ECEN 5463 HW 4

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Problem 1

Question 1. Consider the nonlinear system

$$\dot{x}_1 = x_1^2 - x_2 \qquad \dot{x}_2 = \tanh(x_1) + x_3 + x_1 x_2^2 \qquad \dot{x}_3 = x_1 x_2 + x_2 x_3 + (2 + \sin(x_1))\, u$$

where $x_i \in \mathbb{R}$ for $i = 1, 2, 3$, and $u \in \mathbb{R}$.

(A) Design a the controller u such that the closed-loop system has a globally asymptotically stable equilibrium point at the origin. (Hint: textbook chapter on backstepping)

(B) Implement you control design in MATLAB and plot the response starting from the initial condition $x_1^0 = x_2^0 = x_3^0 = 5$.

1-1 Design a controller u such that the closed-loop system has a globally asymptotically stable equilibrium point at the origin

We follow the recursive application as outlined in the book.

$\dot{x} = f_0(x) + g_0(x) z_1$

$\dot{z_1} = f_1(x, z_1) + g_1(x, z_1) z_2$

$\dot{z_2} = f_2(x, z_1, z_1) + g_2(x, z_1, z_2) z_3$

where the following relations hold:

$z_3 = u$

$f_0(x) = x^2 \qquad g_0(x) = -1$

$f_1(x, z_1) = \tanh(x) + \mathrm{x}\mathrm{z}_1^2 \quad g_2(x, z_1) = 1$

$f_2(x,z_1,z_2) = \mathrm{xz}_1 + z_1 z_2 \quad g_2(x,z_1,z_2) = 2+\sin(x)$

Define $V_0(x) = \frac{1}{2}x^2$ and $\phi_0(x) = x^2 + x$ such that $\dot{V}_0$ is negative definite.

Now let $\phi_1(x,z_1) = \frac{\partial\phi_0}{\partial x}(f_0 + g_0 z_1) - \frac{\partial V_0}{\partial x} g_0 - k_1(z_1 - \phi_0) - f_1$ for some $k_1 > 0$

Expanding, $\phi_1(x,z_1) = (2x+1)\left(x^2 - z_1\right) + x - k_1\left(z_1 - x^2 - x\right) - \tanh(x) + \mathrm{xz}_1^2$

$\phi_1(x,z_1) = 2x^3 - 2\mathrm{xz}_1 + x^2 - z_1 + x - k_1 z_1 + k_1 x^2 + k_1 x - \tanh(x) + \mathrm{xz}_1^2$

Now let $\phi_2(x,z_1,z_2) = -\frac{\partial\phi_1}{\partial x}(f_0 + g_0 z_1) - \frac{\partial\phi_1}{\partial z_1}(f_1 + g_1 z_2) + \frac{\partial V_1}{\partial z_1} g_1 + k_2(z_2 - \phi_1) + f_2$ for some $k_2 > 0$

and let $V_2(x,z_1,z_2) = V_1(x,z_1) + \frac{1}{2}(z_2 - \phi_2(x,z_1))^2$

As this is a third order system, $\phi_2 = u$. Proving GAS requires that each candidation function $V_{0..2}$ be positive definite and radially unbounded, which they are. We also need $\dot{V_{0...2}}$ to be negative definite. As $\dot{V}_0$ is negative definite, picking an arbitrarily large $k_1, k_2 > 0$ will force V_1 and V_2 to be negative definite as well.

```
clear all;

syms x
syms z [3 1]

f = [x^2-z1;
     tanh(x)+z2+x*z1^2;
     x*z1+z1*z2+(2+sin(x))*z3]
```

f =

$$\begin{pmatrix} x^2 - z_1 \\ x\,{z_1}^2 + z_2 + \tanh(x) \\ z_3\,(\sin(x) + 2) + x\,z_1 + z_1\,z_2 \end{pmatrix}$$

```
f0 = x^2;
g0 = -1;

f1 = tanh(x)+x*z1^2;
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```
g1 = 1;

f2 = x*z1+z1*z2;
g2 = (2+sin(x));

V0 = 1/2*x^2;
phi0 = x^2+0.1*x;

k1 = 800;
phi1 = 1/g1 * (diff(phi0,x)*(f0+g0*z1) - diff(V0,x)*g0 -k1*(z1-phi0) - f1);

k2 = 1000;
V1 = V0 + 1/2*(z1-phi0)^2;
phi2 = 1/g2 * (diff(phi1,x)*(f0+g0*z1) + diff(phi1,z1)*(f1+g1*z2) - diff(V1,z1)*g1 - k2*(z2-ph
V2 = V1 + 1/2*(z2-phi2)^2;

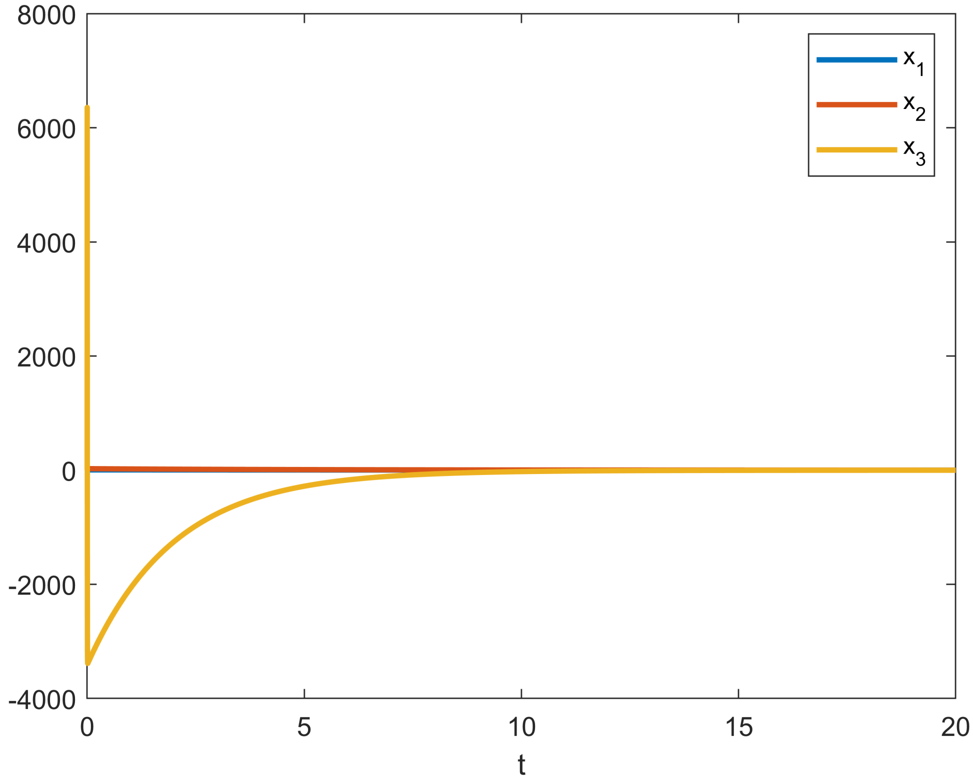
dV0 = diff(V0,x)*(f0+g0*phi0)
```

dV0 =

$$-\frac{x^2}{10}$$

1-2 Simulate to show that the controller works at the initial condition $x_1^0 = x_2^0 = x_3^0 = 5$

```
F = matlabFunction(f,"Vars", {[x;z1;z2],z3}, "File", "F");
cntrl = matlabFunction(phi2, "Vars", {[x;z1;z2]});
[t,y] = ode45(@(t,x) F(x,cntrl(x)), [0 20], [5;5;5]);
plot(t,y, "LineWidth", 2)
xlabel("t")
legend("x_1", "x_2", "x_3")
```



8000
6000
4000
2000
0
-2000
-4000
0
5
10
15
20
t
x1
x2
x3