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Multigrid technique for non-eliminated boundary conditions in arbitrary domain

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Poisson equation in arbitrary domain (possibly with moving boundary) is central to many applications, such as diffusion phenomena,fluid dynamics, charge transport in semiconductors, crystal growth, electromagnetism and many others. We present a rather simple iterative method [3] to solve the Poisson equation in arbitrary domain Ω, identified by a level set function ϕ in such a way $\Omega = \left\{x \in \mathbb{R}^d \colon \phi(x) < 0\right\}$, and mixed boundary conditions. Such iterative scheme is just the building-block for a proper multigrid approach [4].

The method is based on ghost-cell technique for finite difference discretization on a regular Cartesian grid. The structure of ghost points is complex and elimination of discrete boundary conditions from the system is hard to perform. In addition, a simple Gauss-Seidel scheme for the whole system does not converge. Therefore, in order to provide a good smoother for the multigrid approach, we relax the whole problem introducing a fictitious time and looking for the steady-state solution.

The fictitious time-dependence of the problem including boundary conditions leads us to an iterative scheme for the set of all unknowns (internal points and ghost points), which is proved to converge, at least for first order accurate discretization. The smoothing procedure of the multigrid approach in the interior is again Gauss-Siedel-like, while the iterations on the boundary are performed in order to provide smooth errors.

Multigrid techniques for ghost points are well-studied in literature, but just in the case of rectangular domain, where a restriction operator is defined separately for the interior of the domain and for the boundary, and the restriction of the boundary is performed using a restriction operator of codimension 1, since ghost points are aligned with the Cartesian axis. In the case of arbitrary domain, ghost points have an irregular structure and we provide a reasonable definition of the restriction operator for the boundary.

We also show that a proper treatment of the boundary iterations can improve the rate of convergence of the multigrid and the cost of this extra computational work is negligible, i.e. tends to zero as the dimension of the problem increases.

The method can also be extended to the interesting case of discontinuous coefficients across an interface. Some preliminary numerical results are provided in this talk.

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