The η-compounding random walk

1 Appendix

The following identities satisfied by the binomial coefficients are useful:

$$\sum_{n=0}^{T}\binom{T}{n} = 2^T \tag{1}$$

$$\sum_{n=0}^{T}\binom{T}{n} n = \sum_{n=0}^{T}\binom{T}{n}(T-n) = T\,2^{T-1} \tag{2}$$

$$\sum_{n=0}^{T}\binom{T}{n} n^2 = \sum_{n=0}^{T}\binom{T}{n}(T-n)^2 = T\,(T+1)\,2^{T-2}. \tag{3}$$

2 Definition of η-compounding

Adopting the notation in [3] to fit the way we usually write the isoelastic utility function, we define the generalised exponential and logarithm as

$$\exp_\eta(x) = \begin{cases} (1+(1-\eta)\,x)^{\frac{1}{1-\eta}} & 0 \le \eta < 1 \\ \exp(x) & \eta = 1 \end{cases} \tag{4}$$

$$\log_\eta(x) = \begin{cases} \frac{1}{1-\eta}\left(x^{1-\eta}-1\right) & 0 \le \eta < 1 \\ \log(x) & \eta = 1 \end{cases}. \tag{5}$$

Following [1] we define the generalised compounding operator, $\otimes$, as

$$x \otimes y = \exp_\eta\left[\log_\eta(x)+\log_\eta(y)\right]. \tag{6}$$

An η-compounding process with growth factor, g, is one where the initial value is η-compounded by g at each step:

$$x_{t+1} = x_t \otimes g, \tag{7}$$

with $x_0 = X0$. For $T \ge 1$, we have

$$\begin{aligned} x_T &= X_0 \otimes \underbrace{g \otimes \ldots \otimes g}_{T\text{-times}} \\ &= X_0 \otimes \exp_\eta\left[\underbrace{\log_\eta(g)+\ldots+\log_\eta(g)}_{T\text{-times}}\right] \\ &= X_0 \otimes \exp_\eta\left[\log_\eta(g)\,T\right]. \end{aligned} \tag{8}$$

From this we see that the growth rate (the quantity with a dimension of time^{-1}) is $\log_\eta(g)$. From Eq. (4), we see that $\exp_\eta(x)$ is only well defined for

$$\begin{aligned} x &\ge -\frac{1}{1-\eta} & 0 \le \eta < 1 \\ x &> -\infty & \eta = 1. \end{aligned}$$

Hence, for general values of $\eta \ne 1$, Eq. (8) is only well-defined for all $T \ge 1$ if $\log_\eta(g) > 0$. This implies that $g \ge 1$. Some confusion can arise for some rational values of η for which the branch point of Eq. (4) at $x = -\frac{1}{1-\eta}$ disappears. For example, if $\eta = \frac{1}{2}$, we have

$$\begin{aligned} x_T &= \exp_{\frac{1}{2}}\left[\log_{\frac{1}{2}}(X_0) + T\log_{\frac{1}{2}}(g)\right] \\ &= \left[\sqrt{X_0} + \frac{T}{2}\,\log_{\frac{1}{2}}(g)\right]^2. \end{aligned}$$

Although this is well defined for all T even when $\log_{\frac{1}{2}}(g) < 0$, it is not continuously reachable from $\eta = \frac{1}{2} \pm \epsilon$. Furthermore, this results in an unnatural model where a negative growth rate corresponds to increasing x_T. In what follows we will respect the constraint $g \ge 1$ in order to avoid such pathologies.

3 The η-compounding random walk

We are interested in studying the η-compounding random walk in discrete time. At each step, there are now two possible growth factors, $g+r$ and $g-r$, which we assume occur with equal probability:

$$x_{t+1} = \begin{cases} x_t \otimes (g+r) & \text{with probability } \frac{1}{2} \\ x_t \otimes (g-r) & \text{with probability } \frac{1}{2}. \end{cases} \tag{9}$$

with $x_0 = X_0$. In order to keep everything well-defined we need $g - r > 1$, assuming that $g > 0$ and $r > 0$.

We can generalise some of the calculations for the multiplicative random walk in [2] to the η-compounding random walk. If, after playing T rounds of the game, we experience n "wins" (and $T-n$ "losses"), then x_T will take the value

$$\begin{aligned} x_T &= X_0 \otimes \underbrace{(g+r)\otimes\ldots\otimes(g+r)}_{n\text{-times}} \otimes \underbrace{(g-r)\otimes\ldots\otimes(g-r)}_{T-n\text{-times}} \\ &= X_0 \otimes \exp_\eta\left[n\,\log_\eta(g+r)\right] \otimes \exp_\eta\left[(T-n)\,\log_\eta(g-r)\right] \\ &= X_0 \otimes \exp_\eta\left[n\,\log_\eta(g+r) + (T-n)\log_\eta(g-r)\right]. \end{aligned}$$

The probability of this value is

$$p(n) = \binom{T}{n}\left(\frac{1}{2}\right)^T, \tag{10}$$

where $\binom{T}{n}$ is the binomial coefficient - the number of ways in which n wins can occur in a sequence of T rounds of the game. The expectation value of x_T is therefore

$$\begin{aligned} \mathbb{E}\left[x_T\right] &= \sum_{n=0}^{T}\binom{T}{n}\left(\frac{1}{2}\right)^T X_0 \otimes \exp_\eta\left[n\,\log_\eta(g_1) \right.\\ &\quad \left. +(T-n)\log_\eta(g_2)\right] \\ &= \frac{1}{2^T}\sum_{n=0}^{T}\binom{T}{n}\left[X_0^{1-\eta} - T + n g_1^{1-\eta} + (T-n) g_2^{1-\eta}\right]^{\frac{1}{1-\eta}}, \end{aligned} \tag{11}$$

where, for brevity we have written $g_1 = g + r$ and $g_2 = g - r$.

4 Expectation value for the case of $\eta = \frac{1}{2}$

This sum can be done exactly at least for the case $\eta = \frac{1}{2}$:

$$
\begin{aligned}
\mathbb{E}\left[x_T\right] =& \frac{1}{4}\left(\sqrt{g+r}+\sqrt{g-r}-2\right)^2 T^2 \\
&+\left[\left(\sqrt{g+r}+\sqrt{g-r}-2\right)\sqrt{X_0}\right. \\
&\left.+\frac{1}{4}\left(\sqrt{g+r}-\sqrt{g-r}\right)^2\right] T \\
&+X_0 .
\end{aligned} \tag{12}
$$

For large T we find

$$
\mathbb{E}\left[x_T\right] \sim \frac{1}{4}\left(\sqrt{g+r}+\sqrt{g-r}-2\right)^2 T^2 . \tag{13}
$$

5 Typical value for the η-compounding random walk

The *typical* value of x_T can be found by finding n^*, the value of n that maximises the probability, Eq. (10). This is

$$
\begin{aligned}
n^* &= \arg\max_n \binom{T}{n} \\
&= \frac{T}{2} .
\end{aligned}
$$

Thus the typical value of x_T is

$$
\begin{aligned}
\tilde{x}_T =& X_0 \otimes \exp_\eta\left[\frac{T}{2}\left(\log_\eta(g+r)+\log_\eta(g-r)\right)\right] \\
=& \left(\mathsf{X}_0^{1-\eta}+\left((g+r)^{1-\eta}+(g-r)^{1-\eta}-2\right)\frac{T}{2}\right)^{\frac{1}{1-\eta}} .
\end{aligned} \tag{14}
$$

For large T, we find

$$
\tilde{x}_T \sim \left(\frac{1}{2}\left((g+r)^{1-\eta}+(g-r)^{1-\eta}-2\right)\right)^{\frac{1}{1-\eta}} T^{\frac{1}{1-\eta}} . \tag{15}
$$

Note that for $\gamma = \frac{1}{2}$, this agrees with Eq. (13) which suggests that for the $\frac{1}{2}$-compounding random walk, the expected value is representative. For $\eta = \frac{1}{2}$, it turns out that the difference between the expected value and the typical value is sub-leading in T:

$$
\mathbb{E}\left[x_T\right]-\tilde{x}_T = \frac{1}{2} T\left(g-\sqrt{g-r}\sqrt{g+r}\right) \tag{16}
$$

6 Time-averaged growth rate

From Eq. (9), the quantity $y_t = \log_\eta x_t$ follows a simple additive random walk:

$$
y_{t+1} = \begin{cases} y_t + a & \text{with probability } \frac{1}{2} \\ y_t + b & \text{with probability } \frac{1}{2}, \end{cases} \tag{17}
$$

where

$$
\begin{aligned}
a &= \log_\eta(g+r) \\
b &= \log_\eta(g-r) .
\end{aligned}
$$

If, after playing T rounds of the game, we experience n “wins" (and $T-n$ "losses"), then y_T will take the value

$$
y_T = n\, a + (T-n)\, b .
$$

The corresponding probability is again given by Eq. (10). The expectation value of y_T is

$$
\begin{aligned}
\mathbb{E}\left[y_T\right] &= \sum_{n=0}^{T}\binom{T}{n}\left(\frac{1}{2}\right)^T\left[n\, a+(T-n)\, b\right] \\
&= \frac{T}{2}\left(a\sum_{n=0}^{T}\binom{T}{n} n + b\sum_{n=0}^{T}\binom{T}{n}(T-n)\right) \\
&= \frac{T}{2}(a+b) \\
&= \frac{T}{2}\left(\log_\eta(g+r)+\log_\eta(g-r)\right) .
\end{aligned} \tag{18}
$$

where the second-but-last line follows from the identity Eq. (2). Thus the time averaged growth rate corresponds to the growth rate of the typical trajectory. We then find that

$$
\begin{aligned}
\exp_\eta\left(\mathbb{E}\left[\log_\eta(x_T)\right]\right) &= \exp_\eta\left[\frac{T}{2}\left(\log_\eta(g+r)+\log_\eta(g-r)\right).\right] \\
&= \left(\frac{1}{2}\left((g-r)^{1-\eta}+(g+r)^{1-\eta}-2\right)\right)^{\frac{1}{1-\eta}}
\end{aligned} \tag{19}
$$

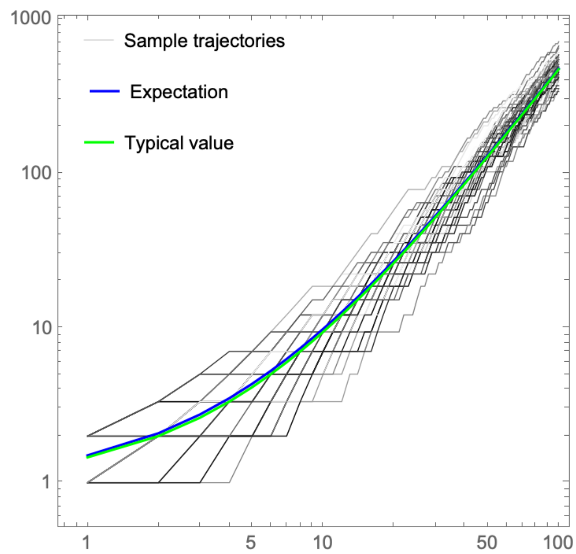

Figure 1: Some sample trajectories for $\eta = \frac{1}{2}$ with $g = \frac{3}{2}$ and $r = \frac{1}{2}$.

References

[1] Peter Carr and Umberto Cherubini. Generalized compounding and growth optimal portfolios reconciling Kelly and Samuelson. *The Journal of Derivatives*, 30(2):74–93, 2022.

[2] Sidney Redner. Random multiplicative processes: An elementary tutorial. *American Journal of Physics*, 58(3):267–273, 1990.

[3] Takuya Yamano. Some properties of q-logarithm and q-exponential functions in Tsallis statistics. *Physica A: Statistical Mechanics and its Applications*, 305(3-4):486–496, 2002.