The η-compounding random walk

1 Appendix - binomial sums

We will need finite sums of powers weighted by binomial coefficients:

$$S_k(T) = \sum_{n=0}^{T} \binom{T}{n} n^k, \tag{1}$$

which are essentially the moments of the binomial distribution with $p = \frac{1}{2}$. For $k = 0$ we have $S_0(T) = 2^T$ from the definition of the binomial distribution. For $k \geq 0$, we can evaluate the sums sequentially by differentiating the binomial theorem,

$$(x+y)^T = \sum_{n=0}^{T} \binom{T}{n} x^n y^{T-n}$$

with respect to x and setting $x = y = 1$. For example, differentiating once with respect to x, we get

$$T\,(x+y)^{T-1} = \sum_{n=0}^{T} \binom{T}{n} n x^{n-1} y^{T-n}$$

and setting $x = y = 1$ gives $S_1(T)$. We can now differentiate again and use the formula for $S_1(T)$ to get $S_2(T)$ and so on. The first few sums are:

$$\begin{aligned}
S_0(T) &= \sum_{n=0}^{T} \binom{T}{n} &&= 2^T && (2)\\
S_1(T) &= \sum_{n=0}^{T} \binom{T}{n} n &&= T\,2^{T-1} && (3)\\
S_2(T) &= \sum_{n=0}^{T} \binom{T}{n} n^2 &&= T\,(T+1)\,2^{T-2} && (4)\\
S_3(T) &= \sum_{n=0}^{T} \binom{T}{n} n^3 &&= T^2\,(T+3)\,2^{T-3}. && (5)
\end{aligned}$$

Due to the symmetry of the binomial coefficients, we can always write

$$S_m(T) = \sum_{n=0}^{T} \binom{T}{T-n} n^m.$$

2 Definition of η-compounding

Adopting the notation in [4] to fit the way we usually write the isoelastic utility function, we define the generalised exponential and logarithm as

$$\exp_\eta(x) = \begin{cases} (1+(1-\eta)\,x)^{\frac{1}{1-\eta}} & 0 \leq \eta < 1 \\ \exp(x) & \eta = 1 \end{cases} \tag{6}$$

$$\log_\eta(x) = \begin{cases} \frac{1}{1-\eta}\left(x^{1-\eta} - 1\right) & 0 \leq \eta < 1 \\ \log(x) & \eta = 1 \end{cases}. \tag{7}$$

Following [2] we define the generalised compounding operator, $\otimes$, as

$$x \otimes y = \exp_\eta\left[\log_\eta(x) + \log_\eta(y)\right]. \tag{8}$$

An η-compounding process with growth factor, g, is one where the initial value is η-compounded by g at each step:

$$x_{t+1} = x_t \otimes g, \tag{9}$$

with $x_0 = X_0$. For $T \geq 1$, we have

$$\begin{aligned}
x_T &= X_0 \otimes \underbrace{g \otimes \ldots \otimes g}_{T\text{-times}} \\
&= X_0 \otimes \exp_\eta\left[\underbrace{\log_\eta(g) + \ldots + \log_\eta(g)}_{T\text{-times}}\right] \\
&= X_0 \otimes \exp_\eta\left[\log_\eta(g)\,T\right].
\end{aligned} \tag{10}$$

From this we see that the growth rate (the quantity with a dimension of time^{-1}) is $\log_\eta(g)$. From Eq. (6), we see that $\exp_\eta(x)$ is only well defined for

$$\begin{aligned}
x &\geq -\frac{1}{1-\eta} && 0 \leq \eta < 1 \\
x &> -\infty && \eta = 1.
\end{aligned}$$

Hence, for general values of $\eta \neq 1$, Eq. (10) is only well-defined for all $T \geq 1$ if $\log_\eta(g) > 0$. This implies that $g \geq 1$. Some confusion can arise for some rational values of η for which the branch point of Eq. (6) at $x = -\frac{1}{1-\eta}$ disappears. For example, if $\eta = \frac{1}{2}$, we have

$$\begin{aligned}
x_T &= \exp_{\frac{1}{2}}\left[\log_{\frac{1}{2}}(X_0) + T\log_{\frac{1}{2}}(g)\right] \\
&= \left[\sqrt{X_0} + \frac{T}{2}\,\log_{\frac{1}{2}}(g)\right]^2.
\end{aligned}$$

Although this is well defined for all T even when $\log_{\frac{1}{2}}(g) < 0$, it is not continuously reachable from $\eta = \frac{1}{2} \pm \epsilon$. Furthermore, this results in an unnatural model where a negative growth rate corresponds to increasing x_T. In what follows we will respect the constraint $g \geq 1$ in order to avoid such pathologies.

3 The η-compounding random walk

We are interested in studying the η-compounding random walk in discrete time. At each step, there are now two possible growth factors, $g+r$ and $g-r$, which we assume occur with equal probability:

$$x_{t+1} = \begin{cases} x_t \otimes (g+r) & \text{with probability } \frac{1}{2} \\ x_t \otimes (g-r) & \text{with probability } \frac{1}{2}. \end{cases} \tag{11}$$

with $x_0 = X_0$. In order to keep everything well-defined we need $g - r > 1$, assuming that $g > 0$ and $r > 0$.

We can generalise some of the calculations for the multiplicative random walk in [3] to the η-compounding random walk. If, after playing T rounds of the game, we experience n “wins" (and $T - n$ "losses"), then x_T will take the value

$$
\begin{aligned}
x_T =& X_0 \otimes \underbrace{g_1 \otimes \ldots \otimes g_1}_{n\text{-times}} \otimes \underbrace{g_2 \otimes \ldots \otimes g_2}_{T-n\text{-times}} \\
=& X_0 \otimes \exp_\eta \left[n \log_\eta(g_1)\right] \otimes \exp_\eta \left[(T-n) \log_\eta(g_2)\right] \\
=& X_0 \otimes \exp_\eta \left[n \log_\eta(g_1) + (T-n) \log_\eta(g_2)\right],
\end{aligned}
$$

where for brevity we write $g_1 = g + r$ and $g_2 = g - r$. The probability of this value is

$$
p(n) = \binom{T}{n} \left(\frac{1}{2}\right)^T, \tag{12}
$$

where $\binom{T}{n}$ is the binomial coefficient - the number of ways in which n wins can occur in a sequence of T rounds of the game. The expectation value of x_T is therefore

$$
\begin{aligned}
\mathbb{E}\left[x_T\right] =& \sum_{n=0}^{T} \binom{T}{n} \left(\frac{1}{2}\right)^T X_0 \otimes \exp_\eta \left[n \log_\eta(g_1)\right. \\
&\left.+(T-n) \log_\eta(g_2)\right] \\
=& \frac{1}{2^T} \sum_{n=0}^{T} \binom{T}{n} \left[X_0^{1-\eta} - T + n g_1^{1-\eta}\right. \\
&\left.+(T-n) g_2^{1-\eta}\right]^{\frac{1}{1-\eta}} \\
=& \frac{1}{2^T} \sum_{n=0}^{T} \binom{T}{n} \left[X_0^{1-\eta} + G_2 T + (G_1 - G_2) n\right]^{\frac{1}{1-\eta}}
\end{aligned} \tag{13}
$$

where we write $G_{1,2} = g_{1,2}^{1-\eta} - 1$ to keep the notation compact.

4 Exact expectation value for $\eta = \frac{1}{2}$

This sum in Eq. (13) can be done exactly at least for the case $\eta = \frac{1}{2}$ where we have

$$
\mathbb{E}\left[x_T\right] = \frac{1}{2^T} \sum_{n=0}^{T} \binom{T}{n} \left[\sqrt{X_0} + G_2 T + (G_1 - G_2) n\right]^2
$$

After expanding the square in the summand and using Eqs. (2)-(4) some algebra leads to

$$
\begin{aligned}
\mathbb{E}\left[x_T\right] =& \frac{1}{4}(G_1 + G_2)^2 T^2 \\
&+ \left[\frac{1}{4}(G_1 - G_2)^2 + \sqrt{X_0}(G_1 + G_2)\right] T \\
&+ X_0.
\end{aligned}
$$

In the original notation this is:

$$
\begin{aligned}
\mathbb{E}\left[x_T\right] =& \frac{1}{4} \left(\sqrt{g+r} + \sqrt{g-r} - 2\right)^2 T^2 \\
&+ \left[(\sqrt{g+r} + \sqrt{g-r} - 2)\sqrt{X_0}\right. \\
&\left.+\frac{1}{4}(\sqrt{g+r} - \sqrt{g-r})^2\right] T \\
&+ X_0.
\end{aligned} \tag{14}
$$

For large T we find

$$
\mathbb{E}\left[x_T\right] \sim \frac{1}{4} \left(\sqrt{g+r} + \sqrt{g-r} - 2\right)^2 T^2. \tag{15}
$$

5 Most likely value for the η-compounding random walk

The *most likely* value of x_T can be found by finding n^*, the value of n that maximises the probability, Eq. (12). This is

$$
\begin{aligned}
n^* =& \arg\max_n \binom{T}{n} \\
=& \frac{T}{2}.
\end{aligned}
$$

Thus the most likely value of x_T is

$$
\begin{aligned}
\tilde{x}_T =& X_0 \otimes \exp_\eta \left[\frac{T}{2} \left(\log_\eta(g+r) + \log_\eta(g-r)\right)\right] \\
=& \left(X_0^{1-\eta} + \left((g+r)^{1-\eta} + (g-r)^{1-\eta} - 2\right) \frac{T}{2}\right)^{\frac{1}{1-\eta}}.
\end{aligned} \tag{16}
$$

For large T, we find

$$
\tilde{x}_T \sim \left(\frac{1}{2}\left((g+r)^{1-\eta} + (g-r)^{1-\eta} - 2\right)\right)^{\frac{1}{1-\eta}} T^{\frac{1}{1-\eta}}. \tag{17}
$$

Note that for $\gamma = \frac{1}{2}$, this agrees with Eq. (15) which suggests that for the $\frac{1}{2}$-compounding random walk, the expected value is representative. For $\eta = \frac{1}{2}$, it turns out that the difference between the expected value and the most likely value is sub-leading in T:

$$
\mathbb{E}\left[x_T\right] - \tilde{x}_T = \frac{1}{2} T \left(g - \sqrt{g-r}\sqrt{g+r}\right) \tag{18}
$$

6 Time-averaged growth rate

From Eq. (11), the quantity $y_t = \log_\eta x_t$ follows a simple additive random walk:

$$
y_{t+1} = \begin{cases} y_t + a & \text{with probability } \frac{1}{2} \\ y_t + b & \text{with probability } \frac{1}{2}, \end{cases} \tag{19}
$$

where

$$
\begin{aligned}
a =& \log_\eta(g+r) \\
b =& \log_\eta(g-r).
\end{aligned}
$$

If, after playing T rounds of the game, we experience n “wins" (and $T-n$ "losses"), then y_T will take the value

$$y_T = n\,a + (T-n)\,b.$$

The corresponding probability is again given by Eq. (12). The expectation value of y_T is

$$\begin{aligned}\mathbb{E}[y_T] =& \sum_{n=0}^{T}\binom{T}{n}\left(\frac{1}{2}\right)^T [n\,a + (T-n)\,b]\\ =& \frac{T}{2}\left(a\sum_{n=0}^{T}\binom{T}{n} n + b\sum_{n=0}^{T}\binom{T}{n}(T-n)\right)\\ =& \frac{T}{2}(a+b)\\ =& \frac{T}{2}\left(\log_\eta(g+r)+\log_\eta(g-r)\right).\end{aligned} \tag{20}$$

where the second-but-last line follows from the identity Eq. (3). Thus the time averaged growth rate corresponds to the growth rate of the most likely trajectory. We then find that

$$\begin{aligned}\exp_\eta\left(\mathbb{E}\left[\log_\eta(x_T)\right]\right) =& \exp_\eta\left[\frac{T}{2}\left(\log_\eta(g+r)+\log_\eta(g-r)\right).\right]\\ =& \left(\frac{1}{2}\left((g-r)^{1-\eta}+(g+r)^{1-\eta}-2\right)\right)^{\frac{1}{1-\eta}} T^{\frac{1}{1-\eta}}.\end{aligned} \tag{21}$$

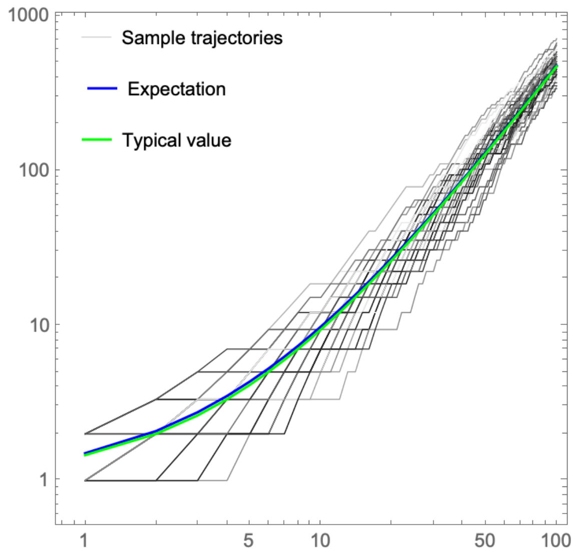

Figure 1: Some sample trajectories for $\eta=\frac{1}{2}$ with $g=\frac{3}{2}$ and $r=\frac{1}{2}$.

7 Exact results for $\eta=\frac{2}{3}$

Similar calculations to $\eta=\frac{1}{2}$ with a lot more algebra were done by H. Reynolds for $\eta=\frac{2}{3}$. The expectation value is

$$\begin{aligned}\mathbb{E}[x_T] =& \frac{1}{8}(G_1+G_2)^3\,T^3\\ &+\frac{3}{8}(G_1+G_2)\Big[(G_1-G_2)^2\\ &+2X_0^{\frac{1}{3}}(G_1+G_2)\Big]\,T^2\\ &+\frac{3}{4}X_0^{\frac{1}{3}}\Big[2X_0^{\frac{1}{3}}(G_1+G2)\\ &+(G_1-G_2)^2\Big]\,T\\ &+X_0.\end{aligned} \tag{22}$$

The most likely value is

$$\begin{aligned}\widetilde{x}_T =& \frac{1}{8}(G_1+G_2)^3\,T^3\\ &+\frac{3}{4}X_0^{\frac{1}{3}}(G_1+G_2)^2\,T^2\\ &+\frac{3}{2}X_0^{\frac{2}{3}}(G_1+G2)T\\ &+X_0.\end{aligned} \tag{23}$$

The difference between the two now grows quadratically in time:

$$\begin{aligned}\mathbb{E}[x_T]-\widetilde{x}_T =& \frac{3}{8}(G_1+G_2)(G_1-G_2)^2\,T^2\\ &+\frac{3}{4}X_0^{\frac{1}{3}}(G_1-G_2)^2\,T.\end{aligned} \tag{24}$$

8 Asymptotic analysis for $0\le\eta\le1$

We can adapt the analysis of Redner [3] to calculate the leading order behaviour for $0\le\eta\le1$. Write the random multiplicative process in its more general form as

$$x_{t+1} = \begin{cases} x_t\otimes z_1 & \text{with probability } p\\ x_t\otimes z_2 & \text{with probability } q.\end{cases} \tag{25}$$

with $x_0 = 1$. The binomial distribution is

$$P(T,n) = \frac{T!}{(T-n)!\,n!}\,p^n q^{T-n}, \tag{26}$$

where $q = 1-p$. We use Stirling's approximation,

$$n! = \sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n \left(1 + \frac{1}{12\,n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right), \tag{27}$$

to approximate the factorials. We then take the continuum limit, $T \to \infty$ with $x = \frac{n}{T}$ fixed, to obtain

$$P(x,T) = \frac{1}{\sqrt{2\pi T\,x\,(1-x)}}\,\mathrm{e}^{T\,\phi(x)}\left(1 + \frac{1}{12}\left(1 - \frac{1}{x} - \frac{1}{1-x}\right)\frac{1}{T} + \mathcal{O}\left(\frac{1}{T^2}\right)\right), \tag{28}$$

where

$$\phi(x) = x\log(p) + (1-x)\log(q) - x\log(x)(1-x)\log(1-x). \tag{29}$$

$$\mathbb{E}\left[x_T^k\right] \approx T\int_{-1}^{1} \mathrm{e}^{-T\,h(x,T,\eta,k)}\,dx, \tag{30}$$

where

$$h(x,T,\eta,k) = f(x,T) + g(x,T,\eta,k)$$

where

$$f(x,T) = x\log(x) + (1-x)\log(1-x) - x\log(p) - (1-x)\log(q) + \frac{1}{2T}\log\left[2\pi T\,x(1-x)\right] \tag{31}$$

and

$$g(x,T,\eta,k) = \begin{cases} -\frac{1}{T}\frac{1}{1-\eta}\log\left[1 + T\left(x\left[z_1^{k(1-\eta)} - 1\right] + (1-x)\left[z_2^{k(1-\eta)} - 1\right]\right)\right] & \text{if } 0 \le \eta < 1. \\ -kx\log(z_1) - k(1-x)\log(z_2) & \text{if } \eta = 1. \end{cases} \tag{32}$$

The standard form of a Laplace integral is

$$I(T) = \int_{-\infty}^{\infty} F(x)\,\mathrm{e}^{T\,\phi(x)}\,dx, \tag{33}$$

The integral (33) has the following asymptotic behaviour as $T \to \infty$ [1]:

$$I(T) \sim \sqrt{\frac{2\pi}{-T\,\phi''(x^*)}}\,\mathrm{e}^{T\,\phi(x^*)}\left(F(x^*) + B(x^*)T^{-1} + \mathcal{O}\left(T^{-2}\right)\right), \tag{34}$$

where x^* is the maximum of $\phi(x)$ and

$$B(x^*) = -\frac{F''(x^*)}{2\,\phi''(x^*)} + \frac{F(x^*)\phi''''(x^*)}{8\,\phi''(x^*)^2} + \frac{F'(x^*)\phi'''(x^*)}{2\,\phi''(x^*)^2} - \frac{5\,F(x^*)\phi'''(x^*)^2}{24\,\phi''(x^*)^3} \tag{35}$$

To compress the notation, define

$$Z_{1,2} = z_{1,2}^{1-\eta} - 1.$$

The final answer is

$$\mathbb{E}\left[x_T\right] = T^{\frac{1}{1-\eta}}\left(pZ_1 + qZ_2\right)^{\frac{1}{1-\eta}}\left[1 + \frac{1}{T}\left(\frac{pq}{2}\frac{\eta}{(1-\eta)^2}\left(\frac{Z_1 - Z_2}{pZ_1 + qZ_2}\right)^2 + \frac{X_0^{1-\eta}}{1-\eta}\frac{1}{pZ_1 + qZ_2}\right) + \mathcal{O}\left(\frac{1}{T^2}\right)\right]$$

References

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