Convergence of a Sixth Order Compact Difference Scheme for the Convection Diffusion Equation Using Multiscale Multigrid Method

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Abstract

We examine an explicit sixth order compact finite difference scheme for numerical solutions of the two dimensional convection diffusion equation with variable coefficients similar to the one presented by Wang and Zhang [18]. The sixth order scheme uses a multiscale multigrid method to compute fourth order solutions on two scale grids, then applies the Richardson extrapolation and an operator based interpolation scheme to approximate the sixth order solution on the fine grid. Convergence analysis is provided in this paper to prove that the sixth order method will achieve the sixth order solutions for certain values of the Reynolds number.

Keywords: convection diffusion equation, Reynolds number, multigrid method, Richardson extrapolation.

1 Introduction

We consider the two dimensional (2D) convection diffusion equation with the Dirichlet boundary condition, which can be written as

$$\begin{aligned} u_{xx} + u_{yy} + p(x,y)u_x + q(x,y)u_y &= f(x,y), \qquad (x,y) \in \Omega, \\ u(x,y) &= g(x,y), \qquad (x,y) \in \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a rectangular domain in R^2 and $\partial\Omega$ is the boundary of Ω. We assume that the convection coefficients $p(x,y)$ and $q(x,y)$ are sufficiently smooth on Ω.

Eq. (1) can be discretized by some finite difference scheme to result in a system of linear equations

$$A^h u^h = f^h, \tag{2}$$

where h is the uniform grid spacing of the discretized domain Ω^h. The magnitudes of $p(x,y)$ and $q(x,y)$ determine the ratio of the convection to diffusion which can be characterized by

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the cell Reynolds number (Re) in the form of

$$Re = max(\sup_{(x,y)\in\Omega} |p(x,y)|, \sup_{(x,y)\in\Omega} |q(x,y)|)h. \tag{3}$$

Solutions to the convection diffusion equation are of practical interest to scientific and engineering modeling problems involving fluid flows and heat transfer. For convection dominated problems, finite difference schemes such as the five-point second order central difference scheme (truncation error $O(h^2)$) and the upwind difference scheme (truncation error $O(h)$) are unable to obtain satisfactory results [21].

In recent decades many fourth order compact (FOC) schemes for the 2D convection diffusion equation have been developed [9, 13, 14]. Sixth order schemes for a variety of related problems have also been proposed. Chu and Fan [5, 6] have proposed a three point combined compact difference scheme for solving the 2D Stommel Ocean model, which is a special case of the convection diffusion equation. Sun and Zhang [16] have proposed a sixth order discretization scheme for solving the 2D convection diffusion equation; however, their solution is not scalable with mesh size. Wang and Zhang [17, 18] have developed a sixth order explicit compact scheme for solving the 2D Poisson equation and the 2D convection diffusion equation. In the present paper we provide a proof that a compact scheme similar to [18] converges to a sixth order solution for certain values of the Reynolds number.

2 Sixth Order Compact Approximations

Our explicit sixth order compact scheme is based on the fourth order compact discretization on the two scale grids. FOC schemes have been proposed by several authors [9, 13, 14] in different ways. In the present paper we use the FOC scheme by Gupta *et al.* [9].

2.1 The FOC finite difference scheme

We use u_0 to denote the approximate value of $u(x,y)$ at a mesh point (x,y). For a particular mesh point, the approximate values at its eight immediate neighboring points are denoted by u_i, $i = 1, 2, ..., 8$. The nine-point compact grid points are labeled as follows

$$\begin{pmatrix} u_6 & u_2 & u_5 \\ u_3 & u_0 & u_1 \\ u_7 & u_4 & u_8 \end{pmatrix}.$$

Similarly, we use p_i, q_i and f_i, $(i = 0, 1, ..., 4)$ to denote the value of $p(x,y)$ and $q(x,y)$ at the neighboring grid points. The fourth order nine-point compact finite difference formula for the mesh point (x,y) can be written as

$$\sum_{j=0}^{8} \alpha_j u_j = \frac{h^2}{2}[8f_0 + f_1 + f_2 + f_3 + f_4] + \frac{h^3}{4}[p_0(f_1 - f_3) + q_0(f_2 - f_4)], \tag{4}$$

where h is the mesh spacing, $\alpha_i (i = 0, 1, ..., 8)$ are the coefficients as

$$\begin{aligned}
\alpha_0 &= -[20 + h^2(p_0^2 + q_0^2) + h(p_1 - p_3) + h(q_2 - q_4)], \\
\alpha_1 &= 4 + \frac{h}{4}[4p_0 + 3p_1 - p_3 + p_2 + p_4] + \frac{h^2}{8}[4p_0^2 + p_0(p_1 - p_3) + q_0(p_2 - p_4)], \\
\alpha_2 &= 4 + \frac{h}{4}[4q_0 + 3q_2 - q_4 + q_1 + q_3] + \frac{h^2}{8}[4q_0^2 + p_0(q_1 - q_3) + q_0(q_2 - q_4)],
\end{aligned}$$

$$
\begin{aligned}
\alpha_3 &= 4-\frac{h}{4}[4p_0-p_1+3p_3+p_2+p_4]+\frac{h^2}{8}[4p_0^2-p_0(p_1-p_3)-q_0(p_2-p_4)],\\
\alpha_4 &= 4-\frac{h}{4}[4q_0-q_2+3q_4+q_1+q_3]+\frac{h^2}{8}[4q_0^2-p_0(q_1-q_3)-q_0(q_2-q_4)],\\
\alpha_5 &= 1+\frac{h}{2}(p_0+q_0)+\frac{h}{8}(q_1-q_3+p_2-p_4)+\frac{h^2}{4}p_0q_0,\\
\alpha_6 &= 1-\frac{h}{2}(p_0-q_0)-\frac{h}{8}(q_1-q_3+p_2-p_4)-\frac{h^2}{4}p_0q_0,\\
\alpha_7 &= 1-\frac{h}{2}(p_0+q_0)+\frac{h}{8}(q_1-q_3+p_2-p_4)+\frac{h^2}{4}p_0q_0,\\
\alpha_8 &= 1+\frac{h}{2}(p_0-q_0)-\frac{h}{8}(q_1-q_3+p_2-p_4)-\frac{h^2}{4}p_0q_0.
\end{aligned}
$$

When $q(x,y)$ and $p(x,y)$ are set to be zero, Eq. (1) is reduced to the 2D Poisson equation, and Eq. (4) will become the well-known Mehrstellen formula [8].

2.2 Fourth order to sixth order

A multigrid method combined with the FOC scheme is used to obtain a fourth order solution on fine (Ω_h) and coarse (Ω_{2h}) grids. We use the Richardson extrapolation technique to obtain a sixth order coarse grid solution. The Richardson extrapolation formula can be written in the form of [4]

$$
\tilde{u}^{2h}_{i,j}=\frac{(2^m\hat{u}^h_{2i,2j}-\hat{u}^{2h}_{i,j})}{2^{m-1}}, \tag{5}
$$

where m is the actual order of solution accuracy from fourth order approximation. $\hat{u}^h$ and $\hat{u}^{2h}$ are the fourth order solutions from grid Ω_h and Ω_{2h}, respectively.

Algorithm 1 Operator based interpolation iteration combined with the sixth order Richardson extrapolation technique.

1: Update every $(even, even)$ grid point on Ω_h.
From $\hat{u}^{2h,k}_{i,j}\in\Omega^4_{2h}$ and $\hat{u}^{h,k}_{2i,2j}\in\Omega^4_h$, we first compute $\tilde{u}^{2h,k+1}_{i,j}\in\Omega^6_{2h}$ by Eq. (5), then use direct interpolation to get $\tilde{u}^{h,k+1}_{2i,2j}\in\Omega^6_h$.

2: Let $u^h_{old}=\tilde{u}^{h,k}$.

3: Update every (odd, odd) grid point on Ω_h.
From Eq. (4), for each (odd, odd) point (i,j), the updated solution is

$$
\begin{aligned}
\tilde{u}^{h,k+1}_{i,j} &= [F_{i,j}-A_{i,j}(1)\tilde{u}^{h,k}_{i+1,j}-A_{i,j}(2)\tilde{u}^{h,k}_{i,j+1}-A_{i,j}(3)\tilde{u}^{h,k}_{i-1,j}-A_{i,j}(4)\tilde{u}^{h,k}_{i,j-1}\\
&\quad -A_{i,j}(5)\tilde{u}^{h,k}_{i+1,j+1}-A_{i,j}(8)\tilde{u}^{h,k}_{i+1,j-1}-A_{i,j}(6)\tilde{u}^{h,k}_{i-1,j+1}-A_{i,j}(7)\tilde{u}^{h,k}_{i-1,j-1}]/A_{i,j}(0).
\end{aligned}
$$

Here, $F_{i,j}$ represents the right-hand side part of Eq. (4).

4: Update every $(odd, even)$ grid point on Ω_h.
From Eq. (4), the idea is similar to the (odd, odd) grid point.

5: Update every $(even, odd)$ grid point on Ω_h.
From Eq. (4), the idea is similar to the $(odd, even)$ grid point.

6: Compute the 2-norm $R=||\tilde{u}^{h,k+1}-u^h_{old}||_2$. If not converged, go back to Step 2.

After the extrapolation, Algorithm 1 is used to approximate the sixth order solution on the fine grid. This algorithm first directly copies the sixth order coarse grid solutions to the corresponding $(even, even)$ fine grid points, then a mesh-refinement [11] interpolation scheme is used to approximate other fine grid points.

In Algorithm 1, we use $\tilde{u}_{i,j}^{h,k}$ to denote the approximate value of $u(x,y)$ at iteration k and grid point $(i,j) \in \Omega_h$. Similarly, the exact value of $u(x,y)$ at grid point $(i,j) \in \Omega_h$ is denoted by $u_{i,j}^h$. $A_{i,j}(l)$ denotes the value of α_l at grid point (i,j). $F_{i,j}$ denotes the value of the right hand side of Eq. (4) at grid point (i,j).

3 Convergence Analysis

We will now demonstrate that Algorithm 1 converges to a sixth order solution for certain Reynolds numbers. Initially we assume Algorithm 1 uses a Jacobi method for updating points; however, we show the results also hold if Algorithm 1 uses a Gauss-Seidel method instead.

3.1 Preliminaries

Before proving the convergence result it will be helpful to establish some properties of our discretization scheme. First we note some useful relations of the α coefficients following Karaa and Zhang [12].

Lemma 3.1

$$-\alpha_0 = \sum_{i=1}^{8} \alpha_i \tag{6}$$

Proof From Eq. (4) we obtain the following identities:

$$\alpha_1 + \alpha_3 = -8 + h(p_1 - p_3) - h^2 p_0^2 \tag{7}$$
$$\alpha_2 + \alpha_4 = -8 + h(q_2 - p_4) - h^2 q_0^2 \tag{8}$$
$$\alpha_5 + \alpha_6 = -2 + hq_0 \tag{9}$$
$$\alpha_7 + \alpha_8 = -2 - hq_0 \tag{10}$$

In summation we obtain

$$\sum_{i=1}^{8} \alpha_i = -20 - h^2(p_0^2 + q_0^2) + h(p_1 - p_3) + h(q_2 - q_4) = -\alpha_0,$$

which is what we wanted to show. ∎

Now define $\tilde{u}^{h,k+1} \pm \tau + O(h^6) = u^h$ so that τ is our fourth order truncation error and the sign before τ is chosen such that τ is positive. For convenience we also define constants γ, γ_1, and γ_2 as follows:

$$\gamma = h(p_1 - p_3 + q_2 - q_4) - h^2(p_0^2 + q_0^2) \tag{11}$$
$$\gamma_1 = h(q_2 - q_4) - h^2 q_0^2 \tag{12}$$
$$\gamma_2 = h(p_1 - p_3) - h^2 p_0^2 \tag{13}$$

We are now ready to examine the upper bound on the fourth order truncation error at each step of Algorithm 1.

Lemma 3.2 *Consider Step 3 in Algorithm 1. Let $\tau_{old,max}$ be an upper bound for the truncation error on the previous iteration for the $(even, odd)$ and $(odd, even)$ grid points. Similarly let τ_{max} be an upper bound for the truncation error on the current iteration for the (odd, odd) grid points. If $\gamma < 18$ and the upper bounds on the truncation errors for Steps 4 and 5 are also strictly decreasing, then $\tau_{max} < \tau_{old,max}$.*

Proof We have a sixth order solution on $(even, even)$ grid points so

$$\tilde{u}^{h,k}_{i+1,j+1}, \tilde{u}^{h,k}_{i-1,j+1}, \tilde{u}^{h,k}_{i+1,j-1}, \tilde{u}^{h,k}_{i-1,j-1} \in \Omega^6_h.$$

Therefore the only contributions to the fourth order truncation error will be from $\tilde{u}^h_{i+1,j}$, $\tilde{u}^h_{i,j+1}$, $\tilde{u}^h_{i-1,j}$, and $\tilde{u}^h_{i,j-1}$. Let $\tau_{1,old}$, $\tau_{2,old}$, $\tau_{3,old}$, and $\tau_{4,old}$ be the fourth order truncation errors from the previous iteration on points 1, 2, 3, and 4 respectively. Using this notation we obtain the following power series expansions:

$$\begin{aligned}
u_{i+1,j} &= \tilde{u}^{h,k}_{i+1,j} \pm \tau_{1,old} + O(h^6) & u_{i,j+1} &= \tilde{u}^{h,k}_{i,j+1} \pm \tau_{2,old} + O(h^6)\\
u_{i-1,j} &= \tilde{u}^{h,k}_{i-1,j} \pm \tau_{3,old} + O(h^6) & u_{i,j-1} &= \tilde{u}^{h,k}_{i,j-1} \pm \tau_{4,old} + O(h^6)\\
u_{i+1,j+1} &= \tilde{u}^{h,k}_{i+1,j+1} + O(h^6) & u_{i-1,j+1} &= \tilde{u}^{h,k}_{i-1,j+1} + O(h^6)\\
u_{i+1,j-1} &= \tilde{u}^{h,k}_{i+1,j-1} + O(h^6) & u_{i-1,j-1} &= \tilde{u}^{h,k}_{i-1,j-1} + O(h^6)\\
u_{i,j} &= \tilde{u}^{h,k+1}_{i,j} + \tau + O(h^6)
\end{aligned}$$

Substituting these power series expansions into Eq. (4) and noting that the α, p, q, and f terms do not contribute to the truncation error, we obtain an expression for the fourth order truncation error of $\tilde{u}^{h,k+1}_{i,j}$ given by

$$\tau = \left| \frac{\alpha_1\tau_{1,old} + \alpha_2\tau_{2,old} + \alpha_3\tau_{3,old} + \alpha_4\tau_{4,old}}{\alpha_0} \right|.$$

We can now obtain an upper bound for τ since

$$\tau = \left| \frac{\alpha_1\tau_{1,old} + \alpha_2\tau_{2,old} + \alpha_3\tau_{3,old} + \alpha_4\tau_{4,old}}{\alpha_0} \right| \leq \left| \frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{\alpha_0} \right| \tau_{old,max}.$$

From Lemma 3.1 Eqs. (7) and (8) we have

$$(\alpha_1 + \alpha_3) + (\alpha_2 + \alpha_4) = -16 + h(p_1 - p_3 + q_2 - q_4) - h^2(p_0^2 + q_0^2).$$

Recall our definition of γ from Equation (11) so that

$$\frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{\alpha_0} = \frac{-16 + \gamma}{20 - \gamma}.$$

Our upper bound becomes

$$\tau = \left| \frac{\alpha_1\tau_1 + \alpha_2\tau_2 + \alpha_3\tau_3 + \alpha_4\tau_4}{\alpha_0} \right| \leq \left| \frac{-16 + \gamma}{20 - \gamma} \right| \tau_{old,max}.$$

Notice that

$$\left| \frac{-16 + \gamma}{20 - \gamma} \right| < 1$$

provided $\gamma < 18$. The upper bound can now be written as

$$\tau \leq \tau_{max} < \tau_{old,max}$$

so the upper bound for the truncation error is strictly decreasing at each iteration. ∎

Lemma 3.3 *Consider Step 4 in Algorithm 1. Let $\tau_{old,max}$ be an upper bound for the truncation error on the previous iteration for the $(even, odd)$ and (odd, odd) grid points. Similarly let τ_{max} be an upper bound for the truncation error on the current iteration for the $(odd, even)$ grid points. If $|-12+\gamma_1| < |20-\gamma_1-\gamma_2|$ and the upper bounds on the truncation errors for Steps 3 and 5 are also strictly decreasing, then $\tau_{max} < \tau_{old,max}$.*

Proof We have a sixth order solution on $(even, even)$ grid points so

$$\tilde{u}^{h,k}_{i+1,j}, \tilde{u}^{h,k}_{i-1,j} \in \Omega^6_h.$$

Let $\tau_{2,old}$, $\tau_{4,old}$, $\tau_{5,old}$, $\tau_{6,old}$, $\tau_{7,old}$, and $\tau_{8,old}$ be the fourth order truncation errors from the previous iteration on points 2, 4, 5, 6, 7, and 8 respectively. The fourth order truncation error for $\tilde{u}^h_{i,j}$ is then

$$\tau = \left| \frac{\alpha_2\tau^2_{2,old} + \alpha_4\tau^2_{4,old} + \alpha_5\tau^2_{5,old} + \alpha_6\tau^2_{6,old} + \alpha_7\tau^2_{7,old} + \alpha_8\tau^2_{8,old}}{\alpha_0} \right|.$$

We can now obtain an upper bound for τ since

$$\begin{aligned} \tau &= \left| \frac{\alpha_2\tau_{2,old} + \alpha_4\tau_{4,old} + \alpha_5\tau_{5,old} + \alpha_6\tau_{6,old} + \alpha_7\tau_{7,old} + \alpha_8\tau_{8,old}}{\alpha_0} \right| \\ &\leq \left| \frac{\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8}{\alpha_0} \right| \tau_{old,max}. \end{aligned}$$

From Lemma 3.1 Eqs. (8), (9), and (10) we have

$$(\alpha_2 + \alpha_4) + (\alpha_5 + \alpha_6) + (\alpha_7 + \alpha_8) = -12 + h(q_2 - q_4) - h^2(q_0^2).$$

Recall our definitions of γ_1 and γ_2 from Eqs. (12) and (13) so that

$$\frac{\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8}{\alpha_0} = \frac{-12 + \gamma_1}{20 - \gamma_1 - \gamma_2}.$$

The upper bound becomes

$$\tau = \left| \frac{\alpha_2\tau_{2,old} + \alpha_4\tau_{4,old} + \alpha_5\tau_{5,old} + \alpha_6\tau_{6,old} + \alpha_7\tau_{7,old} + \alpha_8\tau_{8,old}}{\alpha_0} \right| \leq \left| \frac{-12 + \gamma_1}{20 - \gamma_1 - \gamma_2} \right| \tau_{old,max}.$$

Notice that

$$\left| \frac{-12 + \gamma_1}{20 - \gamma_1 - \gamma_2} \right| < 1$$

provided $|-12 + \gamma_1| < |20 - \gamma_1 - \gamma_2|$. The upper bound can now be written as

$$\tau \leq \tau_{max} < \tau_{old,max}$$

so the upper bound for the truncation error is strictly decreasing at each iteration. ∎

Lemma 3.4 *Consider Step 5 in Algorithm 1. Let $\tau_{old,max}$ be an upper bound for the truncation error on the previous iteration for the $(odd, even)$ and (odd, odd) grid points. Similarly let τ_{max} be an upper bound for the truncation error on the current iteration for the $(even, odd)$ grid points. If $|-12+\gamma_2| < |20-\gamma_1-\gamma_2|$ and the upper bounds on the truncation errors for Steps 3 and 4 are also strictly decreasing, then $\tau_{max} < \tau_{old,max}$.*

Proof This can be proved using the techniques similar to Lemma 3.3. ∎

Corollary 3.5 *Lemmas 3.2, 3.3, and 3.4 are equally valid if we use a Gauss-Seidel method instead of a Jacobi method.*

Proof Define $\tau_{old,max}$ to be an upper bound for the truncation errors on current iteration points for points that have already been updated and previous iteration points for points that have yet to be updated. The same results we obtained using the Jacobi method will follow for the Gauss-Seidel method. ∎

3.2 Convergence results

We now prove sufficient conditions for Algorithm 1 to converge to a sixth order solution.

Theorem 3.6 *If the conditions*

$$\gamma < 18 \tag{14}$$

$$|-12+\gamma_1| < |20-\gamma_1-\gamma_2| \tag{15}$$

$$|-12+\gamma_2| < |20-\gamma_1-\gamma_2| \tag{16}$$

are satisfied on Ω_h, then Algorithm 1 will converge to a sixth order solution.

Proof If (14), (15), and (16) are satisfied then Lemmas 3.2, 3.3, and 3.4 can be applied. The fourth order truncation error is strictly decreasing at each iteration for (odd, odd), $(odd, even)$, $(even, odd)$ points and the $(even, even)$ points already have sixth order accuracy through Richardson Extrapolation. Since the fourth order truncation errors are strictly decreasing, the solution will converge to sixth order. ∎

We now give a convergence result in terms of the Reynolds number. The results in Theorem 3.6 are more general; however, the conditions on the Reynolds number used in the next theorem are much easier to verify in practice.

Theorem 3.7 *Algorithm 1 will converge to a sixth order solution if*

$$0 \le Re < -1+\sqrt{\frac{11}{3}}. \tag{17}$$

Proof It suffices to show (14), (15), and (16). Now recall that

$$\gamma = h(p_1-p_3+q_2-q_4)-h^2(p_0^2+q_0^2)$$

so

$$|\gamma| < 4Re+2Re^2.$$

It follows that (14) is satisfied if

$$2Re+Re^2 < 9. \tag{18}$$

Notice that

$$|-12+\gamma_1| \le 12+|\gamma_1|.$$

So (15) is satisfied if

$$12+|\gamma_1| < |20-\gamma_1-\gamma_2|$$

$$0 < |8 - \gamma_1 - \gamma_2| - |\gamma_1|$$
$$2|\gamma_2| + |\gamma_1| < 8.$$

In terms of the Reynolds number this becomes

$$2Re + Re^2 < \frac{8}{3}.$$

Similarly, (16) is satisfied by

$$2|\gamma_1| + |\gamma_2| < 8.$$

In terms of the Reynolds number this is

$$2Re + Re^2 < \frac{8}{3}.$$

Therefore, in order for Algorithm 1 to converge to a sixth order solution it is sufficient that $2Re + Re^2 < \frac{8}{3}$. Solving for Re we obtain Eq. (17). ∎

4 Numerical Results

In this section, we test the sixth order method (SOC) and compare the numerical results with the fourth order scheme (FOC). The codes are written in Fortran 77 programming language and tested on a PC with 2GB of memory and 2.0GHZ processor.

The 2D convection diffusion equation we tested is as follows:

$$\begin{cases} u(x,y) &= x^2y^2(1-x)(1-y), \\ p(x,y) &= Px(1-y), \\ q(x,y) &= Py(1-x), \end{cases}$$

where $(x,y) \in [0,1]^2$.

The initial guess for the V-Cycle on $4h$ grid is the zero vector. The stopping criteria for the operator based interpolation and the V-Cycle on $2h$ and h grid is 10^{-10}. The errors reported are the maximum absolute errors over the discrete grid of the finest level. For the SOC method, the number of iterations contains three parts. They are the number of V-Cycles for Ω_{2h}, the number of V-Cycles for Ω_h, and the number of iterations for the iterative interpolation combined with the Richardson extrapolation.

Table 1 shows the number of iterations, maximum error and the order of accuracy for different solution strategies with different mesh size. For $P = 10$, the cell Reynolds number is small, and we can see that the order of accuracy for the FOC scheme is nearly 4 as we expected. The numerical results illustrate that the SOC method solves the problem with better accuracy than the FOC scheme, and the order of accuracy is almost 6. By using the line relaxation in the multigrid method, the convergence rate of the FOC method and SOC method is independent of the grid size. We can also see that, by using our new sixth order compact scheme, the number of V-Cycles for Ω_h and Ω_{2h} are reduced, compared to the standard FOC scheme.

When the magnitude of the convection coefficients increases, i.e., when $P = 1000$, we find that the order of accuracy from the FOC scheme is reduced, especially for $n = 16$. It is clear that the SOC scheme can still increase the accuracy when Re increases. Since Re is a function of the mesh size h, the convergence is improved when h is refined. We can see that the number of iterations with the SOC scheme decreases when n increases.

Table 1: Comparison of CPU cost and solution accuracy with different mesh size with fixed P.

		P =10				P=1000			
n	strategy	# it	CPU	error	order	# it	CPU	error	order
16	FOC	7	0.003	3.45e-5	4.06	16	0.005	6.17e-3	2.85
	SOC	(5,6),18	0.002	2.41e-6	5.66	(15,15),96	0.003	4.19e-3	3.17
32	FOC	8	0.008	2.13e-6	4.02	17	0.013	4.15e-4	3.89
	SOC	(6,7),17	0.007	4.36e-8	5.79	(14,15),64	0.014	1.36e-4	4.95
64	FOC	8	0.028	1.33e-7	4.00	19	0.054	2.57e-5	4.01
	SOC	(7,7),15	0.029	7.36e-10	5.89	(15,15),35	0.057	2.97e-6	5.51
128	FOC	8	0.115	8.29e-9	4.00	19	0.239	1.60e-6	4.01
	SOC	(7,8),13	0.131	1.25e-11	5.87	(16,15),21	0.235	5.71e-8	5.70
256	FOC	8	0.746	5.18e-10	4.00	18	1.452	1.00e-7	4.00
	SOC	(8,8),21	0.968	1.97e-13	5.98	(15,14),18	1.441	9.95e-10	5.84

5 Concluding Remarks

We presented a sixth order compact difference scheme for the 2D convection diffusion equation. Theorem 3.7 proved that $0 \leq Re < -1 + \sqrt{\frac{11}{3}}$ is a sufficient condition for our scheme to converge to a sixth order solution. We also proved a more general condition in terms of the FOC scheme coefficients in Theorem 3.6. Testing numerically, we found that our sixth order scheme obtains higher accuracy results than the FOC scheme for Reynolds numbers significantly larger than the upper bound in Theorem 3.7.

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