# INHERITANCE OF SPECTRAL EQUIVALENCE IN ALGEBRAIC MULTILEVEL METHODS

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**Abstract.** In this paper we consider the inheritance of different types of spectral equivalence in algebraic multilevel methods. This leads to new condition number bounds for multilevel methods applied as preconditioner. For specific C.B.S. constants we show that our bounds improve well-known bounds.

**Key words.** Multilevel Methods, AMLI Methods, Spectral Equivalence

**AMS subject classifications.** 65F10, 65F50, 65N22

## 1. Introduction and definitions.

In the last two decades, algebraic multigrid methods became a powerful tool for solving nonsingular linear systems of equations

$$Ax = b.$$

The idea of algebraic multigrid methods is to use only information on the matrix structure and the matrix entries. Algebraic multigrid methods are used as a solver or as a preconditioner of a Krylov subspace method. Most of the algebraic multilevel methods start with a partitioning of the unknowns into fine and coarse grid unknowns. Related to this ordering, the $n \times n$ system matrix $A$ can be permuted in a block $2 \times 2$ form such that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \tag{1.1}$$

Here we assume that this coarsening is already done in some way. Moreover, we assume that the matrix $A$ is symmetric positive definite (spd).

One class of multilevel methods or multilevel preconditioner can be derived by a block factorization of the matrix $A$ partitioned as in (1.1).

$$A = \begin{bmatrix} I & 0 \\ A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix},$$

where $S := (A/A_{11}) := A_{22} - A_{21}A_{11}^{-1}A_{12}$ is the Schur complement. The inverse $A^{-1}$ is given by

$$A^{-1} = \begin{bmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix}. \tag{1.2}$$

Now we use an approximation $\widetilde{A}_{11}$ of $A_{11}$ and an approximation $\widetilde{S}$ of $S$.

This leads to the *two-level preconditioner*:

DEFINITION 1.1. *Let $\widetilde{A}_{11}$ and $\widetilde{S}$ be approximations of $A_{11}$ and $(A/\widetilde{A}_{11}) := A_{22} - A_{21}\widetilde{A}_{11}^{-1}A_{12}$ so that $\widetilde{A}_{11}$, and $\widetilde{S}$ are spd. Define the two level*

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*preconditioner by*

$$C_{2L} = \begin{bmatrix} \widetilde{A}_{11}^{-1} + \widetilde{A}_{11}^{-1}A_{12}\widetilde{S}^{-1}A_{21}\widetilde{A}_{11}^{-1} & -\widetilde{A}_{11}^{-1}A_{12}\widetilde{S}^{-1} \\ -\widetilde{S}^{-1}A_{21}\widetilde{A}_{11}^{-1} & \widetilde{S}^{-1} \end{bmatrix}. \tag{1.3}$$

This is the classical two-level method or two-level preconditioner as proposed by Bank and Dupont [5] and studied in Axelsson and Gustafsson [1] and Vassilevski [15]. This two-level method can be seen as a basis of some multilevel methods and AMLI methods, see sections 2.2 and 2.3.

As described by Eijkhout and Vassilevski [6] the main tool in the analysis of multigrid or multilevel methods for spd matrices is the extended Cauchy-Bunyakowski-Schwarz inequality. If the spd matrix $A$ is partitioned as in (1.1) then there exists a constant $0 < \gamma < 1$, so that for all $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^n$:

$$\left| x_1^T A_{12} x_2 \right| \leq \gamma (x_1^T A_{11} x_1)^{\frac{1}{2}} (x_2^T A_{22} x_2)^{\frac{1}{2}}. \tag{1.4}$$

The constant $\gamma$ is called C.B.S. constant of $A$.

Another major tool or assumption made in the analysis of multigrid or multilevel methods is the concept of spectral equivalence of two spd matrices.

Definition 1.2. *Let $A, B \in \mathbb{R}^{n,n}$ with $A, B \succ 0$. Then $B$ is called spectrally equivalent to $A$ (short: $\sigma-$*general*) with $0 \leq \alpha \leq 1 \leq \beta$, if*

$$\alpha A \preceq B \preceq \beta A. \tag{1.5}$$

*Here, we write $A \succ 0$ ($A \succeq 0$), if $A$ is symmetric positive (semi) definite. We say that $A \succ B$ ($A \succeq B$) if $A - B \succ 0$ ($A - B \succeq 0$).*

In this paper we are interested in the inheritance of different types of spectral equivalence in multilevel methods. So we ask e.g. for the two-level method, if $\widetilde{A}_{11}$ and $\widetilde{S}$ are spectrally equivalent to $A_{11}$ and $(A/\widetilde{A}_{11})$ respectively, is $C_{2L}^{-1}$ then spectrally equivalent to $A$? We answer this question not only for the two-level case but also for the multilevel and AMLI methods. We will show this inheritance of spectral equivalence on each level of these methods. We focus on inheritance of properties from the coarsest level to the finest level. Therewith, we will establish some new condition number bounds for the two-level, multilevel and AMLI methods. We get an upper bound for the condition number which depends on assumptions on every level. To do so we need some more detailed versions of spectral equivalence.

Definition 1.3. *Let $A, B \in \mathbb{R}^{n,n}$ with $A, B \succ 0$.*

1. *$B$ is called* negative spectrally equivalent *(short: $\sigma-$negative) to $A$ with $0 \leq \alpha \leq 1$, if*

$$\alpha A \preceq B \preceq A. \tag{1.6}$$

2. *$B$ is called* positive spectrally equivalent *(short: $\sigma-$positive) to $A$ with $\beta \geq 1$, if*

$$A \preceq B \preceq \beta B. \tag{1.7}$$

3. *$B$ is called* mixed spectrally equivalent *(short: $\sigma-$mixed) to $A$ with $0 \leq \alpha \leq 1 \leq \beta$, if*

$$\alpha A \preceq B \preceq \beta A,\ A \npreceq B,\ B \npreceq A. \tag{1.8}$$

In this paper we will see that this differentiation has several advantages. First, in the literature different types of spectral equivalence are already assumed without having used the terminology (see Section 3). Our new notation helps to distinguish between these assumptions. Second and more important, using the different types of spectral equivalence one can derive condition number bounds in a clear way. Moreover, precious statements of the inheritance of different types of spectral equivalence can be formulated for multilevel methods. Note that the difference between Definition 1.2 and Definition 1.3 (3) is: If $B$ is $\sigma-$mixed to $A$ with $\alpha$ and $\beta$, we can ensure that $B^{-1}A$ has at least one eigenvalue less than or equal to one and one greater than or equal to one. So we can set $\alpha = \left(\lambda_{max}(B^{-1}A)\right)^{-1}$ and $\beta = \left(\lambda_{min}(B^{-1}A)\right)^{-1}$. If $B$ is $\sigma-$general to $A$ with $0 \leq \alpha \leq 1$ and $\beta \geq 1$, it is only $\alpha = min\{1, \left(\lambda_{max}(B^{-1}A)\right)^{-1}\}$ and $\beta = max\{1, \left(\lambda_{min}(B^{-1}A)\right)^{-1}\}$. Here $\lambda_{max}(B^{-1}A)$ and $\lambda_{min}(B^{-1}A)$ denote the largest and smallest eigenvalues of $B^{-1}A$.

The paper is organized as follows. In Section 2 we consider the inheritance of different types of spectral equivalence, which leads to new bounds for the condition numbers. In Subsection 2.1 we start with two-level methods, while Subsections 2.2 and 2.3 contain results for the multilevel and AMLI method. Section 3 describes known results and compares them with our new bounds.

**2. Inheritance of spectral equivalence properties.** In this section we are interested in inheritance of different types of spectral equivalence in multilevel methods. This means for the two-level method, if $\widetilde{A}_{11}$ and $\widetilde{S}$ satisfy a specific spectral equivalence to $A_{11}$ and $(A/\widetilde{A}_{11})$, what holds for $C_{2L}^{-1}$ to $A$? From these results we obtain easily condition number bounds using the following lemma.

LEMMA 2.1. *Let $A, B \in \mathbb{R}^{n,n}$ with $A, B \succ 0$.*

1. *If $B$ is $\sigma-$negative to $A$ with $\alpha$, then*
$$\sigma(B^{-1}A) \subset [1, \frac{1}{\alpha}] \text{ and } \kappa_2(B^{-1}A) \leq \frac{1}{\alpha},$$
2. *If $(A, B)$ is $\sigma-$positive with $\beta$, then*
$$\sigma(B^{-1}A) \subset [\frac{1}{\beta}, 1] \text{ and } \kappa_2(B^{-1}A) \leq \beta,$$
3. *If $(A, B)$ is $\sigma-$mixed with $\alpha$ and $\beta$, then*
$$\sigma(B^{-1}A) \subset [\frac{1}{\beta}, \frac{1}{\alpha}] \text{ and } \kappa_2(B^{-1}A) \leq \frac{\beta}{\alpha}.$$

*Proof.* The proof is a straight forward computation. ∎

**2.1. Two level methods.** Before we state our main result, we list some properties of the C.B.S. constant. These properties are needed in the proof of Theorem 2.2 For each vector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^n$ partitioned as $A$ in (1.1) we get

$$x^T A x \geq (1 - \gamma^2) x_1^T A_{11} x_1, \tag{2.1}$$
$$x^T A x \geq (1 - \gamma^2) x_2^T A_{22} x_2 \quad \text{and} \tag{2.2}$$
$$x^T A x \geq x_2^T (A/A_{11}) x_2 \tag{2.3}$$

see Axelsson and Vassilevski [4]. Notay established in [13] that one can compute the C.B.S. constant $\gamma$ by $1 - \gamma^2 = \lambda_{min}\left(A_{22}^{-1}(A/A_{11})\right) = \lambda_{min}\left(A_{11}^{-1}(A/A_{22})\right)$ resp. $\gamma^2 = \lambda_{max}\left(A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}\right)$.

Next, consider the two-level preconditioner defined by (1.3). Therefor we need the Inverse of $C_{2L}$ defined by (1.3), which is given by

$$C_{2L}^{-1} = \begin{bmatrix} \widetilde{A}_{11} & A_{12} \\ A_{21} & \widetilde{S} + A_{21}\widetilde{A}_{11}^{-1}A_{12} \end{bmatrix}. \tag{2.4}$$

THEOREM 2.2. *Let $A$ be spd and partitioned as in (1.1), $C_{2L}$ be defined by (1.3) and $\gamma, \gamma_C$ the C.B.S. constant of $A$ resp. $C_{2L}^{-1}$.*

1. *If $\widetilde{A}_{11}$ is $\sigma-$negative to $A_{11}$ with $\alpha$ and $\widetilde{S}$ is $\sigma-$negative to $(A/\widetilde{A}_{11})$ with $\xi$, then $C_{2L}^{-1}$ is $\sigma-$negative to $A$ with $\xi_{2L} = \left(\frac{\frac{1}{\alpha}-1}{1-\gamma_C^2} + \frac{1}{\xi}\right)^{-1}$.*
2. *If $\widetilde{A}_{11}$ is $\sigma-$positive to $A_{11}$ with $\beta$ and $\widetilde{S}$ is $\sigma-$positive to $(A/\widetilde{A}_{11})$ with $\theta$, then $C_{2L}^{-1}$ is $\sigma-$positive to $A$ with $\theta_{2L} = \frac{\beta-\gamma^2}{1-\gamma^2} \cdot \frac{\beta+\theta-1}{\beta}$.*
3. *If $\widetilde{A}_{11}$ is $\sigma-$mixed to $A_{11}$ with $\alpha$ and $\beta$ and $\widetilde{S}$ is $\sigma-$mixed to $(A/\widetilde{A}_{11})$ with $\xi$ and $\theta$, then $C_{2L}^{-1}$ is $\sigma-$mixed to $A$ with $\xi_{2L} = \left(\frac{\frac{1}{\alpha}-1}{1-\gamma_C^2} + \frac{1}{\xi}\right)^{-1}$ and $\theta_{2L} = \frac{\beta-\gamma^2}{1-\gamma^2} \cdot \frac{\beta+\theta-1}{\beta}$.*

*Proof.* The first step of the proof is compute the upper and lower bounds $\theta_{2L}$, $\xi_{2L}$, if we assume $\widetilde{A}_{11}$ is to $\sigma-$general to $A_{11}$ with $\alpha$ and $\beta$ and $\widetilde{S}$ is $\sigma-$general to $(A/\widetilde{A}_{11})$ with $\xi$ and $\theta$. Afterwards we show the specific spectral properties.

For the upper bound consider for $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^n$

$$\begin{aligned} x^T(C_{2L}^{-1} - A)x &= x^T \left( \begin{bmatrix} \widetilde{A}_{11} & A_{12} \\ A_{21} & \widetilde{S} + A_{21}\widetilde{A}_{11}^{-1}A_{12} \end{bmatrix} - \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \right) \\ &= x^T \left( \begin{bmatrix} \widetilde{A}_{11} - A_{11} & 0 \\ 0 & \widetilde{S} - (A_{22} - A_{21}\widetilde{A}_{11}^{-1}A_{12}) \end{bmatrix} \right) x \\ &= x_1^T(\widetilde{A}_{11} - A_{11})x_1 + x_2^T(\widetilde{S} - (A/\widetilde{A}_{11}))x_2. \end{aligned} \tag{2.5}$$

We obtain with (2.1)

$$\begin{aligned} x^T(C_{2L}^{-1} - A)x &= x_1^T(\widetilde{A}_{11} - A_{11})x_1 + x_2^T(\widetilde{S} - (A/\widetilde{A}_{11})x_2 \\ &\le (\beta-1)x_1^TA_{11}x_1 + (\theta-1)x_2^T(A/\widetilde{A}_{11})x_2 \\ &\le \frac{\beta-1}{1-\gamma^2}x^TAx + (\theta-1)x_2^T(A_{22} - A_{21}\widetilde{A}_{11}^{-1}A_{12})x_2. \end{aligned}$$

Because $\widetilde{A}_{11}$ is $\sigma-$general to $A_{11}$ we have $x_1^T\widetilde{A}_{11}x_1 \le \beta x_1^TA_{11}x_1$ and $-x_2^TA_{21}\widetilde{A}_{11}^{-1}A_{12}x_2 \le -\frac{1}{\beta}x_2^TA_{21}A_{11}^{-1}A_{12}x_2$. Therewith, (2.2) and (2.3)

$$\begin{aligned} x^T(C_{2L}^{-1} - A)x &\le \frac{\beta-1}{1-\gamma^2}x^TAx + (\theta-1)x_2^T(A_{22} - \frac{1}{\beta}A_{21}A_{11}^{-1}A_{12})x_2 \\ &= \frac{\beta-1}{1-\gamma^2}x^TAx + \frac{\theta-1}{\beta}x_2^T(A/A_{11})x_2 + (\theta-1)\left(1 - \frac{1}{\beta}\right)x_2^TA_{22}x_2 \\ &\le \left(\frac{\beta-1}{1-\gamma^2} + \frac{\theta-1}{\beta} + \frac{(\theta-1)(\beta-1)}{\beta(1-\gamma^2)}\right)x^TAx. \end{aligned}$$

So we found an upper bound independent of $\alpha$ and $\xi$:

$$x^TC_{2L}^{-1}x \le \theta_{2L}x^TAx \tag{2.6}$$

with

$$\theta_{2L} = 1 + \frac{\beta-1}{1-\gamma^2} + \frac{\theta-1}{\beta} + \frac{(\theta-1)(\beta-1)}{\beta(1-\gamma^2)} = \left(1 + \frac{\beta-1}{1-\gamma^2}\right)\left(1+\frac{\theta-1}{\beta}\right)$$
$$= \frac{\beta-\gamma^2}{1-\gamma^2}\cdot\frac{\beta+\theta-1}{\beta} \tag{2.7}$$

For the lower bound consider the assumption $\alpha A_{11} \preceq \widetilde{A}_{11}$ and $\xi(A/\widetilde{A}_{11}) \preceq \widetilde{S}$ which is equivalent to $A_{11} - \widetilde{A}_{11} \preceq (\frac{1}{\alpha}-1)\widetilde{A}_{11}$ and $(A/\widetilde{A}_{11}) - \widetilde{S} \preceq (\frac{1}{\xi}-1)\widetilde{S}$. With (2.5) we have for all $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$x^T(A - C_{2L}^{-1})x = x_1^T(A_{11} - \widetilde{A}_{11})x_1 + x_2T((A/\widetilde{A}_{11}) - \widetilde{S})x_2$$
$$= (\frac{1}{\alpha}-1)x_1^T\widetilde{A}_{11}x_1 + (\frac{1}{\xi}-1)x_2^T\widetilde{S}x_2. \tag{2.8}$$

Since

$$C_{2L} = \begin{bmatrix} I & 0 \end{bmatrix}^T \widetilde{A}_{11}^{-1} \begin{bmatrix} I & 0 \end{bmatrix} + \begin{bmatrix} A_{21}\widetilde{A}_{11}^{-1} & I \end{bmatrix}^T \widetilde{S}^{-1} \begin{bmatrix} A_{21}\widetilde{A}_{11}^{-1} & I \end{bmatrix},$$

$C_{2L}^{-1}$ is spd. Let $\gamma_C$ be the C.B.S. constant of $C_{2L}^{-1}$ then (2.1) gives $x_1^T\widetilde{A}_{11}x_1 \leq \frac{1}{1-\gamma_C^2}x^TC_{2L}^{-1}x$. Furthermore we obtain

$$(C_{2L}^{-1}/\widetilde{A}_{11}) = \widetilde{S} + A_{21}\widetilde{A}_{11}^{-1}A_{12} - A_{21}\widetilde{A}_{11}^{-1}A_{12} = \widetilde{S} \tag{2.9}$$

i.e. $\widetilde{S}$ is the Schur complement of $C_{2L}^{-1}$ respective $\widetilde{A}_{11}$. With (2.3) we have $x^TC_{2L}^{-1}x \geq x_2\widetilde{S}x_2$. By (2.8) we obtain

$$x^T(A - C_{2L}^{-1})x \leq \left(\frac{\frac{1}{\alpha}-1}{1-\gamma_C^2} + \frac{1}{\xi} - 1\right)x^TC_{2L}^{-1}x. \tag{2.10}$$

Thus

$$x^TC_{2L}^{-1}x \geq \left(\frac{\frac{1}{\alpha}-1}{1-\gamma_C^2} + \frac{1}{\xi}\right)^{-1}\cdot x^TAx. \tag{2.11}$$

As we mentioned in the beginning we proved $C_{2L}^{-1}$ is $\sigma-$general to $A$ with $\xi_{2L} = \left(\frac{\frac{1}{\alpha}-1}{1-\gamma_C^2} + \frac{1}{\xi}\right)^{-1}$ and $\theta_{2L} = \frac{\beta-\gamma^2}{1-\gamma^2}\cdot\frac{\beta+\theta-1}{\beta}$. To get the $\sigma-$mixed property we have to show $C_{2L}A$ has an eigenvalue which is greater than or equal to one and one which is less than or equal to one. Because $\widetilde{A}_{11}$ and $\widetilde{S}$ are $\sigma-$ mixed to $A_{11}$ and $(A/\widetilde{A}_{11})$, it exists $x_1 \neq 0$ and $\lambda \leq 1$ as well as $x_2 \neq 0$ and $\mu \leq 1$ (the other case follows analogously), so that

$$\widetilde{A}_{11}^{-1}A_{11}x_1 = \lambda x_1, \quad \widetilde{S}^{-1}(A/\widetilde{A}_{11})x_2 = \mu x_2. \tag{2.12}$$

So we have

$$\frac{x_1^TA_{11}x_1}{x_1^T\widetilde{A}_{11}x_1} = \lambda \leq 1 \text{ and } \frac{x_2^T(A/\widetilde{A}_{11})x_2}{x_2^T\widetilde{S}x_2} = \mu \leq 1 \tag{2.13}$$

respectively

$$x_1^T A_{11} x_1 \leq x_1^T \widetilde{A}_{11} x_1 \text{ and } x_2^T (A/\widetilde{A}_{11}) x_2 \leq x_2^T \widetilde{S} x_2. \tag{2.14}$$

Define $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^n$, then

$$\begin{aligned}
\frac{x^T A x}{x^T C_{2L}^{-1} x} = & \frac{x^T A x}{x^T \widetilde{A} x} \cdot \frac{x^T \widetilde{A} x}{x^T C_{2L}^{-1} x} \\
&= \left(1 + \frac{x_1^T (A_{11} - \widetilde{A}_{11}) x_1}{x^T \widetilde{A} x}\right) \cdot \left(1 + \frac{x^T ((A/\widetilde{A}_{11}) - \widetilde{S}) x}{x^T C_{2L}^{-1} x}\right) \\
&= \left(1 - \frac{x_1^T \widetilde{A}_{11} x_1}{x^T \widetilde{A} x} + \frac{x_1^T A_{11} x_1}{x^T \widetilde{A} x}\right) \cdot \left(1 - \frac{x_2^T \widetilde{S} x_2}{x^T C_{2L}^{-1} x} + \frac{x_2^T (A/\widetilde{A}_{11}) x_2}{x^T C_{2L}^{-1} x}\right) \\
&\leq \left(1 - \frac{x_1^T \widetilde{A}_{11} x_1}{x^T \widetilde{A} x} + \frac{x_1^T \widetilde{A}_{11} x_1}{x^T \widetilde{A} x}\right) \cdot \left(1 - \frac{x_2^T \widetilde{S} x_2}{x^T C_{2L}^{-1} x} + \frac{x_2^T \widetilde{S} x_2}{x^T C_{2L}^{-1} x}\right) = 1.
\end{aligned}$$

So we have seen $\lambda_{min} \leq \frac{v^T A v}{v^T C_{2L}^{-1} v} \leq 1$. As we mentioned before in the same way we get $\lambda_{max}(C_{2L} A) \geq 1$. At last we have to prove the inheritance of $\sigma-$negativity and $\sigma-$positivity. For the first we have to set $\beta = 1$ and get the lower bound. We have to check wether $A \succeq C_{2L}^{-1}$. With the assumptions $A_{11} \succeq \widetilde{A}_{11}$ and $(A/\widetilde{A}_{11}) \succeq \widetilde{S}$ it follows

$$A - C_{2L}^{-1} = \begin{bmatrix} A_{11} - \widetilde{A}_{11} & 0 \\ 0 & (A/\widetilde{A}_{11}) - \widetilde{S} \end{bmatrix} \succeq 0. \tag{2.15}$$

In the same way we get the inheritance of the $\sigma-$positivity. ◻

**2.2. Multilevel methods.** In this section we consider the *multilevel method*.

Definition 2.3. *Let $A^{(1)} \in \mathbb{R}^{n,n}$ be a symmetric positive definite matrix. For $k = 1, \ldots, L-1$, where $L-1$ is the coarsest level, we define the submatrices $A^{(k+1)}$ by $A^{(k+1)} = (A^{(k)}/\widetilde{A}_{11}^{(k)})$, which are partitioned as in (1.1). Given approximations $\widetilde{A}_{11}^{(k)}$ for $A_{11}^{(k)}$ and $C_{ML}^{(L)}$ for $A^{(L)^{-1}}$, s.t. $\widetilde{A}_{11}^{(k)} \succ 0$ and $C_{ML}^{(L)} \succ 0$, we define for $k = 1, \ldots, L-1$ recursively the multilevel preconditioner by*

$$C_{ML}^{(k)} = \begin{bmatrix} A_{11}^{(k)^{-1}} + A_{11}^{(k)^{-1}} A_{12}^{(k)} C_{ML}^{(k+1)} A_{21}^{(k)} A_{11}^{(k)^{-1}} & -A_{11}^{(k)^{-1}} A_{12}^{(k)} C^{(k+1)} \\ -C_{ML}^{(k+1)} A_{21}^{(k)} A_{11}^{(k)^{-1}} & C_{ML}^{(k+1)} \end{bmatrix}.$$

*In addition $\gamma^{(k)}, \gamma_C^{(k)}$ are the C.B.S. constants of $A^{(k)}$ resp. $C_{ML}^{(k)^{-1}}$.*

With Theorem 2.2 we get the following inheritance properties.

Theorem 2.4. *[8]*

1. *Let be $j \in \{1, \ldots, L-1\}$, $\widetilde{A}_{11}^{(k)}$ $\sigma-$negative to $A_{11}^{(k)}$ with $\alpha^{(k)}$ for every $k = j, \ldots, L-1$ and $C_{ML}^{(L)^{-1}}$ $\sigma-$negative to $A^{(L)}$ with $\xi_{ML}^{(L)}$, then $C_{ML}^{(j)^{-1}}$ is $\sigma-$negative to $A^{(j)}$ with $\xi_{ML}^{(j)} = \left(\sum_{k=j}^{L-1} \frac{\alpha^{(k)^{-1}} - 1}{1 - \gamma_C^{(k)}} + \xi^{(L)^{-1}}\right)^{-1}$.*
2. *Let be $j \in \{1, \ldots, L-1\}$, $\widetilde{A}_{11}^{(k)}$ $\sigma-$positive to $A_{11}^{(k)}$ with $\beta^{(k)}$ for every $k = j, \ldots, L-1$ and $C_{ML}^{(L)^{-1}}$ $\sigma-$positive to $A^{(L)}$ with $\theta_{ML}^{(L)}$, then $C_{ML}^{(j)^{-1}}$ is $\sigma-$positive to $A^{(j)}$ with $\theta_{ML}^{(j)} = \prod_{k=j}^{L-1} \frac{\beta^{(k)} - \gamma^{(k)^2}}{1 - \gamma^{(k)^2}} \cdot \theta_{ML}^{(L)}$.*

3. *Let be $j \in \{1,\dots,L-1\}$, $\widetilde{A}_{11}^{(k)}$ $\sigma-$mixed to $A_{11}^{(k)}$ with $\alpha^{(k)}$ and $\beta^{(k)}$ for every $k = j,\dots,L-1$ as well as ${C_{ML}^{(L)}}^{-1}$ is $\sigma-$mixed to $A^{(L)}$ with $\xi_{ML}^{(L)}$ and $\theta_{ML}^{(L)}$, then ${C_{ML}^{(j)}}^{-1}$ is $\sigma-$mixed to $A^{(j)}$ with $\xi_{ML}^{(j)} = \left(\sum_{k=j}^{L-1} \frac{{\alpha^{(k)}}^{-1}-1}{1-\gamma_C^{(k)}} + {\xi^{(L)}}^{-1}\right)^{-1}$ and*

*$\theta_{ML}^{(j)} = \prod_{k=j}^{L-1} \frac{\beta^{(k)}-{\gamma^{(k)}}^2}{1-{\gamma^{(k)}}^2} \cdot \theta_{ML}^{(L)}$.*

**2.3. AMLI method.** Now we analyze the *AMLI method* introduced by Axelsson and Vassilevski in [3, 4]. The AMLI method is a stabilized version of the multilevel method discussed in Subsection 2.2.

DEFINITION 2.5. *Let $A^{(1)} \in \mathbb{R}^{n,n}$ be a symmetric positive definite matrix. For $k = 1,\dots,L-1$, where $L-1$ is the coarsest level, we define the submatrices $A^{(k+1)}$ by $A^{(k+1)} = (A^{(k)}/\widetilde{A}_{11}^{(k)})$, which are partitioned as in (1.1). Given approximations $\widetilde{A}_{11}^{(k)}$ for $A_{11}^{(k)}$ and $C_{AMLI}^{(L)}$ for ${A^{(L)}}^{-1}$, so that $\widetilde{A}_{11}^{(k)} \succ 0$and $C_{AMLI}^{(L)} \succ 0$, we define for $k = 1,\dots,L-1$ the AMLI preconditioner by*

$$C_{AMLI}^{(k)} = \begin{bmatrix} {\widetilde{A}_{11}^{(k)}}^{-1} + {\widetilde{A}_{11}^{(k)}}^{-1} A_{12}^{(k)} {S_{AMLI}^{(k+1)}}^{-1} A_{21}^{(k)} {\widetilde{A}_{11}^{(k)}}^{-1} & -{\widetilde{A}_{11}^{(k)}}^{-1} A_{12}^{(k)} {S_{AMLI}^{(k+1)}}^{-1} \\ -{S_{AMLI}^{(k+1)}}^{-1} A_{21}^{(k)} {\widetilde{A}_{11}^{(k)}}^{-1} & {S_{AMLI}^{(k+1)}}^{-1} \end{bmatrix}$$

*where*

$$S_{AMLI}^{(k+1)} = A^{(k+1)} \left[ I - P_{\nu_{k+1}}^{(k+1)} \left( C_{AMLI}^{(k+1)} A^{(k+1)} \right) \right]^{-1}. \tag{2.16}$$

*Here $P_{\nu_{k+1}}^{(k+1)} \in \mathbb{R}_{\le \nu_{k+1}}[t]$ is a polynomial of degree $\nu_{k+1}$ satisfying $P_{\nu_{k+1}}^{(k+1)}(0) = 1$ and $P_{\nu_{k+1}}^{(k+1)}(t) < 1$ for $t \in [\underline{t}^{(k+1)}, \overline{t}^{(k+1)}]$, where $[\underline{t}^{(k+1)}, \overline{t}^{(k+1)}]$ contains the eigenvalues of $C_{AMLI}^{(k+1)} A^{(k+1)}$. In addition $\gamma^{(k)}, \gamma_C^{(k)}$ are the C.B.S. constants of $A^{(k)}$ resp. ${C_{AMLI}^{(k)}}^{-1}$.*

REMARK 2.6. *The best choice of the polynomial in (2.16) is one which is closed to zero. If we choose $P_{\nu_{k+1}}^{(k+1)}(t) = 1-t$ for every $k = 1,\dots,L-1$ we get $C_{AMLI}^{(k)} = C_{ML}^{(k)}$.*

Using Theorem 2.2 we obtain different results depending on the choice of the polynomial. Here we state a result for monotone polynomials for others see [8].

THEOREM 2.7. *[8] Let $Q_{\nu_{k+1}-1}^{(k+1)} \in \mathbb{R}_{\le \nu_{k+1}-1}[t]$ be defined by $Q_{\nu_{k+1}-1}^{(k+1)}(t) = \frac{1-P_{\nu_{k+1}}^{(k+1)}(t)}{t}$ for every $k = 1,\dots,L-1$ and*

1. *$\widetilde{A}_{11}^{(k)}$ be $\sigma-$negative to $A_{11}^{(k)}$ with $\alpha^{(k)}$, ${C_{AMLI}^{(L)}}^{-1}$ be $\sigma-$negative to $A^{(L)}$ with $\xi^{(L)}$ and $P_{\nu_{k+1}}^{(k+1)}(t) \le 0$ monotonically decreasing for $t \in [1, {\xi^{(k)}}^{-1}]$, then ${C_{AMLI}^{(j)}}^{-1}$ is $\sigma-$negative to $A^{(j)}$ with*
$$\xi^{(j)} = \left( \frac{{\alpha^{(j)}}^{-1}-1}{1-\gamma_C^{(j)}} + \sum_{k=j+1}^{L-1} \frac{{\alpha^{(k)}}^{-1}-1}{1-\gamma_C^{(k)}} \cdot Q_{\nu_k-1}^{(k)}({\xi^{(k)}}^{-1}) + {\xi^{(L)}}^{-1} \cdot Q_{\nu_L-1}^{(L)}({\xi^{(L)}}^{-1}) \right)^{-1}$$
*for every $j \in \{1,\dots,L-1\}$.*

2. *$\widetilde{A}_{11}^{(k)}$ be $\sigma-$positive to $A_{11}^{(k)}$ with $\beta^{(k)}$, ${C_{AMLI}^{(L)}}^{-1}$ be $\sigma-$positive to $A^{(L)}$ with $\theta^{(L)}$ and $P_{\nu_{k+1}}^{(k+1)}(t) \ge 0$ monotonically decreasing for $t \in [{\theta^{(k)}}^{-1}, 1]$, then ${C_{AMLI}^{(j)}}^{-1}$ is $\sigma-$positive to $A^{(j)}$ with $\theta^{(j)} = \prod_{k=j}^{L-1} \frac{\beta^{(k)}-{\gamma^{(k)}}^2}{1-{\gamma^{(k)}}^2} \cdot (Q_{\nu_{k+1}-1}^{(k+1)}({\theta^{(k+1)}}^{-1}))^{-1} \cdot \theta^{(L)}$ for every $j \in \{1,\dots,L-1\}$.*

3. *$\widetilde{A}_{11}^{(k)}$ be $\sigma-$mixed to $A_{11}^{(k)}$ with $\alpha^{(k)}$ and $\beta^{(k)}$, ${C_{AMLI}^{(L)}}^{-1}$ be $\sigma-$mixed to $A^{(L)}$ with $\xi^{(L)}$ and $\theta^{(L)}$, $t = 1$ is the only real root in $I := [{\theta^{(k)}}^{-1}, {\xi^{(k)}}^{-1}]$ of $P_{\nu_k}^{(k)}$ and $P_{\nu_{k+1}}^{(k+1)}$*

*monotonically decreasing on $I$, then $C_{AMLI}^{(j)^{-1}}$ is $\sigma-$mixed to $A^{(j)}$ with*
$\xi^{(j)} = \left( \frac{\alpha^{(j)^{-1}}-1}{1-\gamma_C^{(j)}} + \sum_{k=j+1}^{L-1} \frac{\alpha^{(k)^{-1}}-1}{1-\gamma_C^{(k)}} \cdot Q_{\nu_k-1}^{(k)}(\xi^{(k)^{-1}}) + \xi^{(L)^{-1}} \cdot Q_{\nu_L-1}^{(L)}(\xi^{(L)^{-1}}) \right)^{-1}$ *and*
$\theta^{(j)} = \prod_{k=j}^{L-1} \frac{\beta^{(k)}-\gamma^{(k)^2}}{1-\gamma^{(k)^2}} \cdot (Q_{\nu_{k+1}-1}^{(k+1)}(\theta^{(k+1)^{-1}}))^{-1} \cdot \theta^{(L)}$ *for every $j \in \{1, \ldots, L-1\}$ .*

**3. Comparison.** We start this section with a brief discussion of some well-known results for the AMLI method. In [2, 3, 4] the AMLI method is studied. In detail, Axelsson, Vassilevski [3] analyzed the method for $\widetilde{A}_{11}^{(k)} = A_{11}^{(k)}$ and the $\sigma-$positive property. Then in [4] an approximation for $A_{11}^{(k)}$ is used. Again only positive spectral equivalent approximations are used. No bounds are given for other approximations. Another difference to our approach is the use of the "coarse grid matrix" $A_{22}^{(k)}$ for the "next level matrix" instead of $(A/\widetilde{A}_{11}^{(k)})$ used here. Axelsson and Neytcheva consider in [2] the same methods as in [4]. But there the approximation of $A_{11}^{(k)}$ has a $\sigma-$negative property.

Thus we compare our bounds with well-known bounds, in detail we compare our bounds with the bounds given in [4, 2] and [16] for the two-level method.

First we consider the assumptions $\widetilde{A}_{11}$ is $\sigma-$positive to $A_{11}$ with $\beta$ and $\widetilde{S}$ is $\sigma-$positive to $(A/\widetilde{A}_{11})$ with $\theta$. Following the same computation as given in [4] we get: $C_{2L}^{-1}$ is $\sigma-$positive to $A$ with

$$\theta_{2L}^{[4]} = 1 + \frac{1}{1-\gamma^2} \left\{ \frac{1}{2}(\beta+\theta-2) + \left[ \frac{1}{4}(\beta-\theta)^2 + (\beta-1)(\theta-1)\gamma^2 \right]^{\frac{1}{2}} \right\}. \tag{3.1}$$

In Figure 3.1 we compare our new bound $\theta_{2L}$ with $\theta_{2L}^{[4]}$. For fixed $\beta$ we plot over $\gamma \in [0.1, 0.9]$ and $\theta \in [1, 5]$. We see that the new bound is better for large $\theta$ and $\gamma$ if $\gamma$ is close to one.

Figure 3.2 shows the differences of $\theta_{2L}$ and $\theta_{2L}^{[4]}$ for three C.B.S constants. The first plot is a comparison with $\gamma = \sqrt{\frac{3}{4}}$. This is the C.B.S. constant of an anisotropic orthotropic elliptic partial differential operator in 2D with the corresponding bilinear form $a(u,v) = \int_\Omega \alpha \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \beta \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} dx\, dy$. For details see [14]. The second plot based on $\gamma = \frac{16}{25}$ which is the C.B.S. constant for some Graph-Laplacians, see [10]. Last we consider the C.B.S. constant $\gamma = \sqrt{1-(\frac{9}{14})^2}$ of multilevel preconditioning of elliptic problems discretized by a class of discontinuous Galerkin methods, see [9]. The three plots below show in which area our new bound $\theta_{2L}$ improves $\theta_{2L}^{[4]}$ - marked in blue.

Next we are interested in the $\sigma$-negative spectral properties and compare our results with results given in [2]. There a factorization

$$\frac{x^T A x}{x^T C_{2L}^{-1} x} = \frac{x^T A x}{x^T \widetilde{A} x} \cdot \frac{x^T \widetilde{A} x}{x^T C_{2L}^{-1} x} \tag{3.2}$$

is used. Here $\widetilde{A}$ is defined by $\widetilde{A} = \begin{bmatrix} \widetilde{A}_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and assumed $\widetilde{A} \succ 0$. The resulting bound is $\xi_{2L}^{[2]} = \alpha_A \cdot \xi$ where $\alpha_A \geq 1$ satisfy the relation $\alpha_A x^T \widetilde{A} x \geq x^T A x \geq x^T \widetilde{A} x$. With similar arguments as in the proof of Theorem 2.2 we obtain $\xi_{2L}^{[2]} = \frac{1-\widetilde{\gamma}^2}{\frac{1}{\alpha}-\widetilde{\gamma}^2} \cdot \xi$

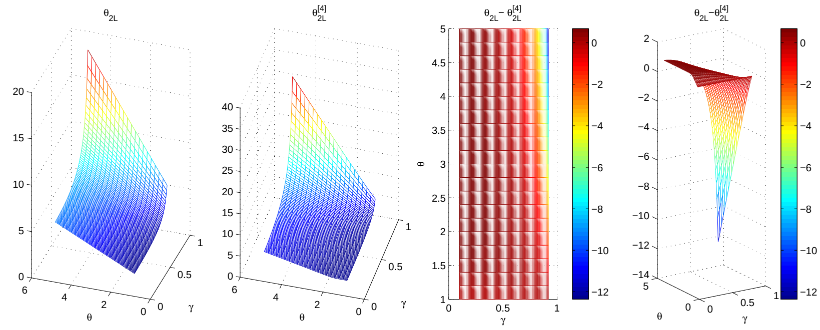


FIG. 3.1. *Comparison of* $\theta_{2L}$ *and* $\theta_{2L}^{[4]}$ *with* $\beta = 1.7$

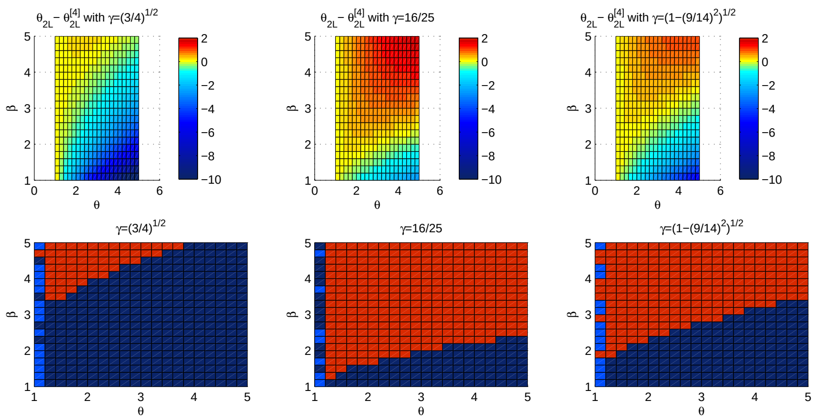


FIG. 3.2. *Comparison of* $\theta_{2L}$ *and* $\theta_{2L}^{[4]}$ *with* $\gamma = \sqrt{\frac{3}{4}}$, $\gamma = \frac{16}{25}$ *and* $\gamma = \sqrt{1-(\frac{9}{14})^2}$

where $\widetilde{\gamma}$ is the C.B.S. constant of $\widetilde{A}$. If we assume $\widetilde{\gamma} \approx \gamma_C$ it is possible to show

$$\xi_{2L} \geq \xi_{2L}^{[2]}. \tag{3.3}$$

At last we compare our results with [7, 16]. There the preconditioner $B$ defined by

$$I - B^{-1}A := (I - JM^{-1}J^TA)(I - PD^{-1}P^TA)(I - JM^{-1}J^TA) \tag{3.4}$$

is analyzed, where $J$ and $P$ are two rectangular matrices such that their number of rows equals $n$ and $M$ and $D$ are approximations of $J^TAJ$ and $P^TAP$. We define $\widetilde{M} = M^T(M + M^T - J^TAJ)^{-1}M$. In [7] Theorem 4.2 it is shown that under the assumptions $M \succ 0$, $\widetilde{M}$ is $\sigma-$positive to $J^TAJ$ with $\beta$ and $D$ is $\sigma-$positive to $P^TAP$ with $\theta$ then $B$ is $\sigma-$positive to $A$ with

$$\theta_{2L}^{[7]} := \frac{\beta + \theta - 1}{1 - \gamma^2}. \tag{3.5}$$

This is a very nice and general result. If we choose some specific matrices for $J$ and $P$ we get the *SMAMLI method* of [11, 12] or the *SMAMLI preconditioner* $C_{SMAMLI}$

defined by

$$I - C_{SMAMLI}A := \left( I - \begin{bmatrix} \widetilde{A}_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} A \right) \cdot \left( I - \begin{bmatrix} -\widetilde{A}_{11}^{-1}A_{12} \\ I \end{bmatrix} \widetilde{S}^{-1} \begin{bmatrix} -A_{21}\widetilde{A}_{11}^{-1} & I \end{bmatrix} A \right) \cdot \left( I - \begin{bmatrix} \widetilde{A}_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} A \right), \tag{3.6}$$

where $\widetilde{A}_{11}$ and $\widetilde{S}$ are approximations of $A_{11}$ and $(A/\widetilde{A}_{11})$.

With $\widetilde{A}_{11}^M := \widetilde{A}_{11}(2\widetilde{A}_{11} - A_{11})^{-1}\widetilde{A}_{11}$ the SMAMLI method can be written as a specific two-level method by using the approximation $\widetilde{A}_{11}^M$ of $A_{11}$ (see [11, 12]). If we now assume $\widetilde{A}_{11}^M$ is $\sigma-$positive to $A_{11}$ with $\beta$ and $\widetilde{S}$ is $\sigma-$positive to $\begin{bmatrix} -A_{21}\widetilde{A}_{11}^{-1} & I \end{bmatrix} A \begin{bmatrix} -\widetilde{A}_{11}^{-1}A_{12} \\ I \end{bmatrix}$ with $\theta$, we obtain with Theorem 2.2 that $C_{SMAMLI}^{-1}$ defined by (3.6) is $\sigma-$positive to $A$ with $\theta_{2L}^{SMAMLI} := \frac{\beta-\gamma^2}{1-\gamma^2} \cdot \frac{\beta+\theta-1}{\beta}$.
Since $\frac{\beta-\gamma^2}{\beta} = 1 - \frac{\gamma^2}{\beta} < 1$ we see

$$\theta_{2L}^{SMAMLI} \leq \theta_{2L}^{[7]}.$$

Hence our new bound improve the general bound given in [7, 16] applied to the SMAMLI method.

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