Lecture 08

Thursday, February 3, 2022 9:00 AM

Exercise: Is $f(t,x) = tx^2$ locally Lipschitz continuous, uniformly in t?

Facts: ① If $f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable on $D \subseteq \mathbb{R}^n$ and if $\frac{\partial f}{\partial x} : \mathbb{R}^n \to \mathbb{R}^{m \times n}$ is continuous then f is locally Lipschitz cts on D

② In addition, if $\left\| \frac{\partial f}{\partial x}(x) \right\| \leq M$ then f is Lipschitz continuous on D

③ If $f: C \to \mathbb{R}^m$, $C \subseteq \mathbb{R}^n$ is locally Lipschitz continuous on C and if C is compact then f is Lipschitz continuous on C

Definition: A set K is compact if every open cover of K has a finite subcover.

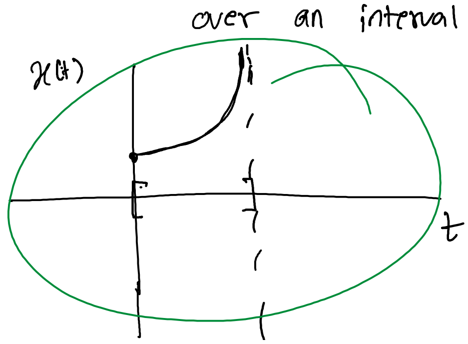
That is if $\{U_1, U_2, \dots\}$ are open and $K \subseteq \bigcup_{i=1}^{\infty} U_i$ then we can select finitely many open sets out of the collection $\{U_i\}_{i=1}^{\infty}$ (say $\{U^1 \dots U^N\}$.) such that $K \subseteq \bigcup_{j=1}^{N} U^j$ and for all j $\exists$ i such that $U^j = U_i$.

In $\mathbb{R}^n$, we have

Theorem: (Heine-Borel, 1895) A set $K \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.

C is closed if C^c is open, C is bounded if $\exists$ $M < \infty$ such that $\forall$ $c \in C$, $\|c\| \leq M$

Exercise: Show that $\dot{x} = x^2$ can have solutions that do not exist over an interval of infinite length



Asymptotic stability: Trajectories tend to an eq. pt. as $t \to \infty$

Theorem: (Existence of global solutions) The system $\dot{x} = f(t,x)$ has solutions starting from $x^0 \in \mathbb{R}^n$ over any given interval I (can be of infinite length) if either:

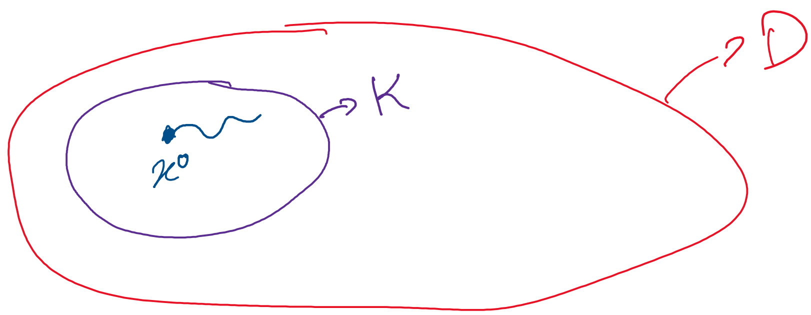
① The function $f: I \times \mathbb{R}^n \to \mathbb{R}^n$ is (globally) Lipschitz continuous in x, over $\mathbb{R}^n$, uniformly in $t \in I$, and piecewise cts in t for all $x \in \mathbb{R}^n$

OR

② The function $f: I \times D \to \mathbb{R}^n$ is locally Lipschitz cts in x over D, uniformly in t and piecewise cts in t over I for all $x \in D$ and

if is known that for some compact set $K \subseteq D$ with $x^0 \in K$ and for every closed interval $J \subset I$ where a solution $t \mapsto \phi(t, t_0, x^0)$ exists, we have $\phi(t, t_0, x^0) \in K$ for all $t \in J$

$t \mapsto \phi(t, t_0, x^0)$ is pre-compact



Given (t_0, x^0), $\exists\ \delta$ so that solutions to $\dot{x} = f(t,x)$ exist on $[t_0, t_0+\delta]$

Fact: If $f: I \times \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous, uniformly in t over $\mathbb{R}^n$ and piecewise continuous in t for all $x \in \mathbb{R}^n$ then any solution $t \mapsto \phi(t, t_0, x^0)$ of $\dot{x} = f(t,x)$ either

① exists over I or

② goes to ∞ in finite time

Stability of equilibrium points.

Autonomous systems : $\dot{x} = f(x)$

Assume $x = 0$ is an equilibrium point $(f(0) = 0)$. (i.e. $f(x^*) = 0$)

If $x^* \neq 0$ is the " " of interest then we can let $z = x - x^*$ and $g(z) = f(z + x^*)$ to get the model

$$\dot{z} = g(z) \quad \text{with} \quad g(0) = 0$$

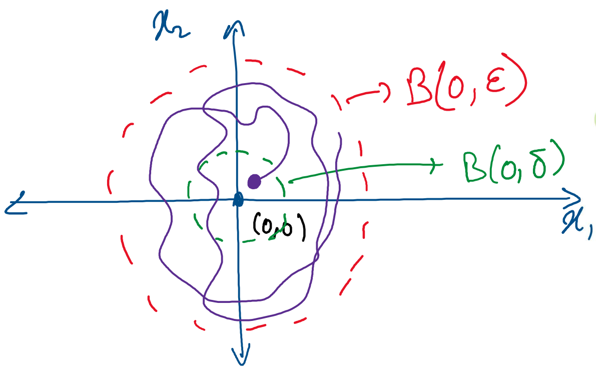
then

If $\dot{z} = g(z)$ has a stable eq. pt. at $z=0$, $\dot{x} = f(x)$ has a stable eq. pt. at $x = x^*$.

Definitions: WLOG let $x=0$ be an eq. pt. of $\dot{x} = f(x)$ where f is locally Lipschitz continuous over some open and connected set $D \subseteq \mathbb{R}^n$ that contains $x=0$. The eq. pt. is

① (locally) stable (or Lyapunov stable) if $\forall\, \epsilon > 0$, $\exists\, \delta > 0$ such that

$x^0 \in B(0,\delta) \implies$ (a) the solution $t \mapsto \phi(t, x^0)$ exists for all $t \geq 0$ and (b) $\forall\, t \geq 0$, $\phi(t,x^0) \in B(0,\epsilon)$

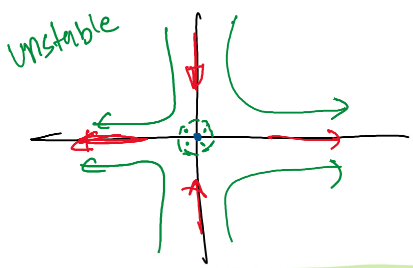


② unstable if not stable

③ (locally) asymptotically stable if it is (locally) stable and $\exists\, \delta > 0$ such that $x^0 \in B(0,\delta) \implies$

(a) the solution $t \mapsto \phi(t,x^0)$ exists for all $t \geq 0$

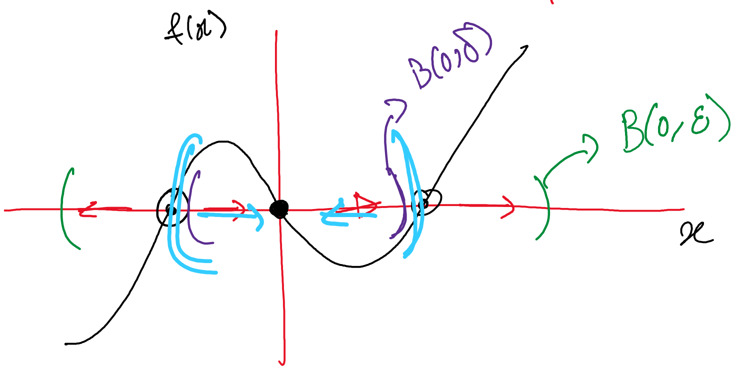
(b) $\lim_{t\to\infty} \|\phi(t,x^0)\| = 0$



Example: $\dot{r} = r(1-r)$, $\dot{\theta} = \sin^2\left(\frac{\theta}{2}\right)$

$r = \sqrt{x^2+y^2}$ $\quad \theta = \text{atan2}(y, x)$

eq. pt. at $(0,1)$ is attractive but not stable also not asymptotically stable.



Def: Let $x=0$ be an eq. pt. of $\dot{x} = f(x)$. A set $D \subseteq \mathbb{R}^n$ is called a domain of attraction of attraction of the eq. pt. if for all $x^0 \in D$

(a) $t \mapsto \phi(t,x^0)$ exists for all t

(b) $\lim_{t\to\infty} \|\phi(t,x^0)\| = 0$.

$x=0$

Definition: an eq. pt. ' is called globally asymptotically stable if it is stable and its domain of attraction is $\mathbb{R}^n$

If $f(x) = Ax$ $(\dot{x} = Ax)$ $\mathrm{re}(\lambda_i(A)) < 0$ for all i then $x=0$ is a GAS eq. pt. In fact, $x(t) = x(0) e^{At}$

__Exercise:__ read about Lyapunov stability of linear systems (Thm 3.1, Ex 3-1)

Definition: an eq. pt. $x=0$ is called exponentially stable if there exist constants $c, k, \lambda > 0$ such that

$$\underbrace{\|x^0\| < c}_{x^0 \in B(0,c)} \Rightarrow \text{for all } t \geq 0, \; \|\phi(t, x^0)\| \leq k \|x^0\| e^{-\lambda t}.$$

globally exponentially stable if for all $x^0 \in \mathbb{R}^n$ and all $t \geq 0$,
$\|\phi(t, x^0)\| \leq k\|x^0\| e^{-\lambda t}$

$\dot{x} = f(x)$