2

The basic equations

2.1. Units and symbols

Electromagnetic theory is often based on the results, verifiable directly or indirectly by experiments, that the force F_q between two electric charges q_1, q_2, and the force F_m between two magnetic poles m_1, m_2, are given respectively by

$$F_q = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}, \; F_m = \frac{4\pi m_1 m_2}{\mu_0 r^2} \tag{2.1}$$

where r is the separation in each case. Some authors argue that free magnetic poles do not exist, and base their definitions of magnetic quantities on the forces between currents, or between moving charges. Others use a different version of the second equation (2.1) in which μ_0 is in the numerator, thereby implying a different meaning for the term 'magnetic pole'. These considerations are unimportant in the ionosphere and magnetosphere whose magnetic permeabilities are always taken to be the same as in a vacuum.

The constants ε_0 and μ_0 are known, respectively, as the permittivity and the permeability of a vacuum. Their numerical values depend on which system of units is being used. The factor 4π appears in (2.1) for rationalised systems of units. When ε_0 and μ_0 are left in a formula as symbols, that formula is valid in any self-consistent system of units. It is often implied that such formulae are restricted to the International System of Units (SI), but this is incorrect. Equations (2.1), and all other formulae in this book that do not use specific numerical values, are valid in any self-consistent rationalised units, including SI units. The recommendations in *Quantities, Units and Symbols* (The Royal Society, 1975) are followed as far as possible. The symbol c is used for the speed of electromagnetic waves in a vacuum. It is occasionally used with other meanings when there should be no risk of confusion. In the International System $\mu_0 = 4\pi \times 10^{-7}$ and $\varepsilon_0 = 1/(\mu_0 c^2)$.

Some mathematical works still use Gaussian units, which are often defined by saying that electrical quantities, including the electric intensity $\boldsymbol{E}$ or $\boldsymbol{e}$ are in (unrationalised) electrostatic units, and magnetic quantities, including the magnetic intensity $\boldsymbol{H}$ or $\boldsymbol{h}$ are in (unrationalised) electromagnetic units. But such a system of mixed units is, in a sense, not self-consistent, and many physicists regard it as bad practice to use symbols with different systems of units in the same formula. Gaussian units have some advantages in mathematical work, but may not be familiar to radio engineers, and their use is declining. The advantages of the Gaussian system are here achieved in a different way by using for magnetic intensity a measure $\mathscr{H}$ defined in §2.10.

In most of this book the electromagnetic fields are those of waves which are harmonic in time with angular frequency ω, and the symbols used for the field vectors are capital letters, for example $\boldsymbol{E}$, $\boldsymbol{H}$, $\boldsymbol{D}$, $\boldsymbol{B}$. These are complex vectors as explained in § 2.5. In the present chapter, however, the discussion is not restricted to harmonically varying fields. The actual field vectors are, in general, functions of time t and of the space coordinates, and small letters are used for them, for example, $\boldsymbol{e}$, $\boldsymbol{h}$, $\boldsymbol{d}$, $\boldsymbol{b}$ which are real vectors. In most of this book the complex $\boldsymbol{B}$ in a wave is not needed. The symbol $\boldsymbol{B}$ is used to denote the real value of the magnetic induction of the earth's magnetic field, assumed to be independent of time.

2.2. Definitions of electric intensity $\boldsymbol{e}$ and magnetic intensity $\boldsymbol{h}$

The electric intensity $\boldsymbol{e}$ at a point in free space is defined as follows. A small charge δq is placed at the point and the force $\delta \boldsymbol{f}$ acting on it is measured. Then

$$\boldsymbol{e} = \lim_{\delta q \to 0} (\delta \boldsymbol{f}/\delta q).$$

In the ionosphere or magnetosphere there are on the average N free electrons and ions per unit volume. The electric intensity defined as above must vary markedly in the free space between them, that is over a distance of the order $N^{-\frac{1}{3}}$. But when $\boldsymbol{e}$ and the other field variables are used in Maxwell's equations, they are assumed to represent vector fields continuously distributed in space and approximately constant over distances which are large compared with $N^{-\frac{1}{3}}$ but small compared to a wavelength. The plasma is thus treated as a continuous medium, and the use of Maxwell's equations implies a 'smoothing out' process over a distance large compared with $N^{-\frac{1}{3}}$. Within the plasma, therefore, definitions of $\boldsymbol{e}$ and $\boldsymbol{h}$ must be found which effect this smoothing out.

For a medium which is not free space, the electric intensity $\boldsymbol{e}$ is usually defined in terms of a long thin cavity. The component of $\boldsymbol{e}$ in a given direction is defined by the following imagined experiment. A long thin cavity is cut in the medium parallel to the given direction. Its length must be so small that the electric state of the medium does not change appreciably within it. For waves in a homogeneous plasma, this means that the length is small compared to a wavelength. An infinitesimal test charge δq is placed at the centre of the cavity, and the component δF of the force acting on it in the direction of the cavity is measured. The component of $\boldsymbol{e}$ in this direction is $\lim_{\delta q \to 0} \delta F/\delta q$. For a medium which contains discrete ions and electrons. this definition can still be used provided that the area of cross-section of the cavity is very large compared with $N^{-\frac{2}{3}}$. It is the electric intensity so defined that is to be used in Maxwell's equations. The cavity definition thus effects the 'smoothing out' process mentioned above.

The magnetic intensity $\boldsymbol{h}$ might be defined in a similar way by using a small magnetic pole in the cavity. But since magnetic poles do not exist, it is better first to define the magnetic induction $\boldsymbol{b}$, as in § 2.4, and then to use

$$\boldsymbol{b} = \mu_0 \boldsymbol{h} \tag{2.2}$$

as the definition of $\boldsymbol{h}$.

2.3. The current density j and the electric polarisation p

In the field of a radio wave in the ionosphere or magnetosphere there is a current density $\boldsymbol{j}$ which arises from the motion of charges. In many problems it will be sufficiently accurate to consider $\boldsymbol{j}$ as arising from the movement of electrons only, but in some cases the movement of heavy ions may also contribute (see §§ 3.11, 3.13, 4.9, 13.8, 13.9, 17.5).

Let $\boldsymbol{r}$ be the average vector displacement of an electron from the position it would have occupied if there were no field. Then the average electron velocity is $\partial \boldsymbol{r}/\partial t$ and the current density is

$$\boldsymbol{j} = Ne\partial \boldsymbol{r}/\partial t \tag{2.3}$$

where e is the charge on one electron, and is a negative number. The average concentration of electrons, N, can only be defined for a volume large enough to contain many electrons, and (2.3) therefore implies a 'smoothing out' over a distance large compared with $N^{-\frac{1}{3}}$. The volume used to define the concentration N should be greater than l_D^3 where l_D is the Debye length, discussed in § 3.5.

Now it is convenient to define the electric polarisation $\boldsymbol{p}$ thus:

$$\boldsymbol{p} = Ne\boldsymbol{r}, \tag{2.4}$$

so that

$$\boldsymbol{j} = \partial \boldsymbol{p}/\partial t. \tag{2.5}$$

The electric polarisation must not be confused with the wave polarisation described in §§ 4.3–4.5.

For dielectrics the electric polarisation is usually defined to be the electric dipole moment per unit volume, but this definition is not suitable for a medium containing free electrons. For if the electrons are randomly distributed and are then all given small equal displacements in the same direction, they remain randomly distributed and it is not obvious that the medium has become polarised. The definition (2.4) of $\boldsymbol{p}$

is therefore somewhat ambiguous because the origin for $\boldsymbol{r}$ can never be specified and it would be impossible to devise an experiment to measure $\boldsymbol{p}$ at a given point. But the velocity $\partial\boldsymbol{r}/\partial t$ has a definite meaning and an experiment to measure $\boldsymbol{j}$, in (2.3) or (2.5), could easily be suggested. Strictly speaking, therefore, the whole theory should be formulated in terms of $\boldsymbol{j}$. But $\boldsymbol{p}$ appears in the last Maxwell equation (2.23) only through its time derivative, so the ambiguity is removed. Hence we use $\boldsymbol{p}$ because it makes the formulae simpler.

2.4. The electric displacement $\boldsymbol{d}$ and magnetic induction $\boldsymbol{b}$

The electric displacement $\boldsymbol{d}$ is defined thus

$$\boldsymbol{d} = \varepsilon_0 \boldsymbol{e} + \boldsymbol{p} \tag{2.6}$$

where $\boldsymbol{e}$ is defined as in § 2.2. The time derivative $\partial\boldsymbol{d}/\partial t$ is called the total current density. It includes $\partial\boldsymbol{p}/\partial t$ which is the contribution (2.5) from the moving electrons and ions, and $\varepsilon_0\partial\boldsymbol{e}/\partial t$ which is the displacement current density contributed by the electric field.

An alternative definition of $\boldsymbol{d}$ equivalent to (2.6) uses a flat plate-like cavity. The component of $\boldsymbol{d}$ in any given direction is defined by the following imagined experiment. In the undisturbed plasma, before the fields are applied, a thin flat cavity is cut with its plane perpendicular to the given direction. Its thickness must be large compared with $N^{-\frac{1}{3}}$ but small compared with a wavelength. An infinitesimal test charge δq is placed in the centre of the cavity and the electromagnetic fields are then applied to the plasma. The component δF of the force on the charge in the direction normal to the cavity is measured. The component of $\boldsymbol{d}$ in this direction is then $\varepsilon_0 \lim_{\delta q \to 0} \delta F/\delta q$. It is the electric displacement $\boldsymbol{d}$ so defined that is used in Maxwell's equations. When this definition is used for a plasma it is important to remember that the cavity is imagined to be cut before the electrons and ions are displaced by the fields; no electrons or ions must cross the central plane of the cavity.

The magnetic induction $\boldsymbol{b}$ may also be defined in terms of a plate-like cavity. The component of $\boldsymbol{b}$ in a given direction is found as follows. Before the electrons and ions are displaced by the fields, a thin flat plate-like cavity is imagined to be cut in the medium perpendicular to the given direction. A straight wire carrying an infinitesimal test current δI is placed in the central plane of the cavity and the component ∂f of the force on unit length of it, in a direction perpendicular to the wire and in the central plane, is measured. The component of $\boldsymbol{b}$ normal to the cavity is $\lim_{\delta I \to 0} \delta f/\delta I$. These definitions effect the 'smoothing out' that is implied when Maxwell's equations are used.

2.5. Harmonic waves and complex vectors

In nearly all the problems discussed in this book, the field is assumed to arise from a wave that varies harmonically with time t, which means that all field variables, that is, components of $\boldsymbol{e}, \boldsymbol{h}, \boldsymbol{p}, \boldsymbol{j}, \boldsymbol{d}$ and $\boldsymbol{b}$, oscillate with the same angular frequency ω. Let $F_0(t)$ be some field component. It is a real variable. Then

$$F_0 = A\cos(\omega t + \phi) = \mathrm{Re}(F\mathrm{e}^{\mathrm{i}\omega t}) \tag{2.7}$$

where

$$F = A\mathrm{e}^{\mathrm{i}\phi}. \tag{2.8}$$

and A and ϕ are real. The complex number F determines the field component completely. It is multiplied by $\mathrm{e}^{\mathrm{i}\omega t}$ and the real part is then taken. Its modulus $A = |F|$ is called the amplitude of F_0 and $\phi = \arg F$ is called the phase.

The same argument applies to each component of every field vector so that, for example,

$$\boldsymbol{e}(t) = \mathrm{Re}(\boldsymbol{E}\mathrm{e}^{\mathrm{i}\omega t}) \tag{2.9}$$

where $\boldsymbol{E}$ is a complex vector. In a similar way capital letters are used to denote the complex vectors representing the other field components. But $\boldsymbol{B}$ and $\boldsymbol{Y}$ are exceptions; they are used for real vectors independent of time. In this book vectors are printed in bold type, their magnitudes in italic type and their components in italic type with subscripts. For example the electric intensity $\boldsymbol{E}$ has Cartesian components E_x, E_y, E_z which in general are complex, and magnitude $E = (E_x^2 + E_y^2 + E_z^2)^{\frac{1}{2}}$ which also, in general, is complex.

The differential equations (2.20) to (2.23) satisfied by the field vectors, are linear and homogeneous, and it is shown in ch. 3 that the relation between $\boldsymbol{D}$ or $\boldsymbol{P}$ or $\boldsymbol{J}$ and $\boldsymbol{E}$ is also linear and homogeneous. Hence if the complex representations, $\boldsymbol{E}\mathrm{e}^{\mathrm{i}\omega t}$, $\boldsymbol{H}\mathrm{e}^{\mathrm{i}\omega t}$ etc., for the field vectors are substituted, the equations may be separated into their real and imaginary parts which must be satisfied separately. Further, the factor $\mathrm{e}^{\mathrm{i}\omega t}$ appears in every term and so it may be cancelled. It follows that in all these equations, the actual real vectors $\boldsymbol{e}$, $\boldsymbol{h}$ etc. may be replaced by $\boldsymbol{E}$, $\boldsymbol{H}$ etc. It is implied that, to interpret the equations, the factor $\mathrm{e}^{\mathrm{i}\omega t}$ is restored in each term, and the real part of the term is then used.

This device of the omitted complex time factor $\mathrm{e}^{\mathrm{i}\omega t}$ is widely used in electrical engineering and in many branches of physics. It cannot be used without modification in problems that involve non-linear terms. For example, problems of energy flow involve products of field quantities, and the complex representations cannot be used immediately because the real part of the product of two complex numbers is not the same as the product of their real parts. It is then possible, however, to extend the use of complex numbers so that products of two complex vectors can be handled. This is done in the theory of energy input to the plasma, §2.11, and of the Poynting vector, §2.12.

To illustrate further the physical meaning of a complex vector, a description will now be given for the vector $\boldsymbol{E}\mathrm{e}^{\mathrm{i}\omega t}$. The same results apply for any of the other field vectors. The complex scalar product of $\boldsymbol{E}$ with itself is

$$\boldsymbol{E}\cdot\boldsymbol{E} = |\boldsymbol{E}\cdot\boldsymbol{E}|\mathrm{e}^{2\mathrm{i}\alpha} \tag{2.10}$$

where $\alpha = \frac{1}{2}\arg(\boldsymbol{E}\cdot\boldsymbol{E})$ is a real angle. Let

$$\boldsymbol{E}_0 = \boldsymbol{E}\mathrm{e}^{-\mathrm{i}\alpha} = \boldsymbol{u} + \mathrm{i}\boldsymbol{v} \tag{2.11}$$

where $\boldsymbol{u}$ and $\boldsymbol{v}$ are real vectors. Then

$$\boldsymbol{E}_0\cdot\boldsymbol{E}_0 = |\boldsymbol{E}\cdot\boldsymbol{E}| = u^2 - v^2 + 2\mathrm{i}\boldsymbol{u}\cdot\boldsymbol{v} \tag{2.12}$$

is real so that $\boldsymbol{u}\cdot\boldsymbol{v} = 0$. Thus $\boldsymbol{u}$ and $\boldsymbol{v}$ are at right angles. Now the electric intensity vector is

$$\mathrm{Re}(\boldsymbol{E}\mathrm{e}^{\mathrm{i}\omega t}) = \mathrm{Re}\{\boldsymbol{E}_0\mathrm{e}^{\mathrm{i}(\omega t+\alpha)}\} = \boldsymbol{u}\cos(\omega t+\alpha) - \boldsymbol{v}\sin(\omega t+\alpha). \tag{2.13}$$

This has two components at right angles and oscillating in quadrature. If the vector (2.13) is drawn from a fixed origin, its other end traces out an ellipse with $\boldsymbol{u}$, $\boldsymbol{v}$ as principal semi-axes. They are, from (2.11), (2.10) the real and imaginary parts of

$$\boldsymbol{E}\cdot\mathrm{e}^{-\mathrm{i}\alpha} = \boldsymbol{E}\{\boldsymbol{E}\cdot\boldsymbol{E}/|\boldsymbol{E}\cdot\boldsymbol{E}|\}^{-\frac{1}{2}}. \tag{2.14}$$

Now (2.11) gives (a star * denotes a complex conjugate)

$$\boldsymbol{E}\cdot\boldsymbol{E}^* = \boldsymbol{E}_0\cdot\boldsymbol{E}_0^* = u^2 + v^2. \tag{2.15}$$

Hence, from (2.12), the squares of the lengths of the principal semi-axes are

$$u^2, v^2 = \tfrac{1}{2}\{\boldsymbol{E}\cdot\boldsymbol{E}^* \pm |\boldsymbol{E}\cdot\boldsymbol{E}|\}. \tag{2.16}$$

The ellipse is a circle if $u = v$ that is, from (2.12), (2.10)

$$\boldsymbol{E}\cdot\boldsymbol{E} = 0. \tag{2.17}$$

It is a straight line if either $u = 0$ or $v = 0$, that is, from (2.16)

$$\boldsymbol{E}\cdot\boldsymbol{E}^* = |\boldsymbol{E}\cdot\boldsymbol{E}|. \tag{2.18}$$

Finally, from (2.11)

$$\mathrm{i}\boldsymbol{E}\wedge\boldsymbol{E}^* = \mathrm{i}(\boldsymbol{u}+\mathrm{i}\boldsymbol{v})\wedge(\boldsymbol{u}-\mathrm{i}\boldsymbol{v}) = 2\boldsymbol{u}\wedge\boldsymbol{v}. \tag{2.19}$$

This is a real vector of length $2uv$ at right angles to $\boldsymbol{u}$ and $\boldsymbol{v}$. It is therefore normal to the plane of the ellipse and its length is $2/\pi$ times the area of the ellipse.

2.6. Maxwell's equations

The electromagnetic field in a plasma is governed by the four Maxwell equations:

$$\mathrm{div}\,\boldsymbol{d} = 0, \tag{2.20}$$

$$\mathrm{div}\,\boldsymbol{b} = 0, \tag{2.21}$$

$$\mathrm{curl}\,\boldsymbol{e} = -\frac{\partial\boldsymbol{b}}{\partial t} = -\mu_0\frac{\partial\boldsymbol{h}}{\partial t}, \tag{2.22}$$

$$\mathrm{curl}\,\boldsymbol{h} = \frac{\partial\boldsymbol{d}}{\partial t}. \tag{2.23}$$

The first, (2.20), is a differential form of Gauss's theorem and results from the inverse square law of force in electrostatics. It is assumed that there is no permanent space charge, but that the contribution to the electric polarisation $\boldsymbol{p}$ of every electron and ion is included. The second equation, (2.21) is similarly a result of the inverse square law of force in magnetism.

The third equation, (2.22), is a differential form of Faraday's law of electromagnetic induction, which states that $\int \boldsymbol{e}\cdot d\boldsymbol{l}$ for a closed circuit is minus the rate of change of the magnetic flux linking the circuit. The integral is proportional to the work done in taking a small charge round the circuit. This could be found by imagining a long thin cavity to be cut along the line of the circuit. The cross section must be large compared with $N^{-\frac{1}{3}}$ but may still be so small that a negligible amount of material is removed, so that the disturbance of the fields is inappreciable. The test-charge is then moved in the cavity right round the circuit and the work measured. This argument shows that the electric intensity $\boldsymbol{e}$ used in (2.22) must be as defined by a long thin cavity, as in § 2.2.

The fourth equation, (2.23), is a differential form of Ampère's circuital theorem using the total current density $\partial\boldsymbol{d}/\partial t$. This is made up of a part $\partial\boldsymbol{p}/\partial t$ arising from the movement of electrons (and possibly ions) and a part $\varepsilon_0(\partial\boldsymbol{e}/\partial t)$ which is the contribution to the displacement current from the changing electric intensity.

2.7. Cartesian coordinate system

Let x, y, z be right-handed Cartesian coordinates, and let $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, be unit vectors in the directions of the x, y, z axes. Subscripts x, y, z are used to denote the x, y, z components respectively of a vector. For example, the components of $\boldsymbol{F}$ are written F_x, F_y, F_z. There is a very useful expression for the operator curl, which may be written in the form of a determinant

$$\operatorname{curl}\boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix}. \tag{2.24}$$

When this is evaluated the operators in the second row must always precede the field components in the third row.

For harmonic waves, the complex representation of § 2.5 is used, so that capital letters are now used for the field vectors and their components, and all field variables contain the time t only through the factor $e^{i\omega t}$, (2.2). Hence the operator $\partial/\partial t$ is equivalent to multiplication by $i\omega$. If this and (2.2), (2.24) are used, the last two Maxwell equations (2.22), (2.23) become:

$$\left.\begin{aligned} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= -i\omega\mu_0 H_x, & \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} &= i\omega D_x \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= -i\omega\mu_0 H_y, & \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} &= i\omega D_y \end{aligned}\right\} \tag{2.25}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\mathrm{i}\omega\mu_0 H_z, \quad \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \mathrm{i}\omega D_z \Bigg\}$$

In most of chs. 2–5 we consider only a single plane wave in a homogeneous medium, and the z axis is chosen to be the direction of the wave normal. For ch. 6 onwards there may be several plane waves with wave normals in different directions and the z axis is then usually chosen to be vertically upwards.

2.8. Progressive plane waves

A plane wave is defined to be a disturbance in which there is no variation of any field component in any plane parallel to a fixed plane. The z axis may be chosen to be normal to this fixed plane, and is called the 'wave normal'. The derivatives $\partial/\partial x$, $\partial/\partial y$ are then zero for all field components, so that (2.25) become

$$\frac{\partial E_y}{\partial z} = \mathrm{i}\omega\mu_0 H_x, \quad \frac{\partial H_x}{\partial z} = \mathrm{i}\omega D_y \tag{2.26}$$

$$\frac{\partial E_x}{\partial z} = -\mathrm{i}\omega\mu_0 H_y, \quad \frac{\partial H_y}{\partial z} = -\mathrm{i}\omega D_x \tag{2.27}$$

$$H_z = 0, \quad D_z = 0. \tag{2.28}$$

The equations (2.28) show that, for plane waves, $\boldsymbol{D}$ and $\boldsymbol{H}$ are perpendicular to the wave normal, and they are therefore said to be 'transverse'. In an isotropic medium $\boldsymbol{D}$ is proportional to $\boldsymbol{E}$, so that $\boldsymbol{E}$ also is transverse. But a magnetoplasma is not isotropic and it is shown later, §§ 4.3, 4.7, that $\boldsymbol{E}$ often has a longitudinal component, that is, a component in the direction of the wave normal.

For a homogeneous isotropic medium we may write

$$\boldsymbol{D} = \varepsilon_0 n^2 \boldsymbol{E} = \varepsilon_0 \varepsilon \boldsymbol{E} \tag{2.29}$$

where n is the refractive index, derived later, chs. 4 and 5, and is a constant (in general complex) at a given frequency. This may be substituted in (2.26), (2.27). The equations (2.26) then involve E_y, H_x only, and (2.27) involve E_x, H_y only. These pairs are therefore independent, and the variables in one pair can vanish without affecting the other. If this happens, the wave is said to be linearly polarised. Elimination of H_y from (2.27) gives

$$\frac{\partial^2 E_x}{\partial z^2} + k^2 n^2 E_x = 0 \tag{2.30}$$

where

$$k = \omega/c, \quad c = (\varepsilon_0\mu_0)^{-\frac{1}{2}}. \tag{2.31}$$

This meaning for k appears in many texts and papers on radio propagation, and it is used throughout this book. It is different from the usage in some other branches of physics, where k means $\omega n/c$.

Two independent solutions of (2.30) are

$$E_x = E_x^{(1)}\,\mathrm{e}^{-ikn z} \tag{2.32}$$

$$E_x = E_x^{(2)}\,\mathrm{e}^{+ikn z} \tag{2.33}$$

where $E_x^{(1)}$, $E_x^{(2)}$ are independent of z and t. These two solutions represent waves travelling in the direction of positive and negative z, respectively. Any other solution of (2.30) can be expressed as the sum of multiples of the two independent solutions (2.32), (2.33).

Substitution of (2.32) into (2.27) gives

$$n\varepsilon_0{}^{\frac{1}{2}}E_x = \mu_0{}^{\frac{1}{2}}H_y. \tag{2.34}$$

Thus the ratio of E_x to H_y is a constant so that E_x and H_y both depend on z only through the factor e^{-iknz}. A plane wave of this kind, in which the dependence of all field quantities upon z is the same, is called a 'progressive plane wave'. The operator $\partial/\partial z$ must then be equivalent to multiplication by a constant (in this case $\partial/\partial z \equiv -\mathrm{i}kn$), The ratio E_x/H_y is called the 'wave impedance', so that a progressive wave is one for which the wave impedance is independent of z.

Similarly, substitution of (2.33) into (2.27) gives

$$n\varepsilon_0{}^{\frac{1}{2}}E_x = -\mu_0^{\frac{1}{2}}H_y, \tag{2.35}$$

so that E_x and H_y both vary with z only through the factor e^{iknz}, and $\partial/\partial z = \mathrm{i}kn$ for both field components. Thus (2.33) is also a progressive plane wave. Consider the expression

$$E_x = E_0 \cos(knz) \tag{2.36}$$

which is a solution of (2.30). It is a plane wave, and can be expressed as the sum of terms like (2.32) and (2.33) with equal modulus. Substitution of (2.36) in (2.27) gives

$$\mu_0^{\frac{1}{2}}H_y = -\mathrm{i}\varepsilon_0{}^{\frac{1}{2}}nE_0 \sin(knz). \tag{2.37}$$

It is at once clear that this does not have the properties of a progressive wave. In this case the two component progressive waves have equal modulus and the wave is called a 'standing wave'. If the moduli of the component progressive wave were unequal, the wave would be called a 'partial standing wave'.

It can be shown in a similar way that (2.26) leads to two linearly polarised progressive plane waves, with the electric vector parallel to the y axis. For the one travelling in the positive z direction, (2.32) and (2.34) are replaced by

$$E_y = E_y^{(1)}\mathrm{e}^{-iknz} \tag{2.38}$$

and

$$n\varepsilon_0{}^{\frac{1}{2}}E_y = -\mu_0{}^{\frac{1}{2}}H_x. \tag{2.39}$$

Similarly (2.33) and (2.35) give the same expressions with the sign of n reversed. It has thus been shown that for progressive plane waves travelling in, say, the positive z direction, there are two independent solutions, each linearly polarised with their electric vectors at right angles. They may be combined with any moduli and relative phase to give a resultant wave which is, in general, elliptically polarised. Any polarisation ellipse can be produced by a suitable combination of the component plane waves and, once established, the ellipse does not change as long as the wave remains in the same homogeneous isotropic medium. In contrast to this, it will be shown later, § 4.3, that in a homogeneous cold magnetoplasma, progressive waves can only have one of two possible polarisations. A wave of any other polarisation must in general be made up of two component waves travelling with different velocities.

2.9. Plane waves in free space

In free space there are no electrons and $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E}$ so that $n = 1$. Then the progressive plane wave solution (2.32) gives

$$E_x = E_x^{(1)} e^{-ikz} \tag{2.40}$$

where $c = (\varepsilon_0 \mu_0)^{-\frac{1}{2}}$ is the speed of the waves, and (2.34) gives

$$\varepsilon_0^{\frac{1}{2}} E_x = \mu_0^{\frac{1}{2}} H_y. \tag{2.41}$$

Thus the ratio E_x/H_y is real which shows that E_x and H_y are in phase, and the vectors $\boldsymbol{E}$, $\boldsymbol{H}$ and the wave normal, in that order, form a right-handed system. The wave impedance is

$$E_x/H_y = (\mu_0/\varepsilon_0)^{\frac{1}{2}} = Z_0, \tag{2.42}$$

which is called the 'characteristic impedance of free space'.

2.10. The notation $\mathscr{H}$ and H

It is now convenient to adopt for the magnetic intensity $\boldsymbol{H}$ a different measure which simplifies the equations and will be used throughout this book. Take

$$\mathscr{H} = Z_0 \boldsymbol{H}. \tag{2.43}$$

Thus $\mathscr{H}$ measures the magnetic field in terms of the electric field that would be associated with it in a progressive plane wave in free space. It has the same effect as if $\boldsymbol{E}$ and $\boldsymbol{H}$ were measured in Gaussian units (see § 2.1).The vector $\mathscr{H}$ has the same physical dimensions as the electric intensity $\boldsymbol{E}$ (which is not true for $\boldsymbol{E}$ and $\boldsymbol{H}$ in Gaussian units). The last two Maxwell equations (2.22), (2.23) now become, for harmonic waves

$$\operatorname{curl} \boldsymbol{E} = -ik\mathscr{H}, \quad \operatorname{curl} \mathscr{H} = ik\varepsilon_0^{-1} \boldsymbol{D} \tag{2.44}$$

or written in full, in Cartesian coordinates:

$$\left.\begin{aligned} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= -ik\mathscr{H}_x, & \frac{\partial \mathscr{H}_z}{\partial y} - \frac{\partial \mathscr{H}_y}{\partial z} &= ik\varepsilon_0^{-1} D_x \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= -ik\mathscr{H}_y, & \frac{\partial \mathscr{H}_x}{\partial z} - \frac{\partial \mathscr{H}_z}{\partial x} &= ik\varepsilon_0^{-1} D_y \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= -ik\mathscr{H}_z, & \frac{\partial \mathscr{H}_y}{\partial x} - \frac{\partial \mathscr{H}_x}{\partial y} &= ik\varepsilon_0^{-1} D_z \end{aligned}\right\}. \tag{2.45}$$

These equations are the starting-point for much of the later work in this book.

2.11. The power input to the plasma from a radio wave

It is shown in books on electromagnetic theory that when a small change $\delta \boldsymbol{d}$ is made in the electric displacement, the electric forces must supply energy to unit volume of the medium, equal to

$$\delta\mathscr{S}_{\mathrm{E}} = \boldsymbol{e}\cdot\delta\boldsymbol{d}. \tag{2.46}$$

A simple way of showing this is to imagine that a homogeneous sample of the plasma is enclosed in a condenser with parallel plates of area A separated by a distance s. Let the charge per unit area on the plates be σ. Then $\sigma = d$ where d is the component of $\boldsymbol{d}$ perpendicular to the plates. The electric intensity $\boldsymbol{e}$ cannot have a component parallel to the conducting plates. It is therefore normal to the plates. Let its magnitude be e. Then the potential difference between the plates is es. If σ is now increased by a small amount $\delta\sigma$, the electric energy supplied is $esA\delta\sigma$ that is energy $e\delta\sigma$ per unit volume. This is equal to $\boldsymbol{e}\cdot\delta\boldsymbol{d}$ as in (2.46). A more general proof is given by Stratton (1941, § 2.8). Thus the power input to unit volume of the plasma is

$$\partial\mathscr{S}_{\mathrm{E}}/\partial t = \boldsymbol{e}\cdot\partial\boldsymbol{d}/\partial t. \tag{2.47}$$

The part of this that arises from the conduction current is dissipated as heat. The remainder gives stored energy in the plasma. The discussion of this topic is continued in § 2.12. See also problems 3.2, 3.3.

It can be shown that energy is also supplied by the magnetic field, given by

$$\delta\mathscr{S}_{\mathrm{M}} = \boldsymbol{h}\cdot\delta\boldsymbol{b} \tag{2.48}$$

per unit volume. In a vacuum, or in any medium for which $\boldsymbol{b} = \mu_0\boldsymbol{h}$, including the plasmas of the earth's upper atmosphere, (2.48) gives $\delta\mathscr{S}_{\mathrm{M}} = \frac{1}{2}\mu_0\delta(\boldsymbol{h}^2)$. There is no mechanism by which magnetic energy is dissipated as heat, so that $\mathscr{S}_{\mathrm{M}}$ is stored energy per unit volume, given by

$$\mathscr{S}_{\mathrm{M}} = \tfrac{1}{2}\mu_0\boldsymbol{h}^2 = \tfrac{1}{2}\boldsymbol{h}\cdot\boldsymbol{b}. \tag{2.49}$$

For a radio wave in which the fields vary harmonically in time, it is of interest to use (2.47) to find the average value $(\partial\mathscr{S}_{\mathrm{E}}/\partial t)_{\mathrm{av}}$ of the power input over a time very long compared to the period $2\pi/\omega$. The complex representations $\boldsymbol{E}$, $\boldsymbol{D}$ of $\boldsymbol{e}$, $\boldsymbol{d}$ cannot immediately be used because (2.47) involves a product. But

$$\boldsymbol{e} = \tfrac{1}{2}(\boldsymbol{E}\mathrm{e}^{\mathrm{i}\omega t} + \boldsymbol{E}^*\mathrm{e}^{-\mathrm{i}\omega t}) \tag{2.50}$$

and there is a similar expression for $\boldsymbol{d}$. Hence (2.47) gives

$$\boldsymbol{e}\cdot\partial\boldsymbol{d}/\partial t = \tfrac{1}{4}\mathrm{i}\omega(\boldsymbol{E}\mathrm{e}^{\mathrm{i}\omega t} + \boldsymbol{E}^*\mathrm{e}^{-\mathrm{i}\omega t})\cdot(\boldsymbol{D}\mathrm{e}^{\mathrm{i}\omega t} - \boldsymbol{D}^*\mathrm{e}^{-\mathrm{i}\omega t}). \tag{2.51}$$

Here there is a term containing $\mathrm{e}^{2\mathrm{i}\omega t}$, and another containing $\mathrm{e}^{-2\mathrm{i}\omega t}$. When the average over a period is taken, the contribution from these oscillatory terms is zero. The remaining terms give

$$(\partial\mathscr{S}_{\mathrm{E}}/\partial t)_{\mathrm{av}} = \tfrac{1}{4}\mathrm{i}\omega(\boldsymbol{E}^*\cdot\boldsymbol{D} - \boldsymbol{E}\cdot\boldsymbol{D}^*) = \tfrac{1}{2}\omega\,\mathrm{Im}(\boldsymbol{E}\cdot\boldsymbol{D}^*). \tag{2.52}$$

It would be expected that the electric energy stored in the medium is proportional to the square of the amplitudes of the fields, that is to $|E|^2$, $|D|^2$ (compare (2.49) for the magnetic energy). These amplitudes are constant. The stored energy may oscillate in a time comparable to the period $2\pi/\omega$, but it cannot continue to increase in a much longer time period. The power (2.52) must therefore be the average rate at which energy is dissipated as heat in the medium.

For most radio waves the fields are small enough to permit the assumption that $\boldsymbol{D}$ depends linearly on $\boldsymbol{E}$. The relation is given by

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{\varepsilon} \boldsymbol{E} \tag{2.53}$$

where $\boldsymbol{\varepsilon}$ is called the relative electric permittivity. Expressions for it are derived in the following chapter. For an isotropic plasma $\boldsymbol{\varepsilon}$ is a scalar ε, in general complex, and (2.52) gives $-\frac{1}{2}\omega \boldsymbol{E}\cdot\boldsymbol{E}^* \operatorname{Im}(\varepsilon)$. This is zero if ε is real. The medium cannot then dissipate energy as heat, and is said to be 'loss free'. For an anisotropic plasma $\boldsymbol{\varepsilon}$ is a tensor of rank 3, represented by a 3×3 matrix. It is then convenient to write (2.53) with suffix notation

$$D_i = \varepsilon_0 \varepsilon_{ij} E_j, \quad i, j = x, y, z \tag{2.54}$$

where it is implied that any term containing the same suffix twice is equal to sum of terms with all possible values of that suffix. This is called the 'summation convention for repeated suffixes'. (See, for example, Jeffreys, 1931). Then (2.52) gives

$$(\partial \mathscr{S}_{\mathrm{E}}/\partial t)_{\mathrm{av}} = \tfrac{1}{4}\mathrm{i}\omega E_i^* E_j(\varepsilon_{ij} - \varepsilon_{ji}^*). \tag{2.55}$$

If the medium is loss free, this must be zero for all possible fields E_i, which requires that

$$\varepsilon_{ij} = \varepsilon_{ji}^*. \tag{2.56}$$

Then $\boldsymbol{\varepsilon}$ is said to be Hermitian, that is the matrix representing $\boldsymbol{\varepsilon}$ is the complex conjugate of its transpose.

2.12. The flow of energy. The Poynting vector

Equations (2.46), (2.48) show that the power being supplied per unit volume of a plasma is

$$\partial \mathscr{S}/\partial t = \partial(\mathscr{S}_{\mathrm{E}} + \mathscr{S}_{\mathrm{M}})/\partial t = \boldsymbol{e}\cdot\partial \boldsymbol{d}/\partial t + \boldsymbol{h}\cdot\partial \boldsymbol{b}/\partial t. \tag{2.57}$$

This may be integrated over some volume V of the medium. Then

$$\frac{\partial}{\partial t}\int_V \mathscr{S}\,\mathrm{d}V = \int_V \{\boldsymbol{e}\cdot\partial \boldsymbol{d}/\partial t + \boldsymbol{h}\cdot\partial \boldsymbol{b}/\partial t\}\,\mathrm{d}V \tag{2.58}$$

which is the rate at which the electric and magnetic forces are supplying energy to the volume. If, now, the Maxwell equations (2.22), (2.23) are used, (2.58) becomes

$$\int_V (\boldsymbol{e}\cdot\operatorname{curl}\boldsymbol{h} - \boldsymbol{h}\cdot\operatorname{curl}\boldsymbol{e})\,\mathrm{d}V = -\int_V \operatorname{div}(\boldsymbol{e}\wedge\boldsymbol{h})\,\mathrm{d}V \tag{2.59}$$

from a well known identity of vector analysis. The last integral can be expressed as a surface integral by using the 'divergence theorem'. The result is

$$\frac{\partial}{\partial t}\int_V \mathscr{S}\,\mathrm{d}V = -\int_S (\boldsymbol{e}\wedge\boldsymbol{h})\cdot\mathrm{d}\boldsymbol{S} \tag{2.60}$$

where the second integral is evaluated over the whole of the surface S enclosing V, and $\mathrm{d}\boldsymbol{S}$ is a vector in the direction of the outward normal, of magnitude equal to an element of surface area.

Equation (2.60) is Poynting's theorem. It suggests that the Poynting vector

$$\boldsymbol{\Pi} = \boldsymbol{e}\wedge\boldsymbol{h} \tag{2.61}$$

gives the flux of energy in the electromagnetic field. This result cannot be regarded as proved, and the difficulties are discussed in books on electromagnetic theory; see for example Stratton (1941, § 2.19). In this book it is assumed that $\boldsymbol{\Pi}$ gives the flux of energy. There are no known cases where this assumption leads to incorrect results.

The Poynting vector (2.61) is the product of two field quantities. For a harmonic wave, these cannot simply be replaced by their complex representations. Instead we have

$$\boldsymbol{\Pi} = \tfrac{1}{4}(\boldsymbol{E}\mathrm{e}^{\mathrm{i}\omega t} + \boldsymbol{E}^*\mathrm{e}^{-\mathrm{i}\omega t})\wedge(\boldsymbol{H}\mathrm{e}^{\mathrm{i}\omega t} + \boldsymbol{H}^*\mathrm{e}^{-\mathrm{i}\omega t}). \tag{2.62}$$

Now take the average of this over one period. The terms containing factors $\mathrm{e}^{2\mathrm{i}\omega t}$ and $\mathrm{e}^{-2\mathrm{i}\omega t}$ are oscillatory and give zero. The remaining terms give

$$\boldsymbol{\Pi}_{\mathrm{av}} = \tfrac{1}{4}(\boldsymbol{E}\wedge\boldsymbol{H}^* + \boldsymbol{E}^*\wedge\boldsymbol{H}) = \tfrac{1}{2}\mathrm{Re}(\boldsymbol{E}\wedge\boldsymbol{H}^*). \tag{2.63}$$

The product $\frac{1}{2}\boldsymbol{E}\wedge\boldsymbol{H}^*$ for a harmonic wave is called the complex Poynting vector. Its real part is assumed to be the time average of the flux of energy. It is extensively used in later chapters for waves in stratified media. Its component $\boldsymbol{\Pi}_z$ perpendicular to the strata is given by

$$4Z_0\boldsymbol{\Pi}_z = E_x\mathscr{H}_y^* - E_y\mathscr{H}_x^* - E_y^*\mathscr{H}_x + E_x^*\mathscr{H}_y. \tag{2.64}$$

2.13. Complex refractive index

In § 2.8 a progressive plane wave in a homogeneous medium was defined to be one in which all field quantities vary in space only through a factor $\mathrm{e}^{-\mathrm{i}knz}$ where z is measured in the direction of the wave normal. The refractive index n need not be real and it is shown in ch. 4 that in general n is complex. Hence let

$$n = \mu - \mathrm{i}\chi \tag{2.65}$$

where μ and χ are real. Then for the progressive wave (2.38)

$$E_y = E_y^{(1)}\mathrm{e}^{-\mathrm{i}k\mu z}\mathrm{e}^{-k\chi z}. \tag{2.66}$$

This represents a linearly polarised wave travelling with wave velocity c/μ and attenuated at a rate depending on χ. The wave must be losing energy which is absorbed by the medium (see problem 2.1). If χ were negative, the wave would grow in amplitude as it travelled. This can happen in some plasmas where there is a source of energy that can be supplied to the wave. It may occur, for example, when there are charged particles streaming through the plasma or when other waves are present. This is part of the subject of 'plasma instabilities', which is discussed in books on plasma physics. In this book, however, it is assumed that there is no source of wave energy in the medium and the plasmas are said to be 'passive'. Then it is expected that μ and χ always have the same sign, and this is shown to be true for the refractive indices derived later, ch. 4. When μ and χ are both negative, the wave (2.66) travels in the direction of negative z and is attenuated as it travels.

In a variable medium the refractive index n is a function of the space coordinates. For some purposes, §§7.19, 8.21, 14.9–14.12, 16.5–16.7, the process of analytic continuation is used to find n at complex values of these coordinates. The restriction that μ and χ have the same sign does not apply when the coordinates are complex, even though the medium is passive.

2.14. Evanescent waves

If n^2 is negative and real, n is purely imaginary, so that $\mu = 0$, and (2.66) becomes

$$E_y = E_y^{(1)} e^{-k\chi z}. \tag{2.67}$$

This appears to represent a wave travelling with infinite wave-velocity. Every field component varies harmonically in time, but there is no harmonic variation in space. The phase is the same for all values of z. A disturbance of this kind is called an 'evanescent' wave. It is still a 'progressive' wave according to the definition adopted here. Other examples of such waves are given in §7.11.

The expressions (2.34), (2.39) are true whether n is real or complex. If it is real, then the electric and magnetic fields of a linearly polarised wave are in phase or antiphase. In an evanescent wave n is purely imaginary, so that the electric and magnetic fields are in quadrature. This means that the time average value of the Poynting vector (2.63) is zero. Hence there is no net energy flow for such a wave in an isotropic medium. For an evanescent wave in an anisotropic medium there can be some flow of energy perpendicular to the wave normal (see §4.8). If n has the general complex value (2.65), the phase difference between the electric and magnetic fields is given by $\arctan(\chi/\mu)$.

If n^2 is real and positive, a progressive plane wave is unattenuated, and there is no absorption of energy. If n^2 is real and negative, the only possible progressive wave is evanescent, and there is no energy flow and again no absorption of energy. A medium for which n^2 is real, whether positive or negative, is said to be 'loss-free'.

2.15. Inhomogeneous plane waves

For the progressive waves considered in previous sections it was convenient to choose the z axis to be in the direction of the wave normal. When the wave normal has some other direction, with direction cosines $\boldsymbol{l} = l_x, l_y, l_z$, the exponential in (2.32) or (2.38) is replaced by

$$\exp\{-ikn(l_x x + l_y y + l_z z)\}. \tag{2.68}$$

The vector $\boldsymbol{l}$ need not be real. Let

$$\boldsymbol{l} = \boldsymbol{a} + i\boldsymbol{b} \tag{2.69}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are real vectors of lengths a and b respectively. Since $\boldsymbol{l}$ is a unit vector it follows that

$$a^2 - b^2 = 1, \quad \boldsymbol{a}\cdot\boldsymbol{b} = 0 \tag{2.70}$$

Thus $\boldsymbol{a}$ and $\boldsymbol{b}$ are at right angles. Now choose new real Cartesian axes with the x, y, z axes parallel to $\boldsymbol{a}, \boldsymbol{a} \wedge \boldsymbol{b}, \boldsymbol{b}$ respectively. In these axes the direction cosines of the wave normal are $(a, 0, ib)$ and the exponential (2.68) becomes

$$\exp\{-ikn(ax + ibz)\} = \exp(-iknax)\exp(knbz). \tag{2.71}$$

A wave of this kind is called an 'inhomogeneous plane wave'. Nearly all the component waves considered in chs. 6, 15–19, are of this kind.

Now suppose that n is real. Then in (2.71) the second exponential is real. The first exponential alone represents a plane wave with wave fronts $x = \text{constant}$. These real planes are not true wave fronts, except when $b = 0$, and it is better to call them 'planes of constant phase'. The second exponential in (2.71) shows that the waves change in amplitude when z varies, and the planes $z = \text{constant}$ are 'planes of constant amplitude'. Next suppose that n is purely imaginary. Then the planes $x = \text{constant}$ are planes of constant amplitude, and the planes $z = \text{constant}$ are planes of constant phase. It has thus been shown that for a loss free medium, in which n^2 must be real and either positive or negative, the planes of constant phase and the planes of constant amplitude are at right angles.

Consider the special case where the wave is linearly polarised with $\boldsymbol{E}$ parallel to the y axis. Then (2.45) shows that

$$\mathscr{H}_x = inbE_y, \mathscr{H}_z = naE_y. \tag{2.72}$$

This result can be used to study the time averaged energy flux $\boldsymbol{\Pi}_{\text{av}}$ (2.63) (see problem 2.1). It shows that $\mathscr{H}_x$ and $\mathscr{H}_z$ are in quadrature. Thus if $\boldsymbol{\mathscr{H}}$ is drawn from a fixed origin its other end traces out a real ellipse, that is in a plane containing both the real and imaginary parts $\boldsymbol{a}, \boldsymbol{b}$ of the complex wave normal vector $\boldsymbol{l}$. It has a very different meaning from the polarisation ellipse of an elliptically polarised wave. A similar result can be derived for a linearly polarised wave with $\boldsymbol{\mathscr{H}}$ parallel to the y axis. Then (2.72) is replaced by

$$E_x = -\frac{ib}{n}\mathscr{H}_y, \quad E_z = \frac{a}{n}\mathscr{H}_y \tag{2.73}$$

For a wave of any other polarisation, a combination of (2.72) and (2.73) can be used.

If n is complex and given by (2.65), the exponential in (2.71) becomes

$$\exp\{-\mathrm{i}k(\mu ax + \chi bz) + k(\mu bz - \chi ax)\}. \tag{2.74}$$

Here the planes of constant phase and of constant amplitude are no longer at right angles. The angle between them is, from (2.74),

$$\arctan\{-ab(\mu^2 + \chi^2)/\mu\chi\}. \tag{2.75}$$

PROBLEMS 2

2.1. For a progressive wave in an isotropic homogeneous medium whose refractive index is complex and whose wave normal is in a real direction, find the time average $\mathbf{\Pi}_{\mathrm{av}}$ of the flux of energy, and find the rate P at which energy is being dissipated as heat in the medium. Verify that $P = -\operatorname{div} \mathbf{\Pi}_{\mathrm{av}}$. Do the same for an inhomogeneous plane wave.

2.2. In a certain radio wave, the electric field at a given fixed point varies harmonically with time as given by a factor $\mathrm{e}^{\mathrm{i}\omega t}$, and its components have the values given below. In each case find its state of polarisation, and, where appropriate, find the area of the polarisation ellipse, the axis ratio, and the direction cosines of the normal to its plane. (a) 1, i, 0. (b) i, $-$i, 2. (c) $2-\mathrm{i}\sqrt{3}$, $2+\mathrm{i}\sqrt{3}$, 4. (d) 2, 2, $\frac{1}{2}$. (e) $1-\mathrm{i}$, $1+\mathrm{i}$, 0. (f) ib, 0, a.

2.3. An inhomogeneous plane wave of angular frequency ω travels in a vacuum. Its wave normal has direction cosines $a, 0, \mathrm{i}(a^2-1)^{\frac{1}{2}}$ where a and the square root are real and positive. It is elliptically polarised with $E_y = \mathscr{H}_y \mathrm{e}^{\mathrm{i}\alpha}$ where α is real. Thus $\mathrm{Re}(\boldsymbol{E}\mathrm{e}^{\mathrm{i}\omega t})$ and $\mathrm{Re}(\mathscr{H}\mathrm{e}^{\mathrm{i}\omega t})$, when drawn from a fixed origin, both trace out ellipses. Prove that these two ellipses have the same size and shape and that their planes make equal and opposite angles β with the plane $y=0$, where $\tan^2\beta = a^2(a^2-1)/(a^2-\cos^2\alpha)$. Prove that the ratio of their principal axes is $(a^2-\cos\alpha)/(a^2+\cos\alpha)$.

2.4. For an inhomogeneous plane wave in a loss free isotropic medium, the electric field is in a real direction perpendicular to both the real and imaginary parts of the complex wave normal vector and is given by $E_0\mathrm{e}^{\mathrm{i}\omega t}$. Prove that when the magnetic intensity vector $\mathrm{Re}(\boldsymbol{H}\mathrm{e}^{\mathrm{i}\omega t})$ is drawn from a fixed origin, it traces out an ellipse whose principal axes are parallel and perpendicular to the planes of constant phase. Show that the component of $\boldsymbol{H}$ parallel to the planes of constant amplitude is in quadrature with E_0. Hence show that the time averaged Poynting vector $\frac{1}{2}\mathrm{Re}(\boldsymbol{E} \wedge \boldsymbol{H}^*)$ is parallel to the planes of constant amplitude. Investigate also the behaviour of the instantaneous Poynting vector $\mathrm{Re}(\boldsymbol{E}\mathrm{e}^{\mathrm{i}\omega t}) \wedge \mathrm{Re}(\boldsymbol{H}\mathrm{e}^{\mathrm{i}\omega t})$. Consider the two cases where the refractive index n is (i) real, and (ii) purely imaginary.