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Efficient AMG block preconditioning of the Boussinesq problem

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In this presentation we describe the design and examine the effectiveness of a block preconditioner, based on a robust AMG solver, for the iterative solution of linear systems that arise in the finite element discretisation of the Boussinesq problem. The Boussinesq problem represents a model of a thermally-driven flow, i.e. the flow of a Newtonian fluid under a temperature gradient. Such a model is valid under the assumption that the temperature gradient is small, which allows fluid density to change linearly under the action of gravity. The applications of this model are considerable: various heat exchange systems (such as nuclear reactor cooling system, forced cooling of electronic equipment), semiconductor fabrication (crystal growth), superconductivity (e.g. Rayleigh-Benard flow of liquid helium), etc.

The Boussinesq flow represents a coupled system of PDEs for the unknown velocity, pressure and temperature fields. It consists of the Navier-Stokes equations describing the fluid flow coupled with the convection-diffusion equation for energy balance. The two classes of problem that are the most frequently studied in this context are time-dependent problems (describing the transient behaviour of the system), and steady-state problems (essential in the linear stability analysis).

The finite element discretisation of both classes of problem can be performed using the same combinations of velocity-pressure approximation spaces (as in the case of Navier-Stokes equations), while the usual choice for the temperature approximation space is the piecewise quadratic finite element space. The time domain in transient problems is usually discretised using modern adaptive fully implicit time-stepping algorithms (such as the trapesoidal rule), resulting in a sequence of large non-linear problems that need to be solved. Similar situations arise in linear stability analysis if some of the continuation algorithms (e.g. Keller's pseudo-arclength method) are used to compute the solution paths. Standard linearisation procedures (e.g. Picard's or Newton's method) are used to solve these non-linear systems, resulting in a sequence of linear problems that need to be solved. From this argument it is obvious that accurate solution of the Boussinesq problem (both transient and steady-state) requires the solution of potentially large number of linear systems, with different coefficient matrices. The size of such linear systems needed for accurate prediction of certain effects (e.g. critical values of the bifurcation parameters) can be very large, especially in 3D. Thus, robust and efficient preconditioned iterative solvers are absolutely essential in this context.

In this presentation we discuss the development and study the efficiency of a preconditioner for a fully coupled Boussinesq system. The preconditioner is based on existing efficient preconditioners for the Navier-Stokes equations (we choose the scaled commutator preconditioner in this case). In the case of Picard's linearisation, the Boussinesq coefficient matrix has an inherent block triangular form, which allows the natural extension of the Navier-Stokes preconditioner. When Newton's linearisation is used, the block triangular form of the coefficient matrix is lost, however the use of the same preconditioner as in Picard's case (with the necessary modifications in the treatment of the momentum block, as in the Navier-Stokes case) proved to be the effective choice. We test our preconditioning strategy on a range of non-trivial Boussinesq problems recently studied in literature in various contexts (differentially heated cavities, Rayleigh-Benard convection). We consider both the transient and the steady-state problems, and demonstrate numerically robustness of our preconditioner with respect to the time step size, diffusivity parameters, spatial discretisation parameter, grid stretching and the shape of the domain.