,\mathrm{i} \\ -1.6 - 7.3\text{e-}12\,\mathrm{i} \\ -2.0 - 7.3\text{e-}12\,\mathrm{i} \end{pmatrix}$$

$\mathbb{R}\{\lambda_1\} < 0$, so it is **stable**.

$\mathbb{R}\{\lambda_2\}$ contains both positive & negative values, so it is **unstable**. $\mathbb{I}\{\lambda_2\}$ is negligible.

$\mathbb{R}\{\lambda_3\}$ contains both positive & negative values, so it is **unstable**. $\mathbb{I}\{\lambda_3\}$ is negligible.

3-2-2 $r > 1$ and $b, \sigma > 0$ where $\sigma > b + 1$

```
for i = 1:3
    eigs{i,1} = eig(subs(jac, X, eq_points(i,:)'));
    eigs{i,1} = subs(eigs{i,1}, [r,b,s], [2,3,5]);
    vpa(eigs{i,1},2)
end
```

```
ans =
```

$$\begin{pmatrix} -3.0 \\ -6.7 \\ 0.74 \end{pmatrix}$$

```
ans =
```

$$\begin{pmatrix} -1.3 - 1.7\,\mathrm{i} \\ -1.3 + 1.7\,\mathrm{i} \\ -6.5 - 2.3\text{e-}10\,\mathrm{i} \end{pmatrix}$$

```
ans =
```

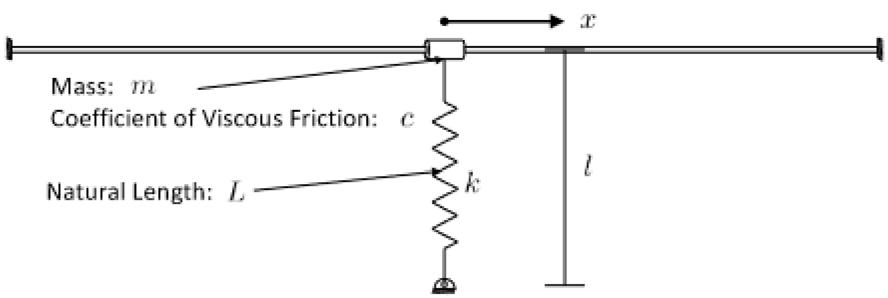
$$\begin{pmatrix} -1.3 - 1.7\,\mathrm{i} \\ -1.3 + 1.7\,\mathrm{i} \\ -6.5 - 2.3\text{e-}10\,\mathrm{i} \end{pmatrix}$$

$\mathbb{R}\{\lambda_1\}$ contains both positive & negaitve values, so it is **unstable**.

$\mathbb{R}\{\lambda_2\} < 0$ and $\mathbb{I}\{\lambda_2\}$ contains both positive & negative values, so it is a **stable focus**.

$\mathbb{R}\{\lambda_3\} < 0$ and $\mathbb{I}\{\lambda_3\}$ contains both positive & negaitve values, so it is a **stable focus**.

Problem 4



$m = 1kg \quad b = 0.1Nsm^{-1} \quad k = 1Nm^{-1} \quad L = 1m$

4-1 Derive equations of motion wrt x

Newtons 2nd: $m\ddot{x} = \sum_i F_i$

Def: $d = \sqrt{x^2 + l^2}$ (spring distance)

Derive forces:

- Spring force: $f_s = -k(d - x)$
- Along x-axis: $f_s^{(x)} = -k(d - L)sin(\theta) = -k(d - L)\frac{x}{d} = -kx\left(1 - \frac{L}{d}\right)$
- Friction force: $f_f^{(x)} = -c\dot{x}$

Equation of motion:

- $m\ddot{x} + c\dot{x} + kx\left(1 - \frac{L}{d}\right) = 0$

Statespace form:

- $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$
- $f(t, x) = \dot{x} = \begin{bmatrix} x_2 \\ -\dfrac{\mathrm{cx}_2 + \mathrm{kx}_1\left(1 - \dfrac{L}{\sqrt{x_1^2 + l^2}}\right)}{m} \end{bmatrix}$

4-2 Show the system undergoes a supercritical pitchfork bifurcation at r=0

Rewrite state space in terms of $r = l - L$

$$f(t,x) = \begin{bmatrix} x_2 \\ -\dfrac{\mathrm{c}x_2 + \mathrm{k}x_1\left(1 - \dfrac{L}{\sqrt{x_1^2 + (r+L)^2}}\right)}{m} \end{bmatrix}$$

```
syms x1 x2 c k m L r

x = [x1;x2];
f = [x2;-(c*x2+k*x1*(1-L/sqrt(x1^2+(r+L)^2)))/m];
f = subs(f, [c,k,m,L], [0.1,1,1,1]);

eq_points = solve(f==0, x, 'ReturnConditions', true);
eq_points = [eq_points.x1, eq_points.x2]
```

eq_points =

$$\begin{pmatrix} 0 & 0 \\ \sqrt{-r\ (r+2)} & 0 \\ -\sqrt{-r\ (r+2)} & 0 \end{pmatrix}$$

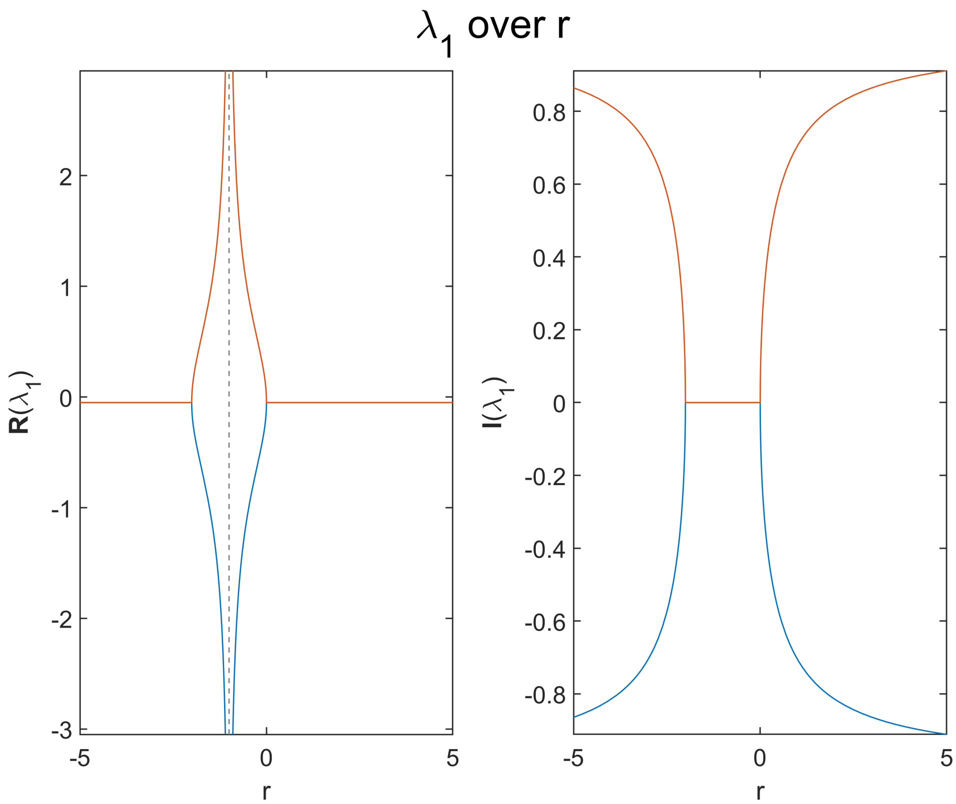
1st eq. point ([0,0]) is always valid.

2nd & 3rd eq. points only valid on $r \in (-2,0)$

<u>This implies a pitchfork bifurcation</u>

Now, must determine whether this is super- or sub- critical

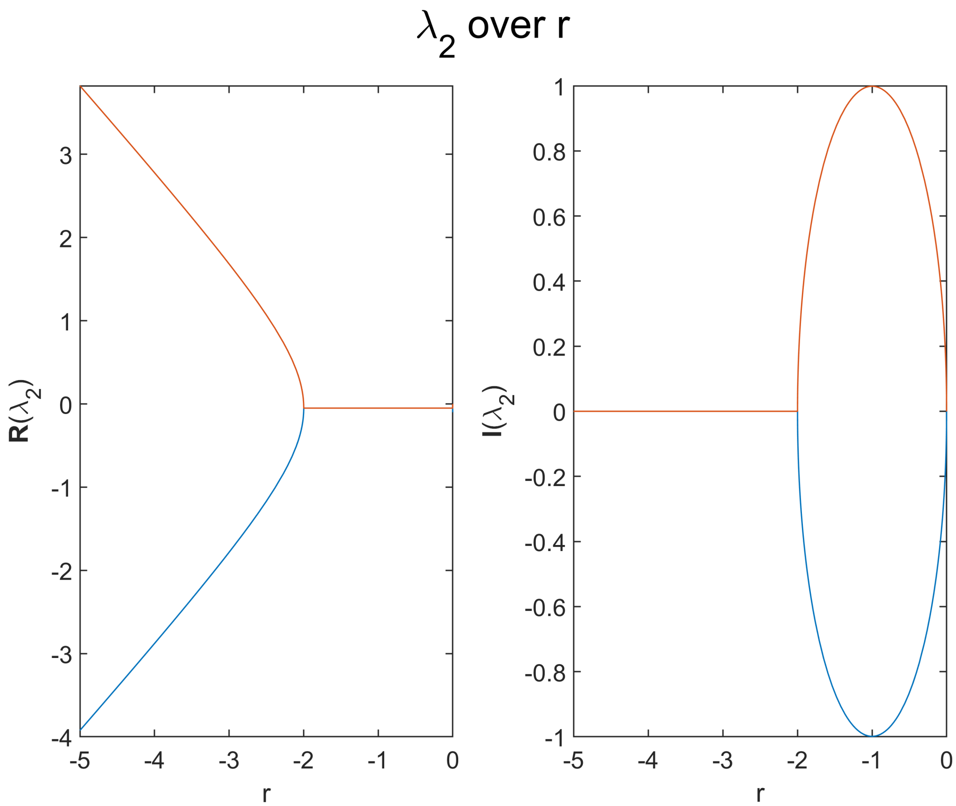
```
jac = jacobian(f,x);
eig1 = eig(subs(jac, x, eq_points(1,:)'));
figure()
sgtitle("\lambda_1 over r")

subplot(1,2,1)
fplot(real(eig1), [-5,5])
xlabel("r")
ylabel("\bf{R}\rm(\lambda_1)")

subplot(1,2,2)
fplot(imag(eig1), [-5,5])
xlabel("r")
ylabel("\bf{I}\rm(\lambda_1)")
```



$\mathbb{R}\{\lambda_1(r), r \in (-2,0)\}$ contains both a positive and negative component while $\mathbb{I}\{\lambda_1(r), r \in (-2,0)\} = 0$. **This is a saddle point - not stable.**

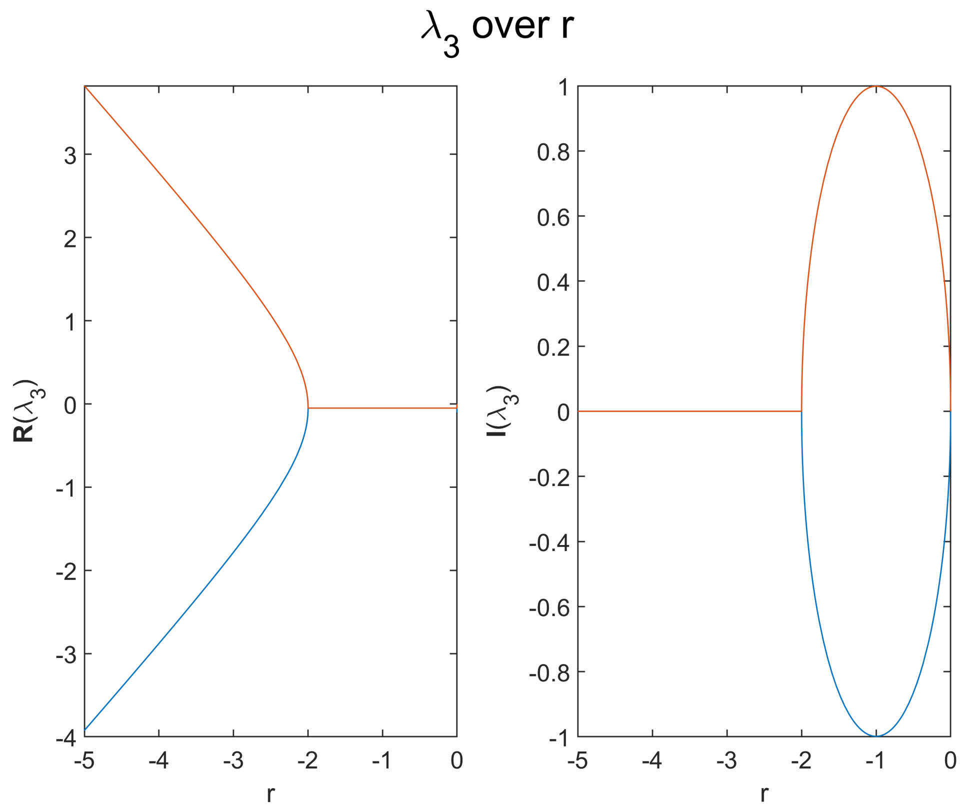
$\mathbb{I}\{\lambda_1(r), r \in (0,5)\}$ contains both a positive and negative component while $\mathbb{R}\{\lambda_1(r), r \in (0,5)\} = 0$. **This is a stable focus - stable.**

```
eig2 = eig(subs(jac, x, eq_points(2,:)'));
figure()
sgtitle("\lambda_2 over r")

subplot(1,2,1)
fplot(real(eig2), [-5,0])
xlabel("r")
ylabel("\bf{R}\rm(\lambda_2)")

subplot(1,2,2)
fplot(imag(eig2), [-5,0])
xlabel("r")
ylabel("\bf{I}\rm(\lambda_2)")
```



$\mathbb{R}\{\lambda_2(r), r \in (-2,0)\} = 0$ while $\mathbb{I}\{\lambda_2(r), r \in (-2,0)\}$ contains both positive and negative components. **This is a stable focus - stable.**

```
eig3 = eig(subs(jac, x, eq_points(3,:)'));
figure()
sgtitle("\lambda_3 over r")

subplot(1,2,1)
fplot(real(eig3), [-5,0])
xlabel("r")
ylabel("\bf{R}\rm(\lambda_3)")

subplot(1,2,2)
fplot(imag(eig3), [-5,0])
xlabel("r")
ylabel("\bf{I}\rm(\lambda_3)")
```



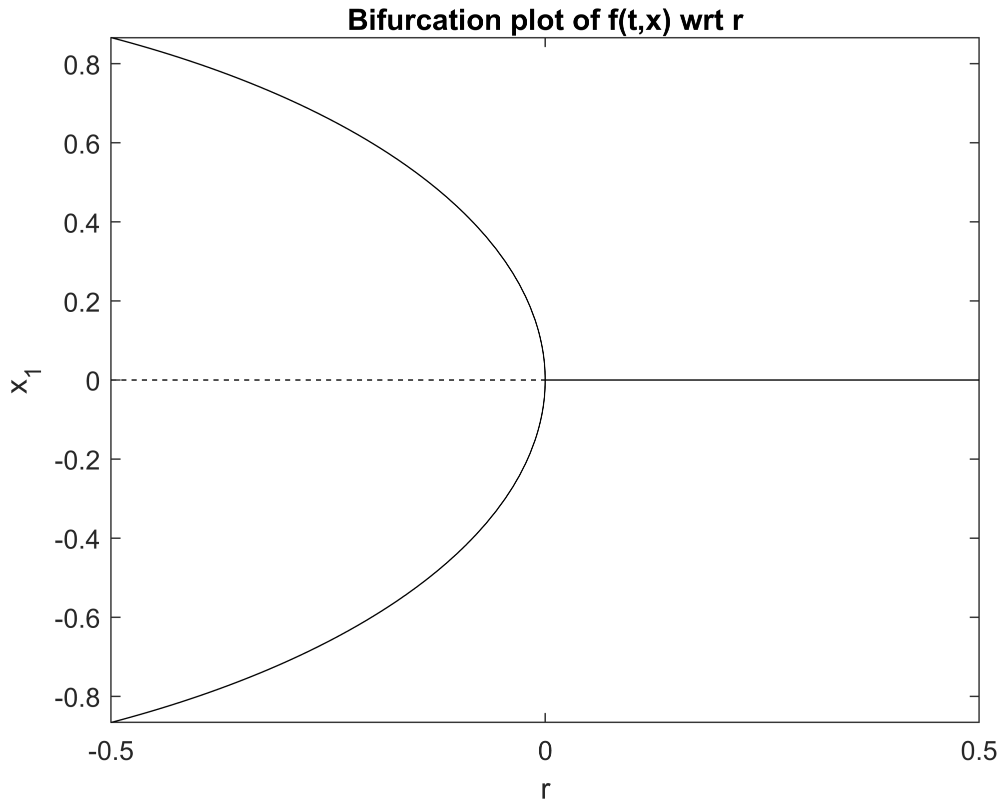
$\mathbb{R}\{\lambda_3(r), r \in (-2,0)\} = 0$ while $\mathbb{I}\{\lambda_3(r), r \in (-2,0)\}$ contains both positive and negative components. **This is a stable focus - stable.**

Summary:

- λ_1 is unstable on $r \in (-2,0)$ & stable on $r \in (0,2)$
- $\lambda_{2,3}$ are stable on $r \in (-2,0)$ & DNE on $r \in (0,2)$
- Thus, there exists a supercritical pitchfork bifurcation at $r = 0$.

4-3 Plot the bifurcation on $r \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

```
figure()

fplot(eq_points(1,1), [-0.5,0], '--k')
hold on;
fplot(eq_points(2:3,1), [-0.5,0], '-k')
fplot(eq_points(1,1), [0,0.5], '-k')
title("Bifurcation plot of f(t,x) wrt r")
xlabel("r")
ylabel("x_1")
```



Note that x_2 is always stable at 0