11

Reflection and transmission coefficients

11.1. Introduction

The later chapters are largely concerned with the derivation and the properties of the reflection coefficient of a stratified ionosphere, and to a lesser extent with the transmission coefficient. The purpose of this chapter therefore, is to set out their definitions and to discuss some of their properties in special cases of importance in radio propagation.

The coordinate system is as defined in § 6.1. The z axis is perpendicular to the strata of the ionosphere, so that all properties of the plasma, especially the electron concentration $N(z)$ and collision frequency $\nu(z)$, are functions only of z. The origin is often taken to be at the ground so that z is height above the earth's surface. Sometimes, however, it is convenient to use a different origin.

Up to § 11.11 it is assumed that the incident wave is plane and travelling upwards, in general obliquely with its wave normal in the x–z plane at an angle θ to the z axis, measured clockwise. Then the x–z plane is called the 'plane of incidence'. If F is any field component of the incident wave then in the free space below the ionosphere

$$F = F_1 \exp\{-\mathrm{i}k(Sx + Cz)\} \tag{11.1}$$

where $S = \sin\theta$, $C = \cos\theta$ and F_1 is a constant, in general complex.

11.2. The reference level for reflection coefficients

Consider, first, a wave vertically incident on the ionosphere, so that $\theta = 0$, and let some field component be given, in the free space below the ionosphere, by

$$F = F_1 \exp(-\mathrm{i}kz) \tag{11.2}$$

where z is height above the ground, and F_1 is a complex constant. This gives rise to a reflected wave travelling downwards for which

$$F = F_2 \exp(\mathrm{i}kz) \tag{11.3}$$

where F_2 is another complex constant. If the ratio of these two field components is measured at the ground the result is

$$R_0 = F_2/F_1, \tag{11.4}$$

and the complex number R_0 is called the reflection coefficient 'at the ground' or 'with the ground as reference level', or 'referred to the level $z = 0$'. The particular field components F are specified later, § 11.4.

If the ratio were measured at some level $z = z_1$, the result would be

$$R_1 = \{F_2 \exp(ikz_1)\}/\{F_1 \exp(-ikz_1)\} = R_0 \exp(2ikz_1) \tag{11.5}$$

where R_1 is the reflection coefficient referred to the level $z = z_1$. Thus the effect of changing the reference level is to alter the argument but not the modulus of the reflection coefficient. Similarly, the reflection coefficient referred to some other level z_2 is

$$R_2 = R_0 \exp(2ikz_2) = R_1 \exp\{2ik(z_2 - z_1)\}. \tag{11.6}$$

This gives the rule for changing the reference level.

The downgoing wave (11.3) may arise by reflection in an ionosphere in which N varies gradually with height and at a level, $z = z_3$ say, where N is appreciably different from zero, the upgoing and downgoing waves are no longer given by (11.2) and (11.3). Yet it is still possible to use the level $z = z_3$ as the reference level, and to write

$$R_3 = \{F_2 \exp(ikz_3)\}/\{F_1 \exp(-ikz_3)\} = R_0 \exp(2ikz_3). \tag{11.7}$$

Here R_3 is the reflection coefficient referred to the level $z = z_3$ and is calculated *as though this level were in free space*, even though the ratio in (11.7) could never be observed.

For many purposes the ground is the best reference level since this is where the reflection coefficients are measured. But in some cases other levels are more convenient. For example, in §§ 11.7–11.11 where the ionosphere is assumed to be a sharply bounded homogeneous medium, the reference level is taken at the boundary.

Suppose next that the incident wave is oblique so that F is given by (11.1). Then the reflected wave is

$$F = F_2 \exp\{-ik(Sx - Cz)\}. \tag{11.8}$$

Both incident and reflected waves depend on x through the same factor $\exp(-ikSx)$, see § 6.2. The reflection coefficient is defined to be the ratio of these fields *measured at the same point*, whose coordinates are (x, z_1), say, so that

$$R_1 = \frac{F_2}{F_1} \exp(2ikCz_1). \tag{11.9}$$

Thus R_1 is independent of the horizontal coordinate x, but it is still necessary to specify the reference level for the vertical coordinate. If this is changed to a new level z_2 then the new reflection coefficient R_2 is

$$R_2 = R_1 \exp\{2ikC(z_2 - z_1)\}. \tag{11.10}$$

If C is real, the effect of changing the reference level is to alter the argument but not the modulus of the reflection coefficient. Problems sometimes arise in which C is complex, for example when the space between the earth and the ionosphere is treated as a wave guide (Budden 1961b). The incident and reflected waves are then inhomogeneous plane waves (§2.15) and changing the reference level can change both the modulus and argument of the reflection coefficient.

In observations at oblique incidence the incident and reflected waves are usually measured at different points on the earth's surface. Suppose that these are separated by a horizontal distance D. Then the ratio of the fields in the reflected and incident waves at the ground is

$$R_D = \{F_2 \exp(-ikSD)\}/F_1 = R_0 \exp(-ikSD). \tag{11.11}$$

11.3. The reference level for transmission coefficients

Suppose that there is free space above the ionosphere and that the wave (11.1) incident from below gives rise to a wave above the ionosphere whose field component F is

$$F = F_3 \exp\{-ik(Sx + Cz)\}. \tag{11.12}$$

These two waves could never be observed at the same point because the ionosphere intervenes between them. The field (11.12), however, can be extrapolated backwards to some level $z = z_1$ below the top of the ionosphere, as though the wave were travelling in free space. Similarly, the field (11.1) can be extrapolated upwards to the same level $z = z_1$. The transmission coefficient T is defined to be the ratio of the fields (11.12) to (11.1) extrapolated in this way to the same point. Thus

$$T = F_3/F_1. \tag{11.13}$$

This ratio is independent of x and z, so that in this example it is unnecessary to specify a reference level for the transmission coefficient.

In some problems however there is, above the ionosphere, a homogeneous medium which is not free space. Then the transmitted field is no longer given by (11.12) but by

$$F = F_3 \exp\{-ik(Sx + qz)\} \tag{11.14}$$

where q is a constant different from C. This field is now extrapolated down to the level $z = z_1$, as though the wave were travelling in the same homogeneous medium throughout, and the transmission coefficient is

$$T = \{F_3 \exp(-ikqz_1)\}/\{F_1 \exp(-ikCz_1)\} = \frac{F_3}{F_1}\exp\{ik(C - q)z_1\}. \tag{11.15}$$

Since this depends on z_1, the reference level must now be specified. Often q is complex even when C is real, so that a change of reference level changes both the modulus and the argument of T.

11.4. The four reflection coefficients and the four transmission coefficients

The most important reflection and transmission coefficients are those deduced when the incident wave is linearly polarised with its electric vector either parallel to the plane of incidence or perpendicular to it. The reflected wave, in general, is elliptically polarised, but may be resolved into linearly polarised components whose electric vectors also are parallel and perpendicular to the plane of incidence. It is convenient to introduce four coefficients $R_{11}, R_{12}, R_{21}, R_{22}$, each equal to the complex ratio of a specified electric field component in the wave after reflection to a specified electric field component in the wave before reflection. The second subscript denotes whether the electric field component specified in the incident wave is parallel, 1, or perpendicular, 2, to the plane of incidence, and the first subscript refers in the same way to the electric field component in the reflected wave.

It is necessary to adopt some convention with regard to sign of the electric fields of the linearly polarised components. For the component with $\boldsymbol{E}$ perpendicular to the plane of incidence the electric field is E_y and its sign is the sign of E_y. For the component with $\boldsymbol{E}$ in the plane of incidence the sign of E is taken to be the sign of $\mathscr{H}_y$.

This sign convention means, for example, that if $R_{11} = 1$ at normal incidence, the electric fields E_x in the incident and reflected waves have opposite directions because the magnetic fields $\mathscr{H}_y$ have the same direction. On the other hand if $R_{22} = 1$ at normal incidence, the electric fields E_y in the incident and reflected waves have the same direction but the magnetic fields $\mathscr{H}_x$ have opposite directions.

These definitions are based on the way in which most radio propagation engineers visualise the incident and reflected waves. They were first formulated in a similar form for the study of the ionospheric reflection of radio waves of very low frequency (Bracewell *et al.*, 1951). When the waves are in the free space below the ionosphere the component of the electric field of one wave in the plane of incidence, that is the x–z plane, is equal to $\mathscr{H}_y$. This is shown from equations (4.2), (4.4), (4.6) by transforming them to the axis system of the present chapter, with $n = 1$. It is therefore useful to use $\mathscr{H}_y$ instead of the electric field component in the plane of incidence. This leads to the following expressions for the reflection coefficients. The superscripts (I), (R), (T) are used to denote the incident, reflected and transmitted waves respectively. Let $E_y^{(\mathrm{I})} = \mathscr{H}_x^{(\mathrm{I})} = 0$. Then

$$R_{11} = \mathscr{H}_y^{(\mathrm{R})}/\mathscr{H}_y^{(\mathrm{I})}, \quad R_{21} = E_y^{(\mathrm{R})}/\mathscr{H}_y^{(\mathrm{I})}. \tag{11.16}$$

Let $E_x^{(\mathrm{I})} = \mathscr{H}_y^{(\mathrm{I})} = 0$. Then

$$R_{12} = \mathscr{H}_y^{(\mathrm{R})}/E_y^{(\mathrm{I})}, \quad R_{22} = E_y^{(\mathrm{R})}/E_y^{(\mathrm{I})}. \tag{11.17}$$

If the incident wave is in a plasma that is not free space, the relation between $\boldsymbol{E}$ and $\mathscr{H}$ is different. For an isotropic plasma, if n is the refractive index, $\mathscr{H}_y$ is n times the component of $\boldsymbol{E}$ in the x–z plane. For an anisotropic plasma the waves are in general combinations of ordinary and extraordinary components and the relation between $\mathscr{H}_y$ and $\boldsymbol{E}$ is complicated and depends on θ. It is convenient and simplest now to retain the definitions (11.16), (11.17), but it must be remembered that they do not now in general use the component of $\boldsymbol{E}$ in the plane of incidence.

In an exactly similar way the four transmission coefficients $T_{11}, T_{12}, T_{21}, T_{22}$ are introduced to indicate the ratios of the field components $\mathscr{H}_y^{(\mathrm{T})}, E_y^{(\mathrm{T})}$ in the transmitted wave to $\mathscr{H}_y^{(\mathrm{I})}, E_y^{(\mathrm{I})}$ in the incident wave. Thus:
Let $E_y^{(\mathrm{I})} = 0$. Then

$$T_{11} = \mathscr{H}_y^{(\mathrm{T})}/\mathscr{H}_y^{(\mathrm{I})}, \quad T_{21} = E_y^{(\mathrm{T})}/\mathscr{H}_y^{(\mathrm{I})}. \tag{11.18}$$

Let $\mathscr{H}_y^{(\mathrm{I})} = 0$. Then

$$T_{12} = \mathscr{H}_y^{(\mathrm{T})}/E_y^{(\mathrm{I})}, \quad T_{22} = E_y^{(\mathrm{T})}/E_y^{(\mathrm{I})}. \tag{11.19}$$

In these definitions, (11.16)–(11.19), the first subscript always refers to the emerging reflected or transmitted wave, and the second subscript refers to the incident wave.

In some older papers on the reflection of radio waves of very low frequency, the subscripts 1, 2, were replaced by $\|$, $\perp$, and the second subscript was placed in front of the symbol. Thus R_{11} was ${}_{\|}R_{\|}$, R_{12} was ${}_{\perp}R_{\|}$ and so on. This notation does not accord with the standard conventions of matrix notation, and has now been almost entirely abandoned.

11.5. Reflection and transmission coefficient matrices

The expressions (11.16)–(11.19) can now be written in matrix notation. Let the field components of the incident wave be written as a column matrix of two elements, and similarly for the reflected and transmitted waves, thus

$$\boldsymbol{e}^{(\mathrm{I})} = \begin{pmatrix} \mathscr{H}_y^{(\mathrm{I})} \\ E_y^{(\mathrm{I})} \end{pmatrix}, \quad \boldsymbol{e}^{(\mathrm{R})} = \begin{pmatrix} \mathscr{H}_y^{(\mathrm{R})} \\ E_y^{(\mathrm{R})} \end{pmatrix}, \quad \boldsymbol{e}^{(\mathrm{T})} = \begin{pmatrix} \mathscr{H}_y^{(\mathrm{T})} \\ E_y^{(\mathrm{T})} \end{pmatrix}. \tag{11.20}$$

Let $\boldsymbol{R}$ and $\boldsymbol{T}$ be the 2×2 square matrices

$$\boldsymbol{R} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \tag{11.21}$$

$$\boldsymbol{T} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}. \tag{11.22}$$

Then (11.16)–(11.19) can be written

$$\boldsymbol{e}^{(\mathrm{R})} = \boldsymbol{R}\boldsymbol{e}^{(\mathrm{I})} \tag{11.23}$$

$$\boldsymbol{e}^{(\mathrm{T})} = \boldsymbol{T}\boldsymbol{e}^{(\mathrm{I})} \tag{11.24}$$

and $\boldsymbol{R}$, $\boldsymbol{T}$ are called the reflection and transmission coefficient matrices respectively.

If a wave is produced by two successive reflection processes whose reflection coefficient matrices are $\boldsymbol{R}_a$, $\boldsymbol{R}_b$, then it follows from (11.23) that the fields in the second reflected wave are $\boldsymbol{R}_b\boldsymbol{R}_a\boldsymbol{e}^{(I)}$. The resultant reflection coefficient matrix is the matrix product of the matrices for the separate reflections. Its value depends on the order in which the reflections occur. This result can clearly be extended to any number of reflections.

11.6. Alternative forms of the reflection coefficient matrix

Instead of resolving the incident and reflected waves into linearly polarised components, they might be resolved into components in a different way to give a different representation of the reflection coefficient matrix. To illustrate this we shall suppose that the waves are resolved into right-handed and left-handed circularly polarised components.

For the incident wave let $E_{\parallel}$ be the component of $\boldsymbol{E}$ in the plane of incidence. Then (4.4) with $n=1$ shows that $E_{\parallel} = \mathscr{H}_y$. If the wave has right-handed circular polarisation, an observer looking in the direction of the wave normal sees the electric vector rotating clockwise. This requires that

$$E_{yr} = -\mathrm{i}E_{\parallel r} = -\mathrm{i}\mathscr{H}_{yr} \tag{11.25}$$

where the second subscript r indicates right-handed circular polarisation. Similarly for left-handed circular polarisation

$$E_{yl} = \mathrm{i}\mathscr{H}_{yl}. \tag{11.26}$$

The total fields E_y, $\mathscr{H}_y$ are the sum of (11.25), (11.26). We now use a superscript (I) to indicate the incident wave. Then

$$\mathscr{H}_y^{(I)} = \mathrm{i}E_{yr}^{(I)} - \mathrm{i}E_{yl}^{(I)}, \quad E_y^{(I)} = E_{yr}^{(I)} + E_{yl}^{(I)}. \tag{11.27}$$

Define the two column matrices

$$\boldsymbol{e}_0^{(I)} = \begin{pmatrix} E_{yr}^{(I)} \\ E_{yl}^{(I)} \end{pmatrix}, \quad \boldsymbol{e}_0^{(R)} = \begin{pmatrix} E_{yr}^{(R)} \\ E_{yl}^{(R)} \end{pmatrix}. \tag{11.28}$$

Then (11.27) can be written in matrix form, with (11.20),

$$\boldsymbol{e}^{(I)} = 2^{\frac{1}{2}}\boldsymbol{U}\boldsymbol{e}_0^{(I)} \tag{11.29}$$

where $\boldsymbol{U}$ is the unitary matrix

$$\boldsymbol{U} = 2^{-\frac{1}{2}}\begin{pmatrix} \mathrm{i} & -\mathrm{i} \\ 1 & 1 \end{pmatrix}. \tag{11.30}$$

Similarly for the reflected wave

$$\boldsymbol{e}^{(R)} = 2^{\frac{1}{2}}\boldsymbol{U}\boldsymbol{e}_0^{(R)}. \tag{11.31}$$

Substitution of (11.29) and (11.31) into (11.23) now gives

$$\boldsymbol{U}\boldsymbol{e}_0^{(R)} = \boldsymbol{R}\boldsymbol{U}\boldsymbol{e}_0^{(I)} \tag{11.32}$$

whence

$$e_0^{(\mathrm{R})} = \boldsymbol{R}_0 e_0^{(\mathrm{I})}, \quad \boldsymbol{R}_0 = \boldsymbol{U}^{-1}\boldsymbol{R}\boldsymbol{U} = \begin{pmatrix} R_{\mathrm{rr}} & R_{\mathrm{rl}} \\ R_{\mathrm{lr}} & R_{\mathrm{ll}} \end{pmatrix}. \tag{11.33}$$

The reflection coefficient matrix $\boldsymbol{R}_0$ for circularly polarised components has thus been obtained from $\boldsymbol{R}$ by the unitary transformation (11.33). It is used in § 11.10.

Some of the properties of $\boldsymbol{R}$ and $\boldsymbol{R}_0$ are illustrated in problems 11.1–11.3.

11.7. Wave impedance and admittance

Let E and H be some components of the electric and magnetic intensities respectively of a wave, at the same point. Then the ratio E/H is a wave impedance. For SI units E is measured in volts per metre and H in amperes per metre so that E/H is measured in ohms and has the dimensions of an electrical impedance. Similarly H/E is a wave admittance and would be measured in siemens. Instead of H we use the measure $\mathscr{H}$ (§ 2.10) for the magnetic intensity. Then the ratios $E/\mathscr{H}$ and $\mathscr{H}/E$ are dimensionless. It will sometimes be necessary to use formulae containing both types of ratio and a single phrase is needed to apply to both. The term 'wave admittance' will be used for this, as is customary. For a discussion of the concept of wave admittance and impedance see Schelkunoff (1938), Booker (1947).

In this book wave admittances are used for a stratified ionosphere with the z axis perpendicular to the strata, and they are therefore defined only in terms of the field components E_x, E_y, $\mathscr{H}_x$, $\mathscr{H}_y$ parallel to the strata. This has the advantage that if the medium has a discontinuity at some plane $z = $ constant, these components are the same on the two sides. Thus the wave admittances are continuous at such a discontinuity.

The concept is now to be applied to a system of waves in the ionosphere whose fields are independent of y and depend on x only through a factor $\exp(-\mathrm{i}kSx)$, where $S = \sin\theta$. These might arise from an incident wave in free space with its wave normal in the x–z plane at an angle θ to the z axis. Then in any region where the medium is homogeneous or sufficiently slowly varying there are in general four waves, and the field components of any one of them contain an exponential factor

$$\exp\left\{-\mathrm{i}k\left(Sx + \int^z q\,\mathrm{d}z\right)\right\} \tag{11.34}$$

where q is one of the four roots of the Booker quartic equation, ch. 6. As a simple illustration suppose that the medium is isotropic and that only one wave (11.34) is present in a range of height where the medium is homogeneous. Let this wave be a vertically polarised upgoing wave, as in § 7.2, so that $\mathscr{H}_x = E_y = 0$ and from (7.4)

$$q\mathscr{H}_y = n^2 E_x, \quad A_{11} = E_x/\mathscr{H}_y = q/n^2 \tag{11.35}$$

where q is given by (7.2) with the $+$ sign. Here A_{11} is a 'wave admittance'. It is independent of the amplitude of the wave and depends only on θ and on the properties of the medium. For a homogeneous medium it would be independent of z. It is called a 'characteristic admittance' of the medium. If a downgoing vertically polarised wave is also present, the ratio A_{11} is no longer independent of z, and it depends on the relative amplitudes of the two waves.

Next suppose that, in the same height range, the only wave present is a horizontally polarised upgoing wave so that $E_x = \mathscr{H}_y = 0$. Now A_{11} cannot be used, but instead from (7.3) we use

$$-qE_y = \mathscr{H}_x, \quad A_{22} = -\mathscr{H}_x/E_y = q, \tag{11.36}$$

where q is the same as in (11.35). Here A_{22} is another wave admittance. Again, in a homogeneous medium, it is independent of z and is a characteristic admittance of the medium. In these two examples A_{11} is actually an impedance, but A_{22} is an admittance.

The concept of wave admittance must now be generalised so as to apply for an anisotropic medium. For any level consider two different wave systems, with the same θ. Their horizontal field components are

$$E_{x1}, -E_{y1}, \mathscr{H}_{x1}, \mathscr{H}_{y1} \quad \text{and} \quad E_{x2}, -E_{y2}, \mathscr{H}_{x2}, \mathscr{H}_{y2}. \tag{11.37}$$

These are the elements of two column matrices $\mathbf{e}_1$ and $\mathbf{e}_2$ respectively, as in § 7.13. The reason for using $-E_y$ was explained in § 7.14 (3), (8). Now define a 2×2 wave admittance matrix $\boldsymbol{A}$ as follows. Let

$$\boldsymbol{P} = \begin{pmatrix} E_{x1} & E_{x2} \\ -\mathscr{H}_{x1} & -\mathscr{H}_{x2} \end{pmatrix}, \quad \boldsymbol{Q} = \begin{pmatrix} \mathscr{H}_{y1} & \mathscr{H}_{y2} \\ E_{y1} & E_{y2} \end{pmatrix}, \tag{11.38}$$

and let

$$\boldsymbol{A} = \boldsymbol{P}\boldsymbol{Q}^{-1}. \tag{11.39}$$

since $\boldsymbol{AQ} = \boldsymbol{P}$ it follows that $\boldsymbol{A}$ has the property

$$\boldsymbol{A}\begin{pmatrix} \mathscr{H}_y \\ E_y \end{pmatrix} = \begin{pmatrix} E_x \\ -\mathscr{H}_x \end{pmatrix} \tag{11.40}$$

where $E_x, -E_y, \mathscr{H}_x, \mathscr{H}_y$ may be any linear combination of the two sets (11.37). There are other ways of defining an admittance matrix, by placing the elements of (11.37) in a different order and with different signs. See, for example, Barron and Budden (1959), Budden (1961a). The choice (11.40) is made to achieve some algebraic simplicity in later applications §§ 18.9, 18.12.

Some of the main properties of the wave admittance matrix may now be listed.

(1) Consider a range of z where the medium is isotropic and slowly varying. Let the two wave systems (11.37) be upgoing progressive waves, one vertically and the other horizontally polarised. Then the elements of $\mathbf{e}_1$, $\mathbf{e}_2$ are

$$\mathbf{e}_1 = E_x, 0, 0, \frac{n^2}{q}E_x; \quad \mathbf{e}_2 = 0, -E_y, -qE_y, 0 \tag{11.41}$$

and it follows from (11.38), (11.39) that

$$A_{12} = A_{21} = 0, \quad A_{11} = q/n^2, \quad A_{22} = q. \tag{11.42}$$

Thus (11.35), (11.36) are the diagonal elements of A. The z dependence of both fields (11.41) includes the factor $\exp(-\mathrm{i}k\int^z q\,\mathrm{d}z)$ but this factor cancels and does not appear in A.

(2) Consider a range of z where the medium is slowly varying but anisotropic. Let each of the two wave systems (11.37) be a progressive wave given by one of the W.K.B. solutions, §7.15, so that all elements of $\mathbf{e}_1$ depend on z through the same factor $\exp(-\mathrm{i}k\int^z q_1 z)$ and similarly for $\mathbf{e}_2$, with the factor $\exp(-\mathrm{i}k\int^z q_2\,\mathrm{d}z)$. These factors cancel in (11.39). In a homogeneous medium A would be independent of z. It is called a 'characteristic' admittance matrix of the medium. It depends on θ and on which two of the four possible waves are chosen for (11.37).

(3) If more than two of the four possible progressive waves are used in choosing $\mathbf{e}_1$, $\mathbf{e}_2$ in (11.37), then A depends on height z, whether or not the medium is homogeneous. The differential equation satisfied by A is studied later, § 18.9.

(4) Consider the special case where $\theta = 0$, so that the wave normals are all parallel to the z axis. Let the two waves chosen for (11.37) be the upgoing ordinary and extraordinary waves so that $\mathbf{e}_1$ and $\mathbf{e}_2$ are given by (7.145), (7.147). Then it can be shown that

$$A = \frac{1}{n_\mathrm{O} - \rho_\mathrm{O}^2 n_\mathrm{E}} \begin{pmatrix} 1 - \rho_\mathrm{O}^2 & \rho_\mathrm{O}(n_\mathrm{O} - n_\mathrm{E}) \\ \rho_\mathrm{O}(n_\mathrm{O} - n_\mathrm{E}) & n_\mathrm{O} n_\mathrm{E}(1 - \rho_\mathrm{O}^2) \end{pmatrix} \tag{11.43}$$

and, as expected from (2), the z dependent factors have cancelled.

(5) A is continuous across any boundary plane $z = \text{constant}$.

(6) A depends on the ratios of field components at one point. This applies also to the reflection coefficient matrix $\boldsymbol{R}$, § 11.5. Thus there is a relation between $\boldsymbol{R}$ and A. An example is given at the end of this section. One important use of the continuity property (5) of A is in the derivation of reflection coefficients.

(7) Transmission coefficients depend on the ratios of field components at different points. A knowledge of A at different points is not sufficient to determine the transmission coefficients.

Suppose that A is known in the free space below the ionosphere. Here the fields may be separated into upgoing and downgoing component waves. For fields of the upgoing component we use a superscript I as in §§ 11.4–11.6. Let $\mathscr{H}_y^{(\mathrm{I})} = a$, $E_y^{(\mathrm{I})} = b$ so that $E_x^{(\mathrm{I})} = Ca$, $\mathscr{H}_x^{(\mathrm{I})} = -Cb$ where C is $\cos\theta$. Similarly for the fields of the downgoing component, superscript R, let $\mathscr{H}_y^{(\mathrm{R})} = c$, $E_y^{(\mathrm{R})} = d$, so that $E_x^{(\mathrm{R})} = -Cc$, $\mathscr{H}_x^{(\mathrm{R})} = Cd$. Then the total fields are given by

$$\begin{pmatrix} E_x \\ -E_y \\ \mathscr{H}_x \\ \mathscr{H}_y \end{pmatrix} = \begin{pmatrix} C & 0 & -C & 0 \\ 0 & -1 & 0 & -1 \\ 0 & -C & 0 & C \\ 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \tag{11.44}$$

whence by inversion of the 4×4 matrix

$$2\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1/C & 0 & 0 & 1 \\ 0 & -1 & -1/C & 0 \\ -1/C & 0 & 0 & 1 \\ 0 & -1 & 1/C & 0 \end{pmatrix} \begin{pmatrix} E_x \\ -E_y \\ \mathscr{H}_x \\ \mathscr{H}_y \end{pmatrix}. \tag{11.45}$$

From the definition (11.20), (11.23) of the reflection coefficient matrix $\boldsymbol{R}$ it follows that

$$\boldsymbol{R}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}. \tag{11.46}$$

Now (11.45) may be partitioned to give

$$\left.\begin{aligned} 2\begin{pmatrix} a \\ b \end{pmatrix} &= \frac{1}{C}\begin{pmatrix} E_x \\ -\mathscr{H}_x \end{pmatrix} + \begin{pmatrix} \mathscr{H}_y \\ E_y \end{pmatrix}, \\ 2\begin{pmatrix} c \\ d \end{pmatrix} &= -\frac{1}{C}\begin{pmatrix} E_x \\ -\mathscr{H}_x \end{pmatrix} + \begin{pmatrix} \mathscr{H}_y \\ E_y \end{pmatrix}. \end{aligned}\right\} \tag{11.47}$$

If (11.46) and (11.40) are now used we obtain

$$\boldsymbol{R}(C + \boldsymbol{A}) = C - \boldsymbol{A} \tag{11.48}$$

where the arbitrary column $(\mathscr{H}_y, E_y)$ on the right has been omitted and C is assumed to be multiplied by the 2×2 unit matrix $\mathbf{1}$. Hence

$$\begin{aligned} \boldsymbol{R} &= (C - \boldsymbol{A})(C + \boldsymbol{A})^{-1} = 2C(C + \boldsymbol{A})^{-1} - \mathbf{1} \\ \boldsymbol{A} &= C(\mathbf{1} - \boldsymbol{R})(\mathbf{1} + \boldsymbol{R})^{-1} = 2C(\mathbf{1} + \boldsymbol{R})^{-1} - C. \end{aligned} \tag{11.49}$$

This shows that if $\boldsymbol{A}$ is transposed then $\boldsymbol{R}$ also is transposed. These results are used in § 18.12. In particular if $\boldsymbol{A}$ and $\boldsymbol{R}$ are diagonal:

$$R_{11} = \frac{C - A_{11}}{C + A_{11}}, \quad A_{11} = C\frac{1 - R_{11}}{1 + R_{11}}; \; R_{22} = \frac{C - A_{22}}{C + A_{22}}, \quad A_{22} = C\frac{1 - R_{22}}{1 + R_{22}}. \tag{11.50}$$

11.8. Reflection at a sharp boundary 1. Isotropic plasma

The formulae for the reflection and transmission coefficients at a plane boundary between two homogeneous media are useful, even though there are no sharp boundaries in the ionosphere and magnetosphere. For example some authors have adopted a simple model ionosphere consisting of a homogeneous plasma with a lower plane boundary and free space below. This has led with good success to explanations of many of the features of the propagation of radio waves of very low frequency. See for example, Yokoyama and Namba (1932); Rydbeck (1944); Bremmer (1949); Barber and Crombie (1959). The formulae have already been used in § 7.7 to study propagation in a medium comprised of discrete strata.

Suppose that in the region $z > 0$ there is a homogeneous medium with refractive index n_2, and in the region $z < 0$ another homogeneous medium with refractive index

n_1. In this section it is assumed that these media are isotropic. The incident plane wave is in the medium where $z<0$ with its wave normal in the $x-z$ plane at an angle θ to the positive z axis, and $0 \leqslant \theta < \frac{1}{2}\pi$. It gives rise to a reflected plane wave in $z<0$ and to a transmitted plane wave in $z>0$. The field components satisfy some boundary conditions so that all field components must depend on x and y in the same way. This leads at once to the requirement that the wave normals of the reflected and transmitted waves are also in the $x-z$ plane. Let them make angles θ_R, θ_T respectively with the z axis. All three angles θ, θ_R, θ_T are measured clockwise from the positive z axis, for an observer looking in the direction of the positive y axis. Since the x dependence of all fields is the same it then follows that

$$\theta_R = \pi - \theta, \quad n_1 \sin\theta = n_2 \sin\theta_T. \tag{11.51}$$

The first is the reflection law, and the second is Snell's law; compare § 6.2. Note that if the medium in $z<1$ is anisotropic, the refractive index n_R for the reflected wave is in general different from that for the incident wave, n_1. Then the first equation (11.51) must be replaced by

$$n_1 \sin\theta = n_R \sin\theta_R. \tag{11.52}$$

It is now required to find the reflection and transmission coefficients, and the boundary plane $z=0$ will be used as the reference level. They are derived from the boundary conditions that the four components E_x, E_y, $\mathscr{H}_x$, $\mathscr{H}_y$ parallel to the boundary are the same on the two sides of the boundary plane $z=0$. The details are set out in textbooks on optics and on electromagnetic wave theory. They were given, in a notation very similar to that used here, by Budden (1961a), who used a definition of the transmission coefficients different from (11.18), (11.19). In writing the results it is convenient to use.

$$q_I = -q_R = n_1 \cos\theta, \quad q_T = n_2 \cos\theta_T = (n_2^2 - n_1^2 \sin^2\theta)^{\frac{1}{2}}. \tag{11.53}$$

These qs are the same as those defined and used in ch. 6. Then

$$R_{12} = R_{21} = T_{12} = T_{21} = 0, \tag{11.54}$$

$$R_{11} = \frac{n_2 \cos\theta - n_1 \cos\theta_T}{n_2 \cos\theta + n_1 \cos\theta_T} = \frac{n_2^2 q_I - n_1^2 q_T}{n_2^2 q_I + n_1^2 q_T}, \tag{11.55}$$

$$T_{11} = \frac{2n_2 \cos\theta}{n_2 \cos\theta + n_1 \cos\theta_T} = \frac{2n_2^2 q_I}{n_2^2 q_I + n_1^2 q_I}, \tag{11.56}$$

$$R_{22} = \frac{n_1 \cos\theta - n_2 \cos\theta_T}{n_1 \cos\theta + n_2 \cos\theta_T} = \frac{q_I - q_T}{q_I + q_T}, \tag{11.57}$$

$$T_{22} = \frac{2n_1 \cos\theta}{n_1 \cos\theta + n_2 \cos\theta_T} = \frac{2q_1}{q_1 + q_T}. \tag{11.58}$$

The above formulae for the elements of $\boldsymbol{R}$ are special cases of (11.66) which gives $\boldsymbol{R}$ for the more general case when the medium in $z>0$ is anisotropic. The full derivation is given in § 11.10.

Formulae (11.55), (11.57) and formulae equivalent to (11.56), (11.58) were given by Fresnel. They are often used in textbooks of optics for real n_1 and n_2, but they are not restricted to real variables, and will be applied here to cases where the refractive indices and the angles are in general complex.

They are often needed for normal incidence, $\theta = \theta_T = 0$ and they then become

$$R_{11} = -R_{22} = \frac{n_2 - n_1}{n_2 + n_1}, \tag{11.59}$$

$$T_{11} = \frac{2n_2}{n_1 + n_2}, \qquad T_{22} = \frac{2n_1}{n_1 + n_2}. \tag{11.60}$$

The above formula for R_{11} is based on the use of $\mathscr{H}_y$ as in (11.16). It is sometimes more convenient at normal incidence to use, instead of R_{11}, the ratio of the values of E_x. The resulting reflection coefficient is $-R_{11}$ in (11.59).

11.9. Properties of the Fresnel formulae

In this section the medium where $z < 1$ is assumed to be free space so that $n_1 = 1$, and $q_I = \cos\theta$. The refractive index of the medium where $z > 0$ will be denoted simply by n, with the subscript 2 omitted.

The reflection coefficient R_{11}, (11.55) is zero when $n^2\cos\theta = q_T$, that is when

$$\theta = \theta_B, \quad \tan\theta_B = n. \tag{11.61}$$

When n is real θ_B is a real angle called the 'Brewster angle'. For real n, when θ is near to θ_B, R_{11} is real. Arg R_{11} changes discontinuously by π when θ increases and passes through θ_B.

When n is complex θ_B is also complex and is called the 'complex Brewster angle'. When $\theta = \theta_B$ the incident and transmitted waves are both inhomogeneous plane waves; § 2.15. When θ is real R_{11} is complex. Then when θ increases, arg R_{11} changes continuously, and the rate of change is greatest, and $|R_{11}|$ is a minimum, when θ is near to $\mathrm{Re}(\theta_B)$.

The formula (11.57) shows that for R_{22} to be zero it is necessary that $\cos\theta = (n^2 - \sin^2\theta)^{\frac{1}{2}}$. This cannot be satisfied for any real or complex θ if $n \neq 1$. Hence in this case there is no Brewster angle.

When $\cos\theta_T$ is zero, $\sin\theta_T$ is unity and Snell's law (11.51) gives

$$\theta = \theta_C, \quad \sin\theta_C = n. \tag{11.62}$$

Then $R_{11} = R_{22} = 1$, and θ_C is called the 'critical angle', by analogy with the corresponding case in optics. If n is complex, the critical angle is also complex.

Suppose that θ and n are real and $\theta_C \leqslant \theta < \frac{1}{2}\pi$, or that θ is real and n is purely imaginary. Then q_T (11.53) is purely imaginary and $|R_{11}| = 1$, $|R_{22}| = 1$, and from (11.55), (11.57)

$$\arg R_{11} = 2\arctan\{(\sin^2\theta - n^2)^{\frac{1}{2}}/n^2\cos\theta\} \tag{11.63}$$

$$\arg R_{22} = 2\arctan\{(\sin^2\theta - n^2)^{\frac{1}{2}}/\cos\theta\} \tag{11.64}$$

where the square root takes its positive value, and the arctangents are between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$. The reader should verify that the signs of (11.63), (11.64) are correct. These are examples where the reflection is total. They are analogous to the phenomenon of 'total internal reflection' in optics.

Curves showing how the coefficients (11.55)–(11.58) depend on the angle θ and on the refractive indices are given in many books. Some examples, where the medium in $z>0$ is an isotropic electron plasma without or with collisions, were given by Budden (1961a). Since then, however, programmable pocket calculators have become readily available and the interested reader can now quickly plot such curves for himself. See problems 11.5, 11.6, 11.8, 11.9 for some examples of the use of these and related formulae.

11.10. Reflection at a sharp boundary 2. Anisotropic plasma

When one or both of the media on the two sides of the boundary plane $z=0$ is anisotropic, the derivation of the reflection coefficients is more complicated. In this section the theory is given only for the case where the medium in $z<0$ is free space, so that $n_1=1$. The wave normal of the incident wave is again assumed to be in the $x-z$ plane at an angle θ to the z axis, and we use $S=\sin\theta$, $C=\cos\theta$. In general it gives a reflected wave where $z<0$ and two transmitted waves, ordinary and extraordinary, in the anisotropic medium where $z>0$. The quantities associated with these waves will be indicated by superscripts (O), (E), which replace the (T) of § 11.4. They are obliquely upgoing plane waves and their fields are determined by their two values $q^{(O)}$ and $q^{(E)}$ of the roots of the Booker quartic (6.15) with coefficients (6.23). These values of q for upgoing waves are chosen as described in § 6.4. To describe the fields of any of the plane waves, it is convenient to use the column matrix $\mathbf{e}$ with four elements E_x, $-E_y$, $\mathscr{H}_x$, $\mathscr{H}_y$ as defined in § 7.13 and used in (7.80). They are given by (7.118) or (7.134). They will now be written $\mathrm{e}_1^{(O)}$, $\mathrm{e}_2^{(O)}$, $\mathrm{e}_3^{(O)}$, $\mathrm{e}_4^{(O)}$ for the ordinary wave, and the same with (O) replaced by (E) for the extraordinary wave. Then the admittance matrix in the medium where $z>0$ is from (11.38) (11.39)

$$\boldsymbol{A}=\begin{pmatrix} \mathrm{e}_1^{(O)} & \mathrm{e}_1^{(E)} \\ -\mathrm{e}_3^{(O)} & -\mathrm{e}_3^{(E)} \end{pmatrix}\begin{pmatrix} \mathrm{e}_4^{(O)} & \mathrm{e}_4^{(E)} \\ -\mathrm{e}_2^{(O)} & -\mathrm{e}_2^{(E)} \end{pmatrix}^{-1} \tag{11.65}$$

This is independent of z (§11.7 item (2)) and is a characteristic admittance matrix. Now $\boldsymbol{A}$ must be the same on the two sides of the bounday plane $z=0$, (§ 11.7 item (5)), and hence in the medium where $z<0$, at the boundary, $\boldsymbol{A}$ is equal to (11.65). But here it is z dependent and not a characteristic admittance matrix. At the boundary, $\boldsymbol{R}$ is found from (11.49), (11.65). Thus the reflection coefficient with reference level at $z=0$ is

$$\boldsymbol{R}=\begin{pmatrix} C\mathrm{e}_4^{(O)}-\mathrm{e}_1^{(O)} & C\mathrm{e}_4^{(E)}-\mathrm{e}_1^{(E)} \\ -C\mathrm{e}_2^{(O)}+\mathrm{e}_3^{(O)} & -C\mathrm{e}_2^{(E)}+\mathrm{e}_3^{(E)} \end{pmatrix}\begin{pmatrix} C\mathrm{e}_4^{(O)}+\mathrm{e}_1^{(O)} & C\mathrm{e}_4^{(E)}+\mathrm{e}_1^{(E)} \\ -C\mathrm{e}_2^{(O)}-\mathrm{e}_3^{(O)} & -C\mathrm{e}_2^{(E)}-\mathrm{e}_3^{(E)} \end{pmatrix}^{-1} \tag{11.66}$$

A method of finding $\boldsymbol{R}$ that used solutions of the Booker quartic equation, and that is in essentials the same as that used here, was given by Yabroff (1957), and by Johler and Walters (1960). They gave curves to show how the elements of $\boldsymbol{R}$ depend on angle of incidence and on the parameters of the medium.

A completely different method for finding $\boldsymbol{R}$ is described in § 18.10, where it is shown that $\boldsymbol{R}$ is the solution of a matrix equation that does not contain $q^{(\mathrm{O})}$ and $q^{(\mathrm{E})}$. See also Barron and Budden (1959, § 10).

11.11. Normal incidence. Anisotropic plasma with free space below it

For normal incidence $\theta = 0$, two cases will be studied. For the first, in this section, it is assumed, as in § 11.10, that the medium where $z < 1$ is free space, and the homogeneous anisotropic plasma where $z > 0$ is treated as a simple model of the ionosphere with its lower boundary at $z = 0$, and with z measured vertically upwards. The second case, in § 11.12, treats the problem when both media are anisotropic plasmas. For both cases it is convenient to choose the x and y axes so that the vector $\boldsymbol{Y}$, anti-parallel to the earth's magnetic field, is in the x–z plane.

For the field components $\mathbf{e}^{(\mathrm{O})}$, $\mathbf{e}^{(\mathrm{E})}$ in the upper medium we may use, instead of (7.118), the simpler forms given in (7.145), (7.147). Thus the elements are
For $\mathbf{e}^{(\mathrm{O})}$:

$$1, \quad -\rho_{\mathrm{O}}, \quad -\rho_{\mathrm{O}} n_{\mathrm{O}}, \quad n_{\mathrm{O}} \tag{11.67}$$

For $\mathbf{e}^{(\mathrm{E})}$:

$$\rho_{\mathrm{O}}, \quad -1, \quad -n_{\mathrm{E}}, \quad \rho_{\mathrm{O}} n_{\mathrm{E}} \tag{11.68}$$

The subscripts O, E, for ordinary and extraordinary, now replace the superscripts (O), (E). If these are now used in (11.66) with $C = 1$, it gives

$$\boldsymbol{R} = \begin{pmatrix} n_{\mathrm{O}} - 1 & \rho_{\mathrm{O}}(n_{\mathrm{E}} - 1) \\ -\rho_{\mathrm{O}}(n_{\mathrm{O}} - 1) & -(n_{\mathrm{E}} - 1) \end{pmatrix} \begin{pmatrix} n_{\mathrm{O}} + 1 & \rho_{\mathrm{O}}(n_{\mathrm{E}} + 1) \\ \rho_{\mathrm{O}}(n_{\mathrm{O}} + 1) & n_{\mathrm{E}} + 1 \end{pmatrix}^{-1} \tag{11.69}$$

and on multiplying out

$$R_{11} = \frac{1}{\rho_{\mathrm{E}} - \rho_{\mathrm{O}}} \left\{ \rho_{\mathrm{E}} \frac{n_{\mathrm{O}} - 1}{n_{\mathrm{O}} + 1} - \rho_{\mathrm{O}} \frac{n_{\mathrm{E}} - 1}{n_{\mathrm{E}} + 1} \right\}, \tag{11.70}$$

$$R_{22} = \frac{1}{\rho_{\mathrm{E}} - \rho_{\mathrm{O}}} \left\{ \rho_{\mathrm{O}} \frac{n_{\mathrm{O}} - 1}{n_{\mathrm{O}} + 1} - \rho_{\mathrm{E}} \frac{n_{\mathrm{E}} - 1}{n_{\mathrm{E}} + 1} \right\}, \tag{11.71}$$

$$R_{12} = R_{21} = \frac{2}{\rho_{\mathrm{E}} - \rho_{\mathrm{O}}} \left\{ \frac{1}{n_{\mathrm{O}} + 1} - \frac{1}{n_{\mathrm{E}} + 1} \right\}. \tag{11.72}$$

The relation $\rho_{\mathrm{O}} \rho_{\mathrm{E}} = 1$, (4.29), has been used here.

If the earth's magnetic field is horizontal, the wave polarisations are, from (4.82): $\rho_{\mathrm{O}} = 0$ and $\rho_{\mathrm{E}} = \infty$. Then the elements of $\boldsymbol{R}$ are

$$R_{11} = \frac{n_{\mathrm{O}} - 1}{n_{\mathrm{O}} + 1}, \quad R_{22} = -\frac{n_{\mathrm{E}} - 1}{n_{\mathrm{E}} + 1}, \quad R_{12} = R_{21} = 0. \tag{11.73}$$

The first two are just the Fresnel formulae (11.59) for normal incidence with $n_1 = 1$.

Suppose now that the earth's magnetic field is vertical and directed downwards, as at the north magnetic pole. Then the vector $\boldsymbol{Y}$ is directed upwards and its direction cosine is $l_z = +1$ for upgoing waves. For the ordinary wave when $Y < 1$, $\rho_{\mathrm{O}} = -\mathrm{i}$ which corresponds to right-handed circular polarisation. Similarly $\rho_{\mathrm{E}} = +\mathrm{i}$. Hence

$$n_{\mathrm{O}}^2 = 1 - X/(U+Y), \quad n_{\mathrm{E}}^2 = 1 - X/(U-Y). \tag{11.74}$$

If $Y > 1$, the order of the terms ordinary and extraordinary and of the subscripts O, E must be reversed. The elements of R are now

$$\begin{aligned} R_{11} = -R_{22} &= \frac{1}{2}\left\{\frac{n_{\mathrm{O}}-1}{n_{\mathrm{O}}+1} + \frac{n_{\mathrm{E}}-1}{n_{\mathrm{E}}+1}\right\}, \\ R_{12} = R_{21} &= \mathrm{i}\left\{\frac{1}{n_{\mathrm{E}}+1} - \frac{1}{n_{\mathrm{O}}+1}\right\}. \end{aligned} \tag{11.75}$$

These may be simplified by using the alternative representation $\boldsymbol{R}_0$ of § 11.6, in which the incident and reflected waves are resolved into circularly polarised components. As in § 11.6 we use subscripts r, l to denote right-handed and left-handed components, so that

$$\boldsymbol{R}_0 = \begin{pmatrix} R_{\mathrm{rr}} & R_{\mathrm{rl}} \\ R_{\mathrm{lr}} & R_{\mathrm{ll}} \end{pmatrix}. \tag{11.76}$$

Then the transformation (11.33), with (11.30), when used with (11.75) gives

$$R_{\mathrm{rr}} = R_{\mathrm{ll}} = 0, \quad R_{\mathrm{rl}} = \frac{1-n_{\mathrm{E}}}{1+n_{\mathrm{E}}}, \quad R_{\mathrm{lr}} = \frac{1-n_{\mathrm{O}}}{1+n_{\mathrm{O}}}. \tag{11.77}$$

These results show that the two waves normally incident on a sharply bounded ionosphere are independently reflected. For example an incident wave that is circularly polarised with a right-handed sense would give a wave in the ionosphere with the same polarisation, and the reflected wave would be circularly polarised with a left-handed sense. This reversal of the sense occurs because the directions of the wave normals are opposite for the incident and reflected waves; compare end of § 4.4. The absolute direction of rotation is the same for all three waves, namely clockwise when seen by an observer looking upwards. Because the wave polarisations are the same for all three waves, the boundary conditions are satisfied with this one polarisation, and no wave with the other polarisation is produced at the boundary.

11.12. Normal incidence. Two anisotropic plasmas

To study the reflection at normal incidence when the media on both sides of the boundary are anisotropic, we shall from the start use a representation in which the waves are resolved, not into linearly polarised components, but into ordinary and extraordinary components. The values of n_{O}, ρ_{O}, n_{E}, ρ_{E} are different for the two media, and this will be indicated by using superscripts (1) for $z < 0$, and (2) for $z > 0$. Thus the incident wave is resolved into an ordinary component whose fields are the column matrix $\mathbf{e}_{\mathrm{O}}^{(\mathrm{I})}$ with elements

$$a_{\mathrm{O}}(1, -\rho_{\mathrm{O}}^{(1)}, -\rho_{\mathrm{O}}^{(1)} n_{\mathrm{O}}^{(1)}, n_{\mathrm{O}}^{1}) \tag{11.78}$$

and an extraordinary wave whose $\mathbf{e}_{\mathrm{E}}^{(\mathrm{I})}$ has elements

$$a_{\mathrm{E}}(1, -\rho_{\mathrm{E}}^{(1)}, -\rho_{\mathrm{E}}^{(1)} n_{\mathrm{E}}^{(1)}, n_{\mathrm{E}}^{(1)}). \tag{11.79}$$

The reflected wave is similarly resolved into an ordinary component whose $\mathbf{e}_{\mathrm{O}}^{(\mathrm{R})}$ has elements

$$b_{\mathrm{O}}(1, -\rho_{\mathrm{O}}^{(1)}, \rho_{\mathrm{O}}^{(1)} n_{\mathrm{O}}^{(1)}, -n_{\mathrm{O}}^{(1)}) \tag{11.80}$$

and an extraordinary component whose $\mathbf{e}_{\mathrm{E}}^{(\mathrm{R})}$ has elements

$$b_{\mathrm{E}}(1, -\rho_{\mathrm{E}}^{(1)}, \rho_{\mathrm{E}}^{(1)} n_{\mathrm{E}}^{(1)}, -n_{\mathrm{E}}^{(1)}). \tag{11.81}$$

These formulae are given in (7.146), (7.148). The polarisations $\rho_{\mathrm{O}}, \rho_{\mathrm{E}}$ are defined with the same axes for upgoing and downgoing waves and therefore have the same two values. For an explanation of this see end of § 4.4. Finally the fields of the transmitted waves in the upper medium are

$$c_{\mathrm{O}}(1, -\rho_{\mathrm{O}}^{(2)}, -\rho_{\mathrm{O}}^{(2)} n_{\mathrm{O}}^{(2)}, n_{\mathrm{O}}^{(2)}) \tag{11.82}$$

and

$$c_{\mathrm{E}}(1, -\rho_{\mathrm{E}}^{(2)}, -\rho_{\mathrm{E}}^{(2)} n_{\mathrm{E}}^{(2)}, n_{\mathrm{E}}^{(2)}). \tag{11.83}$$

The boundary conditions require that the sum of the fields (11.78)–(11.81) is equal to the sum of (11.82) and (11.83). This can be written in matrix form

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -\rho_{\mathrm{O}}^{(1)} & -\rho_{\mathrm{E}}^{(1)} & -\rho_{\mathrm{O}}^{(1)} & -\rho_{\mathrm{E}}^{(1)} \\ -\rho_{\mathrm{O}}^{(1)} n_{\mathrm{O}}^{(1)} & -\rho_{\mathrm{E}}^{(1)} n_{\mathrm{E}}^{(1)} & \rho_{\mathrm{O}}^{(1)} n_{\mathrm{O}}^{(1)} & \rho_{\mathrm{E}}^{(1)} n_{\mathrm{E}}^{(1)} \\ n_{\mathrm{O}}^{(1)} & n_{\mathrm{E}}^{(1)} & -n_{\mathrm{O}}^{(1)} & -n_{\mathrm{E}}^{(1)} \end{pmatrix} \begin{pmatrix} a_{\mathrm{O}} \\ a_{\mathrm{E}} \\ b_{\mathrm{O}} \\ b_{\mathrm{E}} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -\rho_{\mathrm{O}}^{(2)} & -\rho_{\mathrm{E}}^{(2)} \\ -\rho_{\mathrm{O}}^{(2)} n_{\mathrm{O}}^{(2)} & -\rho_{\mathrm{E}}^{(2)} n_{\mathrm{E}}^{(2)} \\ n_{\mathrm{O}}^{(2)} & n_{\mathrm{E}}^{(2)} \end{pmatrix} \begin{pmatrix} c_{\mathrm{O}} \\ c_{\mathrm{E}} \end{pmatrix}. \tag{11.84}$$

Now the matrices are partitioned thus

$$\boldsymbol{a} = \begin{pmatrix} a_{\mathrm{O}} \\ a_{\mathrm{E}} \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} b_{\mathrm{O}} \\ b_{\mathrm{E}} \end{pmatrix}, \quad \boldsymbol{c} = \begin{pmatrix} c_{\mathrm{O}} \\ c_{\mathrm{E}} \end{pmatrix}, \tag{11.85}$$

and

$$\boldsymbol{M}_1 = \begin{pmatrix} 1 & 1 \\ -\rho_{\mathrm{O}} & -\rho_{\mathrm{E}} \end{pmatrix}, \quad \boldsymbol{M}_2 = \begin{pmatrix} -\rho_{\mathrm{O}} & -\rho_{\mathrm{E}} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} n_{\mathrm{O}} & 0 \\ 0 & n_{\mathrm{E}} \end{pmatrix} \tag{11.86}$$

where appropriate superscripts (1), (2) are to be attached. Then (11.84) gives

$$\boldsymbol{M}_1^{(1)}(\boldsymbol{a} + \boldsymbol{b}) = \boldsymbol{M}_1^{(2)} \boldsymbol{c}, \quad \boldsymbol{M}_2^{(1)}(\boldsymbol{a} - \boldsymbol{b}) = \boldsymbol{M}_2^{(2)} \boldsymbol{c}. \tag{11.87}$$

Next, by analogy with § 11.4, define reflection and transmission coefficient matrices

$$\boldsymbol{R}_c = \begin{pmatrix} R_{\mathrm{OO}} & R_{\mathrm{OE}} \\ R_{\mathrm{EO}} & R_{\mathrm{EE}} \end{pmatrix}, \quad \boldsymbol{T}_c = \begin{pmatrix} T_{\mathrm{OO}} & T_{\mathrm{OE}} \\ T_{\mathrm{EO}} & T_{\mathrm{EE}} \end{pmatrix} \tag{11.88}$$

so that

$$\boldsymbol{b} = \boldsymbol{R}_c \boldsymbol{a}, \quad \boldsymbol{c} = \boldsymbol{T}_c \boldsymbol{a}. \tag{11.89}$$

These are substituted in (11.87). The equations must be true for any $\boldsymbol{a}$, so the factor $\boldsymbol{a}$ on the right may be omitted, whence

$$\boldsymbol{M}_1^{(1)}(1 + \boldsymbol{R}_c) = \boldsymbol{M}_1^{(2)}\boldsymbol{T}_c, \quad \boldsymbol{M}_2^{(1)}(1 - \boldsymbol{R}_c) = \boldsymbol{M}_2^{(2)}\boldsymbol{T}_c. \tag{11.90}$$

Then elimination of $\boldsymbol{T}_c$ gives

$$(\boldsymbol{M}_1^{(2)})^{-1}\boldsymbol{M}_1^{(1)}(1 + \boldsymbol{R}_c) = (\boldsymbol{M}_2^{(2)})^{-1}\boldsymbol{M}_2^{(1)}(1 - \boldsymbol{R}_c) \tag{11.91}$$

which is a matrix equation for finding $\boldsymbol{R}_c$. After some reduction it becomes

$$\begin{pmatrix} n_{\mathrm{O}}^{(2)} & 0 \\ 0 & n_{\mathrm{E}}^{(2)} \end{pmatrix} \boldsymbol{G}(1 + \boldsymbol{R}_c) = \boldsymbol{G}\begin{pmatrix} n_{\mathrm{O}}^{(1)} & 0 \\ 0 & n_{\mathrm{E}}^{(1)} \end{pmatrix}(1 - \boldsymbol{R}_c) \tag{11.92}$$

where

$$\boldsymbol{G} = \begin{pmatrix} \rho_{\mathrm{O}}^{(1)} - \rho_{\mathrm{E}}^{(2)} & \rho_{\mathrm{E}}^{(1)} - \rho_{\mathrm{E}}^{(2)} \\ \rho_{\mathrm{O}}^{(2)} - \rho_{\mathrm{O}}^{(1)} & \rho_{\mathrm{O}}^{(2)} - \rho_{\mathrm{E}}^{(1)} \end{pmatrix}. \tag{11.93}$$

Now consider the special case where $\rho_{\mathrm{O}}^{(2)} = \rho_{\mathrm{O}}^{(1)}$, $\rho_{\mathrm{E}}^{(2)} = \rho_{\mathrm{E}}^{(1)}$. Then $\boldsymbol{G}$ is diagonal and is just a constant multiplier $\rho_{\mathrm{O}} - \rho_{\mathrm{E}}$ that can be cancelled from (11.92). It follows that $\boldsymbol{R}_c$ is then diagonal and

$$R_{\mathrm{OO}} = \frac{n_{\mathrm{O}}^{(1)} - n_{\mathrm{O}}^{(2)}}{n_{\mathrm{O}}^{(1)} + n_{\mathrm{O}}^{(2)}}, \quad R_{\mathrm{EE}} = \frac{n_{\mathrm{E}}^{(1)} - n_{\mathrm{E}}^{(2)}}{n_{\mathrm{E}}^{(1)} + n_{\mathrm{E}}^{(2)}}. \tag{11.94}$$

When both characteristic wave polarisations are the same on the two sides of the boundary there is no intermode conversion at the boundary.

If the lower medium is free space, any wave may be resolved into two waves whose polarisations can be chosen in any way we please; see problem 4.2. They can therefore be chosen to be $\rho_{\mathrm{O}}^{(2)}$, $\rho_{\mathrm{E}}^{(2)}$. Then the formulae (11.94) apply with $n_{\mathrm{O}}^{(1)} = n_{\mathrm{E}}^{(1)} = 1$. Two examples of this have already been given at (11.73) and (11.77).

11.13. Probing the ionosphere by the method of partial reflection

The ionosonde technique, § 1.7, uses vertically incident pulses of radio waves for studying the ionosphere. The main observed reflected pulses come from levels where n_{O} or n_{E} is zero, in the E- and F-regions. But the lower parts of the ionosphere, in the height range 60–100 km can contain weak irregularities of electron concentration N that can give discontinuities of N. Thus there can be quite sharp boundaries between two plasmas with slightly different N, and those boundaries that are horizontal or nearly so can give observable reflections. In temperate latitudes, for waves with vertical wave normals, the two wave polarisations ρ_{O}, ρ_{E} are very close to $\mp \mathrm{i}$, that is circular polarisation, and do not change appreciably when N changes. Thus there is no appreciable intermode conversion at these reflections. The ordinary and extraordinary waves are reflected independently and the reflection coefficients R_{OO}, R_{EE} are given by (11.94) with good accuracy. They are each proportional to the small change δN of N at the reflecting boundary, but the ratio R_{OO}/R_{EE} is independent of δN and depends only on the average composition of the plasma.

The idea that these reflections arise from horizontal sharp boundaries is oversimplified. A better idea is to regard them as produced by scattering from weak random irregularities of N. The theory for this version was given by Booker (1959) who showed that the effective ratio R_{OO}/R_{EE} is the same as for the simpler sharp boundary model.

The values of R_{OO}, R_{EE} are of order 10^{-7} to 10^{-3} so the reflections are extremely weak and it is necessary to make observations in a location where the extraneous electrical noise level is very small. The method was first used by Gardner and Pawsey (1953) and later by numerous other workers, particularly Belrose and Burke (1964), Fejer and Vice (1959a), Holt (1963), Belrose (1970), Belrose, Burke, Coyne and Reed (1972). The frequencies used were typically 2.28 MHz (Gardner and Pawsey), 2.66 and 6.27 MHz (Belrose and Burke).

In some of these experiments the transmitted pulse was linearly polarised. In the ionosphere it splits into two circularly polarised components with opposite senses and equal amplitudes. The reflected pulses were received on two aerials designed to respond to the two circularly polarised components, and the ratio of the amplitudes was measured. In other experiments the transmitting aerial was designed to emit circularly polarised waves and it was arranged that alternate pulses had a right-handed and a left-handed sense.

The reflections occur where N and therefore X is very small so that both refractive indices are close to unity. The ratio of the reflection coefficients (11.94) is then

$$R_{OO}/R_{EE} = \delta(n_O^2)/\delta(n_E^2) \tag{11.95}$$

where δ indicates the small change when the reflecting boundary is crossed. For small X, (4.112) shows that

$$n^2 \approx 1 + X\frac{\frac{1}{2}Y^2\sin^2\Theta - U^2 \pm Y(U^2\cos^2\Theta + \frac{1}{4}Y^2\sin^4\Theta)^{\frac{1}{2}}}{U(U^2 - Y^2)} \tag{11.96}$$

(upper sign for ordinary wave when $Y < 1$), whence

$$\frac{R_{OO}}{R_{EE}} = \frac{\frac{1}{2}Y^2\sin^2\Theta - U^2 + Y(U^2\cos^2\Theta + \frac{1}{4}Y^2\sin^4\Theta)^{\frac{1}{2}}}{\frac{1}{2}Y^2\sin^2\Theta - U^2 - Y(U^2\cos^2\Theta + \frac{1}{4}Y^2\sin^4\Theta)^{\frac{1}{2}}}. \tag{11.97}$$

This depends on the electron collision frequency ν. The main use of the partial reflection method is as a way of measuring ν at heights that are accurately known. For (11.97) it has been assumed that ν is independent of electron velocity. Some workers have used the Sen–Wyller form (3.84) of the constitutive relations so that their version of (11.97) contains the $\mathscr{C}$ integrals (3.83) (Belrose and Burke, 1964).

The waves that travel up to the reflection level and down again are attenuated, and the attenuations are different for the ordinary and extraordinary waves. In measuring R_{OO}/R_{EE} this must be allowed for, and it is particularly important for the greater heights of reflection because the paths through the ionosphere are large. The attenuation depends on X and thence on N so that this method also gives some information about $N(z)$.

11.14. Spherical waves. Choice of reference level

For the definitions of the reflection coefficients given in the preceding sections it was assumed that the incident wave was a plane wave. In practice the incident wave originates at a source of small dimensions, the transmitting aerial, so that the wave front is approximately spherical, but by the time it reaches the ionosphere the radius of curvature is so large that the wave is very nearly a plane wave. In treating it as exactly plane an approximation is used, whose validity must now be examined.

The field emanating from a transmitter of small dimensions at the origin of coordinates can be written as an angular spectrum of plane waves (10.1). It is now assumed that the receiver is in the plane $y = 0$. The field component F in (10.1) may then be E_y when the transmitter is a vertical magnetic Hertzian dipole, or $-\mathscr{H}_y/n$ when it is a vertical electric Hertzian dipole. Then the factor A in (10.1) is given by (10.53) and

$$F = \iint S_1 C^{-1} \exp\{-\mathrm{i}k(S_1 x + Cz)\}\, \mathrm{d}S_1\, \mathrm{d}S_2. \tag{11.98}$$

The factor S_1 gives the angle dependence of the radiated field of the vertical dipole. The integral is proportional to $r^{-1}\mathrm{e}^{-\mathrm{i}kr}\sin\theta$ where r, θ are the polar coordinates of the receiving point. The limits of both integrals are $\pm\infty$ but they are to be treated as contour integrals in the complex planes of S_1 and S_2. The integrand represents a plane wave whose wave normal has direction cosines S_1, S_2, C. For other types of transmitting aerial, the first factor S_1 is replaced by some other combination of S_1, S_2, C.

Each component plane wave in the integrand of (11.98) is reflected from the ionosphere and gives another plane wave travelling obliquely downwards, whose wave normal has direction cosines S_1, S_2, $-C$. In this wave we consider a field component F_R which may or may not be the same as F. The ratio F_R/F is some element or combination of the elements of $\boldsymbol{R}$ which we denote simply by R. This reflection coefficient R is in general a function of S_1 and S_2, though in simple cases it depends only on $\theta = \arccos C$. Suppose that its reference level (see § 11.2) is where $z = z_1$. Then the reflected field is

$$F_R = \iint R(S_1, S_2) S_1 C^{-1} \exp[-\mathrm{i}k\{S_1 x + C(2z_1 - z)\}]\, \mathrm{d}S_1\, \mathrm{d}S_2. \tag{11.99}$$

This may be evaluated by the method of double steepest descents, § 9.10. Suppose first that $R(S_1, S_2)$ is a very slowly varying function of S_1, S_2, so that the double saddle points S_{10}, S_{20} may be found from the exponential alone. Then R is set equal to $R(S_{10}, S_{20})$ and placed outside the integral. But the integral that remains is the same as (11.98) except that the sign of z is reversed and the source is at the point $x = y = 0$, $z = 2z_1$. This is just the geometrical image point of the origin in the reference plane $z = z_1$.

When (9.63) is applied to (11.99) it gives

$$S_{20} = 0, \quad S_{10}/C = x/(2z_1 - z) = \tan\theta_0. \tag{11.100}$$

These give the direction cosines of the wave normal of the predominant plane wave reaching the receiving point. They show that the reflection is just as for simple geometrical optics from the plane $z = z_1$, with reflection coefficient as for a plane wave at angle of incidence θ_0. In this case therefore, the plane wave reflection coefficient may be used, even though the incident wave front is spherical. The justification for it is that $R(S_1, S_2)$ must be sufficiently slowly varying.

Suppose next that we use a different reference level $z = z_2$, so that, from (11.10), the reflection coefficient is now

$$R_2 = R(S_1, S_2)\exp\{2ikC(z_2 - z_1)\}. \tag{11.101}$$

Then (11.99) is replaced by

$$F_R = \iint R_2 S_1 C^{-1} \exp[-ik\{S_1 x + C(2z_2 - z)\}]\,\mathrm{d}S_1\,\mathrm{d}S_2. \tag{11.102}$$

If the presence of the exponential factor in (11.101) is not known, and if it is assumed that R_2 is a slowly varying function of S_1, S_2, the method used for (11.99) would now give a wrong result. It would show a wave reaching the receiver with the wrong angle of incidence $\theta_2 = \arctan\{x/(2z_2 - z)\}$ instead of (11.100), it would use the reflection coefficient for $\theta = \theta_2$ instead of θ_0, and it would give the wrong phase for the received signal.

If the exponentials in (11.101) and (11.102) had been combined before finding the saddle point, the correct result (11.100) would have been obtained. It is now clear that the saddle point for (11.99) must be found by writing the integrand

$$S_1 C^{-1} \exp[-ik\{S_1 x + C(2z_1 - z)\} + \ln\{R(S_1, S_2)\}] \tag{11.103}$$

and using this exponent in (9.63). If R is a function only of θ so that it is independent of azimuth, it can be shown that $\partial(\ln R)/\partial S_2$ has a factor S_2 so that still $S_{20} = 0$ as in (11.100). But the equation for finding S_{10} is now

$$-ik\{Cx - (2z_1 - z)S_{10}\} + \frac{1}{R}\frac{\partial R}{\partial\theta} = 0. \tag{11.104}$$

This gives (11.100) only if

$$\left|\frac{1}{R}\frac{\partial R}{\partial\theta}\right| \ll |kCx| \approx |kS_1(2z_1 - z)| \tag{11.105}$$

and this is the condition that must be satisfied when the plane wave reflection coefficient is used for spherical waves. It is amply satisfied in many practical cases of radio reflection from the ionosphere or from the earth's surface. This applies even for radio waves of very low frequency (VLF) for which k is small. It is important that the reference level is correctly chosen so that R does not contain a hidden exponential factor like the one in (11.101)

In general R is complex and

$$\frac{1}{R}\frac{\partial R}{\partial \theta}=\frac{\partial}{\partial \theta}(\ln|R|)+\mathrm{i}\frac{\partial}{\partial \theta}(\arg R). \tag{11.106}$$

The second term is imaginary, and is large if the reference level has been wrongly chosen. Now (11.101) shows that

$$\frac{\partial}{\partial \theta}(\arg R_2)=\frac{\partial}{\partial \theta}(\arg R)-2kS(z_2-z_1) \tag{11.107}$$

so that a new reference level z_2 can always be found to make (11.107) zero. This rule would in general make the reference level z_2 depend on θ. In practical studies of reflection coefficients it is usual and simpler to use the same reference level for all θ. It is usually chosen so that (11.107) is zero somewhere within the range of θ that is of interest.

The first term of (11.106) is real so that the solution S_{10} of (11.104) is complex. This topic is discussed further in § 11.15. Very often, however, this term is ignored in (11.102) and S_{10} is taken as real.

The function $R(S_1, S_2)$ in (11.99) in general has poles and zeros and branch points. When a contributing saddle point is near to one of these, the simple form of the method of steepest descents cannot be used, and a more elaborate treatment is needed. Examples of these problems in radio propagation have been mentioned in § 9.6. They are outside the scope of this book.

When the reflection coefficient $R(S_1, S_2)$ in (11.99) is not independent of the azimuth angle, $\arctan(S_2/S_1)$, the integrals can still be evaluated by steepest descents, and it is found that the image transmitter is not at the geometrical image point but displaced from it laterally. This is the phenomenon of lateral deviation. Practical cases of it usually occur at high frequencies and are more easily handled by ray tracing methods. Examples have been given in §§ 10.13, 10.14 and figs. 10.12, 10.13.

11.15. Goos–Hänchen shifts for radio waves

In this section the formula (11.99) is used to give the signal reaching a receiver on the ground $z=0$, in the plane $y=0$ at distance x from the transmitter at the origin. It is assumed that the reflection coefficient R is a function only of the angle of incidence $\theta=\arccos C$, as would occur, for example, if the ionosphere were isotropic. To indicate this we write it $R(\theta)$. If it is so slowly varying that it can be set equal to its value at the saddle point and placed outside the integral, the factor that remains is an integral representing a spherical wave coming from a point source at the geometrical image point $x = y = 0$, $z = 2z_1$, of the transmitter in the reference plane $z = z_1$. The line from this image point to the receiver is then the wave normal of the predominant wave front at the receiver. It makes with the z axis an angle θ_0, where

$$\tan\theta_0 = x/2z_1. \tag{11.108}$$

The wave has, in effect, travelled a distance

$$r_0 = (x^2 + 4z_1^2)^{\frac{1}{2}} \tag{11.109}$$

from the image point. If the reflector at $z = z_1$ had been perfect, $R = 1$, the predominant wave at the receiver would contain a factor $\mathrm{e}^{-\mathrm{i}\varphi}$ where the phase φ is kr_0. If the factor R is allowed for, the complex phase is

$$\varphi_0 = kr_0 + \mathrm{i}\ln R(\theta_0) = k(x\sin\theta_0 + 2z_1\cos\theta_0) - \arg R(\theta_0) + \mathrm{i}\ln|R(\theta_0)|. \tag{11.110}$$

The real part of this now includes the real phase change $-\arg R(\theta_0)$ at reflection. The imaginary part gives an amplitude reduction factor $|R(\theta_0)|$.

These results are modified if the dependence of R on θ is allowed for. The integral (11.99) is evaluated by double steepest descents, §9.10, but the second derivatives $\partial^2/\partial S_1\partial S_2$ of the exponent are zero so the S_1 and S_2 integrations can be done independently. The double saddle point is where $S_2 = 0$, $\theta = \theta_g$ with

$$\tan\theta_g = \tan\theta_0 - \frac{\sec\theta_g}{2kz_1}\left(\frac{\partial(\arg R)}{\partial\theta} - \frac{\mathrm{i}}{|R|}\frac{\partial|R|}{\partial\theta}\right)_{\theta=\theta_g}. \tag{11.111}$$

This shows that the predominant wave normal at the receiver no longer has the direction θ_0 of the geometrical image point. There is an angular shift $\theta_g - \theta_0$. This is also the shift of the direction of the virtual source point beyond the reference plane at $z = z_1$. Equation (11.111) has more than one solution, so that there are more than one saddle point contributing to the integral (11.99), each corresponding to a different virtual image source. Usually it is only the contribution of greatest amplitude that is important, but cases can occur where there are two reflected waves of approximately equal amplitudes and different directions arriving at the receiver. For an example see the results at the end of this section. The following theory applies for only one of the solution θ_g of (11.111).

The phase φ_g of the wave reaching the receiver is now

$$\varphi_g = k(x\sin\theta_g + 2z_1\cos\theta_g) - \arg R(\theta_g) + \mathrm{i}\ln|R(\theta_g)| \tag{11.112}$$

A term $-\mathrm{i}\ln\{r_0\sin\theta_g/(r_g\sin\theta_0)\}$ should be added to (11.112) but it is very small and is neglected here. Thus there is a shift $\varphi_g - \varphi_0$ of the complex phase. The second derivatives of the exponent of (11.99) are

$$\partial^2/(\partial S_1)^2\colon \frac{2\mathrm{i}kz_1}{C^3} + \frac{1}{C}\frac{\partial}{\partial\theta}\left(\frac{1}{CR}\frac{\partial R}{\partial\theta}\right), \tag{11.113}$$

$$\partial^2/(\partial S_2)^2\colon \frac{2ikz}{C} + \frac{1}{S_1 CR}\frac{\partial R}{\partial \theta}, \tag{11.114}$$

where θ takes the value θ_g. Thus the formulae (9.64), (9.65) show that the double integral (11.99) has a factor $1/r_g$ where

$$r_g^2 = C^{-2}\left\{z_1 - \frac{iC^2}{2k}\frac{\partial}{\partial\theta}\left(\frac{1}{CR}\frac{\partial R}{\partial\theta}\right)\right\}\left\{z_1 - \frac{i}{2kS_1R}\frac{\partial R}{\partial\theta}\right\} \tag{11.115}$$

and the factor C^{-1} in (11.99) has been included. If R is independent of θ, this gives $r_g = z_1/C$ and since C is now $\cos\theta_0$ from (11.108), this is just r_0 (11.109). The wave front is spherical with its centre at the geometrical image point. But now the two factors of (11.115) show that the S_1 and S_2 integrations respectively give different ranges. The wave front does not come from a single virtual point source and is no longer exactly spherical. The wave normals of the reflected wave front form an astigmatic pencil of rays (Deschamps, 1972). But we shall regard the geometric mean r_g as a new effective range. Thus the angle dependence of R gives a range shift $r_g - r_0$.

Suppose now that there is some range of θ for which $|R| = 1$ when θ is real. This is a case of total reflection, and examples have been given in § 11.9, equation (11.63), (11.64). In optics it occurs when a wave in a transparent medium such as glass is reflected at a boundary where the medium beyond it has a smaller refractive index, and the angle of incidence exceeds the critical angle. It is then known as total internal reflection. In this case $\partial|R|/\partial\theta$ is zero in (11.111) and (11.115). Then θ_g and r_g are both real, and the angular shift $\theta_g - \theta_0$ and the range shift $r_g - r_0$ are real. These two shifts are known as the Goos–Hänchen shifts (Goos and Hänchen, 1947) of the optical image of a point source in a reflecting plane. They have been fully studied in optics. A very complete account has been given in four papers by Lotsch (1970).

For radio waves, the θ dependence of R is important in the study of reflections from the ionosphere, especially for very low frequencies (VLF). Now $\partial|R|/\partial\theta$ is not zero so the angular and range shifts given from (11.111), (11.115) are complex, and there is a shift $\varphi_g - \varphi_0$ of the complex phase, given by (11.110), (11.112). A complex θ_g means that the predominant wave at the receiver is an inhomogeneous plane wave; §2.15. A complex effective range r_g means that the wave front at the receiver has a complex curvature. The physical significance of a wave from a point source whose space coordinates are complex has been discussed by Deschamps (1972) and by Felsen and his co-workers (see, for example, Ra, Bertoni and Felsen, 1973).

In VLF studies measurements are often made of the signal amplitude at the ground for various distances from the transmitter. There are two main contributions to this signal, the direct or ground wave and the wave reflected from the ionosphere. Their relative phase depends on distance so that the curve of signal amplitude versus

Table 11.1

Horizontal Range x (km)	θ_0 (deg)	Complex angle shift $\theta_g - \theta_0$ (deg)	r_0 (km)	Complex range shift $r_g - r_0$ (km)	Complex phase shift $\varphi_g - \varphi_0$ (deg)	$\lvert r_0/r_g \rvert \exp\{\mathrm{Im}(\varphi_g)\}$
100	35.5	0.27 − 0.71i	172.0	− 2.2 + 4.8i	− 0.23 − 0.23i	0.392
200	55.0	1.01 − 1.85i	244.1	− 17.6 + 21.3i	− 1.20 − 2.46i	0.261
		13.25 − 3.88i		− 4.1 − 405.5i	− 63.03 + 72.69i	0.117
250	60.8	2.13 − 3.62i	286.5	− 77.4 + 86.2i	− 0.68 + 0.23i	0.212
		6.40 − 2.81i		− 109.6 − 199.0i	− 0.06 + 0.41i	0.169
260	61.7	2.08 − 4.54i	295.3	− 75.1 + 142.2i	0.23 + 0.28i	0.171
		5.51 − 1.27i		− 77.8 − 146.1i	5.27 + 0.21i	0.168
270	62.6	5.02 − 0.53i	304.1	− 46.0 − 123.1i	9.11 + 0.09i	0.155
		1.68 − 5.29i		− 44.8 + 186.1i	− 0.40 + 0.28i	0.127
300	65.0	4.11 + 0.68i	333.1	2.2 − 92.2i	15.12 − 3.84i	0.143
		0.20 − 6.55i		64.1 + 249.1i	− 5.9 − 141.2i	0.060
400	70.7	2.34 + 1.72i	423.8	39.5 − 40.0i	6.47 + 11.3i	0.175

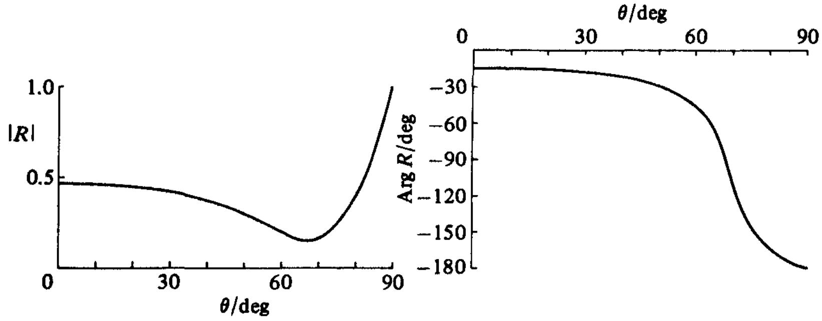
distance shows an interference effect. It is called a 'Hollingworth interference pattern' since Hollingworth (1926) was one of the first to use this method. Much work of this kind has been done for frequencies of 16 kHz and near (see references in § 19.7). Conversely, for a given model of the ionosphere when $R(\theta)$ is known, it is required to calculate the Hollingworth interference pattern, so that it can be compared with the observed pattern.

It is important to check whether the complex Goos–Hänchen shifts are likely to be important in these calculations. Equations (11.111), (11.115) show that the terms that give these shifts are inversely proportional to k, that is to frequency, so that as the frequency gets smaller, the shifts get more important.

To test this, some calculations were made with $R(\theta)$ for a sharply bounded isotropic medium, given by R_{11} in (11.55) with $n_1 = 1$. The function $R(\theta)$ used is shown in fig. 11.1 and is of similar form to the calculated reflection coefficient for model ionospheres (Budden, 1955b; Deeks, 1966a, b; Pitteway, 1965). This example has, however, been deliberately chosen to have a pronounced minimum of $|R|$ and an associated steep gradient of $\arg R$, so as to illustrate the double image point effect. The reflector was assumed to be at height $z_1 = 70$ km and this was used as the reference level. The receiver on the ground was assumed to be at various distances from the transmitter, and the earth's curvature was neglected. Table 11.1 shows some examples of the calculated shifts for a frequency of 16 kHz. The angle θ_0 and the range r_0 are given by (11.108), (11.109), and the phase angles φ_0, φ_g are given by (11.110), (11.112). The signal amplitude is determined mainly by $|r_0/r_g|\exp\{\mathrm{Im}(\varphi_g)\}$ which appears in the last column.

The table shows that, at distances near 260 km from the transmitter, there are two

Fig. 11.1. Dependence on angle of incidence θ of the modulus and argument of the reflection coefficient R, as given by (11.55), for a sharply bounded medium with refractive index $n = 2.39 - 0.84i$. The electric fields of the waves are in the plane of incidence. This example is used in the adjacent table to study the Goos–Hänchen shifts.



contributing reflected waves with comparable amplitudes. For a function $R(\theta)$ with a less pronounced minimum, however, this effect is absent. These and other similar calculations show that, for 16 kHz, the shifts are small compared with errors of measurement and with uncertainties that arise from variability of the ionosphere. The phase shift of 15° at 300 km, shown in the table, might just be large enough to be important.

11.16. The shape of a pulse of radio waves

In the earlier sections of this chapter it has been assumed that the transmitter emits a continuous wave of constant amplitude and fixed frequency. If the transmitted wave is modulated so that, for example, it consists of pulses, then the emitted signal is a combination of many frequencies, and to find the signal reflected from the ionosphere it is necessary to allow for the frequency dependence of the reflection coefficient. Pulsed radio signals, called wave packets, were studied in §§ 5.8, 10.3 and represented by Fourier integrals (5.62), (10.8). But the exact form of the pulse played no part. It was simply assumed that the pulse had a constant frequency f_1 and roughly constant amplitude, and that it lasted for a time T long compared with $1/f_1$. The spectrum function $M(\omega)$ of (5.62) with $\omega = 2\pi f$, was then large near $f = f_1$ and $f = -f_1$, and became very small at other frequencies. It is now of interest to investigate how the form of a pulse of radio waves is modified by a reflection from the ionosphere. It is convenient to use the actual frequency f instead of the angular frequency ω, or the wave number k.

The signal from the transmitter is to be Fourier analysed into its component frequencies. Each frequency contributes an angular spectrum of plane waves, such as (10.1) or (11.98). After reflection from the ionosphere, the signal reaching the receiver is a predominant wave such as (10.4). If the corresponding wave in the emitted angular spectrum has an electric field proportional to $\exp(2\pi i f t)$, the field of the received signal from this wave is proportional to $\exp\{2\pi i f(t - P/c)\}$ where P is the phase path, defined in § 10.16. Here $2\pi f P/c$ is the total change of phase of the signal as it travels from transmitter to receiver. It may be given by (10.39) for a ray path, or it may be the sum of contributions from the paths up to and down from the ionosphere and the phase change on reflection. See for example the exponent in (11.103). In the examples given later in this section, figs. 11.2, 11.3, $P(f)$ is a real function and is best regarded as arising from a ray path as in (10.39). But in general $P(f)$ has an imaginary part arising from attenuation or because $|R| < 1$.

Let the electric field of the predominant wave in the pulse emitted by the transmitter be

$$E(t) = m(t)\cos(2\pi f_1 t) \tag{11.116}$$

where t is time and $m(t)$ is appreciable only in a range $|t| < T$, and varies very slowly compared with $\cos(2\pi f_1 t)$. If this were delineated on the screen of a cathode ray oscillograph with a linear time base, it would appear as a cosine wave $\cos(2\pi f_1 t)$, the carrier (radio) wave, modulated by the function $m(t)$. This function is therefore called the 'shape' of the pulse, and describes how the envelope of the pulse varies with time.

The function $m(t)$ and its Fourier transform $M(f)$ are related thus

$$m(t) = \int_{-\infty}^{\infty} M(f)\exp(2\pi\mathrm{i}ft)\,\mathrm{d}f, \quad M(f) = \int_{-\infty}^{\infty} m(t)\exp(-2\pi\mathrm{i}ft)\,\mathrm{d}t \tag{11.117}$$

and $M(f)$ is appreciable only where $|f|$ is very small compared with the carrier frequency f_1. Since $m(t)$ is necessarily real, $|M(f)|$ is a symmetric function of f, and $\arg M(f)$ is antisymmetric. Then a well-known theorem in Fourier analysis shows that

$$E(t) = \mathrm{Re}\int_{-\infty}^{\infty} M(f-f_1)\exp(2\pi\mathrm{i}ft)\,\mathrm{d}f. \tag{11.118}$$

This signal is emitted from the transmitter. The symbol Re is usually omitted but will be retained in this section. Hence the signal reaching the receiver is

$$E_{\mathrm{r}}(t) = \mathrm{Re}\int_{-\infty}^{\infty} M(f-f_1)\exp\{2\pi\mathrm{i}f(t-P/c)\}\,\mathrm{d}f. \tag{11.119}$$

Now $M(f-f_1)$ is only appreciable when $|f-f_1|$ is small. This suggests that the term $P(f)$ in (11.119) should be expanded in a Taylor series about $f=f_1$. Let

$$f - f_1 = \sigma. \tag{11.120}$$

Then, with (10.40),

$$fP(f) = f_1 P(f_1) + \sigma P'(f_1) + \tfrac{1}{2}\sigma^2 P'_1(f_1) + \tfrac{1}{6}\sigma^3 P'_2(f_1) + \cdots \tag{11.121}$$

where P'_1, P'_2, are written for $\partial P'/\partial f, \partial^2 P'/\partial f^2$ respectively when $f=f_1$.

It was shown in § 10.16 that when P' is real, the time for the pulse to travel from transmitter to receiver is P'/c where P' is the equivalent path, given by (10.40). In general P' is complex, so let

$$t = \tau + P'(f_1)/c, \quad \tau = u + \mathrm{i}v \tag{11.122}$$

where $u = \mathrm{Re}(\tau)$ is a new measure of time.

The integral (11.119) is now

$$\begin{aligned} E_{\mathrm{r}}(t) &= \mathrm{Re}\exp[2\pi\mathrm{i}f_1\{t - P(f_1)/c\}] \\ &\quad\times \int_{-\infty}^{\infty} M(\sigma)\exp\{2\pi\mathrm{i}(\sigma\tau - \tfrac{1}{2}\sigma^2 P'_1/c - \tfrac{1}{6}\sigma^3 P'_2/c \ldots)\}\,\mathrm{d}\sigma. \end{aligned} \tag{11.123}$$

The first exponential is the high frequency oscillation. If $P(f_1)$ has an imaginary part this gives an amplitude reduction factor $\exp[2\pi f_1\{\mathrm{Im}\,P(f_1)\}/c]$. The integral is the function that modulates the high frequency and therefore gives the pulse shape. Clearly if P'_1, P'_2 and higher derivatives are negligible, and if P' is real, the integral is just $m(\tau)$, from (11.117), that is the original pulse, undistorted but delayed by a time P'/c.

Two special cases are now of interest, both for real P'. The first is when the carrier frequency f_1 is so chosen that the curve of $P'(f)$ has a positive slope and only small curvature. There is an example in fig. 12.5(b) at a frequency f_1; this figure gives $h'(f)$ for vertical incidence, and $P' = 2h'$. Then P'_1 is positive and P'_2 is small. If the original pulse is wide enough, $M(\tau)$ falls quickly to zero as σ increases from zero, and we therefore neglect the σ^3 term in (11.123). Then the pulse shape is given by

$$\int_{-\infty}^{\infty} M(\sigma)\exp(-\pi \mathrm{i} P'_1\sigma^2/c)\exp(2\pi\mathrm{i}\sigma\tau)\,\mathrm{d}\sigma. \tag{11.124}$$

This can be transformed by the convolution theorem of Fourier analysis. The Fourier transform of the first exponential in (11.124) is

$$(P'_1/c)^{-\frac{1}{2}}\exp(-\tfrac{1}{4}\pi\mathrm{i})\exp(\pi\mathrm{i}ct^2/P'_1) \tag{11.125}$$

and hence (11.124) becomes, apart from a constant factor

$$\int_{-\infty}^{\infty} m(\tau - t)\exp(\pi\mathrm{i}ct^2/P'_1)\,\mathrm{d}t. \tag{11.126}$$

Integrals of this type are used in optics in finding the Fresnel diffraction pattern of an aperture. Suppose that a parallel beam of monochromatic light of wavelength λ is incident on a slit which allows light to pass with amplitude $m(x)$ where x is measured at right angles to the length of the slit. A screen is placed at a distance y from the slit, and z is distance measured parallel to x in the plane of the screen. Then the amplitude of light arriving at the screen is proportional to

$$\int_{-\infty}^{\infty} m(z-\xi)\exp(2\pi\mathrm{i}\xi^2/\lambda y)\,\mathrm{d}\xi \tag{11.127}$$

and this is called the Fresnel diffraction pattern of the slit. Hence we may say that when a pulse of radio waves is reflected from the ionosphere, in conditions where the $P'(f)$ curve has a finite slope and small curvature, the pulse shape is modified to that of its own Fresnel diffraction pattern. Fig. 11.2 shows this modification for a pulse which is initially rectangular. The distorted pulses show oscillations whose scale in time depends on the value of P'_1. Further examples are given by Rydbeck (1942b).

The second special case is when the $P'(f)$ curve has zero slope, but appreciable curvature, so that $P'(f)$ is a maximum or a minimum, for example, at a frequency F_N or f_2 in fig. 12.5(b). Then the pulse shape is given by

$$\int_{-\infty}^{\infty} M(\sigma)\exp(-\tfrac{1}{3}\pi\mathrm{i}P'_2\sigma^3/c)\exp(2\pi\mathrm{i}\sigma\tau)\,\mathrm{d}\sigma. \tag{11.128}$$

The Fourier transform of the exponential is

$$\int_{-\infty}^{\infty} \exp\{\pi\mathrm{i}(2\sigma t - \tfrac{1}{3}P'_2\sigma^3/c)\}\,\mathrm{d}\sigma = (2\pi^2 c/P'_2)^{\frac{1}{3}}\mathrm{Ai}\{-2t(c\pi^2/P'_2)^{\frac{1}{3}}\} \tag{11.129}$$

from (8.18) where Ai is the Airy integral function. The convolution theorem may now be used as before and gives for (11.128) (apart from a constant factor)

$$\int_{-\infty}^{\infty} m(\tau - t)\,\mathrm{Ai}\{-2t(c\pi^2/P_2')^{\frac{1}{3}}\}\,\mathrm{d}t. \tag{11.130}$$

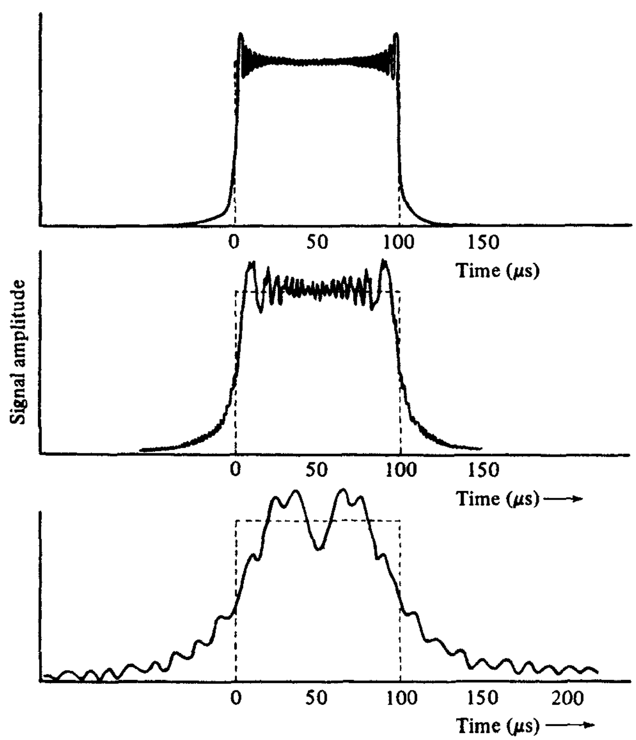
This expression has no simple interpretation like the Fresnel diffraction pattern used in the preceding example. Suppose that the original pulse is rectangular, that is $m(t)=1$ for $0<t<T$, and $m(t)=0$ for all other values of t. Then (11.130) becomes

$$\int_{\tau-T}^{\tau} \mathrm{Ai}\{-2t(c\pi^2/P_2')^{\frac{1}{3}}\}\,\mathrm{d}t. \tag{11.131}$$

Fig. 11.3 shows an example of this function. (Another example is given by Rydbeck, 1942b). If P_2' is positive, as at $f=f_2$ in fig. 12.5(b), then the curve is as shown. If P_2' is negative as at $f=F_N$ in fig. 12.5(b), then the curve of fig. 11.3 is reversed in time and the 'tail' becomes a precursor. This occurs because the 'side-band' frequencies arrive before the predominant frequency.

Suppose now that P and P' are complex. Equations (11.119)–(11.126) are still true but τ is complex. The imaginary part of P simply gives attenuation of the pulse as

Fig. 11.2. Distortion of a rectangular pulse after reflection when the $P'(f)$ curve has a constant slope. The original pulse is shown by broken lines. The values of $\mathrm{d}P'/\mathrm{d}f$ in km MHz^{-1} for the three diagrams are: top 4, middle 40, bottom 400.



shown by the first factor of (11.123). If P'_1 and P'_2 are small enough to be neglected, (11.124) shows that the received pulse shape is $m(\tau) = m(u + \mathrm{i}v)$ from (11.117). This requires that we interpret the original pulse shape $m(t)$ for complex time t.

Suppose that $m(t) = \exp(-\gamma t^2)$. Then

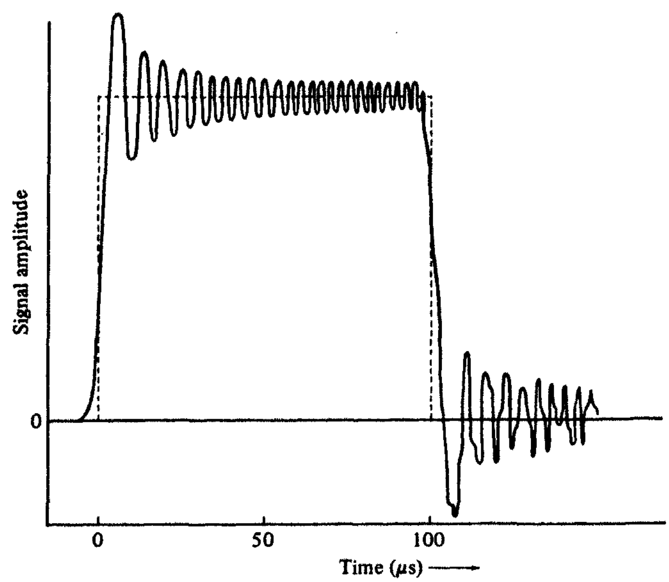
$$m(\tau) = \exp(-\gamma u^2)\exp(\gamma v^2)\exp(-2\mathrm{i}\gamma v u) \qquad (11.132)$$

Here u, given by (11.122), is the new measure of time. The received pulse has maximum amplitude when $u = 0$, that is $t = \mathrm{Re}\{P'(f_1)\}/c$. Thus there is a time delay $\mathrm{Re}(P')/c$. The last factor of (11.132) gives a phase change that is equivalent to a shift of the high frequency carrier to $f_1 - \gamma\,\mathrm{Im}\{P'(f_1)\}/\pi c$. The factor $\exp(\gamma v^2)$ gives an increase of amplitude which must partially compensate the decrease caused by $\mathrm{Im}(P)$.

If $m(t)$ is a rectangular pulse or some similar function with discontinuities, it is not simple to interpret $m(\tau)$ for complex τ. One method now is to use some analytic approximation for $m(t)$. It is then found that $\mathrm{Im}(\tau)$ gives rise to a distortion of the pulse and a frequency shift, but the centre of a symmetric pulse still arrives when $u = 0$, so that the time delay is still $\mathrm{Re}(P')/c$.

If P'_2 is negligible and P'_1 is constant as in (11.124), the result (11.126) for the received pulse shape $m_r(u)$ can still be used with complex τ. By changing the variable

Fig. 11.3. Distortion of a rectangular pulse after reflection when the $P'(f)$ curve has a minimum and $\mathrm{d}^2P'/\mathrm{d}f^2 = 2.34 \times 10^{-11}\,\mathrm{km\,s^2} = 23.4\,\mathrm{km\,MHz^{-2}}$. The original pulse is shown by broken lines.



of integration and using (11.122) it can be written

$$m_r(u) = \int_{-\infty}^{\infty} m(s) \exp\{\pi i c(s - u - iv)^2 / P'_1\} \, ds. \tag{11.133}$$

Suppose that $m(t)$ is a rectangular pulse

$$m(t) = 1 \text{ for } -\tfrac{1}{2}T \leqslant t \leqslant \tfrac{1}{2}T \tag{11.134}$$

and zero outside this range. Then (11.133) becomes, apart from a constant multiplier,

$$m_r(u) = \int_A^B \exp(-x^2) \, dx, \quad A = \exp(-\tfrac{1}{4}i\pi)(\pi/P'_1)^{\frac{1}{2}}(-\tfrac{1}{2}T - iv - u),$$
$$B = \exp(-\tfrac{1}{4}i\pi)(\pi/P'_1)^{\frac{1}{2}}(\tfrac{1}{2}T - iv - u). \tag{11.135}$$

This integral can be expressed in terms of the error function for a complex argument, and there are published tables of this function.

This discussion has shown that there is no simple physical interpretation for a complex P', but that the time of travel of a wave packet is found from $\mathrm{Re}(P')$. This can be applied to the time of travel T of a wave packet along a ray path of length L in a homogeneous medium. Then $P' = cL/\mathscr{U}$ where $\mathscr{U}$ is the complex group velocity given by (5.68), and

$$T = \mathrm{Re}(P')/c = \mathrm{Re}(n' \cos\alpha)L/c. \tag{11.136}$$

The effective velocity of a wave packet in an attenuating medium is $c/\mathrm{Re}(n' \cos\alpha)$. For further discussion of this topic see Suchy (1972a, b, 1974a, § vii).

PROBLEMS 11

11.1. It is observed that a wave reflected from the ionosphere has the same polarisation no matter what the polarisation of the incident wave. Show that the reflection coefficient matrix $\boldsymbol{R}$ (§ 11.5) is singular, i.e. det $\boldsymbol{R} = 0$, and find the polarisation of the reflected wave in terms of the elements of $\boldsymbol{R}$. Show that for one polarisation of the incident wave there is no reflection.

11.2. The surface of the sea may be regarded as a perfect conductor, so that the horizontal components of the total electric field close to the surface are zero. Show that for all angles of incidence the reflection coefficient matrix at the surface as reference level, is

$$\boldsymbol{R} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

What is the value of $\boldsymbol{R}_0$ (§11.6)?

11.3. The polarisation of the incident wave is to be adjusted so that the reflected wave has the same polarisation. Show that in general this can be done in two different ways. What condition does $\boldsymbol{R}$ satisfy when there is only one way?

11.4. State Maxwell's electromagnetic equations in vector form for free space. Express them in terms of the components E_x, E_y, E_z, H_x, H_y, H_z of the electric and magnetic fields respectively in a right-handed system of Cartesian coordinates x, y, z, for a plane electromagnetic wave travelling with its normal parallel to the z axis.

A plane wave in free space is moving from left to right along the z axis and is incident on the plane surface $z = a$, where a is positive, and is partially reflected there. In the plane $z = 0$ the components E_x, E_y, H_x, H_y of the total field are measured and found to have the amplitudes and phases given by the following table:

	Amplitude	Phase
E_x	13	arctan (5/12)
E_y	5	0
Z_0H_x	5	0
Z_0H_y	13	− arctan (5/12)

Here Z_0 is the characteristic impedance of free space. Show that the incident wave is linearly polarised, and find its amplitude and the plane of its electric field. Find also the amplitude and polarisation of the reflected wave, and hence deduce the reflection coefficient.
(From Natural Sciences Tripos, 1955. Part II. Physics: Theoretical Option.)

11.5. A plane electromagnetic wave in free space is normally incident on the plane boundary of a homogeneous loss-free medium with magnetic permeability $\mu = 2$ and electric permittivity $\varepsilon = \frac{1}{2}$. Find the reflection coefficient.
(From Natural Sciences Tripos, 1955 Part II. Physics).

11.6. A piece of glass is partially silvered. Find the reflection coefficient for electromagnetic waves incident normally on the surface if the thickness of the silver is negligible compared with one wavelength. Assume that the resistivity of silver is the same at high frequencies as for steady currents. Find how thick the silver must be to make the (intensity) reflection coefficient equal to 0.8. (Refractive index of glass = 1.5. Resistivity of silver = 1.5×10^{-8} ohm m.)
(From Natural Sciences Tripos, 1953 Part II. Physics).

11.7. Find an expression for the reflection coefficient of a sharply bounded homogeneous ionosphere for propagation from (magnetic) east to west or west to east at the magnetic equator, when the electric vector is in the plane of incidence. Show that the expression is not reciprocal, i.e. it is changed when the sign of the angle of incidence is reversed.
(For the solution see Barber and Crombie, 1959).

11.8. As a rough simple model, an ionospheric layer can be pictured as a homogeneous isotropic plasma of refractive index n with two horizontal boundaries a distance d apart. Show that for angle of incidence θ the reflection and transmission coefficients R_{11}, T_{11} are given by

$$R_{11} = \sin(knd\cos\phi)/\sin(\beta - knd\cos\phi)$$

$$T_{11} = \sin\beta\exp(ikd\cos\phi)/\sin(\beta - knd\cos\phi)$$

where $\sin\theta = n\sin\phi$ and $\tan\frac{1}{2}\beta = in\cos\theta/\cos\phi$. The lower boundary is used as the reference level for R_{11}. Show that, if the plasma is loss-free so that n^2 is real, $|R_{11}|^2 + |T_{11}|^2 = 1$.
(For the solution see Budden, 1961a, § 8.10).

11.9. In problem 11.8 for normal incidence $\theta = 0$, show that

$$R_{11} = (n^2 - 1)\tan(knd)/\{(n^2 + 1)\tan(knd) - 2in\}$$

$$T_{11} = -2in\sec(knd)\exp(ikd)/\{(n^2 + 1)\tan(knd) - 2in\}.$$

Suppose that the medium is a collisionless electron plasma so that $n^2 = 1 - \omega_N^2/\omega^2$. Sketch curves to show how $|R_{11}|$ depends on frequency ω, especially when ω is near to ω_N. Show that when $\omega = \omega_N$, $|R_{11}|^2 = (kd)^2/\{4 + (kd)^2\}$, $|T_{11}|^2 = 4/\{4 + (kd)^2\}$.
(For curves giving results see Budden, 1961a, § 8.11.)

11.10. A plane surface has reflection coefficient $R = 1$ for all angles of incidence θ. It is placed at a height $z = z_0$ and there is a transmitter at the origin of coordinates. In calculating the properties of the reflected wave a reference level $z = z_1 < z_0$ is used where $z_0 - z_1$ is small. Thus the effective reflection coefficient is $R_1 = \exp\{2ik(z_1 - z_0)\cos\theta\}$ (see(11.10)). The geometrical image point A of the transmitter in the reference plane is at $x = y = 0$, $z = 2z_1$. Find the Goos–Hänchen shifts of the true image source point from A, in angle and range. Hence show that the true image source is at $x = y = 0$, $z = 2z_0$, that is at the geometrical image point in the actual reflector at $z = z_0$.