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HyeonGae Park  
**Physics-Based Preconditioning of the rDG-JFNK method  
for All-Speed Fluid Flows with Heat Conduction and  
Viscosity**

Idaho National Laboratory  
Advanced Nuclear Energy Systems  
Multiphysics Methods Group  
P O Box 1625  
Idaho Falls  
ID 83415-3840  
USA  
ryosuke.park@inl.gov  
Robert Nourgaliev  
Vincent Mousseau  
Dana Knoll

A novel high-order-accurate, fully-implicit solution algorithm for all-speed, Navier-Stokes equations is introduced. It combines the “*recovery Discontinuous Galerkin*” (rDG) method for spatial discretization of the hyperbolic and diffusion operators with the “*Explicit, Singly Diagonal Implicit Runge-Kutta*” (ESDIRK) scheme for time integration. Both algorithms are implemented under the “*Jacobian-free Newton-Krylov*” (JFNK) framework. Since hyperbolic and diffusion operators are fully-coupled, temporal discretization errors due to operator-splitting are completely eliminated. The algorithm is demonstrated to be high-(up to 5<sup>th</sup>)-order time-accurate in a wide range of Mach, Reynolds and Péclet numbers even when the pressure-gradient, viscous and heat-conduction operators become of comparable size. The rDG formulation exhibits nearly-spectral accuracy (shown up to the 12<sup>th</sup>-order) in space. The Newton-Krylov (NK) solution is formulated in conservative variables (i.e. density,  $\rho$ ; total energy,  $E$ ; and momentum,  $m$ ), which allows one to stop Newtons iteration any time (if locked in the limit cycle) without losing conservation. The ESDIRK and rDG discretizations provide natural utilities to compute a high-order error estimate in both time and space, which is especially beneficial for Adaptive Mesh Refinement (AMR) and time step control.

The main challenge is to ensure the efficiency of the JFNK linear solver, for which we employ the Jacobian-free version of the right-preconditioned GMRES method. The focus of this paper is the *Physics-Based Preconditioning* (PBP) of our rDG-JFNK algorithm. It is designed to cluster eigenvalues of the Jacobian matrix, ensuring convergence of the GMRES within a few Krylov iterations. Two PBP strategies, specifically designed for DG-type of spatial discretizations, are introduced and extensively tested.

The first PBP capitalizes on explicit availability of the higher-order derivatives in the DG local solution vector. The Jacobian matrix  $\mathbb{J}$  can be rearranged in the block-vector form. Each element  $\hat{\mathbb{J}}_{km}$  of the Jacobian matrix  $\mathbb{J}$  is a  $[3(p+1) \times 3(p+1)]$  matrix, and  $p$  is the highest order of the basis functions, representing non-local coupling of solutions between  $k^{th}$  and  $m^{th}$  cells (i.e.  $\delta\mathbb{U}_k$  and  $\delta\mathbb{U}_m$ ).  $\delta\mathbb{U}_j = \left( \delta\rho_j^{(0)}, \dots, \delta\rho_j^{(p)}, \delta m_j^{(0)}, \dots, \delta m_j^{(p)}, \delta E_j^{(0)}, \dots, \delta E_j^{(p)} \right)^\top$  is a local solution vector at cell  $j$ . Elimination of the off-diagonal blocks ( $\hat{\mathbb{J}}_{km} = 0, k \neq m$ ) can be interpreted as freezing the nonlocal coupling at the previous Newton iteration. When used for preconditioning, this block-diagonal approximation not only provides a tight coupling of the local solution vector (which is very effective when the reaction terms are stiff), but also takes into account *some non-local coupling* since the spatial derivatives are the part of the local solution vector. We refer to this as “*Block-Diagonal*” (BD) preconditioner, and its efficiency is demonstrated here in terms of *a*) the patterns of eigenvalues for the Jacobian matrix (“*Eigenscopy*”); and *b*) the number of the Krylov iterations/vectors required to converge the GMRES to a given linear tolerance.

The second PBP incorporates non-local effects. We split preconditioning into three steps, each targeting eigenvalues associated with a specific physics (i.e., heat conduction, stiff pressure waves and viscous operator). First, we transform the NK solution vector from conservative variables to primitive variables (i.e., pressure,  $P$ ; velocity,  $u$ ; and internal energy,  $e$ ). A consistent transformation of the higher order derivatives is carried out by enforcing the conservation in the weak sense. The resulting system can be cast into the following block-vector form,

$$\begin{bmatrix} \mathbb{J}_{PP} & \mathbb{J}_{Pu} & \mathbb{J}_{Pi} \\ \mathbb{J}_{uP} & \mathbb{J}_{uu} & \mathbb{J}_{ui} \\ \mathbb{J}_{iP} & \mathbb{J}_{iu} & \mathbb{J}_{ii} \end{bmatrix} \begin{pmatrix} \delta\mathcal{P} \\ \delta\mathcal{U} \\ \delta\mathcal{I} \end{pmatrix} = - \begin{pmatrix} \mathbf{res}_P \\ \mathbf{res}_u \\ \mathbf{res}_\mathcal{I} \end{pmatrix},$$

where each element  $\mathbb{J}_{km}$  is a  $N_{\text{cells}}(p+1) \times N_{\text{cells}}(p+1)$  matrix, representing non-linear coupling between the  $k^{th}$  and  $m^{th}$  primitive variables.

In order to solve this coupled system, we first decouple the internal energy from the pressure-velocity matrix, solving for  $\delta\mathcal{I}$ . Step I targets the heat conduction operator. Next, we formulate the pressure-Poisson equation as

$$\underbrace{\left( \mathbb{J}_{PP} - \mathbb{J}_{Pu} \mathbb{J}_{uu}^{-1} \mathbb{J}_{uP} \right)}_{\text{Laplacian, } \mathbb{L}} \delta\mathcal{P} = - \underbrace{\left( \mathbf{res}_P + \mathbb{J}_{Pi} \delta\mathcal{I} - \mathbb{J}_{Pu} \mathbb{J}_{uu}^{-1} (\mathbf{res}_u + \mathbb{J}_{ui} \delta\mathcal{I}) \right)}_{\mathbf{b}},$$

where the Laplacian can be shown to be closely related to the Schur complement of the pressure-velocity matrix. This parabolic system can be solved efficiently

using a multigrid method. Step II targets stiff pressure waves. Finally, we solve for velocity, accounting for viscous stress terms. In the 1D representation, this Step III can be written as

$$\delta\mathcal{U} = -\mathbb{J}_{uu}^{-1} (\mathbf{res}_{\mathcal{U}} + \mathbb{J}_{uP} \delta\mathcal{P} + \mathbb{J}_{ui} \delta\mathcal{I}).$$

It can be easily seen that this PBP is closely related to the classical “operator-split” solution algorithms for incompressible and weakly-compressible flows (e.g., ICE, SIMPLE, Projection). Therefore, we referred to this as “*Operator-Split*” (OS) preconditioner. We demonstrate that the OS preconditioner is very effective in clustering generally-complex eigenvalues near the real axis. It is also found to be very effective in a wide range of Mach ( $M$ ) and Reynolds ( $Re$ ) numbers, keeping the number of GMRES iterations  $\sim 10$ , even at the extremes of  $M = 10^{-4}$  and very low  $Re$  numbers.