Rakhim Aitbayev

Multilevel Preconditioners for Nonselfadjoint or Indefinite Orthogonal Spline Collocation Problems

Department of Mathematics
New Mexico Tech
Socorro
NM 87801
aitbayev@nmt.edu
Bernard Bialecki

TheoremWe develop and study symmetric multilevel preconditioners for the computation ofthe orthogonal spline collocation (OSC) solution of a Dirichlet boundary value problem(BVP) with a nonselfadjoint or an indefinite operator.The OSC solution is sought in the space of piecewise Hermite bicubic spline functionsdefined on a uniform partition.We consider an additive and a multiplicative multilevel preconditionersthat are used with the preconditioned conjugate gradient (PCG) method.Our results and algorithms are closely related to those in[1], [2], [3],and [4].Let Ω be a unit square $(0,1)\times(0,1)$ with the boundary $\partial\Omega$, andlet $x=(x_1,x_2)$. We consider a BVP

$$Lu \equiv \sum_{i,j=1}^{2} a_{ij}(x)u_{x_ix_j} + \sum_{i=1}^{2} b_i(x)u_{x_i} + c(x)u = f(x), \quad x\in\Omega, \quad u=0 \text{ on } \partial\Omega. \tag{1}$$

Operator L could be non-selfadjoint or indefinite in L^2 inner product.We assume that the principal part of L satisfies the uniform ellipticity conditionand that BVP (1) has a unique solution in $H^2(\Omega)$.Let π_0 be a uniform coarsest rectangular partition of Ω. We obtain a set of partitions$\{\pi_k\}_{k=0}^K$ by standard coarsening, andlet $V_0\subset V_1\subset\ldots\subset V_K\equiv V_h$ be the set of corresponding nested spaces ofpiecewise Hermite bicubics that vanish on $\partial\Omega$.Let $\sum$ denote the two-dimensional composite Gauss quadrature corresponding to partitionπ_h with 4 nodes in each element. Let $\mathcal{G}_h$ denote the corresponding set of Gauss points.The OSC discretization of BVP (1) is defined by

$$u_h\in V_h, \quad Lu_h(\xi)=f(\xi), \quad \xi\in\mathcal{G}_h, \tag{2}$$

and it can be written as the operator equation $L_hu_h=f_h$ in the Hilbert space V_h with theinner product $(v,w)_h=\sum vw$.We define and study multilevel additiveB_{a} and multiplicative B_{m} preconditionersfor solving the normal equation $L_h^*L_hu_h=L_h^*f_h$, where L_h^* is the adjoint to L_h.The implementation of B_{a} and $B_{\mathrm{m}}$is based on relationships between basis functions for two consecutive partitionsand the implementation of B_{m} is similar to that forV(1,1)-cycle with the Gauss-Seidel smoothing.A problem on the coarsest partition is assumed sufficiently small, and it is solved exactly.The computational cost of the preconditioning algorithms is $\mathrm{O}(N_K)$.The following is our main result.There are positive independent of h and K constants$\alpha_{\rm a}$,$\beta_{\rm a}$,$\alpha_{\rm m}$,and$\beta_{\rm m}$,such that

$$\begin{aligned} &\alpha_{\rm a}\,(B_{\rm a}v,v)_h \le (L_h^* L_h v,v)_h \le \beta_{\rm a}\,(B_{\rm a}v,v)_h, \quad v\in V_h, \\ &\alpha_{\rm m}\,(B_{\rm m}v,v)_h \le (L_h^* L_h v,v)_h \le \beta_{\rm m}\,(B_{\rm m}v,v)_h, \quad v\in V_h. \end{aligned} \tag{3}$$

To obtain this result, we prove the key assumptions in the general theory of Schwarz methodsformulated in [4] and use the inequalities

$$C^{-1}\|v\|^2_{H^2(\Omega)} \le a_h(v,v) \le C\|\Delta v\|^2_{L^2(\Omega)}, \quad v\in V_h,$$

obtained in [2].We present numerical results that demonstrate the efficiency ofour preconditioning algorithms.

Bibliography

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