Lecture 07

Tuesday, February 1, 2022 9:00 AM

Theorem: (Extension of Picard-Lindelöf theorem 1894)

Let:

H2: $f: \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ is piecewise continuous in t over $I \subseteq \mathbb{R}_{\geq 0}$ for all $x \in D \subseteq \mathbb{R}^n$

H2: f is locally Lipschitz continuous in x, uniformly in t

then:

C1: There exists $\delta > 0$ such that for any $t_0 \in I$ and $x^0 \in D$ $\dot{x} = f(t, x)$ has a unique solution starting from (t_0, x^0) over $[t_0, t_0 + \delta]$

$$\operatorname{sgn}(t) = \begin{cases} +1 & t \geq 0 \\ -1 & t < 0 \end{cases}$$

$f(t, x) = x^2 \operatorname{sgn}(t-1)$ f is not continuous

$$= \begin{cases} -x^2 & t < 1 \\ x^2 & t \geq 1 \end{cases}$$

whenever $x = 0$, f is continuous in t $\iff$ $t \mapsto f(t, 0)$ (or $f(\cdot, 0)$) is cont

" $x \neq 0$, f is not " in t $\iff$ $t \mapsto f(t, 1)$ (or $f(\cdot, 1)$) is not

For any fixed t, f is continuous in x $\iff$ $x \mapsto f(t, x)$ (or $f(t, \cdot)$) is continuous for all t

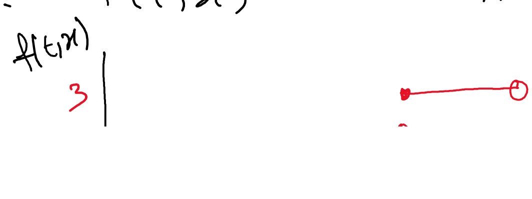
The statement "if t is fixe then f, as a function of is continuous" is true for all t.

Definition: $f: I \times D \to \mathbb{R}^n$ is called piecewise continuous in t over I for all $x \in D$ if for every $x \in D$ and every <u>bounded</u> interval $J \subseteq I$,

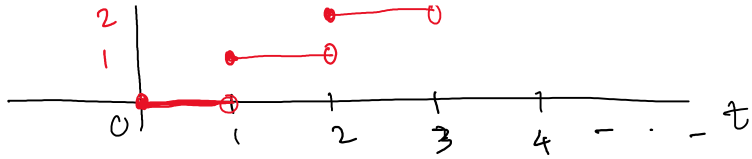
① $t \mapsto f(t, x)$ is continuous at all but finitely many $t \in J$

② at every point of discontinuity, $\lim_{h \to 0} f(t+h, x)$ and $\lim_{h \to 0} f(t-h, x)$ exist and are finite

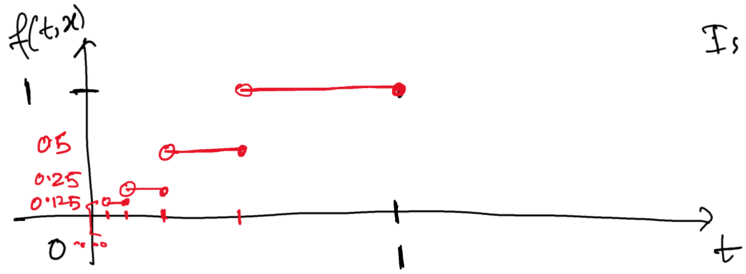
Ex: $f(t, x) = n$ if $n \leq t < n+1$, $n \in \mathbb{N}$



Is f piecewise cts over $\mathbb{R}_{\geq 0} \times \mathbb{R}^n$

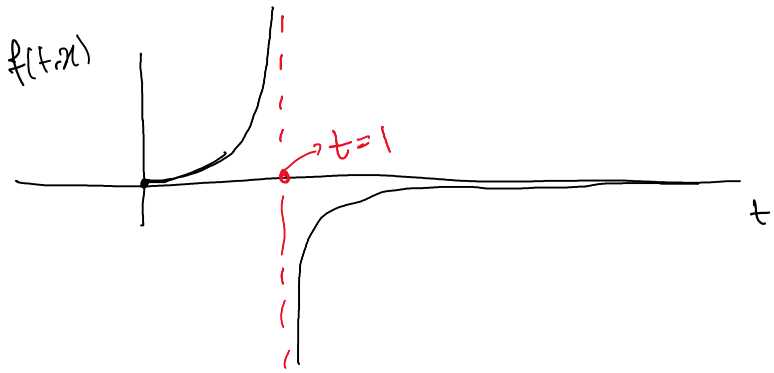


"$n \geq 0$ " " n" Yes!



Is f piec. cts. over $[0,1] \times \mathbb{R}^n$

No!



Not piecewise cts over $\mathbb{R}_{\geq 0}$

Definition: $f : I \times D \to \mathbb{R}^n$ is called

① Lipschitz cts in x over D, uniformly in t, if $\exists\, L > 0$ such that

$\forall\, x, y \in D$, and $\forall\, t \in I$ $\|f(t,x) - f(t,y)\| \leq L\|x - y\|$

② locally Lipschitz continuous in x over D, uniformly in t, if

$\forall\, x \in D$, $\exists\, L^{(x)}, \delta^{(x)} > 0$ such that

$\forall\, z, y \in B(x,\delta)$ and $\forall\, t \in I$, $\|f(t,z) - f(t,y)\| \leq L\|z - y\|$

$\exists\, L$ such that $\forall\, x, y$ "Statement" (one L for all x, y)

$\forall\, x, y$, $\exists\, L$ such that "Statement" (L depends on x, y)

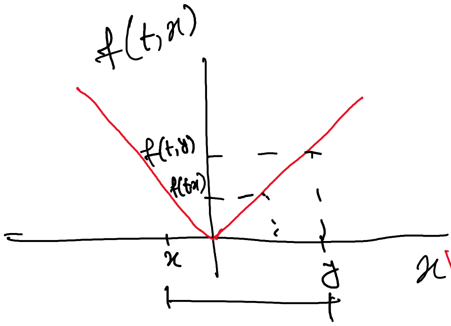
$f(t,x) = |x| \quad x \in \mathbb{R}$ Is this Lip Cts over $I \times \mathbb{R}$, uniformly in t

$x, y \in \mathbb{R}$. $\quad \|f(t,x) - f(t,y)\| = \big| |x| - |y| \big|$

$$\|x - y\| = |x - y|$$

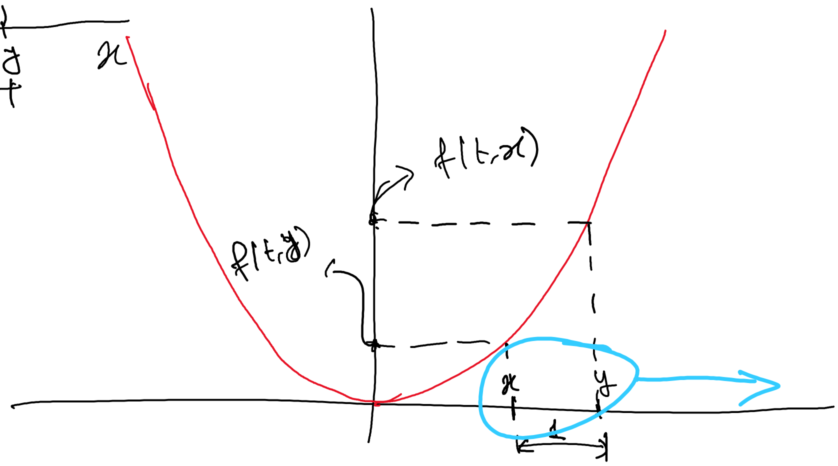
want L so that $\big| |x| - |y| \big| \le L |x - y|$

$L = 1$!



$$\|f(t,x) - f(t,y)\| \le L \underbrace{\|x - y\|}_{1}$$

Ex: $f(t,x) = x^2$



If f were Lip Cts over $I \times \mathbb{R}$ then

$\|f(t,x) - f(t,y)\| \le L$ whenever $\|x - y\| \le 1$

This is a contradiction. If $\|x - y\| = 1$ then

$$\lim_{x \to \infty} \|f(t,x) - f(t,y)\| = \infty$$

$\forall x, y \ \exists L$ s.t. $\| \quad \| \le L \| \quad \|$ $\leftarrow$ Lip Cts

Found x, y such that $\forall L \ \| \quad \| \not\le L \| \quad \|$

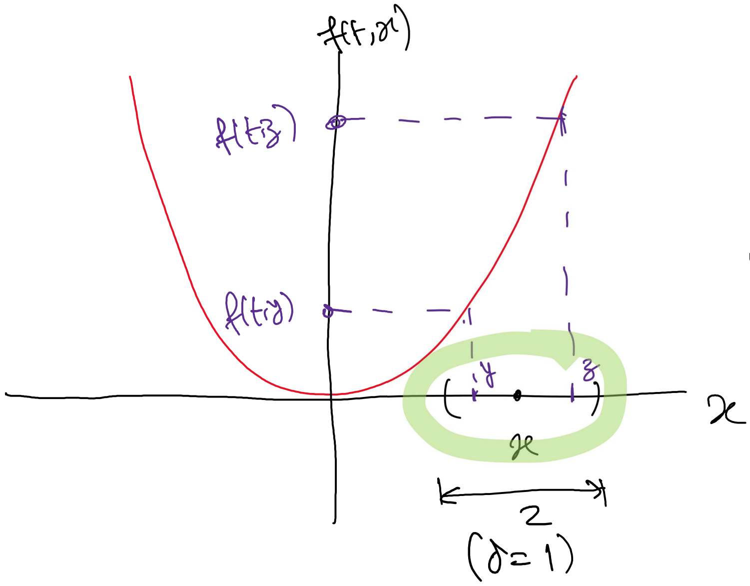
Not Lip. Cts.

Is $f(t,x) = x^2$ locally lip Cts on $I \times \mathbb{R}$

(2) locally Lipschitz continuous in x over D, uniformly in t, if

$\forall x \in D$, $\exists$ $L(x)$, $\delta(x) > 0$ such that

$\forall z, y \in B(x, \delta)$ and $\forall t \in I$, $\|f(t,z) - f(t,y)\| \le L \|z - y\|$



$$|z^2 - y^2| \le L\,|z-y|$$

$$\| $$

$$|(z-y)(z+y)| = |z+y|\,|z-y|$$

$$\le$$

$$L = 2|x(t)|$$

Yes to localy Lipschitz cts over $I \times \mathbb{R}$