MULTIGRID METHODS FOR A BIHARMONIC PROBLEM

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Abstract. We develop multigrid algorithms for a quadratic C^0 interior penalty method for a biharmonic problem with essential and natural boundary conditions of the Cahn-Hilliard type. A multigrid solve for the Poisson problem with homogeneous Neumann boundary condition is used as a preconditioner in the smoothing steps for the fourth order problem, which significantly improves the performance of the multigrid methods. Numerical results are presented.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain, $V = \{v \in H^2(\Omega) : \partial v/\partial n = 0 \text{ on } \partial\Omega\}$, and $f \in L_2(\Omega)$. Consider the following biharmonic problem with essential and natural boundary conditions:
Find $u \in V$ such that

$$\int_\Omega D^2u : D^2v\,dx = \int_\Omega fv\,dx \qquad \forall\, v \in V, \tag{1.1}$$

where $D^2w : D^2v = \sum_{i,j=1}^2 w_{x_ix_j}v_{x_ix_j}$ denotes the inner product of the Hessian matrices of w and v.

The biharmonic problem (1.1) is the weak form of the following boundary value problem:

$$\Delta^2 u = f \qquad \text{in } \Omega, \tag{1.2a}$$

$$\partial u/\partial n = \frac{\partial \Delta u}{\partial n} = 0 \qquad \text{on } \partial\Omega, \tag{1.2b}$$

which is associated with the Cahn-Hilliard model for phase separation phenomena [12].

Under the compatibility condition $\int_\Omega fv\,dx = 0$, the biharmonic problem (1.1) is solvable and the solution is unique up to an additive constant.

Let $\hat{V} = \{v \in V : \int_\Omega v\,dx = 0\}$. Denote the unique solution of (1.1) in $\hat{V}$ by $\hat{u}$. Then we have

$$\|\hat{u}\|_{H^{2+\alpha}(\Omega)} \leq C_\Omega \|f\|_{L_2(\Omega)}.$$

The positive constant C_Ω depends only on Ω and the positive constant α, which will be referred to as the index of elliptic regularity, depends only on the interior angles of Ω. In contrast to the biharmonic problem with homogeneous Dirichlet boundary conditions, the index α is not always greater than $1/2$. In particular $\alpha = 1/3$ for L-shaped domains (cf. [2, 14, 15]).

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Let $\mathcal{T}_h$ be a quasi-uniform triangulation and $V_h \subset H^1(\Omega)$ be the associated $\mathbb{P}_2$ Lagrange finite element space [9], i.e., $V_h = \{v \in C^0(\bar{\Omega}) : v|_T \in \mathbb{P}_2(T) \quad \forall\, T \in \mathcal{T}_h\}$. We define a bilinear form on V_h by

$$\mathcal{A}_h(w_h, v_h) = \sum_{T\in\mathcal{T}_h} \int_T D^2 w_h : D^2 v_h \, dx + \sum_{e\in\mathcal{E}_h} \int_e \left\{\!\!\left\{ \frac{\partial^2 w_h}{\partial n_e^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial v_h}{\partial n_e} \right]\!\!\right] ds$$
$$+ \sum_{e\in\mathcal{E}_h} \int_e \left\{\!\!\left\{ \frac{\partial^2 v_h}{\partial n_e^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial w_h}{\partial n_e} \right]\!\!\right] ds + \sum_{e\in\mathcal{E}_h} \frac{\sigma}{|e|} \int_e \left[\!\!\left[\frac{\partial w_h}{\partial n_e} \right]\!\!\right] \left[\!\!\left[\frac{\partial v_h}{\partial n_e} \right]\!\!\right] ds,$$

where $\mathcal{E}_h$ is the set of all the edges in $\mathcal{T}_h$, σ (≥ 1) is a penalty parameter which only depends on the shape regularity of $\mathcal{T}_h$. The mean $\{\!\{\cdot\}\!\}$ and the jump $[\![\cdot]\!]$ are defined as follows.

Let e be an interior edge shared by two neighboring triangles T_+ and T_-, and n_e be the unit normal of e pointing from T_- to T_+. We define

$$\left\{\!\!\left\{ \frac{\partial^2 v_h}{\partial n_e^2} \right\}\!\!\right\} = \frac{1}{2}\left(\frac{\partial^2 v_h|_{T_+}}{\partial n_e^2} + \frac{\partial^2 v_h|_{T_-}}{\partial n_e^2} \right) \quad \text{and} \quad \left[\!\!\left[\frac{\partial v_h}{\partial n_e} \right]\!\!\right] = \frac{\partial v_h|_{T_+}}{\partial n_e} - \frac{\partial v_h|_{T_-}}{\partial n_e}.$$

For a boundary edge e, we take n_e to be the outward pointing unit normal of e and define

$$\left\{\!\!\left\{ \frac{\partial^2 v_h}{\partial n_e^2} \right\}\!\!\right\} = \frac{\partial^2 v_h}{\partial n_e^2} \quad \text{and} \quad \left[\!\!\left[\frac{\partial v_h}{\partial n_e} \right]\!\!\right] = -\frac{\partial v_h}{\partial n_e}.$$

Note that the value of $\mathcal{A}_h(w_h, v_h)$ is independent of the choice of T_+ and T_-.

Define $\hat{V}_h$ by

$$\hat{V}_h = \{v \in V_h : \int_\Omega v \, dx = 0\}.$$

The bilinear form $\mathcal{A}_h(\cdot,\cdot)$ is positive definite on $\hat{V}_h$ for σ sufficiently large [10], which is assumed to be the case. A C^0 interior penalty method [6] can be formulated for (1.1) as follows:
Find $\hat{u}_h \in \hat{V}_h$ such that

$$\mathcal{A}_h(\hat{u}_h, \hat{v}_h) = (f, \hat{v}_h) \quad \forall\, \hat{v}_h \in \hat{V}_h. \tag{1.3}$$

C^0 interior penalty methods [13, 10] have certain advantages over other finite element methods for fourth order problems. They are simpler than conforming methods which require C^1 elements. They come in a natural hierarchy that can capture smooth solutions efficiently, which is not the case for classical nonconforming methods. Unlike mixed methods they preserve the positive definiteness of the continuous problem and are easier to develop for more complicated problems [13].

Both *a priori* and *a posteriori* error analysis have been carried out for C^0 interior penalty methods [10, 8, 6]. In particular we have the following error estimate (cf. [6]):

$$\|\hat{u} - \hat{u}_h\|_h \leq C h^\alpha \sqrt{\sigma} \|f\|_{L_2(\Omega)},$$

where the constant C depends only on the shape regularity of $\mathcal{T}_h$ and

$$\|w\|_h = \sum_{T\in\mathcal{T}_h} |w|^2_{H^2(T)} + \sum_{e\in\mathcal{E}_h} \frac{\sigma}{|e|} \left\| \left[\!\!\left[\frac{\partial w}{\partial n_e} \right]\!\!\right] \right\|^2_{L_2(e)} \qquad \forall\, w \in V_h + H^2(\Omega).$$

Another important advantage of C^0 interior penalty methods comes from the fact that the underlying finite element spaces are standard spaces for second order problems. Therefore multigrid solves for second order problems can be easily implemented as a preconditioner. By using such a preconditioner in the smoothing steps of multigrid algorithms for fourth order problems, the performance of the smoother and hence the overall performance of the multigrid algorithms can be significantly improved. This approach was carried out in [11] for the biharmonic problem with the boundary conditions of the clamped plate. In this work we apply this approach to solve the more subtle biharmonic problem (1.2).

We will define the multigrid algorithms and discuss some theoretical issues in Section 2. Numerical results are presented in Section 3. The Appendix contains the description of a V-cycle algorithm for a second order singular Neumann boundary value problem that we use as a preconditioner in the smoothing steps,

2. Multigrid Algorithms

Let $\mathcal{T}_k$ $(k = 0, 1, \cdots)$ be a sequence of simplicial and quasi-uniform triangulations. We assume $\mathcal{T}_k$ is obtained from $\mathcal{T}_{k-1}$ by a uniform refinement (i.e., by connecting the midpoints). To better reflect the mesh hierarchy in our notations, from here on we use k instead of h to denote the subscript of the discrete finite element spaces, the functions, the linear functionals and the operators associated with $\mathcal{T}_k$.

We have $V_k \subsetneq V_{k+1}$ $(k \geq 0)$ and $\hat{V}_k \subsetneq \hat{V}_{k+1}$ $(k \geq 0)$. We denote the dual space of V_k by V_k', the natural injection from V_{k-1} to V_k by I_{k-1}^k and its transpose from V_k' to V_{k-1}' by I_k^{k-1}.

The dual space $\hat{V}_k'$ of $\hat{V}_k$ can be identified as a subspace of V_k':

$$\hat{V}_k' = \{\gamma_k \in V_k' : \langle \gamma_k, 1\rangle = 0\}, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ is the canonical bilinear form on the product of a vector space and its dual, and 1 is the unit constant function.

We define $A_k : V_k \to \hat{V}_k'$ by

$$\langle A_k v_k, w_k\rangle = \mathcal{A}_k(v_k, w_k) \quad \forall v_k, w_k \in V_k.$$

We can then rewrite the discrete problem (1.3) as $A_k \hat{u}_k = \psi_k$, where $\psi_k \in \hat{V}_k'$ satisfies $\langle \psi_k, v\rangle = \int_\Omega f v\, dx$ for all $v \in V_k$. Below we will define multigrid algorithms for the general equation (where $\phi_k \in \hat{V}_k'$)

$$A_k \hat{q}_k = \phi_k. \tag{2.2}$$

We need a preconditioner $S_k^{-1} : \hat{V}_k' \to \hat{V}_k$ in the smoothing steps of the multigrid algorithms. Given $\theta_k \in \hat{V}_k'$, $S_k^{-1}\theta_k$ is the output of a V-cycle multigrid algorithm (cf. Appendix A) for the following discrete singular Neumann problem: Find $\hat{s}_k \in \hat{V}_k$ such that

$$\int_\Omega \nabla \hat{s}_k \cdot \nabla \hat{v}_k\, dx = \langle \theta_k, \hat{v}_k\rangle \quad \forall\, \hat{v}_k \in \hat{V}_k. \tag{2.3}$$

The k^{th} level multigrid algorithms for the equation (2.2) with m pre-smoothing and m post-smoothing steps and initial guess $\hat{z}_0$ are defined as follows.

Algorithm 2.1. (V-cycle multigrid algorithm with output $MG_V(k, m, \hat{z}_0, \phi_k)$)

Input: integer k, integer m, $\hat{z}_o \in \hat{V}_k$, and $\phi_k \in \hat{V}_k'$

Output: $\hat{z}_{out} \in \hat{V}_k$

if $k = 0$, take $\hat{z}_{out}$ to be the exact solution of (2.2) in $\hat{V}_k$ and return;

otherwise proceed as follows:

pre-smoothing: for $j = 1, 2, \ldots, m$,

$$\hat{z}_j \leftarrow \hat{z}_{j-1} + \lambda_k S_k^{-1}(\phi_k - A_k \hat{z}_{j-1})$$

λ_k = a pre-defined damping factor of the form Ch_k^2 such that the spectral radius $\rho(\lambda_k S_k^{-1} A_k) < 2$

coarse grid correction:

$$\hat{z}_{m+1} \leftarrow \hat{z}_m + I_{k-1}^k MG_V(k-1, m, 0, r_{k-1})$$

$r_{k-1} = I_k^{k-1}(\phi_k - A_k \hat{z}_m)$

post-smoothing: for $j = m+2, m+3, \ldots, 2m+1$,

$$\hat{z}_j \leftarrow \hat{z}_{j-1} + \lambda_k S_k^{-1}(\phi_k - A_k \hat{z}_{j-1})$$

final output:

$$\hat{z}_{out} \leftarrow \hat{z}_{2m+1}.$$

Algorithm 2.2. (W-cycle multigrid algorithm with output $MG_W(k,m,\hat{z}_0,\phi_k)$)
The same as Algorithm 2.1 except that the coarse grid correction step is replaced by

$$\tilde{z} \leftarrow MG_W(k-1,m,0,r_{k-1}) \quad \text{and} \quad \hat{z}_{m+1} \leftarrow \hat{z}_m + I_{k-1}^k MG_W(k-1,m,\tilde{z},r_{k-1})$$

Algorithm 2.3. (F-cycle multigrid algorithm with output $MG_F(k,m,\hat{z}_0,\phi_k)$)
The same as Algorithm 2.1 except that the coarse grid correction step is replaced by

$$\tilde{z} \leftarrow MG_F(k-1,m,0,r_{k-1}) \quad \text{and} \quad \hat{z}_{m+1} \leftarrow \hat{z}_m + I_{k-1}^k MG_V(k-1,m,\tilde{z},r_{k-1}).$$

Since the preconditioned operator $S_k^{-1}A_k$ behaves like a second order discrete differential operator, the effect of the smoothing steps is similar to that for a second order problem. Based on this observation and numerical results, we believe the following result holds for the multigrid algorithms.

Let $MG(k,m,\hat{z}_0,\phi_k)$ be the output of the k^{th} level V-cycle, W-cycle or F-cycle algorithm for (2.2) with m pre-smoothing and m post-smoothing steps and initial guess $\hat{z}_0$. There exists a positive integer m_*, independent of k, such that

$$\|\hat{q}_k - MG(k,m,\hat{z}_0,\phi_k)\|_{\mathcal{A}_k} \le Cm^{-\alpha}\|\hat{q}_k - \hat{z}_0\|_{\mathcal{A}_k} \tag{2.4}$$

provided $m \ge m_*$. Here $\|\cdot\|_{\mathcal{A}_k} = \sqrt{\mathcal{A}_k(\cdot,\cdot)}$ is the energy norm and the constant C in (2.4) is also independent of k. In particular, the multigrid algorithms are contractions with contraction numbers uniformly bounded away from 1 on all levels. The convergence analysis of the multigrid algorithms will be carried out in a separate paper [7] using the theories developed in [1, 3, 4, 5].

Even though the asymptotic behavior of the contraction numbers of all three multigrid algorithms are identical, the W-cycle algorithm is more robust in the sense that the W-cycle algorithm requires a smaller number of smoothing steps to become a contraction. The F-cycle algorithm is less robust than the W-cycle algorithm but more robust than the V-cycle algorithm. When both W-cycle and F-cycle algorithms are contractions, their performances are almost identical, which makes the less expensive F-cycle algorithm very attractive.

Remark 2.4. The contraction number of a standard multigrid algorithm for a fourth order problem decreases at the rate of $m^{-\alpha/2}$. Thus the effect of m smoothing steps with the preconditioner is roughly equivalent to the effect of m^2 smoothing steps without the preconditioner. This improvement in the performance of the multigrid algorithms is observed in the numerical experiments.

3. Numerical Results

In the following experiments, the preconditioner is taken to be a V-cycle multigrid solve for the discrete Neumann problem (2.3) with one pre-smoothing step and one post-smoothing step.

The first set of experiments is for the unit square where the initial mesh has two elements. We report the contraction numbers in Tables 3.1–3.3 for V-cycle, W-cycle and F-cycle algorithms (Algorithms 2.1–2.3). These contraction numbers are similar to the contraction numbers for second order problems. It is observed that the V-cycle (resp. W-cycle and F-cycle) algorithm is a contraction for $m \ge 4$ (resp. $m \ge 2$ and $m \ge 3$). Moreover, the performances of the W-cycle and F-cycle algorithms are almost identical for $m \ge 3$.

For comparison, we also report the contraction numbers for the multigrid algorithms that do not use a preconditioner in the smoothing steps (Tables 3.4–3.6). The contraction numbers in Table 3.1 (resp. Tables 3.2 and 3.3) for m smoothing steps are comparable to the contraction numbers in Table 3.4 (resp. Tables 3.5 and 3.6) for m^2 smoothing steps.

Asymptotic rates of decrease of the contraction numbers on level 7 for the V-cycle algorithms with and without a preconditioner are plotted as a log-log graph in Figure 3.1. The rate of decrease of the preconditioned scheme looks like m^{-1} while that of the un-preconditioned scheme looks like $m^{-1/2}$. This agrees with the estimate (2.4) since the index of elliptic regularity α equals 1 for a square. It also agrees with Remark 2.4.

k \ m	4	5	6	7	8	9	10	11	12	13
1	0.212	0.126	0.0813	0.0594	0.0442	0.0332	0.0252	0.0192	0.0147	0.0114
2	0.329	0.223	0.190	0.164	0.142	0.124	0.109	0.0967	0.0861	0.0771
3	0.412	0.342	0.308	0.279	0.255	0.234	0.217	0.203	0.190	0.179
4	0.479	0.420	0.386	0.357	0.334	0.314	0.296	0.282	0.266	0.257
5	0.537	0.467	0.434	0.408	0.386	0.367	0.351	0.336	0.324	0.312
6	0.578	0.494	0.462	0.436	0.415	0.396	0.380	0.366	0.353	0.341
7	0.619	0.503	0.472	0.446	0.425	0.406	0.391	0.376	0.364	0.351

TABLE 3.1. Contraction numbers for the V-cycle algorithm (Algorithm 2.1) on the square.

k \ m	2	3	4	5	6	7	8	9	10	11
1	0.661	0.368	0.212	0.126	0.0813	0.0594	0.0442	0.0332	0.0252	0.0192
2	0.483	0.360	0.291	0.241	0.203	0.172	0.148	0.128	0.112	0.0983
3	0.475	0.375	0.335	0.282	0.263	0.229	0.215	0.195	0.182	0.171
4	0.455	0.383	0.335	0.308	0.287	0.270	0.256	0.244	0.233	0.223
5	0.456	0.384	0.344	0.315	0.297	0.279	0.267	0.255	0.245	0.237
6	0.455	0.384	0.344	0.316	0.297	0.280	0.268	0.256	0.248	0.239
7	0.455	0.384	0.344	0.317	0.297	0.281	0.269	0.258	0.248	0.240

TABLE 3.2. Contraction numbers for the W-cycle algorithm (Algorithm 2.2) on the square.

k \ m	3	4	5	6	7	8	9	10	11	12
1	0.368	0.212	0.126	0.0813	0.0594	0.0442	0.0332	0.0252	0.0192	0.0147
2	0.360	0.291	0.241	0.203	0.172	0.148	0.128	0.112	0.0983	0.0872
3	0.375	0.334	0.282	0.262	0.229	0.215	0.195	0.182	0.171	0.161
4	0.383	0.336	0.308	0.287	0.270	0.256	0.244	0.233	0.223	0.215
5	0.384	0.344	0.315	0.297	0.279	0.267	0.255	0.245	0.237	0.229
6	0.385	0.344	0.316	0.297	0.280	0.268	0.256	0.248	0.239	0.230
7	0.386	0.345	0.317	0.297	0.281	0.269	0.258	0.248	0.240	0.232

TABLE 3.3. Contraction numbers for the F-cycle algorithm (Algorithm 2.3) on the square.

k \ m	21	22	23	24	25	26	27	28	29	30
1	0.428	0.410	0.392	0.376	0.361	0.346	0.332	0.320	0.307	0.296
2	0.646	0.614	0.583	0.555	0.529	0.504	0.481	0.459	0.439	0.420
3	0.770	0.728	0.690	0.654	0.621	0.591	0.562	0.535	0.510	0.487
4	0.844	0.797	0.753	0.713	0.676	0.641	0.609	0.579	0.551	0.525
5	0.895	0.843	0.795	0.752	0.711	0.674	0.639	0.607	0.577	0.548
6	0.931	0.876	0.826	0.780	0.737	0.697	0.661	0.627	0.595	0.565
7	0.960	0.902	0.849	0.801	0.757	0.715	0.677	0.642	0.609	0.578

TABLE 3.4. Contraction numbers for the V-cycle algorithm with no preconditioner on the square.

k \ m	7	8	9	10	11	12	13	14	15	16
1	0.967	0.893	0.830	0.774	0.725	0.682	0.642	0.607	0.575	0.545
2	0.824	0.692	0.604	0.541	0.498	0.463	0.437	0.414	0.395	0.377
3	0.696	0.534	0.462	0.447	0.435	0.424	0.415	0.405	0.397	0.389
4	0.514	0.484	0.474	0.457	0.448	0.437	0.429	0.420	0.413	0.406
5	0.527	0.500	0.489	0.471	0.461	0.449	0.440	0.431	0.424	0.417
6	0.530	0.501	0.490	0.472	0.463	0.450	0.442	0.433	0.425	0.418
7	0.532	0.505	0.492	0.474	0.464	0.451	0.442	0.433	0.425	0.418

TABLE 3.5. Contraction numbers for the W-cycle algorithm with no preconditioner on the square.

k \ m	14	15	16	17	18	19	20	21	22	23
1	0.607	0.575	0.545	0.518	0.493	0.470	0.448	0.428	0.410	0.392
2	0.414	0.395	0.377	0.362	0.348	0.335	0.323	0.312	0.302	0.292
3	0.406	0.398	0.390	0.382	0.375	0.369	0.363	0.357	0.351	0.346
4	0.418	0.413	0.406	0.400	0.395	0.390	0.385	0.380	0.376	0.372
5	0.431	0.424	0.417	0.410	0.405	0.399	0.394	0.390	0.386	0.382
6	0.434	0.425	0.418	0.412	0.406	0.401	0.396	0.391	0.387	0.383
7	0.546	0.425	0.418	0.412	0.406	0.400	0.395	0.391	0.387	0.383

TABLE 3.6. Contraction numbers for F-cycle algorithm with no preconditioner on the square.

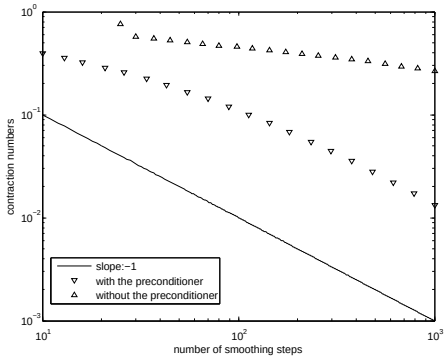

FIGURE 3.1. Asymptotic rate of decrease for the V-cycle algorithms (level 7) on the square.

We also carry out numerical experiments for the L-shaped domain with vertices $(0,0)$, $(1,0)$, $(1,1)$, $(-1,1)$, $(-1,-1)$ and $(0,-1)$. The initial mesh consists of six isosceles triangles sharing $(0,0)$ as a common vertex. We report the contraction numbers for the W-cycle algorithm with/without the preconditioner in Tables 3.7 and 3.8. Again the contraction numbers in Table 3.7 for m smoothing steps are similar to the contraction numbers in Table 3.8 for m^2 smoothing steps.

The asymptotic rates of decrease for the W-cycle algorithms on level 7 are plotted in a log-log graph in Figure 3.2. The rate of decrease of the preconditioned scheme looks like $m^{-1/3}$, which agrees with the

estimate (2.4) since the index of elliptic regularity for the L-shaped domain is $\frac{1}{3}$. The rate of decrease of the un-preconditioned scheme looks like $m^{-1/6}$, which also agrees with Remark 2.4.

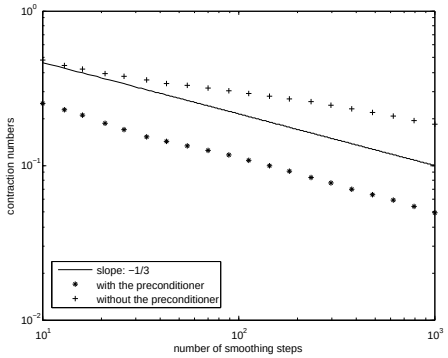

FIGURE 3.2. Asymptotic rate of decrease for the W-cycle algorithm (level 7) on the L-shaped domain.

k \ m	3	5	7	9	11	13	15	17	19	21	23
1	0.319	0.187	0.125	0.105	0.0913	0.0798	0.0699	0.0614	0.0540	0.0476	0.0420
2	0.383	0.273	0.206	0.161	0.139	0.132	0.125	0.119	0.113	0.108	0.103
3	0.390	0.302	0.238	0.208	0.182	0.163	0.152	0.148	0.144	0.141	0.137
4	0.386	0.309	0.271	0.245	0.224	0.208	0.193	0.181	0.170	0.161	0.153
5	0.384	0.315	0.279	0.255	0.237	0.222	0.209	0.198	0.189	0.180	0.172
6	0.384	0.316	0.281	0.257	0.240	0.226	0.213	0.203	0.193	0.185	0.177
7	0.387	0.317	0.281	0.258	0.240	0.226	0.214	0.203	0.194	0.186	0.178

TABLE 3.7. Contraction numbers for the W-cycle algorithm with the preconditioner on the L-shaped domain.

k \ m	5	7	9	11	13	15	17	19	21	23
1	0.943	0.788	0.680	0.600	0.537	0.486	0.443	0.407	0.375	0.347
2	0.790	0.585	0.505	0.459	0.426	0.394	0.375	0.358	0.342	0.328
3	0.666	0.512	0.469	0.456	0.434	0.416	0.400	0.386	0.373	0.362
4	0.580	0.519	0.484	0.454	0.434	0.418	0.405	0.394	0.385	0.376
5	0.581	0.527	0.491	0.465	0.444	0.427	0.414	0.402	0.392	0.384
6	0.587	0.531	0.494	0.467	0.446	0.429	0.415	0.404	0.394	0.386
7	0.587	0.530	0.493	0.467	0.446	0.429	0.415	0.404	0.394	0.386

TABLE 3.8. Contraction numbers for the W-cycle algorithm with no preconditioner on the L-shaped domain.

As we mentioned earlier, the boundary value problem (1.2) is more subtle than the biharmonic problem with the boundary conditions of the clamped plate. This can be seen from the third set of numerical

experiments for a nonstandard quadrilateral domain with vertices $(-1,1)$, $(0,0)$, $(1,1.2)$ and $(0,1)$ depicted in Figure 3.3 (together with an initial mesh). We report the contraction numbers of the W-cycle algorithm with/without a preconditioner in Tables 3.9 and 3.10. By comparing these tables with Tables 3.7 and 3.8, we see that the computation is more demanding for this domain, which agrees with the fact that its elliptic regularity index is smaller than that of the L-shaped domain.

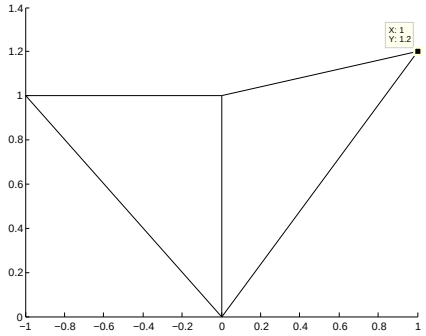

FIGURE 3.3. A nonstandard quadrilateral domain with an initial mesh

k \ m	11	13	15	17	19	21	23	25	27
1	0.825	0.628	0.484	0.377	0.296	0.233	0.185	0.147	0.117
2	0.216	0.192	0.217	0.219	0.205	0.185	0.163	0.143	0.125
3	0.495	0.405	0.333	0.285	0.250	0.222	0.198	0.178	0.160
4	0.367	0.337	0.312	0.285	0.259	0.235	0.222	0.213	0.205
5	0.436	0.371	0.325	0.291	0.269	0.259	0.250	0.242	0.234
6	0.391	0.355	0.322	0.290	0.277	0.268	0.259	0.251	0.244
7	0.416	0.363	0.323	0.291	0.280	0.270	0.262	0.254	0.247
8	0.401	0.359	0.323	0.291	0.280	0.271	0.263	0.255	0.248

TABLE 3.9. Contraction numbers for the W-cycle algorithm on the domain in Figure 3.3

Finally we compare the computational cost between the preconditioned schemes and the un-preconditioned schemes. On the square, the contraction numbers for the preconditioned V-cycle algorithm with $m = 4$ (cf. Table 3.1) are about the same as the contraction numbers for the un-preconditioned V-cycle algorithm with $m = 29$ (cf. Table 3.4). For $k = 7$, the former takes 1.4×10^8 floating point operations and 0.55 second while the latter takes 3.2×10^8 floating point operations and 1.2 seconds.

On the L-shaped domain, the contraction numbers for the preconditioned W-cycle algorithm with $m = 3$ (cf. Table 3.7) are about the same as the contraction numbers for the un-preconditioned W-cycle algorithm with $m = 23$ (cf. Table 3.8). For $k = 7$, the former takes 4.7×10^8 floating point operations and 2.1 seconds while the latter takes 1.1×10^9 floating point operations and 4.7 seconds.

APPENDIX A. A V-CYCLE ALGORITHM FOR THE SINGULAR NEUMANN PROBLEM

Let $L_k : V_k \to \hat{V}_k'$ (cf. (2.1)) be defined by $\langle L_k v_k, w_k \rangle = \int_\Omega \nabla v_k \cdot \nabla w_k \, dx$ for all $v_k, w_k \in V_k$. Then we can rewrite the discrete Neumann problem (2.3) as $L_k \hat{s}_k = \theta_k$. Below we define the V-cycle algorithm for

k \ m	86	89	92	95	98	101	104	107
1	0.977	0.937	0.900	0.864	0.830	0.797	0.766	0.737
2	0.780	0.698	0.625	0.559	0.500	0.446	0.398	0.355
3	0.620	0.516	0.437	0.379	0.338	0.312	0.295	0.284
4	0.414	0.359	0.356	0.354	0.352	0.350	0.349	0.347
5	0.380	0.378	0.376	0.374	0.373	0.371	0.370	0.368
6	0.386	0.384	0.382	0.381	0.379	0.378	0.376	0.375
7	0.388	0.386	0.385	0.383	0.382	0.380	0.379	0.377
8	0.388	0.387	0.385	0.384	0.382	0.381	0.379	0.378

TABLE 3.10. Contraction numbers for the W-cycle algorithm on the domain in Figure 3.3 without a preconditioner

the general equation ($\phi_k \in \hat{V}_k'$)

$$L_k \hat{q}_k = \phi_k \tag{A.1}$$

The V-cycle solve with 1 pre-smoothing step and 1 post-smoothing step is the preconditioner in the smoothing steps of Algorithms 2.1–2.3.

In order to describe the smoothing steps in the V-cycle algorithm for the singular Neumann problem, we first define an inner product $(\cdot,\cdot)_k$ on V_k by

$$(v,w)_k = \frac{1}{3}\sum_{p\in\mathcal{N}_k}\Big(\sum_{T\in\mathcal{T}_p}|T|\Big)v(p)w(p) \quad \forall v,w\in V_k,$$

where $\mathcal{N}_k$ is the set of all nodes (vertices and midpoints) in $\mathcal{T}_k$, $\mathcal{T}_p$ is the set of the triangles sharing p as a common node and $|T|$ denotes the area of the triangle T.

Let $B_k : V_k \to V_k'$ be defined by

$$\langle B_k w, v\rangle = (v,w)_k \quad \forall v,w\in V_k.$$

Note that the matrix representing the operator B_k with respect to the natural nodal basis in V_k and its dual basis in V_k' is a diagonal matrix. Therefore B_k^{-1} can be evaluated with optimal complexity.

Finally, let $\hat{P}_k : V_k \to \hat{V}_k$ be the projection with respect to the inner product $(\cdot,\cdot)_k$. We can compute $\hat{P}_k v_k$ with optimal complexity by the formula

$$\hat{P}_k v_k = v_k - ((v_k,\chi_k)_k/(\chi_k,\chi_k)_k)\,\chi_k,$$

where $\chi_k \in V_k$ is given by

$$\chi_k(p) = \begin{cases} 1 & \text{if } p\in\mathcal{M}_k \\ 0 & \text{otherwise} \end{cases},$$

and $\mathcal{M}_k$ is the set of the midpoints of the edges of the triangles in $\mathcal{T}_k$.

The V-cycle multigrid algorithm for (A.1) with initial guess $\hat{z}_0$ and m pre-smoothing steps and m post-smoothing steps is defined as follows.

Algorithm A.1. (V-cycle multigrid algorithm with output $MG_V^N(k,m,\hat{z}_0,\psi_k)$)

Input: integer k, integer m, $\hat{z}_0$ in $\hat{V}_k$, and ϕ_k in $\hat{V}_k'$

Output: $\hat{z}_{out}$ in $\hat{V}_k$

if $k=0$, then take $\hat{z}_{out}$ to be the exact solution of (A.1) in $\hat{V}_k$ and return;

otherwise proceed as follows:

pre-smoothing: for $j = 1, 2, \ldots, m$,

$$\hat{z}_j \leftarrow \hat{z}_{j-1} + \gamma_k \hat{P}_k B_k^{-1}(\phi_k - L_k \hat{z}_{j-1})$$

γ_k = a pre-defined damping factor of the form Ch_k^2 such that $\rho(\gamma_k B_k^{-1} L_k) < 2$
coarse grid correction:

$$\hat{z}_{m+1} \leftarrow \hat{z}_m + I_{k-1}^k MG_V^N(k-1, m, 0, r_{k-1})$$

$r_{k-1} = I_k^{k-1}(\phi_k - L_k \hat{z}_m)$
post-smoothing: for $j = m+2, m+3, \ldots, 2m+1$,

$$\hat{z}_j \leftarrow \hat{z}_{j-1} + \gamma_k \hat{P}_k B_k^{-1}(\phi_k - L_k \hat{z}_{j-1})$$

final output

$$\hat{z}_{out} \leftarrow \hat{z}_{2m+1}$$

Remark A.2. Algorithm A.1 can be implemented using the natural nodal basis of V_k. No explicit basis for $\hat{V}_k$ is needed.

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