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On the Reuse of Standard Preconditioners for Higher Order Time Discretizations of Parabolic PDEs

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In this abstract we will describe a preconditioner for some higher order time discretizations of parabolic problems. The preconditioner is optimal with respect to the spatial discretization parameters, that typically are the characteristic mesh size parameter h and the polynomial degree p. The preconditioner is also order optimal with respect to Δt. The only assumption is that there exists a preconditioner for the low order time discretization schemes such as Crank-Nicholson or implicit Euler. Such preconditioners are standard, c.f. e.g., [1], [2] and [3].

We study the model problem

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u, \quad \text{in } \Omega, t > 0 \\
u &= 0, \quad \text{on } \partial\Omega, t > 0 \\
u &= u_0, \quad \text{in } \Omega, t = 0.
\end{aligned}
$$

This equation is discretized in space and time to give the following linear system to be solved for each time level

$$Q_{kj}(\Delta t A)u^n = P_{kj}(\Delta t A)u^{n-1},$$

where Δt is the time stepping parameter, the two polynomials Q_{kj} and P_{kj} are the $(k, j)-$ Padé approximation to the exponential function and A is a discrete Laplacian. The polynomials are given by (c.f. [3]):

$$
\begin{aligned}
P_{kj}(\Delta t A) &= \sum_{i=0}^{k} \begin{pmatrix} k \\ i \end{pmatrix} \frac{(k+j-i)!}{(k+j)!}(\Delta t A)^i \\
Q_{kj}(\Delta t A) &= P_{jk}(-\Delta t A).
\end{aligned}
$$

$j\backslash k$	j	$j-1$	$j-2$
2	1.07	1.10	1.17
6	1.49	1.56	1.66
10	2.08	2.20	2.34

Table 1: Upper bound on the condition number for various values of j and k.

The proposed (exact) preconditioner is

$$R_{kj}(\Delta t A) = \left(I - \sqrt[j]{\frac{j!}{(j+k)!}} \Delta t A \right)^j .$$

Hence, R_{kj} is a standard preconditioner for a low order time discretization of a parabolic PDE, used j times. The coefficient before $\Delta t A$ is choosen such that the highest order term of $R_{kj}(\Delta t A)$ and $Q_{kj}(\Delta t A)$ are equal.

In Table 1 we show an upper bound on the condition number of the preconditioned system, using an exact preconditioner for R_{kj}. Further we proove the following lemma.

Lemma 1 *The condition number of the preconditioned system is bounded by*

$$\kappa \left(\left(R_{kj}(\Delta t A) \right)^{-1} Q_{kj}(\Delta t A) \right) < 1.8 \cdot 1.09^j .$$

Bibliography

[1] R. E. Bank and T. Dupont, An optimal order process for solving finite element equations, Math. Comp., Vol. 36, pp. 35–51, 1981.

[2] Maxim A. Olshanskii and Arnold Reusken, On the Convergence of a Multigrid Method for Linear Reaction-Diffusion Problems, Computing, Vol. 65(3), pp. 193–202, 2000.

[3] V. Thomée, Galerkin Finite Element Methods for Parabolic Problems, Springer-Verlag, 2nd edition, 1997.

Lemma 2 *The condition number of the preconditioned system is bounded by*

$$\kappa\left((R_{kj}(\Delta tA))^{-1}\,Q_{kj}(\Delta tA)\right) < 1.8 \cdot 1.09^{j}.$$

Lemma 3 *The condition number of the preconditioned system is bounded by*

$$\kappa\left((R_{kj}(\Delta tA))^{-1}\,Q_{kj}(\Delta tA)\right) < 1.8 \cdot 1.09^{j}.$$

Lemma 4 *The condition number of the preconditioned system is bounded by*

$$\kappa\left((R_{kj}(\Delta tA))^{-1}\,Q_{kj}(\Delta tA)\right) < 1.8 \cdot 1.09^{j}.$$

Lemma 5 *The condition number of the preconditioned system is bounded by*

$$\kappa\left((R_{kj}(\Delta tA))^{-1}\,Q_{kj}(\Delta tA)\right) < 1.8 \cdot 1.09^{j}.$$

Lemma 6 *The condition number of the preconditioned system is bounded by*

$$\kappa\left((R_{kj}(\Delta tA))^{-1}\,Q_{kj}(\Delta tA)\right) < 1.8 \cdot 1.09^{j}.$$