On Optimally Partitioning a Text to Improve Its Compression [1]

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Problem: Text Partitioning

We have a **compressor** $\mathscr{C}$ and a **text** T of size n, we want to **divide** T into $k \leq n$ parts, $T[1..i_1 - 1]T[i_1..i_2 - 1]...T[i_{k-1}..n]$ and **compress each** of them individually with $\mathscr{C}$ to improve the overall compression

Note: We do **not** *permute* the string.
We are only interested in *partitioning* it.

Text Partitioning Example

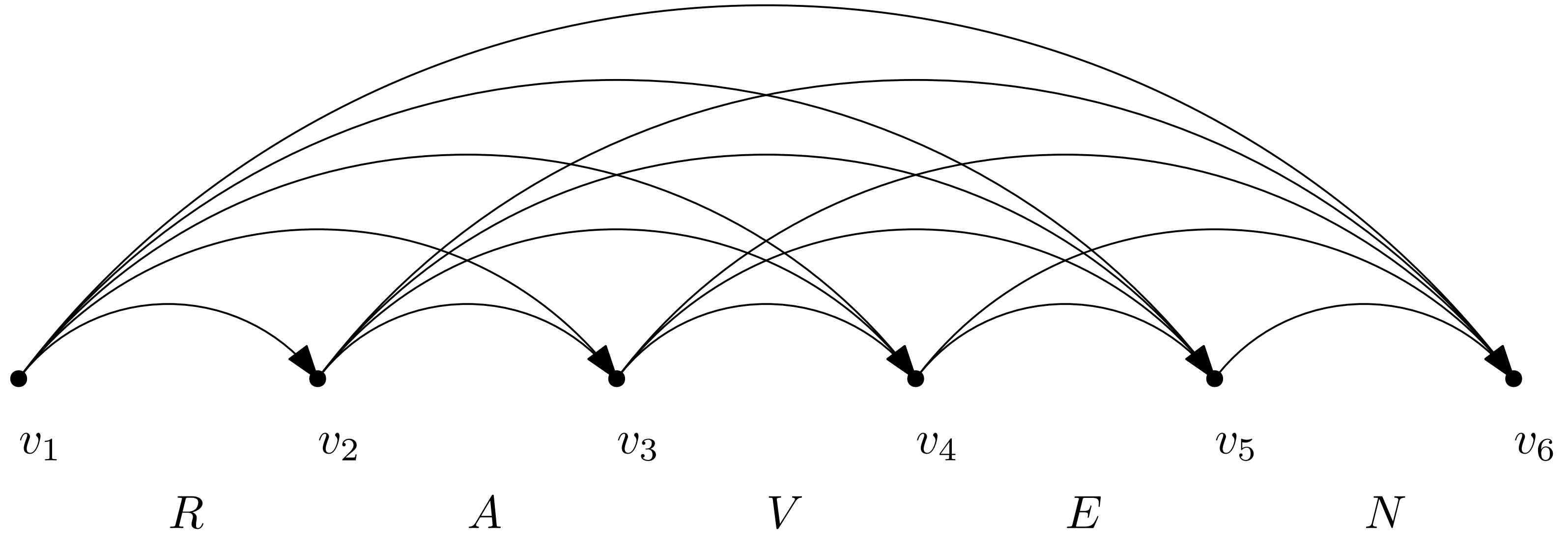
Suppose we have the text $T = \mathrm{a}^n\mathrm{b}^n$.

If we compress the entire text at once we should use one bit per symbol, or $\boldsymbol{O(n)}$ **bits**.

If instead we partition the text to compress a^n and b^n separately we can compress the whole string using only $\boldsymbol{O(\log_2(n))}$ **bits** indicating just the length of each substring.

Reduction to SSSP

We can model the partition problem as a **directed graph** with $n+1$ *ordered* vertices, where an edge exists between v_i and v_j only if $1 \leq i < j \leq n+1$



Reduction to SSSP - Bijection between paths and partitions

In this graph each **edge** represents a **substring** of the text.

We can then show that there exists a **bijection** from each **path** $\pi = \left(v_1, v_{i_1}\right)...\left(v_{i_k}, v_{n+1}\right)$ in the graph, and a **partitioning** of the text T in the form $T[1..i_1 - 1]T[i_1..i_2 - 1]...T[i_k..n]$

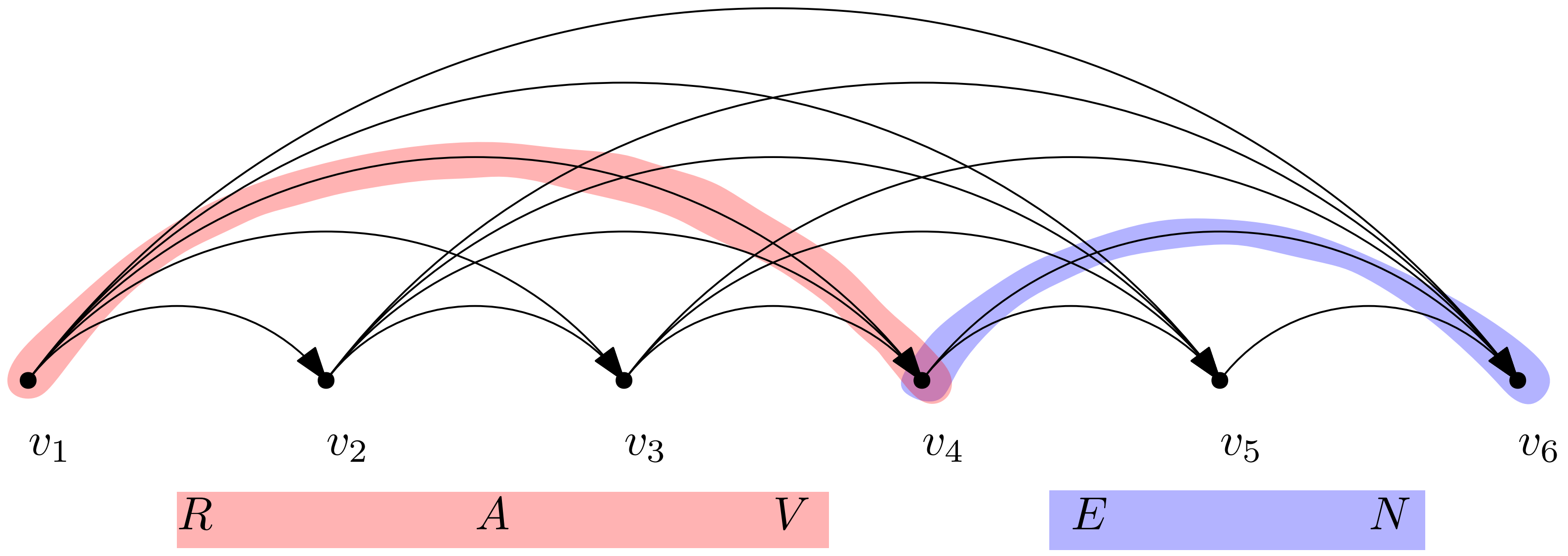


Figure 2: We can map the path $\pi = (v_1, v_4)\ (v_4, v_6)$ to the partitioning of the string $T[1,3]\ T[4,5]$

Reduction to SSSP - Bijection between paths and partitions

If we weight each edge (i, j) of the graph by the cost of compressing the corresponding text segment $w(i, j) = |\mathscr{C}(T[i, j-1])|$, we can solve the partitioning problem *optimally* computing the **Single Source Shortest Path (SSSP)**

It can be computed efficiently in $O(|E|)$ time using a classic dynamic programming algorithm.

Problems:

1. Our graph has $O(n^2)$ nodes by construction
2. To initialize the weight $w(i,j)$ we should execute $\mathscr{C}$ on every substring of the text

Assumption on $\mathscr{C}$

- Our compressor is *monotonic*: the compressed output on a suffix or a prefix of the string is always smaller than the compression on the whole string:

$$|\mathscr{C}(T[i,j])| \geq |\mathscr{C}(T[i,j-1])|$$

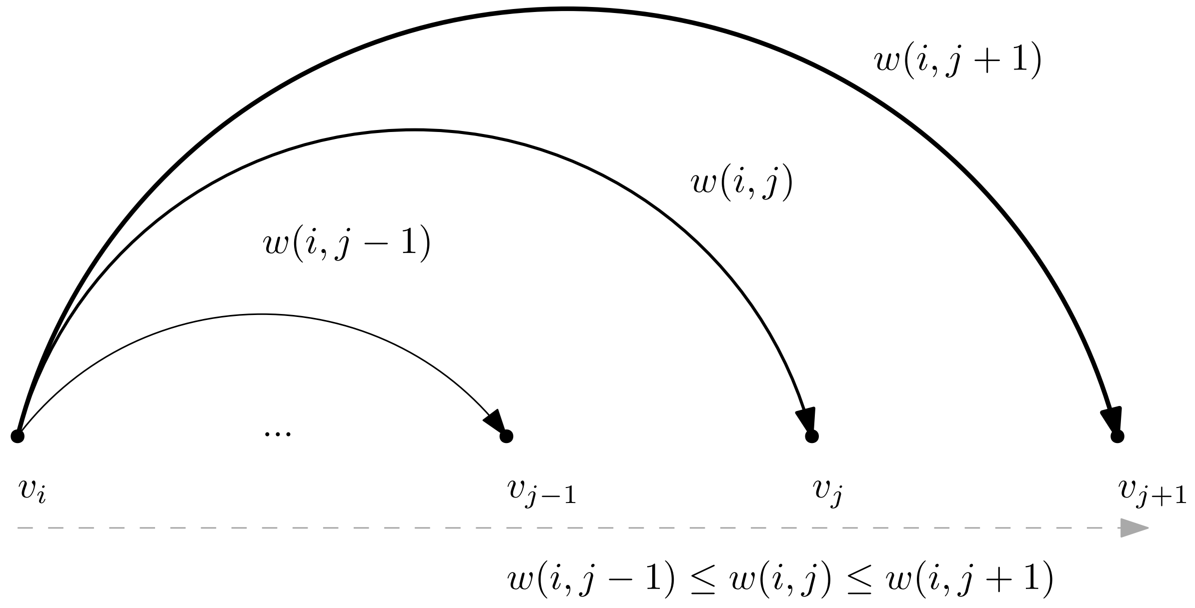
$$|\mathscr{C}(T[i,j])| \geq |\mathscr{C}(T[i+1,j])|$$

- We can compute the size of the compressed output incrementally: computing $|\mathscr{C}(T[i,j])|$ from the state of $\mathscr{C}(T[i-1,j])$ or $\mathscr{C}(T[i,j-1])$ takes constant time

How the property of monotonicity affect the topology of our DAG?

Monotonicity of w

Due to the monotonicity of the compressor for every node $1 \leq i < k < j \leq n+1$ we have that $w(i,k) \leq w(i,j)$



Sparsification of the DAG

Thanks to this property we can obtain an approximated algorithm by **sparsifying** the graph thus selecting only some edges.

We are able to obtain a $\boldsymbol{(1+\varepsilon)}$**-approximation**, for every $\varepsilon \geq 0$, with a time complexity of $\boldsymbol{O(n \log_{1+\varepsilon} L)}$

where $L = w(1, n)$, so the cost of compressing the entire text.

This algorithm can be applied to every dynamic programming algorithm in the form $E[j] = \min_{1 \leq i < j}(E[i] + w(i, j))$ when w is *monotone*!

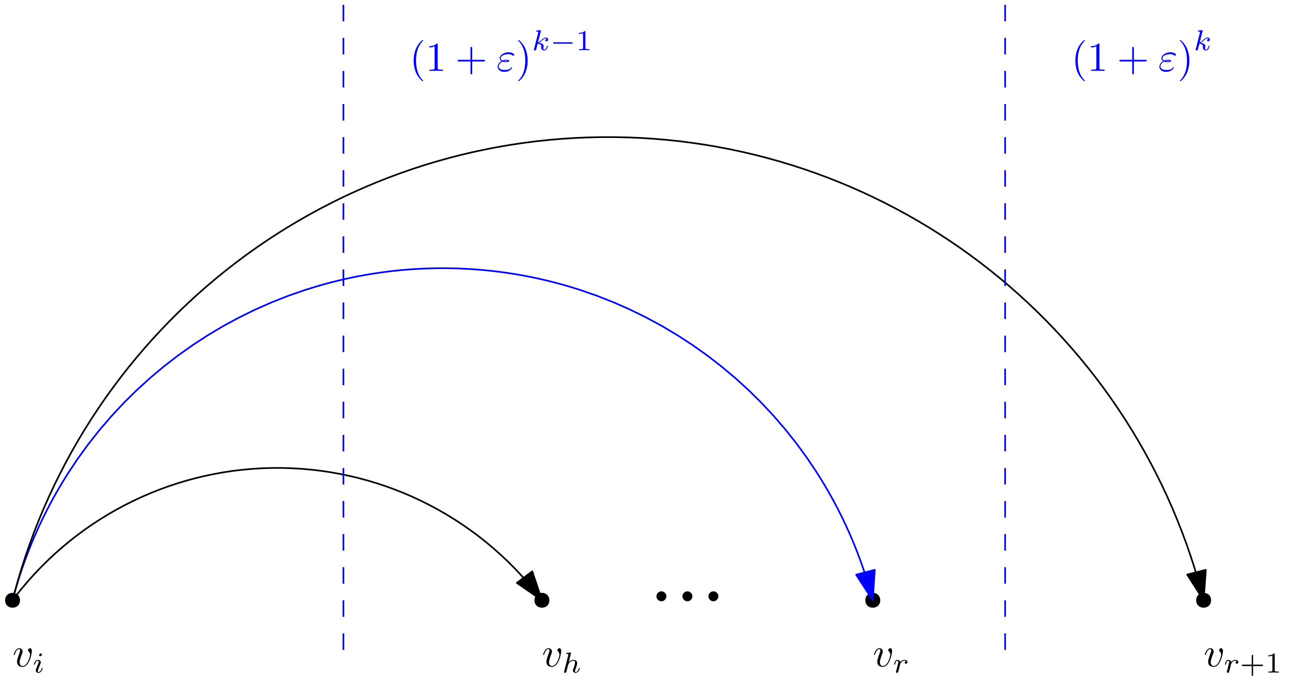
Key Idea: ε-maximal edges

How we can select some edges to obtain the $(1+\varepsilon)$ approximation factor?

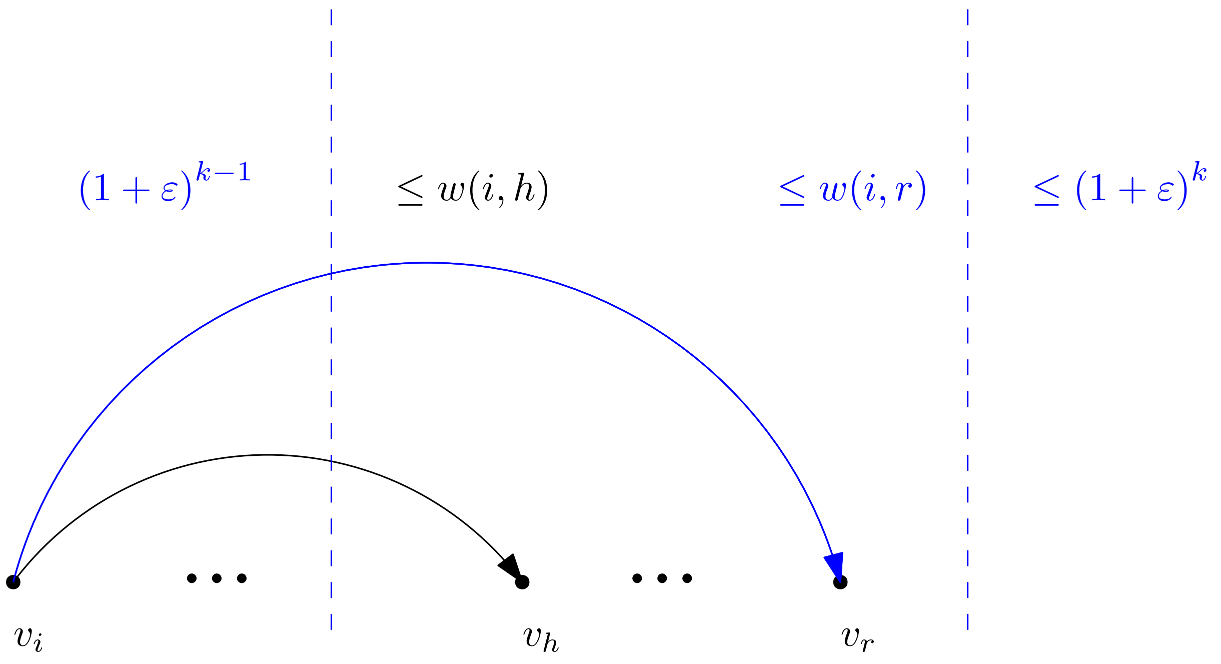
For each node i select the ε-**maximal** edges, so the outgoing edge from i that satisfy one of these conditions:

- The edges (i, j) such that $w(i, j) \leq (1+\varepsilon)^k < w(i, j+1)$ for any integer $k \geq 1$
- The last outgoing edge: $(i, n+1)$

So we select the best approximations of the powers of $(1+\varepsilon)$ from below: We then have at most $\log_{1+\varepsilon} L$ outgoing edges for each node.



Each edge is then *“covered”* by an ε-maximal edge: The weight of the edge is then approximated by $(1+\varepsilon)$ times the weight of the maximal edge that covers it.



$$\frac{w(i,r)}{w(i,h)} \leq \frac{(1+\varepsilon)^k}{(1+\varepsilon)^{k-1}}$$

$$\frac{w(i,r)}{w(i,h)} \leq (1+\varepsilon)$$

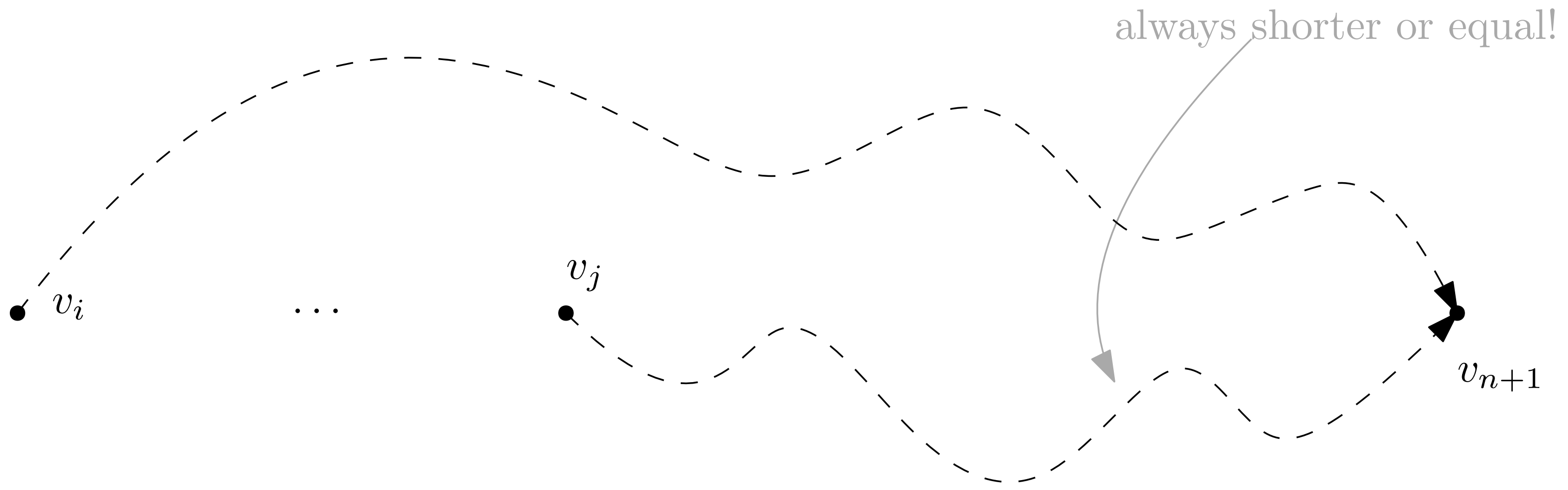
$$w(i,r) \leq (1+\varepsilon)w(i,h)$$

Our edges are increasing and can be approximated but what can we say about *paths* in this graph?

Lemma 1

Let $d_{\mathscr{G}}(i)$ be the cost of the shortest path π_i in our graph $\mathscr{G}$ from v_i to v_{n+1} then

for all the vertices $i, j : 1 \leq i < j \leq n+1$, $d_{\mathscr{G}}(i) \geq d_{\mathscr{G}}(j)$


always shorter or equal!
v_i
$\cdots$
v_j
v_{n+1}

Theorem

Let $\mathscr{G}$ be the full graph and $\mathscr{G}_\varepsilon$ be the graph containing only ε-maximal edges, then $d_{\mathscr{G}_\varepsilon}(i) \leq (1+\varepsilon)d_{\mathscr{G}}(i)$ for every integer $1 \leq i \leq n+1$.

Proof by induction on $\pi(i)$:

- **Base**, trivial case for $n+1$
- Let $\pi(i) = \left(v_i, v_{t_1}\right)..\left(v_{t_h}, v_n\right)$ the shortest path starting from node v_i and let $d_{\mathscr{G}}(i) = w(i, t_1) + d_{\mathscr{G}}(t_1)$ be its cost. We choose the ε-maximal node r that covers t_1: So $r > t_1$ and we already know (by our "key idea") that

  $$w(i,r) \leq (1+\varepsilon) w(i, t_1)$$

  By *Lemma 1*:

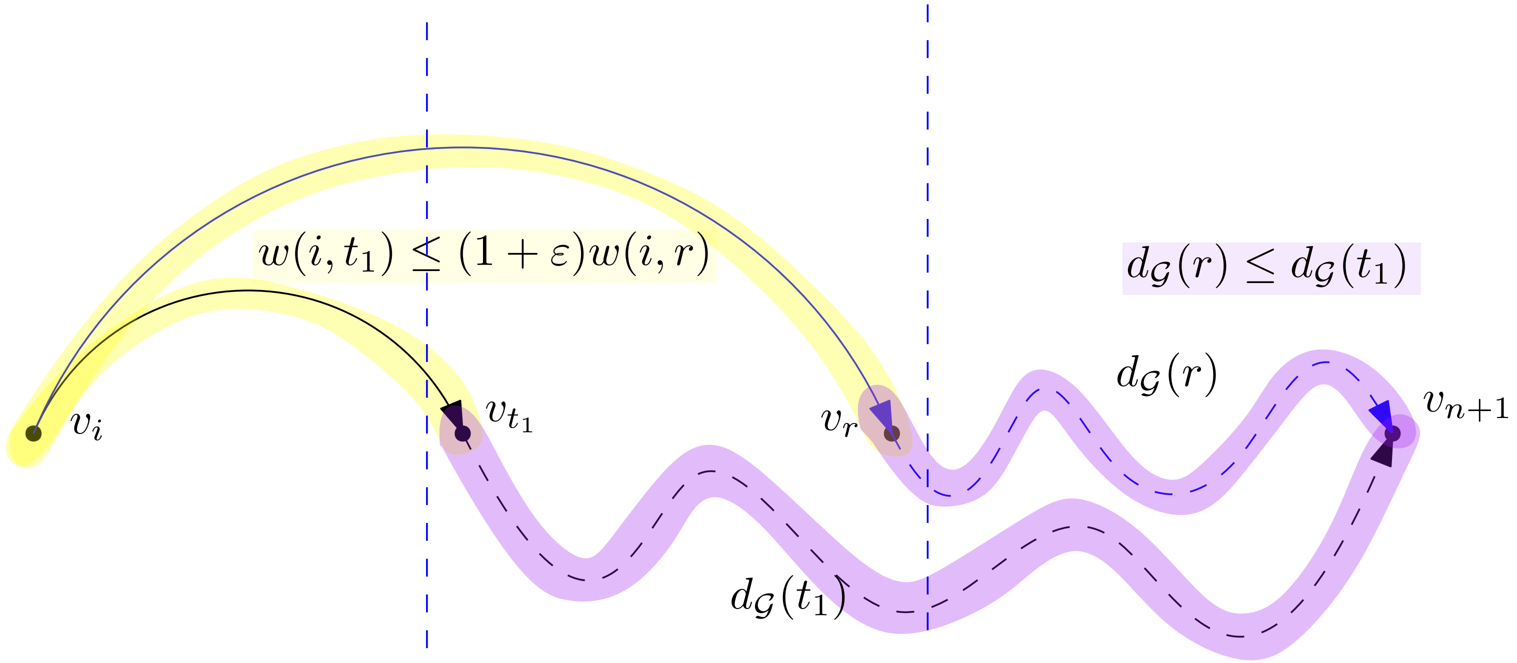
  $$d_{\mathscr{G}}(r) \leq d_{\mathscr{G}}(t_1)$$

By inductive hypothesis:

$$d_{\mathscr{G}_\varepsilon}(r) \leq (1+\varepsilon) d_{\mathscr{G}}(r) \leq (1+\varepsilon) d_{\mathscr{G}}(t_1)$$

In the end

$$d_{\mathscr{G}_\varepsilon}(i) = w(i,r) + d_{\mathscr{G}_\varepsilon}(r) \leq (1+\varepsilon)(w(i,t_1) + d_{\mathscr{G}}(t_1)))$$


$w(i, t_1) \leq (1+\varepsilon) w(i, r)$
$d_{\mathcal{G}}(r) \leq d_{\mathcal{G}}(t_1)$
$d_{\mathcal{G}}(r)$
v_i
v_{t_1}
v_r
v_{n+1}
$d_{\mathcal{G}}(t_1)$

Problem: DAG Construction

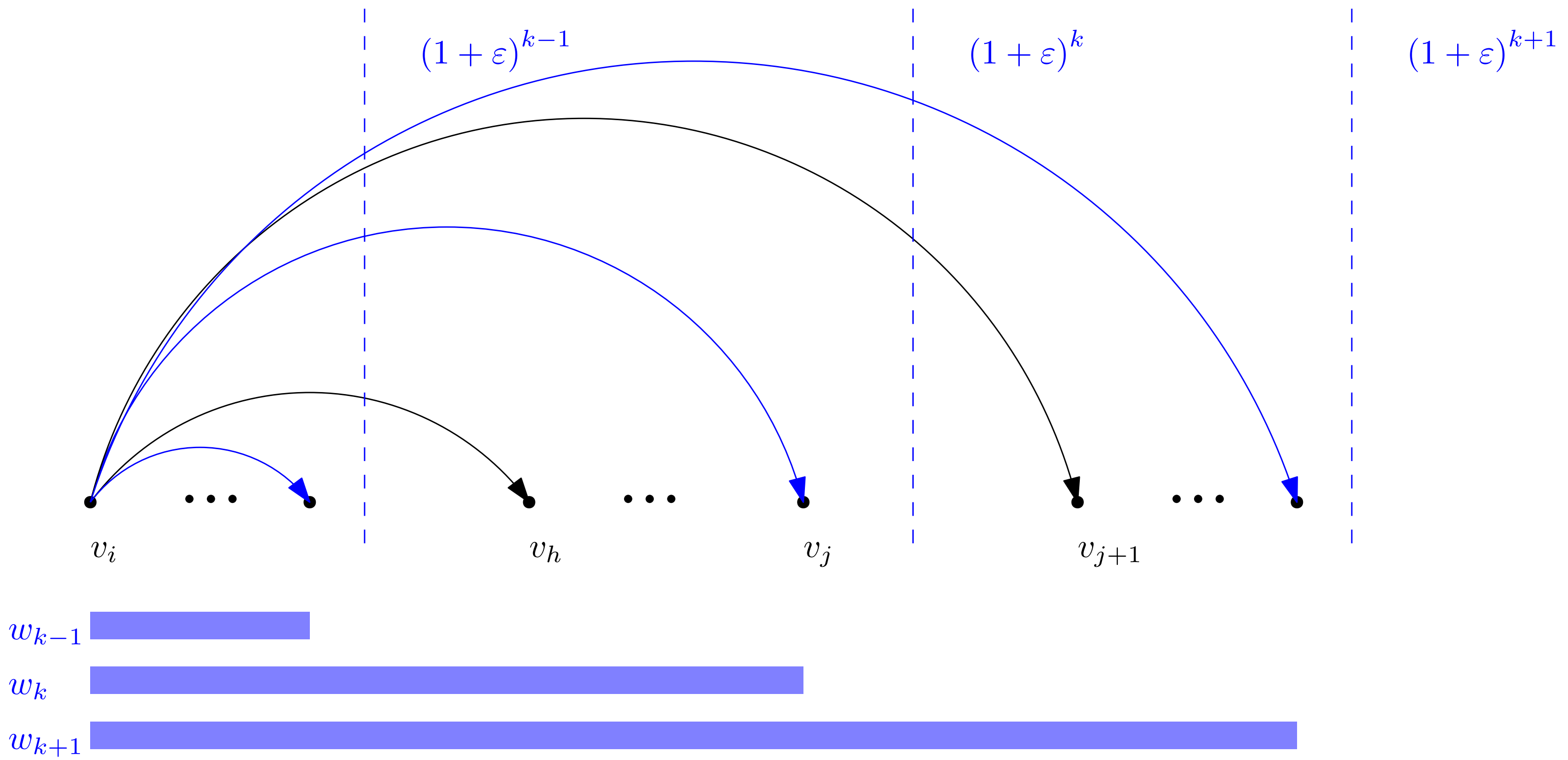
We still have two problems:

1. if we construct *naively* this graph we should remove edges from a $O(n^2)$ graph
2. and we should compute the weight of each edge of the graph

We can solve both these problems efficiently at once: We can find the ε-maximal edges efficently on the fly!

Sliding windows

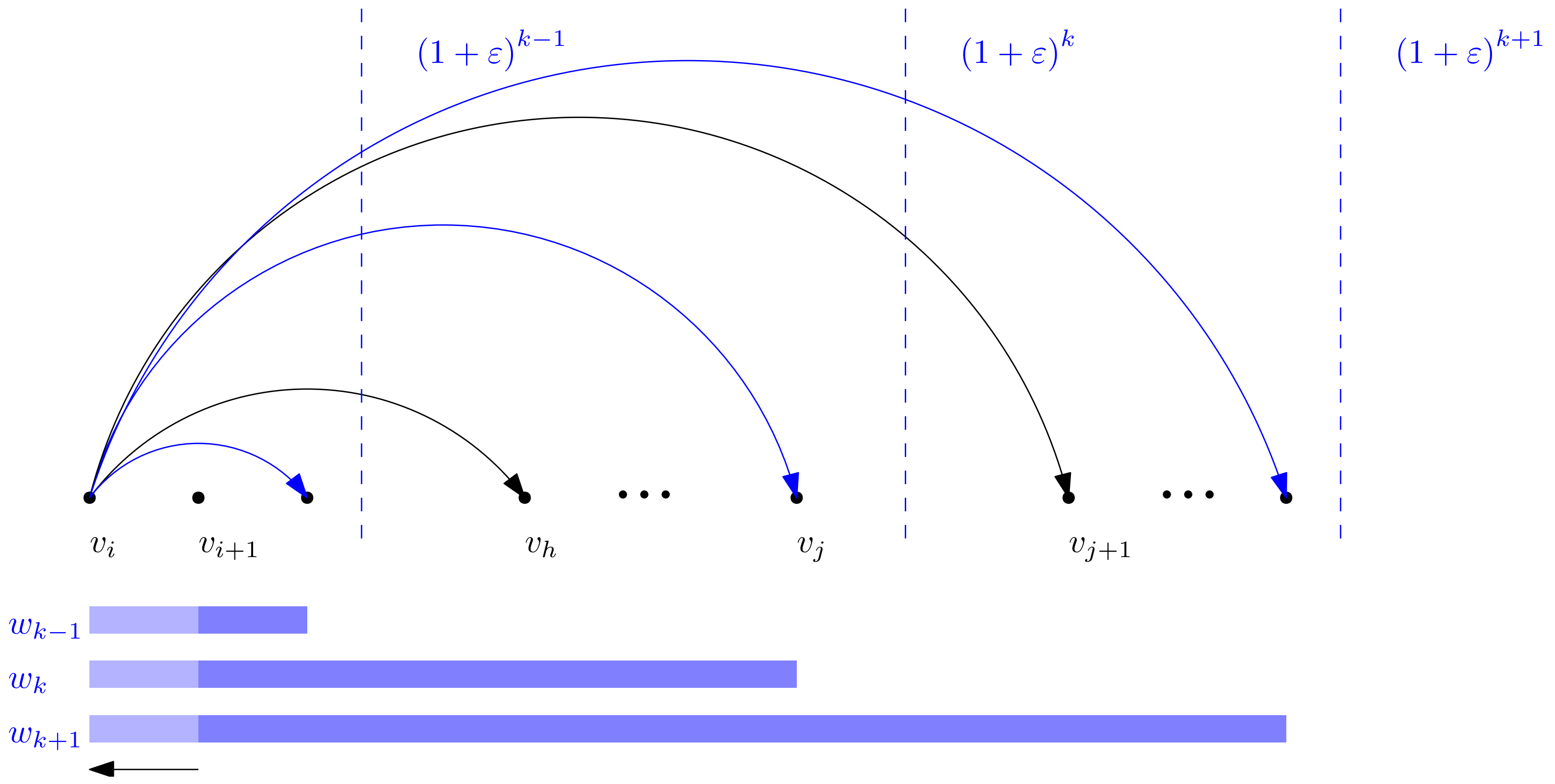
We keep $\log_{1+\varepsilon} L$ sliding windows all starting at v_i, but ending in a different position. The k-th window find the k-th ε-maximal edge.


$(1+\varepsilon)^{k-1}$
$(1+\varepsilon)^{k}$
$(1+\varepsilon)^{k+1}$
v_i
v_h
v_j
v_{j+1}
w_{k-1}
w_k
w_{k+1}

Sliding windows

For each compressor we should implement 2 operations on the windows `advance_left`, `advance_right`: The first operation advances the start of **all** the windows to the left.

The cost of all the windows increase or stay the same.

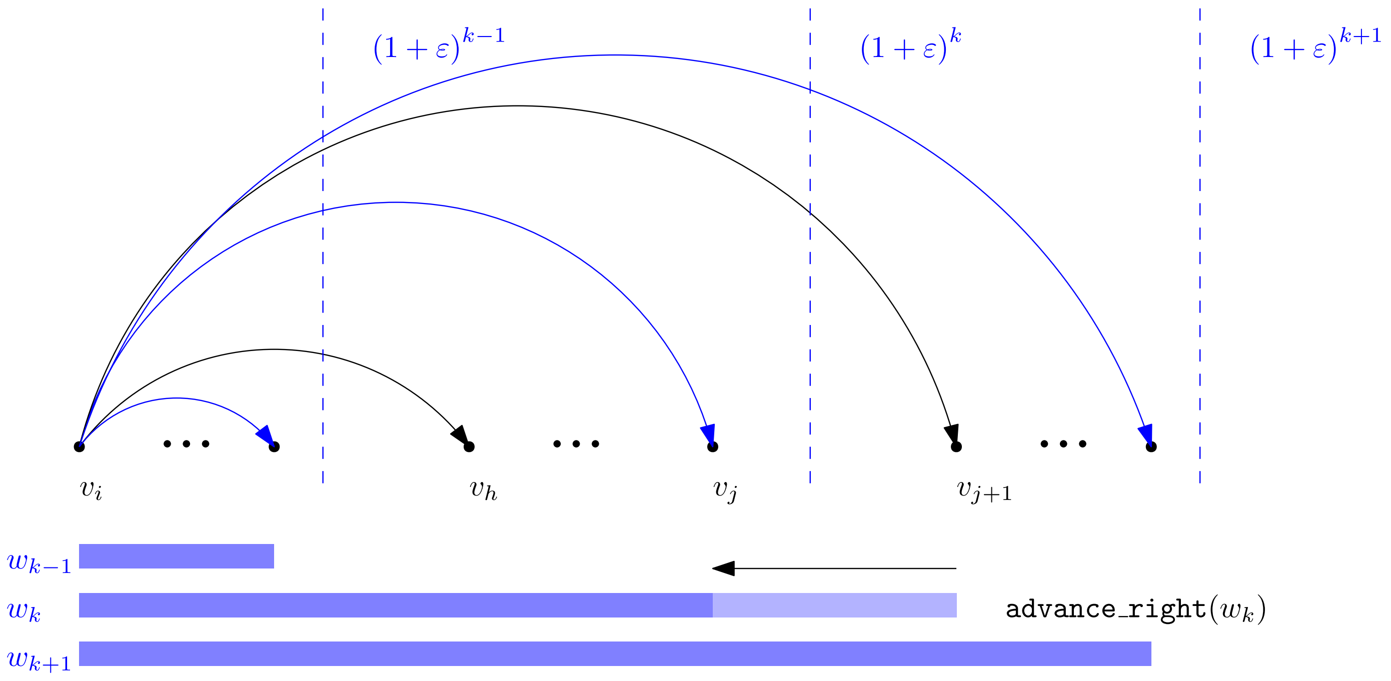


$(1+\varepsilon)^{k-1}$
$(1+\varepsilon)^{k}$
$(1+\varepsilon)^{k+1}$
v_i
v_{i+1}
v_h
v_j
v_{j+1}
w_{k-1}
w_k
w_{k+1}

Sliding windows

`advance_right` advance the end of the k-th window of one position to the left.

We call this function until we reach the first edge smaller than $(1+\varepsilon)^k$, so until we find the k-th maximal edge starting from node i.



if the operations `advance_left` and `advance_right` have respectively a complexity of $O(L)$ and $O(R)$ our algorithm execute asymptotically $O(Ln + Rn \log_{1+\varepsilon} n)$ steps

The authors provide several implementations of the sliding windows framework to estimate the size of different compressors, among the others statistical compressors (using 0-th order and k-order entropy)

Computing Zero Order Entropy

Zero-th order entropy is a well-known lower bound for the performance of statistical compressors.

For each windows w_k that covers the substring $T[i..j]$, we maintain a histogram, $A_k[c]$, indexed by the symbol $c \in \sum$ and the value

$$E_k = \sum_{c \in \sum} A_k[c] \log_2 A_k[c]$$

Using E_k, we can calculate a lower bound on the output of the statistical compressor, $|\mathscr{C}(T[i..j])|$ based on the zero-th order entropy as

$$|T[i..j]|\ H_0(T[i..j]) = |T[i..j]|\log_2|T[i..j]| - E_k$$

From this we can calculate incrementally the value of E_{k+1} removing the old term from the summation and adding the new one:

Let $c = T[j+1]$ then

$$E_{k+1} = E_k - A_k[c]\log_2 A_k[c] + (A_k[c]+1)(\log_2 A_k[c]+1)$$

Thank You!

Bibliography

[1] P. Ferragina, I. Nitto, and R. Venturini, “On Optimally Partitioning a Text to Improve Its Compression,” in *Algorithms - ESA 2009*, A. Fiat and P. Sanders, Eds., Berlin, Heidelberg: Springer Berlin Heidelberg, 2009, pp. 420–431.

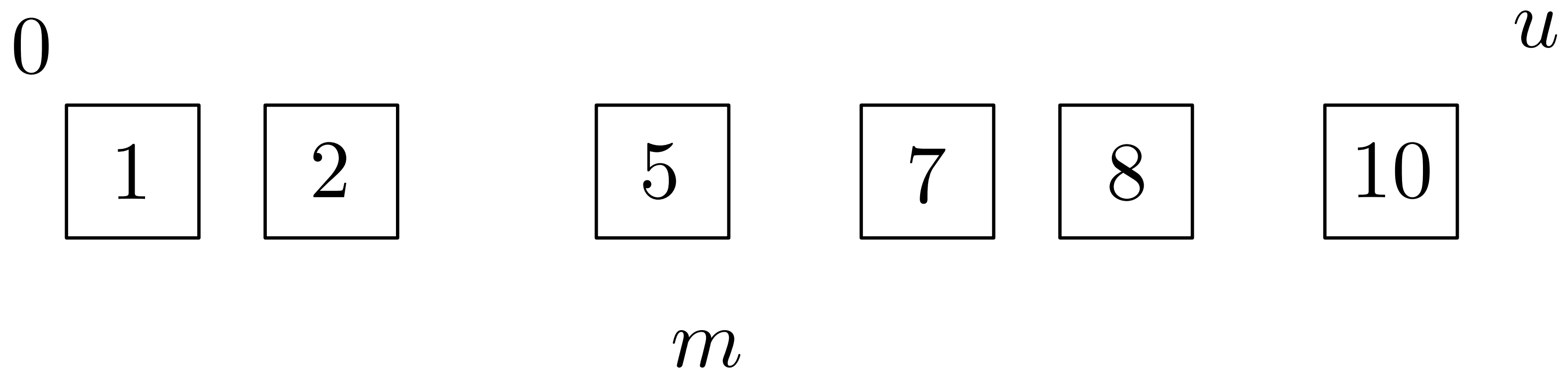
[2] G. Ottaviano and R. Venturini, “Partitioned Elias–Fano Indexes,” in *Proceedings of the 37th International ACM SIGIR Conference*, in SIGIR '14. New York, NY, USA: ACM, 2014, pp. 273–282. doi: **10.1145/2600428.2609615**.

Bonus Slides: Partitioned Elias–Fano

Elias–Fano Data Structure

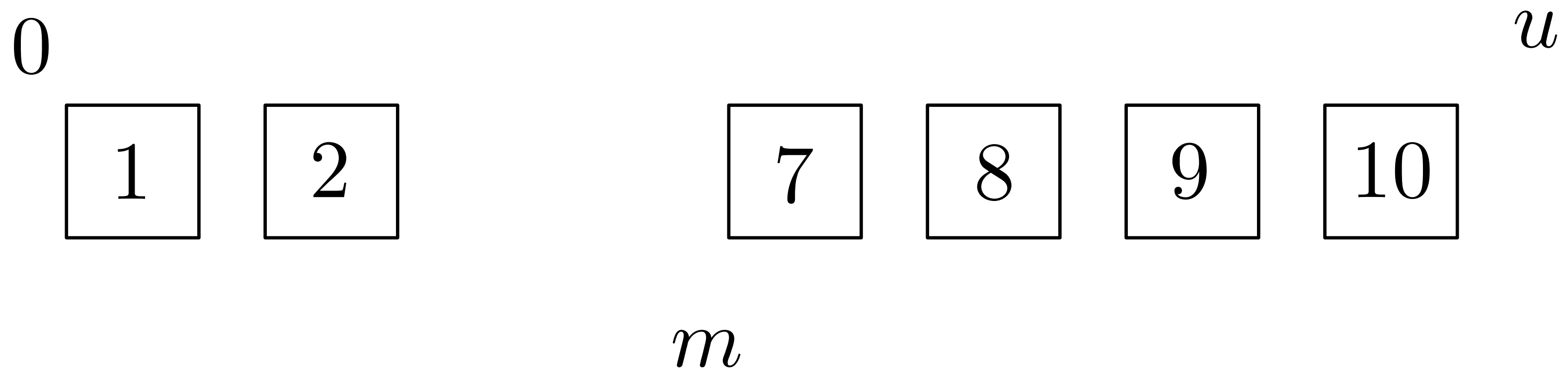
A compact data structure to store a set of m monotonically increasing integers upper-bounded by u.

It uses $\approx \left\lceil \log_2 \frac{u}{m} \right\rceil + 2$ bits per element.



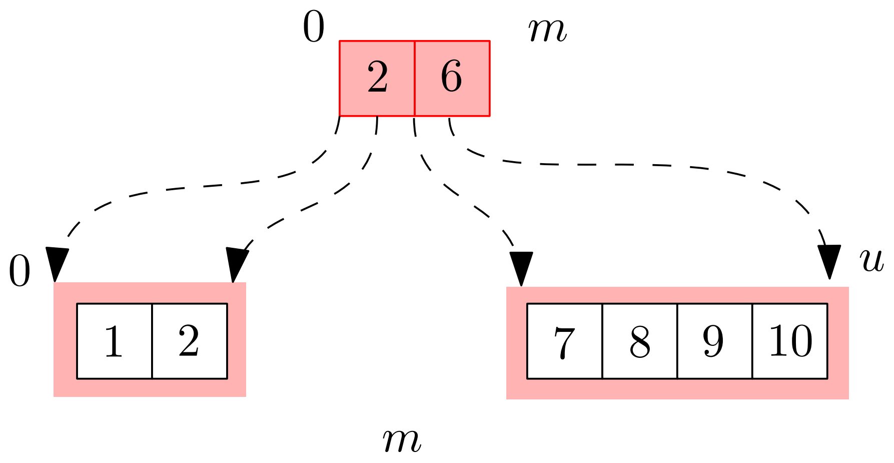
Note that $\frac{u}{m}$ is the average distance between consecutive elements. It doesn't exploit the distribution of the data, but denser lists require fewer bits.

Some sequence are more compressible than others



Partitioned Elias-Fano [2]

We can improve compression by exploiting clusters of data with a two-level structure. The first level determines the bounds of the b clusters, and the second level contains smaller Elias-Fano lists.



How can we find the best partitioning to minimize the space occupancy of both levels?

We can use our partitioning algorithm, assigning a weight to each edge based on the number of bits required to represent the partition in the first level and the Elias-Fano structure in the second level.

The authors also improved the bound by showing that substituting an edge in the path with two sub-edges is always bounded by a constant factor.