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A Primal-Dual Formulation for the Bingham Flow

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The Bingham flow is an example of a Stokes-type equation with shear-dependent viscosity. If $\mathbf{Du} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ and $|\mathbf{Du}| = \sqrt{\mathrm{tr}(\mathbf{Du}^2)}$, the equations read

$$\begin{cases} -\nabla\cdot\tau + \nabla p & = \mathbf{f}, \\ -\nabla\cdot\mathbf{u} & = 0 \\ +B.C., \end{cases}$$

and

$$\begin{cases} \tau = 2\mu\mathbf{Du} + \tau_s\frac{\mathbf{Du}}{|\mathbf{Du}|}, & \text{if } |\mathbf{Du}| \neq 0, \\ |\tau| \leq \tau_s, & \text{if } |\mathbf{Du}| = 0, \end{cases}$$

where the velocity $\mathbf{u} \in \mathbb{R}^n$, $n = 2, 3$ and $p \in \mathbb{R}$ are the unknowns and μ, τ_s are given constants. Due to its non-differentiability for $\mathbf{Du} = 0$, a regularization of the form $\mathbf{Du} = \sqrt{\mathrm{tr}(\mathbf{Du}^2) + \varepsilon}$ $(\varepsilon > 0)$ is necessary. It is a well-known fact that applying a nonlinear solver such as Newton or Picard to these equations results in a high number of outer iterations, especially for small choices of ε [1]. In this talk we suggest an alternative approach inspired by [6]: We introduce a dual variable $\mathbf{W} = \frac{\mathbf{Du}}{|\mathbf{Du}|}$, the equations for the Bingham flow are then reformulated as

$$\begin{cases} -\nabla\cdot(2\mu\mathbf{Du} + \tau_s\mathbf{W}) + \nabla p & = \mathbf{f}, \\ -\nabla\cdot\mathbf{u} & = 0, \\ \mathbf{W}|\mathbf{Du}| & = \mathbf{Du} \\ +B.C. \end{cases}$$

We address a few properties of this formulation and its numerical solution. Moreover, we perform several numerical experiments for solving the Bingham equations in this formulation, including the lid-driven cavity test and an example where the analytical solution is known. These experiments indicate a significant reduction in the number of nonlinear iterations over the nonlinear solvers of the equations in primal variables.

Acknowledgement:
We thank M. Olshanskii for fruitful discussions.

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