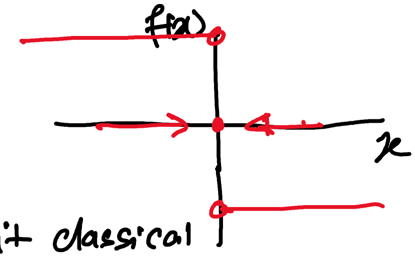
L

Tuesday, February 8, 2022 9:00 AM

$$\dot{x} = f(x) \qquad f(x)$$

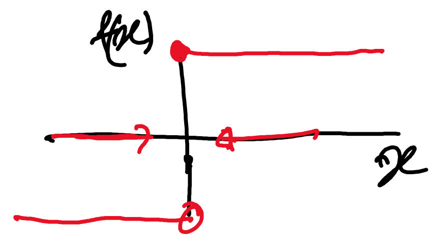
Krasovskii

$$\dot{x} = \begin{cases} -1 & x > 0 \\ 0 & x = 0 \\ 1 & x < 0 \end{cases}$$



Ex: show that $\dot{x} = f(x)$ does not admit classical solutions starting from any initial condition other than $x=0$ but admits caratheodory solutions starting from all initial conditions

$$\dot{x} = \begin{cases} -1 & x < 0 \\ 1 & x \geq 0 \end{cases}$$



Analysis of stability of eq. pts:

① Linearization.

Given $\dot{x} = f(x)$ eq. pt. at $x = x^*$, Linearize about the trajectory $x(t) = x^*$ for all t to get the linear system

$$\dot{\delta}_x = A\,\delta_x \qquad A = \left.\frac{\partial f}{\partial x}\right|_{x=x^*}$$

Theorem: the eq. pt. $x = x^*$ is (Lyapunov's first theorem)

① (locally) Asymptotically stable if all eigenvalues of A have strictly negative real parts

② unstable if at least one eigenvalue has a strictly positive real part

③ if eigenvalues of A are on the imaginary axis then no conclusions

Example: Pendulum:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \underbrace{-\frac{b}{ml^2}x_2}_{f_1} - \underbrace{\frac{g}{l}\sin(x_1)}_{f_2} + r \end{aligned}$$

Eq. pt.: $x_2 = 0$

$\sin(x_1) = 0$

$x_1 = \pm n\pi$

$$x^* = \begin{bmatrix}0\\0\end{bmatrix}$$

$n = 0, 1, 2, \ldots$

$$A = \frac{\partial f}{\partial x}\Big|_{x=0} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\Bigg|_{x=0} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos(x_1) & -\frac{b}{ml^2} \end{bmatrix}}_{\text{evaluate } x=0}$$

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{b}{ml^2} \end{bmatrix} \qquad \lambda_{1,2} = \frac{-b}{ml^2} \pm \sqrt{\frac{b^2}{m^2 l^4} - \frac{4g}{l}}$$

always strictly negative real part!

$\dot{x} = f(x,u)$ linearize about (x^*, u^*) to get

$K: m \times n$

$$\delta\dot{x} = A\delta x + B\delta u \qquad \underset{\mathbb{R}^m}{\delta u} = K \underset{\mathbb{R}^n}{\delta x}$$

$$\delta\dot{x} = (A + BK)\delta x$$

$$\delta u = u - u^*$$

$$u = u^* + K\delta x$$

Exercise: Show that $x^* = \begin{bmatrix}\pi\\0\end{bmatrix}$ is unstable!

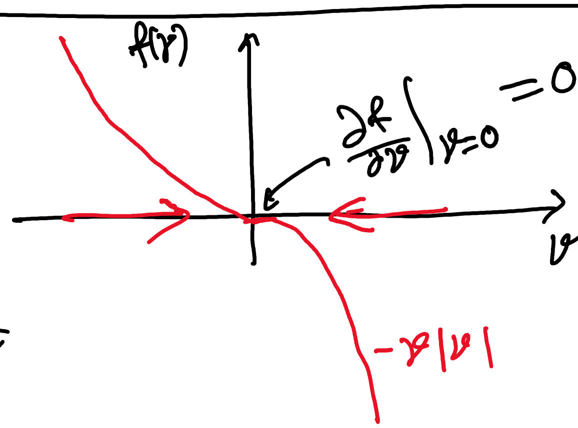
Ex: $\dot{v} = -v|v|$

$v \in \mathbb{R}$

Exercise: linearization about

$v^*(t) = 0$ for all t

is $\delta\dot{v} = 0$



Lyapunov's direct method.

Definition: Let $D \subseteq \mathbb{R}^n$ be any set that contains the origin. $V: D \to \mathbb{R}$ is called

① Positive semi definite on D if $V(x) \geq 0$ for all $x \in D$

② Positive definite on D if $V(x) > 0$ for all $x \in D \setminus \{\vec{0}\}$ and $V(\vec{0}) = 0$

Ex: $V : \mathbb{R}^2 \to \mathbb{R}$, $V\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1^2 + x_2^2$ PSD + PD

$$V\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1^2 \quad \text{PSD}$$

$$V\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 0 \quad \text{Not PD}$$

Theorem: If

H1: $D \subseteq \mathbb{R}^n$ is open, connected, and $\vec{0} \in D$

H2: $f(\vec{0}) = 0$

H3: f is locally Lieschitz continuous on D

H4: $V : D \to \mathbb{R}$ is continuously differentiable and Positive definite

then the eq. pt. $x = 0$ of $\dot{x} = f(x)$ is

C1: stable if $\left(\frac{\partial V}{\partial x}(x)\right) f(x) \leq 0 \quad \forall x \in D$

C2: locally asymptotically stable if

$$\left(\frac{\partial V}{\partial x}(x)\right) f(x) < 0 \text{ for all } x \in D \setminus \{\vec{0}\}$$

$$\dot{x} = -x + x^2$$

$V(x) = \frac{1}{2} x^2$ (P.D), (Cts. diff)

$$\left(\frac{\partial V}{\partial x}(x)\right) f(x) = (x)(-x+x^2) = -x^2+x^3$$

set $D = \{x \in \mathbb{R} \mid |x|<1\}$

then $-x^2+x^3 \leq 0$ for all $x \in D$

and $-x^2+x^3 < 0$ for all $x \in D \setminus \{0\}$

$$\left.\begin{aligned} \dot{x}_1 &= \frac{x_1}{3}(x_1^2+x_2^2-2) - \frac{4}{3}x_1x_2^2 \\ \dot{x}_2 &= 4x_1x_2 + x_2(x_1^2+x_2^2-2) \end{aligned}\right\}$$

$\boxed{V(x) = 3x_1}$

$V(x) =$

Exercise $\boxed{\frac{\partial V}{\partial x}(x) f(x)}$ D?

Cts-Piecewise affine

$V(x_1, x_2)$