ECEN 5463 | EX 2

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Problem 1

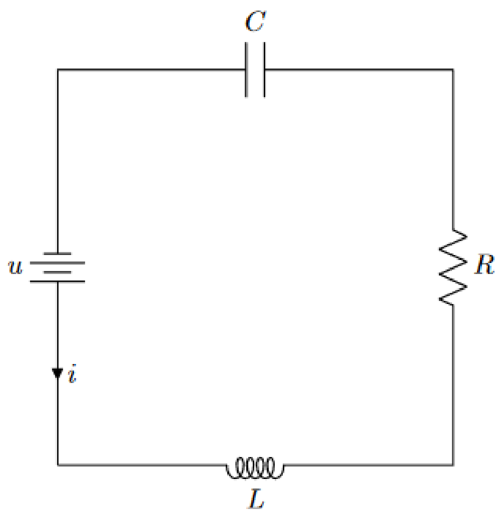


Figure 1: RLC circuit.

$$L\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} + R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{C}q = u,$$

where u is the applied voltage, $R > 0$, $L > 0$, and $C > 0$ are **unknown**.

1.1 Design an implementable adaptive controller to drive the charge across the capacitor to 5 C.

Closed Loop Dynamics:

$-L\ddot{e} - R\dot{e} + \frac{1}{C}(q_d - e) = u$, where $e = q_d - q$ and $q_d = 5$

Let $r = \dot{e} + \alpha e$, $\theta = \begin{bmatrix} \frac{1}{C} \\ R \\ L \end{bmatrix}$, and $L\dot{r} = Y(e,r)\theta - u = L\ddot{e} + L\alpha\dot{e}$

Thus, $Y(e,r) = \begin{bmatrix} q_d - q & -\dot{e} & \alpha\dot{e} \end{bmatrix}$.

Let $V(z) = \frac{1}{2}e^2 + \frac{L}{2}r^2 + \frac{1}{2}\tilde{\theta}^T\tilde{\theta}$, where $z = \begin{bmatrix} e \\ r \\ \tilde{\theta} \end{bmatrix}$ and $\tilde{\theta} = \theta - \hat{\theta}$ and $\hat{\theta}$ is the estimate of θ.

$$\dot{V}(z) = \frac{\partial V(z)}{\partial z}\dot{z} = \begin{bmatrix} e & \mathrm{Lr} & \tilde{\theta}^T \end{bmatrix} \begin{bmatrix} \dot{e} \\ \dot{r} \\ \dot{\tilde{\theta}} \end{bmatrix} = e\dot{e} + \mathrm{rL}\dot{r} + \tilde{\theta}^T\dot{\tilde{\theta}}$$

$$\dot{V}(z) = e(r - \alpha e) + r(Y(e,r)\theta - u) - \tilde{\theta}^T\dot{\hat{\theta}}$$

$$\dot{V}(z) = \mathrm{er} - e^2 a + r(Y(e,r)\theta - u) - \tilde{\theta}^T\dot{\hat{\theta}}$$

We need to define u and $\dot{\hat{\theta}}$ such that $\dot{V}(z)$ is either negative definite or negative semidefinite.

Try $u = Y(e,r)\hat{\theta} + e + r$ and $\dot{\hat{\theta}} = \mathrm{rY}^T(e,r)$

$$\dot{V}(z) = er - e^2 a + \mathrm{rY}(e,r)\left(\theta - \hat{\theta}\right) - r^2 - er - \tilde{\theta}^T\mathrm{rY}^T(e,r)$$

$$\dot{V}(z) = -e^2 a - r^2 + \mathrm{rY}(e,r)\tilde{\theta} - \tilde{\theta}^T\mathrm{rY}^T(e,r)$$

$$\dot{V}(z) = -e^2 a - r^2$$

$\dot{V}(z) \leq 0$ on the line $z = \begin{bmatrix} 0 \\ 0 \\ \tilde{\theta} \end{bmatrix}$. Now we must apply the invariance principle to further evaluate the equilibrium point. See below for an analysis of the hypotheses of Corollary 90.

1.) $D \subset \mathbb{R}^5$ is an open set. Here, $D = \mathbb{R}^5$ as $V(z) > 0 \quad \forall z \in \mathbb{R}^5$ and $\dot{V}(z) \leq 0 \quad \forall z \in \mathbb{R}^5$

2.) $\dot{z} = g(z)$ is locally Lipschitz. This is true as all components of $g(z) = \begin{bmatrix} \dot{e} & \dot{r} & \dot{\tilde{\theta}} \end{bmatrix}^T$ are locally Lipschitz.

3.) $V(z)$ is continuously differentiable

4.) By Claim 64 and acknowledging that $V(z)$ is radially unbounded, given any $\beta > 0$, Ω'_β is a filled ellipse, compact, connected, and in $D = \mathbb{R}^5$ and is positively invariant. Let $\Omega = \Omega'_\beta$ for any $\beta > 0$.

5.) $\dot{V}(z) := \frac{\partial V(z)}{\partial z}g(z) \leq 0, \quad \forall x \in \Omega$. This is true as $\dot{V}(z)$ was designed to be negative semidefinite.

6.) $E = \left\{ z \in \Omega \mid \frac{\partial V(z)}{\partial z} g(z) = 0 \right\} = \Omega \cap \left\{ \begin{bmatrix} e \\ r \\ \widetilde{\theta} \end{bmatrix} \in R^5 \mid e, r = 0 \right\}$

7.) Find $M \subset E$, where M is the largest invariant set in E.

Let's rewrite the dynamics:

$$\dot{e} = r - \alpha e$$

$$\dot{r} = \frac{(Y(e,r)\theta - u)}{L} = \frac{Y(e,r)\widetilde{\theta} - e - r}{L}$$

$$\dot{\widetilde{\theta}} = -\dot{\widehat{\theta}} = -\mathrm{r}\mathrm{Y}^T(e,r)$$

Note: $Y(e,r) = \begin{bmatrix} q_d & \alpha e - r & \alpha(r - \alpha e) \end{bmatrix}$

We know that, as the origin is an equilibrium point, M must include the set $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$. To see if we can find a larger invariant set, let $\zeta^0 = \begin{bmatrix} 0 \\ 0 \\ \widetilde{\theta} \end{bmatrix}$. Then at $t = 0$, $g(z) = \begin{bmatrix} r - \alpha e \\ \frac{Y(e,r)\widetilde{\theta} - e - r}{L} \\ -\mathrm{r}\mathrm{Y}^T(e,r) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{q_d \widetilde{\theta}_1}{L} \\ 0 \end{bmatrix}$. The second term is clearly only 0 if $q_d \widetilde{\theta}_1 = 0$.

Thus $M = \left\{ \begin{bmatrix} e \\ r \\ \widetilde{\theta}_1 \\ \widetilde{\theta}_2 \\ \widetilde{\theta}_3 \end{bmatrix} \in \Re^5 \mid e, r, \widetilde{\theta}_1 = 0 \right\}$ and $M \subset E$. Given any $\zeta^0 \in \Omega$, $\lim_{t \to \infty} \mathrm{dist}(\phi(t, \zeta^0), M) = 0$ and $\lim_{t \to \infty} \phi_e(t, \zeta^0) = 0$. In other words, the the angular position will approach the setpoint when $\zeta^0 \in \Omega$.

As $V(z)$ is radially unbounded, we can pick an arbitrarily large β such that $\zeta^0 \in \Omega'_\beta$. Letting $\Omega = \Omega'_\beta$ shows **global asymptotic stability (GAS).**

1.2 Bonus

In order to show that $\widehat{\theta}_1$ converges to $\frac{1}{C}$, let us use $Y(e,r)$. We defined $Y(e,r)$ such that $L\dot{r} = Y(e,r)\theta - u$.

We've already shown GAS of the origin. In order for r to stay at the origin, $\dot{r}$ must also be 0. This implies that

$$0 = Y(e,r)\theta - Y(e,r)\widehat{\theta} - e - r$$

$$0 = Y(e,r)\widetilde{\theta} \qquad \text{As } e, r = 0$$

$$0 = \begin{bmatrix} q_d & \alpha e - r & \alpha(r - \alpha e) \end{bmatrix} \widetilde{\theta}$$

$$0 = \begin{bmatrix} q_d & 0 & 0 \end{bmatrix} \widetilde{\theta} \qquad \text{As } e, r = 0$$

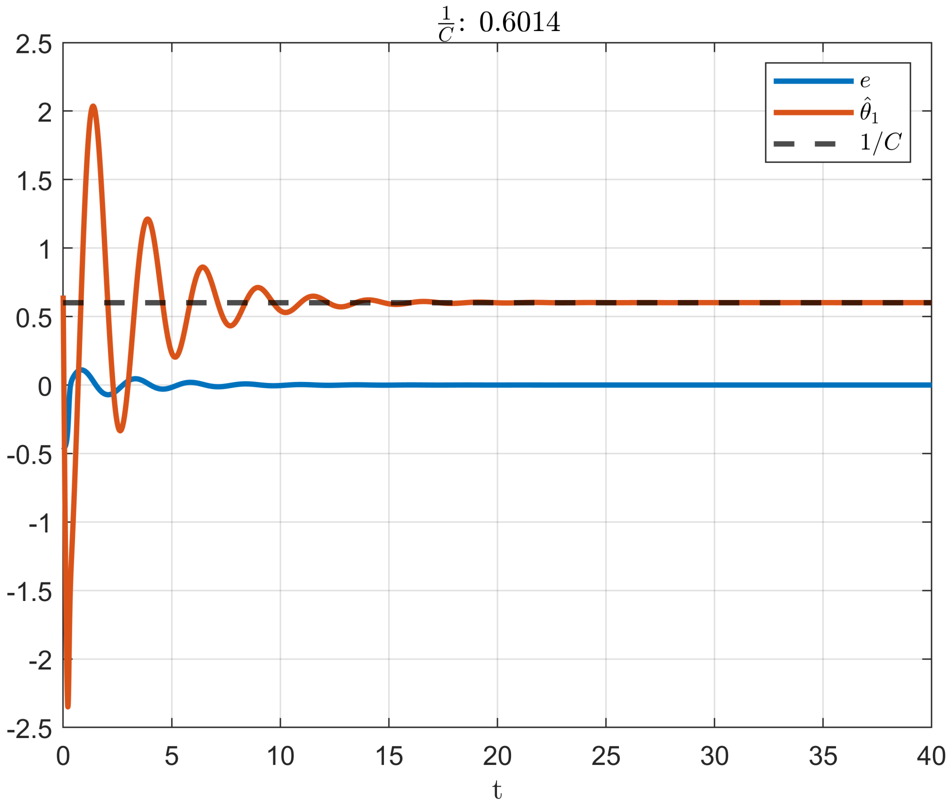
Thus, we have proven that $\widetilde{\theta}_1 q_d = 0$ and $\widetilde{\theta}_1$ will converge to 0 as the system approaches equilibrium. If $q_d \neq 0$, this shows that $\widehat{\theta}_1$ converges to $\frac{1}{C}$.

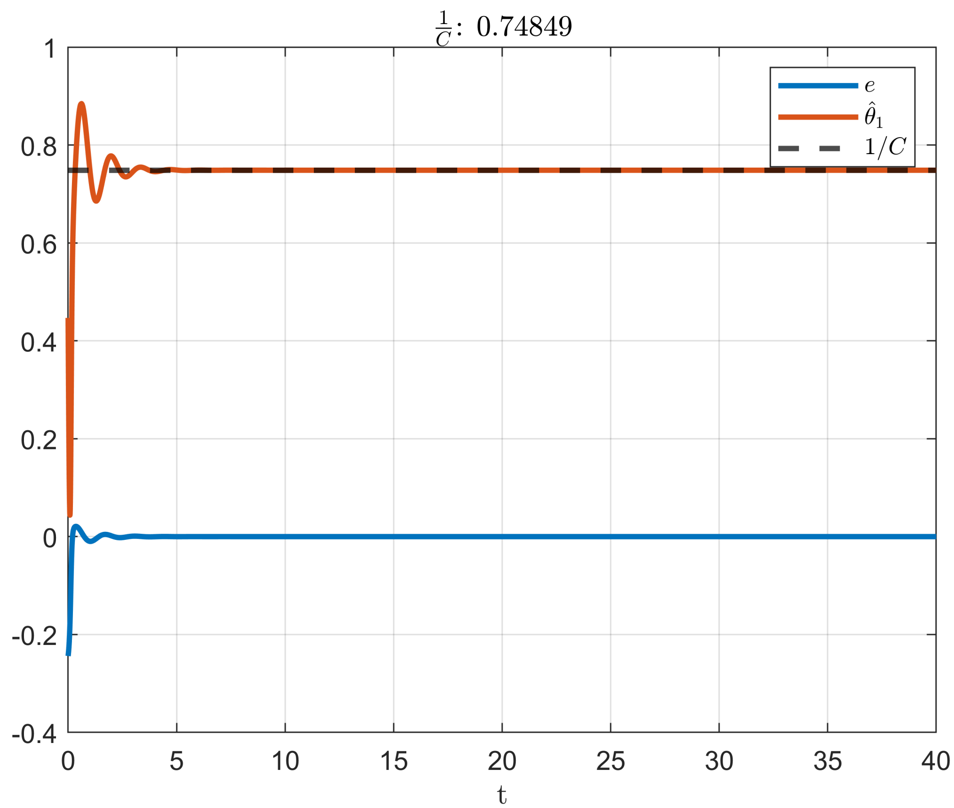
1.3 Simulate the controller to show that it works as expected.

This will generate two plots show the controller working with randomized parameter values, estimates, and initial conditions. Notice that $\widetilde{\theta} \to 0$ over the course of the simuation. This proves that $\widehat{\theta} \to \frac{1}{C}$ and that the bonus is satisfied.

```
clearvars;
close all;

for i=1:2
    p.theta = rand(3,1);
    p.alpha = 8;     % alpha > 0
    p.qd = 5;

    % x0 = [q; dq]
    x0 = (rand(2,1)-0.5);   % Pick a random starting charge & current
    theta_hat_dot0 = rand(3,1); % Start with zeros for estimate

    y0 = [x0; theta_hat_dot0];

    tspan = [0 40];
```

```
    [t,y] = ode45(@(t,y) closedLoopDynamicsRLC(t,y,p), tspan, y0);

    figure()
    plot(t,y(:,1), t, y(:,3), "LineWidth", 2);
    hold on;
    yline(p.theta(1), '--k', "LineWidth", 2)

    xlabel("t", "Interpreter", "latex")
    legend("$e$", "$\hat\theta_1$", "$1/C$", "Interpreter", "latex");

    title("$\frac{1}{C}$: " + p.theta(1), "Interpreter", "latex")
    grid on;
end
```





Problem 2

$$M(q)\ddot{q} + V_m(q,\dot{q})\dot{q} + F_s(\dot{q}) + F_d\dot{q} + Sq = \tau,$$

where $q = [q_1 \ q_2]^T$ and $\dot{q} = [\dot{q}_1 \ \dot{q}_2]^T$ are the angular positions (rad) and angular velocities (rad/s) of the two links, respectively, $\tau = [\tau_1 \ \tau_2]^T$ is the torque (N m) produced by the motors that drive the joints, $M(q)$ is the inertia matrix, and $V_m(q,\dot{q})$ is the centripetal-Coriolis matrix, defined as

$$M(q) := \begin{bmatrix} p_1 + 2p_3c_2(q) & p_2 + p_3c_2(q) \\ p_2 + p_3c_2(q) & p_2 \end{bmatrix}, \qquad V_m(q,\dot{q}) = \begin{bmatrix} p_3s_2(q)\dot{q}_2 & -p_3s_2(q)(\dot{q}_1 + \dot{q}_2) \\ p_3s_2(q)\dot{q}_1 & 0 \end{bmatrix},$$

where $p_1 = 3.473\,\mathrm{kg\,m^2}$, $p_2 = 0.196\,\mathrm{kg\,m^2}$, $p_3 = 0.242\,\mathrm{kg\,m^2}$, $c_2(q) = \cos(q_2)$, $s_2(q) = \sin(q_2)$, and $F_d\dot{q} = \begin{bmatrix} f_{d1} & 0 \\ 0 & f_{d2} \end{bmatrix}\dot{q}$ N m and $F_s(\dot{q}) = [f_{s1}\tanh(\dot{q}_1),\ f_{s2}\tanh(\dot{q}_2)]^T$ N m are the models for dynamic and static friction, respectively, where $f_{d1} = 5.3\,\mathrm{kg\,m/s}$, $f_{d2} = 1.1\,\mathrm{kg\,m/s}$, $f_{s1} = 8.45\,\mathrm{kg\,m/s}$, and $f_{s2} = 2.35\,\mathrm{kg\,m/s}$. The matrix $S = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}$ contains the torsional spring coefficients $s_1 = 0.5\,\mathrm{N\,m/rad}$ and $s_2 = 0.25\,\mathrm{N\,m/rad}$.

2.1 Assuming that the parameters are unkown, design a controller such that starting from any initial condition, the robot comes to rest at $q_d = [\pi \quad \pi/2]^T$

Closed loop dynamics:

$$-M(q_d - e)\ddot{e} - V_m(q_d - e, -\dot{e})\dot{e} + F_s(-\dot{e}) - F_d\dot{e} + S(q_d - e) = \tau$$

$$\ddot{e} = M^{-1}(q_d - e)\left(F_s\left(-\dot{e}\right) + S(q_d - e) - V_m\left(q_d - e, -\dot{e}\right)\dot{e} - F_d\dot{e} - \tau\right)$$

The system parameters $\theta = \begin{bmatrix} p_1 & p_2 & p_3 & f_{d1} & f_{d2} & f_{s1} & f_{s2} & s_1 & s_2 \end{bmatrix}^T$ are unkown.

Now, let $r = \dot{e} + \alpha e$ and $\widetilde{\theta} = \theta - \widehat{\theta}$, where $\widehat{\theta}$ is our estimate of the sysem parameters. Define $z = \begin{bmatrix} e & r & \widetilde{\theta} \end{bmatrix}^T$ and $M(q_d - e)\dot{r} = Y_1(e, r)\theta - \tau$

$$g(z) = \dot{z} = \begin{bmatrix} \dot{e} \\ \dot{r} \\ \dot{\widetilde{\theta}} \end{bmatrix} \begin{bmatrix} r - \alpha e \\ M^{-1}(q_d - e)(Y_1(e, r)\theta - \tau) \\ -\widehat{\theta} \end{bmatrix}$$

Let us define $V(z) = \frac{1}{2}e^T e + \frac{1}{2}r^T M(q_d - e)r + \frac{1}{2}\widetilde{\theta}^T\widetilde{\theta}$

$$\frac{\partial V(z)}{\partial z} = \begin{bmatrix} e^T + \frac{1}{2}r^T \frac{\partial(M(q_d - e)r)}{\partial e} & r^T M(q_d - e) & \widetilde{\theta}^T \end{bmatrix}$$

$$\dot{V}(z) = \frac{\partial V(z)}{\partial z} g(z) = \left(e^T + \frac{1}{2}r^T \frac{\partial(M(q_d - e)r)}{\partial e}\right)(r - \alpha e) + r^T M(q_d - e)\dot{r} + \widetilde{\theta}^T\dot{\widetilde{\theta}}$$

$$= e^T(r - \alpha e) + r^T\left(\frac{1}{2}\frac{\partial(M(q_d - e)r)}{\partial e}(r - \alpha e) + M(q_d - e)\dot{r}\right) - \widetilde{\theta}^T\widehat{\theta}$$

Now define $Y_1(e, r)\theta = M(q_d - e)\dot{r} + \tau$ and $Y_2(e, r)\theta = \frac{1}{2}\frac{\partial(M(q_d - e)r)}{\partial e}(r - \alpha e)$

$$\dot{V}(z) = e^T(r - \alpha e) + r^T(Y_2(e, r)\theta + Y_1(e, r)\theta - \tau) - \widetilde{\theta}^T\widehat{\theta}$$

$$\dot{V}(z) = e^T r - e^T\alpha e + r^T(Y_1(e, r)\theta + Y_2(e, r)\theta - \tau) - \widetilde{\theta}^T\widehat{\theta}$$

Now we need to design τ and $\widehat{\theta}$ such that $\dot{V}(z)$ is negative semidefinite or negative definite. Let us choose $\tau = Y_1(e, r)\widehat{\theta} + Y_2(e, r)\widehat{\theta} + e + r$.

$\dot{V}(z) = e^T r - e^T \alpha e + r^T \left(Y_1(e,r)\widetilde{\theta} + Y_2(e,r)\widetilde{\theta} - e - r \right) - \widetilde{\theta}^T \hat{\theta}$

$= e^T r - r^T e - e^T \alpha e - r^T r + r^T (Y_1(e,r) + Y_2(e,r))\widetilde{\theta} - \widetilde{\theta}^T \hat{\theta}$

$= -e^T \alpha e - r^T r + r^T (Y_1(e,r) + Y_2(e,r))\widetilde{\theta} - \dot{\hat{\theta}}^T \widetilde{\theta}$

By defining $\hat{\theta}^T = r^T (Y_1(e,r) + Y_2(e,r))$, we show that $\dot{V}(z) \leq 0 \ \ \forall z \in \mathfrak{R}^{13}$

$\dot{V}(z) = -e^T \alpha e - r^T r$

Conclusion for controller design:

Defining $\tau = (Y_1(e,r) + Y_2(e,r))\hat{\theta} + e + r$ and $\hat{\theta} = (Y_1(e,r) + Y_2(e,r))^T r$ makes $\frac{\partial V(z)}{\partial z} g(z)$ negative semidefinite and shows the origin to be at least Lyupanov stable. Next, we will apply the invariance principle to conclusively determine the type of stability.

$\dot{V}(z) \leq 0$ on the line $z = \begin{bmatrix} 0 & 0 & 0 & 0 & \widetilde{\theta} \end{bmatrix}^T$. See below for an analysis of the hypotheses of Corollary 90.

1.) $D \subset \mathbb{R}^{13}$ is an open set. Here, $D = \mathbb{R}^{13}$ as $V(z) > 0 \quad \forall z \in \mathbb{R}^{13}$ and $\dot{V}(z) \leq 0 \quad \forall z \in \mathbb{R}^{13}$

2.) $\dot{z} = g(z)$ is locally Lipschitz. This is true as all components of $g(z) = \begin{bmatrix} \dot{e} & \dot{r} & \dot{\widetilde{\theta}} \end{bmatrix}^T$ are locally Lipschitz.

3.) $V(z)$ is continuously differentiable

4.) By Claim 64 and acknowledging that $V(z)$ is radially unbounded, given any $\beta > 0$, Ω'_β is a filled ellipse, compact, connected, and in $D = \mathbb{R}^{13}$ and is positively invariant. Let $\Omega = \Omega'_\beta$ for any $\beta > 0$. We can show that $V(z)$is radially unbounded as $M(q_d - e)$ is positive definite thus $z \to \infty \implies V(z) \to \infty$.

5.) $\dot{V}(z) := \frac{\partial V(z)}{\partial z} g(z) \leq 0, \quad \forall x \in \Omega$. This is true as $\dot{V}(z)$ was designed to be negative semidefinite.

6.) $E = \left\{ z \in \Omega \mid \frac{\partial V(z)}{\partial z} g(z) = 0 \right\} = \Omega \cap \left\{ \begin{bmatrix} e \\ r \\ \widetilde{\theta} \end{bmatrix} \in R^{13} \mid e, r = 0 \right\}$

7.) Find $M \subset E$, where M is the largest invariant set in E.

First, we need to compute $Y_1(e,r)$ and $Y_2(e,r)$. The following code snippets were adapted from Handout 3.

Compute $Y_1(e,r)$ and save to a file

```
clearvars;
syms theta [9 1]
syms x [4 1]
syms tau [2 1]
syms alpha
syms qd [2 1]

e = x(1:2);
r = x(3:4) + alpha*e;

q = qd-e;

M = [theta1+2*theta3*cos(q(2)),  theta2+theta3*cos(q(2));
     theta2+theta3*cos(q(2)),    theta2                  ];
Vm = [theta3*sin(q(2))*-x4,  -theta3*sin(q(2))*(-x3-x4);
      theta3*sin(q(2))*-x3,   0                        ];
Fd = diag(theta(4:5));
Fs = theta(6:7).*tanh(-x(3:4));
S = diag(theta(8:9));


f1 = Fs+S*q(1:2)-Vm*x(3:4)-Fd*x(3:4)+M*alpha*x(1:2);

Y1 = sym(zeros(numel(f1),numel(theta)));

for i=1:numel(f1)
    for j=1:numel(theta)
        temp = coeffs(f1(i,:),theta(j),'All');
        if numel(temp) == 2
            Y1(i,j) = temp(1);
        end
    end
end

matlabFunction(simplify(Y1), 'File', 'Y1', 'Vars', {x,alpha,qd});
```

Now compute $Y_2(e,r)$ and save to a file

```
f2 = 1/2*[diff(M*r, x1), diff(M*r, x2)]*(r-alpha*e);
Y2 = sym(zeros(numel(f2),numel(theta)));

for i=1:numel(f2)
    for j=1:numel(theta)
        temp = coeffs(f2(i,:),theta(j),'All');
        if numel(temp) == 2
```

```
            Y2(i,j) = temp(1);
        end
    end
end

matlabFunction(simplify(Y2), 'File', 'Y2', 'Vars', {x,alpha,qd});
```

Now rewrite the system dynamics to include our control terms:

$$\dot{e} = r - \alpha e$$

$$\dot{r} = M^{-1}(q_d - e)(Y_1(e,r)\theta - \tau) = M^{-1}(q_d - e)\left(Y_1(e,r)\tilde{\theta} - Y_2(e,r)\hat{\theta} - e - r\right)$$

$$\dot{\tilde{\theta}} = -\dot{\hat{\theta}} = -(Y_1(e,r) + Y_2(e,r))^T r$$

As the origin is an equilibrium point, we know that M must include the set $\left\{\vec{0}\right\}$. To see if we can find a larger invariant set, let $\zeta^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \tilde{\theta} \end{bmatrix}$. Then at $t = 0$,

$$g(z) = \begin{bmatrix} r - \alpha e \\ M^{-1}(q_d - e)\left(Y_1(e,r)\tilde{\theta} - Y_2(e,r)\hat{\theta} - e - r\right) \\ -(Y_1(e,r) + Y_2(e,r))^T r \end{bmatrix} = \begin{bmatrix} 0 \\ M^{-1}(q_d - e)\left(Y_1(e,r)\tilde{\theta} - Y_2(e,r)\hat{\theta}\right) \\ 0 \end{bmatrix}.$$ We need to find a ζ^0 which will set $g(z) = \vec{0}$ at $t = 0$. The following snippet of code shows $Y_1(e,r)\tilde{\theta}$ and $Y_2(e,r)\hat{\theta}$ at $\begin{bmatrix} e \\ r \end{bmatrix} = \vec{0}$.

```
clearvars;
syms z [13 1]
syms qd [2 1]
syms that [9 1]
syms alpha
e = z(1:2);
r = z(3:4);
y1 = Y1([e;r-e*alpha], alpha, qd);
y2 = Y2([e;r-e*alpha], alpha, qd);

y1 = subs(y1, z(1:4), zeros(4,1))*z(5:13)
```

y1 =

$$\begin{pmatrix} \mathrm{qd}_1\, z_{12} \\ \mathrm{qd}_2\, z_{13} \end{pmatrix}$$

```
y2 = subs(y2, z(1:4), zeros(4,1))*that
```

y2 =

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

It is shown that $Y_2(e,r)\widehat{\theta} = \vec{0}$ if $\begin{bmatrix} e \\ r \end{bmatrix} = \vec{0}$, however $Y_1(e,r)\widetilde{\theta}$ is still dependent on $q_{d1}\widetilde{\theta}_8$ and $q_{d2}\widetilde{\theta}_9$.

Thus $M = \left\{ \begin{bmatrix} e_1 \\ e_2 \\ r_1 \\ r_2 \\ \widetilde{\theta}_1 \\ \widetilde{\theta}_2 \\ \widetilde{\theta}_3 \\ \widetilde{\theta}_4 \\ \widetilde{\theta}_5 \\ \widetilde{\theta}_6 \\ \widetilde{\theta}_7 \\ \widetilde{\theta}_8 \\ \widetilde{\theta}_9 \end{bmatrix} \in \mathfrak{R}^{13} \mid e, r, \widetilde{\theta}_8, \widetilde{\theta}_9 = 0 \right\}$ and $M \subset E$. Given any $\zeta^0 \in \Omega$, $\lim_{t\to\infty} \mathrm{dist}(\phi(t,\zeta^0), M) = 0$ and $\lim_{t\to\infty} \phi_e(t,\zeta^0) = 0$. In other words, the the angular position will approach the setpoint when $\zeta^0 \in \Omega$.

As $V(z)$ is radially unbounded, we can pick an arbitrarily large β such that $\zeta^0 \in \Omega'_\beta$. Letting $\Omega = \Omega'_\beta$ shows **global asymptotic stability (GAS).**

2.2 Bonus

In order to show that $\begin{bmatrix} \widehat{\theta}_8 & \widehat{\theta}_9 \end{bmatrix}$ converges to $\begin{bmatrix} s_1 & s_2 \end{bmatrix}$, let us use $Y_1(e,r)$. We defined $Y_1(e,r)$ such that $Y_1(e,r)\theta = M(q_d - e)\dot{r} + \tau$. We've already shown GAS of the origin. In order for r to stay at the origin, $\dot{r}$ must also be 0. This implies that

$$0 = Y_1(e,r)\theta - Y_1(e,r)\widehat{\theta} - Y_2(e,r)\widehat{\theta} - e - r$$

$$0 = Y\left(\vec{0}, \vec{0}\right)\tilde{\theta} \qquad \text{As } e, r = 0$$

$$0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_{d1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_{d2} \end{bmatrix}\tilde{\theta}$$

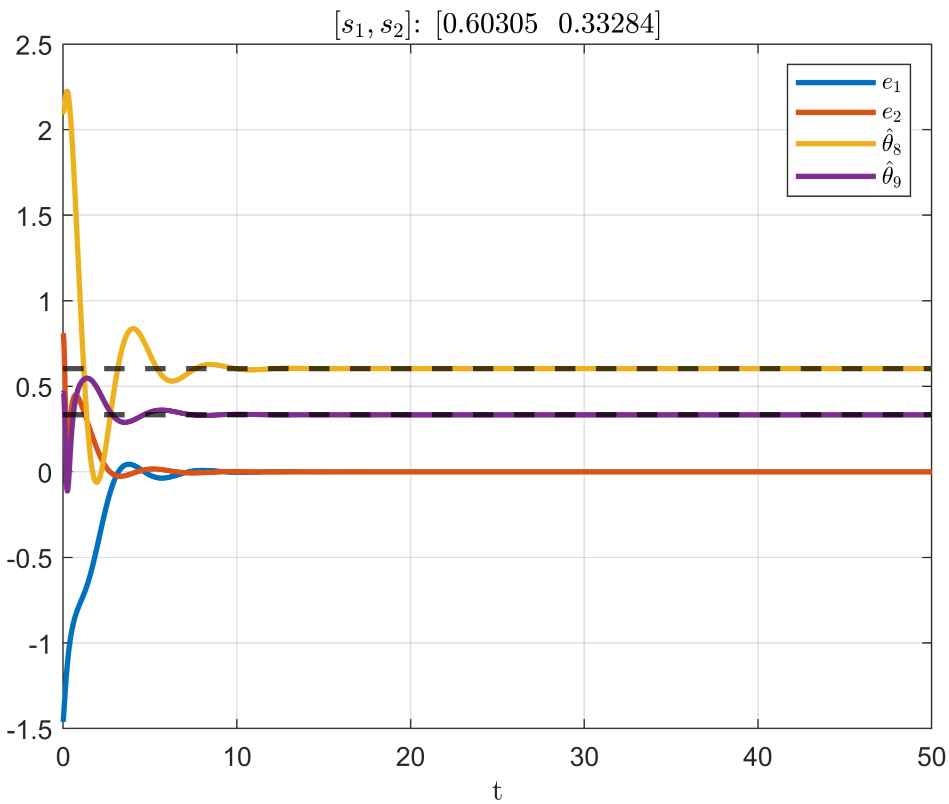
$$\begin{bmatrix} q_{d1}\tilde{\theta}_8 \\ q_{d2}\tilde{\theta}_9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

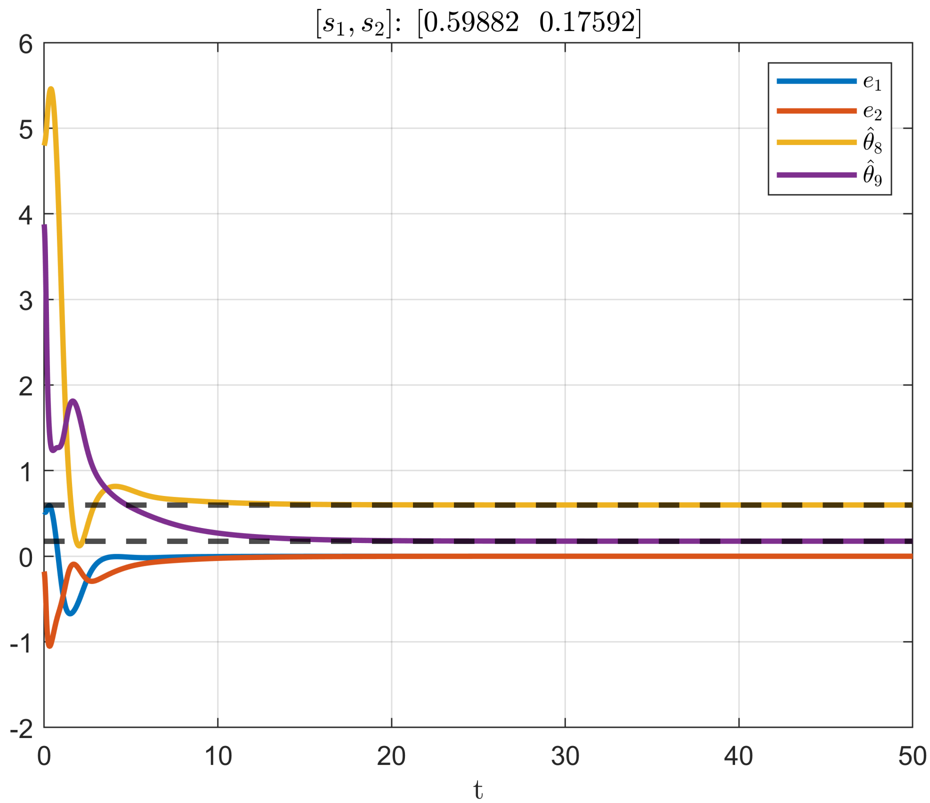
Thus, we have shown that $\begin{bmatrix} q_{d1} \\ q_{d2} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} \tilde{\theta}_8 \\ \tilde{\theta}_9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, as $t \to \infty$. This shows that $\begin{bmatrix} \hat{\theta}_8 \\ \hat{\theta}_9 \end{bmatrix}$ converges to $\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$.

2.3 Simulate the controller to show that it works as expected.

```
clearvars;

for i = 1:2
    % Initialize theta to be a random value.
    p.theta = [3.473;0.196;0.242;5.3;1.1;8.45;2.35;0.5;0.25];

    % Bias theta to be within 50% of the standard parameters
    p.theta  = p.theta + p.theta.*(1.0.*rand(9,1) - 0.5);

    p.alpha = 1;     % alpha > 0


    p.qd = [pi; pi/2];

    % x0 = [q; dq]
    x0 = pi*(rand(4,1)-0.5);     % Pick a random starting positions and velocities

    % Random initial guess for theta_hat
    theta_hat_dot0 = 5*rand(9,1);

    y0 = [x0; theta_hat_dot0];

    tspan = [0 15];
    [t,y] = ode45(@(t,y) closedLoopDynamicsDDR(t,y,p), tspan, y0);

    theta_tilde = mat2str(p.theta - y(end,5:13)', 2);

    figure()
    plot(t,y(:,1:2), t,y(:,12:13), "LineWidth", 2);
    hold on;
    yline(p.theta(8:9), '--k', "LineWidth", 2)

    xlabel("t", "Interpreter", "latex")
```

```
    legend("$e_1$", "$e_2$", "$\hat\theta_8$", "$\hat\theta_9$", "Interpreter", "latex");

    title("$\left[s_1, s_2 \right]$: $\left[" + p.theta(8) + "~~" + p.theta(9)+ "\right]$", "In
    grid on;
end
```





Problem 3

Assume that

1.) $f : \Re^2 \to \Re^2$ is a locally Lipschitz function with $f(0) = 0$.

2.) $V(x) = \dfrac{(x_1 + x_2)^2}{1 + (x_1 + x_2)^2} + (x_1 - x_2)^2$

3.) $\dfrac{\partial V(x)}{\partial x} f(x) = -x_2^2$

4.) whenever $x_1 \neq 0$, we know that $\dot{x_2} \neq 0$.

Find the largest estimate you can for the region of attraction of the origin. you can use a computer to compute sublevel sets and plot contour plots. Submit a plot of the region of attraction.

3.1 Find the largest estimate you can for the region of attraction

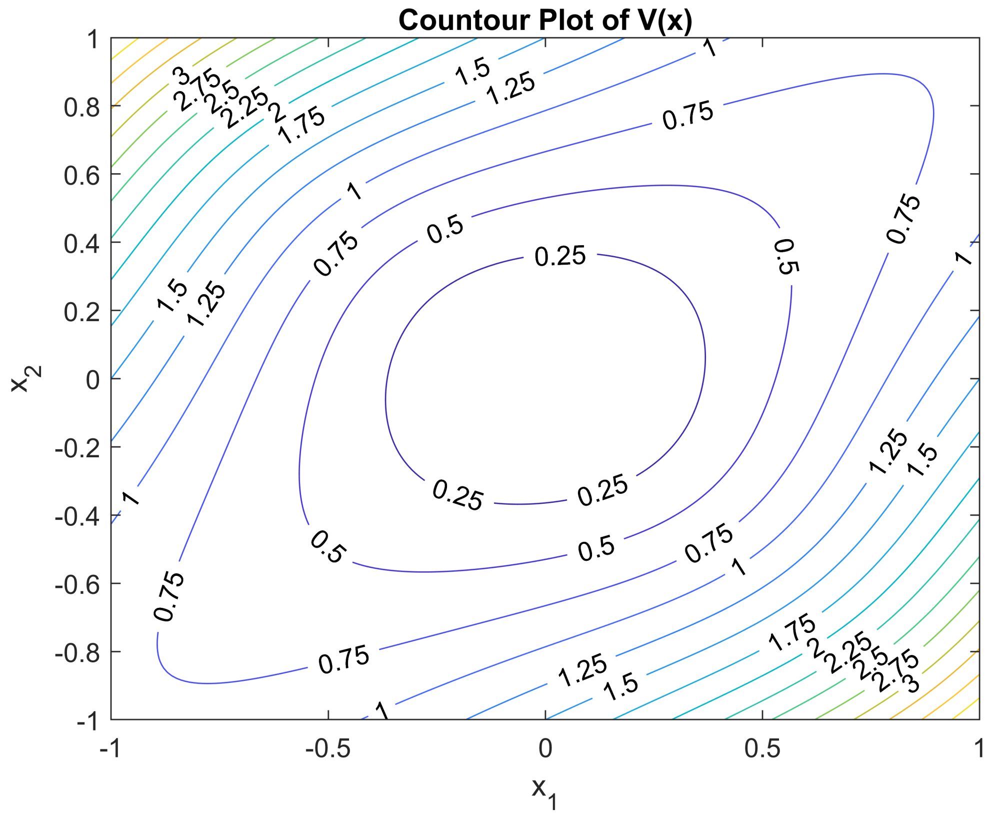
$E = \{x \in \Omega \mid x_2 = 0\}$. As $x_1 \neq 0 \implies \dot{x_2} \neq 0$, $M = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$. This implies asymptotic stability at the origin, though we need to define the region of attraction. The region of attraction Ω is clearly not $\mathbb{R}^2$ as $V(x)$ is not radially unbounded. Let us look at the simple case of the line $\gamma = x_2 = x_1$, on which $V(x) = \frac{4\gamma^2}{1 + 4\gamma^2}$.

$$\lim_{\gamma \to \infty} \frac{4\gamma^2}{1 + 4\gamma^2} = 1$$

Thus, any contours greater than 1 will never reach the line $x_1 = x_2$ and will not include the origin. The region of attraction of the origin is defined as $\Omega = \{x \in \Re^2 \mid V(x) < 1\}$.

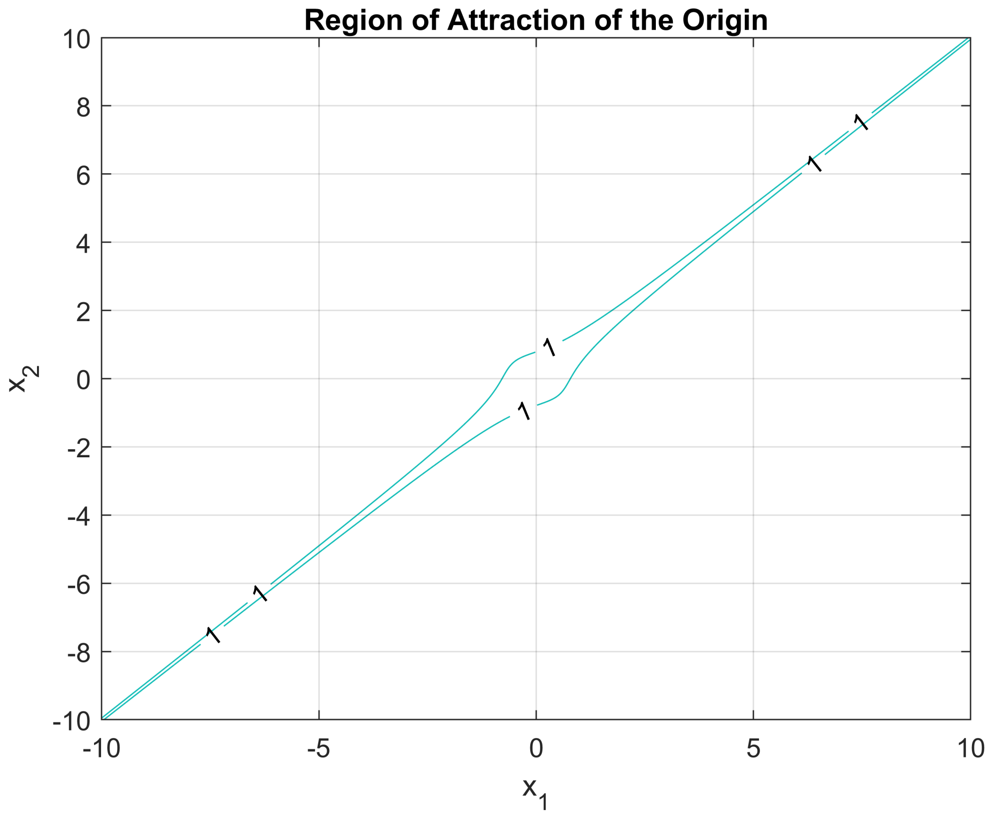
3.2 Plot the contour plots

```
clearvars;

syms V(x,y)
V(x,y) = simplify((x+y)^2/(1+(x+y)^2)+(x-y)^2);
v = matlabFunction(V);

d = 100;
range = 1;
figure()
x = linspace(-range,range,2*range*d);
y = x;

[X,Y] = meshgrid(x,y);

figure()
contour(X,Y,v(X,Y), "LevelStep", 0.25, "ShowText", 'on')
xlabel("x_1")
ylabel("x_2")
title("Countour Plot of V(x)")
```



3.3 Plot the region of attraction

```
range = 10;
figure()
x = linspace(-range,range,2*range*d);
y = x;

[X,Y] = meshgrid(x,y);
figure()
contour(X,Y,v(X,Y), "LevelList", [1], "ShowText", 'on')
xlabel("x_1")
ylabel("x_2")
title("Region of Attraction of the Origin")
grid on
```



Problem 4

The Block Equation, commonly used in nuclear magnetic resonance (NMR) and magnetic resonance imaging (MRI), is a model for the magnetization experinced by nuclei of atoms when exposed to a magnetic field. Assuming a constant magnetic field, B_0, along th elongitudinal axis z, the x, y, and z components of magnetization evolve according to

$$\dot{M}_x = -\frac{1}{T_2} M_x + \gamma B_0 M_y$$

$$\dot{M}_y = -\frac{1}{T_2} M_y - \gamma B_0 M_x$$

$$\dot{M}_z = -\frac{1}{T_1} (M_z - M_0)$$

where γ is the gyromagnetic ratio that depends on the chemical strucutre of the atom. Trajecotries generated by this model precess in the transverse xy plane, decaying with time constant T_2. Simultaneously the longitudinal component returns to the equilibrium magnetizaiton M_0 with time constant T_1

4.1 Find a candidate Lyapunov function to get the best possible stability result for the equilibrium $[M_x \quad M_y \quad M_z]^T = [0 \quad 0 \quad M_0]^T$

Define $x = \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}$ and $f(x) = \dot{x} = \begin{bmatrix} -\frac{1}{T_2}M_x + \gamma B_0 M_y \\ -\frac{1}{T_2}M_y - \gamma B_0 M_x \\ -\frac{1}{T_1}(M_z - M_0) \end{bmatrix}$, where $f\left(\begin{bmatrix} 0 \\ 0 \\ M_0 \end{bmatrix}\right) = 0$.

As the origin is not an equilibrium point, we must perform a linear transformation before applying Lyupanov analysis.

Define $z = \begin{bmatrix} M_x \\ M_y \\ M_z - M_0 \end{bmatrix}$ and $g(z) = \dot{z} = \begin{bmatrix} -\frac{1}{T_2}z_1 + \gamma B_0 z_2 \\ -\frac{1}{T_2}z_2 - \gamma B_0 z_1 \\ -\frac{1}{T_1}z_3 \end{bmatrix}$. Now $g\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = 0$ and Lyupanov analysis may be applied.

Define $V(z)$ of the quadratic form $V(z) = \frac{1}{2}z^T A z$, where $A = \mathrm{diag}\left(\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}\right)$.

$$\frac{\partial V(z)}{\partial z} = z^T A$$

$$\dot{V}(z) = \frac{\partial V(z)}{\partial z} g(z) = z^T A \begin{bmatrix} -\frac{1}{T_2}z_1 + \gamma B_0 z_2 \\ -\frac{1}{T_2}z_2 - \gamma B_0 z_1 \\ -\frac{1}{T_1}z_3 \end{bmatrix} = \begin{bmatrix} z_1 a_1 & z_2 a_2 & z_3 a_3 \end{bmatrix} \begin{bmatrix} -\frac{1}{T_2}z_1 + \gamma B_0 z_2 \\ -\frac{1}{T_2}z_2 - \gamma B_0 z_1 \\ -\frac{1}{T_1}z_3 \end{bmatrix}$$

$$= z_1 a_1\left(-\frac{1}{T_2}z_1 + \gamma B_0 z_2\right) + z_2 \alpha_2\left(-\frac{1}{T_2}z_2 - \gamma B_0 z_1\right) + z_3 a_3\left(-\frac{1}{T_1}z_3\right)$$

$$= -\frac{a_1}{T_2}z_1^2 - \frac{a_2}{T_2}z_2^2 - \frac{a_3}{T_1}z_3^2 + \gamma B_0 z_1 z_2 (a_1 - a_2)$$

$\frac{\partial V(z)}{\partial z} g(z) < 0 \ \ \forall z \in D\backslash\{0\}$ iff $a_1, a_2, a_3 > 0$ and $a_1 = a_2$.

Thus, the simplified equation is given as $V(z) = \frac{1}{2}a_0 z_1^2 + \frac{1}{2}a_0 z_2^2 + \frac{1}{2}a_3 z_3^2$ and $\frac{\partial V(z)}{\partial z} g(z) = -\frac{a_0}{T_2}\left(z_1^2 + z_2^2\right) - \frac{a_3}{T_1} z_3^2$

With the added constraints, $\dot{V}(z)$ is negative definite for all $z \in \mathbb{R}^3$. Thus, $D = \mathbb{R}^3$ and the system is **GAS** by the Barbashin-Krasovskii Theorem**.** See below for a verification of the hypotheses of the theorem.

1.) The origin is an equilibrium point of $\dot{z} = g(z)$ where $g: \ \ \mathbb{R}^n \rightarrow \mathbb{R}^n$ is locally Lipschitz continous. This is true as z transformed such that the origin is an equilibrium point. $g(x)$ is locally Lipschitz continous if $T_{1,2} \neq 0$.

2.) $V: \ \ \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuously differentiable and positive definite. $V(z)$ is of the quadratic form, so this hypothesis is true.

3.) $||z|| \rightarrow \infty \implies V(z) \rightarrow \infty$ ($V(z)$ is radially unbounded). $V(z)$ is of the quadratic form, so this hypothesis is true.

4.) $\frac{\partial V(z)}{\partial z} g(z) < 0 \ \ \ \forall x \in \mathbb{R}^n \backslash \{0\}$. $\dot{V}(z)$ is negative definite if $V(z) = \frac{1}{2}a_0 z_1^2 + \frac{1}{2}a_0 z_2^2 + \frac{1}{2}a_3 z_3^2$.

```
function [dy] = closedLoopDynamicsRLC(t, x, p)
    p.Y = [p.qd, -x(2), p.alpha*x(2)];
    p.r = x(2) + p.alpha*x(1);

    dx = openLoopDynamicsRLC(t, x, controlRLC(t,x,p), p);
    theta_hat_dot = updateLawRLC(t, x, p);
    dy = [dx; theta_hat_dot];
end

function [u] = controlRLC(~, x, p)
    u = p.Y*x(3:5) + x(2) + x(1);
end

% x = [e; e_dot; theta_hat_dot], where e = qd - q
function [dx] = openLoopDynamicsRLC(~,x,u,p)
    dx(1,:) = x(2);
    dx(2,:) = (1/p.theta(3))*((p.qd-x(1))*p.theta(1) - p.theta(2)*x(2) - u);
end

function theta_hat_dot = updateLawRLC(~, ~, p)
    theta_hat_dot = p.r*p.Y';
end
```

```
function [dy] = closedLoopDynamicsDDR(t, x, p)
    e  = x(1:2);
```

```
    de = x(3:4);

    a = p.alpha;
    p.r = de + a*e;

    dx = openLoopDynamicsDDR(t, x, controlDDR(t, x, p), p);
    theta_hat_dot = updateLawDDR(t, x, p);
    dy = [dx;theta_hat_dot];
end

function [u] = controlDDR(~, x, p)
    e = x(1:2);
    r = p.r;

    u = (Y1(x(1:4), p.alpha, p.qd) + Y2(x(1:4), p.alpha, p.qd))*x(5:13) + e + r;
end

% x = [e; e_dot; theta_hat_dot], where e = qd - q
function [dx] = openLoopDynamicsDDR(~,x,u,p)
    p1  = p.theta(1);
    p2  = p.theta(2);
    p3  = p.theta(3);
    fd1 = p.theta(4);
    fd2 = p.theta(5);
    fs1 = p.theta(6);
    fs2 = p.theta(7);
    s1  = p.theta(8);
    s2  = p.theta(9);

    q = p.qd-x(1:2);
    dq = -x(3:4);

    M = [p1+2*p3*cos(q(2)),  p2+p3*cos(q(2));
         p2+p3*cos(q(2)),    p2];

    S = diag([s1, s2]);

    Fs = [fs1*tanh(dq(1)); fs2*tanh(dq(2))];

    Fd = diag([fd1, fd2]);

    sq2 = sin(q(2));
    Vm = [p3*sq2*dq(2),  -p3*sq2*(dq(1)+dq(2));
          p3*sq2*dq(1),  0];

    dx(1:2,:) = x(3:4);
    dx(3:4,:) = M\(Fs+S*q+Vm*dq+Fd*dq-u);
end

function theta_hat_dot = updateLawDDR(~, x, p)
    theta_hat_dot = (Y1(x(1:4),p.alpha,p.qd) + Y2(x(1:4), p.alpha, p.qd))'*p.r;
end
```