Catherine E. Powell

Efficient preconditioning for saddle-point systems arising in mixed finite element approximation of Darcy flow

UMIST Maths Dept.
P.O.Box 88 Sackville Street
Manchester M60 1QD UK.
cp@fire.ma.umist.ac.uk

The main focus of this work is the design of optimal and computationally efficient preconditioning strategies for a particular saddle-point system that arises in the mathematical modelling of flow in porous media. Specifically, we are solving a mixed formulation of a standard scalar diffusion problem, derived from Darcy's law:

$$\begin{aligned} \mathcal{A}^{-1}\vec{u} - \nabla p &= 0, & \\ \nabla \cdot \vec{u} &= -f & \text{in } \Omega, \\ p &= g & \text{on } \partial\Omega_D, \\ \vec{u}\cdot\vec{n} &= 0 & \text{on } \partial\Omega_N. \end{aligned}$$

Discretisation via the lowest-order Raviart-Thomas finite element method yields a saddle-point system matrix,

$$C = \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix},$$

that is highly ill-conditioned with respect to the discretisation parameter *and*, more importantly, to the coefficients embedded in the permeability tensor $\mathcal{A}$.

The system can be solved in a multitude of different ways. However, permeability coefficients frequently exhibit anisotropies and discontinuities and several previously suggested preconditioning schemes are known to lose robustness in such cases. Further, many authors comment on the difficulties of solving indefinite systems, and have developed methodologies to convert the system in question to a positive definite one. We demonstrate that solving the original unmodified saddle-point system using minimal residual schemes is not problematic provided sufficient attention is paid to the coefficient term.

It is well known that the underlying variational problem is well-posed in two pairs of function spaces, $H(div) \times L^2$ and $L^2 \times H^1$, leading to the possibility of two distinct types of block-diagonal preconditioners. We evaluate the efficiency of both types of approach when anisotropic and discontinuous diffusion coefficients are present. The first of these incorporates a known 'specialised' multigrid approximation to a weighted $H(div)$ operator. For the second, we propose a simpler black-box method, the key tools for which are diagonal scaling for a weighted mass matrix and an algebraic multigrid V-cycle applied to a sparse approximation to a generalised diffusion operator.

Eigenvalue bounds and numerical results are presented to illustrate not only the optimality of both preconditioners with respect to the discretisation parameter but also the superior robustness of the black-box scheme with respect to the PDE coefficients. Today, the existence of freely available algebraic multigrid codes makes it a feasible preconditioning strategy for tackling saddle-point problems arising in mixed finite element formulations of a wide range of other second-order elliptic problems.

1. Powell, C.E., Silvester, D., Optimal preconditioning for Raviart-Thomas mixed formulation of second-order elliptic PDEs. *To appear in SIAM J. Matrix Anal., 2004.*

2. Powell, C.E., Silvester, D., Black-box preconditioning for mixed formulation of self-adjoint elliptic PDEs. *Lecture Notes in Computer Science, 35, 'Challenges in Scientific Computing,' 2003.*

3. Powell C.E., Optimal preconditioning for mixed finite element formulation of second-order elliptic problems, *Ph.D. thesis, UMIST, (England), 2003.*