The η-compounding random walk

1 Appendix - binomial sums

We will need finite sums of powers weighted by binomial coefficients:

$$S_k(T) = \sum_{n=0}^{T} \binom{T}{n} n^k, \tag{1}$$

which are essentially the moments of the binomial distribution with $p = \frac{1}{2}$. For $k = 0$ we have $S_0(T) = 2^T$ from the definition of the binomial distribution. For $k \geq 0$, we can evaluate the sums sequentially by differentiating the binomial theorem,

$$(x+y)^T = \sum_{n=0}^{T} \binom{T}{n} x^n y^{T-n}$$

with respect to x and setting $x = y = 1$. For example, differentiating once with respect to x, we get

$$T\,(x+y)^{T-1} = \sum_{n=0}^{T} \binom{T}{n} n x^{n-1} y^{T-n}$$

and setting $x = y = 1$ gives $S_1(T)$. We can now differentiate again and use the formula for $S_1(T)$ to get $S_2(T)$ and so on. The first few sums are:

$$\begin{aligned}
S_0(T) &= \sum_{n=0}^{T} \binom{T}{n} &&= 2^T && (2)\\
S_1(T) &= \sum_{n=0}^{T} \binom{T}{n} n &&= T\,2^{T-1} && (3)\\
S_2(T) &= \sum_{n=0}^{T} \binom{T}{n} n^2 &&= T\,(T+1)\,2^{T-2} && (4)\\
S_3(T) &= \sum_{n=0}^{T} \binom{T}{n} n^3 &&= T^2\,(T+3)\,2^{T-3}. && (5)
\end{aligned}$$

Due to the symmetry of the binomial coefficients, we can always write

$$S_m(T) = \sum_{n=0}^{T} \binom{T}{T-n} n^m.$$

2 Definition of η-compounding

Adopting the notation in [4] to fit the way we usually write the isoelastic utility function, we define the generalised exponential and logarithm as

$$\exp_\eta(x) = \begin{cases} (1+(1-\eta)\,x)^{\frac{1}{1-\eta}} & 0 \leq \eta < 1 \\ \exp(x) & \eta = 1 \end{cases} \tag{6}$$

$$\log_\eta(x) = \begin{cases} \frac{1}{1-\eta}\left(x^{1-\eta}-1\right) & 0 \leq \eta < 1 \\ \log(x) & \eta = 1 \end{cases}. \tag{7}$$

Following [2] we define the generalised compounding operator, $\otimes$, as

$$x \otimes y = \exp_\eta\left[\log_\eta(x) + \log_\eta(y)\right]. \tag{8}$$

An η-compounding process with growth factor, g, is one where the initial value is η-compounded by g at each step:

$$x_{t+1} = x_t \otimes g, \tag{9}$$

with $x_0 = X_0$. For $T \geq 1$, we have

$$\begin{aligned}
x_T &= X_0 \otimes \underbrace{g \otimes \ldots \otimes g}_{T\text{-times}} \\
&= X_0 \otimes \exp_\eta\Big[\underbrace{\log_\eta(g) + \ldots + \log_\eta(g)}_{T\text{-times}}\Big] \\
&= X_0 \otimes \exp_\eta\left[\log_\eta(g)\,T\right]. \qquad (10)
\end{aligned}$$

From this we see that the growth rate (the quantity with a dimension of time^{-1}) is $\log_\eta(g)$. From Eq. (6), we see that $\exp_\eta(x)$ is only well defined for

$$\begin{aligned}
x &\geq -\frac{1}{1-\eta} && 0 \leq \eta < 1 \\
x &> -\infty && \eta = 1.
\end{aligned}$$

Hence, for general values of $\eta \neq 1$, Eq. (10) is only well-defined for all $T \geq 1$ if $\log_\eta(g) > 0$. This implies that $g \geq 1$. Some confusion can arise for some rational values of η for which the branch point of Eq. (6) at $x = -\frac{1}{1-\eta}$ disappears. For example, if $\eta = \frac{1}{2}$, we have

$$\begin{aligned}
x_T &= \exp_{\frac{1}{2}}\left[\log_{\frac{1}{2}}(X_0) + T\log_{\frac{1}{2}}(g)\right] \\
&= \left[\sqrt{X_0} + \frac{T}{2}\,\log_{\frac{1}{2}}(g)\right]^2.
\end{aligned}$$

Although this is well defined for all T even when $\log_{\frac{1}{2}}(g) < 0$, it is not continuously reachable from $\eta = \frac{1}{2} \pm \epsilon$. Furthermore, this results in an unnatural model where a negative growth rate corresponds to increasing x_T. In what follows we will respect the constraint $g \geq 1$ in order to avoid such pathologies.

3 The η-compounding random walk

We are interested in studying the η-compounding random walk in discrete time. At each step, there are now two possible growth factors, $g+r$ and $g-r$, which we assume occur with equal probability:

$$x_{t+1} = \begin{cases} x_t \otimes (g+r) & \text{with probability } \frac{1}{2} \\ x_t \otimes (g-r) & \text{with probability } \frac{1}{2}. \end{cases} \tag{11}$$

with $x_0 = X_0$. In order to keep everything well-defined we need $g - r > 1$, assuming that $g > 0$ and $r > 0$.

We can generalise some of the calculations for the multiplicative random walk in [3] to the η-compounding random walk. If, after playing T rounds of the game, we experience n “wins" (and $T - n$ "losses"), then x_T will take the value

$$\begin{aligned} x_T =& X_0 \otimes \underbrace{g_1 \otimes \ldots \otimes g_1}_{n\text{-times}} \otimes \underbrace{g_2 \otimes \ldots \otimes g_2}_{T-n\text{-times}} \\ =& X_0 \otimes \exp_\eta \left[n \log_\eta(g_1)\right] \otimes \exp_\eta \left[(T-n)\log_\eta(g_2)\right] \\ =& X_0 \otimes \exp_\eta \left[n \log_\eta(g_1) + (T-n)\log_\eta(g_2)\right], \end{aligned}$$

where for brevity we write $g_1 = g + r$ and $g_2 = g - r$. The probability of this value is

$$p(n) = \binom{T}{n} \left(\frac{1}{2}\right)^T, \tag{12}$$

where $\binom{T}{n}$ is the binomial coefficient - the number of ways in which n wins can occur in a sequence of T rounds of the game. The expectation value of x_T is therefore

$$\begin{aligned} \mathbb{E}\left[x_T\right] =& \sum_{n=0}^{T} \binom{T}{n} \left(\frac{1}{2}\right)^T X_0 \otimes \exp_\eta \left[n \log_\eta(g_1) \right. \\ & \left. + (T-n)\log_\eta(g_2)\right] \\ =& \frac{1}{2^T} \sum_{n=0}^{T} \binom{T}{n} \left[X_0^{1-\eta} - T + n g_1^{1-\eta} \right. \\ & \left. + (T-n) g_2^{1-\eta}\right]^{\frac{1}{1-\eta}} \\ =& \frac{1}{2^T} \sum_{n=0}^{T} \binom{T}{n} \left[X_0^{1-\eta} + G_2 T + (G_1 - G_2) n\right]^{\frac{1}{1-\eta}} \end{aligned} \tag{13}$$

where we write $G_{1,2} = g_{1,2}^{1-\eta} - 1$ to keep the notation compact.

4 Exact expectation value for $\eta = \frac{1}{2}$

This sum in Eq. (13) can be done exactly at least for the case $\eta = \frac{1}{2}$ where we have

$$\mathbb{E}\left[x_T\right] = \frac{1}{2^T} \sum_{n=0}^{T} \binom{T}{n} \left[\sqrt{X_0} + G_2 T + (G_1 - G_2) n\right]^2$$

After expanding the square in the summand and using Eqs. (2)-(4) some algebra leads to

$$\begin{aligned} \mathbb{E}\left[x_T\right] =& \frac{1}{4}(G_1 + G_2)^2 T^2 \\ & + \left[\frac{1}{4}(G_1 - G_2)^2 + \sqrt{X_0}(G_1 + G_2)\right] T \\ & + X_0. \end{aligned}$$

In the original notation this is:

$$\begin{aligned} \mathbb{E}\left[x_T\right] =& \frac{1}{4}\left(\sqrt{g+r} + \sqrt{g-r} - 2\right)^2 T^2 \\ & + \left[(\sqrt{g+r} + \sqrt{g-r} - 2)\sqrt{X_0} \right. \\ & \left. + \frac{1}{4}(\sqrt{g+r} - \sqrt{g-r})^2\right] T \\ & + X_0. \end{aligned} \tag{14}$$

For large T we find

$$\mathbb{E}\left[x_T\right] \sim \frac{1}{4}\left(\sqrt{g+r} + \sqrt{g-r} - 2\right)^2 T^2. \tag{15}$$

5 Most likely value for the η-compounding random walk

The *most likely* value of x_T can be found by finding n^*, the value of n that maximises the probability, Eq. (12). This is

$$\begin{aligned} n^* =& \arg\max_n \binom{T}{n} \\ =& \frac{T}{2}. \end{aligned}$$

Thus the most likely value of x_T is

$$\begin{aligned} \tilde{x}_T =& X_0 \otimes \exp_\eta \left[\frac{T}{2}\left(\log_\eta(g+r) + \log_\eta(g-r)\right)\right] \\ =& \left(X_0^{1-\eta} + \left((g+r)^{1-\eta} + (g-r)^{1-\eta} - 2\right)\frac{T}{2}\right)^{\frac{1}{1-\eta}}. \end{aligned} \tag{16}$$

For large T, we find

$$\tilde{x}_T \sim \left(\frac{1}{2}\left((g+r)^{1-\eta} + (g-r)^{1-\eta} - 2\right)\right)^{\frac{1}{1-\eta}} T^{\frac{1}{1-\eta}}. \tag{17}$$

Note that for $\gamma = \frac{1}{2}$, this agrees with Eq. (15) which suggests that for the $\frac{1}{2}$-compounding random walk, the expected value is representative. For $\eta = \frac{1}{2}$, it turns out that the difference between the expected value and the most likely value is sub-leading in T:

$$\mathbb{E}\left[x_T\right] - \tilde{x}_T = \frac{1}{2}T\left(g - \sqrt{g-r}\sqrt{g+r}\right) \tag{18}$$

6 Time-averaged growth rate

From Eq. (11), the quantity $y_t = \log_\eta x_t$ follows a simple additive random walk:

$$y_{t+1} = \begin{cases} y_t + a & \text{with probability } \frac{1}{2} \\ y_t + b & \text{with probability } \frac{1}{2}, \end{cases} \tag{19}$$

where

$$\begin{aligned} a =& \log_\eta(g+r) \\ b =& \log_\eta(g-r). \end{aligned}$$

If, after playing T rounds of the game, we experience n “wins" (and $T-n$ "losses"), then y_T will take the value

$$y_T = n\,a + (T-n)\,b.$$

The corresponding probability is again given by Eq. (12). The expectation value of y_T is

$$\begin{aligned}\mathbb{E}\left[y_T\right] =& \sum_{n=0}^{T}\binom{T}{n}\left(\frac{1}{2}\right)^T\left[n\,a+(T-n)\,b\right]\\ =& \frac{T}{2}\left(a\sum_{n=0}^{T}\binom{T}{n}n+b\sum_{n=0}^{T}\binom{T}{n}(T-n)\right)\\ =& \frac{T}{2}(a+b)\\ =& \frac{T}{2}\left(\log_\eta(g+r)+\log_\eta(g-r)\right). \end{aligned}\tag{20}$$

where the second-but-last line follows from the identity Eq. (3). Thus the time averaged growth rate corresponds to the growth rate of the most likely trajectory. We then find that

$$\begin{aligned}\exp_\eta\left(\mathbb{E}\left[\log_\eta(x_T)\right]\right) =& \exp_\eta\left[\frac{T}{2}\left(\log_\eta(g+r)+\log_\eta(g-r)\right).\right]\\ =& \left(\frac{1}{2}\left((g-r)^{1-\eta}+(g+r)^{1-\eta}-2\right)\right)^{\frac{1}{1-\eta}}T^{\frac{1}{1-\eta}}.\end{aligned}\tag{21}$$

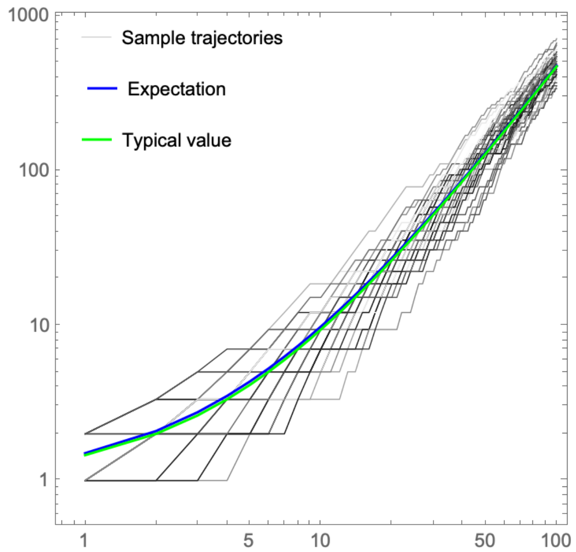

Figure 1: Some sample trajectories for $\eta=\frac{1}{2}$ with $g=\frac{3}{2}$ and $r=\frac{1}{2}$.

7 Exact results for $\eta = \frac{2}{3}$

Similar calculations to $\eta=\frac{1}{2}$ with a lot more algebra were done by H. Reynolds for $\eta=\frac{2}{3}$. The expectation value is

$$\begin{aligned}\mathbb{E}\left[x_T\right] =& \frac{1}{8}\left(G_1+G_2\right)^3 T^3\\ &+\frac{3}{8}\left(G_1+G_2\right)\Big[\left(G_1-G_2\right)^2\\ &+2X_0^{\frac{1}{3}}\left(G_1+G_2\right)\Big]\,T^2\\ &+\frac{3}{4}X_0^{\frac{1}{3}}\Big[2X_0^{\frac{1}{3}}\left(G_1+G2\right)\\ &+\left(G_1-G_2\right)^2\Big]\,T\\ &+X_0.\end{aligned}\tag{22}$$

The most likely value is

$$\begin{aligned}\widetilde{x}_T =& \frac{1}{8}\left(G_1+G_2\right)^3 T^3\\ &+\frac{3}{4}X_0^{\frac{1}{3}}\left(G_1+G_2\right)^2T^2\\ &+\frac{3}{2}X_0^{\frac{2}{3}}\left(G_1+G2\right)T\\ &+X_0.\end{aligned}\tag{23}$$

The difference between the two now grows quadratically in time:

$$\begin{aligned}\mathbb{E}\left[x_T\right]-\widetilde{x}_T =& \frac{3}{8}(G_1+G_2)(G_1-G_2)^2T^2\\ &+\frac{3}{4}X_0^{\frac{1}{3}}(G_1-G_2)^2T.\end{aligned}\tag{24}$$

8 Review of results for multiplicative case, $\eta = 1$

In this section, I summarise the results of Redner [3] for the general multiplicative random walk

$$x_{t+1} = \begin{cases} x_t\,z_1 & \text{with probability } p\\ x_t\,z_2 & \text{with probability } q=1-p.\end{cases}\tag{25}$$

with $x_0 = 1$. The expected value after T steps is

$$\mathbb{E}\left[x_T\right] = X_0 \sum_{n=0}^{T} P(n,T)\, z_1^n z_2^{T-n}, \tag{26}$$

where $P(n,T)$ is the binomial distribution,

$$P(n,T) = \binom{T}{n} p^n q^{T-n} = \frac{T!}{(T-n)!\, n!}\, p^n q^{T-n}. \tag{27}$$

The expected value can be calculated directly as

$$\begin{aligned}\mathbb{E}\left[x_T\right] =& X_0 \sum_{n=0}^{T} \binom{T}{n} (p\, z_1)^n (q z_2)^{T-n} \\ &= X_0 \left(p z_1 + q z_2\right)^T, \end{aligned} \tag{28}$$

which follows from the binomial theorem.

For what follows, it is instructive to learn how to recover the large T behaviour of $\mathbb{E}\left[x_T\right]$ without directly evaluating the sum. The idea is to take a continuum limit in which the sum can be written as an integral which can then be approximated using methods of asymptotic analysis. Starting with the binomial distribution, we use Stirling's approximation,

$$n! = \sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n \left(1 + \frac{1}{12\, n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right), \tag{29}$$

to approximate the factorials. We then take the continuum limit, $T \to \infty$ with $x = \frac{n}{T}$ fixed, to obtain

$$P(x,T) = \begin{cases} \frac{1}{\sqrt{2\pi T\, x\,(1-x)}}\, \mathrm{e}^{T\,\phi(x)} \left(1 + \frac{1}{12}\left(1 - \frac{1}{x} - \frac{1}{1-x}\right)\frac{1}{T} + \mathcal{O}\left(\frac{1}{T^2}\right)\right) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{30}$$

where

$$\phi(x) = x\log(p) + (1-x)\log(q) - x\log(x)(1-x)\log(1-x). \tag{31}$$

Writing $\mathbb{E}\left[x_T\right]$ in the form

$$\mathbb{E}\left[x_T\right] = X_0 \sum_{n=0}^{T} P(n,T)\ \left(z_1^{\frac{n}{T}}\, z_2^{1-\frac{n}{T}}\right)^T,$$

we can replace the sum by an integral in the continuum limit to obtain

$$\begin{aligned}\mathbb{E}\left[x_T\right] &= T\, X_0 \int_{-\infty}^{\infty} P(x,T)\ \left(z_1^x\, z_2^{1-x}\right)^T \\ &= T\, X_0 \int_{-\infty}^{\infty} P(x,T)\, \mathrm{e}^{T(x\log z_1 + (1-x)\log z_2)} \\ &= T^{\frac{1}{2}} X_0 \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\, x\,(1-x)}}\, \mathrm{e}^{T\,\widetilde{\phi}(x)} \left(1 + \frac{1}{12}\left(1 - \frac{1}{x} - \frac{1}{1-x}\right)\frac{1}{T} + \mathcal{O}\left(\frac{1}{T^2}\right)\right) \chi_{[0,1]}(x), \end{aligned} \tag{32}$$

where

$$\widetilde{\phi}(x) = x\log(p) + (1-x)\log(q) - x\log(x)(1-x)\log(1-x) + x\log z_1 + (1-x)\log z_2. \tag{33}$$

$\mathbb{E}\left[x_T\right]$ can therefore be broken up into a series of integrals,

$$\mathbb{E}\left[x_T\right] = T^{\frac{1}{2}}\,\mathcal{I}_1(T) + T^{-\frac{1}{2}}\mathcal{I}_2(T) + \ldots \tag{34}$$

each of which is of the form

$$\mathcal{I}_k(T) = X_0 \int_{-\infty}^{\infty} F_k(x)\, \mathrm{e}^{T\, \widetilde{\phi}(x)}\, dx, \tag{35}$$

with

$$F_0(x) = \frac{1}{\sqrt{2\pi\, x\,(1-x)}}\, \chi_{[0,1]}(x) \tag{36}$$

$$F_1(x) = \frac{1}{12}\, \frac{1}{\sqrt{2\pi\, x\,(1-x)}}\, \left(1 - \frac{1}{x} - \frac{1}{1-x}\right)\, \chi_{[0,1]}(x). \tag{37}$$

Integrals of the form

$$I(T) = \int_{-\infty}^{\infty} F(x)\, \mathrm{e}^{T\, \phi(x)}\, dx, \tag{38}$$

are standard Laplace integrals. Such integrals have the following asymptotic behaviour as $T \to \infty$ [1]:

$$I(T) \sim \sqrt{\frac{2\pi}{-T\, \phi''(x^*)}}\, \mathrm{e}^{T\, \phi(x^*)}\, \left(F(x^*) + B(x^*)T^{-1} + \mathcal{O}\left(T^{-2}\right)\right), \tag{39}$$

where x^* is the maximum of $\phi(x)$ and

$$B(x^*) = -\frac{F''(x^*)}{2\, \phi''(x^*)} + \frac{F(x^*)\phi''''(x^*)}{8\, \phi''(x^*)^2} + \frac{F'(x^*)\phi'''(x^*)}{2\, \phi''(x^*)^2} - \frac{5\, F(x^*)\phi'''(x^*)^2}{24\, \phi''(x^*)^3} \tag{40}$$

To find the maximum of $\widetilde{\phi}$ we solve

$$\begin{aligned} \widetilde{\phi}'(x^*) &= 0 \\ \log pz_1 - \log qz_2 - \log x^* + \log(1-x^*) &= 0 \\ \log\left(\frac{pz_1}{qz_2}\frac{1-x^*}{x^*}\right) &= 0 \end{aligned}$$

which gives

$$x^* = \frac{\xi}{1+\xi} \qquad \text{with } \xi = \frac{pz_1}{qz_2}. \tag{41}$$

The second derivative is

$$\widetilde{\phi}''(x) = -\frac{1}{x} - \frac{1}{1-x}.$$

We can now use Eq. (39) to extract the leading order behaviour of $\mathbb{E}\left[x_T\right]$ in Eq. (34) from the leading order behaviour of $\mathcal{I}_0(T)$. The quantities required are:

$$\widetilde{\phi}(x^*) = \log\left(pz_1 + qz_2\right) \tag{42}$$

$$\widetilde{\phi}''(x^*) = -\frac{(pz_1 + qz_2)^2}{pqz_1z_2} \tag{43}$$

$$F_0(x^*) = \frac{pz_1 + qz_2}{\sqrt{2\pi pqz_1z_2}} \tag{44}$$

Putting all the bits together in Eq. (39) we get the leading order asymptotic behaviour

$$\begin{aligned} \mathbb{E}\left[x_T\right] &\sim T^{\frac{1}{2}}\, X_0 \sqrt{\frac{2\pi pqz_1z_2}{T(pz_1+qz_2)^2}}\, \mathrm{e}^{T\, \log(pz_1+qz_2)}\, \frac{pz_1+qz_2}{\sqrt{2\pi pqz_1z_2}} \\ &= X_0\left(pz_1+qz_2\right)^T. \end{aligned} \tag{45}$$

We recover the exact result, Eq. (28). This tells us that the sub-leading terms in the expansion (34) must vanish. This can either be because they are all zero for some reason or because the corrections coming from using Stirlings formula to approximate the binomial distribution by its continuum limit in Eq. (30) cancel the higher order terms in the Laplace integral in Eq. (39). With some help from Mathematica we can evaluate these terms to show that it is the latter. The sub-leading $\mathcal{O}(T^{-1})$ contribution in Eq. (39) to $\mathcal{I}_0(T)$ is

$$\frac{(pz_1+qz_2)\left(p^2z_1^2+pqz_1z_2+q^2z_2^2\right)}{12\sqrt{2\pi(pqz_1z_2)^3}}$$

The leading $\mathcal{O}(T^{-1})$ contribution to $\mathcal{I}_1(T)$ in Eq. (34) is

$$-\frac{(pz_1+qz_2)\left(p^2z_1^2+pqz_1z_2+q^2z_2^2\right)}{12\sqrt{2\pi(pqz_1z_2)^3}}.$$

Thus we learn that to correctly calculate the sub-leading behaviour, we need to account for cancellations between the corrections to the continuum limit of the binomial distribution and the corrections to the Laplace formula.

9 Asymptotic analysis for $0<\eta<1$

We can adapt the analysis of Redner [3] to calculate the leading order behaviour for $0\leq\eta\leq 1$. Write the random η-compounding random walk in its more general form as

$$x_{t+1}=\begin{cases} x_t\otimes z_1 & \text{with probability } p\\ x_t\otimes z_2 & \text{with probability } q=1-p.\end{cases} \tag{46}$$

with $x_0=1$.

To compress the notation, define

$$Z_{1,2}=z_{1,2}^{1-\eta}-1.$$

The final answer is

$$\mathbb{E}\left[x_T\right]=T^{\frac{1}{1-\eta}}\left(pZ_1+qZ_2\right)^{\frac{1}{1-\eta}}\left[1+\frac{1}{T}\left(\frac{pq}{2}\frac{\eta}{(1-\eta)^2}\left(\frac{Z_1-Z_2}{pZ_1+qZ_2}\right)^2+\frac{X_0^{1-\eta}}{1-\eta}\frac{1}{pZ_1+qZ_2}\right)+\mathcal{O}\left(\frac{1}{T^2}\right)\right]$$

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