In [1]:

```
# Set up environment with correct dependencies
using Pkg
Pkg.activate(".")
Pkg.instantiate()
```

```
 Activating environment at `~/GitHub/MathSys/teaching/MA934/MA934-slid
es/Project.toml`
```

In [2]:

```
using Plots
using LaTeXStrings
pyplot()
# Set default fonts for all plots
fnt = Plots.font("DejaVu Sans", 8.0)
default(titlefont=fnt, guidefont=fnt, tickfont=fnt, legendfont=fnt)
```

MA934

Floating point arithmetic

How computers approximate arithmetic

Representation of unsigned integers

Binary representation of (3 bit) integers:

$$b_2 b_1 b_0 = b_2\, 2^2 + b_1\, 2^1 + b_0\, 2^0$$

Finite maximum and minimum integers. Modern computers use 64 bits but Julia provides smaller (and larger) integer types.

Here are the unsigned integer types:

Type	Bits	Minimum	Maximum
UInt8	8	0	$2^8 - 1$
UInt16	16	0	$2^16 - 1$
UInt32	32	0	$2^{32} - 1$
UInt64	64	0	$2^{64} - 1$
UInt128	128	0	$2^{128} - 1$

In [3]:

```
x=UInt8(2^8 -1)
bitstring(x)
typeof(x+1)
```

Out[3]:

```
Int64
```

Representation of signed integers : two's complement

The negative of $x > 0$ is encoded using *two's complement*,

> $\overline{x}$ = flip all bits of x and add 1.

Signed integer types in Julia:

Type	Bits	Minimum	Maximum
Int8	8	-2^7	$2^7 - 1$
Int16	16	-2^{15}	$2^15 - 1$
Int32	32	-2^{31}	$2^{31} - 1$

Example:
6 = 00000110,
$\overline{6}$ = 11111001 + 1 = 11111010

Type	Bits	Minimum	Maximum
Int64	64	-2^{63}	$2^{63}-1$
Int128	128	-2^{127}	$2^{127}-1$

In [4]:

```
bitstring(Int8(-2^7))
```

Out[4]:

```
"10000000"
```

Advantages of two's complement

Why not just use a "sign" bit?

- Avoids two representations of zero.
- Subtraction can be performed using the same hardware as addition:

> Subtraction of y from x : add two's complement of y to x and drop leading ("overflow") bit.

- Example: $7-6=7+\overline{6}$ = (check)

Representation of real numbers

Use a binary version of *normalised* scientific notation: $x=-1^S\times(1.0+0.M)\times 2^E$

e.g. IEEE 754 32-bit (Float32):

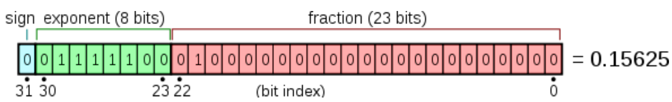
Field	Size	Bits
Sign (S)	1	31
Exponent (E + 127)	8	23 - 30
Mantissa (M)	23	0 - 22

Round-off error

- Finite mantissae introduce errors when truncating real numbers whose binary expansion is longer than 23 (Float32) or 52 (Float64).
- Round-off error is a feature of the hardware and cannot be avoided.

This can lead to counter-intuitive behaviour:

"Bias": if exponent is E, we store E+127 (for Float32). This makes comparisons easier..



In [5]:

```
bitstring(Float32(0.15625))
```

Out[5]:

```
"00111110001000000000000000000000"
```

In [6]:

```
using Printf
a = Float64(0.1);
b = Float64(0.2);
c = Float64(0.3);
@printf("a = %.16e, b = %.16e, c = %.
```

```
a = 1.0000000000000001e-01,
b = 2.0000000000000001e-01,
c = 2.9999999999999999e-01
```

In [7]:

```
a + b == c
```

Out[7]:

```
false
```

Floating point arithmetic

Rules for adding two floating point numbers:

1. Rewrite the smaller number so its exponent matches that of the larger number.
2. Add the mantissae.
3. Normalise the sum.
4. Round the sum.

Machine precision

In [8]:

```
a = Float32(1.0); b = Float32(10.0^(-10));
println(a == 0.0, " ", b == 0.0, " ", a + b == a)

b*10.0^32
```

```
false false true
```

Out[8]:

```
1.0000000133514320e22
```

The *machine precision* (or *machine epsilon*) is the smallest floating point number which when added to 1 gives an answer larger than one.

Machine precision

In Julia `eps()` gives the machine precision:

In [9]:

```
println(eps(Float32))
println(eps(Float64))
```

```
1.1920929e-7
2.220446049250313e-16
```

These values are 2.0^{-23} and 2.0^{-52}, respectively.

Loss of significance

Subtraction of "nearby" numbers leads to loss of precision.

Calculate sum and difference of 2 nearby numbers in 32-bit precision:

In [10]:

```
a32 = Float32(2.0) + Float32(3.0)^-12; b32 = Float32(2.0);
sum32 = a32+b32; diff32 = a32 - b32;
println("32 bit: a = ", a32, ", b = ", b32)
println("32 bit: sum = ", sum32,", difference = ", diff32 )
```

```
32 bit: a = 2.000002, b = 2.0
32 bit: sum = 4.000002, difference = 1.9073486e-6
```

Loss of significance

Now calculate the same sum and difference in 64 bit precision:

In [11]:

```
a64 = Float64(2.0) + Float64(3.0)^-12; b64 = Float64(2.0);
sum64 = a64+b64; diff64 = a64 - b64;
println("64 bit: a = ", a64, ", b = ", b64)
println("64 bit: sum = ", sum64,", difference = ", diff64 )
```

```
64 bit: a = 2.000001881676423, b = 2.0
64 bit: sum = 4.000001881676424, difference = 1.8816764231210925e-6
```

Loss of significance

We can treat the 64 bit answer as "exact" compared to the 32 bit answer and calculate the relative error in the 32 bit result:

In [12]:

```
relErrSum = abs(sum64-sum32)/sum64
relErrDiff = abs(diff64-diff32)/diff64
println("Sum relative error = ", relErrSum)
println("Difference relative error = ", relErrDiff)
```

```
Sum relative error = 6.418049292656560e-9
Difference relative error = 0.013643264790885564
```

Numerical instability

Iterative calculations can lose precision by accumulation of round-off error:

- Assuming inputs ~ 1, each FP addition introduces an error of $\sim \epsilon_m$.
- Might expect, after n iterative steps, total error $\sim \sqrt{n}\,\epsilon_m$?

However, some iterations can produce total error $\sim e^n$ due to *instability*.

Numerical instability: simple example

Consider the following iterative procedures:

P1 : $a_{n+1} = \phi\, a_n$ with $a_0 = 1$,

P2 : $a_{n+1} = a_{n-1} - a_n$ with $a_0 = 1$ and $a_1 = \phi$,

where $\phi = (\sqrt{5} - 1)/2$.

Both have the exact solution (check):

$$a_n = \phi^n.$$

Numerical instability: simple example

However their numerical behaviour is very different for large n:

In [13]:

```
phi = Float32(0.5*(sqrt(5.)-1.)); n = 30
# Allocate some Float32 arrays
P1 = zeros(Float32, n); P2 = zeros(Float32, n);
# Set initial conditions
P1[1] = P2[1] = Float32(1.); P2[2] = phi;
```

In [14]:

```
for i in 2:n
    P1[i] = phi*P1[i-1]
end
```

In [15]:

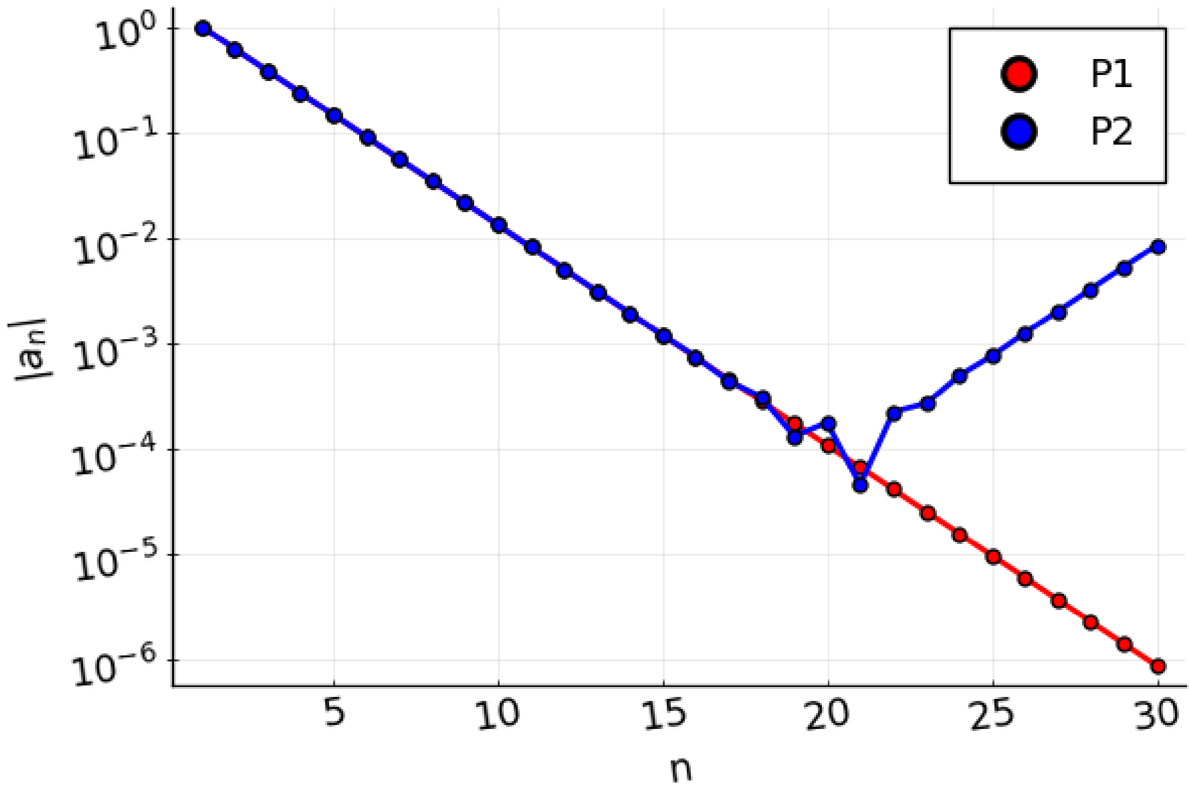
```
for i in 3:n
    P2[i] = P2[i-2] - P2[i-1]
end
```

Numerical instability: simple example

In [16]:

```julia
plot(1:n, P1, yscale=:log10, label="", linewidth=2, linecolor=:red, xlabel="n",
scatter!(1:n, P1, markercolor=:red, label = "P1", markersize=5)
plot!(1:n, abs.(P2), label="", linewidth=2, linecolor=:blue)
scatter!(1:n, abs.(P2), markercolor=:blue, label = "P2", markersize=5)
```

Out[16]:



Numerical instability: simple example

What's going on? Due to round-off we have solved a different problem:

P2 : $a_{n+1} = a_{n-1} - a_n$ with $a_0 = 1$ and $a_1 = \phi + \varepsilon$.

The solution is

$$a_n = \left(1 + \frac{\varepsilon}{\sqrt{5}}\right) \phi^n - \frac{\varepsilon}{\sqrt{5}} \tilde{\phi}^n,$$

where $\tilde{\phi} = (-\sqrt{5} - 1)/2$.
Notice that $\left|\tilde{\phi}\right| > 1$. This is an unstable iteration.

In [ ]:

```
1
```