In [14]:

```
# Set up environment with correct dependencies
using Pkg
Pkg.activate(".")
Pkg.instantiate()
```

```
 Activating environment at `~/GitHub/MathSys/teaching/MA934/MA934-slid
es/Project.toml`
```

In [15]:

```
using Plots
using LaTeXStrings
pyplot()
# Set default fonts for all plots
fnt = Plots.font("DejaVu Sans", 8.0)
default(titlefont=fnt, guidefont=fnt, tickfont=fnt, legendfont=fnt)
```

MA934

Recursive functions and sorting algorithms

How your choice of algorithm can really make a difference

Iteration vs recursion

- An iterative function is one that loops to repeat some part of the code.
- A recursive function is one that calls itself again to repeat the code.

Recursive functions are a natural framework for implementing divide and conquer algorithms.

Anatomy of recursive functions

Every recursive function consists of:

- one or more **recursive cases**: inputs for which the function calls itself
- one or more **base cases**: inputs for which the function returns a (usually simple) value.

In [16]:

```
function f(n)
    if n==1
        return 1
    else
        return n*f(n-1)
    end
end
```

Out[16]:

```
f (generic function with 1 m
ethod)
```

In [17]:

```
print([f(n) for n in 1:10])
```

```
[1, 2, 6, 24, 120, 720, 504
0, 40320, 362880, 3628800]
```

Recursive function calls incur additional computational overheads.

Overheads: call stack and recursion depth

$$
\begin{aligned}
f(4) &= 4 * f(3) \\
&= 4 * (3 * (f(2)) \\
&= 4 * (3 * (2 * f(1)) \\
&= 4 * (3 * (2 * (1 * f(0)))) \\
&= 4 * (3 * (2 * (1 * 1))) \\
&= 4 * (3 * (2 * 1)) \\
&= 4 * (3 * 2) \\
&= 4 * 6 = 24.
\end{aligned}
$$

- Variables and information associated with each call stored on the **call stack** until base case is reached.
- **Recursion depth**: maximum size of the call stack.
- Infinite (or excessive) recursion depth leads to **stack overflow**.

Example : iterative calculation of the Fibonacci sequence

The Fibonacci numbers are defined by the recursion:

$$F_n = F_{n-1} + F_{n-2}$$

with $F_1 = 0$, $F_2 = 1$.

Obvious approach by iteration:

In [18]:

```
function Fib1(n)
    if n==1 || n ==2
        return n-1
    else
    a = zeros(Int64,n)
    a[1] = 0; a[2] = 1
    for i in 3:n
        a[i] = a[i-1] + a[i-2]
    end
    return a[n]
    end
end

```

Out[18]:

```
Fib1 (generic function with
1 method)
```

In [19]:

```
print(Fib1.(1:10))
```

```
[0, 1, 1, 2, 3, 5, 8, 13, 2
1, 34]
```

Example : recursive calculation of the Fibonacci sequence

This can also by done recursively:

In [20]:

```
function Fib2(n)
    if n == 1 || n == 2
        return n-1
    else
        return Fib2(n-1) + Fib2(n-2)
    end
end
```

Out[20]:

```
Fib2 (generic function with 1 method)
```

In [21]:

```
print(Fib2.(1:10))
```

```
[0, 1, 1, 2, 3, 5, 8, 13, 21, 34]
```

Aside : memoization

Memoization is a technique that uses a lookup table to "remember" the values returned by a function for previously evaluated inputs. Avoids repeated evaluations with the same input.

Here is another Fibonacci function that combines memoization with recursion:

In [22]:

```
memo = Dict()
memo[1] = 0
memo[2] = 1

function Fib3(n)
    if !(n in keys(memo))
        memo[n]=Fib3(n-1)+Fib3(n-2)
    end
    return memo[n]
end

```

Out[22]:

```
Fib3 (generic function with
1 method)
```

In [23]:

```
print(Fib3.(1:10))
```

```
[0, 1, 1, 2, 3, 5, 8, 13, 2
1, 34]
```

> Take home : there are often lots of ways of doing the same thing.
> Now let's look at something less trivial.

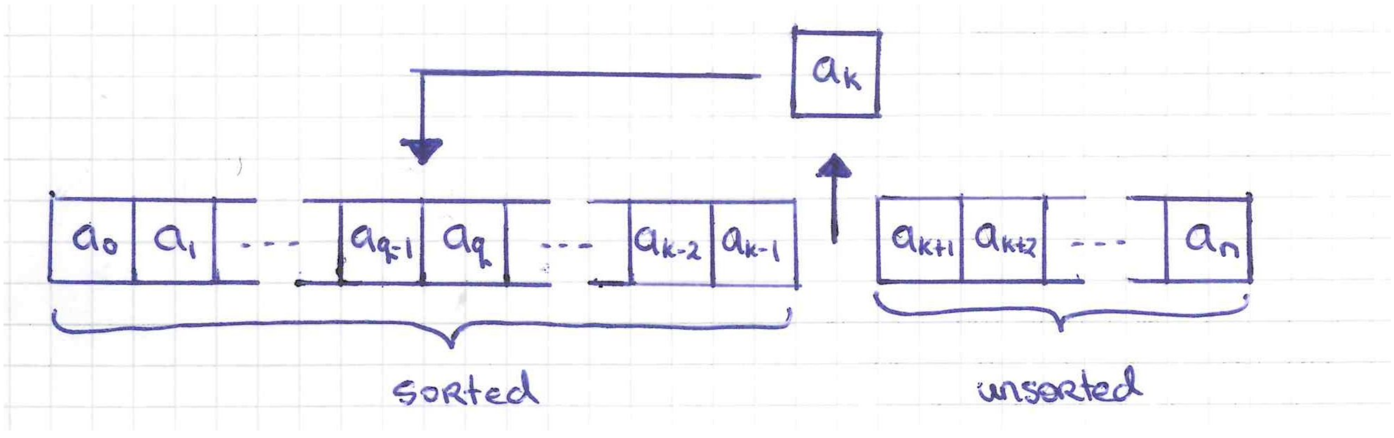
Sorting

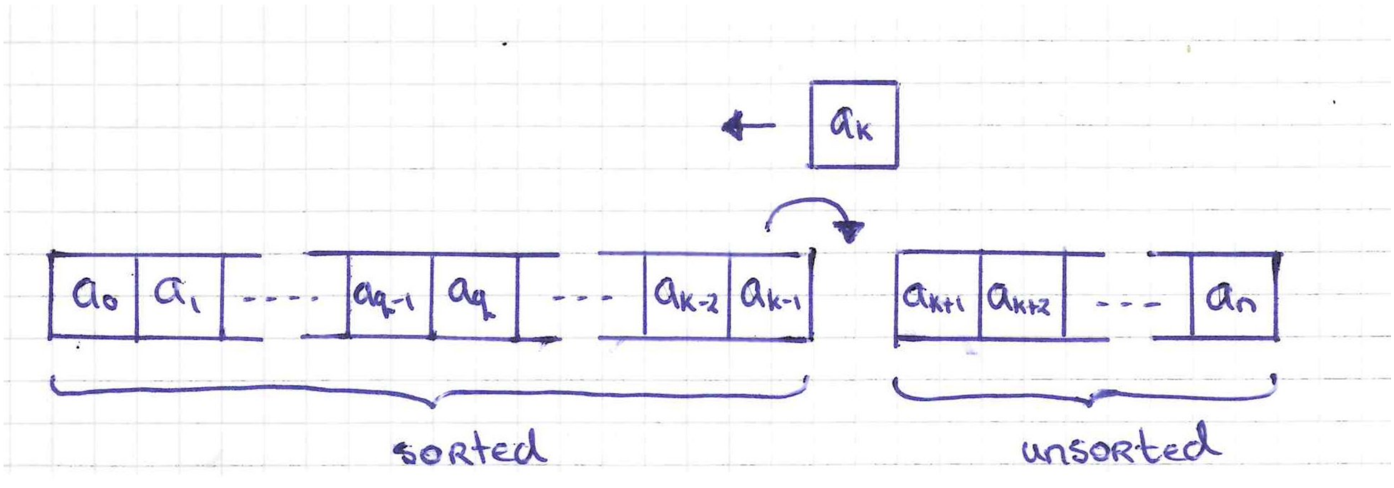
Sorting is the task of placing an unordered list of integers in order with as few comparisons as possible.

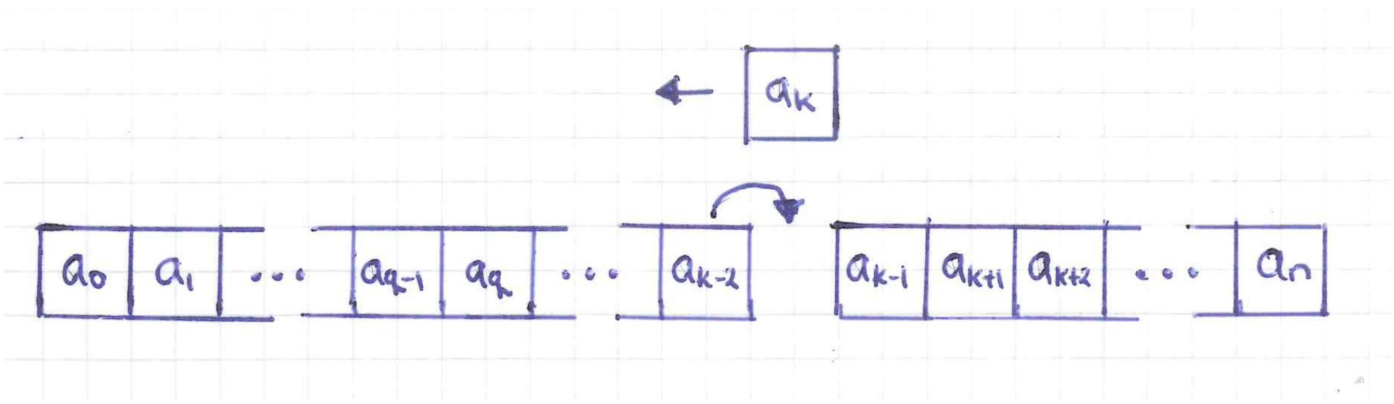
There are **lots** of ways of doing this.

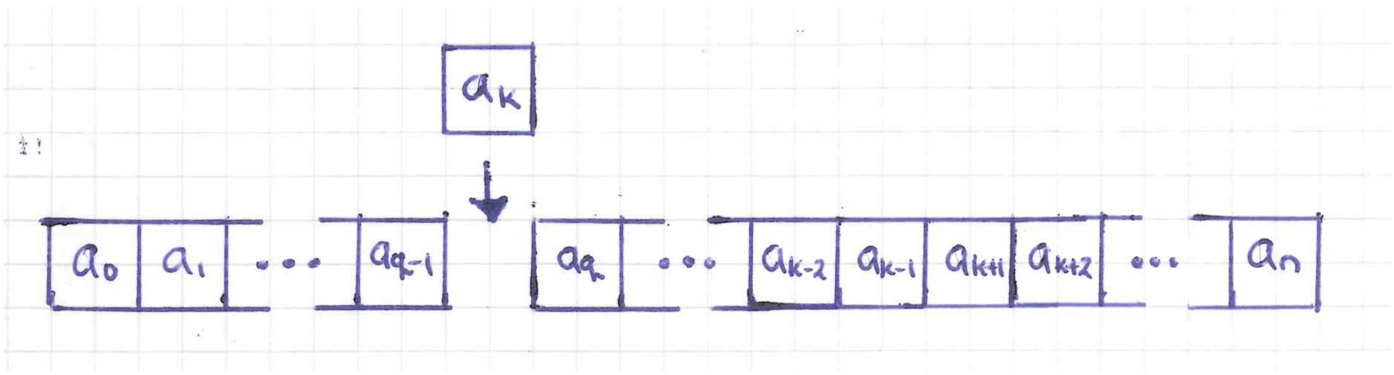
Insertion sort - an iterative sort

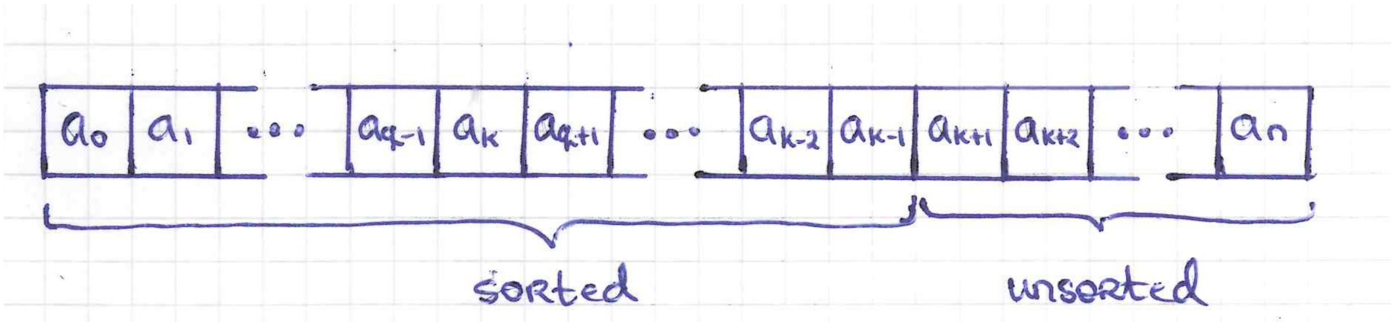
Insertion sort: step through each item in turn, placing it in the appropriate location among the previously examined items:











Computational complexity of insertion sort

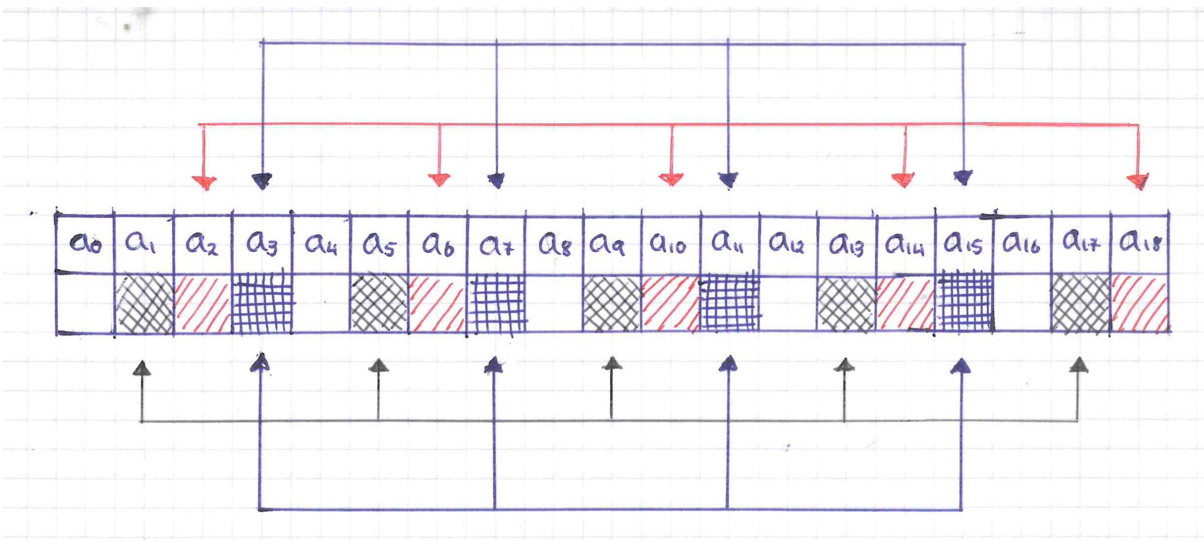
Consider sorting an array of length n.

- **Best case**: if input array is already in order? n comparisons.
- **Worst case**: if input array is in reverse order? $\frac{1}{2}\, n\,(n+1)$ comparisons. Why? Computational complexity of insertion sort is therefore $\mathcal{O}(n^2)$.

Typical case $\sim\, n^2$. Can we do better?

Partial sorts

A **partial q-sort** of a list of numbers is an ordering in which all subsequences with stride q are sorted.



A trivial modification of insertion sort does partial q-sorts

ShellSort - improving on insertion sort

- ShellSort: do a succession of partial q-sorts, with q taken from a pre-specified list, Q.
- Start from a large increment and finish with increment 1, which produces a fully sorted list.
- Performance depends on Q but generally faster than insertion sort.

Example. $Q = \left\{2^i\, :\, i = i_{max}, i_{max} - 1, \ldots, 2, 1, 0\right\}$ where i_{max} is the largest i with $2^i\; <\; \frac{n}{2}$. Typical case $\sim\, n^{\frac{3}{2}}$ (although worst case still n^2 .).

ShellSort - improving on insertion sort

- Surprising (at first) that ShellSort beats insertion sort since the last pass is a full insertion sort. Why is this?
- A better choice of increments is $Q = \left\{ \frac{1}{2}(3^i - 1) : i = i_{max}, i_{max} - 1, \ldots, 2, 1 \right\}$. This gives typical case $\sim n^{\frac{5}{4}}$ and worst case $\sim n^{\frac{3}{2}}$.
- General understanding of the computational complexity of ShellSort is an open problem.

Mergesort - a recursive sort

- divide-and-conquer sorting strategy invented by Von Neumann.
- Mergesort interlaces two **sorted** arrays into a larger sorted array.
- Given the interlace() function, mergesort is very simple:

```
def mergeSort(A):
   n=len(A)
   if n == 1:
      return A  # an array of length 1 is already sorted
   else: m=n/2
      return interlace(mergeSort(A[0:m]),
                       mergeSort(A[m:n]))
```

Mergesort : the interlace() function

In [24]:

```
function interlace(A::Array{Int64,1}, B::Array{Int64,1})
      if length(A) == 0
        return B
      elseif length(B) == 0
        return A
      elseif A[1] < B[1]
        return vcat([A[1]], interlace(A[2:end], B))
      else
        return vcat([B[1]], interlace(A, B[2:end]))
      end
    end
```

Out[24]:

```
interlace (generic function with 1 method)
```

In [25]:

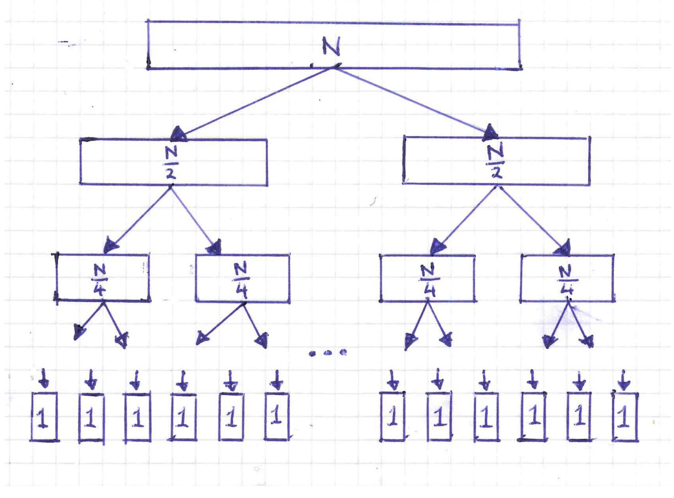
```
print(interlace([1,3,5],[2,4,6,8]))
```

```
[1, 2, 3, 4, 5, 6, 8]
```

Complexity of Mergesort

The `interlace()` function can be shown to be $\mathcal{O}(n)$ where n is the size of the output array.
At level k, there are 2^{k-1} `interlace()` calls of size $\frac{n}{2^{k-1}}$.

Therefore, each level is $\mathcal{O}(n)$.
Number of levels, L, satisfies $n = 2^L$ so
$L = \log_2 n$.



Heuristically, expect
$$F(n) = \mathcal{O}(n \ \log_2 n)$$

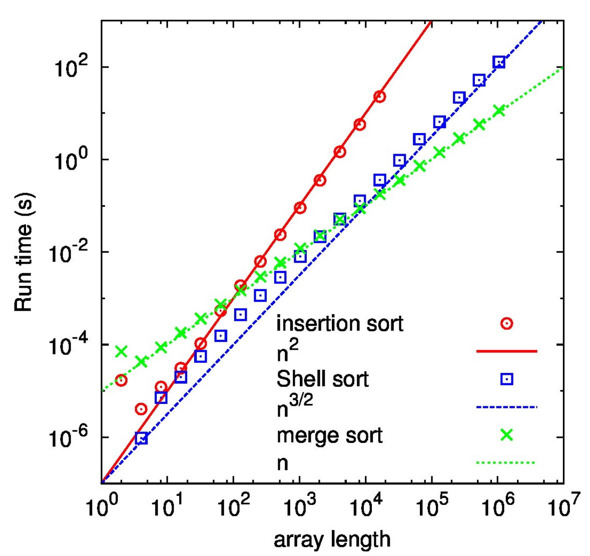
Complexity of Mergesort

We can also write a recursion equation for $F(n)$ based on the function defnition:

$$F(n) = 2 \ F(\frac{n}{2}) + n^1$$

with $F(1) = 1$.

This is the "Master theorem" case 2 so $\mathcal{O}(n \log_2 n)$.



In [ ]:

```
1
```

In [ ]:

```
1
```