Geometric Random Walk

1 Definition of the model

Consider a multiplicative random walk in discrete time with growth factors $g+r$ and $g-r$ occuring with equal probability:

$$x_{t+1} = \begin{cases} (g+r)\ x_t & \text{with probability } \frac{1}{2} \\ (g-r)\ x_t & \text{with probability } \frac{1}{2}. \end{cases} \tag{1}$$

We can think of the parameter g as the average growth factor and r parameter as the variability. We must have $|r| < g$. The choice $g = \frac{21}{20}$ and $r = \frac{9}{20}$ gives the growth factors of 1.5 and 0.6 corresponding to Peters' bet in Ford and Kay. We further assume that $x_0 = 1$.

The following identities satisfied by the binomial coefficients will be useful:

$$\sum_{n=0}^{T} \binom{T}{n} = 2^T \tag{2}$$

$$\sum_{n=0}^{T} \binom{T}{n} n = \sum_{n=0}^{T} \binom{T}{n} (T-n) = T\, 2^{T-1} \tag{3}$$

$$\sum_{n=0}^{T} \binom{T}{n} n^2 = \sum_{n=0}^{T} \binom{T}{n} (T-n)^2 = T\,(T+1)\, 2^{T-2}. \tag{4}$$

2 Expectation value, variance and standard deviation after T steps

If, after playing T rounds of the game, we experience n “wins" (and $T-n$ "losses"), then x_T will take the value

$$x_T = (g+r)^n (g-r)^{T-n}.$$

The probability of this value is

$$p(n) = \binom{T}{n} \left(\frac{1}{2}\right)^T, \tag{5}$$

where $\binom{T}{n}$ is the binomial coefficient - the number of ways in which n wins can occur in a sequence of T rounds of the game. The expectation value of x_T is therefore

$$\begin{aligned} \mathbb{E}\left[x_T\right] &= \sum_{n=0}^{T} \binom{T}{n} \left(\frac{1}{2}\right)^T (g+r)^n (g-r)^{T-n} \\ &= \sum_{n=0}^{T} \binom{T}{n} \left(\frac{g+r}{2}\right)^n \left(\frac{g-r}{2}\right)^{T-n} \\ &= g^T, \end{aligned} \tag{6}$$

where the final step follows from the binomial theorem. Thus the expectation value grows exponentially,

$$\mathbb{E}\left[x_T\right] = \exp(\mu T), \tag{7}$$

with rate $\mu = \log(g)$. Any moment of x_T can be calculated in the same way:

$$\begin{aligned} \mathbb{E}\left[x_T^p\right] &= \sum_{n=0}^{T} \binom{T}{n} \left(\frac{1}{2}\right)^T (g+r)^{p\,n} (g-r)^{p\,(T-n)} \\ &= \left[\frac{(g+r)^p}{2} + \frac{(g-r)^p}{2}\right]^T. \end{aligned} \tag{8}$$

In particular, the variance is

$$\mathbb{E}\left[x_T^2\right] = \left(g^2 + r^2\right)^T.$$

The standard deviation is then

$$\begin{aligned} \text{std}\left[x_T\right] &= \sqrt{\mathbb{E}\left[x_T^2\right] - \mathbb{E}\left[x_T\right]^2} \\ &= \sqrt{\left(g^2+r^2\right)^T - g^{2T}}, \end{aligned} \tag{9}$$

which also grows exponentially at a rate which is faster than that of the expectation value, Eq. (6). Thus the uncertainty in the outcome after T rounds grows even in relative terms.

3 Typical value after T steps

The *typical* value of x_T can be found by finding n^*, the value of n that maximises the probability, Eq. (5). This is

$$\begin{aligned} n^* &= \arg\max_n \binom{T}{n} \\ &= \frac{T}{2}. \end{aligned}$$

Thus the typical value of x_T is

$$\begin{aligned} \widetilde{x}_T &= (g+r)^{\frac{T}{2}} (g-r)^{\frac{T}{2}} \\ &= \left(\sqrt{g^2 - r^2}\right)^T. \end{aligned} \tag{10}$$

The typical value also shows exponential dependence on T, but since $|r| < g$, it is clear that the typical value of x_T is always exponentially smaller than the expectation value of x_T. For the values of g and r corresponding to Peters' bet, the expectation value *increases* by a factor of 1.05 per round whereas the typical value *decreases* by a factor of $\frac{3}{\sqrt{10}} \approx 0.948683$ per round.

4 Time averaged growth rate

The time averaged growth rate is

$$\gamma_T = \frac{\log x_T}{T}. \tag{11}$$

Let us now examine some of the statistical properties of γ_t. From Eq. (1), the quantity $y_t = \log x_t$ follows a simple additive random walk:

$$y_{t+1} = \begin{cases} y_t + a & \text{with probability } \frac{1}{2} \\ y_t + b & \text{with probability } \frac{1}{2}, \end{cases} \tag{12}$$

where

$$
\begin{aligned}
a &= \log(g+r) \\
b &= \log(g-r).
\end{aligned}
$$

As a random variable, y_t behaves very differently to x_t. If, after playing T rounds of the game, we experience n “wins" (and $T-n$ "losses"), then y_T will take the value

$$
y_T = n\,a + (T-n)\,b.
$$

The corresponding probability is again given by Eq. (5). The expectation value of y_T is

$$
\begin{aligned}
\mathbb{E}\left[y_T\right] &= \sum_{n=0}^{T} \binom{T}{n} \left(\frac{1}{2}\right)^{T} \left[n\,a + (T-n)\,b\right] \\
&= \frac{T}{2} \left(a \sum_{n=0}^{T} \binom{T}{n} n + b \sum_{n=0}^{T} \binom{T}{n} (T-n)\right) \\
&= \frac{T}{2}\,(a+b)
\end{aligned}
\tag{13}
$$

where the last line follows from the identity Eq. (3). It is a bit more work to calculate the second moment of y_T:

$$
\begin{aligned}
\mathbb{E}\left[y_T^2\right] &= \sum_{n=0}^{T} \binom{T}{n} \left(\frac{1}{2}\right)^{T} \left[n\,a + (T-n)\,b\right]^2 \\
&= \frac{(a-b)^2}{2^T} \left(\sum_{n=0}^{T} \binom{T}{n} n^2\right) \\
&\quad + 2\,\frac{(a-b)\,b\,T}{2^T} \left(\sum_{n=0}^{T} \binom{T}{n} n\right) \\
&\quad + \frac{b^2 T^2}{2^T} \left(\sum_{n=0}^{T} \binom{T}{n}\right) \\
&= \frac{1}{4} T \left((a+b)^2 T + (a-b)^2\right),
\end{aligned}
\tag{14}
$$

where the last line uses Eqs. (2)-(4). Going back to Eq. (11) for the growth rate, γ_T, we can now use Eqs. (13) and (14) to write down the expectation value and variance of the growth rate:

$$
\begin{aligned}
\mathbb{E}\left[\gamma_T\right] &= \frac{1}{2}\left(\log(g+r) + \log(g-r)\right) \\
&= \log\sqrt{g^2 - r^2},
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
\mathbb{E}\left[\gamma_T^2\right] &= \frac{1}{4}\left(\log(g+r) + \log(g-r)\right)^2 \\
&\quad + \frac{1}{4}\frac{1}{T}\left(\log(g+r) - \log(g-r)\right)^2 \\
&= \left(\log\sqrt{g^2-r^2}\right)^2 + \frac{1}{T}\left(\log\sqrt{\frac{g+r}{g-r}}\right)^2.
\end{aligned}
\tag{16}
$$

The standard deviation of γ_T is then

$$
\begin{aligned}
\text{std}\left[\gamma_T\right] &= \sqrt{\mathbb{E}\left[\gamma_T^2\right] - \mathbb{E}\left[\gamma_T\right]^2} \\
&= \frac{1}{\sqrt{T}}\left(\log\sqrt{\frac{g+r}{g-r}}\right).
\end{aligned}
\tag{17}
$$

Since std $[\gamma_T]$ goes to zero as T becomes large, γ_T becomes deterministic and tends to the value given by Eq. (15). Eqs. (15) and (17) allow us to determine how many rounds of the game, T^*, are required for the standard deviation of γ_T to become comparable to the expectation value of γ_T. This establishes a natural timescale that quantifies what we mean by “large” time. Setting the right hand sides of Eqs. (15) and (17) equal to each other and solving for T gives a timescale:

$$
T^* = \left(\frac{\log\left(\frac{g+r}{g-r}\right)}{\log\left(g^2 - r^2\right)}\right)^2.
\tag{18}
$$

For the parameters corresponding to Peters' bet, T^* corresponds to 75 rounds of the game. Thus “large time” corresponds to a number of iterations much larger than 75.

To conclude, consider a trajectory that grows exponentially with the long-time growth rate, Eq. (15). This is given by

$$
\begin{aligned}
\bar{x}_T &= \exp\left(\gamma_T T\right) \\
&= \exp\left(\mathbb{E}\left[\log x_T\right]\right) \\
&= \exp\log\left(\sqrt{g^2-r^2}\right)\,T \\
&= \left(\sqrt{g^2-r^2}\right)^T,
\end{aligned}
$$

which corresponds to the typical value of the original process x_T found in Eq. (10). This correspondence explains why analysing repeated multiplicative gambles in terms of grow rates is equivalent to reasoning about representative outcomes rather than expectation values of outcomes.

5 Optimal leverage and Kelly Criterion

Suppose we now consider a combination of a geometric random walk given by Eq.(1) and a deterministic multiplicative growth process

$$
x_{t+1} = h\,x_t,
\tag{19}
$$

with growth factor $h > 0$. In the investment analogy, Eq. (1) describes a risky asset and Eq. (**??**) describes a risk-free asset. The optimal leverage question is to determine what proportion of an investor's wealth should be invested in the risky asset versus the risk-free asset in order to maximise the time-averaged growth rate. Let us denote the leverage by f. That is to say, at every step a proportion f is invested in the risky asset and $1-f$ in the risk-free asset. The dynamics can be written as

$$x_{t+1} = \begin{cases} h\,(1-f)\,x_t + (g+r)\ f\,x_t & \text{with prob. } \frac{1}{2} \\ h\,(1-f)\,x_t + (g-r)\ f\,x_t & \text{with prob. } \frac{1}{2} \end{cases}$$
$$= \begin{cases} (G+R)\ x_t & \text{with prob. } \frac{1}{2} \\ (G-R)\ x_t & \text{with prob. } \frac{1}{2}. \end{cases} \tag{20}$$

where

$$G = (g-h)\ f + h$$
$$R = f\,r.$$

We still have a multiplicative random walk with different growth factors. We can now use the results of the previous section. The time averaged growth factor is

$$\begin{aligned}\Lambda(f) &= \sqrt{G^2 - R^2} \\ &= \sqrt{((g-h)\ f + h)^2 - f^2 r^2}.\end{aligned} \tag{21}$$

Differentiating with respect to f and setting $\Lambda'(f^*) = 0$ we obtain the optimal leverage value (or Kelly Criterion),

$$f^* = \frac{(g-h)\,h}{r^2 - (g-h)^2}. \tag{22}$$

Note that the optimal leverage can be negative which indicates that one should take a short position in the risky asset. For the original parameters, $g = \frac{21}{20}$, $r = \frac{9}{20}$ and $h = 1$, this gives $f^* = \frac{1}{4}$ and $\Lambda(f^*) = \frac{9}{4\sqrt{5}} \approx 1.00623$.