UNIFORM CONVERGENCE OF THE MULTIGRID V-CYCLE ON GRADED MESHES

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Abstract. We prove the uniform convergence of the multigrid V-cycle on graded meshes for corner-like singularities of elliptic equations on a bounded domain $\Omega \subset \mathbb{R}^2$. In particular, using some weighted Sobolev space $K_a^m(\Omega)$ and the method of subspace corrections with the elliptic projection decomposition estimate on $K_a^m(\Omega)$, we show that the multigrid V-cycle converges uniformly for piecewise linear functions with standard smoothers (Richardson, weighted Jacobi, Gauss-Seidel, etc.).

1. Introduction

The multigrid method has proved to be one of the most efficient techniques to solve the large systems of algebraic equations from the finite element discretization of elliptic boundary problems. Many details on the convergence properties of the multigrid method for elliptic equations can be found in monographs and survey papers by Bramble [11], Hackbusch[26], Trottenberg, Oosterlee and Schüller [29], Xu [32] and the references there in.

It is well known that the geometry of the boundary and the change of the boundary condition will influence the regularity of the solution [6, 9, 24, 25]. In particular, if a two-dimensional domain possesses reentrant corners, cracks, or there exist abrupt changes of boundary conditions, the solution of the elliptic boundary problem may have singularities in H^2. We define the singularities of these types the corner-like singularities, since they can be interpreted as corner singularities (artificial vertices are needed when the boundary condition changes) [28]. Graded meshes [3, 7, 9] are needed to obtain better numerical approximations to the solutions in these cases.

It is non-trivial to analyze the convergence rate of multigrid methods on such graded meshes due to the lack of the regularity of the solution and the non-uniformity of the mesh. One result for the uniform convergence of the multigrid method with the full regularity was given by Braess and Hackbusch [10]; in Brenner's paper [22], the analysis of the convergence rate for partial regularity was presented; Bramble, Pasciak, Wang and Xu [15] developed the convergence estimate without regularity assumptions with the L^2-projection decomposition. In addition, on graded meshes, based on the approximation property in [7], Yserentant [37] proved the uniform convergence of the multigrid W-cycle for piecewise linear functions by applying a particular iterative method on each level. There are also classical convergence proofs that use algebraic techniques and derive convergence results based on assumptions related to, but nevertheless different from the regularity of the underlying PDE [17, 30].

In this paper, we shall use the elliptic projection decomposition estimate on the weighted Sobolev space K_a^m and the method of subspace corrections to prove the uniform convergence of the multigrid V-cycle with standard subspace smoothers (Richardson, weighted Jacobi, Gauss-Seidel, etc.). To date, this approach has been known to work only for problems with full elliptic regularity. The reason we are able to obtain the uniform convergence result in cases of less regular solutions is that we use special graded meshes, tuned up to capture the correct behavior of the solutions near singularities.

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Let Ω be a bounded polygonal domain or a domain with cracks in $\mathbb{R}^2$. We consider the following elliptic equation with mixed boundary conditions

$$(1) \qquad \begin{cases} -\Delta u = f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega_D \\ \partial u/\partial n = 0 & \text{on} \quad \partial\Omega_N \end{cases}$$

as a prototype problem, where Ω_D and Ω_N are composed of segments of the boundary. Hence, possible corner-like singularities may appear in the solution. Denote by $H^1_D(\Omega) = \{u \in H^1(\Omega)|\ u = 0 \text{ on } \partial\Omega_D\}$ the space of all the functions in $H^1(\Omega)$ with trace 0 on $\partial\Omega_D$. Let Γ_j, $0 \leq j \leq J$, be a sequence of appropriately graded triangulations with triangles, which are nested on Ω. Denote by $\mathcal{M}_j$, $0 \leq j \leq J$, the finite element space associated to the linear Lagrange triangle [23]. Then,

$$\mathcal{M}_0 \subset \mathcal{M}_1 \subset \ldots \subset \mathcal{M}_j \subset \ldots \subset \mathcal{M}_J \subset H^1_D(\Omega).$$

We solve Equation (1) by looking for an approximation $u_J \in \mathcal{M}_J$, such that

$$a(u_J, v_J) = (Au_J, v_J) = (\nabla u_J, \nabla v_J) = \langle f, v_J \rangle, \quad \forall v_J \in \mathcal{M}_J, \quad f \in (H^1_D(\Omega))'$$

where A is the elliptic operator and $A = -\Delta$ for Equation (1). Let N_J be the dimension of the space $\mathcal{M}_J$. The following quasi-optimal rate of convergence for the finite element approximation $u_J \in \mathcal{M}_J$ can be recovered on Γ_J,

$$||u - u_J||_{H^1(\Omega)} \leq CN_J^{-1/2}||f||_{L^2(\Omega)}.$$

To be more precise, let n be the number of iterations on each level. The main objective of this paper is to prove the uniform convergence of the multigrid V-cycle with standard subspace smoothers (Richardson, weighted Jacobi, Gauss-Seidel, etc.) for piecewise linear functions on graded meshes. Moreover, we shall show that the convergence rate c satisfies

$$c \leq \frac{c_1}{c_1 + c_2 n},$$

where c_1 and c_2 are constants related to the elliptic equation and the smoother, independent of the mesh size. This will also provide a method to estimate the efficiency of other subspace smoothers on graded meshes.

The rest of this paper is organized as follows. In Section 2, we shall introduce the weighted Sobolev space $K^m_a(\Omega)$ for the boundary value problem (1) and the method of subspace corrections. We shall briefly describe the generation of the graded mesh on which the finite element solution converges to the exact solution of (1) quasi-optimally. In Section 3, the approximation and smoothing properties are followed by our main theorem.

2. Weighted Sobolev spaces and the method of subspace corrections

In this section, we shall first introduce the weighted Sobolev space $K^m_a(\Omega)$ and the mesh refinements to recover the quasi-optimal rates of convergence of the finite element solution. Then, we shall describe the method of subspace corrections and the technique for estimating the norm of the product of non-expensive operators.

2.1. **Weighted Sobolev spaces and graded meshes.** Let $(x, y) \in \Omega$ be an arbitrary point and $S = \{S_i\}$ be the set of (artificial) vertices of Ω, on which the solution has singularities in $H^2(\Omega)$. Denote by $r_i(x, y)$ the distance from (x, y) to the vertex $S_i \in S$. Let $\rho(x, y) = \prod_i r_i$ be the smooth function on Ω. Then, the weighted Sobolev space $K^m_a(\Omega)$, $m \geq 0$, is defined as follows [9, 27]

$$K^m_a(\Omega) = \{u \in H^m_{loc}(\Omega)|\ \rho^{i+j-a}\partial^i_x \partial^j_y u \in L^2(\Omega),\ i + j \leq m\}.$$

The K_a^m-norm and seminorm for any function $v \in K_a^m(\Omega)$ are

$$||v||^2_{K_a^m(\Omega)} := \sum_{i+j\le m} ||\rho^{i+j-a}\partial_x^i\partial_y^j v||^2_{L^2(\Omega)}$$
$$|v|^2_{K_a^m(\Omega)} := \sum_{i+j=m} ||\rho^{m-a}\partial_x^i\partial_y^j v||^2_{L^2(\Omega)}.$$

Note that ρ behaves like the distance funciton $r_i(x,y)$ near the vertex S_i. Thus, we have the following proposition and mesh refinements as in [9, 28].

Proposition 2.1. *We have $|v|_{K_1^1(\Omega)} \eqsim |v|_{H^1(\Omega)}$, $||v||_{K_1^0(\Omega)} \ge C||v||_{L^2(\Omega)}$ and the Poincare type inequality $||v||_{K_1^0(\Omega)} \le C|v|_{K_1^1(\Omega)}$ for $v \in K_1^1(\Omega) \cap \{v|_{\partial\Omega_D} = 0\}$.*

By $a \eqsim b$, we mean that there are positive constants C_1, C_2, such that $C_1 b \le a \le C_2 b$.

Let κ be the ratio of decay of triangles near the set S. Then, one can choose $\kappa = 2^{-1/\epsilon}$, for $\forall\epsilon < \min(\pi/\alpha_i)$, where α_i is the interior angle of vertex S_i, and $\alpha_i = 2\pi$ on the artificial vertices where $\partial\Omega_D$ and $\partial\Omega_N$ meet. We assume that no triangle consists of more than one point in S and any S_i is a vertex of some triangle in the initial trianglulation. Let $\Gamma_i = \{T_k\}$ be the triangulation after i refinements. Then, for the $i+1$th refinement, if the function ρ is bounded away from 0 on a triangle, new triangles are generated by connecting the mid-points of the old triangle. However, if S_i is one of the vertices of triangle $\triangle S_iBC$, we pick a point D on $\overline{S_iB}$ and another point E on $\overline{S_iC}$, such that the following holds for the ratios of the lengths

$$\kappa = \overline{S_iD}/\overline{S_iB} = \overline{S_iE}/\overline{S_iC}.$$

Then, triangle $\triangle S_iBC$ is divided into four triangles by connecting D, E, and the mid-point of $\overline{BC}$. (Fig. 2.1 and 2.2)

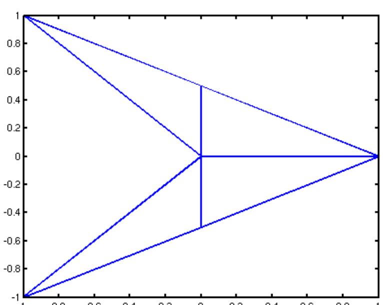

Fig. 2.1. Initial mesh

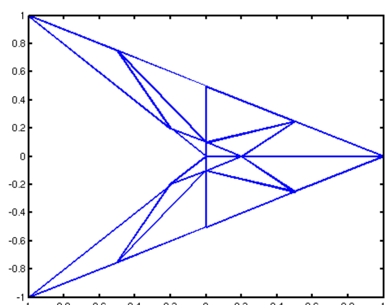

Fig. 2.2. Triangulation after one refinement, $\kappa = 0.2$

We note that other refinements in [3, 7] also satisfy this condition although they follow different constructions. We now conclude this subsection by restating the following theorem derived in [9, 28].

Theorem 2.2. *Let $u_i \in \mathcal{M}_i$ be the finite element solution of Equation (1). There exists a constant $B_1 = B_1(\Omega,\kappa,\epsilon)$, such that*

$$||u-u_i||_{H^1(\Omega)} \le B_1 \dim(\mathcal{M}_i)^{-1/2}||f||_{K^0_{\epsilon-1}(\Omega)} \le B_1\dim(\mathcal{M}_i)^{-1/2}||f||_{L^2(\Omega)},$$

for $\forall f \in K^0_{\epsilon-1}(\Omega)$, where $\mathcal{M}_i$ is the finite element space of linear functions on the graded mesh Γ_i as described in Introduction.

2.2. **The method of subspace corrections.** Let $H^1_D(\Omega) = \{u \in H^1(\Omega)|\ u = 0 \text{ on } \partial\Omega_D\}$ be the Hilbert space for Equation (1). Recall the graded triangulation Γ_j in the last subsection and the finite element space $\mathcal{M}_j \in H^1_D(\Omega)$ of piecewise linear funcitons on Γ_j. In addition, since the meshes are nested, we have

$$\mathcal{M}_0 \subset \mathcal{M}_1 \subset \ldots \subset \mathcal{M}_j \subset \ldots \subset \mathcal{M}_J \subset H^1_D(\Omega).$$

Let $A : H^1_D(\Omega) \to (H^1_D(\Omega))'$ be the differential operator for Equation (1). Then the weak form of Equation (1) is

$$a(u, v_i) = (Au, v_i) = (-\Delta u, v_i) = (\nabla u, \nabla v_i) = \langle f, v_i \rangle, \quad \forall v_i \in \mathcal{M}_i,$$

where the pairing $(\cdot, \cdot)$ is the inner product in $L^2(\Omega)$. In addition, $a(\cdot, \cdot)$ is a continuous bilinear form on $H^1_D(\Omega) \times H^1_D(\Omega)$ and also coercive by the Poincare inequality.

Meanwhile, let $Q_i,\ P_i : H^1_D(\Omega) \to \mathcal{M}_i,\ A_i : \mathcal{M}_i \to \mathcal{M}_i$ be the orthogonal projections and the restriction of A on $\mathcal{M}_i$, respectively by:

$$(Q_i u, v_i) = (u, v_i), \quad \forall u \in H^1_D(\Omega), \forall u_i, v_i \in \mathcal{M}_i,$$
$$a(P_i u, v_i) = a(u_i, v_i), \quad (Au_i, v_i) = (A_i u_i, v_i).$$

Let $\mathcal{N}_i = \{x^i_j\}$ be the set of node points in Γ_i and $\phi_k(x^i_j) = \delta_{j,k}$ be the linear finite element nodal basis function corresponding to node x^i_k. Then the ith level finite element discretization reads as: Find $u_i \in \mathcal{M}_i$, such that

$$A_i u_i = f_i, \tag{2}$$

where $f_i \in \mathcal{M}_i$ satisfying $(f_i, v_i) = \langle f, v_i \rangle$, $\forall v_i \in \mathcal{M}_i$.

The standard multigrid backslash cycle algorithm solves (2) by the iterative method

$$u^l_i = u^{l-1}_i + B_i(f_i - A_i u^{l-1}_i).$$

The operator $B_i : \mathcal{M}_i \to \mathcal{M}_i$, $0 \le i \le J$ is recursively defined as follows [33].

Algorithm 2.1. *Let $R_i \approx A^{-1}_i$, $i > 0$, denote a local relaxation method. For $i = 0$, define $B_0 = A^{-1}_0$. Assume that $B_{i-1} : \mathcal{M}_{i-1} \to \mathcal{M}_{i-1}$ is defined. Then,*

1. *Fine grid smoothing: For $u^0_i = 0$ and $k = 1, 2, \cdots, n$*

$$u^k_i = u^{k-1}_i + R_i(f_i - A_i u^{k-1}_i)$$

2. *Coarse grid correction: Find the corrector $e_{i-1} \in \mathcal{M}_{i-1}$ by the iterator B_{i-1}*

$$e_{i-1} = B_{i-1} Q_{i-1}(f_i - A_i u_i).$$

Then, $B_i f_i = u^n_i + e_{i-1}$.

In addition, Let B^v_J be the corresponding operator defined for the multigrid V-cycle. Then, it satisfies $I - B^v_J A_J = (I - B_J A_J)^*(I - B_J A_J)$ [33] . With the above algorithm, we have

$$\begin{aligned}(I - B_J A_J)u &= u - u^n_J - e_{J-1} = (I - T_J)u - e_{J-1} \\ &= (I - B_{J-1} A_{J-1})(I - T_J)u,\end{aligned}$$

where T_j is a linear operator and $T_i = R_i A_i P_i$ and $T_0 = P_0$ for $n = 1$. A recursive application of the above identity yields

$$(I - B_J A_J) = (I - T_0)(I - T_1) \cdots (I - T_J).$$

Define $||u||^2_a = a(u, u) = (Au, u)$ on Ω. Then, for the uniform convergence of the multigrid V-cycle, we need to show that

$$||I - B_J A_J||^2_a \le c < 1,$$

where c is independent of J.

Associated with each T_i, we introduce its symmetrization

$$\bar{T}_i = T_i + T^*_i - T^*_i T_i,$$

where T_i^* is the adjoint operator of T_i with respect to the inner product $a(\cdot,\cdot)$. By a well-known result of [34], we have the following estimate

$$||I - B_J A_J||_a^2 = \frac{c_0}{1+c_0},$$

where

$$c_0 \le \sup_{||v||_a=1} \sum_{i=1}^{J} a((\bar{T}_i^{-1} - I)(P_i - P_{i-1})v, (P_i - P_{i-1})v).$$

From this starting point to prove the uniform convergence,we will concentrate on the estimate on the constant c_0. One may notice that the above presentation is in terms of operators, while the matrix representation of the iteration is often used in practice. We conclude this section by providing the relation between the operator representation and the matrix representation.

Lemma 2.3. *If R_D is the corresponding matrix representation of the subspace smoother $R : \mathcal{M}_i \to \mathcal{M}_i$ and A_D is the matrix representation of A on $\mathcal{M}_i$, then the following identities hold for $u \in \mathcal{M}_i$ and $f \in \mathcal{M}_i$*

$$R(f) = \sum_j (\sum_k (R_D)_{j,k}(f, \phi_k))\phi_j$$
$$R(Au) = \sum_j (\sum_k (R_D)_{j,k}(A_D u_D^l)_k)\phi_j,$$

where $(R_D)_{j,k}$ represents the common element of the jth row and the kth column of the matrix.

Proof. The subspace correction method in terms of operators on a certain level can be written in the following way. Given the number of iterations n, then for $u^l \in \mathcal{M}_i$ and $l < n$,

$$u^l = u^{l-1} + R(f - Au^{l-1}).$$

Then, the inner product with ϕ_j leads to

$$(u^l, \phi_j) = (u^{l-1}, \phi_j) + (R(f), \phi_j) - (R(Au^{l-1}), \phi_j).$$

By comparison, the corresponding iteration in the matrix representation is

$$u_D^l = u_D^{l-1} + R_D(f_D - A_D u_D),$$

with $u^l = \sum_j (u_D^l)_j \phi_j$, $(f_D)_j = (f, \phi_j)$ and $(A_D)_{j,k} = a(\phi_j, \phi_k)$. For a better presentation, we introduce the mass matrix M, such that $M_{j,k} = (\phi_j, \phi_k)$. Thus, we have the following relations between the functions u^l, u^{l-1} and the vectors u_D^l, u_D^{l-1},

$$(u^l, \phi_j) = (Mu_D^l)_j, \quad (u^{l-1}, \phi_j) = (Mu_D^{l-1})_j.$$

Therefore, taking the matrix representation of the iteration into account, since they are equivalent, we have

$$(R(f), \phi_j) = (MR_D f_D)_j, \quad (R(Au^{l-1}), \phi_j) = (MR_D A_D u_D)_j.$$

Based on the definition of f_D and M, the linear operator R is defined as follows,

$$R(f) = \sum_j (\sum_k (R_D)_{j,k}(f, \phi_k))\phi_j.$$

As for $R(Au^l)$, since $(Au^l, \phi_j) = a(u^l, \phi_j) = (A_D u_D^l)_j$,

$$\begin{aligned} R(Au^l) &= \sum_j (\sum_k (R_D)_{j,k}(Au^l, \phi_k))\phi_j \\ &= \sum_j (\sum_k (R_D)_{j,k}(A_D u_D^l)_k)\phi_j, \end{aligned}$$

which completes the proof. □

3. Uniform convergence of the multigrid method on graded meshes

We now are going to estimate the constant c_0 introduced in Section 2 and to give the main theorem in this paper. For the proof of the uniform convergence, we shall start with some lemmas first. To better explain our results, we assume there is only one vertex S_0 of Ω, on which the solution of Equation (1) has a singularity in $H^2(\Omega)$. The same argument, however, will work on domains with multiple singular vertices. Recall the way we refine the mesh in Section 2. Denote by $T_i^{S_0}$ the initial triangles with vertex S_0. Thus, the mesh generation contains the following. After N refinements, $\cup T_i^{S_0}$ is chopped into $N+1$ sub-domains D_n, $0 \le n \le N$, such that $\rho(x,y) \approxeq \kappa^n$ on D_n for $0 \le n < N$ and $\rho(x,y) \le C\kappa^N$ on D_N. Then, sub-triangles are generated in these layers and the mesh size on D_n is $O(\kappa^n 2^{n-N})$, for $0 \le n \le N$. (Fig. 3.1 and 3.2)

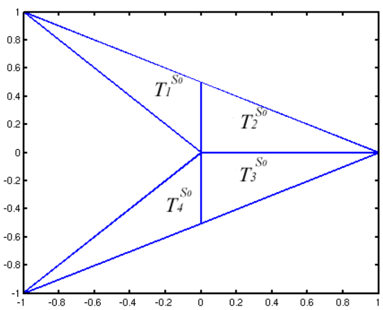

Fig. 3.1. Initial triangles with vertex S_0

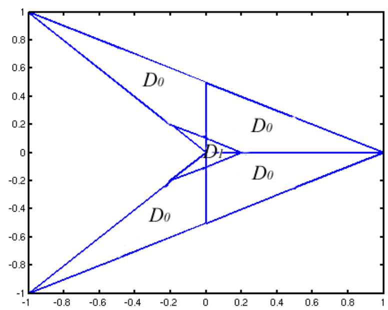

Fig. 3.2. Layer D_0 and D_1 after one refinement, $\kappa = 0.2$

Note that $\Omega = (\cup D_n) \cup (\Omega \backslash \cup D_n)$. Let ∂D_n be the boundary of D_n. Then, we define the continuous function $r_p(x,y)$ on Ω as follows. Let the restriction of r_p on every $T_i^{S_0} \cap D_n$ be a linear function, such that

$$r_p(x,y) = \begin{cases} (1/2\kappa)^n \quad \text{on} \quad \partial D_n \cap \partial D_{n+1} \cap T_i^{S_0}, \quad \text{for} \quad \forall T_i^{S_0} \\ (1/2\kappa)^N \quad \text{on} \quad D_N \\ \quad 1 \quad \text{otherwise,} \end{cases}$$

where N is the number of refinements. Thus $N = 1$ for Γ_1, $N = 2$ for Γ_2, and $N = i$ for Γ_i. Hence, $r_p \approxeq (1/2\kappa)^n$ on D_n. Recall that $\epsilon < 1$ is the parameter in κ, such that $\kappa = 2^{-1/\epsilon}$.

Denote by $(\cdot,\cdot)_{r_p}$ the weighted inner product with respect to r_p,

$$(u,v)_{r_p} = (r_p u, r_p v) = \int_\Omega r_p^2 uv.$$

Then, we have the estimate below.

Lemma 3.1.

$$(u_i - P_{i-1}u_i, u_i - P_{i-1}u_i)_{r_p} \le \frac{c_1}{N_i} a(u_i,u_i), \quad \forall u_i \in \mathcal{M}_i.$$

where $N_i = O(2^{2i})$ is the dimension of $\mathcal{M}_i$

Proof. This lemma can be proved by the duality argument as follows.

Consider the following boundary value problem

$$\begin{cases} -\Delta w = r_p^2(u_i - P_{i-1}u_i) \quad \text{in} \quad \Omega \\ w = 0 \quad \text{on} \quad \partial\Omega_D \\ \partial w/\partial n = 0 \quad \text{on} \quad \partial\Omega_N \end{cases}$$

Then, since $P_{i-1}w \in \mathcal{M}_{i-1}$, we have

$$\begin{aligned}(r_p(u_i - P_{i-1}u_i), r_p(u_i - P_{i-1}u_i)) &= (r_p^2(u_i - P_{i-1}u_i), u_i - P_{i-1}u_i) \\ &= (\nabla w, \nabla(u_i - P_{i-1}u_i)) \\ &= (\nabla(w - P_{i-1}w), \nabla(u_i - P_{i-1}u_i))\end{aligned}$$

Since r_p is continuous, we note that Δw is a continuous function on triangulation Γ_i that is derived after i refinements. Then, based on the arguments in Theorem 2.2, we have

$$\begin{aligned}|w - P_{i-1}w|^2_{H^1(\Omega)} &\leq (C_1/N_{i-1})||\Delta w||^2_{K^0_{\epsilon-1}(\Omega)} \\ &= (C_1/N_{i-1})(\sum_{n=0}^{i} ||\rho^{1-\epsilon}\Delta w||^2_{L^2(D_n)} + ||\rho^{1-\epsilon}\Delta w||^2_{L^2(\Omega\setminus\cup D_n)}) \\ &\leq (C_2/N_{i-1})(\sum_{n=0}^{i} ||\kappa^{n(1-\epsilon)}\Delta w||^2_{L^2(D_n)} + ||\Delta w||^2_{L^2(\Omega\setminus\cup D_n)}) \\ &= (C_2/N_{i-1})(\sum_{n=0}^{i} ||2^n 2^{-\frac{n}{\epsilon}}\Delta w||^2_{L^2(D_n)} + ||\Delta w||^2_{L^2(\Omega\setminus\cup D_n)}) \\ &= (C_2/N_{i-1})(\sum_{n=0}^{i} ||2^n \kappa^n \Delta w||^2_{L^2(D_n)} + ||\Delta w||^2_{L^2(\Omega\setminus\cup D_n)}) \\ &\leq (C/N_{i-1})(\sum_{n=0}^{i} ||r_p^{-1}\Delta w||^2_{L^2(D_n)} + ||\Delta w||^2_{L^2(\Omega\setminus\cup D_n)}) = (C/N_{i-1})||r_p^{-1}\Delta w||^2_{L^2(\Omega)}.\end{aligned}$$

Note that $N_i = O(N_{i-1})$. Therefore, we have the following estimates to complete the proof,

$$\begin{aligned}||u_i - P_{i-1}u_i||^2_{r_p} &\leq \frac{|w - P_{i-1}w|^2_{H^1}|u_i - P_{i-1}u_i|^2_{H^1}}{||(u_i - P_{i-1}u_i)||^2_{r_p}} \\ &= \frac{|w - P_{i-1}w|^2_{H^1}|u_i - P_{i-1}u_i|^2_{H^1}}{||r_p^{-1}\Delta w||^2_{L^2}} \\ &\leq \frac{c_1}{N_i}|u_i - P_{i-1}u_i|^2_{H^1} \leq \frac{c_1}{N_i}a(u_i, u_i).\end{aligned}$$

□

We here state another lemma regarding the smoother $\bar{R}_i$ on $\mathcal{M}_i$, where $\bar{R}_i$ is the symmetrization of R_i, $\bar{R}_i = R_i + R_i^t - R_i^t R_i$.

Lemma 3.2. *For any subspace smoother $\bar{R}_i : \mathcal{M}_i \to \mathcal{M}_i$, assume its corresponding matrix representation $\bar{R}_D$ satisfies*

$$V_D^t \bar{R}_D V_D \geq C V_D^t V_D, \quad \forall V_D \in \mathbb{R}^k, \quad C > 0,$$

on every level i, where C is a constant independent of i, and k is the dimension of $\bar{R}_D$. Then, the following estimate holds for the graded mesh Γ_i,

$$\frac{c_2}{N_i}(v, v) \leq (\bar{R}_i v, v)_{r_p}, \quad \forall v \in \mathcal{M}_i,$$

where N_i is the the dimension of $\mathcal{M}_i$ as in Lemma 3.1.

Proof. For any $v = \sum_j (V_D)_j \phi_j \in \mathcal{M}_i$, from Lemma 2.3, we have

$$(v, v) = (\sum_k (V_D)_k \phi_k, \sum_j (V_D)_j \phi_j) = V_D^T M V_D$$

$$(\bar{R}_i v, v)_{r_p} = (\sum_k (\bar{R}_D M V_D)_k r_p \phi_k, \sum_j (V_D)_j r_p \phi_j) = V_D^T M \bar{R}_D \tilde{M} V_D,$$

where M is the mass matrix and $(\tilde{M})_{j,k} = (r_p\phi_j, r_p\phi_k)$. Hence, $(\tilde{M})_{j,k} \approxeq 1/N_i$, since the mesh size at the support of ϕ_j is $O(\kappa^n 2^{n-i})$ if it is sitting in D_n. Thus, if we set $\xi = M^{1/2}V_D$, since M and $\tilde{M}$ are symmetric positive, it is equivalent to show that

$$\xi^T M^{1/2} \bar{R}_D \tilde{M} M^{-1/2} \xi \geq (C/N_i)\xi^T \xi \qquad (*).$$

Since the eigenvalues of $M^{1/2}\bar{R}_D\tilde{M}M^{-1/2}$ are also the eigenvalues of $\bar{R}_D\tilde{M}$, we have the following estimates for the eigenvalues of $\bar{R}_D\tilde{M}$ by setting $q = \tilde{M}^{1/2}w$,

$$w^T \bar{R}_D \tilde{M} w = q^T \tilde{M}^{-1/2} \bar{R}_D \tilde{M}^{1/2} q \geq C q^T q = C w^T \tilde{M} w \approxeq (C/N_i) w^T w,$$

since $\tilde{M}^{-1/2}\bar{R}_D\tilde{M}^{1/2}$ and $\bar{R}_D$ have the same eigenvalues, and $\tilde{M}^{-1} \approxeq N_i$. Therefore the minimum eigenvalue of $M^{1/2}\bar{R}_D\tilde{M}M^{-1/2}, \lambda_{min} \geq C/N_i$, and $(*)$ holds, which completes the proof. □

Lemma 3.3. *If the matrix representation R_D corresponding to the smoother R_i on $\mathcal{M}_i$ satisfies*

$$V_D^t R_D A_D V_D \leq \omega V_D^t V_D, \quad 0 < \omega < 2, \quad \forall V_D \in \mathbb{R}^k,$$

where A_D is the matrix form for A_i, and k is the dimension of R_D. Then, for $\forall v = \sum_j (V_D)_j \phi_i \in \mathcal{M}_i$,

$$(R_i A v, v)_{r_p} \leq \omega (v, v)_{r_p}.$$

Proof. By Lemma 2.3 and the definition of $(\cdot,\cdot)_{r_p}$, we have the following by setting $w = \tilde{M}^{1/2}V_D$

$$(R_i A v, v)_{r_p} = V_D^T A_D R_D \tilde{M} V_D = w^T \tilde{M}^{-1/2} R_D A_D \tilde{M}^{1/2} w \leq \omega V_D^T \tilde{M} V_D = \omega (v,v)_{r_p},$$

which follows the same argument as in the proof of Lemma 3.2. □

Remark 3.4. Recall the definition of the graded mesh. The triangles are shape-regular elements and the minimum angle of the triangles are bounded from 0. Therefore, the maximum eigenvalue of A_D is bounded and $(A_D)_{i,j} = O(1)$. In fact, one can verify that standard smoothers (Richardson, weighted Jacobi, Gauss-Seidel, etc.) satisfy Lemma 3.2 and 3.3, since $(R_D)_{i,j} = O(1)$.

Before we state our main theorem, we shall introduce some notation that will be used in the proof. Recall the operator T_i. Let R_i^t be the adjoint of R_i with respect to $(\cdot,\cdot)$. We can apply R_i and R_i^t alternatively with n smoothings on the ith level. We also define the operator G_i and G_i^* as follows,

$$G_i = I - R_i A_i, \quad G_i^* = I - R_i^t A_i.$$

With this type of subspace correction, we have

$$T_i = \begin{cases} P_i - (G^* G_i)^{\frac{n}{2}} P_i & \text{for even } n \\ P_i - G_i (G_i^* G_i)^{\frac{n-1}{2}} P_i & \text{for odd } n \end{cases}$$

Therefore, if we define

$$G_{i,n} = \begin{cases} G_i^* G_i & \text{for even } n \\ G_i G_i^* & \text{for odd } n, \end{cases}$$

since $P_i^2 = P_i$, the presentation for $\bar{T}_i$ is

$$\bar{T}_i = T_i + T_i^* - T_i^* T_i = (I - G_{i,n}^n) P_i.$$

With this setting-up, the uniform convergence of the multigrid method can be proved as follows.

Theorem 3.5. *On every triangulation Γ_i, suppose that the smoother on each subspace $\mathcal{M}_i$ satisfies Lemma 3.2 and 3.3. Then we have $||I - B_J A_J||_a^2 = \frac{c_0}{1+c_0} \leq \frac{c_1}{c_1+c_2 n}$.*

Proof. To estimate the constant c_0, we consider the decomposition $v = \sum_i v_i$ for any $v \in \mathcal{M}_J$ with

$$v_i = (P_i - P_{i-1})v.$$

Since $P_{i-1} = P_{i-1}P_i = P_i P_{i-1}$, Lemma 3.1 implies that

$$N_i(v_i, v_i)_{r_p} \leq c_1 a(v_i, v_i). \tag{3}$$

By the identity of Xu and Zikatanov [34] and the conditions in Lemma 3.2 and 3.3, we have

$$\begin{aligned}
a(\bar{T}_i(I-\bar{T}_i)v_i, v_i) &= a((I-G_{i,n}^n)^{-1}G_{i,n}^n v_i, v_i) \\
&= (\bar{R}_i^{-1}\bar{R}_i A_i (I-G_{i,n}^n)^{-1}G_{i,n}^n v_i, v_i) \\
&\leq \frac{N_i}{c_2}((I-G_{i,n})(I-G_{i,n}^n)^{-1}G_{i,n}^n v_i, v_i)_{r_p} \\
&\leq \frac{N_i}{c_2}\max_{t\in[0,1]}(I-t)(I-t^n)^{-1}t^n(v_i, v_i)_{r_p} \\
&\leq \frac{N_i}{c_2 n}(v_i, v_i)_{r_p}.
\end{aligned}$$

On the other hand, from the relation in (3),

$$\sum_{i=0}^{J} a(\bar{T}_i(I-\bar{T}_i)v_i, v_i) \leq \sum_{i=1}^{J} \frac{N_i}{c_2 n}(v_i, v_i)_{r_p} \leq \sum_{i=0}^{J} \frac{c_1}{c_2 n} a(v_i, v_i) = \frac{c_1}{c_2 n} a(v, v).$$

Therefore, $c_0 \leq \frac{c_1}{c_2 n}$ and consequently, the method of subspace corrections has the following convergence estimate for the multigrid V-cycle:

$$||I - B_J A_J||_a^2 = \frac{c_0}{1+c_0} \leq \frac{c_1}{c_1 + c_2 n},$$

which completes the proof. ☐

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