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Multilevel Preconditioners for Nonselfadjoint or Indefinite Orthogonal Spline Collocation Problems

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We develop and study symmetric multilevel preconditioners forthe computation of the orthogonal spline collocation (OSC) solution of aDirichlet boundary value problem (BVP) with a nonselfadjoint or anindefinite operator. The OSC solution is sought in the space ofpiecewise Hermite bicubic spline functions defined on a uniformpartition. We consider an additive and a multiplicative multilevel-preconditioners that are used with the preconditioned conjugategradient (PCG) method. Results and algorithms presented in this paperare closely related to those in [1], [2], [3], and [4].Let Ω be a unit square $(0,1)\times(0,1)$ with the boundary$\partial\Omega$, and let $x=(x_1,x_2)$. We consider a BVP

$$Lu \equiv \sum_{i,j=1}^{2} a_{ij}(x)u_{x_ix_j} + \sum_{i=1}^{2} b_i(x)u_{x_i} + c(x)u = f(x), \quad x\in\Omega, \quad u=0 \text{ on } \partial\Omega. \tag{1}$$

Operator L could be non-selfadjoint or indefinite in L^2 innerproduct. We assume that the principal part of L satisfies theuniform ellipticity condition and that BVP (1) has aunique solution in $H^2(\Omega)$.Let π_0 be a uniform coarsest rectangular partition of Ω.We obtain a set of partitions $\{\pi_k\}_{k=0}^{K}$ by standardcoarsening, and let $V_0\subset V_1\subset\ldots\subset V_K\equiv V_h$be the set of corresponding nested spaces of piecewise Hermitebicubics that vanish on $\partial\Omega$. Let $\sum$ denote thetwo-dimensional composite Gauss quadrature corresponding to partitionπ_h with 4 nodes in each element. Let $\mathcal{G}_h$ denote thecorresponding set of Gauss points. The OSC discretization of BVP(1) is defined by

$$u_h\in V_h, \quad Lu_h(\xi)=f(\xi), \quad \xi\in\mathcal{G}_h, \tag{2}$$

and it can be written as the operator equation $L_hu_h=f_h$ in theHilbert space V_h with the inner product $(v,w)_h=\sum vw$.Let $\{\psi_{k,j}\}_{j=1}^{N_k}$ be the standard finite element basisof V_k consisting of products of one-dimensional value and slopebasis functions. Using space decomposition

$$V_h = V_0 + \sum_{k=1}^{J}\sum_{j=1}^{N_k} V_{k,j}, \quad V_{k,j}=\text{span}(\psi_{k,j}),$$

we define and study multilevel additive $B_{\mathrm{a}}$and multiplicative B_{m} preconditioners forsolving the normal equation $L_h^* L_h u_h = L_h^* f_h$, where L_h^*isthe adjoint to L_h. The implementation of B and B_{m} is based on relationshipsbetween basis functions for two consecutive partitions and theimplementation of B_{m} is similar to thatforV(1,1)-cycle with the Gauss-Seidel smoothing. A problem on thecoarsest partition is assumed sufficiently small, and it is solvedexactly. The computational cost of the preconditioning algorithmsisO(N_K). The following is our main result.**Theorem.***There are positive independent of h and K constants*α_{a}, β_{a},α_{m}, *and* β_{m},*such that*

$$\begin{aligned} \alpha_{\mathrm{a}}\,(B_{\mathrm{a}}v,v)_h \le (L_h^* L_h v,v)_h \le \beta_{\mathrm{a}}\,(B_{\mathrm{a}}v,v)_h, \quad v\in V_h, \\ \alpha_{\mathrm{m}}\,(B_{\mathrm{m}}v,v)_h \le (L_h^* L_h v,v)_h \le \beta_{\mathrm{m}}\,(B_{\mathrm{m}}v,v)_h, \quad v\in V_h. \end{aligned} \tag{3}$$

To obtain these results, we prove the key assumptions in the generaltheory of Schwarz methods presented in [4], and use the inequalities

$$C^{-1}\|v\|^2_{H^2(\Omega)} \le a_h(v,v) \le C\|\Delta v\|^2_{L^2(\Omega)}, \quad v\in V_h,$$

obtained in [2]. We present numerical results that demonstratethe efficiency of our preconditioning algorithms.**References**[1] R. Aitbayev and B. Bialecki, *A preconditioned conjugategradient method for nonselfadjoint or indefinite orthogonal splinecollocation problems*, SIAM J. Numer. Anal., 41 (2003), pp. 589–604.[2]B. Bialecki, *Convergence analysis of orthogonalsplinecollocation for elliptic boundary value problems*, SIAM J. Numer.Anal., 35 (1998),pp. 617–631.[3]B. Bialecki and M. Dryja, *Multilevel additive andmultiplicative methods for orthogonal spline collocation problems*,Numer. Math., 77 (1997), pp. 35–58.[4]B. F. Smith, P. E. Bjørstad, and W. D. Gropp,*Domain Decomposition: Parallel Multilevel Methods for Elliptic PartialDifferential Equations*, Cambridge University Press, New York, 1996.