On algebraic two-level methods for non-symmetric systems

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January 8, 2007

Abstract

Here we analyze algebraic multilevel methods applied to non-symmetric M-matrices. We consider two types of multilevel approximate block factorizations. The first one is known as the AMLI method. The second method is the multiplicative counterpart of the AMLI method which we call multiplicative algebraic multilevel method, the MAMLI method. Closely related to the MAMLI method is the symmetrized MAMLI method, short SMAMLI. We establish convergence results and comparison results between these methods.

Keywords. Algebraic multilevel methods, multilevel approximate block factorizations, algebraic multigrid methods, AMLI method

1 Introduction

In many recent papers algebraic multigrid methods or multilevel methods were designed to solve large sparse linear systems efficiently. Using only information on the matrix structure and the matrix entries, these methods should perform like or even better than the geometric multigrid methods for elliptic problems [13, 31]. Among several algebraic methods the algebraic multigrid method (AMG) and the multilevel approximate block factorization are best-known. The pioneering work on algebraic multilevel methods was done by Brandt, McCormick and Ruge [7] and Ruge and Stüben [26, 28, 29] by introducing the AMG method in the eighties.

Although these multilevel methods work very well in practice for many problems, there is not that much known about theoretical convergence properties, especially for non-symmetric problems. Recently, a theoretical comparison of different algebraic multigrid methods applied to symmetric positive definite systems was given by Notay in [25].

Here we analyze algebraic multilevel methods for non-symmetric systems. We choose the AMLI method introduced by Axelsson and Vassilevski [2, 3] as starting point of our analysis. Convergence results for the AMLI method are given in [2, 3, 22, 1]. However, the results can be applied only to so called Stieltjes matrices, i.e. symmetric positive definite M-Matrices. Notay gives in [23] some results for the multilevel approximate block factorization or AMLI method applied to some special non-symmetric M-matrices which arise from a specific discretization of a certain PDE. So far a general convergence analysis for the AMLI method for a wide class of non-symmetric matrices is still missing. In this paper we give convergence results for the AMLI method applied to arbitrary non-symmetric M-matrices. This results presented here are based on [16, 17].

The AMLI method can be expressed as an inexact additive Schwarz method, see e.g. [16]. We use this connection with Schwarz methods to introduce the multiplicative counterpart of the AMLI method, which we call the MAMLI method (multiplicative algebraic multilevel method). Closely related to the MAMLI method is the symmetrized MAMLI (SMAMLI) method. In particular, the MAMLI method is a multigrid method with one pre-smoothing step, and the SMAMLI method is a multigrid method with one pre- and one post-smoothing step using fine-grid only relaxation. Moreover, there is a relation between the multiplicative AMLI techniques and the AMGe method [8] and the AMGr method [15]. These methods use similar restriction and interpolatation operators. Also the convergence analysis in [8, 15] includes error bounds, but it is given for the symmetric and positive definite case only. Though there are similarities between our methods and AMGe, AMGr, due to length restrictions we focus on the AMLI and (S)MAMLI methods.

In [34] Xu introduced a general framework to consider Schwarz methods and multigrid methods. However, the results in [34] can be applied to symmetric positive definite matrices only. Recently Schwarz methods for non-symmetric matrices were analyzed in [11, 5, 21]. In these papers an algebraic convergence theory for the additive and multiplicative Schwarz was introduced. However, this theory includes only special restriction and interpolation operators which are used in domain decomposition methods. This theory can not be applied to multilevel or multigrid methods.

Here, we provide techniques to analyze the AMLI and (S)MAMLI methods applied to the wide class of non-symmetric M-matrices. Moreover, we give a comparison of our convergence bounds for the AMLI method and the (S)MAMLI methods. Due to length restrictions we consider only the two-level case. A more detailed analysis including multilevel convergence, convergence rates, and more theoretical comparison results can be found in [16, 17, 20, 18, 19].

The paper is organized as follows. In the next section we give some notation. Section 3 gives a short introduction into the multilevel approximate block factorization (AMLI) method and describes the new MAMLI method. In Section 4

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we establish our convergence analysis and our comparison results. In Section 5 our theoretical results are highlighted by numerical experiments.

2 Notation

A matrix B is non-negative (positive), denoted $B \geq 0$ ($B > 0$), if its entries are non-negative (positive). We say that $B \geq C$ if $B - C \geq 0$, and similarly with the strict inequality. These definitions carry over to vectors. A matrix A is a non-singular M-matrix if its off-diagonal elements are non-positive, and it is monotone, i.e., $A^{-1} \geq 0$. It follows that if A and B are non-singular M-matrices and $A \geq B$, then $A^{-1} \leq B^{-1}$ [6, 32]. By $\rho(B)$ we denote the spectral radius of the matrix B.

We say (M, N) is a splitting of A if $A = M - N$ and M is non-singular. A splitting is regular if $M^{-1} \geq 0$ and $N \geq 0$; it is weak regular of the first type if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$. [6, 32, 33].

Next we recall the definition of the weighted max-norm. Given a positive vector $w \in \mathbb{R}^n$, denoted $w > 0$, the weighted max-norm is defined for any $y \in \mathbb{R}^n$ as $\|y\|_w = \max_{j=1,\cdots,n} |\frac{1}{w_j} y_j|$. The corresponding matrix norm is defined as $\|T\|_w = \sup_{\|x\|_w=1} \|Tx\|_w$ and the following lemma holds.

Lemma 2.1. [9, 10] *Let $A \in \mathbb{R}^{n\times n}$, $w \in \mathbb{R}^n$, $w > 0$, and $\gamma > 0$ such that*

$$|A|w \leq \gamma w \tag{1}$$

Then, $\|A\|_w \leq \gamma$. If the inequality in (1) is strict, then the bound on the norm is also strict.

Most of our estimates hold for all positive vectors w of the form $w = A^{-1}e$, where e is any positive vector, i.e., for any positive vector w such that Aw is positive. In particular this would hold for a M-matrix A and $e = (1, \ldots, 1)^T$, i.e., with $w = A^{-1}e$ being the row sums of A^{-1}.

3 The AMLI and the MAMLI method

Most of the algebraic multilevel methods start with a partitioning of the unknowns into fine and coarse grid nodes. Related to this ordering the $n \times n$ system matrix A can be permuted in a block 2×2 form

$$A = \begin{bmatrix} A_{FF} & A_{FC} \\ A_{CF} & A_{CC} \end{bmatrix}. \tag{2}$$

Here F denotes the set of fine grid unknowns, and C denotes the set of coarse grid unknowns with $|F| = n_F$ and $|C| = n_C$. Furthermore, we will denote with I the $n \times n$ identity matrix and with I_F and I_C the $n_F \times n_F$ and $n_C \times n_C$ the identity matrix, respectively.

There are a lot of different algorithms and strategies to perform the above partitioning. The choice of the partitioning has a major influence for the convergence behavior. But here we assume that this partitioning is done already in an arbitrary way. We want to focus on the convergence behavior for general partitioning and will compare different methods starting with the same partitioning.

If the submatrix A_{FF} is non-singular, A can be factorized as

$$A = \begin{bmatrix} I_F & 0 \\ A_{CF}A_{FF}^{-1} & I_C \end{bmatrix} \begin{bmatrix} A_{FF} & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I_F & A_{FF}^{-1}A_{FC} \\ 0 & I_C \end{bmatrix},$$

where $S := (A/A_{FF}) := A_{CC} - A_{CF}A_{FF}^{-1}A_{FC}$ is the Schur complement. If we now use an approximation $\widetilde{A}_{FF}$ of A_{FF} and an approximation $\widetilde{S}$ of S, or approximations of the inverses of these matrices, we obtain the matrix M with

$$M = \begin{bmatrix} I_F & 0 \\ A_{CF}\widetilde{A}_{FF}^{-1} & I_C \end{bmatrix} \begin{bmatrix} \widetilde{A}_{FF} & 0 \\ 0 & \widetilde{S} \end{bmatrix} \begin{bmatrix} I_F & \widetilde{A}_{FF}^{-1}A_{FC} \\ 0 & I_C \end{bmatrix}.$$

This factorization is known as an approximate two-level (multilevel) block factorization [25]. A lot of multilevel methods use this two-level block approximate factorization as a major tool (see e.g. [2, 3, 1, 4, 27] and references in [25]). One of these methods is the AMLI method by Axelsson and Vassilevski [2, 3]. The AMLI method, in its basic form, can be described as the stationary iteration with the iteration matrix $T_{AMLI} = I - M^{-1}A$. If the AMLI method is used as a preconditioner for the cg-method, the preconditioner is M^{-1}. For the iteration matrix we obtain

$$\begin{aligned} T_{AMLI} &= (I - M^{-1}A) = I - \left(\begin{bmatrix} I_F & 0 \\ A_{CF}\widetilde{A}_{FF}^{-1} & I_C \end{bmatrix} \begin{bmatrix} \widetilde{A}_{FF} & 0 \\ 0 & \widetilde{S} \end{bmatrix} \begin{bmatrix} I_F & \widetilde{A}_{FF}^{-1}A_{FC} \\ 0 & I_C \end{bmatrix} \right)^{-1} A \\ &= I - \begin{bmatrix} -\widetilde{A}_{FF}^{-1}A_{FC} \\ I_C \end{bmatrix} \widetilde{S}^{-1} \begin{bmatrix} -A_{CF}\widetilde{A}_{FF}^{-1} & I_C \end{bmatrix} A - \begin{bmatrix} I_F \\ 0 \end{bmatrix} \widetilde{A}_{FF}^{-1} \begin{bmatrix} I_F & 0 \end{bmatrix} A. \end{aligned}$$

Using the following matrices

$$P_1 := \begin{bmatrix} -\widetilde{A}_{FF}^{-1}A_{FC} \\ I_C \end{bmatrix} \widetilde{S}^{-1} \begin{bmatrix} -A_{CF}\widetilde{A}_{FF}^{-1} & I_C \end{bmatrix} A \quad \text{and} \quad P_2 := \begin{bmatrix} I_F \\ 0 \end{bmatrix} \widetilde{A}_{FF}^{-1} \begin{bmatrix} I_F & 0 \end{bmatrix} A \tag{3}$$

we obtain

$$T_{AMLI} = I - P_1 - P_2. \tag{4}$$

Thus, the AMLI method can be written as an additive Schwarz method with inexact local solves (see [34, 14, 30] for details about Schwarz methods).

For some problems it is known that the multiplicative Schwarz method converges faster than the additive Schwarz method [12, 21]. So it is worth considering the multiplicative version of the AMLI method. The multiplicative version, which we call the MAMLI method, is given by:

$$\begin{aligned} T_{MAMLI} &= (I - P_1)(I - P_2) \\ &= \left(I - \begin{bmatrix} -\widetilde{A}_{FF}^{-1}A_{FC} \\ I_C \end{bmatrix} \widetilde{S}^{-1} \begin{bmatrix} -A_{CF}\widetilde{A}_{FF}^{-1} & I_C \end{bmatrix} A\right)\left(I - \begin{bmatrix} I_F \\ 0 \end{bmatrix} \widetilde{A}_{FF}^{-1} \begin{bmatrix} I_F & 0 \end{bmatrix} A\right). \end{aligned} \tag{5}$$

Closely related to the multiplicative version of the AMLI method is the symmetrized MAMLI method (SMAMLI). This variant is defined in terms of their iteration matrices given by

$$\begin{aligned} T_{SMAMLI} &= (I - P_2)(I - P_1)(I - P_2) \\ &= \left(I - \begin{bmatrix} I_F \\ 0 \end{bmatrix} \widetilde{A}_{FF}^{-1} \begin{bmatrix} I_F & 0 \end{bmatrix} A\right) \\ &\quad \cdot \left(I - \begin{bmatrix} -\widetilde{A}_{FF}^{-1}A_{FC} \\ I_C \end{bmatrix} \widetilde{S}^{-1} \begin{bmatrix} -A_{CF}\widetilde{A}_{FF}^{-1} & I_C \end{bmatrix} A\right) \\ &\quad \cdot \left(I - \begin{bmatrix} I_F \\ 0 \end{bmatrix} \widetilde{A}_{FF}^{-1} \begin{bmatrix} I_F & 0 \end{bmatrix} A\right). \end{aligned} \tag{6}$$

Of course the quality of the approximations of A_{FF} and S will be important for the convergence behavior of all these methods.

Assuming that the system matrix A is a non-singular (non-symmetric) M-matrix our convergence analysis given in Section 4 allows approximations $\widetilde{A}_{FF}$ and $\widetilde{S}$ such that the splitting $(\widetilde{A}_{FF}, (\widetilde{A}_{FF} - A_{FF}))$ is weak regular of the first type, i.e.

$$\widetilde{A}_{FF}^{-1} \geq 0 \quad \text{and} \quad I_F - \widetilde{A}_{FF}^{-1}A_{FF} \geq 0; \tag{7}$$

and the splitting $(\widetilde{S}, (\widetilde{S} - (A/\widetilde{A}_{FF})))$ is weak regular of the first type, i.e.

$$\widetilde{S}^{-1} \geq 0 \quad \text{and} \quad I_C - \widetilde{S}^{-1}(A/\widetilde{A}_{FF})) \geq 0. \tag{8}$$

Here $(A/\widetilde{A}_{FF})$ is defined by $(A/\widetilde{A}_{FF}) := A_{CC} - A_{CF}\widetilde{A}_{FF}^{-1}A_{FC}$.

Note that there is a coupling between both approximations, but it is very mild. Indeed, starting with an M-matrix A, the approximations given by the Jacobi and the Gauss-Seidel splittings and the incomplete LU-factorization are admissible approximations for example. Even more, A_{CC} fulfills the assumptions for the approximation $\widetilde{S}$ of S.

4 Convergence Results

We start this section with a fundamental proposition which is the main tool in our convergence analysis.

Proposition 4.1. *Let $A \in \mathbb{R}^{n\times n}$ be a non-singular M-matrix. If $C \in \mathbb{R}^{n\times n}$ is non-negative, $I - CA$ is non-negative, and C has no zero row, then*

$$\rho(I - CA) \leq \|I - CA\|_w < 1 \ ,$$

where $w := A^{-1}e$ and $e \in \mathbb{R}^n$ is an arbitrary positive vector.

Proof: Let $e \in \mathbb{R}^n$ be an arbitrary positive vector. Under the above assumption Ce is also positive. Since A is a non-singular M-matrix the inverse A^{-1} is non-negative and has no zero row. So the vector $w := A^{-1}e$ is as well positive. Using these properties we get

$$0 \leq (I - CA)\, w \;=\; w - CAA^{-1}e = w - Ce < w \ .$$

Due to Lemma 2.1 this leads to

$$\rho(I - CA) \leq \|I - CA\|_w < 1 \ .$$

□

In the following we consider the AMLI iteration matrix as given in (4),

$$T_{AMLI} \;=\; I - P_1 - P_2.$$

Lemma 4.2. *Let A be a non-singular M-matrix partitioned as in* (2). *If the splitting $\left(\widetilde{A}_{FF}, \widetilde{A}_{FF} - A_{FF}\right)$ is a weak regular splitting of A_{FF} of first type and the splitting $\left(\widetilde{S}, \widetilde{S} - \left(A/\widetilde{A}_{FF}\right)\right)$ is also a weak regular splitting of $\left(A/\widetilde{A}_{FF}\right)$ of first type, then*

$$T_{AMLI} \geq 0.$$

Proof: A computation leads to

$$\begin{aligned} T_{AMLI} &= \begin{bmatrix} I_F - \widetilde{A}_{FF}^{-1}A_{FF} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \widetilde{A}_{FF}^{-1}A_{FC}\widetilde{S}^{-1}A_{CF}\left(I_F - \widetilde{A}_{FF}^{-1}A_{FF}\right) & 0 \\ -\widetilde{S}^{-1}A_{CF}\left(I_F - \widetilde{A}_{FF}^{-1}A_{FF}\right) & 0 \end{bmatrix} \\ &+ \begin{bmatrix} 0 & -\widetilde{A}_{FF}^{-1}A_{FC}\left(I_C - \widetilde{S}^{-1}\left(A/\widetilde{A}_{FF}\right)\right) \\ 0 & I_C - \widetilde{S}^{-1}\left(A/\widetilde{A}_{FF}\right) \end{bmatrix} \end{aligned}$$

Using the assumption on the approximations $\widetilde{A}_{FF}$ and $\widetilde{S}$ and the sign pattern of the M-matrix A we get that T_{AMLI} is non-negative. □

Lemma 4.2 states the non-negativity of the AMLI iteration matrix. This property will be used in the convergence analysis.

Theorem 4.3. *Let A be a non-singular M-matrix partitioned as in* (2), *i.e.*

$$A = \begin{bmatrix} A_{FF} & A_{FC} \\ A_{CF} & A_{CC} \end{bmatrix},$$

where $A_{FF} \in \mathbb{R}^{n_F,n_F}$ and $A_{CC} \in \mathbb{R}^{n_C,n_C}$. If $\left(\widetilde{A}_{FF}, \widetilde{A}_{FF} - A_{FF}\right)$ is a weak regular splitting of A_{FF} of first type and $\left(\widetilde{S}, \widetilde{S} - \left(A/\widetilde{A}_{FF}\right)\right)$ is also a weak regular splitting of $\left(A/\widetilde{A}_{FF}\right)$ of first type, then

$$\rho\left(T_{AMLI}\right) \leq \|T_{AMLI}\|_w < 1,$$

where $w = A^{-1}e$ for an arbitrary positive vector e.

Proof: In order to use Proposition 4.1 we first write T_{AMLI} as $T_{AMLI} = I - C_{AMLI}A$. Then we establish that $I - C_{AMLI}A$ satisfies the assumptions of Proposition 4.1. Since with Lemma 4.2 T_{AMLI} is non-negative, it suffices to show that C_{AMLI} is non-negative and C_{AMLI} has no zero row.

We get

$$T_{AMLI} = I - P_1 - P_2 = I - \left(M_S + M_{CG}\right)A = I - C_{AMLI}A \tag{9}$$

with the definition $C_{AMLI} := M_S + M_{CG}$, where

$$M_S := \begin{bmatrix} \widetilde{A}_{FF}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \text{ and } M_{CG} := \begin{bmatrix} -\widetilde{A}_{FF}^{-1}A_{FC} \\ I_C \end{bmatrix} \widetilde{S}^{-1} \begin{bmatrix} -A_{CF}\widetilde{A}_{FF}^{-1} & I_C \end{bmatrix}. \tag{10}$$

Using the M-matrix and splitting properties we see that both terms M_S and M_{CG} are non-negative. As a sum of two non-negative matrices, C_{AMLI} is also non-negative.

Since $\widetilde{A}_{FF}^{-1}$ and $\widetilde{S}^{-1}$ are the inverses of non-singular matrices they do not have zero rows. So the first n_F rows of

$$M_S = \begin{bmatrix} \widetilde{A}_{FF}^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

are not zero rows. Moreover the last n_C rows of

$$M_{CG} = \begin{bmatrix} \widetilde{A}_{FF}^{-1}A_{FC}\widetilde{S}^{-1}A_{CF}\widetilde{A}_{FF}^{-1} & -\widetilde{A}_{FF}^{-1}A_{FC}\widetilde{S}^{-1} \\ -\widetilde{S}^{-1}A_{CF}\widetilde{A}_{FF}^{-1} & \widetilde{S}^{-1} \end{bmatrix}$$

are not zero rows. Since M_{CG} and M_S are non-negative and $C_{AMLI} = M_{CG} + M_S$ the matrix C_{AMLI} has no zero row.

With Lemma 4.2 and Proposition 4.1 we obtain

$$\rho\left(T_{AMLI}\right) \leq \|T_{AMLI}\|_w < 1.$$

□

Starting with an M-matrix A and approximations as in (7) and (8) we proved convergence of the AMLI method for a wide class of non-symmetric matrices. In the convergence proof the non-negativity of the iteration matrices was the major tool.

Next we analyze the MAMLI iteration matrix as given in (5)

$$\begin{aligned} T_{MAMLI} &= (I-P_1)(I-P_2) \\ &= \left(I - \begin{bmatrix} -\widetilde{A}_{FF}^{-1}A_{FC} \\ I_C \end{bmatrix} \widetilde{S}^{-1} \begin{bmatrix} -A_{CF}\widetilde{A}_{FF}^{-1} & I_C \end{bmatrix} A\right)\left(I - \begin{bmatrix} \widetilde{A}_{FF}^{-1} & 0 \\ 0 & 0 \end{bmatrix} A\right) . \end{aligned}$$

Before we consider the product $(I-P_1)(I-P_2)$ we look at each factor separately. It is worth mentioning that not both factors $(I-P_1)$ and $(I-P_2)$ are non-negative in general. This is a major difference form the convergence analysis for special Schwarz methods given in [11, 5, 21]. While factor $I-P_2$ is non-negative the other factor $I-P_1$ need not to be non-negative (see Example 4.5).

Proposition 4.4. *Let A be a non-singular M-matrix partitioned as in* (2). *If* $\left(\widetilde{A}_{FF}, \widetilde{A}_{FF} - A_{FF}\right)$ *is a weak regular splitting of A_{FF} of first type, then*

$$I - P_2 \quad \geq \quad 0 \quad \text{and } \|I-P_2\|_w = 1 ,$$

where $w = A^{-1}e$ for an arbitrary positive vector e.

Proof: Using the splitting properties and the M-matrix sign pattern we get that

$$I - P_2 \quad = \quad I - \begin{bmatrix} \widetilde{A}_{FF}^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A_{FF} & A_{FC} \\ A_{CF} & A_{CC} \end{bmatrix} = \begin{bmatrix} I_F - \widetilde{A}_{FF}^{-1}A_{FF} & -\widetilde{A}_{FF}^{-1}A_{FC} \\ 0 & I_C \end{bmatrix}$$

is non-negative.

Let e be an arbitrary positive vector. Since A is a non-singular M-matrix $w = A^{-1}e$ is also positive. With $\widetilde{A}_{FF}^{-1} \geq 0$ we obtain

$$0 \leq (I-P_2)w = \left(I - \begin{bmatrix} \widetilde{A}_{FF}^{-1} & 0 \\ 0 & 0 \end{bmatrix} A\right) A^{-1}e = w - \begin{bmatrix} \widetilde{A}_{FF}^{-1} & 0 \\ 0 & 0 \end{bmatrix} e \leq w.$$

Using Lemma we obtain 2.1 $\|I-P_2\|_w \leq 1$. But since the last components of $(I-P_2)w$ and w are the same, we also obtain

$$\|I-P_2\|_w = \sup_{\|x\|_w=1} \|(I-P_2)x\|_w \geq \|(I-P_2)w\|_w. = \max_i \frac{((I-P_2)w)_i}{w_i} \geq 1$$

This leads to $\|I-P_2\|_w = 1$. □

Example 4.5. *Consider the matrix*

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \quad \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\end{matrix}}\right\} n_F \\ \left.\vphantom{\begin{matrix}1\\1\end{matrix}}\right\} n_C \end{matrix}$$

and partition it as indicated. We use the approximations

$$\widetilde{A}_{FF} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad \widetilde{S} = \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & 2 \end{bmatrix}$$

of A_{FF} and $(\widetilde{A}/\widetilde{A}_{FF})$. These approximations fulfills (7) *and* (8). *Then $I - P_1$ is given by*

$$I - P_1 = \frac{1}{6}\begin{bmatrix} 6 & 0 & 0 & 0 \\ 1 & 6 & -3 & 2 \\ 2 & 0 & 0 & 4 \\ 0 & 0 & 3 & 0 \end{bmatrix} .$$

Although $(I-P_1)$ is not non-negative in general, we are able to establish the non-negativity of the MAMLI iteration matrix.

Lemma 4.6. *Let A be a non-singular M-matrix partitioned as* (2). *If* $\left(\widetilde{A}_{FF}, \widetilde{A}_{FF} - A_{FF}\right)$ *is a weak regular splitting of A_{FF} of first type and* $\left(\widetilde{S}, \widetilde{S} - \left(A/\widetilde{A}_{FF}\right)\right)$ *is also a weak regular splitting of* $\left(A/\widetilde{A}_{FF}\right)$ *of first type, then*

$$T_{MAMLI} \geq 0.$$

Proof: A computation leads to

$$\begin{aligned} T_{MAMLI} \quad = & \quad \begin{bmatrix} I_F - \widetilde{A}_{FF}^{-1}A_{FF} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \widetilde{A}_{FF}^{-1}A_{FC}\widetilde{S}^{-1}A_{CF}\left(I_F - \widetilde{A}_{FF}^{-1}A_{FF}\right)^2 & 0 \\ -\widetilde{S}^{-1}A_{CF}\left(I_F - \widetilde{A}_{FF}^{-1}A_{FF}\right)^2 & 0 \end{bmatrix} \\ & + \begin{bmatrix} 0 & -\widetilde{A}_{FF}^{-1}A_{FC}\left(I_C - \widetilde{S}^{-1}\left(A/\widetilde{A}_{FF}\right)\right) \\ 0 & I_C - \widetilde{S}^{-1}\left(A/\widetilde{A}_{FF}\right) \end{bmatrix} \\ & + \begin{bmatrix} 0 & -\widetilde{A}_{FF}^{-1}A_{FC}\widetilde{S}^{-1}A_{CF}\left(I_F - \widetilde{A}_{FF}^{-1}A_{FF}\right)\widetilde{A}_{FF}^{-1}A_{FC} \\ 0 & \widetilde{S}^{-1}A_{CF}\left(I_F - \widetilde{A}_{FF}^{-1}A_{FF}\right)\widetilde{A}_{FF}^{-1}A_{FC} \end{bmatrix} \end{aligned}$$

Using the assumption on the approximations $\widetilde{A}_{FF}$ and $\widetilde{S}$ and the sign pattern of the M-matrix A we get that T_{MAMLI} is non-negative. □

Next we prove the convergence of the MAMLI method.

Theorem 4.7. *Let A be a non-singular M-matrix partitioned as in* (2), *i.e.*

$$A = \begin{bmatrix} A_{FF} & A_{FC} \\ A_{CF} & A_{CC} \end{bmatrix},$$

where $A_{FF} \in \mathbb{R}^{n_F,n_F}$ and $A_{CC} \in \mathbb{R}^{n_C,n_C}$. If $\left(\widetilde{A}_{FF}, \widetilde{A}_{FF} - A_{FF}\right)$ is a weak regular splitting of A_{FF} of first type and $\left(\widetilde{S}, \widetilde{S} - \left(A/\widetilde{A}_{FF}\right)\right)$ is also a weak regular splitting of $\left(A/\widetilde{A}_{FF}\right)$ of first type, then

$$\rho\left(T_{MAMLI}\right) \le \|T_{MAMLI}\|_w < 1,$$

where $w = A^{-1}e$ for an arbitrary positive vector e.

Proof: As in the proof of Theorem 4.3 we apply Proposition 4.1. So we first write T_{MAMLI} as $T_{MAMLI} = I - C_{MAMLI}A$. Then we establish that $I - C_{MAMLI}A$ satisfies the assumptions of Proposition 4.1. Since with Lemma 4.6 T_{MAMLI} is non-negative it suffices to show that C_{MAMLI} is non-negative and C_{MAMLI} has no zero row. With

$$\begin{aligned} T_{MAMLI} &= (I - P_1)(I - P_2) \\ &= \left(I - \begin{bmatrix} -\widetilde{A}_{FF}^{-1}A_{FC} \\ I_C \end{bmatrix} \widetilde{S}^{-1} \begin{bmatrix} -A_{CF}\widetilde{A}_{FF}^{-1} & I_C \end{bmatrix} A\right) \\ &\quad \cdot \left(I - \begin{bmatrix} I_F \\ 0 \end{bmatrix} \widetilde{A}_{FF}^{-1} \begin{bmatrix} I_F & 0 \end{bmatrix} A\right), \end{aligned}$$

we easily obtain

$$T_{MAMLI} = I - (M_{CG} + M_S - M_{CG}AM_S)A = I - C_{MAMLI}A. \tag{11}$$

where

$$C_{MAMLI} := M_{CG} + M_S - M_{CG}AM_S$$

and M_S and M_{CG} are as in (10), i.e.

$$M_S = \begin{bmatrix} \widetilde{A}_{FF}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad M_{CG} = \begin{bmatrix} -\widetilde{A}_{FF}^{-1}A_{FC} \\ I_C \end{bmatrix} \widetilde{S}^{-1} \begin{bmatrix} -A_{CF}\widetilde{A}_{FF}^{-1} & I_C \end{bmatrix}.$$

Next we show that C_{MAMLI} is a non-negative matrix. Using the M-matrix and splitting properties we see that both matrices M_S and M_{CG} are non-negative. For $-M_{CG}AM_S$ we obtain

$$-M_{CG}AM_S = \begin{bmatrix} \widetilde{A}_{FF}^{-1}A_{FC}\widetilde{S}^{-1}A_{CF}\left(I_F - \widetilde{A}_{FF}^{-1}A_{FF}\right)\widetilde{A}_{FF}^{-1} & 0 \\ -\widetilde{S}^{-1}A_{CF}\left(I_F - \widetilde{A}_{FF}^{-1}A_{FF}\right)\widetilde{A}_{FF}^{-1} & 0 \end{bmatrix}. \tag{12}$$

Using the M-matrix and splitting properties we obtain that $-M_{CG}AM_S$ is also non-negative. Consequently

$$C_{MAMLI} = M_{CG} + M_S + (-M_{CG}AM_S) \tag{13}$$

is also non-negative.

Next, we prove that C_{MAMLI} has no zero row. We already established that all three terms of C_{MAMLI} in (13) are non-negative. So it suffices to prove that the term $M_{CG} + M_S$ has no zero row, but this was already done in the proof of Theorem 4.3. Now using Proposition 4.1, we get that

$$\rho\left(T_{MAMLI}\right) \le \|T_{MAMLI}\|_w < 1\ .$$

□

Starting with an M-matrix A and approximations as in (7) and (8) we proved convergence of the AMLI and the MAMLI. As far as we know these are the first convergence results for these methods for a wide class of non-symmetric matrices.

Now we will analyze the SMAMLI iteration matrix as given in (6)

$$T_{SMAMLI} = (I - P_2)(I - P_1)(I - P_2)$$

Theorem 4.8. *Let A be a non-singular M-matrix partitioned as in (2). If $\left(\widetilde{A}_{FF}, \widetilde{A}_{FF} - A_{FF}\right)$ is a weak regular splitting of A_{FF} of first type and $\left(\widetilde{S}, \widetilde{S} - \left(A/\widetilde{A}_{FF}\right)\right)$ is also a weak regular splitting of $\left(A/\widetilde{A}_{FF}\right)$ of first type, then*

$$\begin{aligned} T_{SMAMLI} &\geq 0, \\ \rho\left(T_{SMAMLI}\right) \leq \|T_{SMAMLI}\|_w &< 1, \end{aligned}$$

where $w = A^{-1}e$ for an arbitrary positive vector e.

Proof: We easily obtain with Lemma 4.6 and Proposition 4.4

$$T_{SMAMLI} = (I - P_2)(I - P_1)(I - P_2) = (I - P_2)T_{MAMLI} \geq 0. \tag{14}$$

With Theorem 4.7 and Proposition 4.4 we get

$$\rho(T_{SMAMLI}) \leq \|T_{SMAMLI}\|_w \leq \|I - P_2\|_w \|T_{MAMLI}\|_w < 1. \tag{15}$$

□

In the remaining of this section we compare the SMAMLI, MAMLI, and AMLI methods with respect to the weighted maximum norm of their iteration matrices.

Theorem 4.9. *Let A be a non-singular M-matrix partitioned as in (2). If $\left(\widetilde{A}_{FF}, \widetilde{A}_{FF} - A_{FF}\right)$ is a weak regular splitting of A_{FF} of first type and $\left(\widetilde{S}, \widetilde{S} - \left(A/\widetilde{A}_{FF}\right)\right)$ is also a weak regular splitting of $\left(A/\widetilde{A}_{FF}\right)$ of first type, then*

$$\|T_{SMAMLI}\|_w \leq \|T_{MAMLI}\|_w \leq \|T_{AMLI}\|_w < 1,$$

where $w = A^{-1}e$ for an arbitrary positive vector e.

Proof: The inequality $\|T_{AMLI}\|_w < 1$ was proved in Theorem 4.3. Using the matrices M_S and M_{CG} as in (10), i.e.

$$M_S = \begin{bmatrix} \widetilde{A}_{FF}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad M_{CG} = \begin{bmatrix} -\widetilde{A}_{FF}^{-1}A_{FC} \\ I_C \end{bmatrix} \widetilde{S}^{-1} \begin{bmatrix} -A_{CF}\widetilde{A}_{FF}^{-1} & I_C \end{bmatrix}.$$

we have with (9) and (11)

$$\begin{aligned} T_{AMLI} &= I - (M_S + M_{CG})\, A \\ T_{MAMLI} &= I - (M_S + M_{CG} - M_{CG}AM_S)\, A = T_{AMLI} + M_{CG}AM_SA. \end{aligned}$$

Due to the lemmata 4.2 and 4.6 both iteration matrices are non-negative. Since A is a M-matrix and e is a positive vector, $w = A^{-1}e$ is also positive. So both terms $T_{AMLI}w$ and $T_{MAMLI}w$ are positive.

As seen in (12) the term $-M_{CG}AM_S$ is non-negative. So we obtain that the vector $M_{CG}AM_Se$ has to be non-positive. Therefore

$$\begin{aligned} \|T_{MAMLI}\|_w &= \max_{i=1\dots n} \frac{(T_{MAMLI}w)_i}{w_i} = \max_{i=1\dots n} \frac{(T_{AMLI}w + M_{CG}AM_SAA^{-1}e)_i}{w_i} \\ &\leq \max_{i=1\dots n} \frac{(T_{AMLI}w)_i}{w_i} + \min_{i=1\dots n} \frac{(M_{CG}AM_Se)_i}{w_i} = \|T_{AMLI}\|_w + \min_{i=1\dots n} \frac{(M_{CG}AM_Se)_i}{w_i} \\ &\leq \|T_{AMLI}\|_w. \end{aligned}$$

The remaining inequality $\|T_{SMAMLI}\|_w \leq \|T_{MAMLI}\|_w$ follows directly from inequality (15). □

So we expect that the SMAMLI method is the fastest of the presented methods followed by the MAMLI method and then by the AMLI method. Note that Theorem 4.9 is one of few theoretical results which gives a better bound for the convergence rate of the multiplicative Schwarz method compared with a bound for the additive Schwarz method.

In the following we establish a surprising result for the AMLI, MAMLI and SMAMLI methods. All these methods coincide if the block A_{FF} is inverted exactly, i.e. $\widetilde{A}_{FF} = A_{FF}$, independent of the quality of the approximation of the Schur complement S. This results holds for all system matrices A , symmetric or non-symmetric, and not only M-matrices.

Theorem 4.10. *Let A be a non-singular matrix, which is partitioned in the following 2×2 block structure*

$$A = \begin{bmatrix} A_{FF} & A_{FC} \\ A_{CF} & A_{CC} \end{bmatrix}.$$

If A_{FF} is non-singular and $\widetilde{A}_{FF} = A_{FF}$ then

$$T_{AMLI} = T_{MAMLI} = T_{SMAMLI}.$$

holds for any approximation $\widetilde{S}$ of (A/A_{FF}).

Proof:
Using the matrices M_S and M_{CG} as in (10), we have from (9) and (11)

$$T_{AMLI} = I - (M_S + M_{CG}) A \text{ and } T_{MAMLI} = I - (M_S + M_{CG} - M_{CG} A M_S) A.$$

Similarly we obtain for the SMAMLI method

$$T_{SMAMLI} = I - (2M_S + M_{CG} - M_S A M_{CG} - M_S A M_S - M_{CG} A M_S + M_S A M_{CG} A M_S) A$$

But since a computation leads to

$$\begin{aligned} M_{CG} A M_S &= M_S A M_{CG} = M_S A M_{CG} A M_S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ M_S A M_S &= \begin{bmatrix} A_{FF}^{-1} & 0 \\ 0 & 0 \end{bmatrix} = M_S \end{aligned}$$

we obtain $T_{AMLI} = T_{MAMLI} = T_{SMAMLI}$.

$\square$

5 Numerical Examples

We consider the 2D convection-diffusion equation

$$-\nu \triangle u + \bar{\nu} \nabla u = 0$$

in the unit square, with homogeneous Dirichlet boundary conditions everywhere, except on the boundary $0 \leq x \leq 1, y = 1$ where we set $u = 1$. Here $\nu := 10^{-2}$ is a constant diffusion parameter and $\bar{\nu}$ a convective flow. In this paper we tested a rotating flow

$$\bar{\nu} = \begin{bmatrix} sin(\pi x) cos(\pi y) \\ -cos(\pi x) \sin(\pi y) \end{bmatrix}.$$

We discretize the resulting systems using a uniform mesh with constant mesh size h in all directions, a five-point stencil for the second order part, and upwind discretization for the first order part of the equation. This leads to the system

$$Ax = b$$

where A is a non-symmetric M-matrix, see e.g. [13]. To solve the linear system we used the presented AMLI, MAMLI, and SMAMLI methods.

We use the algorithm 1 to split the unknowns in fine and coarse level variables. This algorithm is inspired by [24, Algoritm 3.1]. Only one step of this coarser was needed, since we considered only the two-level case.

Algorithm 1: Fine-Coarse-partitioning

$\mathbf{F}, \mathbf{C} \leftarrow$ **NotayCoarsing(A)**
begin
  $F \leftarrow \emptyset$, $C \leftarrow \emptyset$, $U \leftarrow \{1, \ldots, n\}$
  $N_i \leftarrow \{j \neq i | a_{ij} \neq 0\}$,
  $N_i^T \leftarrow \{j \neq i | a_{ji} \neq 0\}$,
  $S_i \leftarrow \{j \neq i || a_{ij}| \geq 0.25 \max_{k \in N_i} |a_{ik}|\}$
  $S_i^T \leftarrow \{j \neq i || a_{ji}| \geq 0.25 \max_{k \in N_i^T} |a_{ki}|\}$
  $p_i \leftarrow 4|S_i^T \cap U| + 2|S_i \cap U| + |N_i \cap U|$
  while $U \neq \emptyset$ **do**
    $i \leftarrow \arg\max_{i \in U} p_i$
    $C \leftarrow C \cup \{i\}$, $U \leftarrow U \setminus \{i\}$
    forall $j \in S_i^T \cap U$ **do**
      $F \leftarrow F \cup \{j\}$, $U \leftarrow U \setminus \{j\}$
      Update p_k for all $k \in N_j$
    end
  end
end

The approximations $\widetilde{A}_{FF}$ and $\widetilde{S}$ were formed by using the splitting induced by one of the following methods, the Jacobi-Iteration or the Gauss-Seidel-Iteration or the incomplete LU factorization of the corresponding matrix A_{FF} or $\left(A/\widetilde{A}_{FF}\right)$. Of course different choices of approximations leads to different costs per iteration step. But here we focus on the number of iterations.

		AMLI	MAMLI	SMAMLI
	Jacobi	1113 (0.9987)	964 (0.8966)	540 (0.5501)
No. of iter.	Gauss	572 (0.9457)	469 (0.8023)	294 (0.6570)
(iter. time)	LUinc	113 (0.6540)	94 (0.5484)	86 (0.5866)
	Jacobi	0.9931	0.9919	0.9860
spectral radius	Gauss	0.9864	0.9828	0.9740
	LUinc	0.9331	0.9164	0.9094

Table 1: Results for the "rotating flow" problem of dimension $A \in \mathbb{R}^{1024\times 1024}$

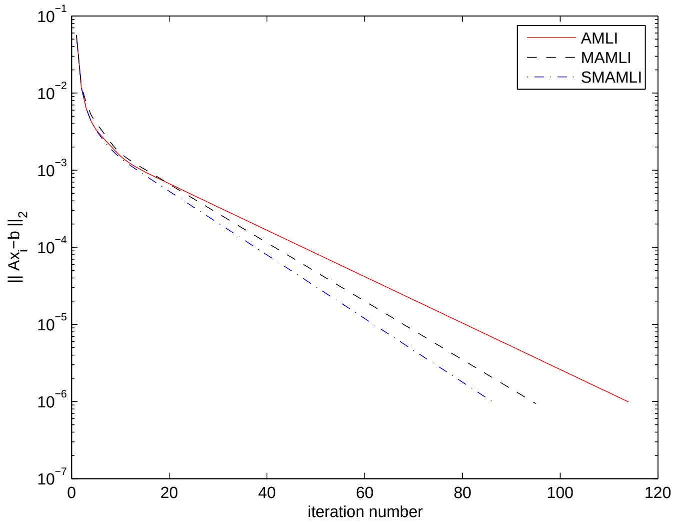

Figure 1: Residuum for the "rotating flow" problem of dimension $A \in \mathbb{R}^{1024\times 1024}$ with LUinc approximation

The given number of iterations were needed to reduce the residuum to $\|Ax-b\|_2 < 10^{-6}$. As a starting approximation we used the zero vector. In the Table 1 we give the number of iterations, iteration time, and spectral radii for the described configurations and approximations. Moreover, we present selected plots of the norm of the residuum with respect to the number of iterations; see Figure 1. These plots show the characteristic behavior of the presented methods.

In conclusion, these simple experiments confirm our theoretical predictions from Theorem 4.9: The SMAMLI method converges faster than the MAMLI method, which in turn converges faster than the AMLI method.

References

[1] O. Axelsson and M. Neytcheva. Algebraic multilevel iteration method for Stieltjes matrices. *Numerical Linear Algebra with Applications*, 1(3):213–236, 1994.

[2] O. Axelsson and P.S. Vassilevski. Algebraic Multilevel Preconditioning Methods, I. *Numerische Mathematik*, 56:157–177, 1989.

[3] O. Axelsson and P.S. Vassilevski. Algebraic Multilevel Preconditioning Methods, II. *SIAM Journal Numerical Analysis*, 27:1569–1590, 1990.

[4] R.E. Bank and C. Wagner. Multilevel ILU decomposition. *Numerische Mathematik*, 82(4):543–576, 1999.

[5] M. Benzi, A. Frommer, R. Nabben, and D.B. Szyld. Algebraic theory of multiplicative schwarz methods. *Numerische Mathematik*, 89:605–639, 2001.

[6] A. Berman and R.J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences*. Academic Press, New York, 1979.

[7] A. Brandt, S. McCormick, and J. Ruge. Algebraic multigrid (AMG) for sparse matrix equations. In *Sparsity and its applications (Loughborough, 1983)*, pages 257–284. Cambridge Univ. Press, Cambridge, 1985.

[8] M. Brezina, A. J. Cleary, R. D. Falgout, V. E. Henson, J. E. Jones, T. A. Manteuffel, S. F. McCormick, and J. W. Ruge. Algebraic multigrid based on element interpolation (AMGe). *SIAM Journal on Scientific Computing*, 22(5):1570–1592 (electronic), 2000.

[9] A. Frommer, H. Schwandt, and D.B. Szyld. Asynchronous weighted additive Schwarz methods. *Electronic Transactions on Numerical Analysis*, 5(June):48–61 (electronic), 1997.

[10] A. Frommer and D.B. Szyld. Weighted max norms, splittings, and overlapping additive Schwarz iterations. *Numerische Mathematik*, 83(2):259–278, 1999.

[11] A. Frommer and D.B. Szyld. An algebraic convergence theory for restricted additive Schwarz methods using weighted max norms. *SIAM Journal on Numerical Analysis*, 39:463–479, 2001.

[12] M. Griebel and P. Oswald. On the abstract theory of additive and multiplicative Schwarz algorithms. *Numerische Mathematik*, 70:163–180, 1995.

[13] W. Hackbusch. *Multigrid methods and applications*, volume 4 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1985.

[14] W. Hackbusch. *Iterative Solution of Large Sparse Systems of Equations*. Springer Verlag, New York – Berlin – Heidelberg, 1994.

[15] S. MacLachlan, T. Manteuffel, and S. McCormick. Adaptive reduction-based AMG. *Numerical Linear Algebra with Applications*, 13(8):599–620, 2006.

[16] C. Mense. Algebraisches Schwarz-Verfahren unter Verwendung des Schurkomplements. Master's thesis, Universität Bielefeld, 2002.

[17] C. Mense. *Analyse von algebraischen Mehrgitter-Verfahren für unsymmetrische Systeme*. PhD thesis, Technical University of Berlin, 2007. in preperation.

[18] C. Mense and R. Nabben. Convergence of algebraic multi-level methods for non-symmetric problems. in preparation, 2007.

[19] C. Mense and R. Nabben. Convergence rates and comparisons of algebraic multi-level methods which use approximated Schur complements. in preparation, 2007.

[20] C. Mense and R. Nabben. On algebraic two-level methods for non-symmetric systems - convergence results. submitted, 2007.

[21] R. Nabben. Comparisons between multiplicative and additive schwarz iterations in domain decompositon methods. *Numerische Mathematik*, 95:145–162, 2003.

[22] M. Neytcheva, O. Axelsson, and K. Georgiev. An application of the AMLI method for solving convection-diffusion problems with potential velocity field. In *Algebraic multilevel iteration methods with applications, Vol. I, II (Nijmegen, 1996)*, pages 197–210. Katholieke Univ. Nijmegen, Dept. Math., Nijmegen, 1996.

[23] Y. Notay. A robust algebraic multilevel preconditioner for non-symmetric M-matrices. *Numerical Linear Algebra with Applications*, 7(5):243–267, 2000.

[24] Y. Notay. Robust parameter-free algebraic multilevel preconditioning. *Numerical Linear Algebra with Applications*, 9(6-7):409–428, 2002.

[25] Y. Notay. Algebraic multigrid and algebraic multilevel methods: a theoretical comparison. *Numerische Linear Algebra with Applications*, 12:419–451, 2005.

[26] J. W. Ruge and K. Stüben. Algebraic multigrid. In *Multigrid methods*, volume 3 of *Frontiers Appl. Math.*, pages 73–130. SIAM, Philadelphia, PA, 1987.

[27] Y. Saad and B. Suchomel. ARMS: an algebraic recursive multilevel solver for general sparse linear systems. *Numerical Linear Algebra with Applications*, 9(5):359–378, 2002.

[28] K. Stüben. Algebraic multigrid (AMG): experiences and comparisons. *Applied Mathematics and Computation*, 13(3-4):419–451, 1983.

[29] K. Stüben. An introduction to algebraic multigrid. In *U. Trottenberg, C. W. Oosterlee and A. Schüller Multigrid*, pages 413–528. Academic Press Inc., 2001.

[30] A. Toselli and O. Widlund. *Domain decomposition methods—algorithms and theory*, volume 34 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 2005.

[31] U. Trottenberg, C.W. Oosterlee, and A. Schüler. *Multigrid*. Academic Press, London, 2001.

[32] R. S. Varga. *Matrix iterative analysis*, volume 27 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, expanded edition, 2000.

[33] Z. I. Woźnicki. Nonnegative splitting theory. *Japan Journal of Industrial and Applied Mathematics*, 11:289–342, 1994.

[34] J. Xu. Iterative methods by space decomposition and subspace correction. *SIAM Review*, 34:581–613, 1992.