The γ-compounding random walk

[2] [1]

Define the generalised exponential and logarithm as

$$\exp_\gamma(x) = \begin{cases} (1+\gamma x)^{\frac{1}{\gamma}} & 0<\gamma\le 1 \\ \exp(x) & \gamma=0 \end{cases} \tag{1}$$

$$\log_\gamma(x) = \begin{cases} \frac{1}{\gamma}(x^\gamma-1) & 0<\gamma\le 1 \\ \log(x) & \gamma=0 \end{cases}, \tag{2}$$

and the generalised compounding operator, $\otimes$, as

$$x\otimes y = \exp_\gamma\left[\log_\gamma(x)+\log_\gamma(y)\right]. \tag{3}$$

We are interested in studying the gamma-compounding random walk in discrete time with growth factors $g+r$ and $g-r$ occurring with equal probability:

$$x_{t+1} = \begin{cases} x_t\otimes(g+r) & \text{with probability } \frac{1}{2} \\ x_t\otimes(g-r) & \text{with probability } \frac{1}{2}. \end{cases} \tag{4}$$

with $x_0 = X_0$.

If, after playing T rounds of the game, we experience n “wins" (and $T-n$ "losses"), then x_T will take the value

$$\begin{aligned} x_T =& X_0\otimes\underbrace{(g+r)\otimes\ldots\otimes(g+r)}_{n\text{-times}}\otimes\underbrace{(g-r)\otimes\ldots\otimes(g-r)}_{T-n\text{-times}} \\ =& X_0\otimes\exp_\gamma\left[n\,\log_\gamma(g+r)\right]\otimes\exp_\gamma\left[(T-n)\,\log_\gamma(g-r)\right] \\ =& X_0\otimes\exp_\gamma\left[n\,\log_\gamma(g+r)+(T-n)\log_\gamma(g-r)\right]. \end{aligned}$$

The probability of this value is

$$p(n) = \binom{T}{n}\left(\frac{1}{2}\right)^T, \tag{5}$$

where $\binom{T}{n}$ is the binomial coefficient - the number of ways in which n wins can occur in a sequence of T rounds of the game. The expectation value of x_T is therefore

$$\mathbb{E}\left[x_T\right] = \sum_{n=0}^{T}\binom{T}{n}\left(\frac{1}{2}\right)^T X_0\otimes\exp_\gamma\left[n\,\log_\gamma(g+r)\right. \tag{6}$$
$$\left.+(T-n)\log_\gamma(g-r)\right].$$

This sum can be done exactly at least for the case $\gamma=\frac{1}{2}$ (details later):

$$x_T = \left(\sqrt{X_0}-1+\sqrt{P(T,\sqrt{g+r},\sqrt{g-r})}\right)^2, \tag{7}$$

where

$$\begin{aligned} P(n,x,y) =& \frac{1}{4}n(n+1)x^2+\frac{1}{2}n(n-1)xy \\ & -n(n-1)x+\frac{1}{4}n(n+1)y^2-n(n-1)y \\ & +(n-1)^2. \end{aligned}$$

References

[1] Peter Carr and Umberto Cherubini. Generalized compounding and growth optimal portfolios reconciling kelly and samuelson. *The Journal of Derivatives*, 30(2):74–93, 2022.

[2] Sidney Redner. Random multiplicative processes: An elementary tutorial. *American Journal of Physics*, 58(3):267–273, 1990.