Table M Construction

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Introduction

The Table of Insurance Charges (commonly known as "Table M") is an essential tool for retrospective rating. The purpose of this Study Aid is to provide the student with a clear understanding of how to do Table M construction problems.

Data Requirements

To construct Table M, we need a sample of experience from a collection of similarly-sized risks (strictly speaking, we should do a separate analysis for each of many size groups, but because we are working through our examples manually, we will focus on a single size group.) For each risk we need actual losses for the year, or the actual loss ratio, or some other measure that will allow us to compare a risk's experience with that of the average risk of its size. A representative sample of data is as follows:

Experience for Risks with Expected Losses of $100,000

Risk	Actual Loss
1	$20,000
2	$50,000
3	$60,000
4	$70,000
5	$80,000
6	$80,000
7	$90,000
8	$100,000
9	$150,000
10	$300,000

It is essential that we know the *expected* loss or loss ratio for the risks under consideration, as well as the actual numbers, so we can compute entry ratios. In this example the expected losses of $100,000 are stated explicitly. We could also use the average of the actual losses; the answer should be similar if the expected losses have been estimated accurately. (If the sample average does not equal the expected amount, we may not have $\phi(0) = 1$ when we finish the problem. This does happen occasionally; we will discuss this issue at the end of this study note.) In some problems we are given premium and an expected loss ratio; in these cases we can compute expected losses by multiplying the two.

Computing Entry Ratios

No matter how the data is presented, we begin by computing the entry ratio for each risk. The entry ratio is the ratio of actual to expected losses, or of actual to expected loss ratios. A new column is added to the data as follows:

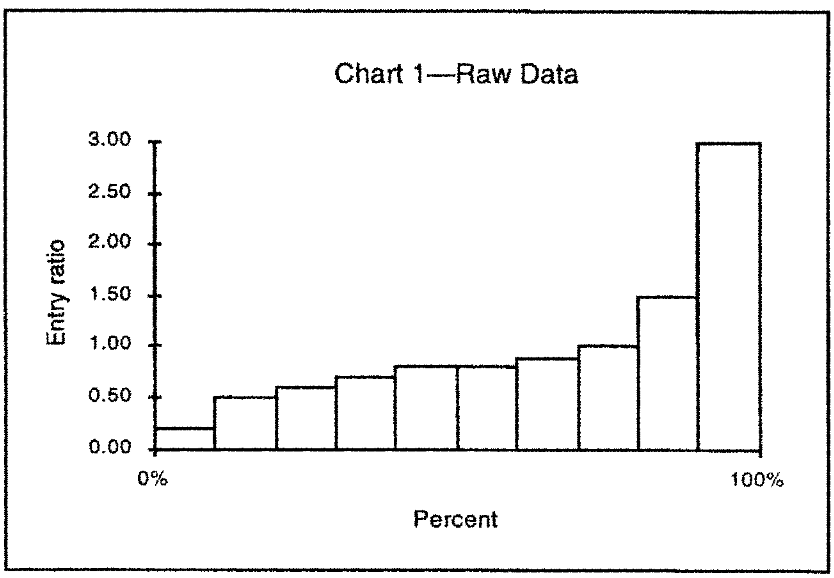
Risk	Actual Loss	Entry Ratio
1	$20,000	0.20
2	$50,000	0.50
3	$60,000	0.60
4	$70,000	0.70
5	$80,000	0.80
6	$80,000	0.80
7	$90,000	0.90
8	$100,000	1.00
9	$150,000	1.50
10	$300,000	3.00

The data in a constructed problem is often chosen so the entry ratios will come out to round numbers. If we are using real life data, this will generally not be the case.

Constructing a Lee Diagram

We can now construct a "Lee Diagram"[1] from the raw data. This is not always essential to the solution, but it can be a big help in understanding the problem. Be prepared to construct such a diagram if one is needed to help understand or explain the problem.

The horizontal axis of the diagram represents cumulative percentages of the number of risks in the sample, from 0% to 100%. (We can also think of this axis as representing the *number* of risks, but this will result in a misleading scale.) In our sample problem there are ten risks, each representing 10% of the total.



[1]See Y. S. Lee, "The Mathematics of Excess of Loss Coverages and Retrospective Rating. A Graphical Approach," Section 4, *PCAS* **75**, 1988, pp. 64–78.

The vertical axis represents the entry ratio. Again, if we use the loss ratio, the scale will be wrong.

Arrange the risks by increasing order of entry ratio and draw a vertical bar for each risk with the height equal to the entry ratio. The width of the bar should correspond to the percentage the one risk represents of the entire population of risks—in this example, 10%. In some cases two or more risks will share the same entry ratio, and in these cases the bars will be of equal height.

The region covered by the bars is the region we will focus on in our work.

Computing an insurance charge from the diagram

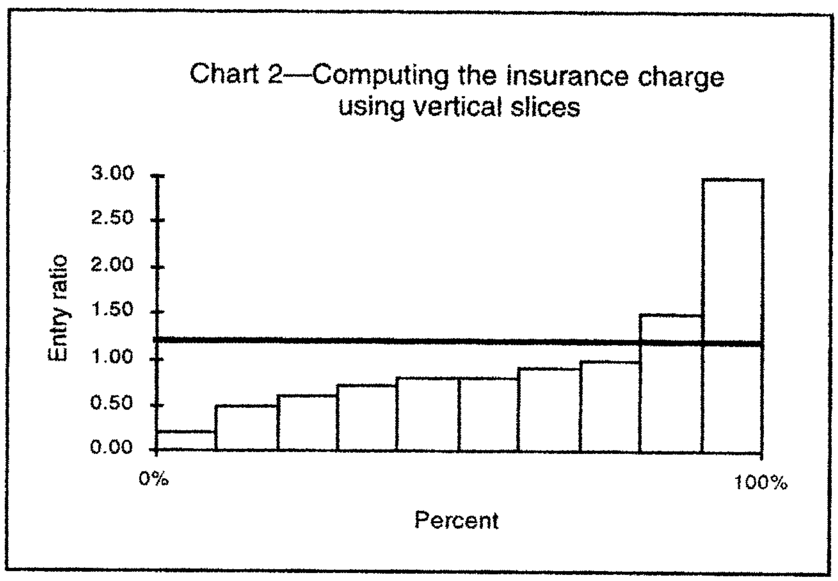
Given an entry ratio r, the insurance charge $\phi(r)$ is simply the area of the portion of the region that extends above height r.

Problem 1

Using the sample data, compute $\phi(1.2)$.

Solution

If we draw a horizontal line at height 1.2, we see that only the last two bars extend above the line. The area of the region above the line is made up of two rectangles. The first rectangle has height $1.5 - 1.2 = 0.3$ and width 10%, so its area is 0.03. The second rectangle has height $3.0 - 1.2 = 1.8$ and width 10%, so its area is 0.18.



We add the two areas together to determine that $\phi(1.2) = 0.03 + 0.18 = 0.21$.

Computing an insurance charge by limiting each risk's losses

An equivalent calculation can be made directly from the table. An entry ratio of 1.2 corresponds to a loss of $120,000. If we limit each risk's entry ratio to r and compute the average difference, we obtain $\phi(r)$. In our example this works as follows:

Risk	Entry Ratio	Limited Entry Ratio	Difference
1	0.20	0.20	0.00
2	0.50	0.50	0.00
3	0.60	0.60	0.00
4	0.70	0.70	0.00
5	0.80	0.80	0.00
6	0.80	0.80	0.00
7	0.90	0.90	0.00
8	1.00	1.00	0.00
9	1.50	1.20	0.30
10	3.00	1.20	1.80

The average difference is $\phi(1.2) = (0.00 + \cdots + 0.00 + 0.30 + 1.80)/10 = 0.21$. This calculation is identical to the one performed in Problem 1; it is only our thinking that is different.

Analysis by vertical slices—the per-risk view

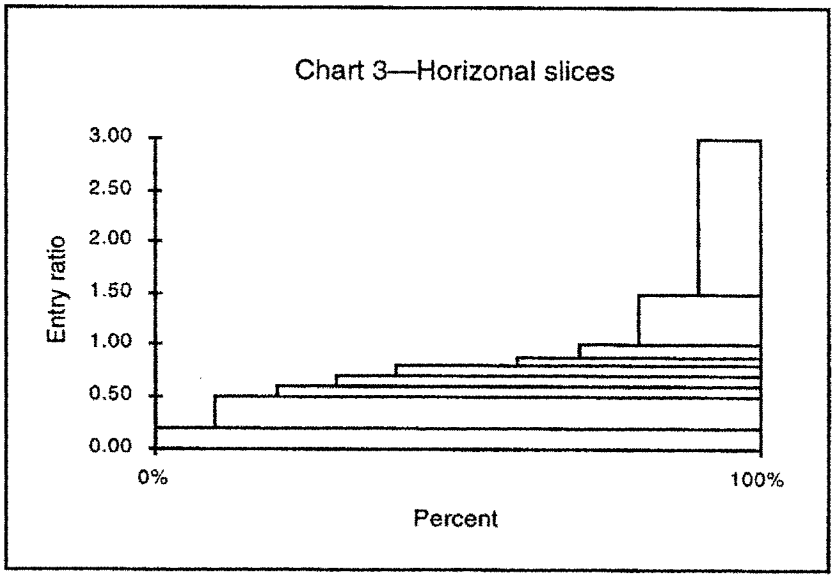
In both of the above calculations we looked at our data from a per-risk viewpoint. Since each risk corresponds to a vertical bar on the Lee diagram, this is equivalent to an analysis by vertical slices.

Such an analysis has some important advantages. It is the more natural approach, since it corresponds most closely to the way in which the data is presented. It is also the easiest approach to understand—if we want to measure the effect of a loss limitation, what could be more natural than to limit the losses risk by risk and see what happens? Finally, it is a quick solution if we want to compute the insurance charge for just one entry ratio.

On the other hand, this approach has some limitations. If we want to compute insurance charges for many different entry ratios, we must begin the process afresh for each entry ratio. In addition, when we have thousands of risks in our sample, as we would in a real-life problem, we must deal with each one individually when we use this approach.

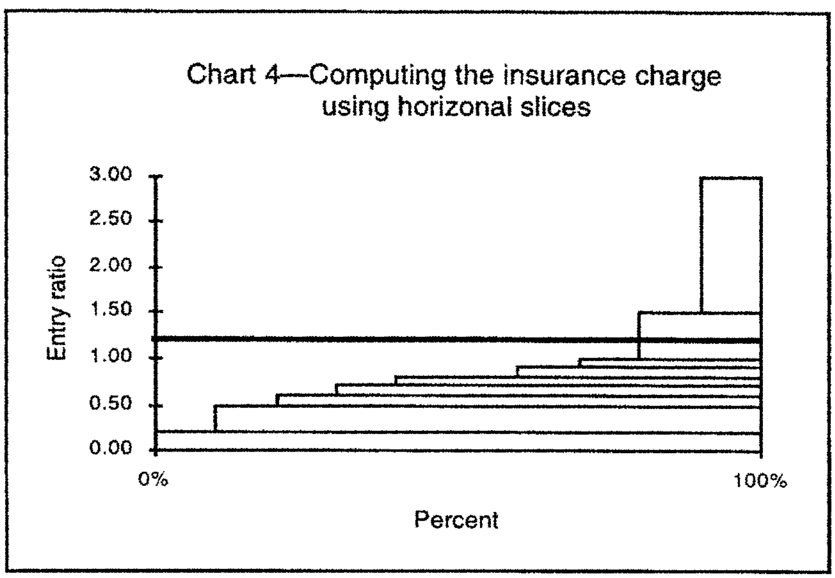
Analysis by horizontal slices—the per-layer view

The alternative approach is to slice the region being analyzed horizontally, or per-layer.



This approach is more difficult for the student to understand, and it requires some initial preparation because the data is not ordinarily presented in this way. Despite these disadvantages, the student should understand this method thoroughly. It is the method used in practice to construct Table M, and it works well when there are many risks.

If we return to our example, we can see that 10% of the risks have loss in the layer $1.5 \leq r \leq 3.0$ while 20% have loss in the layer $1.0 \leq r \leq 1.5$. To compute $\phi(1.2)$, we need only look at these two layers.



The top rectangle has height $3.0 - 1.5 = 1.5$ and width 10%, so its area is 0.15. The second rectangle has height $1.5 - 1.2 = 0.3$ and width 20%, so its area is 0.06 (note that the bottom of the rectangle is the line $r = 1.2$ since we are interested only in the part of the region that lies above that line.)

We add the two areas together to determine that $\phi(1.2) = 0.15 + 0.06 = 0.21$. The answer is of course identical to that obtained by slicing the region vertically, since the region is the same in either case, but the method used to find the answer is different.

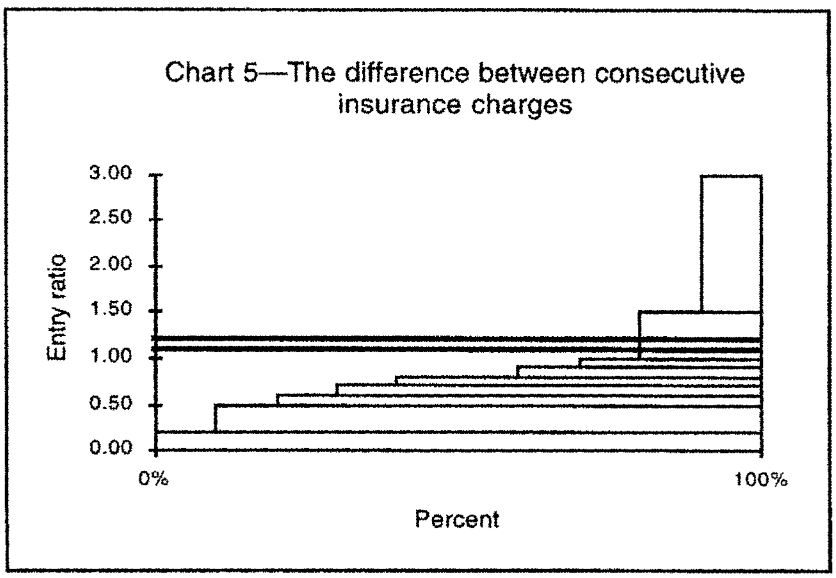
The advantage of this approach can be seen when we compute the insurance charge for a second entry ratio.

Problem 2

In the example, compute $\phi(1.1)$.

Solution

Since we have already computed $\phi(1.2)$, we need only add the area of the rectangular region that lies between $r = 1.2$ and $r = 1.1$. This area has height $1.2 - 1.1 = 0.1$.



The width of the new region is equal to the percentage of risks whose entry ratios exceed 1.1 (this percentage is known as $G(1.1)$ in the terminology used by Lee.) This width is 20%. We conclude that $\phi(1.1) = \phi(1.2) + (1.2 - 1.1)(20\%) = 0.23$.

The solution is made easier because we can build on the work we did for Problem 1. This is the advantage of the per-layer approach.

The technique used to solve Problem 2 depended on the fact that the region between $r = 1.2$ and $r = 1.1$ was a rectangle. If we had a risk in our sample with an entry ratio of 1.15, the region would not have been a rectangle and our answer would have been off.

Constructing Table M by the horizontal slicing method

We can use the ideas from the previous section to construct our own Table M from the sample data. We will begin with the largest entry ratios and work our way down.

The real Table M has rows corresponding to hundreds of different entry ratios and is produced using computers. When doing a problem by hand, we prefer to reduce the number of rows to a manageable level.

How many rows do we need? It is clear that we need a row for each row of the table we wish to produce. We also need a row for each entry ratio represented in our sample; otherwise some of the slices will not be rectangles.

Problem 3

Use the sample data to construct a Table M that lists insurance charges and insurance savings for entry ratios that are multiples of 0.20 from 0.00 to 1.20.

Solution

Based on the above discussion, we need rows for $r = 0.00$, 0.20, 0.40, 0.60, 0.80, 1.00, and 1.20. We also need rows for $r = 0.50$, 0.70, 0.90, 1.50, and 3.00 because we have risks in our sample corresponding to these values. We therefore create a table with 12 rows, as follows:

Table M

r	# Risks	# Risks over	% Risks over	$\phi(r)$	$\psi(r)$
0.00	0				
0.20	1				
0.40	0				
0.50	1				
0.60	1				
0.70	1				
0.80	2				
0.90	1				
1.00	1				
1.20	0				
1.50	1				
3.00	1				

In each row we enter the number of risks in our sample with the corresponding entry ratio. Each risk fits into some row because we made sure to have a row for each entry ratio in the sample. On the other hand, some rows may have no risks. This is not as likely to happen for a real Table M, which is based on many risks, as it is for a sample of only a few risks such as we are using for this example.

We next fill in the column that shows how many risks have entry ratios that are over a given entry ratio. Begin by entering 0 in the bottom row, because no risk can have an entry ratio that exceeds the maximum. Then work up the table; the entry in each row is equal to the entry in the row beneath plus the entry in the previous column of the row beneath. Convert to a percentage basis by dividing by the total number of risks.

Table M

r	# Risks	# Risks over	% Risks over	$\phi(r)$	$\psi(r)$
0.00	0	10	100%		
0.20	1	9	90%		
0.40	0	9	90%		
0.50	1	8	80%		
0.60	1	7	70%		
0.70	1	6	60%		
0.80	2	4	40%		
0.90	1	3	30%		
1.00	1	2	20%		
1.20	0	2	20%		
1.50	1	1	10%		
3.00	1	0	0%		

For example, the value 6 in the "# Risks over" column is obtained by adding the values 2 and 4 from the row beneath. The value 60% in the "% Risks over" column is 6 divided by the number of risks.

Now we are ready to compute the insurance charges $\phi(r)$. Again we start at the bottom of the table.

The insurance charge for the bottom row is zero because no risks in the sample exceeded the entry ratio for that row. This means the region we are measuring does not extend above the horizontal line corresponding to that entry ratio. It also means if losses had been capped at that level for the purpose of retro rating the sample portfolio, the insurance company would have lost no premium as a result of the capping.

Next we move up a row, so that the horizontal line moves down on the Lee diagram to $r = 1.5$. The newly added area is sure to be a rectangle because of the way we selected the rows for our table.

The height of the new rectangle is the difference between the entry ratio for this row and that for the row beneath. The width corresponds to the percentage of risks with $r > 1.5$; we can see from the table that this percentage is 10%. We can therefore compute the insurance charge as the product: $\phi(1.5) = \phi(3.0) + (3.0 - 1.5)(10\%) = 0.00 + 0.15 = 0.15$.

For the row $r = 1.2$, we add a second new rectangle to our region. This rectangle has height $1.5 - 1.2 = 0.3$ and width 20%, so $\phi(1.2) = \phi(1.5) + (1.5 - 1.2)(20\%) = 0.21$.

We continue to move up the table, with each $\phi(r)$ obtained by multiplying the entry ratio difference by the % of risks over the selected entry ratio and adding the result to the charge from the row beneath, as follows:

Table M

r	# Risks	# Risks over	% Risks over	$\phi(r)$	$\psi(r)$
0.00	0	10	100%	1.00	
0.20	1	9	90%	0.80	
0.40	0	9	90%	0.62	
0.50	1	8	80%	0.53	
0.60	1	7	70%	0.45	
0.70	1	6	60%	0.38	
0.80	2	4	40%	0.32	
0.90	1	3	30%	0.28	
1.00	1	2	20%	0.25	
1.20	0	2	20%	0.21	
1.50	1	1	10%	0.15	
3.00	1	0	0%	0.00	

The problem asked only for entry ratios that are multiples of 0.20 from 0.00 to 1.20, but we have computed additional entries as a necessary by-product of our work. We can eliminate these additional entries if we wish. We can also compute the savings for each row by using the formula $\psi(r) = \phi(r)+r-1$. Thus, for example, $\psi(1.2) = \phi(1.2)+1.2-1 = 0.41$.

Table M

r	$\phi(r)$	$\psi(r)$
0.00	1.00	0.00
0.20	0.80	0.00
0.40	0.62	0.02
0.60	0.45	0.05
0.80	0.32	0.12
1.00	0.25	0.25
1.20	0.21	0.41

This is the required solution.

Constructing Table M using horizontal slices of equal height

Many readings that discuss Table M construction[2] use a method that is differs slightly from the method described in the previous section. This alternate method assumes that each slice is of equal height, i.e., that the values in the entry ratio column are in strict arithmetic progression.

When a real Table M is constructed, the entry ratios are usually chosen so as to have intervals of 0.01 between rows. There are hundreds of rows in the resulting table, but this is unavoidable given the large sample sizes involved (in fact, there would be even more rows if not for the fact that each risk's entry ratio is rounded to two decimal places.)

[2]Lee, *op cit,* Gillam and Snader, *Fundamentals of Individual Risk Rating,* National Council on Compensation Insurance (Study Note), 1992, Part II, and Skurnick, "The California Table L," *PCAS* **61**, 1974.

If, on the other hand, we use a small sample size to generate a manageable example, it will often be the case that the entry ratios are not equally spaced (for example, the difference between the row with $r = 3.0$ and the one with $r = 1.5$ in the example discussed above is far larger than any other difference in the table.) When this is true, we can save a considerable amount of time by using slices that vary in height. Of course, we can always convert to an equal-height situation by adding rows, but this can add considerably to the number of rows (and time) needed. For example, if we were to do Problem 3 using slices of equal height, we would need a table with 31 rows (from 0.00 to 3.00 in intervals of 0.10,) and we would need 301 rows if the risk with an entry ratio of 1.00 had an entry ratio of 0.99 instead.

Despite this fact, it is important to understand how the equal-height method works if one wishes to understand the articles that use it. We will illustrate this method using a data sample that is specially chosen to keep the number of rows under control.

Problem 4

Given the following data, construct a Table M that shows insurance charges and savings for entry ratios that are multiples of 0.25 from 0.50 to 2.00.

Risk	Loss Ratio
1	30%
2	45%
3	45%
4	120%

Solution:

We begin by converting to an entry ratio basis. (It may be tempting to begin work immediately using the loss ratios, but this will result in a value for $\phi(0)$ that is not equal to 1.) We will use the sample average of 60% as our expected loss ratio and we compute entry ratios using this average.

Risk	Loss Ratio	r
1	30%	0.50
2	45%	0.75
3	45%	0.75
4	120%	2.00

We can now construct a table with entry ratios equally spaced in intervals of 0.25 using the method we used for Problem 3:

Table M

r	# Risks	# Risks above	% Risks above	$\phi(r)$
0.00	0	4	100%	1.0000
0.25	0	4	100%	0.7500
0.50	1	3	75%	0.5000
0.75	2	1	25%	0.3125
1.00	0	1	25%	0.2500
1.25	0	1	25%	0.1875
1.50	0	1	25%	0.1250
1.75	0	1	25%	0.0625
2.00	1	0	0%	0.0000

Although this computation is not difficult, the work can be simplified further. Each $\phi(r)$ in this table is obtained by adding a constant multiple of the "# Risks above" column to the charge in the row beneath (the multiple is the constant row height divided by the number of risks in the sample, or $0.25/4 = 0.0625$ in this example.)

As a result, we can do the double summation process on an integers-only basis and scale down later. The result looks like this:

Table M

r	# Risks	# Risks above	Upward sum	$\phi(r)$
0.00	0	4	16	1.0000
0.25	0	4	12	0.7500
0.50	1	3	8	0.5000
0.75	2	1	5	0.3125
1.00	0	1	4	0.2500
1.25	0	1	3	0.1875
1.50	0	1	2	0.1250
1.75	0	1	1	0.0625
2.00	1	0	0	0.0000

Here we compute an "Upward sum" column by adding the "# Risks above" entry for each row to the "Upward sum" entry from the row beneath. (Note that this is not quite the same as the method used to compute the "# Risks above" column!)

The "Upward sum" column must be correct up to a constant because the slices are of equal height. Since we know that $\phi(0)$ must be 1.00, we multiply the entire column by 0.0625, or 1/16, to obtain the $\phi(r)$ column. [3]

[3] This method can be understood geometrically: Imagine that the Lee diagram is divided into tiny blocks, each one risk wide and of height equal to the height of the slices. Then the "# Risks Above" column counts the blocks in a slice and the "Upward sum" column counts the blocks above a given horizontal line. The number at the top of this column is the total number of blocks in the entire region; since the region has area 1.00, the area of each block must be the reciprocal of this total number (here the area is 0.0625.) Multiply each entry in the "Upward sum" column by the area of a block to compute the needed area $\phi(r)$.

Table M

r	$\phi(r)$	$\psi(r)$
0.50	0.5000	0.0000
0.75	0.3125	0.0625
1.00	0.2500	0.2500
1.25	0.1875	0.4375
1.50	0.1250	0.6250
1.75	0.0625	0.8125
2.00	0.0000	1.0000

Again we compute the $\psi(r)$ column using the equation $\psi(r) = \phi(r) + r - 1$.

While it is good to understand the equal-height method, one should be aware of its disadvantages. It is more difficult to understand from an intuitive standpoint, and it can add considerably to the time needed to do a problem manually.

Table M Construction and the functions in retro rating

The concepts we have reviewed in this Study Aid provide insight into Lee's discussion of the functions in retrospective rating.

Lee begins by defining $\phi(r) = \int_r^\infty (y-r)dF(y)$. This definition parallels our first computation of $\phi(r)$. It considers all risks whose entry ratios exceed r, and it limits the losses of each to r. The difference between the raw entry ratio y and the limited value r, when summed over all such risks, is the insurance charge $\phi(r)$. This definition is easy to understand, but as we have seen, it does not lead to the most convenient computation.

Lee then discusses the equation $d\phi/dr = -G(r)$. This ties directly to our computation of ϕ using the method of horizontal strips. We see this as follows: In Problem 2 we computed $\phi(1.1) = \phi(1.2) + (1.2 - 1.1)(20\%)$. Rewrite this as $\phi(1.2) - \phi(1.1) = (1.2 - 1.1)(-20\%)$, or $\Delta\phi = \Delta r(-G(r))$, where $G(r)$ is the percentage of risks with entry ratios that exceed r. Given that $\Delta\phi/\Delta r \approx d\phi/dr$, we can derive Lee's equation directly from our computation.

Similarly, we can see from our construction that the difference between two consecutive entries in the "% Risks over" column is simply the percentage of risks in the row; that is, $dG/dr = -f(r)$. We do not actually have a column for $f(r)$ in our construction, but we could obtain one by dividing the "# Risks" column by the total number of risks (10 risks in our example.)

We conclude that Lee's equations

$$\phi'(r) = -G(r),$$

$$\phi''(r) = f(r)$$

are directly evident in our construction.

What if the sample loss ratio does not equal the expected loss ratio?

The average loss ratio of the risks in the sample may not equal the stated expected loss ratio. In real life this can happen because of random fluctuations, but it can also happen if one is not careful in constructing a sample problem.

In theory, the correct response to this situation is to use the actual loss ratio of the sample rather than the nominal expected loss ratio. (Only in this way can one get a Table M that satisfies the identity $\phi(0) = 1$.)

But what do we do in practice when we complete a problem and $\phi(0) \neq 1$? The correct adjustment is to divide both the entry ratio column (the r s) and the insurance charge column (the $\phi(r)$ s) by the calculated $\phi(0)$. This will normalize the table so that $\phi(0) = 1$ as it should.

Gillam and Snader do not address this issue; in their example the actual loss ratio equals the expected loss ratio. Lee and Skurnick both make this adjustment; we can see this from the fact that they normalize the R_i s to get the final entry ratios (they do this when they multiply by the factor $S_{2,0}/S_{1,0}$.) The latter two authors do most of the work with the unnormalized ratios, then adjust the entry ratios at the end.

This normalization process can produce a Table M for which the entry ratios are denominated in odd decimals rather than round numbers. If one desires a table with entry ratios that are denominated in round numbers, one can interpolate to approximate the corresponding insurance charges.