An efficient hp-refinement strategy for singular solutions in 1-D

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Abstract

This paper investigates the behaviour of an efficiency algorithm for adaptive grid refinement. Refinement decisions are based on maximizing efficiency in every step, assuming that multigrid solvers are used such that the work for solving the linear system is proportional to the number of unknowns. We consider an exact solution that has a singularity of $x^{\alpha}-$type. We first consider h-refinement. By comparing the efficiency algorithm with the optimal radical mesh, we confirm that the efficiency algorithm generates a high-accuracy mesh sequence. We find that the process of refinement by using the efficiency algorithm becomes inefficient when the solution is highly singular. Correspondingly, a modified efficiency algorithm is given for this situation. It is shown that the modified efficiency algorithm is efficient by comparing with a threshold-based algorithm. At last, we give a similar efficiency algorithm for h-p refinement.

1 Introduction

1.1 The efficiency algorithm for h-refinement

We first describe an efficiency-based approach that was initially proposed in [1]. We consider using the First-Order System Least Squares (FOSLS) finite element method to solve a PDE, or system of PDEs:

$$L\mathbf{w} = \mathbf{f} \text{ in } \Omega, \tag{1}$$

with appropriate boundary condition. The FOSLS methodology yields a system of first-order PDEs

$$L_i\mathbf{u} = f_i,\ i = 1, ..., M. \tag{2}$$

It has been established that the FOSLS L^2-functional

$$\boldsymbol{G}(\mathbf{u}, \mathbf{f}) = \sum_{i=1}^{M} ||L_i\mathbf{u} - f_i||_0^2 \tag{3}$$

can be used as a local a-posteriori estimate for grid adaption if the continuity and coercivity bounds for the corresponding bilinear form can be verified in some suitable norm. In many cases, this norm is a properly scaled sum of H^1-norms of components of $\mathbf{u}$, see [1].

The proposed efficiency strategy in each step determines a refinement region $R \subset \Omega$ that maximizes the efficiency:

$$\frac{\text{gain in accuracy}}{\text{work to solve the new discrete problem}}. \tag{4}$$

This is the same as minimizing the work per unit accuracy gain:

$$w = \frac{\text{work}}{\text{gain}}. \tag{5}$$

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For h-refinement, a reasonable approximation for the work required can be given by the following expression:

$$\text{work to solve the new discrete problem} \approx an_{old} + bn_{new}, \tag{6}$$

if the solver of choice is a multilevel algorithm. Here a and b are constants, and for simplicity we can choose $a = b$. The symbol n_{old} denotes the number of vertices in the current mesh, and n_{new} denotes the number of vertices that will be added. The gain in accuracy can be measured as follows:

$$\text{gain in accuracy} = \sqrt{\frac{\boldsymbol{G}(\mathbf{u}^h_{old}, \mathbf{f})}{\boldsymbol{G}(\mathbf{u}^h_{new}, \mathbf{f})}}. \tag{7}$$

We cannot calculate $\boldsymbol{G}(\mathbf{u}^h_{new}, \mathbf{f})$ before refining the mesh, but this quantity can be determined approximately if the refinement is accomplished by splitting elements in two. However, the accuracy gain depends on whether the solution is singular.

1.2 Asymptotic behaviour of the error

For simplicity, consider one component of $\mathbf{u}$ denoted by u, and suppose that u is partly smooth in a domain with singularities.

THEOREM 1. *Let $m < s < k$, $m, k \in \boldsymbol{N}, s \in \boldsymbol{R}$, assume $\Omega = \Omega_s \cup \Omega_r$, $\Omega_s \cap \Omega_r = \emptyset$, $u|_{\Omega_s} \in H^s(\Omega_s)$, $u|_{\Omega_r} \in H^k(\Omega_r)$, and let $u_h \in V_h$ be a variational approximation of u. If the error is evaluated by the H^m norm, then we have the following error bounds, see* [2]*:*

$$\begin{aligned} ||u - u_h||_{H^m(\Omega_s)} &\le Ch^{s-m}||u||_{H^s(\Omega_s)}, \\ ||u - u_h||_{H^m(\Omega_r)} &\le Ch^{k-m}||u||_{H^k(\Omega_r)}. \end{aligned} \tag{8}$$

This estimate indicates that the error in the less smooth domain would go to zero slower than $O(h^{k-m})$. Especially when $s \to m$, error convergence becomes very slow.

For elements τ_j in which the solution is smooth (at least in $H^{p+1}(\tau_j)$ if order p elements are used), we have

$$\begin{aligned} \epsilon_j = ||Lu^h - f||^2_{0,\tau_j} = ||L(u^h - u)||^2_{0,\tau_j} &\approx ||u^h - u||^2_{H^1(\tau_j)} \\ &\le Ch_j^{2p}||u||^2_{H^{p+1}(\tau_j)} \\ &\le CM_{p+1}h_j^{2p}h_j. \end{aligned} \tag{9}$$

If τ_j is split in two, then we have 2 new elements τ_{j1}, τ_{j2} satisfying the error reduction ratio formula:

$$\frac{\epsilon_{j1} + \epsilon_{j2}}{\epsilon_j} \approx \left(\frac{1}{2}\right)^{2p}. \tag{10}$$

However, if u is less smooth in some element τ_i, i.e. $u \in H^{q+\sigma}(\tau_i)$ with $q \le p, q \in \mathbf{N},\ 0 < \sigma < 1$, we have:

$$\epsilon_i \le Ch_j^{2(q+\sigma-1)}||u||^2_{q+\sigma,\tau_i}. \tag{11}$$

If again, we split this element into two, assuming τ_{i1} contains the singularity, then $\epsilon_{i1} >> \epsilon_{i2}$ and $||u||_{q+\sigma,\tau_{i1}} \approx ||u||_{q+\sigma,\tau_i}$, and we obtain the approximate error reduction ratio:

$$\frac{\epsilon_{i1} + \epsilon_{i2}}{\epsilon_i} \approx \frac{\epsilon_{i1}}{\epsilon_i} \approx \left(\frac{1}{2}\right)^{2(q+\sigma-1)}. \tag{12}$$

Thus, we can use the approximation

$$\boldsymbol{G}(\mathbf{u}^h_{new}, \mathbf{f}) \approx \left(\frac{1}{2}\right)^{2(q+\sigma-1)} \boldsymbol{G}_{R\cap\Omega_s}(\mathbf{u}^h_{old}, \mathbf{f}) + \left(\frac{1}{2}\right)^{2p} \boldsymbol{G}_{R\cap\Omega_r}(\mathbf{u}^h_{old}, \mathbf{f}) + \boldsymbol{G}_{R^c}(\mathbf{u}^h_{old}, \mathbf{f}). \tag{13}$$

At last, the efficiency algorithm (EA) is as follows:

1) Order the local errors ϵ_i s.t. $\epsilon_1 \geq \epsilon_2 \geq ... \geq \epsilon_{N_l}$, where N_l is the number of elements on current level.
2) Calculate $w_i = \text{work/gain}$, for $i = 1, 2, ..., N_l$.
3) Find $i : 1 \leq i \leq N_l$ that minimizes w_i.
4) Refine elements τ_j: $1 \leq j \leq i$.
5) Repeat.

In this paper, we assume that we know the power and the location of the singularities in advance. In more realistic situations, this information can be deduced by monitoring error reduction in each element during a few steps of uniform refinement.

2 EA for h-refinement and modifications

We applied this refinement strategy to a simple problem in 1-D:

$$u'' = \alpha(\alpha - 1)x^{\alpha-2}, u(0) = 0, u(1) = 1, \tag{14}$$

with the exact solution given by $u = x^\alpha$. Note that $u \in H^{1+\alpha-\frac{1}{2}}((0,1))$. If we choose $\alpha < \frac{3}{2}$, then $u \notin H^2((0,1))$ (there is a x^α-type singularity at $x = 0$). Let $p = 1$, and for simplicity, we choose u_h to be the interpolant of u in the nodes of the mesh without using FOSLS to solve this problem. Let the error be evaluated by the H^1 seminorm of $u - u_h$, i.e., for element τ_i, let $\epsilon_i = ||u' - u_h'||^2_{0,\tau_i}$. Then the error reduction ratio can be approximated by $\frac{1}{4}$ where u is smooth, and $(\frac{1}{2})^{2\alpha-1}$ where u is singular.

First of all, the EA can be considered as a kind of feedback algorithm, so it is convergent, see [1]. Secondly, we want to know if the EA generates a good mesh with a fast rate of global error convergence w.r.t. the DOF.

2.1 Comparison with the optimal mesh (radical mesh)

Assume that the polynomial degree p of all elements is uniform and fixed. Given the number of DOF $n = Np$, N = the number of elements, it has been shown in [3] that for the x^α-type singular solution the rate of convergence is never better than n^{-p}.

THEOREM 2. *Let $E = (\sum \epsilon_i)^{\frac{1}{2}}$. There is a constant $C = C(\alpha, p) > 0$ such that for any mesh $\{0 = x_0 < x_1 < ... < x_N = 1\}$:*

$$E \geq C\, n^{-p}. \tag{15}$$

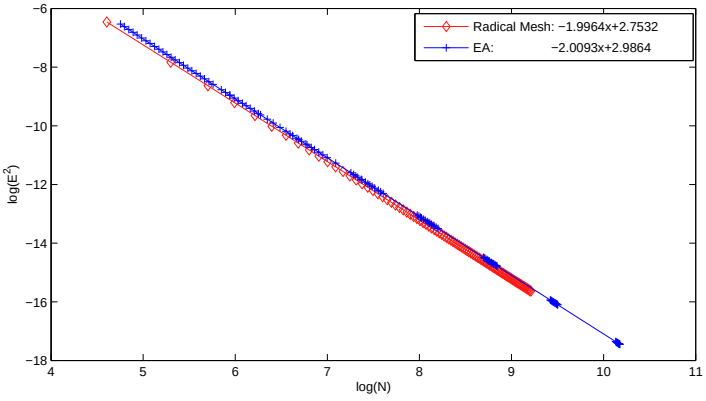

Figure 1: $\log(E^2)$ vs $\log(N)$ curves, $\alpha = 0.6,\ p = 1$.

Furthermore, for our example problem, an asymptotically optimal mesh called the radical mesh is described by Schwab in [4] as

$$x_j = (j/N)^{(p+1/2)/(\alpha-1/2)},\ j = 0, ..., N. \tag{16}$$

Fig. 1 gives the $\log(E^2)$ vs $\log(N)$ curve of the EA and Radical meshes for $p = 1,\ \alpha = 0.6$. It can be observed that the EA does give a high-accuracy mesh sequence.

We are also interested in whether the EA results in a mesh with fast global error convergence w.r.t. refinement level. This is true when u is relatively smooth, e.g. $u \in H^{p+1}$ if p-elements are used. In fact, if we define $f(i) = \frac{\sum_{j\le i}\epsilon_j}{\sum_{j\le N_l}\epsilon_j}$ to be the fraction of error in the elements being refined, then w_i can be given by

$$\begin{aligned} w_i &= C\,(N_l + i)\left(1 - f(i) + \left(\frac{1}{2}\right)^{2p} f(i)\right)^{\frac{1}{2}} \\ &= C\,N_l\,(1 + \frac{i}{N_l})\left(1 - f(i) + \left(\frac{1}{2}\right)^{2p} f(i)\right)^{\frac{1}{2}} \\ &= C_0\,\Phi(i), \end{aligned} \tag{17}$$

with $C_0 = C\,N_l$, where N_l is the number of elements on the current level.

Hence, finding i to minimize w_i is equivalent to minimizing the factor $\Phi(i)$. Apparently, if we want to obtain a fast convergence of global error w.r.t. refinement level, a good refinement strategy should choose large $f(i)$.

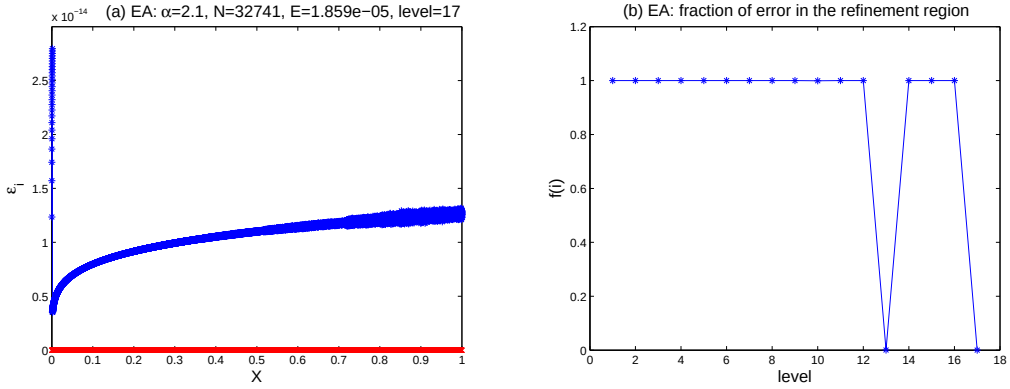

Figure 2: (a) error distribution at the end of refinement, (b) fraction of error in refinement region vs level; EA for smooth case $\alpha = 2.1,\ p = 1$.

Fig. 2 shows the curve of $f(i)$ vs refinement level when the solution $u \in H^2$. It can be observed that for most steps the refinement region contains a large fraction of the error. This confirms that the EA is an efficient strategy when the solution is relatively smooth. In fact, in Fig. 2 it can be observed that local errors in all elements tend to be almost equally distributed.

2.2 Inefficiency of original EA for highly singular solutions

When the solution u is singular, the refinement process of the EA becomes inefficient:

1) The local error in the first element, which contains the singularity, is always the largest. Hence, it is refined by the EA in every step.

2) The refinement process seems to become periodic after some steps (Fig. 3(a)). When the ratio $\frac{\epsilon_1}{\max_{i=2}^{N}\epsilon_i}$ reaches a local minimum, many elements are refined in this step; we call this a big refinement. Then the ratio

reaches a local maximum. Before the ratio reaches a local minimal again, only the first element (possibly with a few other elements) is continuously being refined. We call each such step a small refinement.

3) The fraction of the error in the refinement region decreases w.r.t. refinement level, see Fig. 4(a).

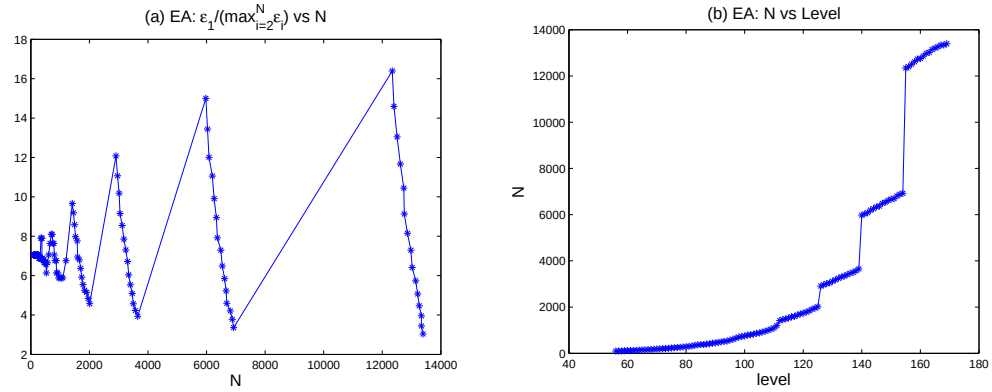

Figure 3: (a) $\frac{\epsilon_1}{\max_{i=2}^{N} \epsilon_i}$ vs N, (b) N vs level, total work=O(386,661); EA for singular case $\alpha = 0.6,\ p = 1$.

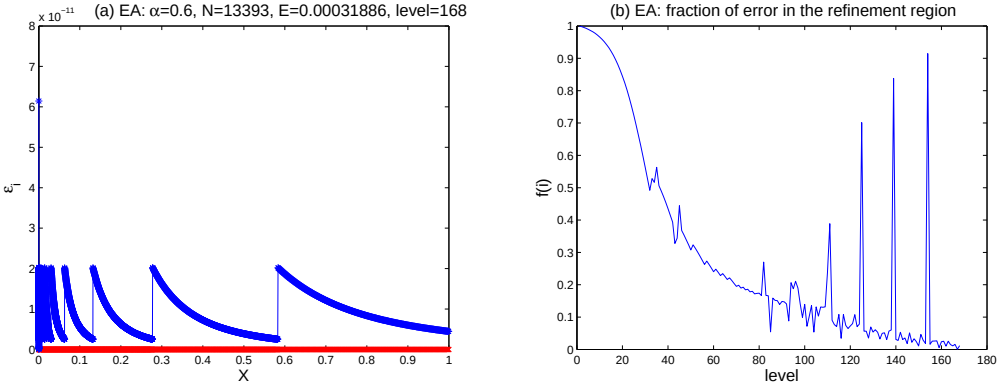

Figure 4: (a) error distribution at the end of refinement, (b) fraction of error in refinement region vs level; EA for singular case $\alpha = 0.6,\ p = 1$.

Many steps of small refinement result in an inefficient algorithm. However, these steps can be replaced by putting a geometrically graded mesh starting from the singular point, which allows to save a lot of work and brings faster global error convergence w.r.t. refinement level.

2.3 Modified efficiency algorithm (MEA)

Assume each element only contains at most one singularity initially, and suppose we knew the location and the power of the singularities. If we do graded mesh refinement for elements containing a singularity such that we obtain the same error reduction speed as in elements in which the solution is smooth, and, correspondingly, we change the work factor in the original efficiency algorithm, then we expect the efficiency algorithm to generate the good grid in an efficient way. This results in the following algorithm (MEA):

1) Order the local errors ϵ_i s.t. $\epsilon_1 \geq \epsilon_2 \geq ... \geq \epsilon_{N_l}$.

2) Perform graded mesh refinement for elements containing a singularity, i.e. if $u \in H^{q_i+\sigma_i}(\tau_i)$, then graded mesh refinement with m_i levels is used for τ_i if τ_i needs to be refined, with m_i, the number of new elements,

satisfying:

$$\left(\frac{1}{2}\right)^{2m_i(q_i-1+\sigma)} \approx \left(\frac{1}{2}\right)^{2p} \Rightarrow m_i = \lceil \frac{p}{q_i - 1 + \sigma_i} \rceil.$$

For elements in which the solution is smooth, set $m_i = 2$, and single refinement is used.

3)

$$w_i = \frac{\text{work}}{\text{gain}} = C\, N_l \frac{\sum_{j \le i} m_j + N_l - i}{N_l} \left(1 - f(i) + \left(\frac{1}{2}\right)^{2p} f(i)\right)^{\frac{1}{2}} = C_0\, \Phi_{\text{MEA}}(i).$$

4) Find i such that i minimizes the factor $\Phi_{\text{MEA}}(i)$, and refine the first i elements.

5) Repeat.

We again choose $\alpha = 0.6$ and $p = 1$ for our example problem. There is only one singularity, at $x = 0$, with error convergence speed $(\frac{1}{2})^{0.2}$. Therefore, for the element containing $x = 0$, we should use 11 refinements $(m = \lceil \frac{1}{0.1} \rceil)$.

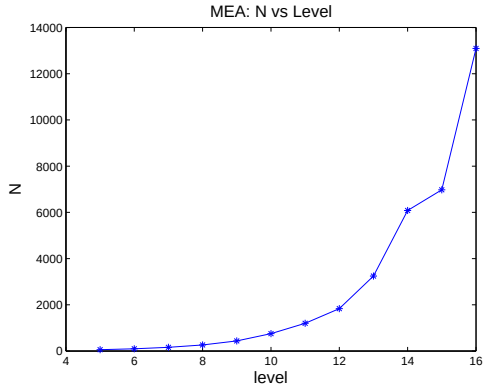

Figure 5: N vs level, total work = O(34271); MEA for singular case $\alpha = 0.6,\ p = 1$.

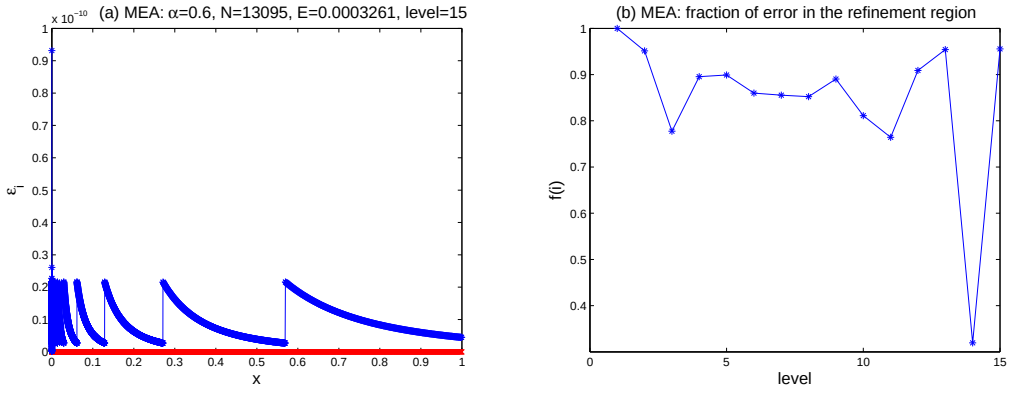

Figure 6: (a) error distribution at the end of refinement, (b) fraction of error in refinement region vs level; MEA for singular case $\alpha = 0.6,\ p = 1$.

By comparing Fig. 5 and Fig. 6 with Fig. 3 and Fig 4, we can see

1) The number of elements increases much faster than the original EA, which means much less work.

2) For almost all steps of the MEA, the fraction of error in the refinement region is much larger than using the EA, which means global error decreases much faster w.r.t. the refinement level.

3) The MEA results in almost the same grid sizes, error distribution and global error as the EA, but uses much less work.

4) The local error in the first element still remains the largest in all steps of the modified algorithm, just like the original algorithm.

Hence we can confirm that the modified algorithm is a significant improvement when the solution is singular.

Furthermore, we give the global error vs N curve (in $\log - \log$ scale) for the radical mesh, the EA and the MEA for comparison in Fig. 7.

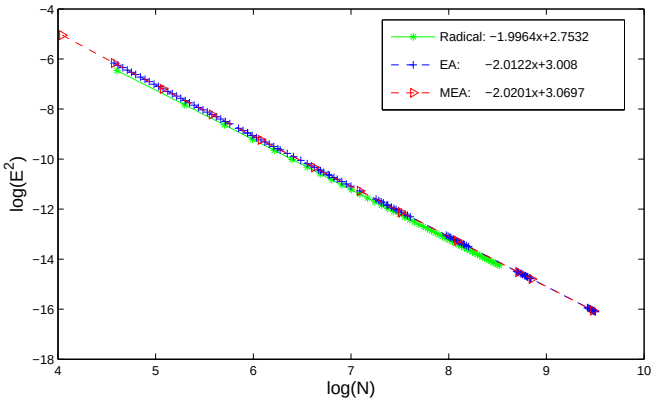

Figure 7: $\log(E^2)$ vs $\log(N)$; $\alpha = 0.6,\ p = 1$.

This confirms that the grid generated by the modified efficiency algorithm is a roughly optimal grid w.r.t. error reduction. Note that the MEA requires $O(34,271)$ total work to reach the pre-specified error tolerance, while the original EA requires $O(386,661)$ total work.

2.4 Comparison with the threshold-based refinement strategy

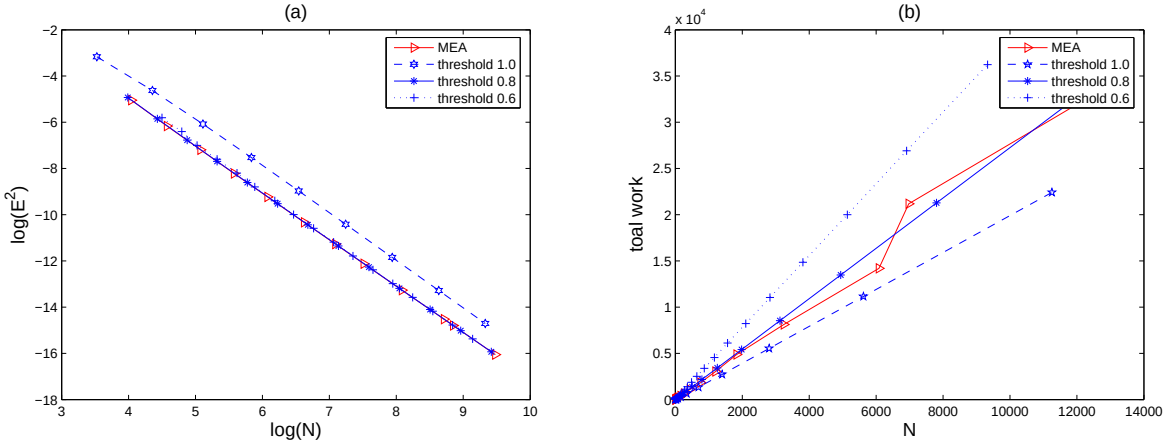

Figure 8: MEA and threshold: (a) $\log(E^2)$ vs $\log(N)$, (b) total work vs N; $\alpha = 0.6,\ p = 1$.

At last, we compare the MEA with the threshold-based refinement strategy which refines a fixed fraction of the global error, i.e. each step refines the first i elements s.t. $f(i)$ defined in step 3 of the MEA is a constant. The same graded mesh refinement strategy is used for the elements containing a singularity. We find for our example problem:
1) If we choose to refine a small amount of the global error (less than the average of $f(i)$ in the MEA), then the global error decreases w.r.t. DOF at nearly the same speed as the MEA. The choice of a smaller fraction only results in a very small improvement of the rate of global error convergence w.r.t. DOF, which is not worthwhile since much more work is required.
2) We obtain a faster rate of global convergence w.r.t. the refinement level if a large fraction is used. However, the global error decreases slower w.r.t. DOF, and it follows that the resulting grid is not highly accurate.

By taking into account the singularity and using graded mesh refinement for the elements containing a singular point, the MEA automatically and adaptively chooses the best fraction for both work and gain in error reduction. As a result, it generates a roughly optimal grid efficiently, i.e. global error decreases fast w.r.t. DOF and refinement level.

We cannot give a qualitative error convergence analysis for the MEA, since it is not easy to find $f(i)$ in each step theoretically. However, we can give a sharp analysis of why the MEA does more big refinements than the EA for singular solutions as follows. Note that for $p = 1,\ \alpha = 0.6$, the EA and the MEA choose i to minimize the following factors:

$$\begin{aligned}\Phi_{\mathrm{EA}}(i) &= \left(1+\frac{i}{N}\right)\left\{1-\frac{3}{4}f(i)+\left[\left(\frac{1}{2}\right)^{0.2}-\frac{1}{4}\right]f(1)\right\}^{\frac{1}{2}};\\ \Phi_{\mathrm{MEA}}(i) &= \left(1+\frac{i+10}{N}\right)\left\{1-\frac{3}{4}f(i)\right\}^{\frac{1}{2}}.\end{aligned} \tag{18}$$

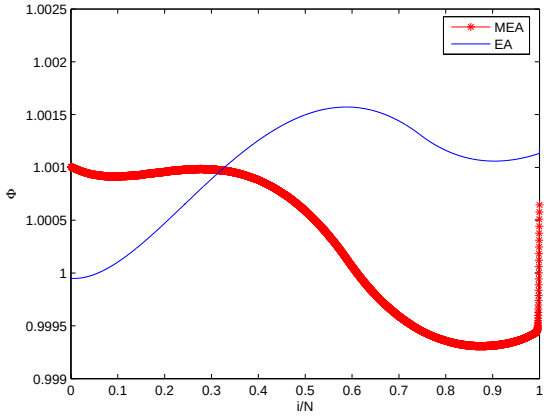

Figure 9: $\Phi(i)$ vs $\frac{i}{N}$ curve for EA and MEA. In the EA, the global minimum lies close to 0 when the error in the first element is large, while for the MEA the global minimum almost always lies close to 1; $\alpha = 0.6,\ p = 1$.

Note that $f(1)$ is the fraction of the error in the first, singular, element. After a few steps of refinement, both algorithms result in almost the same mesh. Since the curves of both factors tend to $(1+\frac{i}{N})(1-\frac{3}{4}\frac{i}{N})^{\frac{1}{2}}$ due to equal distribution of the error, there are two local minimums. One occurs for small i and one occurs for large i. For the small i_1, $\Phi_{\mathrm{EA}}(i_1) \approx 1$ and $\Phi_{\mathrm{MEA}}(i_1) \approx 1+\frac{i_1+m}{N} \gtrsim 1$. For large i_2 s.t. $f(i_2) \approx 1$, $\Phi_{\mathrm{EA}}(i_2) \approx (1+\frac{i_2}{N})\{\frac{1}{4}+[(\frac{1}{2})^{0.2}-\frac{1}{4}]f(1)\}^{\frac{1}{2}}$ and $\Phi_{\mathrm{MEA}}(i_2) \approx \frac{1}{2}(1+\frac{i_2+10}{N})$. Generally, i_2 is not close to N unless local errors are sufficiently equal-distributed. It follows that $\Phi_{\mathrm{MEA}}(i_2) < 1 < \Phi_{\mathrm{MEA}}(i_1)$. However, $\Phi_{\mathrm{EA}}(i_2)$

may be larger than $\Phi_{\rm EA}(i_1)$ when $f(1)$ is large enough. Hence,the EA normally chooses small refinement except when the local errors are nearly equal-distributed.

For implementation, we don't need to solve the linear system on the resulting mesh until the new level is refined enough to have a significant number of additional points in it. We approximately know the local errors in all new elements and can use them for the next step. In this way complexity is never a problem, and we still get an highly accurate mesh sequence. Even if this procedure is followed, putting a graded mesh may still make the whole process much faster, because calculating and maximizing the efficiency in many times may be costly as well.

3 EA for hp-refinement

3.1 Hp-version EA and MEA

For hp-refinement, suppose we know a good approximation of the p-refinement error reduction ratio for each element. Then we can develop a similar hp-version EA and MEA. However, in general, giving an accurate approximate error reduction ratio for any element when we increase p is difficult.

We consider an hp-finite element method for our simple example problem (14). Let $\mathbf{I} = \{0 = x_0 < x_1 < ... < x_N = 1\}$ be the mesh, and $\mathbf{p} = \{p_1, p_2, ..., p_N\}$ be the degree of polynomials on the mesh. Let u_s be the finite element solution of (14) and let $\epsilon_i(p) = ||u' - u_s'||^2_{0,\tau_i}$ be the local error in element $\tau_i = [x_{i-1}, x_i]$ with polynomials of degree p_i. We choose local Legendre polynomials as the modal base functions, see [4]. Then we have the following theorem, see [3]:

THEOREM 3. *Let $\epsilon_i(p)$ be the local error of the finite element solution of problem* (14)*, let*

$$\tau_i = [x_{i-1}, x_i],\ r_i = \frac{\sqrt{x_i} - \sqrt{x_{i-1}}}{\sqrt{x_i} + \sqrt{x_{i-1}}},$$

then

$$\epsilon_1(p_1) \approx \frac{h_1^{2\alpha-1}}{p_1^{4\alpha-2}}. \tag{19}$$

If r_i $(2 \le i \le N)$ is not close to 1*, then*

$$\epsilon_i(p_i) \approx \left\{ h_i^{\alpha-1/2} \left(\frac{1-r_i^2}{2r_i}\right)^{\alpha-1} \frac{r_i^{p_i}}{p_i^{\alpha}} \right\}^2. \tag{20}$$

For the first element, which contains the singularity, if we only consider h-refinement, we have the error reduction ratio $(\frac{1}{2})^{2\alpha-1}$ as (12). For element τ_i without singularity, note that r_i is small, and again we obtain the same h-reduction ratio $(\frac{1}{2})^{2p_i}$ as (10). Moreover, suppose we double the degree of polynomial p_i, then we obtain the p-reduction ratio as follow

$$\frac{\epsilon_i(2p_i)}{\epsilon_i(p_i)} \approx \left(\frac{r_i^{p_i}}{2^{\alpha}}\right)^2. \tag{21}$$

Note that a suitable approximation formula may not be available for the error reduction in the case of general problems different from (14).

Now we develop a hp-version MEA:

1) Let $p_{\max}$ be the maximum polynomial degree.

2) Order the local error ϵ_i s.t. $\epsilon_1 \ge \epsilon_2 \ge ...\epsilon_{N_l}$.

3) Three types of refinements are used depending on the elements: we put a graded mesh with $p = 1$ for the elements containing a singularity such that the error reduction ratio is close to $\frac{1}{4}$; for elements without a singularity, p-refinement (double p) is used if the p-refinement ratio is less than the h-refinement ratio and $p < p_{\max}$; otherwise $h-$refinement is used and the degree p will be inherited by both sub-elements. (Note that

here we assume that the work of solving the problem is proportional to the DOF; then doubling p or splitting the element into two elements with order p has the same computational complexity.)

4) Calculate w_i = work/gain using (10), (12) and (21), for $i = 1, 2, ..., N_l$.

5) Find $i : 1 \leq i \leq N_l$ that minimizes w_i.

6) Refine elements τ_j: $1 \leq j \leq i$.

7) Repeat.

An hp-version EA can be similarly given as the MEA; the only difference is that no graded mesh strategy is used for elements containing a singularity, only h-refinement is allowed for such elements.

3.2 Optimal hp geometric mesh

Just as in the case of h-refinement, we need to seek some kind of optimal mesh for comparison. Suppose the locations of the grid points are given by:

$$x_i = q^{N-i}, \quad 0 < q < 1, \ i = 1, 2, ..., N. \tag{22}$$

Then $r_i = r = \frac{1-\sqrt{q}}{1+\sqrt{q}}, \forall i : 1 \leq i \leq N$. It was shown in [1] that the optimal degree distribution of $\mathbf{p}$ tends to a linear distribution with slope

$$s_o = (\alpha - 1/2)\frac{\log q}{\log r}. \tag{23}$$

Furthermore, the optimal geometric mesh factor q and linear slope s_o combination is given by

$$q_{opt} = (\sqrt{2} - 1)^2, \ s_{opt} = 2\alpha - 1. \tag{24}$$

3.3 Numerical results and comparisons

We apply the hp-version EA/MEA strategy to problem (14) with $\alpha = 0.6$, and compare the numerical results with the geometric mesh with $q = q_{opt}$ and $q = \frac{1}{2}$.

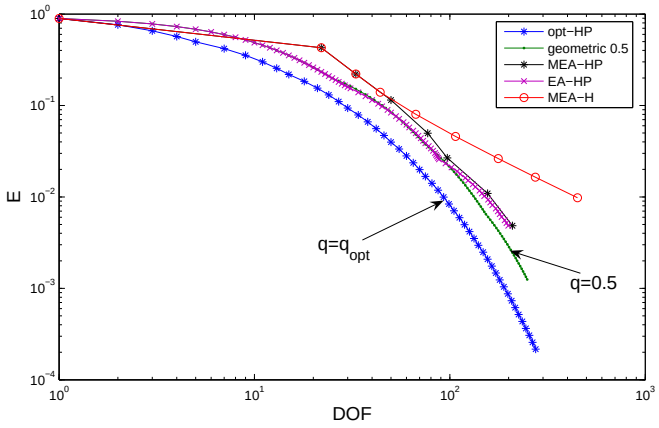

Figure 10: $E = ||u' - u'_s||_0$ vs DOF in $\log - \log$ scale

From Fig. 10, we can see that the hp-finite element methods result in much faster error convergence than the h-finite element method. Both the hp-EA and hp-MEA result in meshes with rate of error convergence very close to the geometrical mesh with grading number $q = 0.5$. Again, the MEA, which uses a graded mesh, saves a lot of work over the EA. Furthermore, the performance of the optimal hp-geometric mesh is better than the EA, MEA and geometrical mesh with $q = 0.5$. Unlike the case of the h-version, the difference is noticeable here.

4 Conclusions

In summary we conclude:

1) Maximizing efficiency is a good strategy to generate a high accuracy mesh sequence. When the solution is smooth, it is also efficient.

2) The process of refinement of the original EA becomes inefficient due to refinement steps in which only a small number of elements are refined. Putting a graded mesh for elements with a singularity to obtain a uniform error convergence speed is a significant improvement, which saves a lot of work and still gives a high-accuracy mesh sequence.

3) For hp-refinement, similar conclusions are obtained as h-refinement. However, for general problems, the difficulty here lies in how to find a good approximation for the p-error reduction ratio.

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