3

The constitutive relations

3.1. Introduction

Before (2.45) can be applied to the theory of wave propagation in a plasma, it is necessary to express the electric displacement $\boldsymbol{D}$, and therefore the electric polarisation $\boldsymbol{P}$, in terms of the electric intensity $\boldsymbol{E}$. The resulting expressions are called the constitutive relations of the plasma and are derived in this chapter. The subject of wave propagation is resumed in ch. 4.

From now on, except in § 3.8, all fields are assumed to vary harmonically in time, and are designated by capital letters representing complex vectors, as explained in § 2.5. Time derivatives of some of the fields appear in the constitutive relations, and since $\partial/\partial t \equiv \mathrm{i}\omega$, it follows that the angular frequency ω appears in the expression for the electric permittivity. When this happens the medium is said to be time dispersive.

In this and the following chapter the plasma is assumed to be homogeneous. In later chapters the results are applied to an inhomogeneous plasma. This is justified provided that the plasma is sufficiently slowly varying. The meaning of 'slowly varying' is discussed in § 7.10. It is further assumed, in the present chapter, that the spatial variation of the fields can be ignored. This means that, over a distance large compared with $N^{-\frac{1}{3}}$, the fields can be treated as uniform. Thus spatial derivatives of the fields do not appear in the expression for the permittivity. When this assumption is not true, the expression for the permittivity contains the refractive index vector $\boldsymbol{n}$; § 5.1. Then the medium is said to have spatial dispersion. This is important in phenomena where $|\boldsymbol{n}| \gg 1$. For example it plays an essential part in the dispersion relations for plasma waves, and ion acoustic waves, which depend on the plasma temperature, and whose wave velocities are small, so that $|\boldsymbol{n}|$ is large. See also problem 13.2. But for radio waves the effect of spatial dispersion is negligible.

3.2. Free, undamped electrons

Consider a volume of plasma as large as possible consistent with the assumption that within it the electric intensity $\boldsymbol{E}$ is uniform (see § 3.5). Then every electron experiences the same electric force $\boldsymbol{E}e$. The electrons have random velocities because of their thermal motions, but these are in all directions and the average over many electrons is zero. Let $\boldsymbol{r}$ be the vector displacement of an electron from the position it would occupy if there were no field $\boldsymbol{E}$. It is a harmonically varying quantity and has the complex representation

$$\boldsymbol{r} = \boldsymbol{R}e^{i\omega t} \tag{3.1}$$

where it is implied that the real part of the right-hand side is used as explained in §2.5. It is now assumed, as a first approximation, that all other forces on the electrons are negligible. Thus collisions and the effect of the earth's magnetic field are ignored. The force exerted by the magnetic field $\mathscr{H}$ of the wave is neglected; the justification for this is discussed in § 3.7. Then the equation of motion of one electron is

$$\boldsymbol{E}e = m\partial^2\boldsymbol{r}/\partial t^2 \tag{3.2}$$

where m is the mass of an electron. Hence from (2.4) and (3.1) on multiplying (3.2) by Ne:

$$\boldsymbol{E}Ne^2 = -\omega^2 m\boldsymbol{P}. \tag{3.3}$$

This gives the contribution to $\boldsymbol{P}$ from the electrons. An exactly similar argument can be used to find the contribution from the ions. Since an ion is of the order of 2000 to 60 000 times more massive than an electron, its contribution to $\boldsymbol{P}$ in (3.3) is smaller in the same proportion. Thus, for radio waves, it is nearly always permissible to neglect the ions' contributions. The discussions in this book are therefore given for electrons, and the ions are ignored, except where specifically stated. The same results can be used for the ions provided that m and e are replaced by the appropriate values. Some examples where the effect of ions is studied are given in §§3.11, 3.13, 4.9, 13.9. If the ions are ignored, (3.3) gives for the constitutive relation

$$\boldsymbol{P} = -\varepsilon_0 X\boldsymbol{E}, \tag{3.4}$$

$$X = Ne^2/(\varepsilon_0 m\omega^2). \tag{3.5}$$

The quantity X is very important and appears throughout the theory in this book. In Appleton's original paper (1932), and in many important early papers on magnetoionic theory, the symbol x was used, but for many years this has now been replaced by X to avoid confusion with the coordinate x. Sometimes X is written

$$X = \omega_N^2/\omega^2 = f_N^2/f^2, \quad \omega_N^2 = Ne^2/\varepsilon_0 m, \quad \omega_N = 2\pi f_N. \tag{3.6}$$

Here f_N is called the 'plasma frequency', and ω_N is the angular plasma frequency. Its square is proportional to the electron concentration N. X also is proportional to N and inversely proportional to the square of the wave frequency f. Note that its value is independent of the sign of the electronic charge.

The following are useful numerical values. The plasma frequency f_N is in hertz and the electron concentration N is in m^{-3}:

$$f_N^2 = 80.61\,N, \quad f_N = 8.98\,N^{\frac{1}{2}}, \quad N = 1.240 \times 10^{-2} f_N^2. \tag{3.7}$$

3.3. The Lorentz polarisation term

In the preceding section it was assumed that the same electric field $\boldsymbol{E}$ acts on all the electrons in the volume of plasma considered, and in later sections it is assumed that this $\boldsymbol{E}$ is the same as the average or 'smoothed out' $\boldsymbol{E}$ as defined, in § 2.2, in terms of a long thin cavity, and as used in Maxwell's equations (2.45). In some types of dielectric this assumption is not true. Lorentz (1909) considered dielectrics in which all the electrons are bound within molecules. When an electric field $\boldsymbol{E}$ is applied, each molecule is distorted so that it acquires an electric dipole moment $\boldsymbol{M}_{\mathrm{E}}$, and the electric polarisation is $\boldsymbol{P} = N_{\mathrm{M}}\boldsymbol{M}_{\mathrm{E}}$ where N_{M} is the concentration of the molecules. Thus $\boldsymbol{P}$ is the electric dipole moment per unit volume, and the difficulty mentioned in § 2.3, connected with the origin of $\boldsymbol{r}$, does not arise. Lorentz showed that if the molecules of the dielectric are arranged on a cubic lattice, or if they are arranged randomly in space, as in a fluid or in glass, for example, then the average electric field acting on a molecule is $\boldsymbol{E}+\frac{1}{3}\boldsymbol{P}/\varepsilon_0$. Here the second term is called the Lorentz polarisation term. Some authors assumed that this expression applied also to the random arrangement of free electrons in a plasma. If this were so the constitutive relation (3.4) would become

$$P = -\varepsilon_0 EX/(1+\tfrac{1}{3}X) \tag{3.8}$$

and there would be corresponding changes in the forms (3.14), (3.23), (3.35), given later. It is now known that for a plasma the Lorentz term must be omitted.

The reasons for this may be summarised as follows. In this discussion the word 'average' implies an average over very many particles, at a fixed time. The electric force acting on any one electron is made up of the electrostatic forces from the other electrons and from the ions, as well as the force $\boldsymbol{E}e$. The particles are randomly distributed in space so that the total electrostatic force is different for different electrons, and it is required to find the average value. To do this consider a very large number N_{c} of 'complexions' of the plasma, and for each one choose the origin of coordinates to be at the position of one test electron. To find the average force on it, we add together the forces in the different complexions and then divide by N_{c}. This is achieved by imagining that the complexions are all superimposed. In any one complexion, two electrons are not likely to be extremely close together because of the electrostatic repulsion. Let s_{e} be the distance where the electrostatic energy is about equal to the average thermal energy so that

$$e^2/4\pi\varepsilon_0 s_{\mathrm{e}} \approx KT \tag{3.9}$$

where K is Boltzmann's constant and T is the temperature. Then for all the complexions there will be very few electrons at distances less than s_{e} from the test electron. Apart from this the electrons are randomly distributed in space, so that when the complexions are superimposed the distribution of electronic charge at distances greater than s_{e} approaches uniformity as $N_{\mathrm{c}} \to \infty$. At smaller distances the spatial charge density gets less but the charge distribution is spherically symmetrical, with the test charge at the centre. A spherical distribution of charge gives no electric intensity at its centre.

Similarly the centre of a positive ion cannot be at a distance less than the ion radius s_i from the test electron. At this distance there are repulsive forces much stronger than the electrostatic attraction. Apart from this the positive ions are randomly distributed in space. When the complexions are superimposed the positive charge of the ions is uniformly distributed but with a reduced charge density at distances less than s_i. Again the positive charge has a spherical distribution and gives no electric intensity at its centre. Thus the average force acting on the test electron is simply $\boldsymbol{E}e$.

The result of applying the field $\boldsymbol{E}$ is that the electrons have displacements that are the same for all, and similarly the ions are displaced all by the same amount, and the ions and electrons move relative to each other. But after these displacements, the spatial distributions of the particles are still random, and so the foregoing argument still applies. Thus, for a plasma, the Lorentz term is absent.

It is useful to see how this kind of argument can give a Lorentz term for other kinds of dielectric. Consider, therefore, a fluid containing molecules which become electrically polarised when the electric field $\boldsymbol{E}$ is applied. For simplicity suppose that the molecules are spherical. Each polarised molecule behaves as though it has an electric dipole of moment $\boldsymbol{M}_E$ at its centre. The molecules are randomly distributed in space, except that no two molecule centres can be closer together than the molecular diameter d_M. It is required to find the average electric field acting on one test molecule, and this is done by imagining a large number N_c of complexions to be superimposed, with the test molecule at the origin. There is then a spatial distribution of electric dipoles, that is a distribution of electric polarisation which, outside the radius d_M, tends to uniformity as $N_c \to \infty$, and has the value $N_c N_M \bar{\boldsymbol{M}}_E$ where $\bar{\boldsymbol{M}}_E$ is the average value of $\boldsymbol{M}_E$. The test molecule is in a spherical cavity of radius d_M within this distribution. Standard electrostatic theory now shows that the electric field within this cavity is uniform and equal to $\frac{1}{3}N_c N_M \bar{\boldsymbol{M}}_E/\varepsilon_0$. Thus its average value is $\frac{1}{3}N_M \bar{\boldsymbol{M}}_E/\varepsilon_0 = \frac{1}{3}\boldsymbol{P}/\varepsilon_0$. This is the Lorentz term.

The difference between the plasma and the molecular dielectric is that for the plasma there are two distributions of charges that are displaced relative to each other and each behaves, on average, as a spherical distribution of charge, with zero electric intensity at the centre. For the dielectric there is only one distribution of polarised molecules which, on average, behaves as a spherical distribution of electric polarisation, with the Lorentz field $\frac{1}{3}\boldsymbol{P}/\varepsilon_0$ at its centre.

Many arguments for and against the inclusion of the Lorentz term have been given. Some are theoretical (Darwin, 1934, 1943) and others are based on observations (Farmer and Ratcliffe, 1936; Ratcliffe, 1939; N. Smith, 1941;

Beynon, 1947). The author is indebted to Professor C.O. Hines for allowing him to see a draft of an unpublished paper in which arguments similar to those given here were very thoroughly reviewed. The strongest observational evidence that the Lorentz term should not be included is probably from the theory of whistlers; see § 13.8 and Eckersley (1935).

3.4. Electron collisions. Damping of the motion

An electron in a stationary volume of plasma is subjected to a force exerted on it by other particles. Its magnitude and direction vary rapidly in time because all particles have thermal motions. For an electron whose average velocity is zero, the time average of this force must be zero because the plasma as a whole is not accelerated. But if the electron is caused to move, by the field of a radio wave, it does on average experience a force caused by encounters with other particles. These encounters are known as collisions.

In the ionosphere the concentration of neutral particles is about $2 \times 10^{22}\,\mathrm{m}^{-3}$ at height 70 km, and $2 \times 10^{18}\,\mathrm{m}^{-3}$ at height 300 km. The concentration of electrons is about 10^{11} to $3 \times 10^{11}\,\mathrm{m}^{-3}$ at the maximum of the F2-layer, that is about 300 km, and is always less than this at other heights. Thus in the ionosphere the concentration of neutrals exceeds the concentration of charged particles by a factor 10^7 or more, so most of the encounters made by an electron are with neutral particles. The force exerted by a neutral particle on an electron is of very short range and therefore the encounter is of short duration, of order $10^{-14}\,\mathrm{s}$ (see problem 3.8). This is a minute fraction of the period of a typical radio wave. Most encounters in the ionosphere can therefore be treated as instantaneous.

For an encounter of an electron with a charged particle, the force is a modified form of the inverse square law of electrostatic force, called the Coulomb force (see (3.18) below). It is of longer range, and these encounters cannot be considered to be instantaneous. An electron experiences electric forces from the other charges at all times. This type of encounter can be dealt with by plasma kinetic theory using a Boltzmann–Vlasov equation of Fokker–Planck type. (See, for example, Shkarofsky, Johnston and Bachynski, 1966). It is found, however, that the final result can still be expressed in the form (3.11) below, provided that a suitable value is used for the effective collision frequency ν. In the magnetosphere the plasma is almost fully ionised so that the concentration of neutral particles is very small, and electron encounters are mainly with charged particles. But the concentrations are small so that the effective collision frequency is small, of order $100\,\mathrm{s}^{-1}$, and for many purposes it can be neglected. In the theory that follows we examine the effect of electron collisions that are assumed to be instantaneous.

Suppose that each electron makes, on average, ν instantaneous collisions per unit time with other particles. If τ is the time between two such successive collisions of one electron then the average value of τ is $1/\nu$. It is shown in books on statistical mechanics that the probability that τ lies in the range τ to $\tau + \mathrm{d}\tau$ is $\nu e^{-\nu\tau}\,\mathrm{d}\tau$. Suppose that each electron is subjected to a steady force f. In time τ an electron then moves a distance $s = \frac{1}{2}(f/m)\tau^2$. This movement is superimposed on the random thermal movements. The average value of s for many electrons is therefore

$$s = \frac{f}{2m}\int_0^\infty \tau^2 \nu e^{-\nu\tau}\mathrm{d}\tau = f/m\nu^2 \tag{3.10}$$

which is the average displacement between two collisions. Since there are, on average, ν collisions per unit time, the average velocity is given by

$$V = f/m\nu. \tag{3.11}$$

This is a steady velocity so that each electron behaves as though f is balanced by a retarding force $-m\nu V$ proportional to its velocity. This concept was first suggested by Lorentz (1909).

In a radio wave f is a sinusoidally varying force. If ν is large compared with the angular wave frequency ω, the above treatment might be expected to be approximately true, but when ν is comparable with, or much less than ω, the analysis is more complicated. It can be shown, however, that the relation can still be expected to hold. This is proved in numerous books and papers that deal with plasma kinetic theory. See, for example, Clemmow and Dougherty (1969, § 9.5), Rawer and Suchy (1967, §§ 2–4), Allis (1956). A version of the theory that uses the notation and methods of this book was given by Budden (1965). A very illuminating discussion of electron collisions was given by Ratcliffe (1959).

Equation (3.11) shows that an electron is subjected to a retarding or damping force, proportional to the velocity $V = \partial \boldsymbol{r}/\partial t$. This is not the total velocity of the electron but only that part of it that is imparted to it by the electric field of a wave. The equation will be assumed to hold for all values of ν. When the damping force is included the equation of motion (3.2) becomes

$$\boldsymbol{E}e = m\partial^2\boldsymbol{r}/\partial t^2 + m\nu\partial\boldsymbol{r}/\partial t. \tag{3.12}$$

Now it is often assumed that the collision frequency ν is the same for all electrons, regardless of their total velocity. If this is done, (3.12) may be multiplied by Ne, and from (2.4), (3.1)

$$\boldsymbol{E}Ne^2 = -\omega^2 m(1 - \mathrm{i}\nu/\omega)\boldsymbol{P}. \tag{3.13}$$

The assumption that ν is independent of electron velocity is not true, and a modification of the theory is needed to deal with this, given in § 3.12. It is shown there, however, that (3.13) is still useful provided that ν is replaced by a suitable 'effective' value, sometimes written ν_{eff}. Here we shall omit the subscript$_{\text{eff}}$ and assume that ν has the appropriate effective value.

Equation (3.13) would be the same as (3.3) if the electron mass m were replaced by the complex value $m(1 - i\nu/\omega)$. Thus the whole of magnetoionic theory, ch. 4, can be treated by deriving the theory of a collisionless plasma and then replacing the electron and ion masses by their appropriate complex values. This idea was first suggested by Smith, R.L. and Brice (1964).

In (3.13) $\boldsymbol{P}$ is the contribution to the electric polarisation from the electrons. If the contribution from the ions is neglected, as in §3.2, (3.13) gives the required constitutive relation when the earth's magnetic field is also neglected. It may be written

$$\boldsymbol{P} = -\varepsilon_0 \frac{X}{1 - iZ} \boldsymbol{E} \tag{3.14}$$

where X is defined by (3.5) and

$$Z = \nu/\omega. \tag{3.15}$$

This Z is a useful measure of the collision frequency. In Appleton's original theory (1932) the symbol z was used, but this is now replaced by Z, to avoid confusion with the coordinate z. In much of this book the notation

$$U = 1 - iZ \tag{3.16}$$

is used. If $\nu = 0$, then $U = 1$.

From (3.14), (3.15) it is clear that the effect of collisions is small when the frequency ω is large. For many purposes it is useful and simpler to ignore collisions altogether. The plasma is then described as a 'collisionless plasma'.

Since $\boldsymbol{D} = \varepsilon_0 \varepsilon \boldsymbol{E}$ it follows from (2.29), (3.14), (3.16) that

$$\varepsilon = 1 - X/U = n^2 \tag{3.17}$$

3.5. The Debye length

In the forms (3.4), (3.14) of the constitutive relation, and in the more complicated forms to be derived later, the plasma is treated as though it is a continuum, and the fields $\boldsymbol{E}$ and $\boldsymbol{P}$ are 'smoothed out' or average values, as discussed in §§2.2–2.4. The discrete nature of the charges in the plasma is ignored. This treatment is used for radio waves throughout this book and the justification for it must be examined. The relations (3.4), (3.14) could not be applied to a small volume of plasma with linear dimensions of order $N^{-\frac{1}{3}}$, which would contain only a few charges. They can only be applied to phenomena whose length scale, usually the wavelength of a radio wave, greatly exceeds some length l_D. An estimate of this length is now required.

In free space the force exerted on one charge by another is given by the inverse square law, that is the Coulomb law. In a plasma this is not true if the two charges are well separated, because the fields are modified by the intervening charges. If one charge is an electron, it repels other electrons and attracts positive charges, so that it is surrounded by a spherical distribution of particles with positive charges predominating. Thus its field is partially screened. Its unscreened field would have the Coulomb potential $\phi = e/(4\pi\varepsilon_0 r)$ at radius r. With screening present the potential is approximately

$$\phi = \frac{e}{4\pi\varepsilon_0 r}\exp(-2^{\frac{1}{2}}r/l_D) \tag{3.18}$$

where

$$l_D = (\varepsilon_0 KT/Ne^2)^{\frac{1}{2}} \tag{3.19}$$

is the Debye length. Here K is Boltzmann's constant and it is assumed that the ions and electrons have the same temperature T. If their temperatures are different, a slightly more complicated expression must be used. The result (3.18) was first derived by Debye and Hückel (1923) for mobile ions in electrolytes. The theory is given in many books on plasma physics; see, for example, Clemmow and Dougherty (1969, pp. 233 ff).

If an electron undergoes a sudden change in its motion, if, for example, it makes a collision, the other charges near it would at once experience a change in the electric force arising from it. For more remote charges this change would be smaller, and for charges at distances exceeding l_D the change would be negligibly small.

In this book the Debye length is important for two reasons. First, it establishes the length scale for the smoothing out of the fields, $\boldsymbol{E}$, $\boldsymbol{P}$, etc. as described in ch. 2. Second, it affects the theory of collisions between charged particles. To calculate the effective collision frequency ν_{eff}, an integral must be evaluated that gives the collision cross section. If the Coulomb law of force is used, this integral diverges and no useful value exists. But with the potential (3.18), the integral exists and a formula can be derived for ν_{eff}. The theory and the formula are given by Ginzburg (1970).

For the ionosphere typical approximate values of l_D are 0.1 m at height 70 km, 3×10^{-3} m at height 100 km, 7×10^{-3} m at height 300 km. For the magnetosphere the value may be 1 or 2 m. The radio propagation problems in this book involve length scales much greater than these values, so that it will only rarely be necessary to make any reference to the Debye length.

3.6. Effect of the earth's magnetic field on the motion of electrons

Let $\boldsymbol{B}$ be the constant magnetic induction of the earth's field. A charge e moving with velocity $\partial\boldsymbol{r}/\partial t$ through it is subjected to a force $e(\partial\boldsymbol{r}/\partial t) \wedge \boldsymbol{B}$. The equation of motion of an electron (3.12) has an additional term thus

$$\boldsymbol{E}e + e(\partial\boldsymbol{r}/\partial t) \wedge \boldsymbol{B} = m\partial^2\boldsymbol{r}/\partial t^2 + m\nu\partial\boldsymbol{r}/\partial t. \tag{3.20}$$

The operator $\partial/\partial t$ is now replaced by iω. If it is assumed that ν is the same for all electrons, (3.20) may be multiplied by $Ne/m\omega^2$ to give

$$\frac{Ne^2}{m\omega^2}\boldsymbol{E} + \frac{\mathrm{i}e}{m\omega}\boldsymbol{P} \wedge \boldsymbol{B} = -\boldsymbol{P}(1 - \mathrm{i}Z) \tag{3.21}$$

where (2.4), (3.1), (3.15) have been used. Let

$$\boldsymbol{Y} = e\boldsymbol{B}/m\omega. \tag{3.22}$$

For electrons e is a negative number, so $\boldsymbol{Y}$ is antiparallel to $\boldsymbol{B}$. Now (3.21) is rearranged and (3.5) is used, to give

$$-\varepsilon_0 X\boldsymbol{E} = U\boldsymbol{P} + \mathrm{i}\boldsymbol{P} \wedge \boldsymbol{Y} \tag{3.23}$$

where U is given by (3.16). If the contributions of the ions to $\boldsymbol{P}$ are negligible, (3.23) is the required constitutive relation. It is used for most of the theory in this book, and in particular it is used in ch. 4 to derive the important formulae of magnetoionic theory. A plasma for which (3.23) can be used is called an 'electrons only' plasma, or simply an 'electron plasma'. Positive ions must be present to ensure that the plasma is electrically neutral, but their movements are so small that they can otherwise be ignored.

The length of the vector $\boldsymbol{Y}$ is denoted by

$$Y = |\boldsymbol{Y}| = \omega_H/\omega, \quad \omega_H = |e\boldsymbol{B}/m| = 2\pi f_H. \tag{3.24}$$

In Appleton's original theory (1932) the symbol y was used instead of Y. An electron moving in the magnetic field $\boldsymbol{B}$ traverses a helical path, and makes one turn of the helix in a time $1/f_H$. This time is independent of the electron's speed provided that this is small enough for relativistic effects to be neglected. The frequency f_{H} is called the 'gyro-frequency' for electrons. Some authors call it the 'cyclotron frequency'. ω_H is the angular gyro-frequency.

The equation (3.23) is a vector equation. When written out in Cartesian components it gives three equations, but these may be written as a single matrix equation. Let l_x, l_y, l_z be the direction cosines of $\boldsymbol{Y}$. Then (3.23) gives

$$-\varepsilon_0 X\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \begin{pmatrix} U & \mathrm{i}Yl_z & -\mathrm{i}Yl_y \\ -\mathrm{i}Yl_z & U & \mathrm{i}Yl_x \\ \mathrm{i}Yl_y & -\mathrm{i}Yl_x & U \end{pmatrix}\begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix}. \tag{3.25}$$

3.7. Effect of the magnetic field of the wave on the motion of electrons

In the preceding sections the force exerted on an electron by the magnetic field $\mathscr{H}$ of the wave has been neglected. To justify this, it is interesting to estimate the order of magnitude of this force by considering the motion of one electron. Suppose a wave of angular frequency ω is travelling in the positive z direction, and is linearly polarised with its electric vector in the x direction. Let x denote the coordinate of one electron, and neglect damping forces. Let the electric field have amplitude E_0. The following argument involves products of harmonically varying quatities, so that complex numbers should be avoided. Hence we write

$$m\frac{\partial^2 x}{\partial t^2} = eE_0 \cos\omega t, \quad \frac{\partial x}{\partial t} = \frac{eE_0}{m\omega}\sin\omega t. \tag{3.26}$$

Since damping is neglected, the refractive index n is real (see ch. 4). Hence the magnetic field is $H_y = (\varepsilon_0/\mu_0)^{\frac{1}{2}} nE_0 \cos \omega t$, from (2.34). Then the instantaneous force exerted on the electron by this field is

$$\mu_0 e H_y \frac{\partial x}{\partial t} = \frac{ne^2 E_0^2}{m\omega c} \sin \omega t \cos \omega t. \tag{3.27}$$

This is in the direction of the z axis, that is, of the wave normal. Notice that it varies with twice the wave frequency, and that its time average value is zero.

If damping forces are included, the magnetic field H_y and the velocity $\partial x/\partial t$ are no longer in quadrature, and the force then has an average value different from zero. There is thus an average force in the direction of the wave normal, which gives rise to radiation pressure. It is usually neglected in the theory of radio waves in the ionosphere, but it is discussed in books on electromagnetic theory (see, for example Clemmow, 1973).

The maximum value of the force (3.27) is $ne^2E_0^2/(2m\omega c) = F_{\mathrm{M}}$. The ratio of this to the maximum electric force F_{E} is

$$F_{\mathrm{M}}/F_{\mathrm{E}} = neE_0/(2m\omega c). \tag{3.28}$$

The value of E_0 at 100 km from an omnidirectional radio transmitter radiating 1 MW is of the order 0.08 V/m. The electric field encountered in the ionosphere from man-made radio transmitters will rarely reach this value, and the frequency is always greater than 10 kHz. The refractive index n is of order unity. Hence the ratio (3.28) will not exceed about 3.7×10^{-4}, and will usually be very much less than this, so that the effect of the magnetic field of the wave can be neglected.

Cases have arisen where very much larger electric fields are encountered in the ionosphere and magnetosphere. For example, when a radio transmitter is carried on a space vehicle, the fields near to it are large. In the early 1970s experiments on heating the ionosphere artificially were made at Platteville, Colorado and shortly afterwards at Arecibo, Puerto Rico. A narrow beam of radiation was sent vertically upwards from a directive aerial on the ground. The powers used were of the order of 2 MW (Platteville) and 160 kW (Arecibo). This arrangement gives wave fields in the ionosphere very much greater than those from a commercial radio transmitter, and the forces arising from the magnetic field of the wave cannot then be neglected. One of the most important results observed in these experiments was the appearance of parametric instabilities which arose because of non-linear effects, including this magnetic force. A special number of *Radio Science* (November 1974) was devoted to this subject, and it includes several excellent summaries. Reviews have been given by Fejer (1975, 1979). The general theory of non-linear effects in plasmas is discussed by Gurevich (1978).

Electromagnetic energy is radiated from a lightning flash, which may give electric fields in the ionosphere much greater than 0.1 V m^{-1}, and magnetic fields which are comparable with the earth's magnetic field. In this case the received signals are known as 'atmospherics', and they are studied with receivers which accept frequencies down to 100 Hz or less. The magnetic field of the wave, and other non-linear effects, may play a part in the mechanism of reflection of these signals from the ionosphere, at points near to the flash.

In commercial radio communication the field intensities rarely reach levels where non-linear effects are important, and in this book they are almost entirely disregarded.

3.8. Electric neutrality of the plasma. Plasma oscillations

It is assumed that, before the plasma is disturbed by a radio wave, it is electrically neutral. The justification for this must be examined. In this section the field variables may have a general time dependence and they are therefore denoted by small letters. If the electrons in a plasma are given small displacements $\boldsymbol{r}$ from their normal positions, and if $\boldsymbol{r}$ depends on the space coordinates of the electrons, there will be an increase of electron concentration at some places and a depletion at others. Thus there is a charge density ρ given approximately by

$$\rho = -\operatorname{div}\boldsymbol{p} = -Ne\operatorname{div}\boldsymbol{r}. \tag{3.29}$$

It is here assumed that $\boldsymbol{p}$, $\boldsymbol{r}$ and ρ are 'smoothed out' as though the plasma is a continuum (see §§ 2.2, 2.3, 3.5). It is also assumed that the resulting change of N is so small that a term $-e\boldsymbol{r}\cdot\operatorname{grad} N$ may be neglected in (3.29).

The first Maxwell equation (2.20) gives

$$\varepsilon_0 \operatorname{div}\boldsymbol{e} = -\operatorname{div}\boldsymbol{p} = \rho. \tag{3.30}$$

Conservation of the charges requires that

$$\operatorname{div}\boldsymbol{j} = -\partial\rho/\partial t = Ne\operatorname{div}(\partial\boldsymbol{r}/\partial t) \tag{3.31}$$

where $\boldsymbol{j}$ is the current density, § 2.3. If the earth's magnetic field is neglected, the equation of motion of any one electron is (3.12). Multiply this equation by Ne/m and take its divergence. With (3.29)–(3.31) and (3.6) this gives

$$\frac{\partial^2\rho}{\partial t^2} + \nu\frac{\partial\rho}{\partial t} + \omega_N^2\rho = 0. \tag{3.32}$$

This applies at every point in the plasma. If ν is small an approximate solution is

$$\rho = \rho_0\cos(\omega_N t + \alpha)\exp(-\tfrac{1}{2}\nu t) \tag{3.33}$$

where ρ_0 and α are independent of the time t but may depend on the space coordinates. The charge density oscillates with the plasma frequency $\omega_N/2\pi$. This is known as a 'plasma oscillation'. It decays with a time constant $2/\nu$. In the ionosphere this is about 1 ms or less. Thus an appreciable charge density cannot be sustained for a time longer than this.

The charge density ρ could be expressed as a three-dimensional Fourier integral representation in space. For any one Fourier component ρ_F we can use the complex representation and include the time dependence (3.33) thus

$$\rho_F \propto \exp\{i(\omega_N t - \boldsymbol{K}\cdot\boldsymbol{x})\}\exp(-\tfrac{1}{2}\nu t) \tag{3.34}$$

where $\boldsymbol{x}$ denotes the space coordinates x, y, z. This is a wave whose wave vector is $\boldsymbol{K}$. Its components may have any real values so any wavelength is possible and any wave normal direction is possible but the angular frequency is always ω_N. The charge density (3.34) is in strata parallel to the wave fronts, and gives an electric field parallel to the wave normal. Maxwell's equations (2.22) show that there is no magnetic field in this wave. In a plasma where the electron temperature is allowed for, a wave of this kind is known as an electron plasma wave or Langmuir wave. Then the frequency is not restricted to ω_N but can take values in a small range near ω_N.

If the earth's magnetic field $\boldsymbol{B}$ is included, it is convenient to use Cartesian coordinates x, y, z with the z axis parallel to $\boldsymbol{B}$. The electron's equation of motion is (3.20) and has the additional term $e(\partial\boldsymbol{r}/\partial t)\wedge\boldsymbol{B}$. If the electron velocities $\partial\boldsymbol{r}/\partial t$ are all parallel to $\boldsymbol{B}$, this term is zero and the equation for ρ is still (3.32). But in general the constituent waves (3.34) now have electric fields which give to $\partial\boldsymbol{r}/\partial t$ components in the x and y directions, so that it does not remain parallel to $\boldsymbol{B}$. In one special case this does not happen. It is when $\boldsymbol{r}$ and $\partial\boldsymbol{r}/\partial t$ are independent of x and y, so that ρ is in strata perpendicular to $\boldsymbol{B}$. The electron motions are then parallel to $\boldsymbol{B}$ and are unaffected by it. The direction of $\boldsymbol{K}$ is now restricted to be parallel to $\boldsymbol{B}$ but the wavelength may have any value. The time dependence is given by (3.33).

In the general case when the term $e(\partial\boldsymbol{r}/\partial t)\wedge\boldsymbol{B}$ is not zero, the Fourier analysis in space may still be done but the frequency depends on $\boldsymbol{K}$ and is found from the dispersion relation. This idea is the basis of some studies of the oscillations of a magnetoplasma. See, for example, Dougherty and Monaghan (1966). A more detailed theory of plasma oscillations is given by Clemmow and Dougherty (1969).

3.9. The susceptibility matrix

In the matrix form (3.25) of the constitutive relations the components of $\boldsymbol{E}$ are expressed in terms of the components of $\boldsymbol{P}$. In later chapters it will be necessary to have the components of $\boldsymbol{P}$ expressed in terms of those of $\boldsymbol{E}$. Equation (3.25) is equivalent to three simultaneous equations for P_x, P_y, P_z. These can be solved, and the process is equivalent to inversion of the 3×3 matrix. The result is

$$\frac{1}{\varepsilon_0}\begin{pmatrix}P_x\\P_y\\P_z\end{pmatrix} = -\frac{X}{U(U^2-Y^2)} \times\begin{pmatrix}U^2-l_x^2Y^2 & -il_zYU-l_xl_yY^2 & il_yYU-l_xl_zY^2\\ il_zYU-l_xl_yY^2 & U^2-l_y^2Y^2 & -il_xYU-l_yl_zY^2\\ -il_yYU-l_xl_zY^2 & il_xYU-l_yl_zY^2 & U^2-l_z^2Y^2\end{pmatrix}\begin{pmatrix}E_x\\E_y\\E_z\end{pmatrix}. \tag{3.35}$$

The expression multiplying (E_x, E_y, E_z) is called the susceptibility matrix of the ionosphere, and is denoted by $\boldsymbol{M}$. It was given in this form by Banerjea (1947). Its components are denoted by $M_{ij}(i,j = x, y, z)$.

It can be shown that the elements of $\boldsymbol{M}$ have the following properties:

$$M_{xx}(M_{yz} + M_{zy}) = M_{xy}M_{zx} + M_{yx}M_{xz}, \tag{3.36}$$

$$M_{xx}M_{yy} - M_{yz}M_{zy} = \frac{X^2}{U^2 - Y^2} \tag{3.37}$$

and four other relations obtained by permuting the suffixes in (3.36) and (3.37).

$$\begin{aligned} M_{xx}M_{yy}M_{zz} + M_{xy}M_{yz}M_{zx} + M_{yx}M_{zy}M_{xz} - M_{xx}M_{yz}M_{zy} \\ - M_{yy}M_{xz}M_{zx} - M_{zz}M_{yx}M_{xy} = -\frac{X^3}{U(U^2 - Y^2)}. \end{aligned} \tag{3.38}$$

These relations are needed later.

If the medium is loss-free, $Z = 0$ and $U = 1$. Then $M_{ij} = M_{ji}^*$ (where a star denotes a complex conjugate) so that the matrix $\boldsymbol{M}$ is Hermitian. The electric relative permittivity is $\boldsymbol{\varepsilon} = \mathbf{1} + \boldsymbol{M}$ where $\mathbf{1}$ is the unit matrix, so $\boldsymbol{\varepsilon}$ is also Hermitian. It was shown in § 2.11 that the time average rate at which energy goes into the plasma from an electromagnetic field is then zero.

3.10. Complex principal axes

To derive the refractive indices of a homogeneous electron plasma, the constitutive relation can be used in the form (3.25), and this is done in §§ 4.3, 4.6. It is not necessary to invert the matrix and get explicit expressions for the components of $\boldsymbol{P}$. But for a plasma in which more than one species of particle contributes to $\boldsymbol{P}$, the constitutive relation cannot be written in the form (3.25). It is necessary first to derive an expression of the form (3.25) for each species. The matrices are then inverted to give the contribution to $\boldsymbol{P}$ from each species. The total $\boldsymbol{P}$ is the sum of these contributions, expressed as a 3×3 matrix multiplying $\boldsymbol{E}$, as in (3.35). This sum is the constitutive relation. Inversion of the matrices in the form (3.35) is algebraically very complicated. Fortunately there is a simpler method which proves to be very powerful and useful, both in magnetoionic theory, ch. 4, and in the more general theory of radio wave propagation.

The values of X, Y, U are different for each species of charged particle and later we shall attach subscripts to indicate the values for different species. For the present, however, (3.25) applies to any one species and the subscripts are omitted.

The relation (3.25) can be simplified by a special choice of the real Cartesian axis system x, y, z. We therefore choose the z axis to be parallel to $\boldsymbol{Y}$, so that $l_x = l_y = 0$, $l_z = 1$, and (3.25) becomes:

$$-\varepsilon_0 X \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \begin{pmatrix} U & \mathrm{i}Y & 0 \\ -\mathrm{i}Y & U & 0 \\ 0 & 0 & U \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix}. \tag{3.39}$$

For many crystalline dielectrics the matrix that expresses $\boldsymbol{E}$ in terms of $\boldsymbol{P}$ is symmetric. It is then possible to choose real orthogonal Cartesian axes so that the matrix is diagonal. These axes are called principal axes. The inverted matrix is then found simply by taking the reciprocals of the elements of the original diagonal matrix. The matrix in (3.39) is not symmetric and so in general a plasma has no real principal axes. It is still possible, however, to transform the axis system so that the matrix is diagonal. The resulting new axis system is complex.

In the old axis system of (3.39) the new complex axes are eigen column vectors of the matrix. Standard matrix theory (see, for example, Graham, 1979) shows that they are

$$\boldsymbol{u}_1 = 2^{-\frac{1}{2}}\begin{pmatrix}1\\-\mathrm{i}\\0\end{pmatrix},\quad \boldsymbol{u}_2 = 2^{-\frac{1}{2}}\begin{pmatrix}1\\\mathrm{i}\\0\end{pmatrix},\quad \boldsymbol{u}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix} \tag{3.40}$$

with eigen values $U+Y$, $U-Y$, U respectively. Each vector in (3.40) might be multiplied by any constant. The constants have been chosen so that (3.40) satisfies the normalisation condition used in (3.44) below.

Note that $\boldsymbol{u}_1 \cdot \boldsymbol{u}_2$ is not zero so that the new axes are not all mutually orthogonal. In a system of oblique axes there are two ways of specifying the components of a vector, namely the contravariant and covariant forms (see, for example, Mathews and Walker 1970, § 15-3). We here use the contravariant form.

The contravariant components E_1, E_2, E_3 of the vector $\boldsymbol{E}$ in the new axis system are a linear combination of E_x, E_y, E_z and are found by multiplication by a transforming matrix $\mathscr{U}$ thus

$$\begin{pmatrix}E_1\\E_2\\E_3\end{pmatrix} = \mathscr{U}\begin{pmatrix}E_x\\E_y\\E_z\end{pmatrix} \tag{3.41}$$

and the same applies for any other vector, including $\boldsymbol{P}$. Now (3.39) may be multiplied on the left by $\mathscr{U}$ to give

$$-\varepsilon_0 X \mathscr{U}\begin{pmatrix}E_x\\E_y\\E_z\end{pmatrix} = \mathscr{U}\begin{pmatrix}U & \mathrm{i}Y & 0\\ -\mathrm{i}Y & U & 0\\ 0 & 0 & U\end{pmatrix}\mathscr{U}^{-1}\mathscr{U}\begin{pmatrix}P_x\\P_y\\P_z\end{pmatrix}. \tag{3.42}$$

For the matrix $\mathscr{U}(\ldots)\mathscr{U}^{-1}$ to be diagonal, the three columns of $\mathscr{U}^{-1}$ must be those in (3.40). Hence

$$\mathscr{U}^{-1} = 2^{-\frac{1}{2}}\begin{pmatrix}1 & 1 & 0\\ -\mathrm{i} & \mathrm{i} & 0\\ 0 & 0 & 2^{\frac{1}{2}}\end{pmatrix},\quad \mathscr{U} = 2^{-\frac{1}{2}}\begin{pmatrix}1 & \mathrm{i} & 0\\ 1 & -\mathrm{i} & 0\\ 0 & 0 & 2^{\frac{1}{2}}\end{pmatrix}. \tag{3.43}$$

The inverse of $\mathscr{U}$ is the transpose of its complex conjugate, and $\mathscr{U}$ is therefore called a unitary matrix. This condition may be written in subscript notation with the summation convention, thus

$$\mathscr{U}_{ij}\mathscr{U}^*_{kj} = \delta_{ik} \tag{3.44}$$

where δ_{ik} is the Kronecker delta.

Now (3.41) with (3.43) shows that

$$E_1 = 2^{-\frac{1}{2}}(E_x + \mathrm{i}E_y), \quad E_2 = 2^{-\frac{1}{2}}(E_x - \mathrm{i}E_y), \quad E_3 = E_z$$
$$E_x = 2^{-\frac{1}{2}}(E_1 + E_2), \quad E_y = -\mathrm{i}2^{-\frac{1}{2}}(E_1 - E_2), \tag{3.45}$$

and there are similar expressions for the components of any other field vector.

The result (3.42) may now be written

$$-\varepsilon_0 X\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = \begin{pmatrix} U+Y & 0 & 0 \\ 0 & U-Y & 0 \\ 0 & 0 & U \end{pmatrix}\begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}. \tag{3.46}$$

The matrix can be inverted at once by taking the reciprocals of its elements. Hence

$$P_1 = -\frac{\varepsilon_0 X}{U+Y}E_1, \quad P_2 = -\frac{\varepsilon_0 X}{U-Y}E_2, \quad P_3 = -\frac{\varepsilon_0 X}{U}E_3. \tag{3.47}$$

For an 'electrons only' plasma, this is the constitutive relation in complex principal axes. The factors

$$\frac{-X}{U+Y}, \quad \frac{-X}{U-Y}, \quad \frac{-X}{U} \tag{3.48}$$

are the elements of the diagonalised susceptibility tensor. We may also use $\boldsymbol{D} = \varepsilon_0\boldsymbol{E} + \boldsymbol{P} = \varepsilon_0\boldsymbol{\varepsilon}\boldsymbol{E}$ whence

$$D_1 = \varepsilon_0\varepsilon_1 E_1, \quad D_2 = \varepsilon_0\varepsilon_2 E_2, \quad D_3 = \varepsilon_0\varepsilon_3 E_3 \tag{3.49}$$

where

$$\varepsilon_1 = 1 - X/(U+Y), \quad \varepsilon_2 = 1 - X/(U-Y), \quad \varepsilon_3 = 1 - X/U. \tag{3.50}$$

These are the principal axis values of the elements of the diagonalised electric permittivity tensor $\boldsymbol{\varepsilon}$ for an 'electrons only' plasma. They were first used for the ionospheric plasma by Westfold (1949). See also Lange-Hesse (1952), Davids (1953). Later in this book the following combinations of ε_1, ε_2, ε_3 appear often and it is useful to introduce the notation

$$G = \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2) - \varepsilon_3, \quad J = \tfrac{1}{2}\varepsilon_3(\varepsilon_1 + \varepsilon_2) - \varepsilon_1\varepsilon_2. \tag{3.51}$$

For an 'electrons only' plasma in which ν is assumed to be independent of velocity, it can be shown from (3.50) that

$$G = -J = -XY^2/U(U^2 - Y^2). \tag{3.52}$$

But this is not true for a plasma with more than one contributing species of ion, nor when ν is velocity dependent; see § 3.12.

When $\varepsilon_1, \varepsilon_2, \varepsilon_3$ have been found, the equations may be transformed back to give the permittivity tensor $\boldsymbol{\varepsilon}$ in any other axis system. For example in the real axis system used for (3.39), it becomes

$$\boldsymbol{\varepsilon} = \mathscr{U}^{-1}\begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}\mathscr{U} = \begin{pmatrix} \frac{1}{2}(\varepsilon_1 + \varepsilon_2) & \frac{1}{2}\mathrm{i}(\varepsilon_1 - \varepsilon_2) & 0 \\ -\frac{1}{2}\mathrm{i}(\varepsilon_1 - \varepsilon_2) & \frac{1}{2}(\varepsilon_1 + \varepsilon_2) & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}. \tag{3.53}$$

In magnetoionic theory, ch. 4, we use a real axis system in which the vector $\boldsymbol{Y}$ is in the x-z plane at an angle Θ to the z axis, where $\tan\Theta$ is positive. In this system it can be shown that

$$\boldsymbol{\varepsilon}=\begin{pmatrix} \frac{1}{2}(\varepsilon_1+\varepsilon_2)\cos^2\Theta+\varepsilon_3\sin^2\Theta & \frac{1}{2}\mathrm{i}(\varepsilon_1-\varepsilon_2)\cos\Theta & G\sin\Theta\cos\Theta \\ -\frac{1}{2}\mathrm{i}(\varepsilon_1-\varepsilon_2)\cos\Theta & \frac{1}{2}(\varepsilon_1+\varepsilon_2) & -\frac{1}{2}\mathrm{i}(\varepsilon_1-\varepsilon_2)\sin\Theta \\ G\sin\Theta\cos\Theta & \frac{1}{2}\mathrm{i}(\varepsilon_1-\varepsilon_2)\sin\Theta & \frac{1}{2}(\varepsilon_1+\varepsilon_2)\sin^2\Theta+\varepsilon_3\cos^2\Theta \end{pmatrix}. \tag{3.54}$$

When studying waves in a horizontally stratified ionosphere, the main subject of this book, we use a coordinate system in which the vector $\boldsymbol{Y}$ has direction cosines l_x, l_y, l_z. In this system it can be shown that

$$\boldsymbol{\varepsilon}=\begin{pmatrix} \frac{1}{2}(\varepsilon_1+\varepsilon_2)-l_x^2G & -l_xl_yG+\frac{1}{2}\mathrm{i}l_z(\varepsilon_1-\varepsilon_2) & -l_xl_zG-\frac{1}{2}\mathrm{i}l_y(\varepsilon_1-\varepsilon_2) \\ -l_xl_yG-\frac{1}{2}\mathrm{i}l_z(\varepsilon_1-\varepsilon_2) & \frac{1}{2}(\varepsilon_1+\varepsilon_2)-l_y^2G & -l_yl_zG+\frac{1}{2}\mathrm{i}l_x(\varepsilon_1-\varepsilon_2) \\ -l_xl_zG+\frac{1}{2}\mathrm{i}l_y(\varepsilon_1-\varepsilon_2) & -l_yl_zG-\frac{1}{2}l_x(\varepsilon_1-\varepsilon_2) & \frac{1}{2}(\varepsilon_1+\varepsilon_2)-l_z^2G \end{pmatrix}. \tag{3.55}$$

Suppose now that the superimposed magnetic field is everywhere reversed in direction, so that $\boldsymbol{Y}$ is reversed and l_x, l_y, l_z all change sign. Then (3.55) shows that $\boldsymbol{\varepsilon}$ is replaced by its transpose.

3.11. Properties of principal axis elements of the permittivity. Effect of ions

All the dispersive and refracting properties of a cold magnetoplasma can be expressed in terms of the three elements ε_1, ε_2, ε_3 of the diagonalised permittivity tensor. It is therefore useful to set out their main properties. This is done below in a numbered sequence (1) to (16). Items (1) to (6) apply to an 'electrons only' plasma. The remaining items allow for ions of non-infinite mass.

In an 'electrons only' plasma, positive ions must be present to ensure that the plasma is electrically neutral, § 3.8, but it is implied that they are infinitely massive. Then (3.5) shows that X is zero for the ions. Consequently, for the ions, the terms (3.48) are zero and the ions make no contribution to $\boldsymbol{P}$. Then, if the electron collision frequency ν is assumed to be independent of velocity, $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are given by (3.50), and this is their meaning in nearly the whole of this book. For many purposes it is useful to neglect collisions and this is done in the rest of this section. The discussion of collisions is resumed in § 3.12.

(1) From (3.50):

$$\varepsilon_1 = 1 - X/(1+Y) = 1 - f_N^2/\{f(f+f_H)\} \tag{3.56}$$

$$\varepsilon_2 = 1 - X/(1-Y) = 1 - f_N^2/\{f(f-f_H)\} \tag{3.57}$$

$$\varepsilon_3 = 1 - X = 1 - f_N^2/f^2. \tag{3.58}$$

With collisions neglected, all three εs are real when X is real. But note that, in some later sections, complex values of X are used. Curves showing how these elements depend on frequency are given in fig. 3.1. The behaviour is slightly different for $f_N < f_H$ and $f_N > f_H$, and examples of both are given.

(2) The zeros of the εs are given by

$$\varepsilon_1{:}X = 1 + Y \quad f = \pm(\tfrac{1}{4}f_H^2 + f_N^2)^{\frac{1}{2}} - \tfrac{1}{2}f_H \tag{3.59}$$

$$\varepsilon_2{:}X = 1 - Y \quad f = \pm(\tfrac{1}{4}f_H^2 + f_N^2)^{\frac{1}{2}} + \tfrac{1}{2}f_H \tag{3.60}$$

$$\varepsilon_3{:}X = 1 \qquad f = f_N. \tag{3.61}$$

The frequencies (3.59) (3.60) are called 'cut-off' frequencies, and (3.61) is called the 'window' frequency (see §§ 17.6–17.9), although some authors call this also a cut-off frequency. In each case there is only one real positive value of f that gives a zero. The three zeros are marked in fig. 3.1.

(3) ε_2 is infinite when $f = f_H$, the electron gyro-frequency. The other two are never infinite when $f > 0$.

(4) All three εs tend to infinity when $f \to 0$. For very small f, ε_1 and ε_3 are negative and ε_2 is positive.

(5) ε_1 and ε_2 are never equal (see item (15) below).

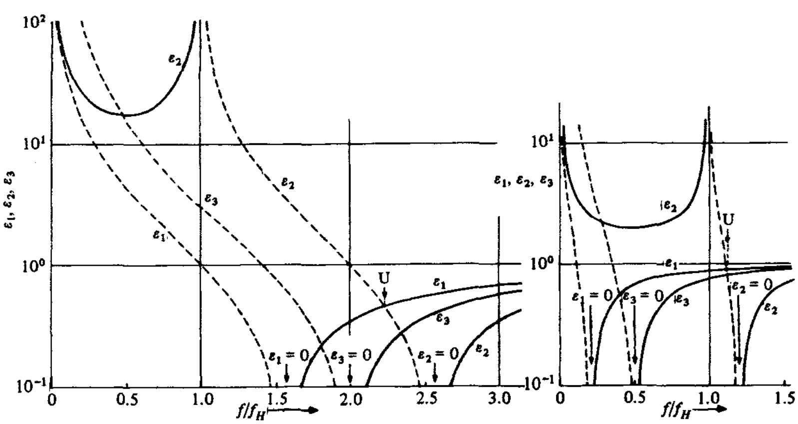
(6) $\varepsilon_1 + \varepsilon_2$ is zero when

$$X = 1 - Y^2; \quad f = \pm f_{\mathrm{U}}; \quad f_{\mathrm{U}} = (f_N^2 + f_H^2)^{\frac{1}{2}} \text{ (positive value)} \tag{3.62}$$

f_{U} is called the 'upper hybrid resonance frequency', and is marked U in fig. 3.1. Its significance is explained in § 5.7 item (7).

When ions of non-infinite mass are allowed for, the contributions to $\boldsymbol{P}$ in (3.47) from electrons and from each species of ion must be added. It is now assumed that all ions have a single positive charge. A subscript is added to X, Y etc. to distinguish the

Fig. 3.1. Frequency dependence of ε_1, ε_2, ε_3 for a cold collisionless electron plasma. In the left-hand figure $f_N/f_H = 2$, and in the right-hand figure $f_N/f_H = 0.5$. The ordinate is the modulus on a logarithmic scale. Continuous curves are used for positive values and broken curves for negative values of the εs. U denotes the upper hybrid resonance. Abscissa scale is linear.



values for electrons and ions. Thus we use the integer $i = 1, 2, 3 \ldots\ldots$ to designate the ion species in order of increasing mass.

The frequencies

$$f_{Ni} = \frac{1}{2\pi}(N_i e^2/\varepsilon_0 m_i)^{\frac{1}{2}}, \quad f_{Hi} = \frac{1}{2\pi}|eB/m_i| \tag{3.63}$$

are called the ion plasma frequency and the ion gyro-frequency respectively. For electrons the subscript i is replaced by e. In most of this section $\nu_i = Z_i = 0$; $U_i = 1$ for all i including $i = \mathrm{e}$.

(7) Let $C_i = N_i/N_\mathrm{e}$. Since the plasma is electrically neutral it is necessary that

$$N_\mathrm{e} = \sum_i N_i, \quad \sum_i C_i = 1. \tag{3.64}$$

In cases where $\sum_i C_i < 1$ it is implied that positive ions of infinite mass are present with concentration $N_\mathrm{e} - \sum_i N_i$.

(8) The expressions for the εs in (1) are replaced by

$$\varepsilon_1 = 1 - X_\mathrm{e}/(U_\mathrm{e} + Y_\mathrm{e}) - \sum_i X_i(U_i - Y_i) \tag{3.65}$$

$$\varepsilon_2 = 1 - X_\mathrm{e}/(U_\mathrm{e} - Y_\mathrm{e}) - \sum_i X_i(U_i + Y_i) \tag{3.66}$$

$$\varepsilon_3 = 1 - X_\mathrm{e}/U_\mathrm{e} - \sum_i X_i/U_i. \tag{3.67}$$

The U_is and U_e are retained here for future reference. As in (1) all three εs are real when collisions are neglected. Curves giving examples of how they depend on frequency are shown in fig. 3.2 for a plasma with two species of positive ion.

(9) Each of (3.65)–(3.67) contains a 'sum over species'. Some authors use a different notation for these. A fairly widely used system is that of Stix (1962), namely

$$\varepsilon_1 = L, \quad \varepsilon_2 = R, \quad \varepsilon_3 = P \tag{3.68}$$

(10)
$$\varepsilon_1(f) = \varepsilon_2(-f) \tag{3.69}$$

This is useful, for example, when calculating ε_1 and ε_2 with a programmable pocket calculator, when collisions are neglected. The same program will do both. It is true also for an 'electrons only' plasma.

(11) (3.5) and (3.24) show that

$$X_i = N_i e^2/(\varepsilon_0 m_i \omega^2) = C_i X_\mathrm{e}(m_\mathrm{e}/m_i) \tag{3.70}$$

$$Y_i = eB/\omega m_i \qquad = Y_\mathrm{e}(m_\mathrm{e}/m_i). \tag{3.71}$$

Now m_e/m_i is very small, about 5.4×10^{-4} if i refers to protons and smaller still for heavier ions. Hence ε_3 in (3.67) is only very slightly different from (3.58) at all frequencies. For (3.65), (3.66), if f is greater than four or five times the greatest ion gyro-frequency f_{H1}, the Σ terms are small and ε_1, ε_2 are very close to their values (3.56) (3.57) for electrons only.

(12) ε_3 has one zero where $f=f_0$ and this is slightly greater than, but very close to f_{Ne}. In practical cases the difference is negligible. f_0 is called the 'window' frequency; Budden and Stott (1980). See §§ 17.6–17.9. ε_1 and ε_2 have zeros very close to (3.59), (3.60) respectively. It can be shown that ε_2 has no other zeros, but the total number of zeros of ε_1, for positive f, is equal to the total number of positive ion species, including ions of infinite mass.

(13) ε_2 is infinite when $f=f_{He}$, the electron gyro-frequency, as in (3). ε_1 is infinite when $f=f_{Hi}$, that is at each of the ion gyro-frequencies.

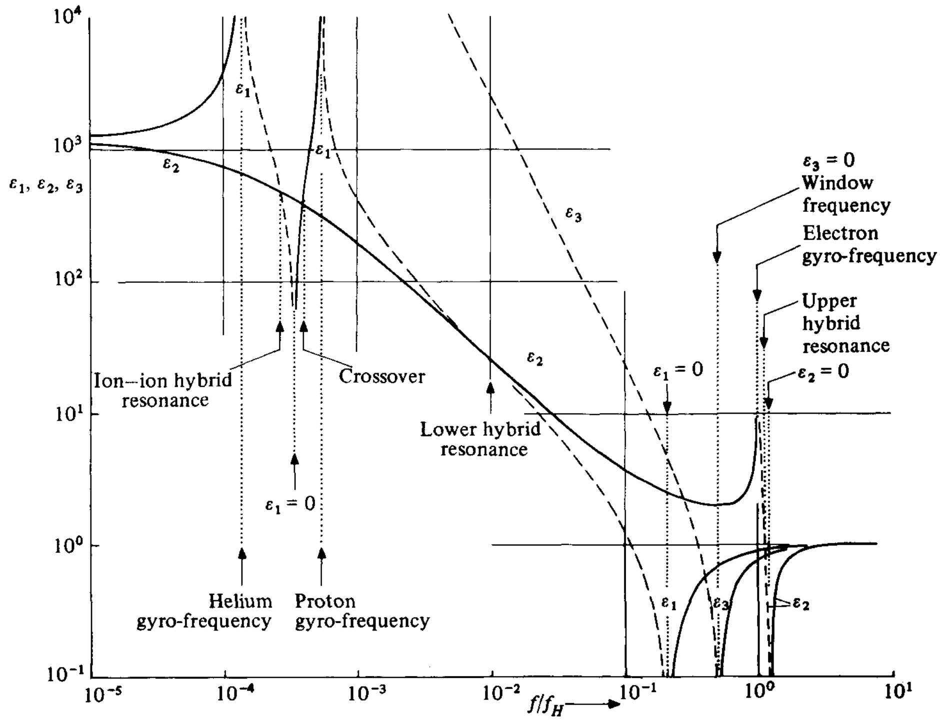
(14) When $f\to 0$, $\varepsilon_3\to\infty$ as in (4). If f is very small and there are no ions of infinite mass, it can be shown, by using (3.64), that

$$\left.\begin{matrix}\varepsilon_1\\ \varepsilon_2\end{matrix}\right\} = 1+\rho/\varepsilon_0 B^2 \pm 2\pi N_e Mf/(\varepsilon_0 e B^3) + O(f^2) \tag{3.72}$$

where

$$\rho = N_e m_e + \sum_i N_i m_i, \quad M = \sum_i C_i m_i^2 - m_e^2. \tag{3.73}$$

Fig. 3.2. Similar to fig. 3.1 but for a plasma in which the positive ions are protons and singly charged helium ions in equal concentrations. $f_{Ne}/f_H = 0.5$. Both scales are logarithmic.



and the unit charge e is here a positive number. Thus as $f \to 0$, ε_1 and ε_2 approach the same non-infinite value. See fig. 3.2.

(15) Frequencies $f \neq 0$ where $\varepsilon_1 = \varepsilon_2$ are called 'crossover' frequencies (Smith and Brice, 1964). It can be shown from (3.65) (3.66) that there are none if there is only one positive ion species. A crossover frequency occurs between each consecutive pair of ion gyro-frequencies. Fig. 3.2 shows an example. When $\varepsilon_1 = \varepsilon_2$, (3.53) shows that $\boldsymbol{\varepsilon}$ is diagonal for the real axes used there, and two of its three elements are equal. A dielectric with this property is said to be 'uniaxial'. For some examples of its properties see §§ 5.5, 19.3, problem 4.7. Such a medium is about the simplest possible kind of anisotropic medium and is most useful for illustrating the theory of wave phenomena. See, for example, Clemmow (1966, ch. 8), Felsen (1964).

(16) Frequencies for which $\varepsilon_1 + \varepsilon_2 = 0$ are called hybrid frequencies. The greatest is the upper hybrid frequency $f_{\rm U}$ and is still given approximately by (3.62). This result is very little affected by the positive ions. The next greatest is denoted by $f_{\rm L}$ and is called the lower hybrid frequency. It exceeds the greatest ion gyro-frequency f_{H1}, and it is given approximately by

$$f_{\rm L} = \frac{f_{H\rm e} f_{N\rm e} m_{\rm e}^{\frac{1}{2}}}{f_{\rm U}} \left(\sum_i \frac{C_i}{m_i} \right)^{\frac{1}{2}}. \tag{3.74}$$

There is also a hybrid frequency, called an ion–ion hybrid frequency, between each pair of ion gyro-frequencies. There is an example in fig. 3.2. The total number of hybrid frequencies is equal to the number of ion species of non-infinite mass, including electrons.

For a more detailed description of the effect of positive ions on the properties of a plasma see Al'pert (1983), Al'pert, Budden *et al.* (1983), Budden and Stott (1980), Booker (1984).

3.12. Collisions. The Sen–Wyller formulae

In this section we again consider an 'electrons only' plasma. If the collision frequency ν is independent of electron velocity, the εs are given by (3.50). But in practice ν is a function of the velocity v and allowance must be made for this. In most of this book ν means the effective collision frequency used in (3.15), but in this section it denotes the velocity dependent collision frequency $\nu(v)$. The effective collision frequency is here temporarily written $\nu_{\rm eff}$. The contribution to $\boldsymbol{P}$ from one species of charged particle is given by (3.47), and the total $\boldsymbol{P}$ is found by adding the contributions for all the species. Consider now, those electrons whose thermal velocity v is in the range v to $v + \delta v$, and let their concentration be δN. It might at first be thought that these various groups of δN electrons can be treated as though they are different species, and that their contributions to $\boldsymbol{P}$ in (3.47) should be added. This, however, is oversimplified. A full kinetic theory shows that the groups δN cannot be treated in this way, as though they are independent of each other.

Collisions are of greatest importance in the lower part of the ionosphere, below about 120 km, where the velocity distribution function of the electrons may with fair accuracy be taken as Maxwellian and isotropic in velocity space. Thus

$$\delta N = 4\pi^{-\frac{1}{2}} N(\tfrac{1}{2}m/KT)^{\frac{3}{2}} v^2 \exp(-\tfrac{1}{2}mv^2/KT)\delta v \tag{3.75}$$

where T is the electron temperature. Instead of v it is convenient to use the variable

$$u = \tfrac{1}{2}mv^2/KT. \tag{3.76}$$

When $\nu(v)$ is expressed as a function of u, let it be denoted by $\check{\nu}(u)$. For the distribution (3.75) the full theory shows that

$$P_1 = -\tfrac{4}{3}\pi^{-\frac{1}{2}}\varepsilon_0 X E_1 \int_0^\infty \frac{u^{\frac{3}{2}} e^{-u}\, du}{U(u) + Y} \tag{3.77}$$

with similar expressions for P_2 and P_3. If the contributions (3.47) were simply added, the factor $u^{\frac{3}{2}}$ in (3.77) would be replaced by $\frac{3}{2}u^{\frac{1}{2}}$. The full theory leading to (3.77) is given by Sen and Wyller (1960), Sen (1967). See also Phelps (1960). A version of it that uses the notation and methods of this book was given by Budden (1965).

Now (3.47) is used to give the three elements of the permittivity tensor. In the expression relating P_2 to E_2, the Y in (3.77) is replaced by $-Y$. For P_3 and E_3 the Y is omitted. Then:

$$\varepsilon_1 = 1 - \tfrac{4}{3}\pi^{-\frac{1}{2}} X \int_0^\infty \frac{u^{\frac{3}{2}} e^{-u} du}{U(u) + Y}, \quad \varepsilon_2 = 1 - \tfrac{4}{3}\pi^{-\frac{1}{2}} X \int_0^\infty \frac{u^{\frac{3}{2}} e^{-u} du}{U(u) - Y},$$

$$\varepsilon_3 = 1 - \tfrac{4}{3}\pi^{-\frac{1}{2}} X \int_0^\infty \frac{u^{\frac{3}{2}} e^{-u} du}{U(u)}. \tag{3.78}$$

If ν is independent of v, so that U is a constant, the integrals in (3.78) can be evaluated at once, and the result (3.50) is obtained.

The collision frequency ν cannot be independent of v, because, if it were, the phenomenon of wave interaction, also called the Luxembourg effect, could not occur; see §§ 13.11–13.13. It has often been assumed, especially in studies of the theory of wave interaction, that the electrons have a constant mean free path λ_e. This would make ν proportional to v thus

$$\nu = v/\lambda_e, \quad \langle \nu \rangle = \left(\frac{8KT}{\pi m}\right)^{\frac{1}{2}} \frac{1}{\lambda_e} \tag{3.79}$$

where the Maxwellian distribution (3.75) has been used to find the average value $\langle \nu \rangle$. Some results for this case have been given by Margenau (1946), Ginzburg (1970). But laboratory experiments (Crompton, Huxley and Sutton, 1953; Phelps and Pack, 1959; Huxley and Crompton, 1974) have shown that for electrons of small energy, down to about 0.1 eV, the collision frequency is more nearly proportional to v^2. It will therefore now be assumed that

$$\nu(v) = \beta v^2, \quad \nu_{av} = 3\beta KT/m, \quad \check{\nu}(u) = \tfrac{2}{3}\nu_{av} u \tag{3.80}$$

where (3.75) was used to find ν_{av}. Then the three integrals in (3.78) can all be written in the form

$$\mathscr{I} = w\int_0^\infty \frac{u^{\frac{3}{2}}\mathrm{e}^{-u}\mathrm{d}u}{s-\mathrm{i}u} \tag{3.81}$$

where

$$w = \tfrac{3}{2}\omega/\nu_{\mathrm{av}} \tag{3.82}$$

and s is $w(1+Y)$, $w(1-Y)$, w for ε_1, ε_2, ε_3 respectively.

The integrals can be separated into their real and imaginary parts, and all six resulting integrals can be expressed in terms of the standard functions

$$\mathscr{C}_p(s) = \frac{1}{p!}\int_0^\infty \frac{u^p\mathrm{e}^{-u}\mathrm{d}u}{u^2+s^2} \tag{3.83}$$

thus

$$\left.\begin{aligned}
\varepsilon_1 &= 1 - X[w^2(1+Y)\mathscr{C}_{\frac{3}{2}}\{w(1+Y)\} + \mathrm{i}w\tfrac{5}{2}\mathscr{C}_{\frac{5}{2}}\{w(1+Y)\}]\\
\varepsilon_2 &= 1 - X[w^2(1-Y)\mathscr{C}_{\frac{3}{2}}\{w(1-Y)\} + \mathrm{i}w\tfrac{5}{2}\mathscr{C}_{\frac{5}{2}}\{w(1-Y)\}]\\
\varepsilon_3 &= 1 - X[w^2\mathscr{C}_{\frac{3}{2}}(w) + \mathrm{i}w\tfrac{5}{2}\mathscr{C}_{\frac{5}{2}}(w)].
\end{aligned}\right\} \tag{3.84}$$

The functions (3.83) appear in various problems of kinetic theory. They have been tabulated by Dingle, Arndt and Roy (1956 b). A way of calculating them that is very useful in computing was given by Hara (1963).

Sen and Wyller (1960), Sen (1967) and others have used (3.84) in magnetoionic theory to calculate the refractive indices and polarisations for radio waves in a cold magnetoplasma. The resulting formulae are known as the Sen–Wyller formulae. Some authors call this theory 'generalised magnetoionic theory' but this is incorrect. The Appleton–Lassen version of the theory is a special case where ν is assumed to be independent of v. The Sen–Wyller version is another special case where ν is assumed to be proportional to v^2.

Instead of using the average value ν_{av} (3.80), some authors use a value $\nu_{\mathrm{m}} = \frac{2}{3}\nu_{\mathrm{av}}$ which they call the 'monoenergetic collision frequency'. This term is misleading and should be abandoned. For a physical variable, the natural and widely understood parameter to use in describing it is the average value.

If ν_{av}/ω is small enough, so that s is large, the factor $(1-\mathrm{i}u/s)^{-1}$ in (3.81) may be expanded by the binomial theorem, and the series may then be integrated term by term. This gives an asymptotic (divergent) expansion for the function $\mathscr{I}$; see § 8.11. Thus

$$\mathscr{I} = \tfrac{3}{4}\pi^{\frac{1}{2}}\frac{w}{s}\left[1+\frac{5\mathrm{i}}{2s}-\frac{35}{4s^2}\cdots\right] = \tfrac{3}{4}\pi^{\frac{1}{2}}\frac{w}{s}\left[1-\frac{5\mathrm{i}}{2s}+\frac{10}{4s^2}\right]^{-1}. \tag{3.85}$$

If s is so large that only the first two terms need to be used, (3.78), with the last expression in (3.85), gives

$$\varepsilon_1 = 1 - X/(1+Y-\mathrm{i}Z_{\mathrm{eff}}),\ \varepsilon_2 = 1 - X/(1-Y-\mathrm{i}Z_{\mathrm{eff}}),\ \varepsilon_3 = 1 - X/(1-\mathrm{i}Z_{\mathrm{eff}}) \tag{3.86}$$

where

$$Z_{\text{eff}} = 5/2w, \quad \nu_{\text{eff}} = \omega Z_{\text{eff}} = \tfrac{5}{3}\nu_{\text{av}}. \tag{3.87}$$

Thus the standard result (3.50) may still be used provided that the correct 'effective' collision frequency ν_{eff} (3.87) is used. This conclusion is satisfactory for nearly all radio propagation problems in the ionosphere and magnetosphere.

Results of calculations that compared the Appleton–Lassen formulae and the Sen–Wyller formulae were presented by several authors. Thus Budden (1965) included a study of very low frequencies where $Z_{\text{eff}} = \nu_{\text{eff}}/\omega$ is large so that failure of (3.86) might be expected for ε_3. Smith (1975) studied the case $f \approx f_H$ so that $Y \approx 1$ and failure of (3.86) must occur for ε_2. Johler and Harper (1962), Deeks (1966b) and others used both formulae in studies of the ionospheric reflection of radio waves of very low frequency. Numerous other cases were given by Sen and Wyller (1960), Sen (1967). The results show that in nearly all cases there is very little difference. The electron collision frequency is not known with sufficient accuracy to be able to distinguish between the two in any actual radio measurements. The author does not know of any radio observations that cannot be explained by the Appleton–Lassen formulae.

The electron collision frequency can be shown to be given by $\nu = N_{\text{m}} A v$ where N_{m} is the concentration of the particles collided with, and A is the 'transport area of cross section'. If ν is proportional to v^2, this shows that A must be very small for slowly moving electrons, and that their mean free path $1/N_{\text{m}}A$ must be very large. But this seems most unlikely on general physical grounds. For an electron temperature of 800 K, typical for a height of about 150 km, three quarters of the electrons have energies less than about 0.1 eV. The assumption (3.80) is based on laboratory measurements, in most of which electron energies of 0.1 to 0.2 eV or more were used. Measurements for lower energies are more difficult but some have been reported. There may be some uncertainty about how well a confined laboratory plasma can simulate the ionospheric plasma. There seems to be no strong evidence that the result $\nu \propto v^2$ can be used for the electrons of lowest energy in the ionosphere. For these reasons it seems preferable at present to retain the simpler formulae (3.47), as is done in this book.

It is possible that, for the small electron thermal velocities of interest in the ionosphere, the dependence of ν on v does not follow any power law. But for the reasons given above it is the author's opinion that the assumption $\nu \propto v$ is likely to be better than $\nu \propto v^2$. It may therefore be useful to give the relevant formulae. Results equivalent to (3.91) below were derived by Margenau (1946). See also Allis (1956, §42). Instead of (3.80) we now use (3.79) and the three integrals in (3.78) can then all be written in the form

$$\mathscr{J} = W^{\frac{1}{2}} \int_0^\infty \frac{u^{\frac{3}{2}} \mathrm{e}^{-u} \, \mathrm{d}u}{t^{\frac{1}{2}} - \mathrm{i}u^{\frac{1}{2}}} \tag{3.88}$$

where

$$W = \frac{4}{\pi}\left(\frac{\omega}{\langle\nu\rangle}\right)^2 = \tfrac{1}{2}\omega^2\lambda_e^2 m/KT \tag{3.89}$$

and t is $(1+Y)^2W, (1-Y)^2W, W$ for $\varepsilon_1, \varepsilon_2, \varepsilon_3$ respectively. The integrals (3.88) can be separated into their real and imaginary parts, and all six resulting integrals can be expressed in terms of the standard functions

$$\mathscr{A}_p(t) = \frac{1}{p!}\int_0^\infty \frac{u^p \mathrm{e}^{-u}\mathrm{d}u}{t+u}. \tag{3.90}$$

These, like the functions (3.83), have been tabulated (Dingle, Arndt and Roy, 1956a). When used in (3.78) they give finally

$$\begin{aligned}
\varepsilon_1 &= 1 - X\left[W(1+Y)\mathscr{A}_{\frac{3}{2}}\{(1+Y)^2W\} + \mathrm{i}W^{\frac{1}{2}}\frac{8}{3\sqrt{\pi}}\mathscr{A}_2\{(1+Y)^2W\}\right] \\
\varepsilon_2 &= 1 - X\left[W(1-Y)\mathscr{A}_{\frac{3}{2}}\{(1-Y)^2W\} + \mathrm{i}W^{\frac{1}{2}}\frac{8}{3\sqrt{\pi}}\mathscr{A}_2\{(1-Y)^2W\}\right] \\
\varepsilon_3 &= 1 - X\left[W\mathscr{A}_{\frac{3}{2}}(W) + \mathrm{i}W^{\frac{1}{2}}\frac{8}{3\sqrt{\pi}}\mathscr{A}_2(W)\right].
\end{aligned} \tag{3.91}$$

If the average collision frequency $\langle\nu\rangle$ is small so that W and t are large, the factor $\{1 - \mathrm{i}(u/t)^{\frac{1}{2}}\}^{-1}$ in (3.88) may be expanded by the binomial theorem, and the series integrated term by term. This gives the asymptotic series

$$\mathscr{I} = \frac{3\sqrt{\pi}}{4}\left(\frac{W}{t}\right)^{\frac{1}{2}}\left\{1 - \frac{8}{3\sqrt{\pi}}\frac{\mathrm{i}}{t^{\frac{1}{2}}} + \frac{0.237}{t} + \dots\right\}^{-1}. \tag{3.92}$$

If t is so large that only two terms need to be used, the result is the same as (3.86) where now

$$Z_{\text{eff}} = \tfrac{4}{3}\langle\nu\rangle/\omega, \quad \nu_{\text{eff}} = \tfrac{4}{3}\langle\nu\rangle = \frac{4}{3\lambda_e}\left(\frac{8KT}{\pi m}\right)^{\frac{1}{2}}, \tag{3.93}$$

compare (3.87).

3.13. Electron–electron collisions. Electron–ion collisions

Two other aspects of collisions in a plasma have received some study and must be briefly mentioned. The first is collisions between like particles, particularly between electrons. In the small volume of plasma considered in this chapter (see § 3.5) the electrons all have, on average, the same ordered velocity. Thus for any two electrons the momentum associated with the ordered motion is the same. If these two electrons collide, the total ordered momentum that they share cannot change. This suggests that electron–electron collisions should not, on average, affect the ordered part of the motion, so that the constitutive relation is also unaffected by them. There is, however, a way in which these collisions can have an effect. In a wave the ordered velocities vary in space over a distance of about a wavelength. The part of the ordered velocity perpendicular to the wave normal thus has a spatial gradient of the same kind as is studied in the theory of viscosity in a gas. Transfer of ordered electron momentum between regions of different transverse velocity gives rise to a viscous force. This can lead to an additional damping effect which has been called 'electron viscosity'. It depends on the spatial variation of the wave fields and is therefore an example of 'spatial dispersion' mentioned in § 3.1. It is unlikely to be important for radio waves, since their wavelength is long, but can be important for other kinds of wave in a plasma. For a discussion of electron–electron collisions see Al'pert (1980b, 1983), Akhiezer, Lapshin and Stepanov (1976), Ginzburg (1970).

The other effect to be mentioned concerns collisions between unlike particles. In § 3.4 it was implied that the particles that an electron collides with have an average velocity of zero. This is not strictly true. If these particles are ions, they have an ordered velocity component imparted to them by the wave. In (3.12) the collision damping force is $mv\partial \boldsymbol{r}_e/\partial t$ where $\partial \boldsymbol{r}_e/\partial t$ is the ordered velocity of the electrons, as now indicated by the subscript e. Consider the contribution to this term from the various ion species $i = 1, 2, \ldots$, and let ν_{ei} be the appropriate effective collision frequencies. Then $\partial \boldsymbol{r}_e/\partial t$ must be replaced by the relative velocity of the electrons and ions, so that the correct damping force is

$$m \sum_i \nu_{ei} \partial(\boldsymbol{r}_e - \boldsymbol{r}_i)/\partial t. \tag{3.94}$$

For many frequencies, including those of the radio waves studied in this book, the ion velocities $\partial \boldsymbol{r}_i/\partial t$ are small enough to be neglected, but this is not true for frequencies near to or less than the ion gyro-frequency. This effect has been studied by Al'pert (1980b, 1983) who gave examples showing that it can be important. Ions and electrons can, through collisions, impart to the neutral particles an ordered velocity $\partial \boldsymbol{r}_n/\partial t$ and the summation in (3.94) should include a term to allow for this. A full theory that does this was given by Al'pert, Budden, Moiseyev and Stott (1983, appendix A).

PROBLEMS 3

3.1. A radio transmitter on the ground, with a power of 100 kW, emits continuous waves of frequency 2 MHz uniformly in all directions above the earth's surface. Its radiated field acts on an electron in the lower ionosphere at 100 km from the transmitter. Find the ordered velocity imparted to the electron if the electric field is the same as it would be in free space. Compare this with the average thermal velocity of the electron if the temperature is 500 K.

3.2. The energy supplied to unit volume of a plasma by an electric field $\boldsymbol{e}$, that is applied starting at time $t = 0$, is

$$\mathscr{S}_E = \int_0^t \boldsymbol{e} \frac{\partial \boldsymbol{d}}{\partial t} \mathrm{d}t.$$

Apply this to a cold electron plasma in which there is no superimposed magnetic field, and collisions are negligible. Since the energy cannot be dissipated as heat it must be stored in the plasma. Show that, when $\boldsymbol{e}$ is the field $E_0 \sin \omega t$ of a linearly polarised radio wave, this stored energy is

$$\tfrac{1}{2}E_0^2\left(\varepsilon_0 \sin^2 \omega t + \frac{Ne^2}{m\omega^2}\cos^2 \omega t\right).$$

Show that the first term is the same as the energy that would be stored in the electric field if it were in a vacuum, and the second term is the kinetic energy of the electrons. Note that the energy $\frac{1}{2}\mu_0 H^2$ supplied by the magnetic field H of the wave is not included.

(Two methods of solving this problem were given by Budden, 1961a, pp. 33–5.)

3.3. A cold plasma contains free electrons. There is no superimposed magnetic field and collisions are negligible. A plane linearly polarised radio wave is present whose electric field at a given point is $E_0 \sin \omega t$. The wave is progressive if $\omega > \omega_N$, or evanescent if $\omega < \omega_N$. Prove that the total stored energy per unit volume, including contributions from both electric and magnetic fields, is

$$\tfrac{1}{2}\varepsilon_0 E_0^2\{1 + (X-1)\cos 2\omega t\} \quad \text{when} \quad \omega > \omega_N,$$

$$\tfrac{1}{2}\varepsilon_0 E_0^2\{X + (X-1)\cos 2\omega t\} \quad \text{when} \quad \omega < \omega_N,$$

where $X = \omega_N^2/\omega^2$.

3.4. A uniform cold electron plasma occupies the space between two parallel planes and there is a vacuum outside. There is no superimposed magnetic field and collisions are negligible. The electrons are all displaced by the same small distance perpendicular to the planes, and are then released from rest. Show that they oscillate with angular frequency ω_N. (Assume that *all* the electrons, including those near the boundary, experience the same electric field as an electron near the centre.)

Repeat this problem for a plasma enclosed in a sphere and show that the oscillation frequency is now $\omega_N/\sqrt{3}$. Try it also for a plasma in a cylinder if the initial electron displacements are perpendicular to the axis. (Oscillation frequency $\omega_N/\sqrt{2}$.)

(Note. These oscillations are sometimes called 'plasma oscillations' but they are very different from those described in § 3.8. For an application to radio propagation, see: Herlofson, 1951; Pitteway, 1958, 1960.)

3.5. In a plasma formed of ionised helium gas, half the ions are singly charged and the other half are doubly charged. Show that this plasma has one crossover frequency, $f = f_{H1}\sqrt{2}$ approximately, where f_{H1} is the gyro-frequency of the singly charged ions.

3.6. A system of principal axis coordinates x_1, x_2, x_3 is given in terms of the Cartesian coordinates x, y, z by $x_1 = 2^{-\frac{1}{2}}(x + \mathrm{i}y)$, $x_2 = 2^{-\frac{1}{2}}(x - \mathrm{i}y)$, $x_3 = z$. Similar relations, (3.41), hold for the contravariant principal axis components of any vector.

V is a scalar function of position. Prove the following results:

(a) $\boldsymbol{a}\cdot\boldsymbol{b} = a_1 b_2 + a_2 b_1 + a_3 b_3$

(b) $\operatorname{div} \boldsymbol{a} = \partial a_1/\partial x_1 + \partial a_2/\partial x_2 + \partial a_3/\partial x_3$

The contravariant principal axis components of the following vectors are:

(c) $\boldsymbol{a} \wedge \boldsymbol{b}$: $\mathrm{i}\{a_3 b_1 - a_1 b_3, a_2 b_3 - a_3 b_2, a_1 b_2 - a_2 b_1\}$

(d) $\boldsymbol{a} \wedge \boldsymbol{a}^*$: $\mathrm{i}\{a_3 a_2^* - a_1 a_3^*, a_2 a_3^* - a_3 a_1^*, a_1 a_1^* - a_2 a_2^*\}$

(e) $\operatorname{curl} \boldsymbol{a}$: $\mathrm{i}\{\partial a_1/\partial x_3 - \partial a_3/\partial x_2, \partial a_3/\partial x_1 - \partial a_2/\partial x_3, \partial a_2/\partial x_2 - \partial a_1/\partial x_1\}$

(f) grad V: $\partial V/\partial x_2, \partial V/\partial x_1, \partial V/\partial x_3$

(Notes. (i) Some of these results are useful in one method of deriving the dispersion relation for a cold magnetoplasma. It is the method that uses Maxwell's equations (2.20)–(2.23) expressed in the principal axis system. See problem 4.12. (ii) The covariant principal axis components of grad V are $\partial V/\partial x_1$, $\partial V/\partial x_2$, $\partial V/\partial x_3$.)

3.7. An electron of mass m and speed v collides with a molecule of mass $M \gg m$ initially at rest. The angle between the electron's paths before and after the collision is χ. Show that the fraction of kinetic energy lost by the electron is $2m/M(1 - \cos\chi) + \mathrm{O}(m/M)^2$. In many such collisions suppose that the final paths can have any possible direction with equal probability. Show that the average fractional loss of kinetic energy is $G \approx 2m/M$.

(Experimental measurements show that in practice G can be considerably larger than this. A full treatment of this problem must make allowance for the thermal velocities of the molecules. See §§ 13.12, 13.13 and the references given there.)

3.8. Make a rough estimate of the duration τ of the encounter of an electron with (a) a massive ion of charge e, (b) a massive neutral molecule such as N_2 or O_2. (Hint: the electron's motion will be appreciably affected by the ion if it is within a range equal to the Debye length l_D. The duration τ will be very roughly the time it takes for an electron with the average thermal velocity to travel a distance l_D. For a diatomic molecule let d be the separation of the atoms. The range of the force that the molecule exerts on an electron may be of the order $2d$. The duration τ will be very roughly the time it takes the electron to travel a distance $4d$. The values of d for N_2 and O_2 are 0.11, 0.12 nm respectively. The mean molecular diameters are about three times this).

3.9. An electron approaches a very massive neutral spherical molecule along a line which, when produced, passes at shortest distance b from the molecule's centre, and b is called the 'impact parameter'. After the encounter, the electron has been deflected through an angle χ, and $A = 2\pi\int_0^\infty (1 - \cos\chi) b\,\mathrm{d}b$ is called the 'transport area of cross section' for the encounter. As a simple model, the molecule can be thought of as an infinitely massive, perfectly conducting small sphere, at rest.

(a) Show that an electron at distance r experiences an attractive force proportional to r^{-5}.
(b) If the electron approaches with velocity v, show that A is proportional to $1/v$. (Hint: the differential equation of the orbit is

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}+u=\frac{Ku^3}{b^2v^2}$$

where $u=1/r$, K is a constant and θ is the polar coordinate angle. Show that χ is given by

$$\pi+\chi=2\int_0^{x_0}\{1-x^2+Kx^4/(2v^2b^4)\}^{-\frac{1}{2}}\,\mathrm{d}x$$

where x_0 is the greatest value of b/r. Hence show that $\cos\chi$ depends on b and v only through the combination b^2v. Use this in the integral for A. It is not necessary to evaluate the integrals).
(c) Hence show that the effective collision frequency is independent of v.