

The η -compounding random walk

1 Appendix - binomial sums

We will need finite sums of powers weighted by binomial coefficients:

$$S_k(T) = \sum_{n=0}^T \binom{T}{n} n^k, \quad (1)$$

which are essentially the moments of the binomial distribution with $p = \frac{1}{2}$. For $k = 0$ we have $S_0(T) = 2^T$ from the definition of the binomial distribution. For $k \geq 0$, we can evaluate the sums sequentially by differentiating the binomial theorem,

$$(x + y)^T = \sum_{n=0}^T \binom{T}{n} x^n y^{T-n}$$

with respect to x and setting $x = y = 1$. For example, differentiating once with respect to x , we get

$$T(x + y)^{T-1} = \sum_{n=0}^T \binom{T}{n} n x^{n-1} y^{T-n}$$

and setting $x = y = 1$ gives $S_1(T)$. We can now differentiate again and use the formula for $S_1(T)$ to get $S_2(T)$ and so on. The first few sums are:

$$S_0(T) = \sum_{n=0}^T \binom{T}{n} = 2^T \quad (2)$$

$$S_1(T) = \sum_{n=0}^T \binom{T}{n} n = T 2^{T-1} \quad (3)$$

$$S_2(T) = \sum_{n=0}^T \binom{T}{n} n^2 = T(T+1) 2^{T-2} \quad (4)$$

$$S_3(T) = \sum_{n=0}^T \binom{T}{n} n^3 = T^2(T+3) 2^{T-3}. \quad (5)$$

Due to the symmetry of the binomial coefficients, we can always write

$$S_m(T) = \sum_{n=0}^T \binom{T}{T-n} n^m.$$

2 Definition of η -compounding

Adopting the notation in [3] to fit the way we usually write the isoelastic utility function, we define the generalised exponential and logarithm as

$$\exp_{\eta}(x) = \begin{cases} (1 + (1 - \eta)x)^{\frac{1}{1-\eta}} & 0 \leq \eta < 1 \\ \exp(x) & \eta = 1 \end{cases} \quad (6)$$

$$\log_{\eta}(x) = \begin{cases} \frac{1}{1-\eta} (x^{1-\eta} - 1) & 0 \leq \eta < 1 \\ \log(x) & \eta = 1 \end{cases}. \quad (7)$$

Following [1] we define the generalised compounding operator, \otimes , as

$$x \otimes y = \exp_{\eta} [\log_{\eta}(x) + \log_{\eta}(y)]. \quad (8)$$

An η -compounding process with growth factor, g , is one where the initial value is η -compounded by g at each step:

$$x_{t+1} = x_t \otimes g, \quad (9)$$

with $x_0 = X_0$. For $T \geq 1$, we have

$$\begin{aligned} x_T &= X_0 \otimes \underbrace{g \otimes \dots \otimes g}_{T\text{-times}} \\ &= X_0 \otimes \exp_{\eta} \left[\underbrace{\log_{\eta}(g) + \dots + \log_{\eta}(g)}_{T\text{-times}} \right] \\ &= X_0 \otimes \exp_{\eta} [\log_{\eta}(g) T]. \end{aligned} \quad (10)$$

From this we see that the growth rate (the quantity with a dimension of time⁻¹) is $\log_{\eta}(g)$. From Eq. (6), we see that $\exp_{\eta}(x)$ is only well defined for

$$\begin{aligned} x &\geq -\frac{1}{1-\eta} & 0 \leq \eta < 1 \\ x &> -\infty & \eta = 1. \end{aligned}$$

Hence, for general values of $\eta \neq 1$, Eq. (10) is only well-defined for all $T \geq 1$ if $\log_{\eta}(g) > 0$. This implies that $g \geq 1$. Some confusion can arise for some rational values of η for which the branch point of Eq. (6) at $x = -\frac{1}{1-\eta}$ disappears. For example, if $\eta = \frac{1}{2}$, we have

$$\begin{aligned} x_T &= \exp_{\frac{1}{2}} \left[\log_{\frac{1}{2}}(X_0) + T \log_{\frac{1}{2}}(g) \right] \\ &= \left[\sqrt{X_0} + \frac{T}{2} \log_{\frac{1}{2}}(g) \right]^2. \end{aligned}$$

Although this is well defined for all T even when $\log_{\frac{1}{2}}(g) < 0$, it is not continuously reachable from $\eta = \frac{1}{2} \pm \epsilon$. Furthermore, this results in an unnatural model where a negative growth rate corresponds to increasing x_T . In what follows we will respect the constraint $g \geq 1$ in order to avoid such pathologies.

3 The η -compounding random walk

We are interested in studying the η -compounding random walk in discrete time. At each step, there are now two possible growth factors, $g + r$ and $g - r$, which we assume occur with equal probability:

$$x_{t+1} = \begin{cases} x_t \otimes (g + r) & \text{with probability } \frac{1}{2} \\ x_t \otimes (g - r) & \text{with probability } \frac{1}{2}. \end{cases} \quad (11)$$

with $x_0 = X_0$. In order to keep everything well-defined we need $g - r > 1$, assuming that $g > 0$ and $r > 0$.

We can generalise some of the calculations for the multiplicative random walk in [2] to the η -compounding random walk. If, after playing T rounds of the game, we experience n "wins" (and $T - n$ "losses"), then x_T will take the value

$$\begin{aligned} x_T &= X_0 \otimes \underbrace{g_1 \otimes \dots \otimes g_1}_{n\text{-times}} \otimes \underbrace{g_2 \otimes \dots \otimes g_2}_{T-n\text{-times}} \\ &= X_0 \otimes \exp_\eta [n \log_\eta(g_1)] \otimes \exp_\eta [(T-n) \log_\eta(g_2)] \\ &= X_0 \otimes \exp_\eta [n \log_\eta(g_1) + (T-n) \log_\eta(g_2)], \end{aligned}$$

where for brevity we write $g_1 = g + r$ and $g_2 = g - r$. The probability of this value is

$$p(n) = \binom{T}{n} \left(\frac{1}{2}\right)^T, \quad (12)$$

where $\binom{T}{n}$ is the binomial coefficient - the number of ways in which n wins can occur in a sequence of T rounds of the game. The expectation value of x_T is therefore

$$\begin{aligned} \mathbb{E}[x_T] &= \sum_{n=0}^T \binom{T}{n} \left(\frac{1}{2}\right)^T X_0 \otimes \exp_\eta [n \log_\eta(g_1) \\ &\quad + (T-n) \log_\eta(g_2)] \\ &= \frac{1}{2^T} \sum_{n=0}^T \binom{T}{n} [X_0^{1-\eta} - T + n g_1^{1-\eta} \\ &\quad + (T-n) g_2^{1-\eta}]^{\frac{1}{1-\eta}} \\ &= \frac{1}{2^T} \sum_{n=0}^T \binom{T}{n} [X_0^{1-\eta} + G_2 T + (G_1 - G_2) n]^{\frac{1}{1-\eta}}, \end{aligned} \quad (13)$$

where we write $G_{1,2} = g_{1,2}^{1-\eta} - 1$ to keep the notation compact.

4 Exact expectation value for $\eta = \frac{1}{2}$

This sum in Eq. (13) can be done exactly at least for the case $\eta = \frac{1}{2}$ where we have

$$\mathbb{E}[x_T] = \frac{1}{2^T} \sum_{n=0}^T \binom{T}{n} [\sqrt{X_0} + G_2 T + (G_1 - G_2) n]^2$$

After expanding the square in the summand and using Eqs. (2)-(4) some algebra leads to

$$\begin{aligned} \mathbb{E}[x_T] &= \frac{1}{4} (G_1 + G_2)^2 T^2 \\ &\quad + \left[\frac{1}{4} (G_1 - G_2)^2 + \sqrt{X_0} (G_1 + G_2) \right] T \\ &\quad + X_0. \end{aligned}$$

In the original notation this is:

$$\begin{aligned} \mathbb{E}[x_T] &= \frac{1}{4} (\sqrt{g+r} + \sqrt{g-r} - 2)^2 T^2 \\ &\quad + \left[(\sqrt{g+r} + \sqrt{g-r} - 2) \sqrt{X_0} \right. \\ &\quad \left. + \frac{1}{4} (\sqrt{g+r} - \sqrt{g-r})^2 \right] T \\ &\quad + X_0. \end{aligned} \quad (14)$$

For large T we find

$$\mathbb{E}[x_T] \sim \frac{1}{4} (\sqrt{g+r} + \sqrt{g-r} - 2)^2 T^2. \quad (15)$$

5 Exact expectation value for $\eta = \frac{2}{3}$

A similar calculation with a lot more algebra was done by H. Reynolds for $\eta = \frac{2}{3}$. The result is

$$\begin{aligned} \mathbb{E}[x_T] &= \frac{1}{8} (G_1 + G_2)^3 T^3 \\ &\quad + \frac{3}{8} (G_1 + G_2) \left[(G_1 - G_2)^2 \right. \\ &\quad \left. + 2 X_0^{\frac{1}{3}} (G_1 + G_2) \right] T^2 \\ &\quad + \frac{3}{4} X_0^{\frac{1}{3}} \left[2 X_0^{\frac{1}{3}} (G_1 + G_2) \right. \\ &\quad \left. + (G_1 - G_2)^2 \right] T \\ &\quad + X_0. \end{aligned} \quad (16)$$

6 Most likely value for the η -compounding random walk

The *most likely* value of x_T can be found by finding n^* , the value of n that maximises the probability, Eq. (12). This is

$$\begin{aligned} n^* &= \arg \max_n \binom{T}{n} \\ &= \frac{T}{2}. \end{aligned}$$

Thus the most likely value of x_T is

$$\begin{aligned} \tilde{x}_T &= X_0 \otimes \exp_\eta \left[\frac{T}{2} (\log_\eta(g+r) + \log_\eta(g-r)) \right] \\ &= \left(X_0^{1-\eta} + ((g+r)^{1-\eta} + (g-r)^{1-\eta} - 2) \frac{T}{2} \right)^{\frac{1}{1-\eta}}. \end{aligned} \quad (17)$$

For large T , we find

$$\tilde{x}_T \sim \left(\frac{1}{2} ((g+r)^{1-\eta} + (g-r)^{1-\eta} - 2) \right)^{\frac{1}{1-\eta}} T^{\frac{1}{1-\eta}}. \quad (18)$$

Note that for $\gamma = \frac{1}{2}$, this agrees with Eq. (15) which suggests that for the $\frac{1}{2}$ -compounding random walk, the expected value is representative. For $\eta = \frac{1}{2}$, it turns out that the difference between the expected value and the most likely value is sub-leading in T :

$$\mathbb{E}[x_T] - \tilde{x}_T = \frac{1}{2}T(g - \sqrt{g-r}\sqrt{g+r}) \quad (19)$$

7 Time-averaged growth rate

From Eq. (11), the quantity $y_t = \log_\eta x_t$ follows a simple additive random walk:

$$y_{t+1} = \begin{cases} y_t + a & \text{with probability } \frac{1}{2} \\ y_t + b & \text{with probability } \frac{1}{2}, \end{cases} \quad (20)$$

where

$$\begin{aligned} a &= \log_\eta(g + r) \\ b &= \log_\eta(g - r). \end{aligned}$$

If, after playing T rounds of the game, we experience n "wins" (and $T - n$ "losses"), then y_T will take the value

$$y_T = n a + (T - n) b.$$

The corresponding probability is again given by Eq. (12). The expectation value of y_T is

$$\begin{aligned} \mathbb{E}[y_T] &= \sum_{n=0}^T \binom{T}{n} \left(\frac{1}{2}\right)^T [n a + (T - n) b] \\ &= \frac{T}{2} \left(a \sum_{n=0}^T \binom{T}{n} n + b \sum_{n=0}^T \binom{T}{n} (T - n) \right) \\ &= \frac{T}{2} (a + b) \\ &= \frac{T}{2} (\log_\eta(g + r) + \log_\eta(g - r)). \end{aligned} \quad (21)$$

where the second-but-last line follows from the identity Eq. (3). Thus the time averaged growth rate corresponds

to the growth rate of the most likely trajectory. We then find that

$$\begin{aligned} \exp_\eta(\mathbb{E}[\log_\eta(x_T)]) &= \exp_\eta \left[\frac{T}{2} (\log_\eta(g + r) + \log_\eta(g - r)) \right] \\ &= \left(\frac{1}{2} ((g - r)^{1-\eta} + (g + r)^{1-\eta} - 2) \right)^{\frac{1}{1-\eta}} \end{aligned} \quad (22)$$

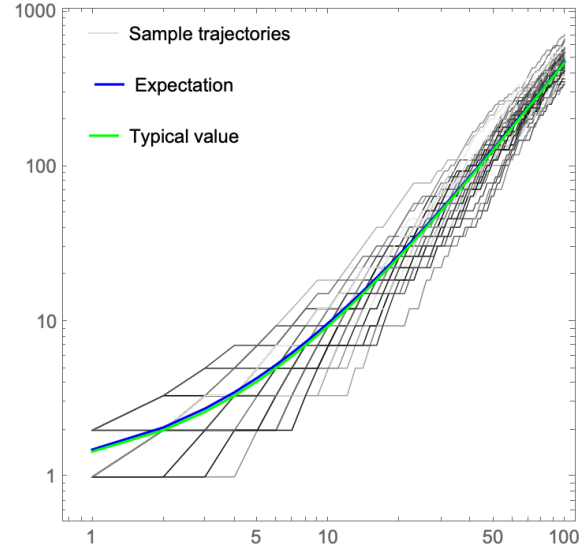


Figure 1: Some sample trajectories for $\eta = \frac{1}{2}$ with $g = \frac{3}{2}$ and $r = \frac{1}{2}$.

References

- [1] Peter Carr and Umberto Cherubini. Generalized compounding and growth optimal portfolios reconciling Kelly and Samuelson. *The Journal of Derivatives*, 30(2):74–93, 2022.
- [2] Sidney Redner. Random multiplicative processes: An elementary tutorial. *American Journal of Physics*, 58(3):267–273, 1990.
- [3] Takuya Yamano. Some properties of q-logarithm and q-exponential functions in Tsallis statistics. *Physica A: Statistical Mechanics and its Applications*, 305(3-4):486–496, 2002.