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Coupled Gauss-Seidel algorithm in multigrid mode for the thin film equation

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In this talk we consider the iterative solution of a nonlinear system arising from a finite element discretisation of the fourth order equation

$$\frac{\partial u}{\partial t} + \nabla \cdot (|u|^\gamma \nabla \Delta u) = 0,$$

where $\gamma > 0$. This equation models a thin liquid film spreading on a solid surface, with u the height of the film. It is well known that for nonnegative initial data, the solution u remains nonnegative for all time. However, this nonnegativity of u is not guaranteed if the equation is discretised in a naive way. Imposing the nonnegativity of u as a constraint leads to a discrete variational inequality to be solved at each time step. Specifically defining S^h to be the space of piecewise linear functions on a uniform mesh and $K^h \subset S^h$ to be the space of nonnegative functions in S^h, given $U^{n-1} \in K^h$ we seek $U^n \in K^h$ and $W^n \in S^h$ such that

$$\begin{aligned} (U^n, \chi)^h + \tau\,(|U^{n-1}|^\gamma \nabla W^n, \nabla \chi) &= (U^{n-1}, \chi)^h \qquad \forall\ \chi \in S^h, \\ (\nabla U^n, \nabla(\chi - U^n)) &\geq (W^n, \chi - U^n)^h \qquad \forall\ \chi \in K^h, \end{aligned}$$

where τ represents the time step, and $(\cdot,\cdot)$ and $(\cdot,\cdot)^h$ represent the L^2 inner product and its trapezoidal rule discretisation respectively.

Well-posedness, stability, unique solvability, and convergence of U^n to u and W^n to $w = -\Delta u$ were established by Barrett, Blowey and Garcke in 1998. To solve the nonlinear system they used an Uzawa algorithm, for which they were able to demonstrate convergence of $U^{n,p} \to U^n$ and of $\int_\Omega |U^{n-1}|^\gamma |\nabla(W^n - W^{n,p})|^2 dx \to 0$, as the number of iterations $p \to \infty$. However, the convergence of this algorithm was found to be extremely slow. Here, we propose instead a coupled Gauss-Seidel algorithm in multigrid mode for the iterative solution of the nonlinear system. Proving convergence for the multigrid algorithm remains an open question, but numerical results indicate mesh independent convergence

to the same solution as that achieved with the Uzawa algorithm in most cases tested, with a greatly reduced computational cost compared to iterating on a single grid.