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AMLS and Spectral Schur Complements

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In the last few decades, Krylov projection methods such as the Lanczos algorithm and its variants, have dominated the scene of algorithms for eigenvalue problems. Recently, an alternative approach has emerged in structural engineering as a competitor to the standard shift-and-invert Lanczos approach. The algorithm, called Automated Multilevel Substructuring method (`AMLS`) is rooted in a domain decomposition framework. It has been reported as being capable of computing thousands of the smallest normal modes of dynamic structures on commodity workstations and of being orders of magnitude faster than the standard approach [5].

A theoretical framework for `AMLS` was recently presented by Benighof and Lehoucq in [3] from the point of view of domain decomposition, using adequate functional spaces and operators on them. The goal of this talk is to present a complementary viewpoint, which is entirely algebraic. `AMLS` is essentially a Schur complement method. Schur complement techniques are well understood for solving linear systems and play a major role in Domain Decomposition techniques. Relatively speaking, the formulation of this method for eigenvalue problems has been essentially neglected so far. One could of course extend the approach used for linear systems in order to compute eigenvalues, by formulating a Schur complement problem for each different eigen-pair, (e.g., by solving the eigenvalue problem as a sequence of linear systems through shift-and-invert). This viewpoint was considered quite early on by Abramov [1, 2] and Chichov [4] who presented what may termed a spectral Schur complement method. It can easily be verified that a scalar λ is an eigenvalue of a matrix A partitioned as

$$A = \begin{pmatrix} B & F \\ E & C \end{pmatrix},$$

if and only if it is an eigenvalue of $S(\lambda) = C - E(B - \lambda I)F$ (this is clearly restricted to those λ's that are not in the spectrum of B). This nonlinear eigenvalue problem may be solved by a Newton-type approach. Alternatively, one can also devise special iterative schemes based on the above observation. An approach of this type is clearly limited by the fact that a Schur complement (or several consecutive ones in an iterative process) is required for each different eigenvalue. It can, however, work well for computing one, or a few, eigenvalues or in some other special situations. For example, this nonlinear viewpoint led to the development of effective shifts of origin for the QR algorithm for tridiagonal matrices [6].

The fundamental premise of `AMLS`, and its attraction, is that it is capable of extracting very good approximations to a large number of the smallest eigenvalues *with only one Schur complement*. To achieve this, `AMLS` relies on clever projection techniques. It builds good bases from one Schur complement, and expands them in an effective way to bigger and bigger domains.

In this talk we adopt a purely algebraic viewpoint and demonstrate that `AMLS` can be viewed as a method which exploits a first order approximation to a nonlinear eigenvalue problem in order to extract a good subspace for a Rayleigh-Ritz projection process. This technique leads to approximations from a single Schur complement derived from a domain decomposition of the physical problem. Exploiting this observation, we have devised several possible enhancements in two main directions. The first introduces Krylov subspaces to the technique, and the second considers a more accurate (second order instead of first order) scheme, which is based on a quadratic eigenvalue problem. Finally, combinations of the above two strategies have been considered with a goal of enhancing robustness.

Currently, `AMLS` is a one-shot algorithm in the sense that certain approximate eigenvectors are build from the last level up to the highest level and no further refinements are made. The current framework does iteratively refine these approximations. We will discuss this issue and will explore the feasibility of an iterative scheme based on `AMLS`.

Bibliography

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