The η-compounding random walk

1 Appendix - binomial sums

We will need finite sums of powers weighted by binomial coefficients:

$$S_k(T) = \sum_{n=0}^{T} \binom{T}{n} n^k, \tag{1}$$

which are essentially the moments of the binomial distribution with $p = \frac{1}{2}$. For $k = 0$ we have $S_0(T) = 2^T$ from the definition of the binomial distribution. For $k \geq 0$, we can evaluate the sums sequentially by differentiating the binomial theorem,

$$(x+y)^T = \sum_{n=0}^{T} \binom{T}{n} x^n y^{T-n}$$

with respect to x and setting $x = y = 1$. For example, differentiating once with respect to x, we get

$$T\,(x+y)^{T-1} = \sum_{n=0}^{T} \binom{T}{n} n x^{n-1} y^{T-n}$$

and setting $x = y = 1$ gives $S_1(T)$. We can now differentiate again and use the formula for $S_1(T)$ to get $S_2(T)$ and so on. The first few sums are:

$$\sum_{n=0}^{T} \binom{T}{n} = 2^T \tag{2}$$

$$\sum_{n=0}^{T} \binom{T}{n} n = \sum_{n=0}^{T} \binom{T}{n} (T-n) = T\,2^{T-1} \tag{3}$$

$$\sum_{n=0}^{T} \binom{T}{n} n^2 = \sum_{n=0}^{T} \binom{T}{n} (T-n)^2 = T\,(T+1)\,2^{T-2}. \tag{4}$$

2 Definition of η-compounding

Adopting the notation in [3] to fit the way we usually write the isoelastic utility function, we define the generalised exponential and logarithm as

$$\exp_\eta(x) = \begin{cases} (1+(1-\eta)\,x)^{\frac{1}{1-\eta}} & 0 \leq \eta < 1 \\ \exp(x) & \eta = 1 \end{cases} \tag{5}$$

$$\log_\eta(x) = \begin{cases} \frac{1}{1-\eta}\left(x^{1-\eta} - 1\right) & 0 \leq \eta < 1 \\ \log(x) & \eta = 1 \end{cases}. \tag{6}$$

Following [1] we define the generalised compounding operator, $\otimes$, as

$$x \otimes y = \exp_\eta\left[\log_\eta(x) + \log_\eta(y)\right]. \tag{7}$$

An η-compounding process with growth factor, g, is one where the initial value is η-compounded by g at each step:

$$x_{t+1} = x_t \otimes g, \tag{8}$$

with $x_0 = X_0$. For $T \geq 1$, we have

$$\begin{aligned} x_T &= X_0 \otimes \underbrace{g \otimes \ldots \otimes g}_{T\text{-times}} \\ &= X_0 \otimes \exp_\eta\left[\underbrace{\log_\eta(g) + \ldots + \log_\eta(g)}_{T\text{-times}}\right] \\ &= X_0 \otimes \exp_\eta\left[\log_\eta(g)\,T\right]. \end{aligned} \tag{9}$$

From this we see that the growth rate (the quantity with a dimension of time^{-1}) is $\log_\eta(g)$. From Eq. (5), we see that $\exp_\eta(x)$ is only well defined for

$$\begin{aligned} x &\geq -\frac{1}{1-\eta} & 0 \leq \eta < 1 \\ x &> -\infty & \eta = 1. \end{aligned}$$

Hence, for general values of $\eta \neq 1$, Eq. (9) is only well-defined for all $T \geq 1$ if $\log_\eta(g) > 0$. This implies that $g \geq 1$. Some confusion can arise for some rational values of η for which the branch point of Eq. (5) at $x = -\frac{1}{1-\eta}$ disappears. For example, if $\eta = \frac{1}{2}$, we have

$$\begin{aligned} x_T &= \exp_{\frac{1}{2}}\left[\log_{\frac{1}{2}}(X_0) + T\log_{\frac{1}{2}}(g)\right] \\ &= \left[\sqrt{X_0} + \frac{T}{2}\,\log_{\frac{1}{2}}(g)\right]^2. \end{aligned}$$

Although this is well defined for all T even when $\log_{\frac{1}{2}}(g) < 0$, it is not continuously reachable from $\eta = \frac{1}{2} \pm \epsilon$. Furthermore, this results in an unnatural model where a negative growth rate corresponds to increasing x_T. In what follows we will respect the constraint $g \geq 1$ in order to avoid such pathologies.

3 The η-compounding random walk

We are interested in studying the η-compounding random walk in discrete time. At each step, there are now two possible growth factors, $g+r$ and $g-r$, which we assume occur with equal probability:

$$x_{t+1} = \begin{cases} x_t \otimes (g+r) & \text{with probability } \frac{1}{2} \\ x_t \otimes (g-r) & \text{with probability } \frac{1}{2}. \end{cases} \tag{10}$$

with $x_0 = X_0$. In order to keep everything well-defined we need $g - r > 1$, assuming that $g > 0$ and $r > 0$.

We can generalise some of the calculations for the multiplicative random walk in [2] to the η-compounding random walk. If, after playing T rounds of the game, we experience n “wins" (and $T-n$ "losses"), then x_T will take the value

$$\begin{aligned} x_T =& X_0 \otimes \underbrace{(g+r)\otimes\ldots\otimes(g+r)}_{n\text{-times}} \otimes \underbrace{(g-r)\otimes\ldots\otimes(g-r)}_{T-n\text{-times}} \\ =& X_0 \otimes \exp_\eta\left[n\,\log_\eta(g+r)\right] \otimes \exp_\eta\left[(T-n)\,\log_\eta(g-r)\right] \\ =& X_0 \otimes \exp_\eta\left[n\,\log_\eta(g+r) + (T-n)\log_\eta(g-r)\right] \end{aligned}$$

The probability of this value is

$$p(n) = \binom{T}{n}\left(\frac{1}{2}\right)^T, \tag{11}$$

where $\binom{T}{n}$ is the binomial coefficient - the number of ways in which n wins can occur in a sequence of T rounds of the game. The expectation value of x_T is therefore

$$\begin{aligned} \mathbb{E}\left[x_T\right] =& \sum_{n=0}^{T}\binom{T}{n}\left(\frac{1}{2}\right)^T X_0 \otimes \exp_\eta\left[n\,\log_\eta(g_1) \right.\\ & \left. +(T-n)\log_\eta(g_2)\right] \\ =& \frac{1}{2^T}\sum_{n=0}^{T}\binom{T}{n}\left[X_0^{1-\eta} - T + n g_1^{1-\eta} + (T-n) g_2^{1-\eta}\right]^{\frac{1}{1-\eta}}, \end{aligned} \tag{12}$$

where, for brevity we have written $g_1 = g+r$ and $g_2 = g-r$.

4 Expectation value for the case of $\eta = \frac{1}{2}$

This sum can be done exactly at least for the case $\eta = \frac{1}{2}$:

$$\begin{aligned} \mathbb{E}\left[x_T\right] =& \frac{1}{4}\left(\sqrt{g+r}+\sqrt{g-r}-2\right)^2 T^2 \\ & + \left[\left(\sqrt{g+r}+\sqrt{g-r}-2\right)\sqrt{X_0} \right.\\ & \left. + \frac{1}{4}\left(\sqrt{g+r}-\sqrt{g-r}\right)^2\right] T \\ & + X_0. \end{aligned} \tag{13}$$

For large T we find

$$\mathbb{E}\left[x_T\right] \sim \frac{1}{4}\left(\sqrt{g+r}+\sqrt{g-r}-2\right)^2 T^2. \tag{14}$$

5 Typical value for the η-compounding random walk

The *typical* value of x_T can be found by finding n^*, the value of n that maximises the probability, Eq. (11). This is

$$\begin{aligned} n^* =& \arg\max_n \binom{T}{n} \\ =& \frac{T}{2}. \end{aligned}$$

Thus the typical value of x_T is

$$\begin{aligned} \widetilde{x}_T =& X_0 \otimes \exp_\eta\left[\frac{T}{2}\,\left(\log_\eta(g+r)+\log_\eta(g-r)\right)\right] \\ =& \left(X_0^{1-\eta} + \left((g+r)^{1-\eta}+(g-r)^{1-\eta}-2\right)\,\frac{T}{2}\right)^{\frac{1}{1-\eta}}. \end{aligned} \tag{15}$$

For large T, we find

$$\widetilde{x}_T \sim \left(\frac{1}{2}\left((g+r)^{1-\eta}+(g-r)^{1-\eta}-2\right)\right)^{\frac{1}{1-\eta}}\,T^{\frac{1}{1-\eta}}. \tag{16}$$

Note that for $\gamma = \frac{1}{2}$, this agrees with Eq. (14) which suggests that for the $\frac{1}{2}$-compounding random walk, the expected value is representative. For $\eta = \frac{1}{2}$, it turns out that the difference between the expected value and the typical value is sub-leading in T:

$$\mathbb{E}\left[x_T\right] - \widetilde{x}_T = \frac{1}{2}T\left(g-\sqrt{g-r}\sqrt{g+r}\right) \tag{17}$$

6 Time-averaged growth rate

From Eq. (10), the quantity $y_t = \log_\eta x_t$ follows a simple additive random walk:

$$y_{t+1} = \begin{cases} y_t + a & \text{with probability } \frac{1}{2} \\ y_t + b & \text{with probability } \frac{1}{2}, \end{cases} \tag{18}$$

where

$$\begin{aligned} a =& \log_\eta(g+r) \\ b =& \log_\eta(g-r). \end{aligned}$$

If, after playing T rounds of the game, we experience n “wins" (and $T-n$ "losses"), then y_T will take the value

$$y_T = n\,a + (T-n)\,b.$$

The corresponding probability is again given by Eq. (11). The expectation value of y_T is

$$\begin{aligned} \mathbb{E}\left[y_T\right] =& \sum_{n=0}^{T}\binom{T}{n}\left(\frac{1}{2}\right)^T\left[n\,a + (T-n)\,b\right] \\ =& \frac{T}{2}\left(a\sum_{n=0}^{T}\binom{T}{n}\,n + b\sum_{n=0}^{T}\binom{T}{n}\,(T-n)\right) \\ =& \frac{T}{2}\,(a+b) \\ =& \frac{T}{2}\,\left(\log_\eta(g+r)+\log_\eta(g-r)\right). \end{aligned} \tag{19}$$

where the second-but-last line follows from the identity Eq. (3). Thus the time averaged growth rate corresponds to the growth rate of the typical trajectory. We then find that

$$\begin{aligned} \exp_\eta\left(\mathbb{E}\left[\log_\eta(x_T)\right]\right) =& \exp_\eta\left[\frac{T}{2}\,\left(\log_\eta(g+r)+\log_\eta(g-r)\right).\right] \\ =& \left(\frac{1}{2}\left((g-r)^{1-\eta}+(g+r)^{1-\eta}-2\right)\right)^{\frac{1}{1-\eta}} \end{aligned} \tag{20}$$

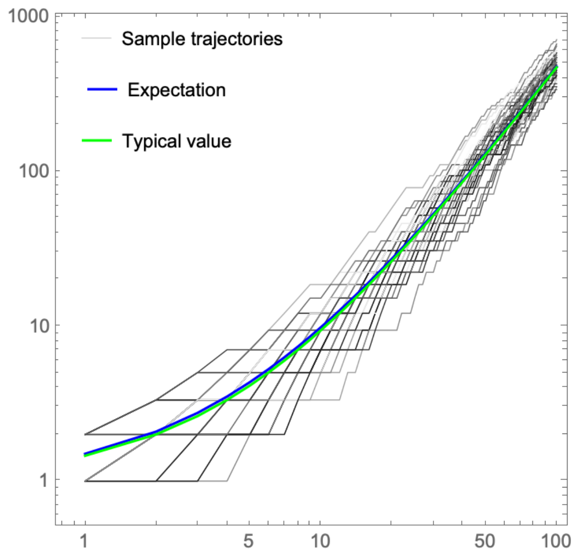

Figure 1: Some sample trajectories for $\eta = \frac{1}{2}$ with $g = \frac{3}{2}$ and $r = \frac{1}{2}$.

References

[1] Peter Carr and Umberto Cherubini. Generalized compounding and growth optimal portfolios reconciling Kelly and Samuelson. *The Journal of Derivatives*, 30(2):74–93, 2022.

[2] Sidney Redner. Random multiplicative processes: An elementary tutorial. *American Journal of Physics*, 58(3):267–273, 1990.

[3] Takuya Yamano. Some properties of q-logarithm and q-exponential functions in Tsallis statistics. *Physica A: Statistical Mechanics and its Applications*, 305(3-4):486–496, 2002.