ECEN5463 | HW2

Collin Thornton

Problem 1

Must find all values of $r \in \mathbb{R}$ such that the ball of radius $\overline{B}(0,r) \subset \mathbb{R}^m$ with resepct to $\dot{x} = f(x), x \in \mathbb{R}^m$ implies $\forall x \in \partial B(0,r), f^T(x)n(x) \leq 0$

```
syms x1 x2 r th

X = [x1;x2];
f = [x1 + x2 - x1*(x1^2+x2^2);
     -2*x1 + x2 - x2*(x1^2 + x2^2)];
```

Equation of a circle: $x^2 + y^2 = r^2$

Normal vector = $< \cos(\theta), \sin(\theta) >$

Convert dynamics to polar coordinates

- $r\,\dot{r} = x_1\dot{x_1} + x_2\,\dot{x_2}$
- $r^2\,\dot{\theta}\ = x_1\,\dot{x_2} - x_2\,\dot{x_1}$

```
n = [cos(th); sin(th)];

rrdot = simplify(x1*f(1)+x2*f(2));
r2the = simplify(x1*f(2)-x2*f(1));

rdot = simplify(subs(rrdot, [x1,x2], [r*cos(th), r*sin(th)])/r);
tdot = simplify(subs(r2the, [x1,x2], [r*cos(th), r*sin(th)])/r^2);
%f_p = [rdot;tdot]
```

Note: Couldn't get the equations for $\dot{r}$ and $\dot{\theta}$ to simplify to an appropriate answer. Simplifying to the following equation:

```
f_p = collect(simplify(subs(f, [x1,x2], [r*cos(th), r*sin(th)])))
```

f_p =

$$\begin{pmatrix} r\ (-\cos(\mathrm{th})\ r^2 + \cos(\mathrm{th}) + \sin(\mathrm{th})) \\ -r\ (\sin(\mathrm{th})\ r^2 + 2\cos(\mathrm{th}) - \sin(\mathrm{th})) \end{pmatrix}$$

```
G = collect(simplify(transpose(f_p)*n))
```

G =

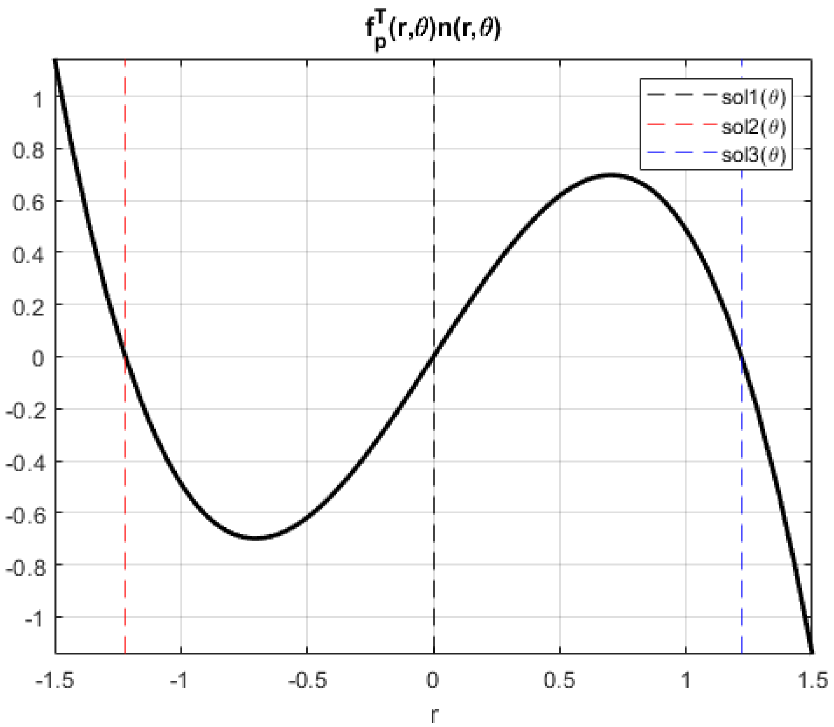
$$-r\left(r^2 + \frac{\sin(2\,\mathrm{th})}{2} - 1\right)$$

```
sol = simplify(solve(G==0,r))
```

sol =

$$\begin{pmatrix} 0 \\ -\frac{\sqrt{2}\ \sqrt{2-\sin(2\,\mathrm{th})}}{2} \\ \frac{\sqrt{2}\ \sqrt{2-\sin(2\,\mathrm{th})}}{2} \end{pmatrix}$$

```
theta =-0.9;
figure()

fplot(subs(G,th,theta), [-1.5,1.5], '-k', "LineWidth", 2)
hold on
xline(double(subs(sol(1),th,theta)), '--k');
xline(double(subs(sol(2),th,theta)), '--r');
xline(double(subs(sol(3),th,theta)), '--b');
legend()
grid on
xlabel("r")
title("f_p^T(r,\theta)n(r,\theta)")
legend("", "sol1(\theta)", "sol2(\theta)", "sol3(\theta)")
```



Thus, $\bar{B}(0,r)$ is positively invariant with respect to $f(x) = \dot{x}$ on the interval $r \geq \frac{\sqrt{2}\,\sqrt{2-\sin(\theta)}}{2}$

Problem 2

2-1 Show that the system $\dot{x} = x^2$ admits solutions that blow up to infinity in finite time

One solution is to prove that the function $f(t, x)$ is not Lipschitz continuous with respect to x.

- $f(t, x) = \dot{x} = x^2$

First step, find the limit of $\frac{d}{\mathrm{d}x} f(t, x) = 2x$ as $x \to \pm\infty$.

- $\lim_{x\to\infty} \frac{\mathrm{d}}{\mathrm{d}x} f(t, x) \ = 2(\infty\ )$
- $\lim_{x\to-\infty} \frac{\mathrm{d}}{\mathrm{d}x} f(t, x) = 2(\infty\ )$

As $\frac{\mathrm{d}}{\mathrm{d}x} f(t, x)$ is not bounded as $x \to \pm\infty$, $||f(t, x) - f(t, x + 1)|| = \infty$, as $x \to \pm\infty$

- **Therefore, $f(t, x)$ is not Lipschitz continuous and thus admits solutions that blow up to infinity in finity time.**

This holds as:

- $$F(t, x) = \int_t f(t, x)\,\mathrm{dt} = \int_t x^2\,\mathrm{dt}$$

Take the case $t = [0, 1]$, $x = \infty$.

- $$F(1, \infty) = \lim_{x\to\infty} \int_0^1 x^2\,\mathrm{dt} \ = \lim_{x\to\infty} x^2 = \infty^2 = \infty$$

Problem 3 Prove whether the following functions are continous from $\mathbb{R}$ to $\mathbb{R}$

Continuous functions $f(x)$ must satisfy the following points to be called piecewise continous:

1. $t \to f(t, x)$ contains a finite number of discontinuties at $t \in J$
2. $\forall\, t \in J,\ \lim_{h\to0^+} f(t + h, x)$ and $\lim_{h\to0^-} f(t + h, x)$ exist and are finite

3-1 $f(x) = \max(0, x)$

```
syms x

f = x*heaviside(x);
```

```
disc = feval(symengine, 'discont', f, x)
```

disc = $\{0\}$

There is only 1 discontinuous point

```
limit(f,x,0,'left')
```

ans = 0

```
limit(f,x,0,'right')
```

ans = 0

```
subs(f,x,0)
```

ans = 0

Both limits exist and are equivalent. Therefore $f(x) = \max(0, x)$ is **continous**.

3-2 $f(x) = \begin{cases} \sin(x) & x > 0 \\ 0 & x \le 0 \end{cases}$

```
syms x
f = sin(x)*heaviside(x);

disc = feval(symengine, 'discont', f, x)
```

disc = $\{0\}$

There is only 1 discontinuous point

```
limit(f,x,0,'left')
```

ans = 0

```
limit(f,x,0,'right')
```

ans = 0

```
subs(f,x,0)
```

ans = 0

Both limits exist and are equivalent. Therefore $f(x) = \begin{cases} \sin(x) & x > 0 \\ 0 & x \le 0 \end{cases}$ is **continous**.

3-3 $f(x) = \mathrm{sgn}(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ 0 & x = 0 \end{cases}$

```
syms x
```

```
f = piecewise(x<0, -1, x==0, 0, x>0, 1);

disc =  feval(symengine, 'discont', f, x)
```

disc = $\{0\}$

There is only one discontinous point

```
limit(f,x,0,'left')
```

ans = -1

```
limit(f,x,0,'right')
```

ans = 1

```
subs(f,x,0)
```

ans = 0

Both limits exist, though they are not equivalent. Therefore $f(x) = \mathrm{sgn}(x)$ is **not continuous**. (Though it is piecewise continuous)

Problem 4

Controller state variable: x

- $x = \begin{bmatrix} q \\ \frac{\mathrm{d}}{\mathrm{d}t} q \\ \int (\hat{q} - q) \, \mathrm{d}t \end{bmatrix}$
- $u = \tau$
- $\ddot{q} = M(q)^{-1} \left(\tau - V_m(q, \dot{q}) \, \dot{q} + F_S(\dot{q})\right)$

```
x0 = zeros(6,1);
tspan = linspace(0,50,1000);

P = cell(3,1);
results = cell(3,2);

P{1} = [3.473;0.196;0.242]; % [p1,p2,p3]'
```

4-1 Develop a PID controll assuming motors can product unlimited torque

```
P{3} = [-1;-1];     % torque bounds
P{2} = [3,   0, 0;     % [kP, kI, kD]
        1.5, 0, 0];
results{1,1} = ode45(@(t,x) closedLoopDynamics(t,x,P), tspan, x0);
```

```
P{2} = [3,   0, 2;
        1.5, 0, 1];
results{2,1} = ode45(@(t,x) closedLoopDynamics(t,x,P), tspan, x0);

P{2} = [3,   0.05, 2;
        1.5, 0.05, 1];
results{3,1} = ode45(@(t,x) closedLoopDynamics(t,x,P), tspan, x0);
```

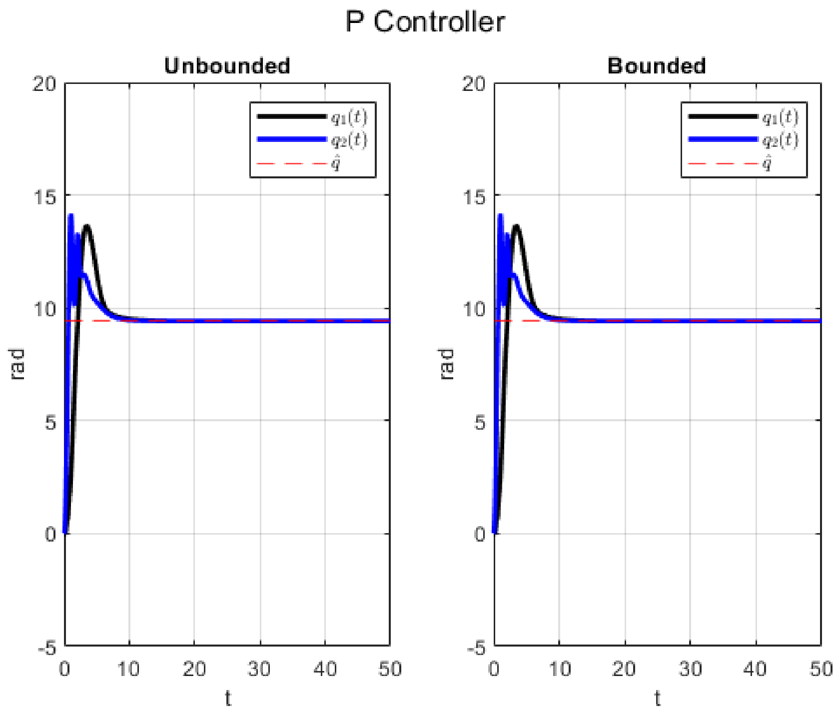
4-2 Limit torque to 60Nm and 20Nm, respectively, and compare output

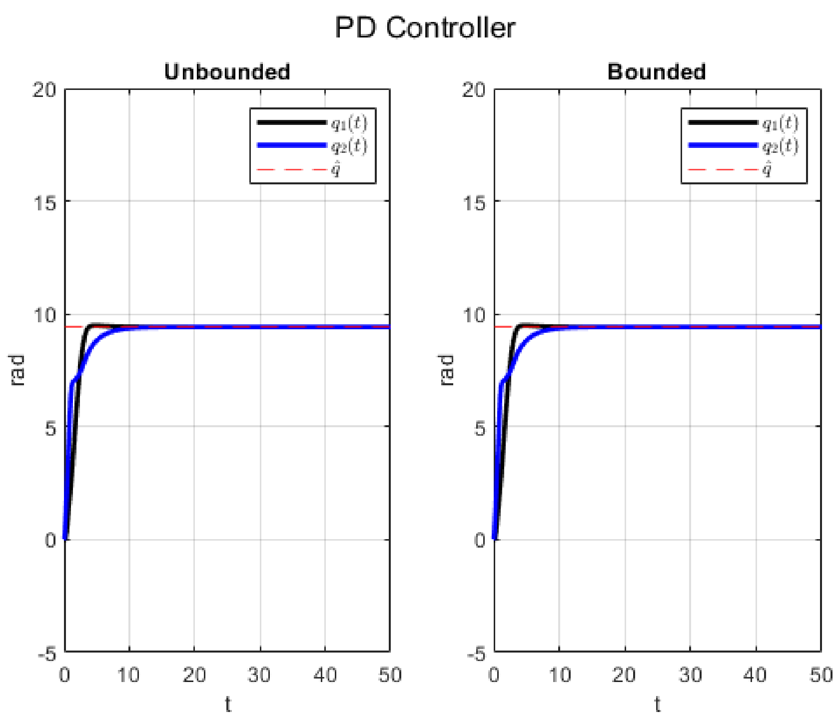
```
P{3} = [60; 20];
P{2} = [3,   0, 0;     % [kP, kI, kD]
        1.5, 0, 0];
results{1,2} = ode45(@(t,x) closedLoopDynamics(t,x,P), tspan, x0);

P{2} = [3,   0, 2;
        1.5, 0, 1];
results{2,2} = ode45(@(t,x) closedLoopDynamics(t,x,P), tspan, x0);

P{2} = [3,   0.05, 2;
        1.5, 0.05, 1];
results{3,2} = ode45(@(t,x) closedLoopDynamics(t,x,P), tspan, x0);
figure()
```

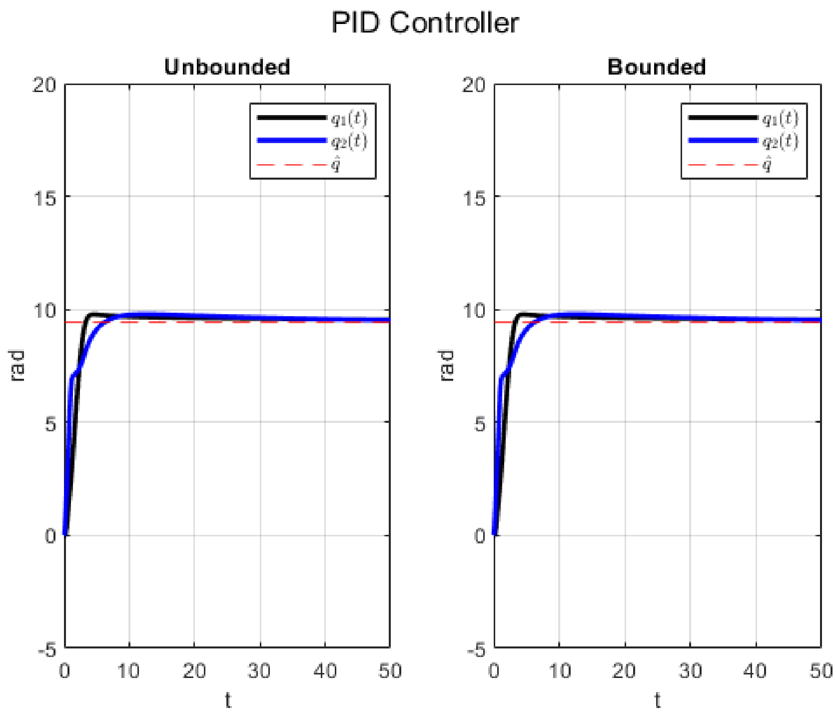
Plot and discuss

```
for i = 1:3
    figure()
    for j = 1:2
        if j == 1
            name = "Unbounded";
        else
            name = "Bounded";
        end
        subplot(1,2,j)
        plot(results{i,j}.x, results{i,j}.y(1,:), '-k', "LineWidth", 2)
        hold on
        plot(results{i,j}.x, results{i,j}.y(2,:), '-b', "LineWidth", 2)
        yline(3*pi, '--r');

        title(name);
        ylim([-5,20])
        xlim([0,50])

        grid on;
        legend("$q_1(t)$", "$q_2(t)$", "$\hat{q}$", "Interpreter", "latex");
        xlabel("t");
        ylabel("rad");
    end
    if i == 1
        sgtitle("P Controller")
    elseif i == 2
        sgtitle("PD Controller")
    else
```

```
        sgtitle("PID Controller")
    end
end
```







Show numerical difference between bounded and unbounded cases

```
norm(results{1,1}.y - results{1,2}.y)
```

```
ans = 0
```

```
norm(results{2,1}.y - results{2,2}.y)
```

```
ans = 0
```

```
norm(results{3,1}.y - results{3,2}.y)
```

```
ans = 0
```

Unbounded Case:

Observations about steady state

- All controllers reach steady state within 30 seoncds, converging on $q(t) = [3\pi, 3\pi]^T$ and minimizing ϵ .
- The P Controller & PD Controller reach steady state within 30 seconds
- The PID requires an additional 10 seconds to reach steady state

Observations about transient response

- The P controller overshoots by more than 5 radians
- The PD controller as a relatively small overshoot in $q_1(t)$ of less than $\frac{1}{2}$ radian
- The PID controller overshoots by about 1 radians

Discussion

- The I term is unecessary as there are no steady state disturbances, thus k_I should be set to 0 in order to minimize oscillation
- $k_I = [0.05, 0.05]^T$ in the PID controller. k_I is useful to minimize small error in steady state, though k_I must be small or oscillation will occur
- Increasing the k_D enables a more agressive transient response while reducing overshoot
- This said, a high k_D will lead to high frequency oscillation in the physical world due to noisy measurements

Bounded Case:

Everything is equivalent to the unbounded case as the controller never demands more than 60 Nm and 20 Nm of torque, respectively.

```
function x_dot = closedLoopDynamics(t, x, P)
    % This function "simulates" the behavior of a two-link robot under the
    % controller given by the 'control' function.
    x_dot = openLoopDynamics(t, x, control(t,x,P), P);
end


function x_dot = openLoopDynamics(t, x, u, P)
    % This function "simulates" the behavior of a two-link robot
    % manipulator using a model in the state space form.
    %
    % TODO: Use the given equations to compute x_dot using t, x, and u
    p = P{1};

    M = [p(1)+2*p(3)*cos(x(2)), p(2)+p(3)*cos(x(2));
         p(2) + p(3)*cos(x(2)), p(2)];
    V = [-p(3)*sin(x(2))*x(4), -p(3)*sin(x(2))*(x(3)+x(4));
         p(3)*sin(x(2))*x(3), 0];
    F = [8.45*tanh(x(3)); 2.35*tanh(x(4))];

    ddq = M\(u - V*x(3:4) - F);
    x_dot = [x(3:4);ddq;3*pi-x(1:2)];
end

function u = control(t, x, P)
    % This function calculates the control signal (a vector containing the
    % torque commands for the two joints of the two-link robot) as a
    % function of time and state
    % Use the given equations to compute u using from t and x here
    K = P{2};
    bounds = P{3};

    e = [3*pi;3*pi] - x(1:2);
    de = -x(3:4);
```

```
    s = [e,x(5:6),de];
    u = zeros(size(bounds));

    for i = 1:length(bounds)
        u(i) = K(i,:)*s(i,:)';

        if bounds(i) > 0
            u(i) = min(u(i), bounds(i));
        end
    end
end
```