On Optimally Partitioning a Text to Improve Its Compression

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Venice, Italy – 23 December 2025

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Problem: Text Partitioning

We have a compressor $\mathcal{C}$ and a Text T of size n, it's possible to divide T into $k \leq n$ parts, $T[1..i_1 - 1]T[i_1..i_2 - 1]...T[i_{k-1}..n]$ and compress each of them individually with $\mathcal{C}$ to improve the overall compression?

Intuitively we can group the most similar parts of the string together so each partition is better compressed by $\mathcal{C}$.

We do **not** permute the string we are only interested on partitioning it.

Text Partitioning Example

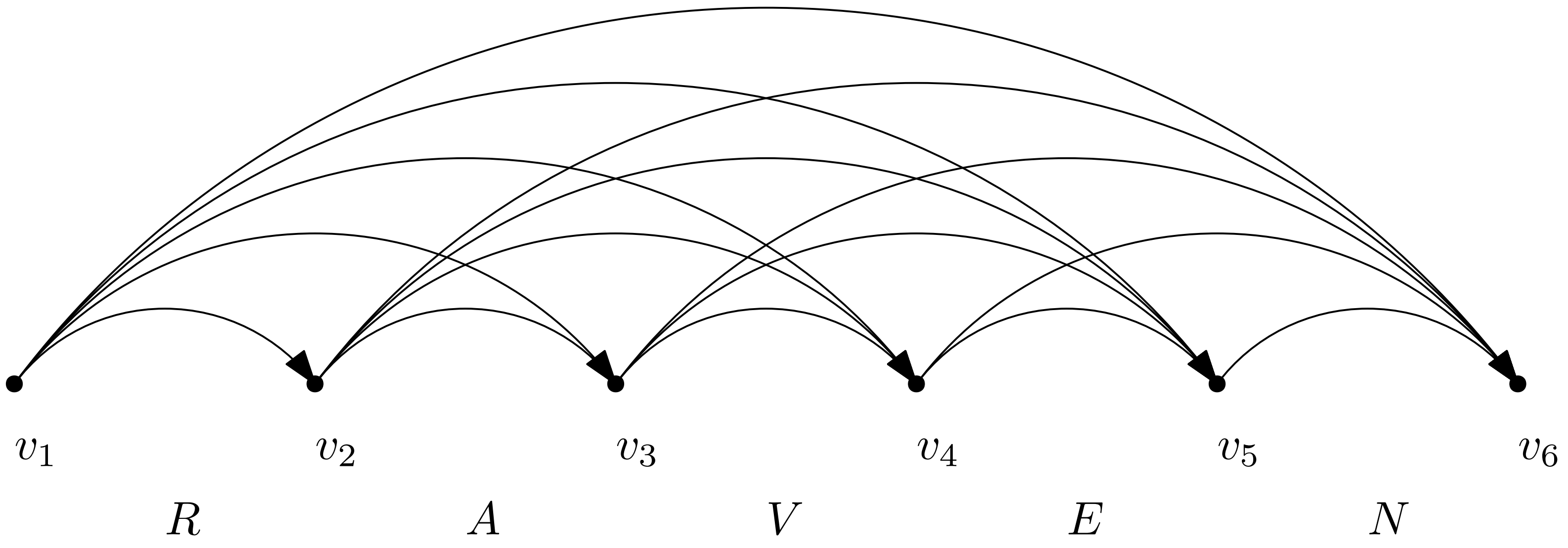
TODO We have a compressor $\mathcal{C}$ and a Text T of size n, it's possible to divide T into $k \leq n$ parts so to improve the overall compression?

Intuitively we can group the most similar parts of the string together so each partition is better compressed by $\mathcal{C}$.

Reduction to SSSP

We can model each partition problem as a directed graph with $n + 1$ vertices, where an edge exists between v_i and v_j only if

$$1 \leq i < j \leq n + 1$$



v_1
v_2
v_3
v_4
v_5
v_6
R
A
V
E
N

Reduction to SSSP - Bijection between paths and partitions

We can then show that there exists a bijection from each path $\pi = \left(v_1, v_{i_1}\right)...\left(v_{i_k}, v_{n+1}\right)$ in the graph, and a partitioning of the text T $T[1..i_1 - 1]T[i_1..i_2 - 1]...T[i_{k-1}..n]$

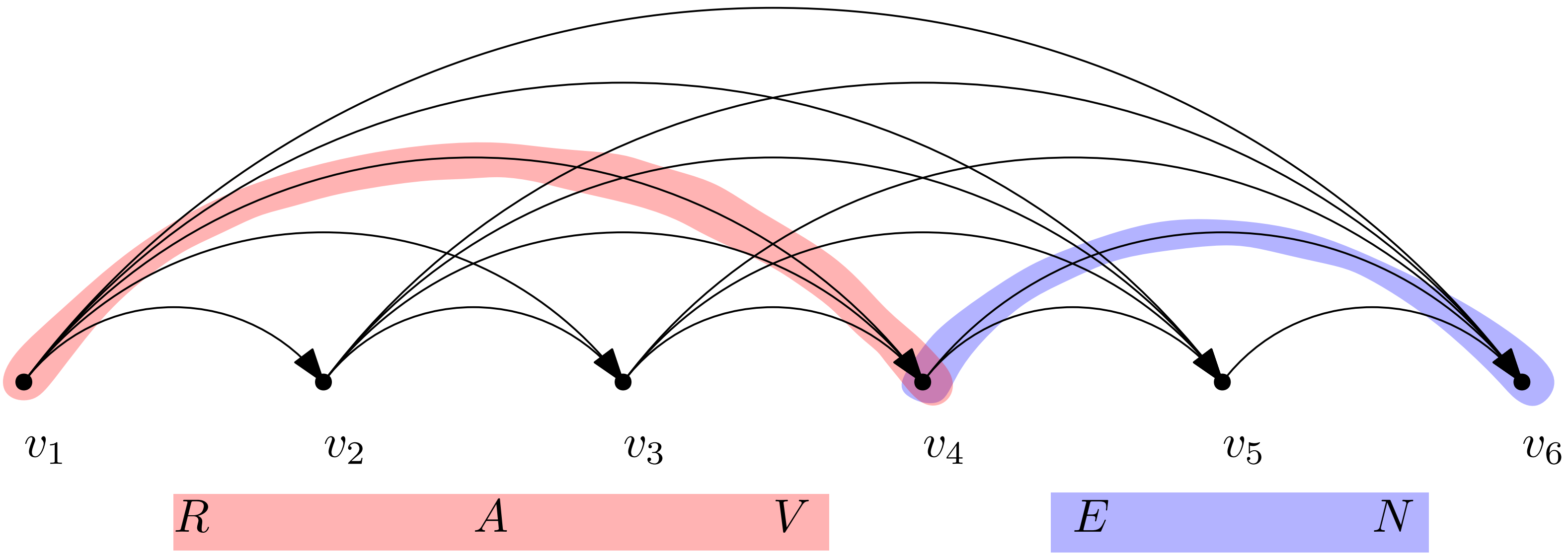


Figure 2: We can map the path $\pi = (v_1, v_4)\ (v_4, v_6)$ to the partitioning of the string $T[1,3]\ T[4,5]$

Reduction to SSSP - Bijection between paths and partitions

If we weight each edge (i, j) of the graph by the cost of compressing the corresponding text segment $w(i, j) = \mathcal{C}(T[i, j-1])$, we can solve the partitioning problem *optimally* computing the **Single Source Shortest Path (SSSP)**

It can be computed efficiently in $O(|E|)$ time using a classic dynamic programming algorithm.

Problems:

1. Our graph has $O(n^2)$ nodes by construction
2. To initialize the weight $w(i, j)$ we should execute $\mathcal{C}$ on every substring of the text

Assumption on $\mathcal{C}$

- Our compressor is *monotonic*: the compressed output on a suffix or a prefix of the string is always smaller than the compression on the whole string:

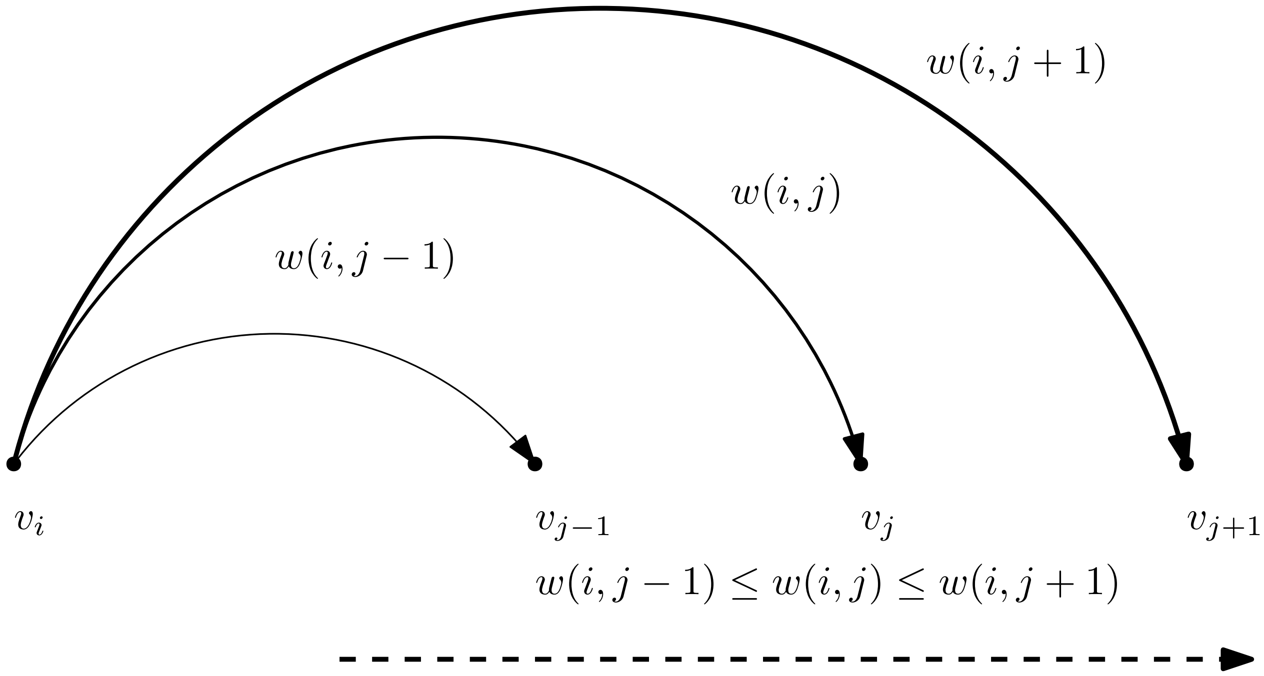
$$\mathcal{C}(T[i,j]) \geq \mathcal{C}(T[i,j-1])$$

$$\mathcal{C}(T[i,j]) \geq \mathcal{C}(T[i+1,j])$$

- We can compute the size of the compressed output incrementally: computing $\mathcal{C}(T[i,j])$ from $\mathcal{C}(T[i-1,j])$ or $\mathcal{C}(T[i,j-1])$ take constant time

Monotonicity of w

Due to the monotonicity of the compressor for every node $1 \leq i < k < j \leq n + 1$ we have that $w(i, k) \leq w(i, j)$



Sparsification of the DAG

Thanks to this property we can obtain an approximated algorithm by **sparsifing** the graph thus selecting only some edges.

We are able to obtain a $(1 + \varepsilon)$-approximation, for every $\varepsilon \geq 0$, with a time complexity of $O(n \log_{1+\varepsilon} L)$

where $L = w(1, n)$, so the cost of compressing the entire text.

This algorithm can be applied to every dynamic programming algorithm in the form $E[j] = \min_{1 \leq i < j}(E[i] + w(i, j))$ when w is *monotone*!

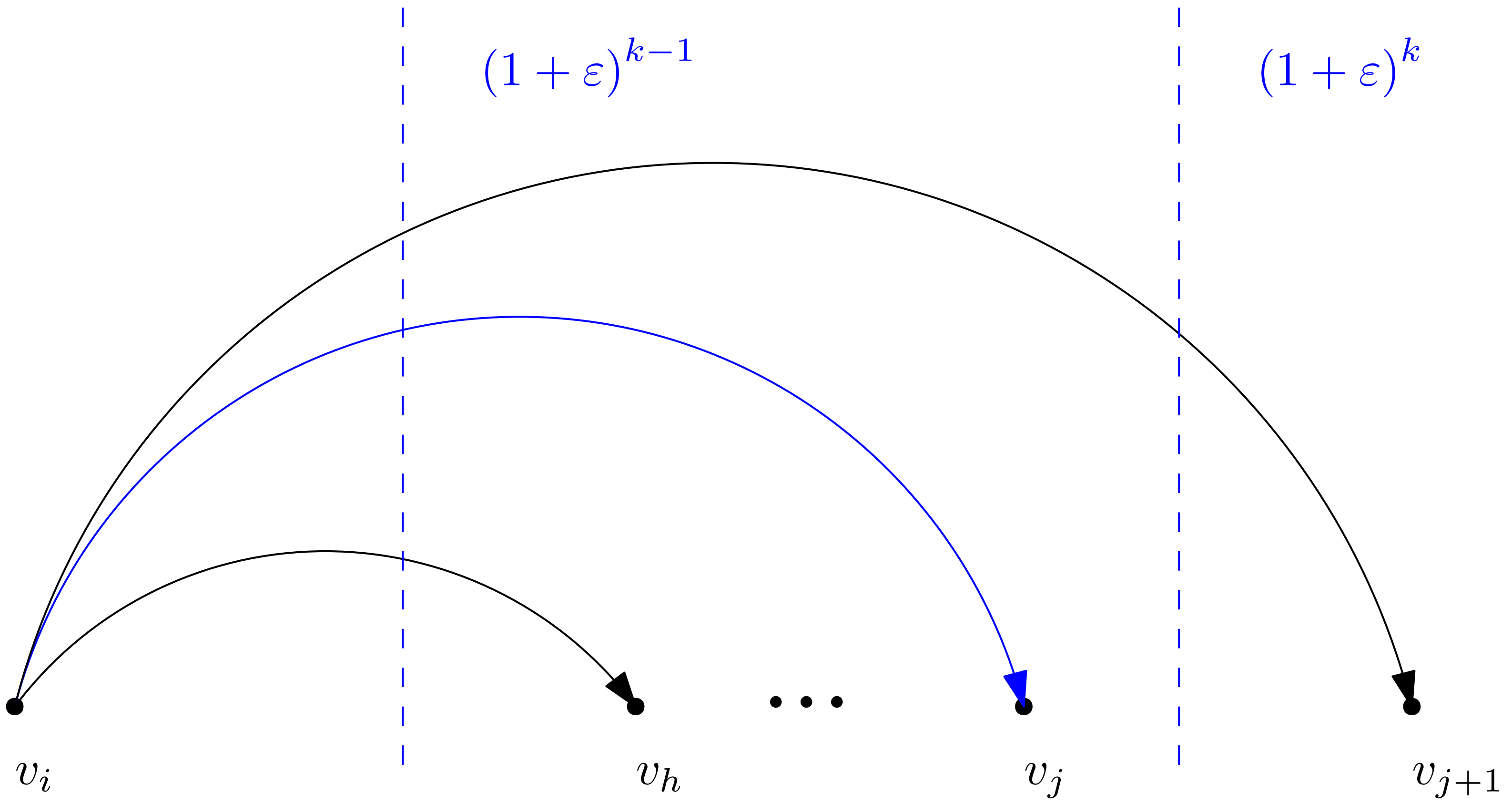
Key Idea: ε-maximal edges

How we can select some edges to obtain the $(1 + \varepsilon)$ approximation factor?

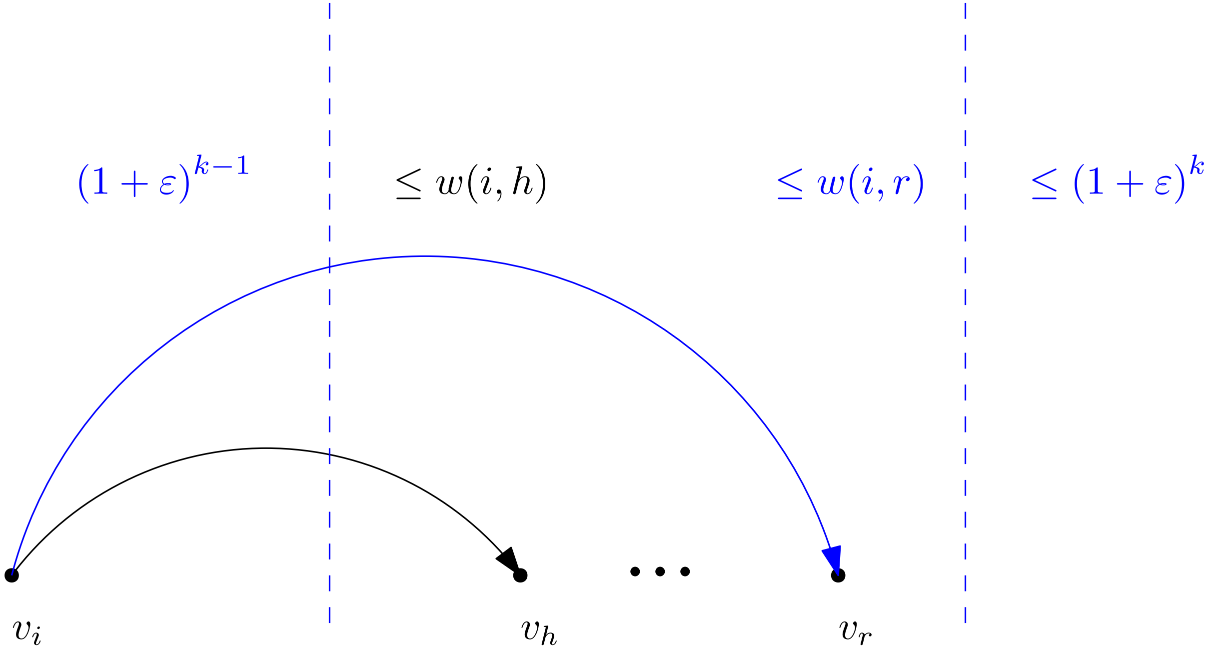
For each node i select the ε-maximal edges, so the outgoing edge from i that satisfy one of these conditions:

- The edges (i, j) such that $w(i, j) \leq (1 + \varepsilon)^k < w(i, j + 1)$ for any integer $k \geq 1$
- The last outgoing edge: $(i, n + 1)$

So we select the best approximations of the powers of $(1 + \varepsilon)$ from below: We then have at most $\log_{1+\varepsilon} L$ outgoing edges for each node.


$(1+\varepsilon)^{k-1}$
$(1+\varepsilon)^{k}$
v_i
v_h
$\dots$
v_j
v_{j+1}

Each edge is then "covered" by an ε-maximal edge: The weight of the edge is then approximated by $(1+\varepsilon)$ times the weight of the maximal edge that covers it.



$$\frac{w(i,r)}{w(i,h)} \leq \frac{(1+\varepsilon)^k}{(1+\varepsilon)^{k-1}}$$

$$\frac{w(i,r)}{w(i,h)} \leq (1+\varepsilon)$$

$$w(i,r) \leq (1+\varepsilon)w(i,h)$$

Lemma 1

Let $d_{\mathcal{G}}(i)$ be the shortest path in our graph $\mathcal{G}$ from v_i to v_{n+1} then

For all the vertices $i, j : 1 \leq i < j \leq n + 1, d_{\mathcal{G}}(i) \geq d_{\mathcal{G}}(j)$

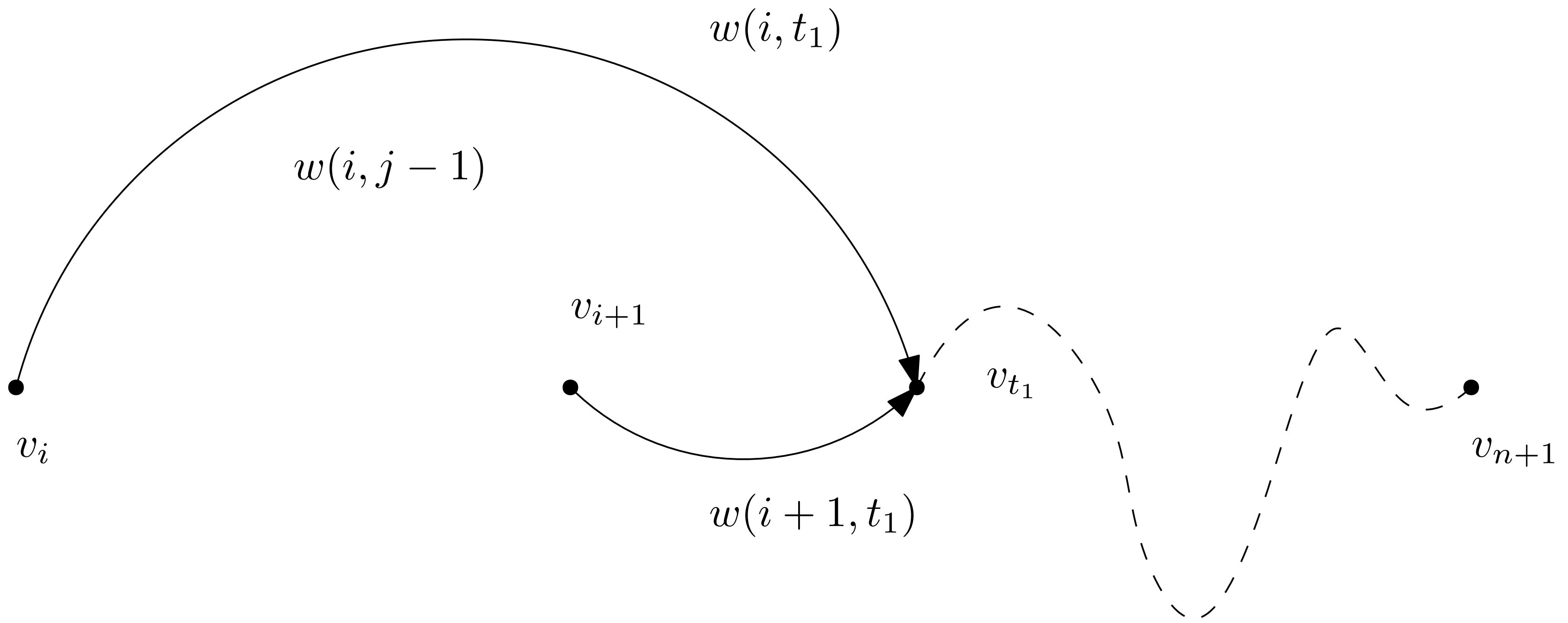
Proof by induction:

- Base, trivial case for $n + 1$
- Then we need to show that $d_{\mathcal{G}}(i) \geq d_{\mathcal{G}}(i + 1)$

Let $d_{\mathcal{G}}(i)$ be $\left(v_i, v_{t_1}\right)\left(v_{t_1}, v_{t_2}\right)\ldots\left(v_{t_k}, v_{n+1}\right)$

- Trivial if $t_1 = i + 1$

- If $t_1 > i+1$ then we can construct a shortest path $\left(v_{i+1}, v_{t_1}\right)\left(v_{t_1}, v_{t_2}\right)\ldots\left(v_{t_k}, v_{n+1}\right)$ because thanks to the definition of *monotonicity* we know that $w(i, t_1) \geq w(i+1, t_1)$



Theorem

Let $\mathcal{G}_\varepsilon$ be the graph containing only ε-maximal edges, then $d_{\mathcal{G}_\varepsilon}(i) \leq (1+\varepsilon)d_{\mathcal{G}}(i)$ for every $1 \leq i \leq n+1$.

Proof by induction:

- Base, trivial case for $n+1$
- Then let $d_{\mathcal{G}}(i) = \left(v_i, v_{t_1}\right)..\left(v_{t_h}, v_n\right) = w(i, t_1) + d_{\mathcal{G}}(t_1)$. We choose the ε-maximal node r that covers t_1: So $r > t_1$ and we already know that $w(i, r) \leq (1+\varepsilon)w(i, t_1)$.
  By *Lemma 1*: $d_{\mathcal{G}}(r) \leq d_{\mathcal{G}}(t_1)$
  By inductive hypothesis $d_{\mathcal{G}_\varepsilon}(r) \leq (1+\varepsilon)d_{\mathcal{G}}(r) \leq (1+\varepsilon)d_{\mathcal{G}}(t_1)$
  In the end $d_{\mathcal{G}_\varepsilon}(i) = w(i, r) + d_{\mathcal{G}_\varepsilon}(r) \leq (1+\varepsilon)\left(w(i, t_1) + d_{\mathcal{G}}(t_1)\right)$

Problem: DAG Construction

We still have two problems:

1. if we construct naively this graph we should remove edges from a $O(n^2)$ graph
2. We should compute the weight of the graph

We can solve both these problems efficiently at once!

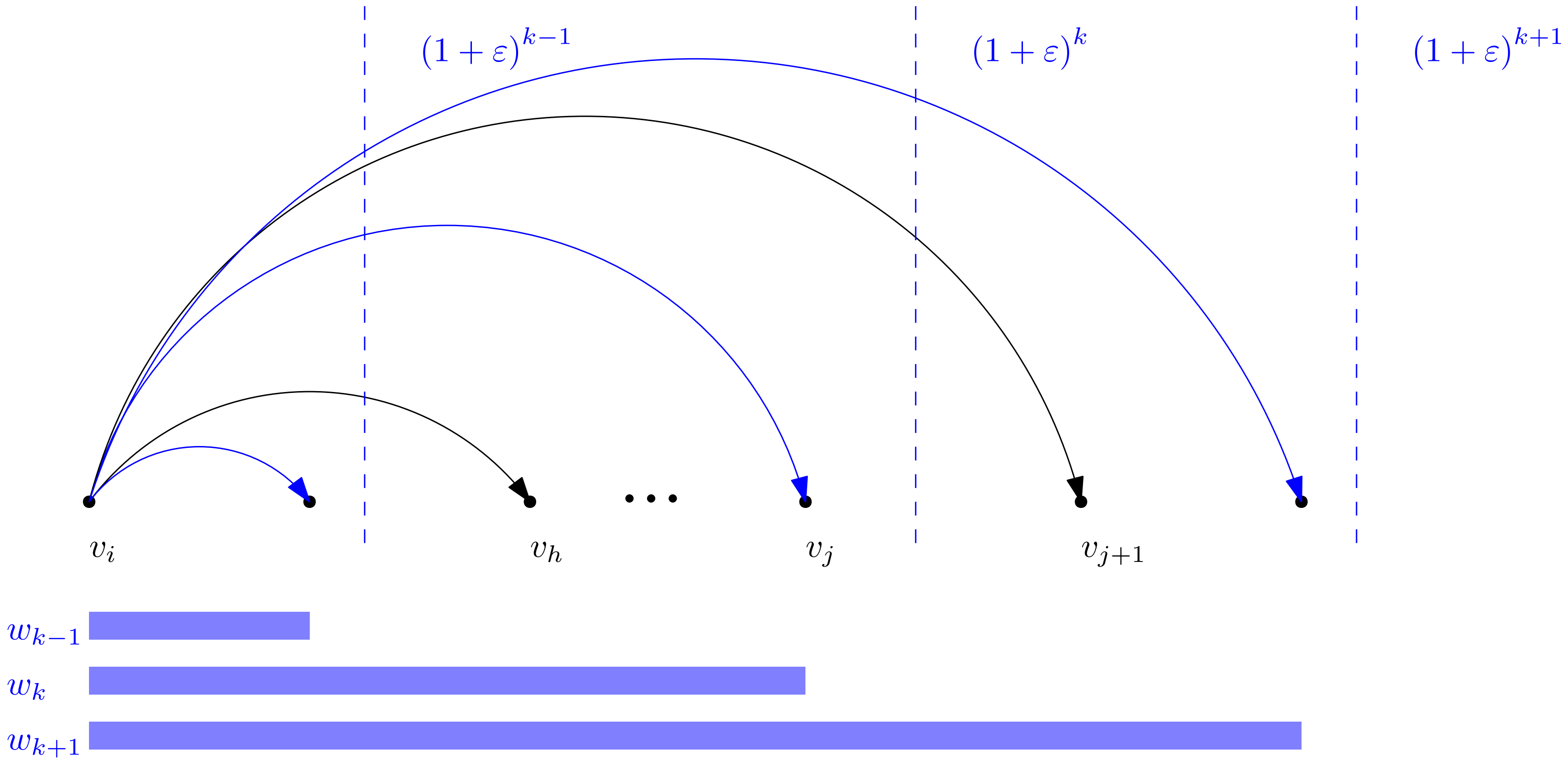
Sliding windows

We keep $\log_{1+\varepsilon} L$ sliding windows all starting at v_i, but ending in a different position. The h-th window advance on the right to reach the last edge smaller than the h-power of $(1+\varepsilon)$ so finding the h-th maximal edge starting from node i.

After completing the dynamic programming step for the node i we advance all the windows from the start.

For each compressor we should implement 2 operations `advance_left`, `advance_right`: if they had respectively a complexity of $O(L)$ and $O(R)$ our algorithm execute asymptotically $O(nL + n \log_{1+\varepsilon} R)$ steps

The authors provide several implementations of the sliding windows approach to estimate the size of different compressors, among the others statistical compressors (using 0-th order and k-order entropy)



$(1+\varepsilon)^{k-1}$
$(1+\varepsilon)^{k}$
$(1+\varepsilon)^{k+1}$
v_i
v_h
v_j
v_{j+1}
w_{k-1}
w_k
w_{k+1}

Thank You!