Rakhim Aitbayev

Multilevel Preconditioners for Nonselfadjoint or Indefinite Orthogonal Spline Collocation Problems

Department of Mathematics
New Mexico Tech
Socorro
NM 87801
aitbayev@nmt.edu
Bernard Bialecki

TheoremWe develop and study symmetric multilevel preconditioners for the computation ofthe orthogonal spline collocation (OSC) solution of a Dirichlet boundary value problem(BVP) with a nonselfadjoint or an indefinite operator.The OSC solution is sought in the space of piecewise Hermite bicubic spline functionsdefined on a uniform partition.We consider an additive and a multiplicative multilevel preconditionersthat are used with the preconditioned conjugate gradient (PCG) method.Our results and algorithms are closely related to those in[1], [2], [3],and [4].Let Ω be a unit square $(0,1)\times(0,1)$ with the boundary $\partial\Omega$, andlet $x=(x_1,x_2)$. We consider a BVP

$$Lu \equiv \sum_{i,j=1}^{2} a_{ij}(x)u_{x_i x_j} + \sum_{i=1}^{2} b_i(x)u_{x_i} + c(x)u = f(x), \quad x\in\Omega, \quad u=0 \text{ on } \partial\Omega. \tag{1}$$

Operator L could be non-selfadjoint or indefinite in L^2 inner product.We assume that the principal part of L satisfies the uniform ellipticity conditionand that BVP (1) has a unique solution in $H^2(\Omega)$.Let π_0 be a uniform coarsest rectangular partition of Ω. We obtain a set of partitions$\{\pi_k\}_{k=0}^{K}$ by standard coarsening, andlet $V_0 \subset V_1 \subset \ldots \subset V_K \equiv V_h$ be the set of corresponding nested spaces ofpiecewise Hermite bicubics that vanish on $\partial\Omega$.Let $\sum$ denote the 2-D composite Gauss quadrature corresponding to partition π_h with 4 nodesin each element. Let $\mathcal{G}_h$ denote the corresponding set of Gauss points.The OSC discretization of BVP (1) is defined by

$$u_h \in V_h, \quad Lu_h(\xi) = f(\xi), \quad \xi \in \mathcal{G}_h, \tag{2}$$

and it can be written as the operator equation $L_h u_h = f_h$ in the Hilbert space V_h with theinner product $(v,w)_h = \sum vw$.We define and study multilevel additiveB_{a} and multiplicative B_{m} preconditionersfor solving the normal equation $L_h^* L_h u_h = L_h^* f_h$, where L_h^* is the adjoint to L_h.The implementation of B_{a} and $B_{\mathrm{m}}$is based on relationships between basis functions for two consecutive partitionsand the implementation of B_{m} is similar to that forV(1,1)-cycle with the Gauss-Seidel smoothing.A problem on the coarsest partition is assumed sufficiently small, and it is solved exactly.The computational cost of the preconditioning algorithms is $\mathrm{O}(N_K)$.The following is our main result.There are positive independent of h and K constantsα_{a},β_{a},α_{m},andβ_{m},such that

$$\begin{aligned} \alpha_{\mathrm{a}}\,(B_{\mathrm{a}}v,v)_h \le (L_h^*L_hv,v)_h \le \beta_{\mathrm{a}}\,(B_{\mathrm{a}}v,v)_h, \quad v\in V_h, \\ \alpha_{\mathrm{m}}\,(B_{\mathrm{m}}v,v)_h \le (L_h^*L_hv,v)_h \le \beta_{\mathrm{m}}\,(B_{\mathrm{m}}v,v)_h, \quad v\in V_h. \end{aligned} \tag{3}$$

In the following table, we present results of our numerical computations; that is,the ratios of spectral constants in (3),the convergence factor $\bar{\rho}$, which is the geometric mean of consecutiveresidual ratios, and the CPU time.

	Additive			*Multiplicative*			*General*		
J	$\beta_{\mathrm{a}}/\alpha_{\mathrm{a}}$	$\bar{\rho}$	$t(s)$	$\beta_{\mathrm{m}}/\alpha_{\mathrm{m}}$	$\bar{\rho}$	$t(s)$	$\beta_{\mathrm{m}}/\alpha_{\mathrm{m}}$	$\bar{\rho}$	$t(s)$
3	3.883	0.072	0.19	1.367	0.005	0.18	925.2	0.094	0.33
4	4.490	0.101	0.59	1.435	0.007	0.90	515.2	0.096	1.80
5	5.016	0.125	2.13	1.476	0.008	3.87	457.5	0.121	8.44
6	5.488	0.142	9.13	1.500	0.009	16.45	402.4	0.166	42.67
7	5.845	0.156	49.93	1.515	0.009	73.43	381.4	0.202	199.40
8	6.162	0.168	278.10	1.524	0.009	334.60	377.3	0.224	995.60

Under *Additive* and *Multiplicative*, we list results for Poisson's equation,and under *General* – results fora PDE with a general nonselfadjoint and indefinite operator L with the coefficients

$$\begin{aligned} &a_{11}(x) = e^{x_1x_2}, \quad a_{12}(x) = 0.5/(1+x_1+x_2), \quad a_{22}(x) = e^{-x_1x_2}, \\ &b_1(x) = x_2e^{x_1x_2} + 10\cos[\pi(x_1+x_2)],_2(x) = -x_1e^{-x_1x_2} + 50\sin(2\pi x_1x_2), \\ &c(x) = 50[1+1/(1+x_1+x_2)]. \end{aligned}$$

The problem with the general equation is solved using the multilevel multiplicative preconditioner.We set $\pi_0 = \Omega$ and reduce the relative residual to less than 10^{-12}.The numerical results demonstrate the efficiency of our preconditioning algorithms.

Bibliography

[1] R. Aitbayev and B. Bialecki, *A preconditioned conjugate gradient-method for nonselfadjoint or indefinite orthogonal spline collocationproblems*, SIAM J. Numer. Anal., 41 (2003), pp. 589–604.

[2] B. Bialecki, *Convergence analysis of orthogonal spline collocationfor elliptic boundary value problems*, SIAM J. Numer. Anal., 35 (1998),pp. 617–631.

[3] B. Bialecki and M. Dryja, *Multilevel additive and multiplicativemethods for orthogonal spline collocation problems*, Numer. Math., 77 (1997),pp. 35–58.

[4] B. F. Smith, P. E. Bjørstad, and W. D. Gropp, *Domain Decomposition:Parallel Multilevel Methods for Elliptic Partial Differential Equations*,Cambridge University Press, New York, 1996.