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Properties determining performance of a multilevel optimization scheme

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We present a multilevel optimization approach (termed ML/Opt). The approach assumes that one has a hierarchy of models, ordered from fine to coarse, of an underlying optimization problem, and that one is interested in finding solutions at the finest level of detail. In this hierarchy of models calculations on coarser levels are less expensive, but also are of less fidelity, than calculations on finer levels. The intent of ML/Opt is to use calculations on coarser levels to accelerate the progress of the optimization on the finest level.

Global convergence (i.e., convergence to a Karush–Kuhn–Tucker point from an arbitrary starting point) is ensured in a straightforward way by requiring a single step of a convergent method on the finest level plus a line-search (or other globalization technique) for incorporating the coarse level corrections. The multilevel approach thus serves primarily as an acceleration technique and practical performance is of central interest.

Performance of the multilevel approach, we argue, is a function of four properties: the nonlinearity of the problem, the consistency of the models across the different levels, complementarity of the problems on different levels, and the degree of separability of the problem across the different model levels. Some of these are features of the hierarchy of models, while others are features of the interaction between the model hierarchy and the methods being used to solve the optimization problems at the various levels. We present inexpensive self-diagnostic tests that shed light on the suitability of the multilevel approach to the problem under consideration, tests that can be performed in the course of applying ML/Opt.

In addition, ML/Opt may be an effective acceleration technique at some stages of the optimization but not at others. For instance, the conditions under which ML/Opt is an effective acceleration technique may not be satisfied when far from a solution. On the other hand, as we explain, we expect these conditions to be satisfied near an optimizer. The self-diagnostic tests can be used to obtain guidance on whether or not to use ML/Opt at different stages of the optimization.

By the nonlinearity of the problem, or, more precisely, the nonquadraticity of the problem, we mean the extent to which the quadratic models used in the optimization fail to capture important features of the problem at any level. This can be judged by checking whether the quadratic model finds approximate minimizers for the subproblems that are solved to compute optimization steps. This test is akin to the mechanisms used to ensure global convergence.

The consistency across levels refers to the extent to which improvements in the merit function computed on a coarser level are manifest in the merit function at a finer level. This can be measured by comparing reduction in the merit function predicted by a coarse level step with the actual reduction attained on a finer grid. This test is similar in spirit to the comparison of predicted and actual decrease in model trust region methods, though the idea is not to regulate the length of the step, but, rather, to judge the efficacy of the model being used to generate steps.

The choice of coarser models is similar to the notion in algebraic multigrid of a coarse grid corresponding to the near nullspace of a fine grid operator. In the optimization context we seek a degree of complementarity of the contributions to the solution that are computed on different levels. The degree of complementarity can be tested via a generalized Rayleigh quotient calculation involving the Hessian of the merit function.

The multilevel approach also requires a degree of separability of the problem across different model levels in order to be effective. Separability can be estimated by comparing corresponding coarse grid and fine grid Hessian-vector products.

We present numerical tests of the approach on a set of synthetic problems involving optimization of systems governed by partial differential equations. Some of these problems are designed to be amenable to the multilevel approach, while others are designed to be resistant. We apply the self-diagnostics to these optimization problems and discuss how the self-diagnostic tests reflect the extent to which the multilevel approach is appropriate for the problems.