Excerpts from: ACHIEVING HIGHER ORDER CONVERGENCE FOR THE PRICES OF EUROPEAN OPTIONS IN BINOMIAL TREES

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Abstract: A new family of binomial trees as approximations to the BlackScholes model is introduced. For this class of trees, the existence of complete asymptotic expansions for the prices of vanilla European options is demonstrated and the rst three terms are explicitly computed. As special cases, a tree with third order convergence is constructed and the conjecture of Leisen and Reimer that their tree has second order convergence is proven.

We work in the Black-Scholes model, and have, as usual, the parameters, r the risk-free rate, T expiry time, σ the volatility, and K the strike. We denote the spot price at time zero by S, and at time t by S_t. The riskless bond has value e^{rt} at time t. The process followed by the stock before discretisation in the risk-neutral measure (i.e. the martingale measure if the bond is numeraire,) is

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{1}$$

and if we take the stock as numeraire, we get in the pricing measure (henceforth, called the *stock measure*)

$$dS_t = (r + \sigma^2)S_t dt + \sigma S_t dW_t \tag{2}$$

A family of recombining binomial trees is a sequence of discrete processes, X_i^n, defined at discrete times which converges to the process in the Black-Scholes model an interval [0,T]. The crucial properties are

- The process X_i^n is defined at the times $T_i = n$ for $i = 0, 1, \dots, n$.
- The processes are Markovian.
- The value of X_i^n takes precisely i + 1 values with non-zero probability.
- The distribution of X_{i+1}^n/X_i^n is a two point distribution independent of X_i; and is the same for each i.
- There exists a choice of probability equivalent to the given distribution such that

$$\mathbb{E}(X_{i+1}^n/X_i^n) = e^{rT/n} \tag{3}$$

In simpler terms, for a given n, the tree is defined by the choice of up and down moves which will be the same in multiplicative terms at every step. So an up move followed by a down move is the same as a down move followed by an up move. The values of up, U_n, and down, D_n, need to be chosen so that

$$D_n < e^{rT/n} < U_n \tag{4}$$

so that

$$p_n = \frac{e^{rT/n} - D_n}{U_n - D_n} \tag{5}$$

References

[1] Mark.S. Joshi “Achieving higher order convergence for the prices of european options in binomial trees “