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Preconditioners for Saddle Point Problems arising in Ocean Flow Computations

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Commonly used finite element and finite difference discretizations of the (Navier-)Stokes equation lead to the saddle point problem

$$K \begin{pmatrix} u \\ p \end{pmatrix} \equiv \begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix},$$

where the unknown vector $(u,p)^T$ contains velocities and pressures respectively. The matrix A is positive definite (not necessarily symmetric). This matrix contains also the essence of matrices arising in ocean flow computations using THCM [4]. In those matrices we have to deal with non-symmetry of A due to the Coriolis force, which plays a very important role in the equations.

In literature one can find several good preconditioners for the matrix K. In general they exploit somehow the structure of the matrix. We present two alternative preconditioners:

$$\hat{K}_a = \begin{pmatrix} A & B \\ B^T & -I/\omega \end{pmatrix} \quad \text{and} \quad \hat{K}_g = \begin{pmatrix} A + \omega BB^T & 0 \\ 0 & -I/\omega \end{pmatrix},$$

the *artificial compressibility* preconditioner and the *grad-div* preconditioner respectively. The preconditioners are related to each other: to solve an equation with $\hat{K}_a$ or $\hat{K}_g$ we have to solve an equation with the matrix $A + \omega BB^T$.

In the table below, we compare the number of iterations of our preconditioners with that of SIMPLER [3] and the preconditioner of Elman et al. [2] on the 2D Stokes problem. In the iterations we used an exact factorization of $A+\omega BB^T$ (in case of $\hat{K}_a$ and $\hat{K}_g$) and A (SIMPLER and Elman). Clearly the preconditioners perform better for ω large. The artificial compressibility preconditioner even shows convergence within one step, simply because for ω large $\hat{K}_a$ tends to K.

	GRAD-DIV			ART. COMPR.			ELMAN	SIMPLER
ω	1	16	256	1	16	256	-	-
ITER	4.5	3.0	3.0	4.0	2.0	1.0	5.5	16.5

Of course, solving $A + \omega BB^T$ by a direct method is to costly and therefore we employ MRILU [1]. We observed that for large ω, the system became harder to solve since then the matrix is far from diagonal dominant. Experiments showed that good preconditioners are constructed for ω ranging from 0 to 2.

For $\omega = 1$, we compared the performance of MRILU to the performance of an exact factorization in MATLAB for a range of problem sizes. For the exact factorization we used an approximate minimum degree ordering to reduce the fill. For a repeated doubling of the number of variables in each direction it appeared that the construction time in the exact factorization grows with almost a factor 8, while the construction time in MRILU grows with a factor close to 4; the latter is also the increase in problem size. For a problem of size about 1 million the fill in the MRILU factorization is more than 4 times less than that in the exact factorization, leading to a more than proportional reduction of a factor 7 for the construction time. In order to gain 6 digits of accuracy 10 to 30 iterations where needed using only a modest amount of time.

In order to study whether the above is also applicable in the presence of a Coriolis force we took a matrix from THCM and stripped it from parts related to the vertical velocity, the salt transport and the energy transport. The table below shows the number of iterations with the grad-div solver using an exact factorization of $A + \omega BB^T$.

	GRAD-DIV			
ω	1/4	1	16	256
iterations	20.5	11.5	3.5	3.0

Applying MRILU to the system $A + \omega BB^T$ in this case showed a behavior similar to that observed for the Stokes equation.

We concluded that for small values of ω, MRILU combined with the grad-div or artificial compressibility preconditioner leads an effective method for the considered saddle point problems.

In the talk we will present results of an eigenvalue analysis for both preconditioners, which shows the effect of the parameter ω. Moreover we intend to show more results on the performance of this approach, especially for the full ocean flow equations.

Bibliography

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