

On Optimally Partitioning a Text to Improve Its Compression [1]

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Problem: Text Partitioning

We have a **compressor** \mathcal{C} and a **text** T of size n , we want to **divide** T into $k \leq n$ parts, $T[1..i_1 - 1]T[i_1..i_2 - 1]\dots T[i_{k-1}..n]$ and **compress each** of them individually with \mathcal{C} to improve the overall compression

Note: We do **not** *permute* the string.
We are only interested in *partitioning* it.

Text Partitioning Example

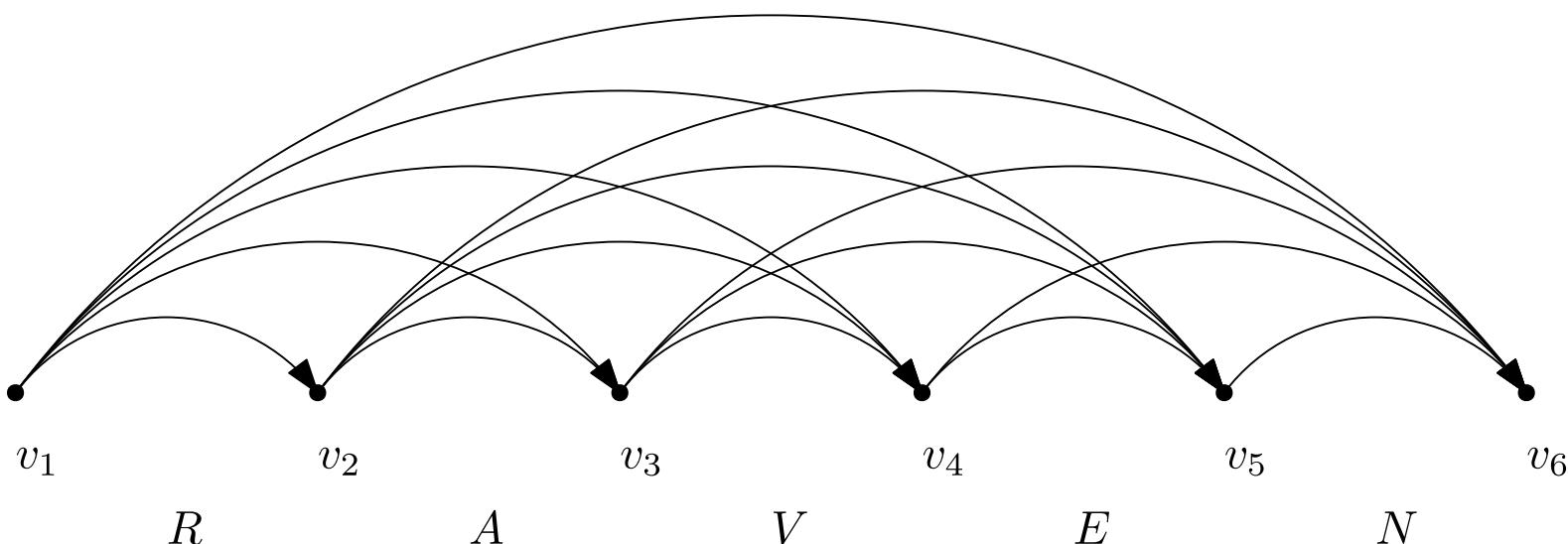
Suppose we have the text $T = a^n b^n$.

If we compress the entire text at once we should use one bit per symbol, or $O(n)$ **bits**.

If instead we partition the text to compress a^n and b^n separately we can compress the whole string using only $O(\log_2(n))$ **bits** indicating just the length of each substring.

Reduction to SSSP

We can model the partition problem as a **directed graph** with $n + 1$ *ordered* vertices, where an edge exists between v_i and v_j only if $1 \leq i < j \leq n + 1$



Reduction to SSSP - Bijection between paths and partitions

In this graph each **edge** represents a **substring** of the text.

We can then show that there exists a **bijection** from each **path** $\pi = (v_1, v_{i_1}) \dots (v_{i_k}, v_{n+1})$ in the graph, and a **partitioning** of the text T in the form $T[1..i_1 - 1]T[i_1..i_2 - 1]\dots T[i_k..n]$

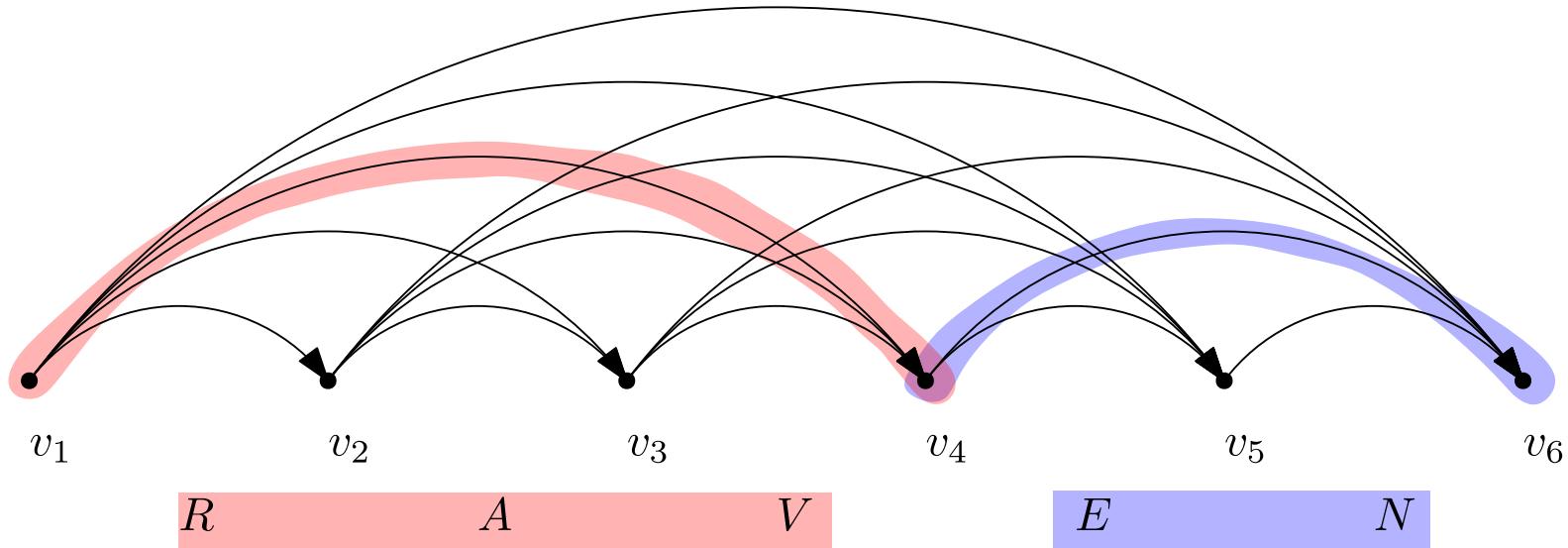


Figure 2: We can map the path $\pi = (v_1, v_4) | (v_4, v_6)$ to the partitioning of the string $T[1, 3] | T[4, 5]$

Reduction to SSSP - Bijection between paths and partitions

If we weight each edge (i, j) of the graph by the cost of compressing the corresponding text segment $w(i, j) = |\mathcal{C}(T[i, j - 1])|$, we can solve the partitioning problem *optimally* computing the **Single Source Shortest Path (SSSP)**

It can be computed efficiently in $O(|E|)$ time using a classic dynamic programming algorithm.

Problems:

1. Our graph has $O(n^2)$ nodes by construction
2. To initialize the weight $w(i, j)$ we should execute \mathcal{C} on every substring of the text

Assumption on \mathcal{C}

- Our compressor is *monotonic*: the compressed output on a suffix or a prefix of the string is always smaller than the compression on the whole string:

$$|\mathcal{C}(T[i, j])| \geq |\mathcal{C}(T[i, j - 1])|$$

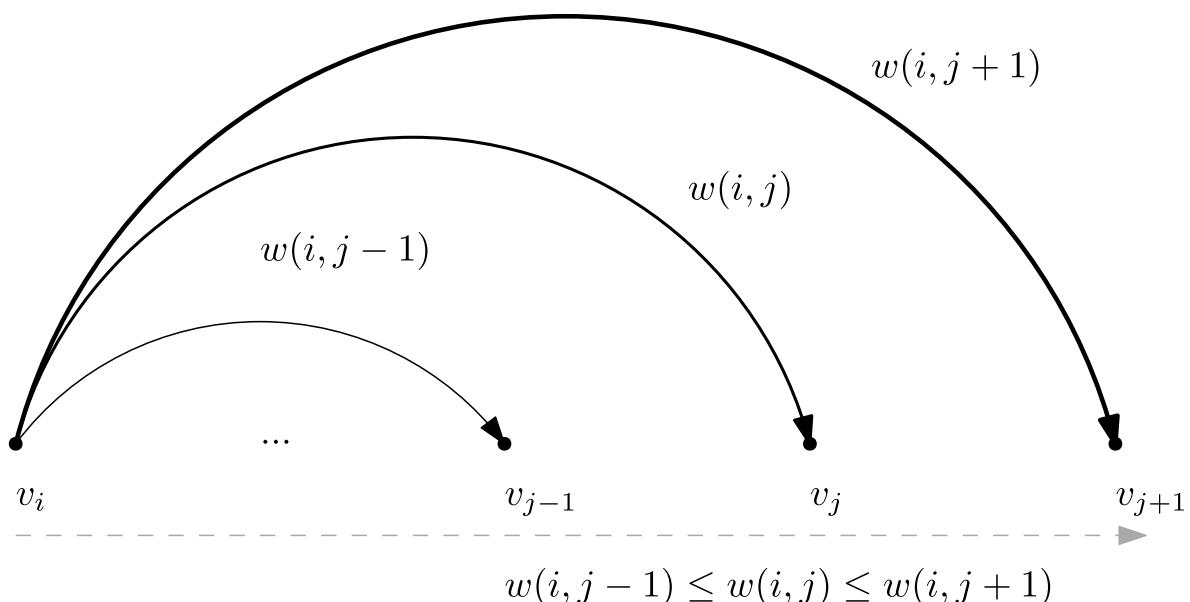
$$|\mathcal{C}(T[i, j])| \geq |\mathcal{C}(T[i + 1, j])|$$

- We can compute the size of the compressed output incrementally: computing $|\mathcal{C}(T[i, j])|$ from the state of $\mathcal{C}(T[i - 1, j])$ or $\mathcal{C}(T[i, j - 1])$ takes constant time

**How the property of monotonicity affect
the topology of our DAG?**

Monotonicity of w

Due to the monotonicity of the compressor for every node $1 \leq i < k < j \leq n + 1$ we have that $w(i, k) \leq w(i, j)$



Sparsification of the DAG

Thanks to this property we can obtain an approximated algorithm by **sparsifying** the graph thus selecting only some edges.

We are able to obtain a **$(1 + \varepsilon)$ -approximation**, for every $\varepsilon \geq 0$, with a time complexity of $O(n \log_{1+\varepsilon} L)$

where $L = w(1, n)$, so the cost of compressing the entire text.

This algorithm can be applied to every dynamic programming algorithm in the form $E[j] = \min_{1 \leq i < j} (E[i] + w(i, j))$ when w is *monotone!*

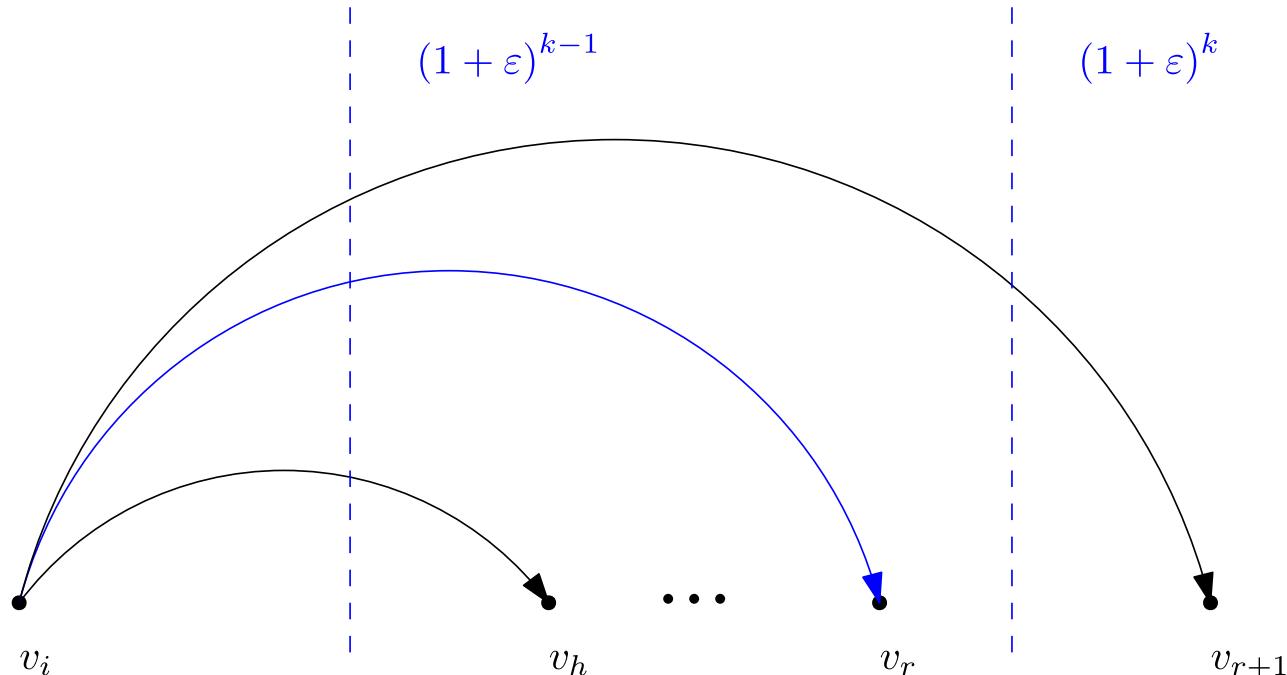
Key Idea: ε -maximal edges

How we can select some edges to obtain the $(1 + \varepsilon)$ approximation factor?

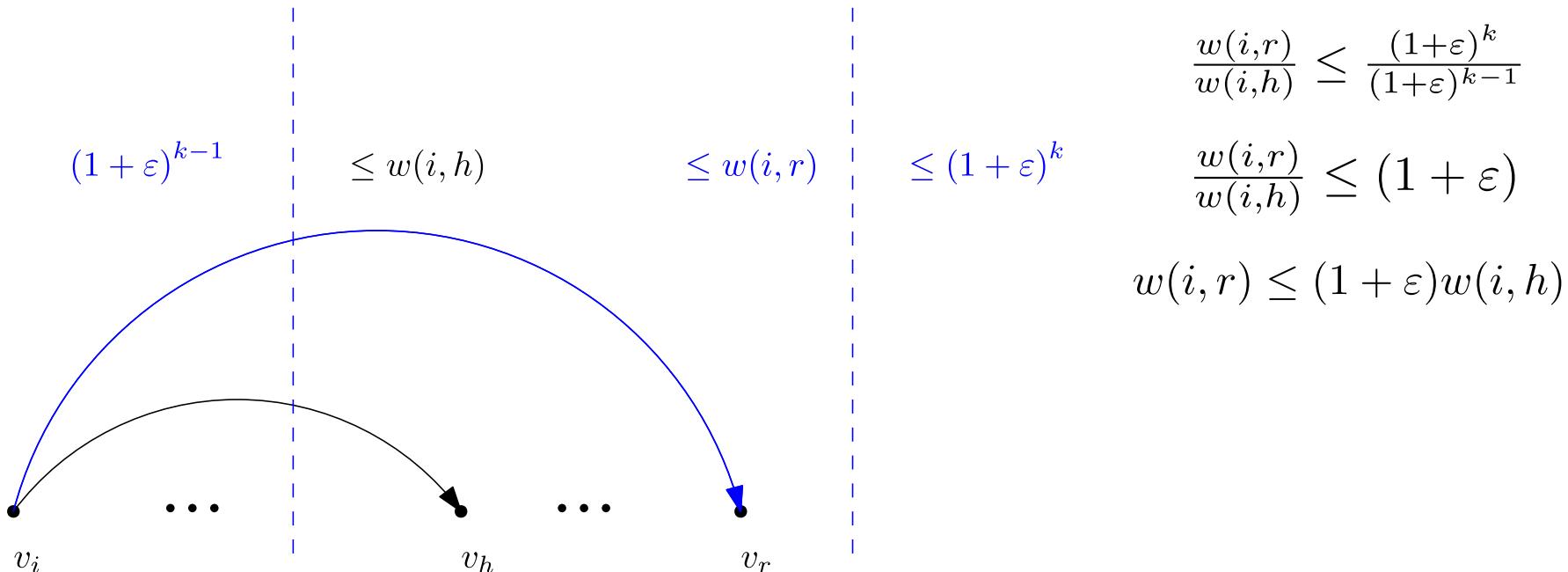
For each node i select the ε -maximal edges, so the outgoing edge from i that satisfy one of these conditions:

- The edges (i, j) such that $w(i, j) \leq (1 + \varepsilon)^k < w(i, j + 1)$ for any integer $k \geq 1$
- The last outgoing edge: $(i, n + 1)$

So we select the best approximations of the powers of $(1 + \varepsilon)$ from below: We then have at most $\log_{1+\varepsilon} L$ outgoing edges for each node.



Each edge is then “*covered*” by an ε -maximal edge: The weight of the edge is then approximated by $(1 + \varepsilon)$ times the weight of the maximal edge that covers it.

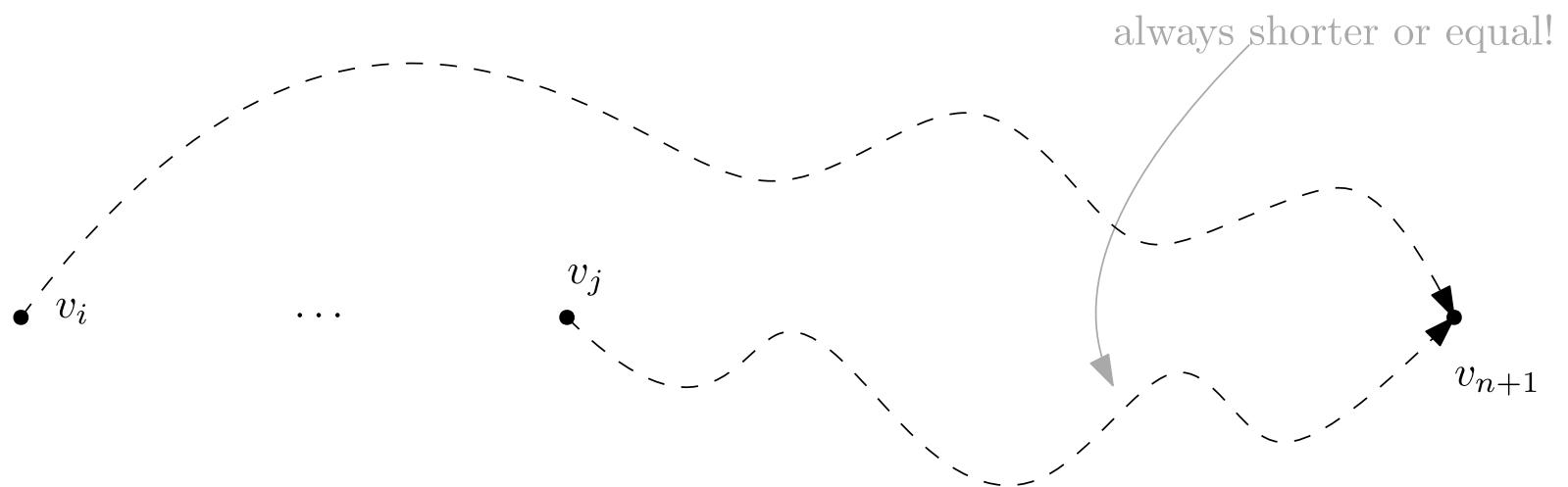


Our edges are increasing and can be approximated but what can we say about *paths* in this graph?

Lemma 1

Let $d_{\mathcal{G}}(i)$ be the cost of the shortest path π_i in our graph \mathcal{G} from v_i to v_{n+1} then

for all the vertices $i, j : 1 \leq i < j \leq n + 1$, $d_{\mathcal{G}}(i) \geq d_{\mathcal{G}}(j)$



Theorem

Let \mathcal{G} be the full graph and \mathcal{G}_ε be the graph containing only ε -maximal edges, then $d_{\mathcal{G}_\varepsilon}(i) \leq (1 + \varepsilon)d_{\mathcal{G}}(i)$ for every integer $1 \leq i \leq n + 1$.

Proof by induction on $\pi(i)$:

- **Base**, trivial case for $n + 1$
- Let $\pi(i) = (v_i, v_{t_1})..(v_{t_h}, v_n)$ the shortest path starting from node v_i and let $d_{\mathcal{G}}(i) = w(i, t_1) + d_{\mathcal{G}}(t_1)$ be its cost. We choose the ε -maximal node r that covers t_1 : So $r > t_1$ and we already know (by our "key idea") that

$$w(i, r) \leq (1 + \varepsilon)w(i, t_1)$$

By *Lemma 1*:

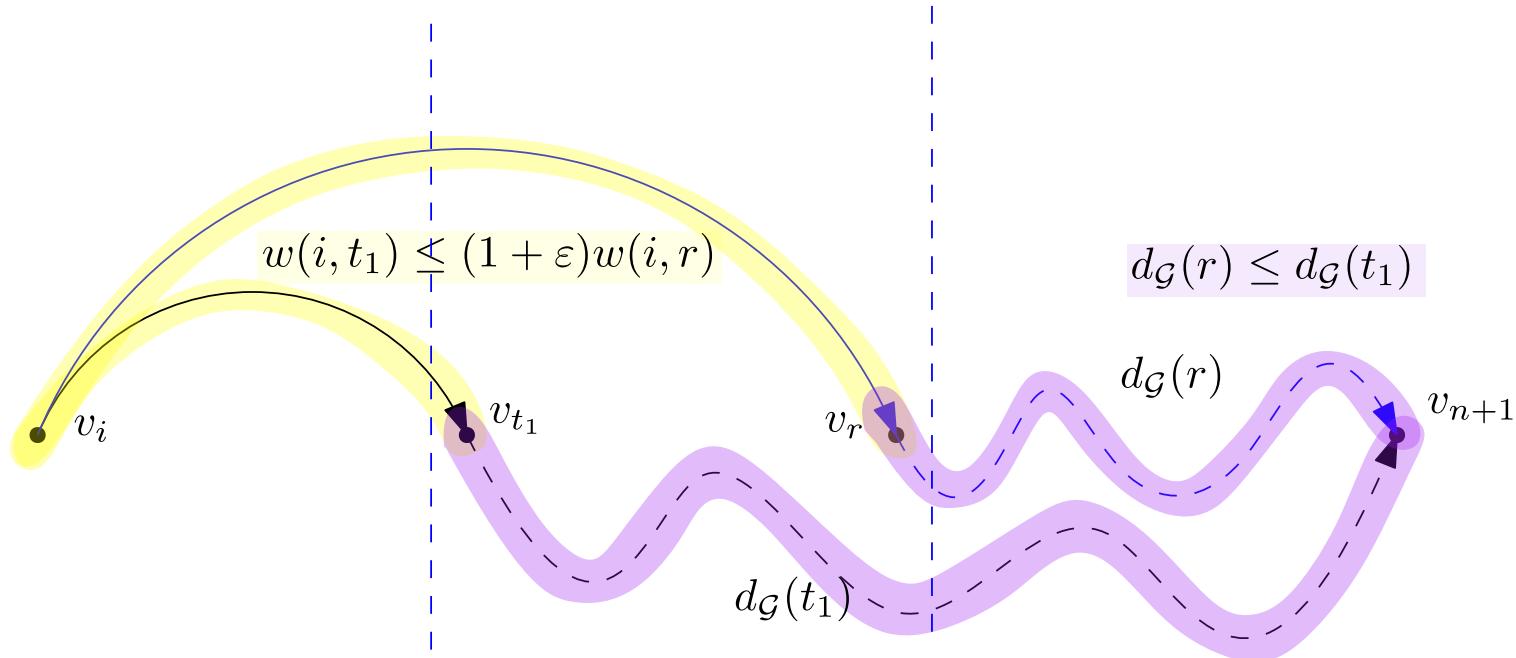
$$d_{\mathcal{G}}(r) \leq d_{\mathcal{G}}(t_1)$$

By inductive hypothesis:

$$d_{\mathcal{G}_\varepsilon}(r) \leq (1 + \varepsilon)d_{\mathcal{G}}(r) \leq (1 + \varepsilon)d_{\mathcal{G}}(t_1)$$

In the end

$$d_{\mathcal{G}_\varepsilon}(i) = w(i, r) + d_{\mathcal{G}_\varepsilon}(r) \leq (1 + \varepsilon)(w(i, t_1) + d_{\mathcal{G}}(t_1)))$$



Problem: DAG Construction

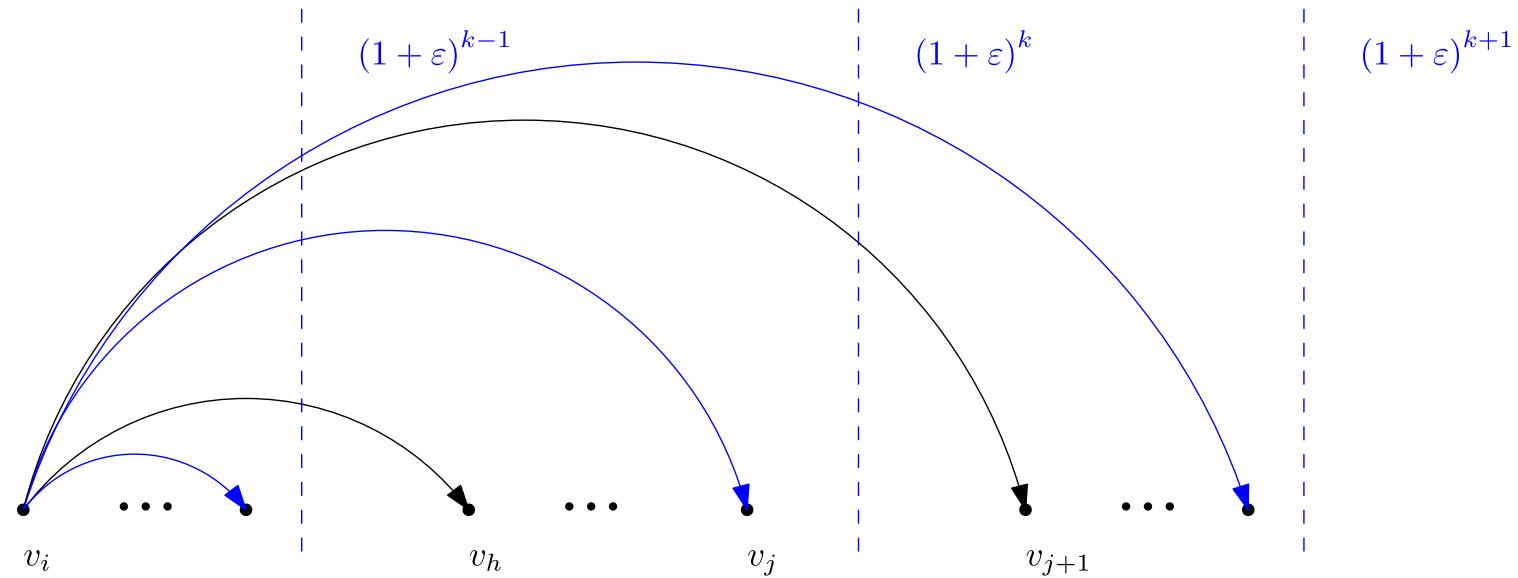
We still have two problems:

1. if we construct *naively* this graph we should remove edges from a $O(n^2)$ graph
2. and we should compute the weight of each edge of the graph

We can solve both these problems efficiently at once: We can find the ε -maximal edges efficiently on the fly!

Sliding windows

We keep $\log_{1+\varepsilon} L$ sliding windows all starting at v_i , but ending in a different position. The k -th window find the k -th ε -maximal edge.



w_{k-1}

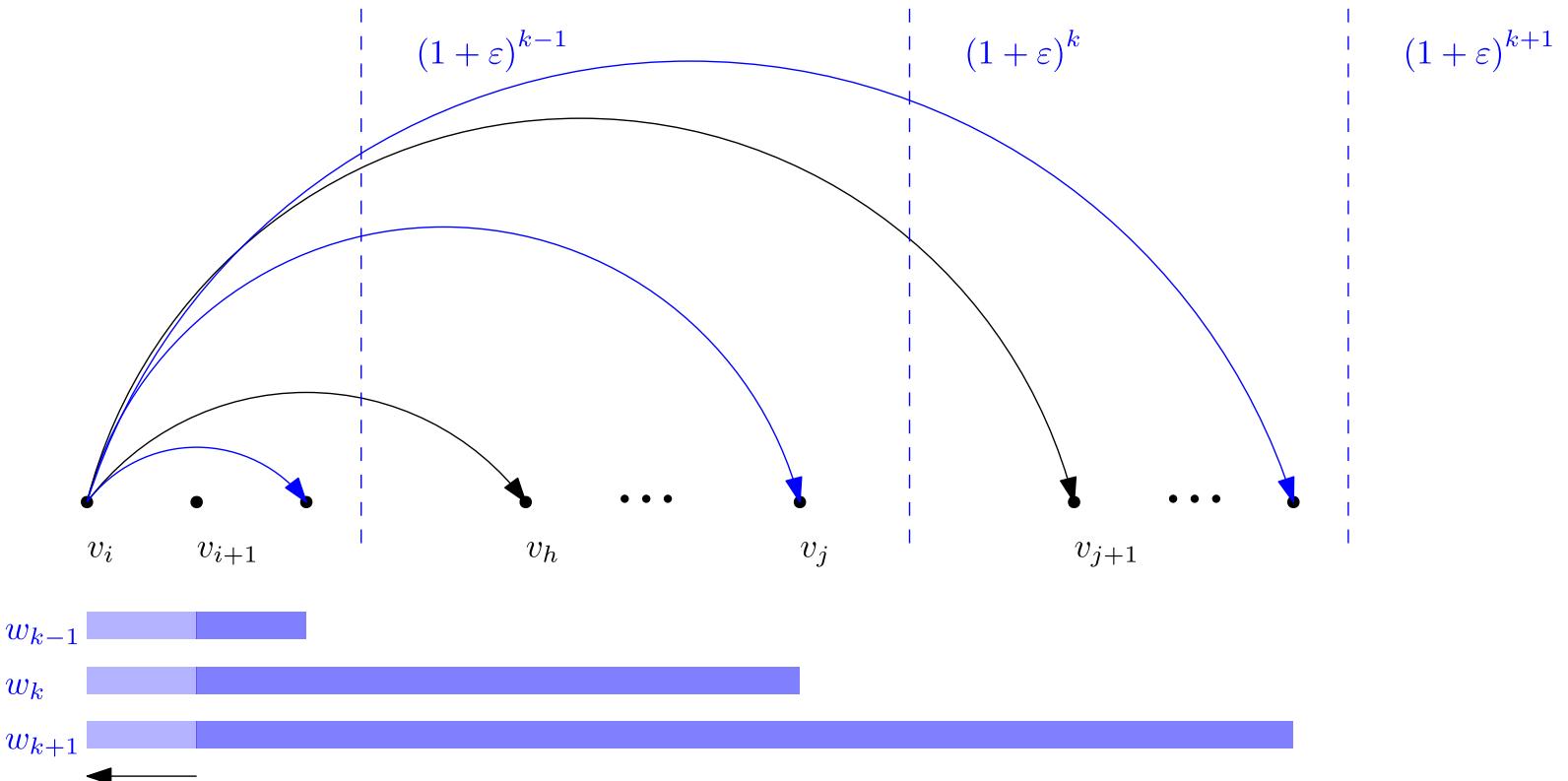
w_k

w_{k+1}

Sliding windows

For each compressor we should implement 2 operations on the windows `advance_left`, `advance_right`: The first operation advances the start of **all** the windows to the left.

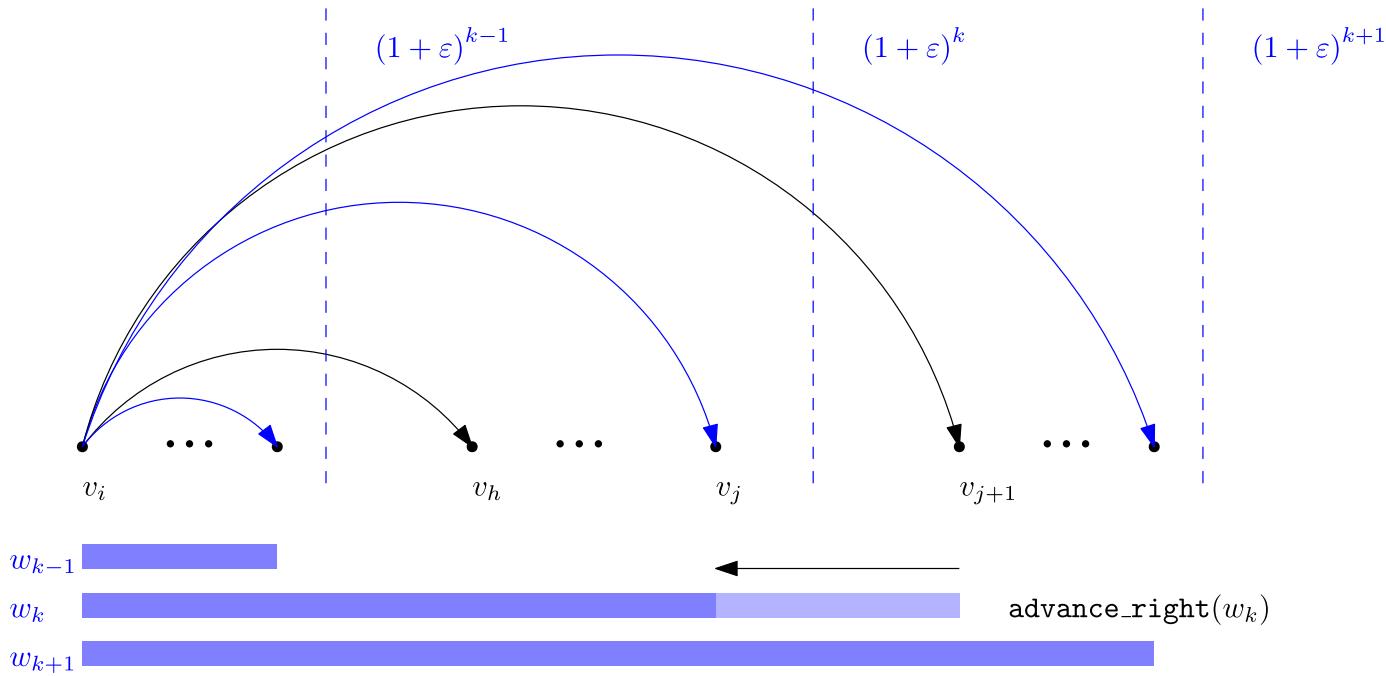
The cost of all the windows increase or stay the same.



Sliding windows

`advance_right` advance the end of the k -th window of one position to the left.

We call this function until we reach the first edge smaller than $(1 + \varepsilon)^k$, so until we find the k -th maximal edge starting from node i .



if the operations `advance_left` and `advance_right` have respectively a complexity of $O(L)$ and $O(R)$ our algorithm execute asymptotically $O(Ln + Rn \log_{1+\varepsilon} n)$ steps

The authors provide several implementations of the sliding windows framework to estimate the size of different compressors, among the others statistical compressors (using 0-th order and k-order entropy)

Computing Zero Order Entropy

Zero-th order entropy is a well-known lower bound for the performance of statistical compressors.

For each windows w_k that covers the substring $T[i..j]$, we maintain a histogram, $A_k[c]$, indexed by the symbol $c \in \Sigma$ and the value

$$E_k = \sum_{c \in \Sigma} A_k[c] \log_2 A_k[c]$$

Using E_k , we can calculate a lower bound on the output of the statistical compressor, $|\mathcal{C}(T[i..j])|$ based on the zero-th order entropy as

$$|T[i..j]| H_0(T[i..j]) = |T[i..j]| \log_2 |T[i..j]| - E_k$$

From this we can calculate incrementally the value of E_{k+1} removing the old term from the summation and adding the new one:

Let $c = T[j + 1]$ then

$$E_{k+1} = E_k - A_k[c] \log_2 A_k[c] + (A_k[c] + 1)(\log_2 A_k[c] + 1)$$

Thank You!

Bibliography

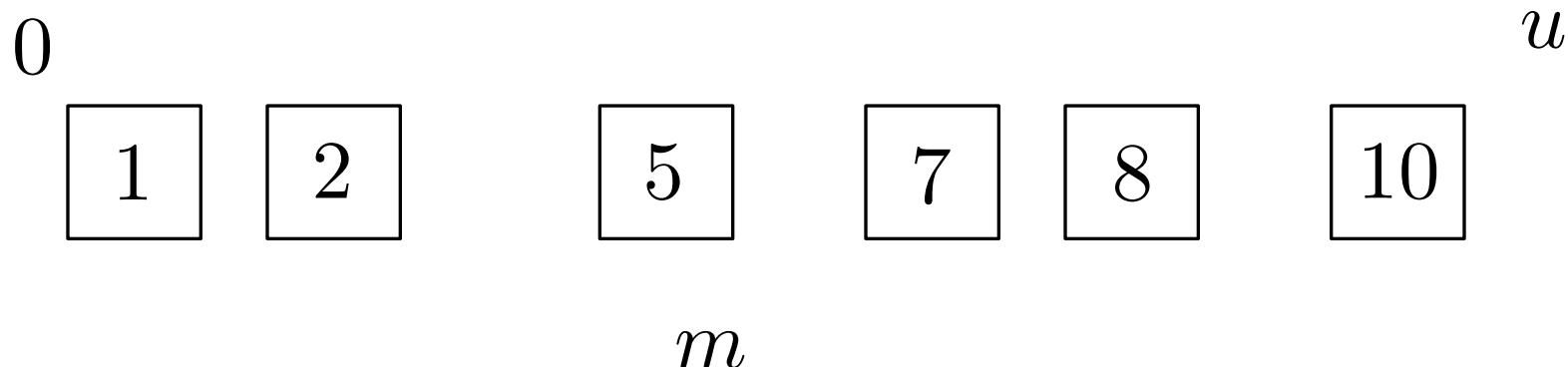
- [1] P. Ferragina, I. Nitto, and R. Venturini, "On Optimally Partitioning a Text to Improve Its Compression," in *Algorithms - ESA 2009*, A. Fiat and P. Sanders, Eds., Berlin, Heidelberg: Springer Berlin Heidelberg, 2009, pp. 420–431.
- [2] G. Ottaviano and R. Venturini, "Partitioned Elias-Fano Indexes," in *Proceedings of the 37th International ACM SIGIR Conference*, in SIGIR '14. New York, NY, USA: ACM, 2014, pp. 273–282. doi: [10.1145/2600428.2609615](https://doi.org/10.1145/2600428.2609615).

Bonus Slides: Partitioned Elias-Fano

Elias-Fano Data Structure

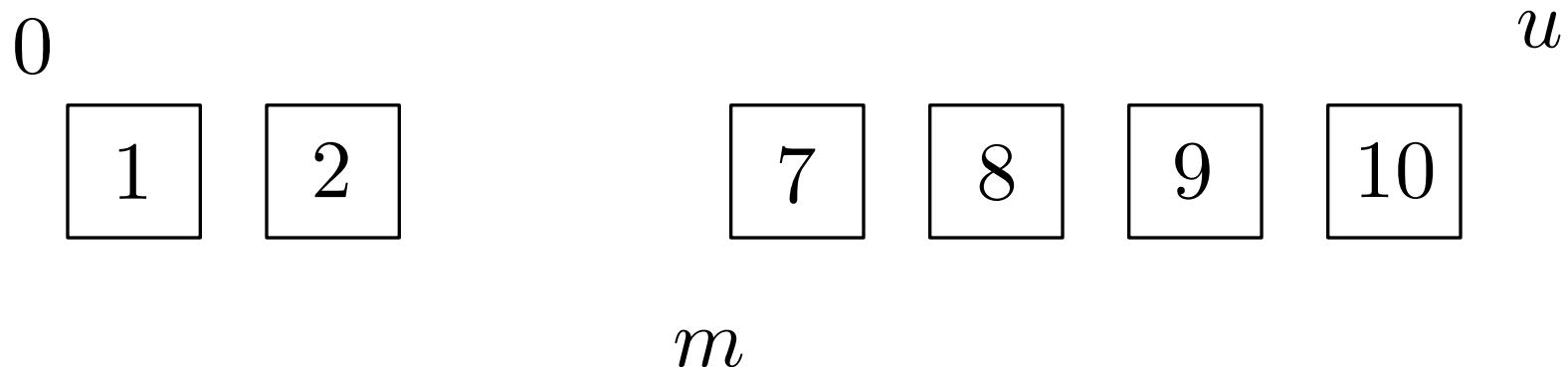
A compact data structure to store a set of m monotonically increasing integers upper-bounded by u .

It uses $\approx \lceil \log_2 \frac{u}{m} \rceil + 2$ bits per element.



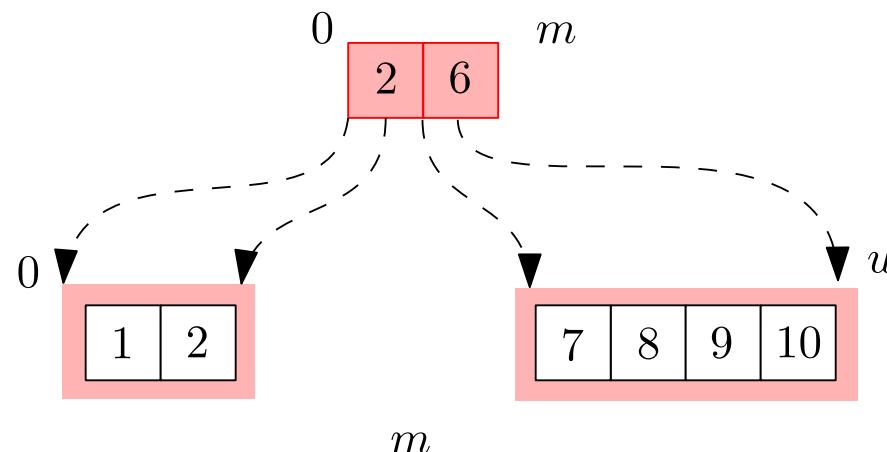
Note that $\frac{u}{m}$ is the average distance between consecutive elements. It doesn't exploit the distribution of the data, but denser lists require fewer bits.

Some sequence are more compressible than others



Partitioned Elias-Fano [2]

We can improve compression by exploiting clusters of data with a two-level structure. The first level determines the bounds of the b clusters, and the second level contains smaller Elias-Fano lists.



How can we find the best partitioning to minimize the space occupancy of both levels?

We can use our partitioning algorithm, assigning a weight to each edge based on the number of bits required to represent the partition in the first level and the Elias-Fano structure in the second level.

The authors also improved the bound by showing that substituting an edge in the path with two sub-edges is always bounded by a constant factor.