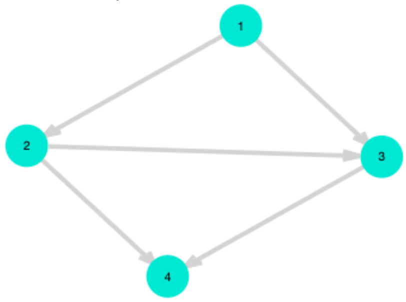
**Deadline: 14 March 2022**

**Problem description**

A model transport network is labelled and parameterised as follows:



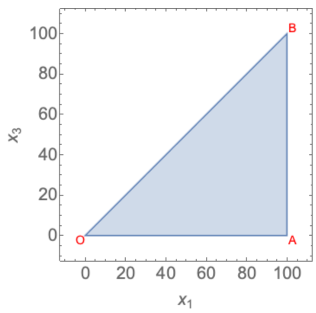

| Link | Link flow | Demand-delay function |
|---|---|---|
| $[1,2]$ | $x_1$ | $\tau_1(x) = x/10$ |
| $[1,3]$ | $x_2$ | $\tau_2(x) = 15$ |
| $[2,3]$ | $x_3$ | $\tau_3(x) = \alpha$ |
| $[2,4]$ | $x_4$ | $\tau_4(x) = 15$ |
| $[3,4]$ | $x_5$ | $\tau_5(x) = x/10$ |

| Path | Path flow |
|---|---|
| $[1,2,4]$ | $h_1$ |
| $[1,3,4]$ | $h_2$ |
| $[1,2,3,4]$ | $h_3$ |

Here $\alpha \geq 0$ is a free parameter. The demand from node 1 to 4 is 100 vehicles per hour. The trip assignment problem is to determine how many vehicles take each available path. The objective is to understand how the trip assignment depends on $\alpha$.

**Analytical work**

1. Write down all constraints the variables $(x_1, x_2, x_3, x_4, x_5, h_1, h_2, h_3)$ must satisfy to represent a valid trip assignment. Write an expression for the average travel time in terms of these variables.
2. Define what it means for a trip assignment to be a social optimum. Define what it means for a trip assignment to be a Wardrop equilibrium.
3. Write down (without proof) constrained optimisation problems whose respective solutions yield the social optimum and Wardrop equilibrium trip assignments for the network shown above. By expressing $x_2$, $x_4$ and $x_5$ in terms of $x_1$ and $x_3$, reduce each of these optimisation problems to two dimensions, including explicit formulae for the objective functions, $F_{\text{SO}}(x_1, x_3)$ and $F_{\text{WE}}(x_1, x_3)$, whose minima yield the social optimum and Wardrop equilibrium respectively.
4. In the $(x_1, x_3)$ plane, the feasible set is a triangle:




Consider the restriction of the optimisation problem for the Wardrop equilibrium to the boundary line AB:

$$\begin{aligned} x_1^* &= 100 \\ x_3^* &= \underset{0 \leq x_3 \leq 100}{\arg\min}\ F_{\text{WE}}(100, x_3). \end{aligned}$$

The solution is

$$(x_1^*, x_3^*) = \begin{cases} (100, 100) & 0 \leq \alpha < 5 \\ (100, 10(15-\alpha)) & 5 \leq \alpha < 15 \\ (100, 0) & 15 \leq \alpha, \end{cases}$$

and the value of the objective function is

$$F_{\text{WE}}(x_1^*, x_3^*) = \begin{cases} 100(10+\alpha) & 0 \leq \alpha < 5 \\ 875 + 150\alpha - 5\alpha^2 & 5 \leq \alpha < 15 \\ 2000 & 15 \leq \alpha. \end{cases}$$

Find the corresponding restricted minima on the other two lines, OA and OB, that bound the feasible set.

5. Find the solution of the two-dimensional optimisation problem corresponding to the Wardrop equilibrium in the absence of constraints. Thus write down the range of $\alpha$ for which the solution of the unconstrained problem lies inside the feasible set.
6. Combine the results of parts (iv) and (v) above to find the Wardrop equilibrium in the $(x_1, x_3)$ plane for $0 \leq \alpha < \infty$. Plot the equilibrium value of $x_3$ and the average travel time as functions of $\alpha$. Briefly describe what your results mean in terms of traffic flows.

**Computational work**

7. Adapt the sample code from class to create an agent-based model to verify the Wardrop equilibrium travel time you found in (6) for a range of values of $\alpha$.
8. Write an implementation of the Frank-Wolfe algorithm that can solve the constrained optimisation problems in (3) and use it to plot the social optimum travel time as a function of $\alpha$.
9. The demand-delay functions used in this example are not very realistic. Write down some more realistic ones assuming that not all routes on the network are identical. Justify your choice of functions and parameter values. Use either your ABM code and your optimisation code to numerically compare the Wardrop equilibrium travel time to the social optimum travel time for your proposed model.

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