ROBUST DOMAIN DECOMPOSITION PRECONDITIONER FOR ELLIPTIC EQUATIONS WITH JUMP COEFFICIENT

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ABSTRACT. This paper proves the robustness of the overlapping domain decomposition preconditioners for the finite element approximation of second order elliptic boundary value problems with strongly discontinuous coefficients. By analyzing the eigenvalue distribution of the preconditioned system, we prove that only a fixed number of eigenvalues may deteriorate with respect to the discontinuous jump or meshsize, and all the other eigenvalues are bounded below and above nearly uniformly with respect to the jump and meshsize. As a result, the asymptotic convergence rate of the preconditioned conjugate gradient methods is uniform with respect to the coefficients and meshsize.

1. INTRODUCTION

Let $\Omega \in \mathbb{R}^d$ $(2 \text{ or } 3)$ be a polygonal or polyhedral domain with $\partial\Omega = \Gamma_D \cup \Gamma_N$. We consider the robustness of the overlapping domain decomposition preconditioned conjugate gradient (PCG) algorithms for the finite element approximation of the following elliptic boundary value problem

$$\begin{cases} -\nabla \cdot (\omega \nabla u) = f & \text{in } \Omega, \\ u = g_D & \text{on } \Gamma_D, \\ \omega \frac{\partial u}{\partial n} = g_N & \text{on } \Gamma_N. \end{cases} \tag{1.1}$$

The coefficient $\omega = \omega(x)$ is a positive and piecewise constant function. More precisely, we assume that there are M open disjointed polygonal or polyhedral subregions Ω_m^0 $(m = 1, \cdots, M)$ satisfying $\cup_{m=1}^M \overline{\Omega_m^0} = \overline{\Omega}$ with $\omega|_{\Omega_m^0} = \omega_m, \ \ m = 1, \ldots, M$, where each $\omega_m > 0$ is a constant. The analysis can be carried through to a more general case when $\omega(x)$ varies moderately in each subregion.

When the above problem is discretized by the finite element method, the conditioning of the resulting discrete system will depend on both the (discontinuous) coefficients and also the meshsize. There has been much interest in the development of iterative methods (such as domain decomposition and multigrid methods) whose convergence rates will be robust with respect to the change of jump size and meshsize (see [3, 6, 11, 17, 21, 30] and the references cited therein). In two dimensions, it is not difficult to see that both domain decomposition ([2, 8, 18, 26]) and multigrid ([4, 28, 30]) methods lead to robust iterative methods. In three dimensions, some nonoverlapping domain decomposition methods have been shown to be robust with respect to both the jump size and meshsize (see [16, 17, 22, 30]). In fact, using the estimates related to weighted L^2-projection in [5], it can be proved that $\kappa(BA) \leq C|\log H|$ in some cases for $d = 3$, where H is the meshsize of the coarse space. For example, if the interface has no cross points [5], or if every subregion touches part of the Dirichlet boundary [19, 25, 29], or if the coefficients satisfy the quasi-monotonicity (c.f. [9, 10]), the multilevel or domain decomposition methods were proved to be robust. But in general, the situations for overlapping domain decomposition methods and multilevel methods are

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still unclear. Technically, the difficulty is due to the lack of uniform or nearly uniform error and stability estimates for weighted L^2-projection, as demonstrated in [19, 27].

Recently in [29], we proved that both the BPX and the multigrid V-cycle preconditioners lead to nearly uniform convergent PCG algorithms, although the resulting condition numbers can deteriorate severely. Our work was motivated by the work of Graham and Hagger [13], where it was proved that a simple diagonal scaling would lead to a preconditioned system that has only a fixed number of small eigenvalues. In particular, they proved that the ratio of the extreme values of the remaining eigenvalues, the *effective condition number* (c.f. [29]), can be bounded by Ch^{-2} where C is a constant independent of the coefficients and meshsize. Similar results can be found in [24].

The goal of this paper is to provide a rigorous proof of the robustness of the overlapping domain decomposition preconditioners. The main idea is to analyze the eigenvalue distribution of the preconditioned systems, and to prove that except for a few "bad" eigenvalues, the effective condition number is bounded uniformly with respect to the coefficients and the meshsize. Thanks to a standard theory for the conjugate gradient method (see [1, 13, 14]), these small eigenvalues will not spoil the efficiency of the method significantly. More specifically, the asymptotic convergent rate of the PCG algorithm will be $1 - \frac{2}{C\sqrt{|\log H|+1}}$, which is uniform with respect to the size of discontinuous jump. When $d = 3$, if each subregion $\Omega_m^0 (m = 1, \cdots, M)$ is assumed to be a polyhedral domain with each edge length of size H_0, then the effective condition number of BA can be bounded by $C\left(1 + \log \frac{H_0}{H}\right)$. As a result, the asymptotic convergence rate of the corresponding PCG algorithm can be bounded by $1 - \frac{2}{C\sqrt{1+\log \frac{H_0}{H}}+1}$. Especially, if the coarse grid satisfies $H \eqsim H_0$, then the asymptotic convergence rate of the PCG algorithm is uniform bounded.

The rest of the paper is organized as follows. In Section 2, we introduce some basic notation, the convergence estimates of PCG algorithm. In Section 3, we discuss the approximation and stability properties of the weighted L^2-projection. Finally, we analyze the eigenvalue distribution of the preconditioned system and prove the convergence rate of the PCG algorithm in Section 4.

For convenience, we will use the following short notation: $x \lesssim y$ means $x \leq Cy$; $x \gtrsim y$ means $x \geq cy$; and $x \eqsim y$ means $cx \leq y \leq Cx$, where c and C are generic positive constants independent of the variables in the inequalities and any other parameters related to mesh, space, and especially the coefficients.

2. PRELIMINARY

In this section, we should introduce the basic notation and the finite element discretization of the model problem (1.1). Then, we describe the standard PCG convergence results.

2.1. **Notation.** The bilinear form of (1.1) reads: $a(u, v) = \sum_{m=1}^{M} \omega_m (\nabla u, \nabla v)_{L^2(\Omega_m^0)}$ for any $u, v \in H_D^1(\Omega)$, where $H_D^1(\Omega) = \{v \in H^1(\Omega) : v|_{\Gamma_D} = 0\}$. The standard H^1-norm and seminorm with respect to any subregion Ω_m^0 are defined by

$$|u|_{1,\Omega_m^0} = \|\nabla u\|_{0,\Omega_m^0}, \quad \|u\|_{1,\Omega_m^0} = \left(\|u\|_{0,\Omega_m^0}^2 + |u|_{1,\Omega_m^0}^2\right)^{\frac{1}{2}}.$$

Thus, $a(u, u)^{\frac{1}{2}} = \left(\sum_{m=1}^{M} \omega_m |u|_{1,\Omega_m^0}^2\right)^{\frac{1}{2}} := |u|_{1,\omega}$ defines the weighted H^1-seminorm. The weighted L^2-inner product is defined by

$$(u, v)_{0,\omega} = \sum_{m=1}^{M} \omega_m (u, v)_{L^2(\Omega_m^0)}, \forall u, v \in L^2(\Omega).$$

The induced weighted L^2- and H^1-norms are defined as follows:

$$\|u\|_{0,\omega} = (u,u)_{0,\omega}^{\frac{1}{2}}, \quad \|u\|_{1,\omega} = \left(\|u\|_{0,\omega}^2 + |u|_{1,\omega}^2\right)^{\frac{1}{2}}.$$

For any subset $O \subset \Omega$, let $|u|_{1,\omega,O}$ and $\|u\|_{0,\omega,O}$ be the restrictions of $|u|_{1,\omega}$ and $\|u\|_{0,\omega}$ on the subset O.

For the distribution of the coefficients, we introduce the index set $I = \left\{m : \text{meas}\left(\partial\Omega_m^0 \cap \Gamma_D\right) = 0\right\}$, where meas$(\cdot)$ is the $d-1$ measure. In other words, I is the index set of all subregions which do not touch the Dirichlet boundary. We assume that the cardinality of I is m_0. Here, we shall emphasize that m_0 is a constant which depends only on the distribution of the coefficients.

2.2. **The Discrete System.** Given a quasi-uniform triangulation $\mathcal{T}_h$ with the meshsize h, let

$$\mathcal{V}_h = \left\{v \in H_D^1(\Omega) : v|_\tau \in \mathcal{P}_1(\tau), \ \forall \tau \in \mathcal{T}_h\right\}$$

be the piecewise linear finite element space, where $\mathcal{P}_1$ denotes the set of linear polynomials. The finite element approximation of (1.1) is the function $u \in \mathcal{V}_h$, such that

$$a(u,v) = (f,v) + \int_{\Gamma_N} g_N v, \ \ \forall v \in \mathcal{V}_h.$$

We define a linear symmetric positive definite (SPD) operator $A : \mathcal{V}_h \to \mathcal{V}_h$ by

$$(Au, v)_{0,\omega} = a(u,v).$$

The related inner product and the induced energy norm are denoted by

$$(\cdot,\cdot)_A := a(\cdot,\cdot), \qquad \|\cdot\|_A := \sqrt{a(\cdot,\cdot)}.$$

Then we have the following operator equation,

$$Au = F, \tag{2.1}$$

where $F \in L^2(\Omega)$ such that $(F,v)_{0,\omega} = (f,v) + \int_{\Gamma_N} g_N v, \ \ \forall v \in \mathcal{V}_h$. Let $\kappa(A)$ be the condition number of A, i.e., the ratio between the largest and the smallest eigenvalues. By the standard finite element theory (c.f. [30]), it is apparent that

$$\kappa(A) \eqsim h^{-2}\mathcal{J}(\omega), \ \ \text{with } \mathcal{J}(\omega) = \frac{\max_m \omega_m}{\min_m \omega_m}.$$

2.3. **Convergence of PCG Algorithm.** The PCG methods can be viewed as a conjugate gradient method applied to the preconditioned system

$$BAu = BF.$$

Here, B is an SPD operator, known as a preconditioner of A. Note that BA is symmetric with respect to the inner product $(\cdot,\cdot)_{B^{-1}}$ (or $(\cdot,\cdot)_A$).

Let $\{u_k : \ k = 0, 1, \cdots\}$ be the solution sequence of the PCG algorithm. It is well known (c.f. [12, 15, 20]) that

$$\|u - u_k\|_A \le 2\left(\frac{\sqrt{\kappa(BA)} - 1}{\sqrt{\kappa(BA)} + 1}\right)^k \|u - u_0\|_A, \tag{2.2}$$

which implies that the PCG method generally converges faster with a smaller condition number $\kappa(BA)$.

However, the estimate given in (2.2) is not sharp. One way to improve the estimate is to look at the eigenvalue distribution of BA (see [1, 13, 14] for more details). Specifically, suppose that we can divide $\sigma(BA)$, the spectrum of BA, into two sets, $\sigma_0(BA)$ and $\sigma_1(BA)$, where σ_0 consists of all "bad" eigenvalues and the remaining eigenvalues in σ_1 are bounded above and below, then we have the following theorem.

Theorem 2.1. *Suppose that $\sigma(BA) = \sigma_0(BA) \cup \sigma_1(BA)$ such that there are m elements in $\sigma_0(BA)$ and $\lambda \in [a, b]$ for each $\lambda \in \sigma_1(BA)$. Then*

$$\|u - u_k\|_A \le 2K \left(\frac{\sqrt{b/a} - 1}{\sqrt{b/a} + 1} \right)^{k-m} \|u - u_0\|_A \tag{2.3}$$

where $K = \max_{\lambda \in \sigma_1(BA)} \prod_{\mu \in \sigma_0(BA)} \left| 1 - \frac{\lambda}{\mu} \right|$.

In particular, if there are only m small eigenvalues in σ_0, say $0 < \lambda_1 \le \cdots \le \lambda_m \ll \lambda_{m+1} \le \cdots \le \lambda_n$, then

$$K = \prod_{i=1}^{m} \left| 1 - \frac{\lambda_n}{\lambda_i} \right| \le \left(\frac{\lambda_n}{\lambda_1} - 1 \right)^m = (\kappa(BA) - 1)^m .$$

In this case, the convergence rate estimate (2.3) becomes

$$\frac{\|u - u_k\|_A}{\|u - u_0\|_A} \le 2 \left(\kappa(BA) - 1\right)^m \left(\frac{\sqrt{b/a} - 1}{\sqrt{b/a} + 1} \right)^{k-m} . \tag{2.4}$$

Based on (2.4), given a tolerance $0 < \epsilon < 1$, the number of iterations of the PCG algorithm needed for $\frac{\|u-u_k\|_A}{\|u-u_0\|_A} < \epsilon$ is given by

$$k \ge m + \left(\log\left(\frac{2}{\epsilon}\right) + m \log(\kappa(BA) - 1) \right) / c_0 \text{ with } c_0 = \log \frac{\sqrt{b/a} + 1}{\sqrt{b/a} - 1}. \tag{2.5}$$

From (2.4), if there are only a few small eigenvalues of BA in $\sigma_0(BA)$, then the asymptotic convergent rate of the algorithm is dominated by the factor $\frac{\sqrt{b/a}-1}{\sqrt{b/a}+1}$, i.e., by b/a where $b = \lambda_n(BA)$ and $a = \lambda_{m+1}(BA)$. We define this quantity as the "effective condition number."

Definition 2.2 ([29]). *Let $\mathcal{V}$ be an n-dimensional Hilbert space. The m-*th effective condition number *of an operator $A : \mathcal{V} \to \mathcal{V}$ is defined by*

$$\kappa_{m+1}(A) = \frac{\lambda_{\max}(A)}{\lambda_{m+1}(A)}$$

where $\lambda_{m+1}(A)$ is the $(m+1)$-th minimal eigenvalue of A.

To estimate the effective condition number, we need to estimate $\lambda_{m+1}(A)$. A fundamental tool is the *Courant-Fisher "minimax" principle* (see, e.g., [12]):

Lemma 2.3. *Let $\mathcal{V}$ be an n-dimensional Hilbert space and $A : \mathcal{V} \to \mathcal{V}$ is an SPD operator on $\mathcal{V}$. Suppose that $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ are the eigenvalues of A, then*

$$\lambda_{m+1}(A) = \max_{dim(S)=m} \min_{0 \ne v \in S^\perp} \frac{(Av, v)}{(v, v)}$$

for $m = 1, 2, \cdots, n$. Especially, for any subspace $\mathcal{V}_0 \subset \mathcal{V}$ with $\dim(\mathcal{V}_0) = n - m$, the following estimation of $\lambda_{m+1}(A)$ holds:

$$\lambda_{m+1}(A) \ge \min_{0 \ne v \in \mathcal{V}_0} \frac{(Av, v)}{(v, v)}. \tag{2.6}$$

Inequality (2.6) is the starting point for our analysis of eigenvalue distribution. It enables us to obtain a lower bound of every eigenvalue if we can estimate $\min_{0 \ne v \in \mathcal{V}_0} \frac{(Av,v)}{(v,v)}$ for some suitable subspace $\mathcal{V}_0$.

3. WEIGHTED L^2-PROJECTION

A major tool to analyze the multilevel and domain decomposition methods is the weighted L^2-projection $Q_H^\omega : L^2(\Omega) \to \mathcal{V}_H$ defined by $(Q_H^\omega u, v_H)_{0,\omega} = (u, v_H)_{0,\omega}$ for any $v_H \in \mathcal{V}_H$. In this section, we shall discuss the approximation and stability properties of Q_H^ω. Most of the results in this section can also be found in [5, 29].

Lemma 3.1 ([5])**.** *Let $\mathcal{V}_H \subset \mathcal{V}_h$ be two nested linear finite element spaces. Then for any $u \in \mathcal{V}_h$, there holds,*

$$\|(I - Q_H^\omega)u\|_{0,\omega} \leq c_d(h,H)H|u|_{1,\omega}, \text{ and } |Q_H^\omega u|_{1,\omega} \lesssim c_d(h,H)|u|_{1,\omega}$$

where $c_d(h,H) = C \cdot \begin{cases} (\log \frac{H}{h})^{1/2}, & \text{if } d = 2 \\ (\frac{H}{h})^{1/2}, & \text{if } d = 3. \end{cases}$

The proof of this lemma is based on the properties of the standard interpolation operator and Sobolev imbedding theorem (for the details, see [5]). The above lemma is not necessarily true for general $H^1(\Omega)$ function. But if we use the full weighted H^1-norm, then we have

Lemma 3.2 ([5])**.** *For all $u \in H_D^1(\Omega)$, we have*

$$\|(I - Q_H^\omega)u\|_{0,\omega} \lesssim H\,|\log H|^{\frac{1}{2}}\,\|u\|_{1,\omega}\,.$$

In general, we can not replace $\|u\|_{1,\omega}$ by the semi-norm $|u|_{1,\omega}$ in the above lemma. For this purpose, we introduce a subspace $\widetilde{H}_D^1(\Omega)$ of $H_D^1(\Omega)$ as follows:

$$\widetilde{H}_D^1(\Omega) = \left\{ v \in H_D^1(\Omega) : \int_{\Omega_m^0} v dx = 0, \forall m \in I \right\}.$$

A important feature of this subspace is that the Poincaré-Friedrichs inequality holds:

$$\|v\|_{0,\omega} \lesssim |v|_{1,\omega}, \quad \forall v \in \widetilde{H}_D^1(\Omega). \tag{3.1}$$

Remark 3.3. *The condition $\int_{\Omega_m^0} v = 0$ is not essential. The main idea is to introduce a subspace such that the Poincaré-Friedrichs inequality (3.1) holds. It can be substituted by some other conditions. For example, we can replace it by*

$$\int_{F_m} v dx = 0, \quad F_m \subset \partial\Omega_m^0 \text{ and } meas(F_m) > 0, \quad \text{for each } m \in I.$$

In this case, the Poincaré-Friedrichs inequality (3.1) is still true (see [11, 30] *for more details).*

Thanks to inequality (3.1), we have the following estimates for the weighted L^2-projection:

Lemma 3.4. *For any $v \in \widetilde{H}_D^1(\Omega)$ we have the following approximation and stability of Q_H^ω :*

$$\|(I - Q_H^\omega)v\|_{0,\omega} \lesssim H\,|\log H|^{\frac{1}{2}}\,|v|_{1,\omega}\,, \tag{3.2}$$

$$|Q_H^\omega v|_{1,\omega} \lesssim |\log H|^{\frac{1}{2}}\,|v|_{1,\omega}\,. \tag{3.3}$$

Proof. See [29, Lemma 3.3]. □

Although it is true for $d = 2$ or 3, Lemma 3.4 is of interest only when $d = 3$. When $d = 2$, Lemma 3.1 is sufficient for our future use. From Lemma 3.4, the approximation and stability of the weighted L^2-projection will deteriorate by $|\log H|$. A sharper estimate can be obtained if we assume that each subregion Ω_m^0 is a polyhedral domain with each edge of length H_0.

Lemma 3.5 ([5])**.** *Assume G is a polyhedral domain in $\mathbb{R}^3$. Then*

$$\|v\|_{L^2(E)} \lesssim |\log H|^{1/2}\|v\|_{1,G}, \quad \forall v \in \mathcal{V}_H(G),$$

where E is any edge of G.

By the Poincaré-Friedrichs inequality (3.1), for each $v \in \widetilde{H}^1_D(\Omega)$, we have

$$\|v\|_{1,\Omega^0_m} \lesssim |v|_{1,\Omega^0_m} \quad \text{for all} \quad \Omega^0_m (m = 1, \cdots, M).$$

Therefore, if Ω^0_m is a polyhedral domain with length$(E) \eqsim H_0$, then by Lemma 3.5 and a standard scaling argument, we have

$$\|v\|_{L^2(E)} \lesssim \left(\log \frac{H_0}{H}\right)^{1/2} |v|_{1,\Omega^0_m}, \quad \forall v \in \mathcal{V}_H(\Omega^0_m) \cap \widetilde{H}^1_D(\Omega). \tag{3.4}$$

In this case, we can get the following approximation and stability properties for the weighted L^2-projection:

Lemma 3.6. *In $\mathbb{R}^3$, assume that each subregion Ω^0_m satisfies $H_0 \eqsim$ length(E) for any edge E of Ω^0_m. Then for all $v \in \widetilde{H}^1_D(\Omega)$, we have*

$$\|(I - Q^\omega_H)v\|_{0,\omega} \lesssim H \left(\log \frac{H_0}{H}\right)^{\frac{1}{2}} |v|_{1,\omega}\,, \tag{3.5}$$

$$|Q^\omega_H v|_{1,\omega} \lesssim \left(\log \frac{H_0}{H}\right)^{\frac{1}{2}} |v|_{1,\omega}\,. \tag{3.6}$$

Proof. Similar to [5, Lemma 4.6]. □

Remark 3.7. *In addition to the condition in Lemma 3.6, if $H \eqsim H_0$ then for all $v \in \widetilde{H}^1_D(\Omega)$, we have*

$$\|(I - Q^\omega_H)v\|_{0,\omega} \lesssim H\,|v|_{1,\omega}\,, \tag{3.7}$$

$$|Q^\omega_H v|_{1,\omega} \lesssim |v|_{1,\omega}\,. \tag{3.8}$$

In fact, in this case, inequality (3.4) *becomes*

$$\|v\|_{L^2(E)} \lesssim |v|_{1,\Omega^0_m}, \quad \forall v \in \mathcal{V}_H(\Omega^0_m) \cap \widetilde{H}^1_D(\Omega). \tag{3.9}$$

Then inequalities (3.7) *and* (3.8) *follow by the same proof as Lemma 3.6.*

4. Overlapping Domain Decomposition Preconditioner

Now, we discuss the robustness of two level overlapping domain decomposition preconditioner. Specifically, we are seeking the solution to (2.1) on a fine grid $\mathcal{T}_h$ with meshsize h as described in Section 2.2. We introduce a coarse grid $\mathcal{T}_H$ with meshsize H, and assume that each element in $\mathcal{T}_H$ aligns with the jump interface, and is a union of some elements in $\mathcal{T}_h$. Let $\mathcal{V} := \mathcal{V}_h$ and $\mathcal{V}_0 := \mathcal{V}_H$ be the piecewise linear continuous finite element spaces on $\mathcal{T}_h$ and $\mathcal{T}_H$, respectively.

The domain Ω is partitioned into L nonoverlapping subdomains Ω_l $(l = 1, \ldots, L)$, such that $\overline{\Omega} = \cup_{l=1}^L \overline{\Omega}_l$. Each subdomain Ω_l is enlarged to Ω'_l in such a way that the restriction of triangulation $\mathcal{T}_h$ on Ω'_l is also a triangulation of Ω'_l itself, and Ω'_l consists of all points in Ω within a distance of CH from Ω_l. Here, we make no assumption on the relationship between this partition and the jump regions Ω^0_m $(m = 1, \ldots, M)$. Based on the partition, a natural decomposition of the finite element space $\mathcal{V}$ is

$$\mathcal{V} = \sum_{l=1}^{L} \mathcal{V}_l, \quad \text{where } \mathcal{V}_l := \{v \in \mathcal{V} : v = 0 \text{ in } \Omega \setminus \Omega'_l\}.$$

As usual, the coarse space $\mathcal{V}_0$ provides the global coupling between subdomains. Obviously, we have the space decomposition $\mathcal{V} = \sum_{l=0}^{L} \mathcal{V}_l$.

For each $l = 0, 1, \ldots, L$, we introduce the projections $P_l,\ Q_l^\omega : \mathcal{V} \to \mathcal{V}_l$ by

$$a(P_l u, v_l) = a(u, v_l), \ \ (Q_l^\omega u, v_l)_{0,\omega} = (u, v_l)_{0,\omega}, \ \ \forall v_l \in \mathcal{V}_l,$$

and define the operator $A_l : \mathcal{V}_l \to \mathcal{V}_l$ by $(A_l u_l, v_l)_{0,\omega} = a(u_l, v_l), \quad \forall u_l, v_l \in \mathcal{V}_l$. For convenience, we denote $A = A_L$ and $Q_{-1}^\omega = 0$. It follows from the definitions that

$$Q_l^\omega A = A_l P_l, \text{ and } Q_l^\omega Q_k^\omega = Q_k^\omega Q_l^\omega = Q_k^\omega \text{ for } k \le l.$$

Then the domain decomposition preconditioner is defined by

$$B = \sum_{l=0}^{L} A_l^{-1} Q_l^\omega. \tag{4.1}$$

Obviously, $BA = \sum_{l=0}^{L} A_l^{-1} Q_l^\omega A = \sum_{l=0}^{L} P_l$. In what follows, we will analyze the eigenvalue distribution of BA and prove the robustness of the preconditioner.

The analysis of the lower bound of eigenvalues relies on certain stable decomposition. Similar to $\widetilde{H}_D^1(\Omega)$ in Section 3, we introduce a subspace $\widetilde{\mathcal{V}}$ of $\mathcal{V}$ by

$$\widetilde{\mathcal{V}} := \widetilde{H}_D^1(\Omega) \cap \mathcal{V} = \left\{ v \in \mathcal{V} : \int_{\Omega_m} v = 0, \text{ for } \ m \in I \right\}.$$

We shall emphasize here that $\dim(\widetilde{\mathcal{V}}^\perp) = m_0$ and the Poincaré-Friedrichs inequality (3.1) holds for any $v \in \widetilde{\mathcal{V}}$. Then we have the following stable decomposition result:

Lemma 4.1. *For any $v \in \mathcal{V}$, there exist $v_l \in \mathcal{V}_l$ such that $v = \sum_{l=0}^{L} v_l$ and*

$$\sum_{l=0}^{L} a(v_l, v_l) \lesssim c_d(h, H)^2 a(v, v). \tag{4.2}$$

For any $v \in \widetilde{\mathcal{V}}$, there are $v_l \in \mathcal{V}_l$ that satisfy $v = \sum_{l=0}^{L} v_l$ and

$$\sum_{l=0}^{L} a(v_l, v_l) \lesssim |\log H| a(v, v). \tag{4.3}$$

Furthermore, if each subregion Ω_m^0 satisfies length$(E) \eqsim H_0$ for any edge E of Ω_m^0, then for any $v \in \widetilde{\mathcal{V}}$, there are $v_l \in \mathcal{V}_l$ that satisfy $v = \sum_{l=0}^{L} v_l$ and

$$\sum_{l=0}^{L} a(v_l, v_l) \lesssim \left(1 + \log \frac{H_0}{H}\right) a(v, v). \tag{4.4}$$

Especially, if $H \eqsim H_0$ then $\sum_{l=0}^{L} a(v_l, v_l) \lesssim a(v, v)$.

Proof. The ideas to prove inequality (4.2)-(4.4) are the same. The main difference is that we use different properties of weighted L^2-projection in Section 3. Here, we follow the idea from [28].

Let $\{\theta_l\}_{l=1}^{L}$ be a partition of unity defined on Ω satisfying $\sum_{l=1}^{L} \theta_l = 1$ and for $l = 1, 2, \cdots, L$,

$$\mathrm{supp}\theta_l \subset \Omega_l' \cup \partial\Omega, \ \ 0 \le \theta_l \le 1, \ \ \|\nabla \theta_l\|_{\infty,\Omega_l} \le CH^{-1}.$$

Here $\|\cdot\|_{\infty,O}$ denote the L^∞ norm of a function defined on a subdomain O.

The construction of such a partition of unity is standard. A partition $v = \sum_{l=0}^{L} v_l$ for $v_l \in \mathcal{V}_l$ can then be obtained by taking $v_0 = Q_0^\omega v$ and

$$v_l = I_h(\theta_l(v - Q_0^\omega v)) \in \mathcal{V}_l,\ l = 1, \cdots, L,$$

where I_h is the nodal value interpolant on $\mathcal{V}$.

From this decomposition, we prove that the inequalities (4.2) and (4.3) hold. For any $\tau \in \mathcal{T}_h$, note that

$$\|\theta_l - \overline{\theta}_{l,\tau}\|_{L^\infty(\tau)} \lesssim h\|\nabla\theta_l\|_{L^\infty(\tau)} \lesssim \frac{h}{H}.$$

Let $w = v - Q_0^\omega v$, and by the inverse inequality

$$\begin{aligned} |v_l|_{1,\tau} &\leq |\overline{\theta}_{l,\tau} w|_{1,\tau} + |I_h(\theta_l - \overline{\theta}_{l,\tau})w|_{1,\tau} \\ &\lesssim |w|_{1,\tau} + h^{-1}\|I_h(\theta_l - \overline{\theta}_{l,\tau})w\|_{0,\tau}. \end{aligned}$$

It is easy to show that $\|I_h(\theta_l - \overline{\theta}_{l,\tau})w\|_{0,\tau} \lesssim \frac{h}{H}\|w\|_{0,\tau}$. Consequently, $|v_l|_{1,\tau}^2 \lesssim |w|_{1,\tau}^2 + \frac{1}{H^2}\|w\|_{0,\tau}^2$. Summing over all $\tau \in \mathcal{T}_h \cap \Omega_l$ with appropriate weights gives

$$|v_l|_{1,\omega}^2 = |v_l|_{1,\omega,\Omega_l}^2 \lesssim |w|_{1,\omega,\Omega_l}^2 + \frac{1}{H^2}\|w\|_{0,\omega,\Omega_l}^2,$$

and

$$\begin{aligned} \sum_{l=1}^{L} a(v_l, v_l) &\lesssim \sum_{l=1}^{L} |v_l|_{1,\omega,\Omega_l}^2 \lesssim \sum_{l=1}^{L} \left(|w|_{1,\omega,\Omega_l}^2 + \frac{1}{H^2}\|w\|_{0,\omega,\Omega_l}^2 \right) \\ &\lesssim \left(|v - Q_0^\omega v|_{1,\omega}^2 + \frac{1}{H^2}\|v - Q_0^\omega v\|_{0,\omega}^2 \right). \end{aligned}$$

From the above inequality, applying Lemma 3.1 for any $v \in \mathcal{V}$, we obtain inequality (4.2)); and applying Lemma 3.4 (or Lemma 3.6) for $v \in \widetilde{\mathcal{V}}$ gives us inequality (4.3) ((4.4) respectively). ☐

Theorem 4.2. *For the additive Schwarz preconditioner B defined by (4.1), the eigenvalues of BA satisfies*

$$\lambda_{\min}(BA) \geq c_d(h,H)^{-2},\quad \lambda_{m_0+1}(BA) \geq C|\log H|^{-1},\quad \textit{and}\quad \lambda_{\max}(BA) \leq C.$$

Moreover, when $d = 3$ and if each subregion Ω_m^0 is a polyhedral domain with each edge of length H_0, then

$$\lambda_{m_0+1}(BA) \geq C\left(1 + \log\frac{H_0}{H}\right)^{-1}.$$

Especially, if $H \eqsim H_0$ then $\lambda_{m_0+1}(BA) \geq C$.

Proof. Noticing that $BA = \sum_{l=0}^{L} P_l$, by a standard coloring argument (c.f. [7, 23]), we have $\lambda_{\max}(BA) \leq C$. For the minimum eigenvalue, for any $v \in \mathcal{V}$, we consider the decomposition $v = \sum_{l=0}^{L} v_l$ as in Lemma 4.1. By the Schwarz inequality, we obtain

$$\begin{aligned} a(v,v) &= \sum_{l=0}^{L} a(v_l, v) = \sum_{l=0}^{L} a(v_l, P_l v) \\ &\leq \left(\sum_{l=0}^{L} a(v_l, v_l)\right)^{1/2} \left(\sum_{l=0}^{L} a\left(P_l v, P_l v\right)\right)^{1/2} \\ &= \left(\sum_{l=0}^{L} a(v_l, v_l)\right)^{1/2} \left(a\left(BAv, v\right)\right)^{1/2}. \end{aligned}$$

Applying inequality (4.2), we get $a(v,v) \le c_d(h,H) a(v,v)^{1/2} a\left(BAv, v\right)^{1/2}$ for any $v \in \mathcal{V}$. This implies $\lambda_{\min}(BA) \ge c_d(h,H)^{-2}$. On the other hand, by (4.3), we have

$$a(v,v) \lesssim |\log H|^{1/2} a(v,v)^{1/2} a\left(BAv, v\right)^{1/2}, \forall v \in \widetilde{\mathcal{V}}.$$

By min-max Lemma 2.3, and noticing that $\dim(\widetilde{\mathcal{V}}^{\perp}) = m_0$, we obtain $\lambda_{m_0+1}(BA) \gtrsim |\log H|^{-1}$. Similarly, from by (4.4) and min-max Lemma 2.3, we get $\lambda_{m_0+1}(BA) \ge C\left(1 + \log \frac{H_0}{H}\right)^{-1}$ if each edge of the subregions satisfies length$(E) \eqsim H_0$. This completes the proof. □

Theorem 4.2 shows that the preconditioned system has only m_0 small eigenvalues, and the effective condition number is bounded by $C|\log H|$, or $C\left(1 + \log \frac{H_0}{H}\right)$ if each subregion is a polyhedral domain with each edge of length H_0. Especially when $H_0 \eqsim H$, the effective condition number is bounded uniformly. Here, we remark that when $d = 2$, $\kappa(BA) \le C\left(1 + \log \frac{H}{h}\right)$ which is quite robust. But, for the worst case in $d = 3$, we have $\kappa(BA) \le C\frac{H}{h}$, which grows rapidly as $h \to 0$. In this case, we have the following convergence estimate for the PCG algorithm.

Theorem 4.3. *In $\mathbb{R}^3$, assume that each subregion Ω_m^0 $(m = 1, \cdots, M)$ is a polyhedral domain with each edge of length H_0. Let $u \in \mathcal{V}$ be the exact solution to equation (2.1) and $\{u_k : k = 0, 1, 2, \dots\}$ be the solution sequence of the PCG algorithm. Then we have*

$$\frac{\|u - u_k\|_A}{\|u - u_0\|_A} \le 2\left(\frac{C_0 H}{h} - 1\right)^{m_0} \rho^{k-m_0} \quad \text{for } k \ge m_0,$$

where $\rho = 1 - \frac{2}{C\sqrt{1+\log\frac{H_0}{H}}+1} < 1$ and C_0, C are constants independent of coefficients and meshsize. Moreover, given a tolerance $0 < \epsilon < 1$, the number of iterations needed for $\frac{\|u-u_k\|_A}{\|u-u_0\|_A} < \epsilon$ satisfies

$$k \ge m_0 + \left(\log\left(\frac{2}{\epsilon}\right) + m_0 \log\left(\frac{C_0 H}{h} - 1\right)\right) / |\log(\rho)|.$$

Especially, if $H \eqsim H_0$ then the asymptotic convergence rate ρ of the PCG algorithm is uniform bounded with respect to both the coefficients and meshsize.

Theorem 4.3 is a direct consequence of inequalities (2.4), (2.5) and Theorem 4.2.

Remark 4.4. *From Theorem 4.3, although the convergence rate will deteriorate slightly by the condition number $\kappa(BA)$, because m_0 is a fixed number, the asymptotic convergence rate can be bounded by $\rho < 1$ which is uniform with respect to the coefficients and the meshsize.*

Without the assumption on the subregions Ω_m^0 $(m = 1, \cdots, M)$, Theorem 4.3 becomes,

$$\frac{\|u - u_k\|_A}{\|u - u_0\|_A} \le 2\left(\frac{C_0 H}{h} - 1\right)^{m_0} \left(1 - \frac{2}{C_1\sqrt{|\log H|} + 1}\right)^{k-m_0} \quad \text{for } k \ge m_0.$$

Remark 4.5. *By similar arguments, the results above can be generated to the* inexact solver additive Schwarz preconditioners *(c.f.* [7]*) and also to the* multilevel additive Schwarz preconditioners *(c.f.* [31]*). For the BPX preconditioner and the multigrid V-cycle preconditioner, similar results can be found in* [29].

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