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On a complex Shifted-Laplace preconditioner for high wavenumber heterogeneous Helmholtz problems

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We are concerned with numerical solutions of the boundary value problem

$$\mathcal{L}u \equiv \left(\partial_{xx} + \partial_{yy} + k^2(x,y)\right) u = f \text{ in } \Omega \in \mathbb{R}^2, \tag{1}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial n} - iku\right) = 0 \text{ on } \Gamma = \partial\Omega. \tag{2}$$

This boundary value problem, so-called the Helmholtz problem, arises in many applications, e.g. acoustics, electromagnetics and geophysics. Here, we consider problems which mimic geophysical applications, in which inhomogeneous property and high wavenumber k are used. Furthermore, we are interested in the solution methods based on the construction of the iterants in the Krylov subspace.

For high wavenumbers, the matrix A obtained from any discretization of (1) and (2) has a property of indefiniteness, where the spectrum is highly distributed in the left and right complex plane. Furthermore, the condition number is also very large. These two properties are not favorable for Krylov subspace iterative methods. So far, there is no iterative method specially developed for indefinite systems. Applying Krylov methods on its normal equations representation, A^*A, does not provide a practical remedy since, even though the linear system can be made definite, the condition number is even worse. Another remedy is by suggesting a preconditioner.

In *[Erlangga et. al., TU Delft-AMA Report 03-01]*, we propose a class of preconditioners, called Shifted-Laplace preconditioner, acting on the Helmholtz problem. We also analyze some properties of these preconditioners. The preconditioners are constructed based on an operator

$$\mathcal{M} \equiv \partial_{xx} + \partial_{yy} - \alpha k^2(x,y), \tag{3}$$

where $\alpha \in \mathbb{C}$. We have previously found that $\alpha = i$, $i^2 = -1$ gives the best convergence for this class of preconditioners. We call the preconditioner (3) with $\alpha = i$ as the Complex Shifted-Laplace (CSL). We have shown also that (i) the spectrum of $M^{-1}A$ is bounded above by one, and (ii) the lower bound of the spectrum of $M^{-1}A$ is $\mathcal{O}(1/k^2)$. From the latter conclusion, we may expect that the convergence rate for increasing k is only determined by the smallest eigenvalue. We have used the preconditioner in BiCGSTAB. A significant reduction in the number of iteration is observed.

In this paper, we discuss another issue in relation with preconditioner solves of any Krylov subspace method. Previously, we solved the preconditioner exactly using direct methods. This process is very costly. Since the preconditioning matrix M is complex, symmetric positive definite (CSPD), several efficient methods can be implemented. We study the use of incomplete LU decomposition of M and multigrid as the preconditioner solver.

For ILU factorizations, level-of-fill ILU acting on M (or ILU(M)) is used. For constructing the LU factors, an algorithm based on regular stencil is used. The algorithm is not only fast in computing the LU factors (so reduce initialization cost) but also requires less storage. Only some diagonals should be stored, the remaining entries are recomputed as needed.

As for multigrid, we implement an algorithm as discussed in *[Oosterlee et. al., SIAM J. Sci. Comput. 19(1) (1998), pp.87–110]*, called MG1. MG1 is originally developed for real-valued matrix. For our applications no modification is made to MG1.

We compare the number of iteration and time to convergence from the two approaches with ILU preconditioner based on A (or ILU(A)) for various wavenumber k. The conclusions are as follows:

(1) ILU(M) results in better performance than ILU(A). The improvement becomes more significant as k increases.

(2) One V(1,1) multigrid iteration is adequate to further improve the computational performance. In comparison with the unpreconditioned and ILU(M)-preconditioned case, for sufficiently large k, the computational time is reduced almost by factor 10 and 3, respectively. Furthermore, the number of iteration can be reduced by factor of 50 and 8, respectively.

(3) Preconditioner solves using multigrid seem to be less sensitive to the inhomogeneity of the media. In comparison with constant media, only less

than 10 % increase of the number iteration is observed for multigrid. For ILU(M) and ILU(A), the number of iteration increases by almost 100 %.

In the future, the research will be geared towards an effective multigrid algorithm for CSPD linear systems, such that the similar reduction factor in the computation time and number of iteration can be achieved.