# The $\gamma$-compounding random walk

[2] [1]

$$\sum_{n=0}^{T}\binom{T}{n} = 2^T \tag{1}$$

$$\sum_{n=0}^{T}\binom{T}{n} n = \sum_{n=0}^{T}\binom{T}{n}(T-n) = T\,2^{T-1} \tag{2}$$

$$\sum_{n=0}^{T}\binom{T}{n} n^2 = \sum_{n=0}^{T}\binom{T}{n}(T-n)^2 = T\,(T+1)\,2^{T-2}. \tag{3}$$

Define the generalised exponential and logarithm as

$$\exp_\gamma(x) = \begin{cases} (1+\gamma\, x)^{\frac{1}{\gamma}} & 0<\gamma\le 1 \\ \exp(x) & \gamma = 0 \end{cases} \tag{4}$$

$$\log_\gamma(x) = \begin{cases} \frac{1}{\gamma}\,(x^\gamma - 1) & 0<\gamma\le 1 \\ \log(x) & \gamma = 0 \end{cases}\,, \tag{5}$$

and the generalised compounding operator, $\otimes$, as

$$x\otimes y = \exp_\gamma\left[\log_\gamma(x) + \log_\gamma(y)\right]. \tag{6}$$

We are interested in studying the gamma-compounding random walk in discrete time with growth factors $g+r$ and $g-r$ occurring with equal probability:

$$x_{t+1} = \begin{cases} x_t\otimes(g+r) & \text{with probability } \frac{1}{2} \\ x_t\otimes(g-r) & \text{with probability } \frac{1}{2}. \end{cases} \tag{7}$$

with $x_0 = X_0$.

If, after playing $T$ rounds of the game, we experience $n$ “wins" (and $T-n$ "losses"), then $x_T$ will take the value

$$\begin{aligned} x_T =& X_0\otimes \underbrace{(g+r)\otimes\ldots\otimes(g+r)}_{n\text{-times}}\otimes\underbrace{(g-r)\otimes\ldots\otimes(g-r)}_{T-n\text{-times}} \\ =& X_0\otimes\exp_\gamma\left[n\,\log_\gamma(g+r)\right]\otimes\exp_\gamma\left[(T-n)\,\log_\gamma(g-r)\right] \\ =& X_0\otimes\exp_\gamma\left[n\,\log_\gamma(g+r) + (T-n)\log_\gamma(g-r)\right]. \end{aligned}$$

The probability of this value is

$$p(n) = \binom{T}{n}\left(\frac{1}{2}\right)^T, \tag{8}$$

where $\binom{T}{n}$ is the binomial coefficient - the number of ways in which $n$ wins can occur in a sequence of $T$ rounds of the game. The expectation value of $x_T$ is therefore

$$\mathbb{E}\left[x_T\right] = \sum_{n=0}^{T}\binom{T}{n}\left(\frac{1}{2}\right)^T X_0\otimes\exp_\gamma\left[n\,\log_\gamma(g+r) + (T-n)\log_\gamma(g-r)\right]. \tag{9}$$

This sum can be done exactly at least for the case $\gamma = \frac{1}{2}$ (details later):

$$x_T = \left(\sqrt{X_0} - 1 + \sqrt{P(T,\sqrt{g+r},\sqrt{g-r})}\right)^2, \tag{10}$$

where

$$\begin{aligned} P(n,x,y) =& \frac{1}{4}n(n+1)x^2 + \frac{1}{2}n(n-1)xy \\ & - n(n-1)x + \frac{1}{4}n(n+1)y^2 - n(n-1)y \\ & + (n-1)^2. \end{aligned}$$

For large enough $T$ we find

$$\mathbb{E}\left[x_T\right] \sim \frac{1}{4}\left(\sqrt{g+r}+\sqrt{g-r}-2\right)^2 T^2. \tag{11}$$

The *typical* value of $x_T$ can be found by finding $n^*$, the value of $n$ that maximises the probability, Eq. (8). This is

$$\begin{aligned} n^* =& \arg\max_n \binom{T}{n} \\ =& \frac{T}{2}. \end{aligned}$$

Thus the typical value of $x_T$ is

$$\begin{aligned} \tilde{x}_T =& X_0\otimes\exp_\gamma\left[\frac{T}{2}\,\left(\log_\gamma(g+r)+\log_\gamma(g-r)\right)\right] \\ =& \left(\frac{T}{2}\left((g-r)^\gamma + (g+r)^\gamma - 2\right) + \mathrm{X0}^\gamma\right)^{1/\gamma}. \end{aligned} \tag{12}$$

For large enough $T$, we find

$$\tilde{x}_T \sim \left(\frac{1}{2}\left((g-r)^\gamma + (g+r)^\gamma - 2\right)\right)^{\frac{1}{\gamma}} T^{\frac{1}{\gamma}}. \tag{13}$$

Note that for $\gamma = \frac{1}{2}$, this agrees with Eq. (11) which suggests that for the $\gamma$-compounding random walk, the expected value is representative.

From Eq. (7), the quantity $y_t = \log_\gamma x_t$ follows a simple additive random walk:

$$y_{t+1} = \begin{cases} y_t + a & \text{with probability } \frac{1}{2} \\ y_t + b & \text{with probability } \frac{1}{2}, \end{cases} \tag{14}$$

where

$$\begin{aligned} a =& \log_\gamma(g+r) \\ b =& \log_\gamma(g-r). \end{aligned}$$

If, after playing $T$ rounds of the game, we experience $n$ “wins" (and $T-n$ "losses"), then $y_T$ will take the value

$$y_T = n\,a + (T-n)\,b.$$

The corresponding probability is again given by Eq. (8). The expectation value of $y_T$ is

$$\begin{aligned}\mathbb{E}\left[y_T\right] =& \sum_{n=0}^{T}\binom{T}{n}\left(\frac{1}{2}\right)^T\left[n\,a+(T-n)\,b\right]\\ =&\frac{T}{2}\left(a\sum_{n=0}^{T}\binom{T}{n}n+b\sum_{n=0}^{T}\binom{T}{n}(T-n)\right)\\ =&\frac{T}{2}\left(a+b\right)\\ =&\frac{T}{2}\left(\log_\gamma(g+r)+\log_\gamma(g-r)\right). \end{aligned} \tag{15}$$

where the second-but-last line follows from the identity Eq. (2). We then find that

$$\begin{aligned}\exp_\gamma\left(\mathbb{E}\left[\log_\gamma(x_T)\right]\right) =& \exp_\gamma\left[\frac{T}{2}\left(\log_\gamma(g+r)+\log_\gamma(g-r)\right).\right]\\ =& \left(\frac{1}{2}\left((g-r)^\gamma+(g+r)^\gamma-2\right)\right)^{\frac{1}{\gamma}}T^{\frac{1}{\gamma}}. \end{aligned} \tag{16}$$

Again, seems to coincide with the typical value. Something to figure out here.

## References


[1] Peter Carr and Umberto Cherubini. Generalized compounding and growth optimal portfolios reconciling Kelly and Samuelson. *The Journal of Derivatives*, 30(2):74–93, 2022.

[2] Sidney Redner. Random multiplicative processes: An elementary tutorial. *American Journal of Physics*, 58(3):267–273, 1990.