
Rakhim Aitbayev
**Multilevel Preconditioners for Nonselfadjoint or Indefinite
 Orthogonal Spline Collocation Problems**

Department of Mathematics
 New Mexico Tech
 Socorro
 NM 87801
 aitbayev@nmt.edu
 Bernard Bialecki

We develop and study symmetric multilevel preconditioners for the computation of the orthogonal spline collocation (OSC) solution of a Dirichlet boundary value problem (BVP) with a nonselfadjoint or an indefinite operator. The OSC solution is sought in the space of piecewise Hermite bicubic spline functions defined on a uniform partition. We consider an additive and a multiplicative multilevel preconditioners that are used with the preconditioned conjugate gradient (PCG) method. Our results and algorithms are closely related to those in [1], [2], [3], and [4]. Let Ω be a unit square $(0, 1) \times (0, 1)$ with the boundary $\partial\Omega$, and let $x = (x_1, x_2)$. We consider a BVP

$$Lu \equiv \sum_{i,j=1}^2 a_{ij}(x)u_{x_i x_j} + \sum_{i=1}^2 b_i(x)u_{x_i} + c(x)u = f(x), \quad x \in \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \quad (1)$$

Operator L could be non-selfadjoint or indefinite in L^2 inner product. We assume that the principal part of L satisfies the uniform ellipticity condition and that BVP (1) has a unique solution in $H^2(\Omega)$. Let π_0 be a uniform coarsest rectangular partition of Ω . We obtain a set of partitions $\{\pi_k\}_{k=0}^K$ by standard coarsening, and let $V_0 \subset V_1 \subset \dots \subset V_K \equiv V_h$ be the set of corresponding nested spaces of piecewise Hermite bicubics that vanish on $\partial\Omega$. Let \sum denote the 2-D composite Gauss quadrature corresponding to partition π_h with 4 nodes in each element. Let \mathcal{G}_h denote the corresponding set of Gauss points. The OSC discretization of BVP (1) is defined by

$$u_h \in V_h, \quad Lu_h(\xi) = f(\xi), \quad \xi \in \mathcal{G}_h, \quad (2)$$

and it can be written as the operator equation $L_h u_h = f_h$ in the Hilbert space V_h with the inner product $(v, w)_h = \sum v w$. We define and study multilevel additive B_a and multiplicative B_m preconditioners for solving the normal equation $L_h^* L_h u_h = L_h^* f_h$, where L_h^* is the adjoint to L_h . The implementation of B_a and B_m is based on relationships between basis functions for two consecutive partitions and the implementation of B_m is similar to that for V(1,1)-cycle with

the Gauss-Seidel smoothing. A problem on the coarsest partition is assumed sufficiently small, and it is solved exactly. The computational cost of the preconditioning algorithms is $O(N_K)$. The following is our main result. There are positive independent of h and K constants $\alpha_a, \beta_a, \alpha_m$, and β_m , such that

$$\begin{aligned}\alpha_a (B_a v, v)_h &\leq (L_h^* L_h v, v)_h \leq \beta_a (B_a v, v)_h, \quad v \in V_h, \\ \alpha_m (B_m v, v)_h &\leq (L_h^* L_h v, v)_h \leq \beta_m (B_m v, v)_h, \quad v \in V_h.\end{aligned}\tag{3}$$

In the following table, we present results of our numerical computations; that is, the ratios of spectral constants in (3), the convergence factor $\bar{\rho}$, which is the geometric mean of consecutive residual ratios, and the CPU time.

| J | <i>Additive</i> | | | <i>Multiplicative</i> | | | <i>General</i> | | |
|-----|--------------------|--------------|--------|-----------------------|--------------|--------|--------------------|--------------|--------|
| | β_a/α_a | $\bar{\rho}$ | $t(s)$ | β_m/α_m | $\bar{\rho}$ | $t(s)$ | β_m/α_m | $\bar{\rho}$ | $t(s)$ |
| 5 | 5.016 | 0.125 | 2.13 | 1.476 | 0.008 | 3.87 | 457.5 | 0.121 | 8.44 |
| 6 | 5.488 | 0.142 | 9.13 | 1.500 | 0.009 | 16.45 | 402.4 | 0.166 | 42.67 |
| 7 | 5.845 | 0.156 | 49.93 | 1.515 | 0.009 | 73.43 | 381.4 | 0.202 | 199.40 |
| 8 | 6.162 | 0.168 | 278.10 | 1.524 | 0.009 | 334.60 | 377.3 | 0.224 | 995.60 |

Under *Additive* and *Multiplicative*, we list results for Poisson's equation, and under *General* – results for a PDE with a general nonselfadjoint and indefinite operator L with the coefficients

$$\begin{aligned}a_{11}(x) &= e^{x_1 x_2}, \quad a_{12}(x) = 0.5/(1 + x_1 + x_2), \quad a_{22}(x) = e^{-x_1 x_2}, \\ b_1(x) &= x_2 e^{x_1 x_2} + 10 \cos[\pi(x_1 + x_2)], \quad b_2(x) = -x_1 e^{-x_1 x_2} + 50 \sin(2\pi x_1 x_2), \\ c(x) &= 50[1 + 1/(1 + x_1 + x_2)].\end{aligned}$$

The problem with the general equation is solved using the multilevel multiplicative preconditioner. We set $\pi_0 = \Omega$ and reduce the relative residual to less than 10^{-12} . The numerical results demonstrate the efficiency of our preconditioning algorithms.

Bibliography

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