ALGEBRAIC MULTIGRID FOR HIGH ORDER HIERARCHICAL EDGE ELEMENTS*

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Abstract. We propose a new algebraic multigrid (AMG) method for the 2D eddy current problem discretized by high order $H(\text{curl})$ conforming elements. Our algorithm extends the work of Reitzinger and Schöberl on edge AMG to high order hierarchical elements. Results from numerical tests are given for bases up to order $p = 9$.

1. Introduction. We are interested in algebraic multigrid methods for solving

$$Ax = b \tag{1.1}$$

resulting from high order finite element discretizations of the eddy current problem using $H_0(curl,\Omega)$ conforming elements on triangular meshes. The space $H_0(curl,\Omega)$ is defined as

$$H_0(curl,\Omega) = \{u \in L^2(\Omega) | \nabla \times u \in L^2(\Omega)\}.$$

$H(\text{curl})$ conforming elements have been known as Nedelec elements or Whitney forms [15]. These elements have degrees of freedom on the edges as opposed to at the nodes and ensure tangential continuity.

The weak formulation of the eddy current problem is given by: Find $u \in H_0(curl,\Omega)$ such that

$$\int_\Omega \mu \nabla \times u \cdot \nabla \times v + \int_\Omega \sigma u \cdot v = \int_\Omega f \cdot v \tag{1.2}$$

for all $v \in H_0(curl,\Omega)$ [8]. We denote the discretization of the left hand side of (1.2) by $H(\text{curl})$ conforming elements as K^e.

Standard AMG methods were developed for elliptic H^1 problems and hence do not work well due to the large kernel of the curl operator which contains the gradient of all differentiable functions, since for any u,

$$\nabla \times \nabla u = 0$$

Reitzinger and Schöberl [10] developed an AMG method for (1.1) using lowest order edge elements and Bochev et. al. [4] obtained almost h independent results using smoothed aggregation techniques. We give a brief overview of their algorithms in section 2.

We are interested in discretizations of 1.2 by high order finite elements, since they are able to obtain more accurate results with less degrees of freedom. However, the equations become more coupled in which case the resulting matrix becomes less sparse. Also, the condition number increases with the order of the elements, which adversely affects the convergence of the iterative method. In the nodal case, [6], [9] and [13] developed AMG methods for high order interpolatory and spectral finite elements.

An interpolatory high order edge element basis for $H(\text{curl})$ was developed by Graglia [5] and Webb [14] introduced a hierarchical basis. [12] give a multigrid approach to solving systems discretized by the high order hierarchical elements of Webb. Using Webb's idea of hierarchical elements, Schöberl and Zaglmayr [11] propose hierarchical H^1 and $H(\text{curl})$ bases

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which satisfy the de Rham complex. This is done by explicitly including the gradients of the high order H^1 functions in the $H(\text{curl})$ space. Zaglmayr [16] also gives an additive Schwarz method for the high order elements. We use this basis as our discretization of (1.2) and give a summary in section 3.

In section 4 we propose an AMG algorithm for solving (1.2) discretized by the basis functions of [11] and section 5 contains numerical tests for our algorithm.

2. AMG for edge elements. We give a brief overview of the edge AMG algorithm of [10]. To define an AMG algorithm, we must specify the interpolation/restriction operator and the smoother. It is not immediately evident how to aggregate edges in order to define an interpolation operator. Reitzinger and Schöberl consider the related nodal problem

$$\int_\Omega \mu \nabla u \cdot \nabla v + \int_\Omega \sigma u v$$

and its nodal element discretization K^n. The main idea is that nodal aggregates induce edge aggregates. Using the standard AMG algorithms, say smoothed aggregation, on K^n we obtain interpolation operators which define nodal aggregates which in turn give rise to nodal interpolation operators P_k^n.

Let W_h^0 and W_H^0 be the fine and coarse nodal finite element spaces respectively and let W_h^1 and W_H^1 be the fine and coarse edge finite element spaces respectively, then we define P_k^e, the edge interpolation operator in such a way that the following de Rham diagram commutes

$$\begin{tikzcd} W_h^0 \arrow[r, "D_k"] \arrow[d, "P_k^n"] & W_h^1 \arrow[d, "P_k^e"] \\ W_H^0 \arrow[r, "D_{k+1}"] & W_H^1 \end{tikzcd} \tag{2.1}$$

that is

$$P_k^e D_k = D_{k+1} P_k^n$$

where D_k is the discrete gradient operator for level k mapping the nodal space into the edge space. D_k is a matrix of where each row represents and edge and contains exactly one -1 and one 1. For example an edge $i \to j$ would have a -1 in column i and a 1 in column j. Once the edge interpolation operators are defined, the coarse grid edge operator is obtained by the Galerkin method, $K_{i+1}^e = (P_i^e)^T K_i^e P_i^e$. The edge interpolation operators P_k^e and the discrete gradients D_k can be obtained using the `coarsen_complex` algorithm defined in [2].

For each level of the multigrid iteration, we use the hybrid smoother of Hiptmair [7]. To obtain almost h-independent results, the interpolation operator P_k^e is smoothed by an appropriate smoother, either Chebyshev or Jacobi [4].

3. High order hierarchical finite element basis for $H(\text{curl})$. We give a brief review of an order p $H(\text{curl})$ basis defined in [11] for triangles.

For each edge (e_1, e_2), the lowest order basis function is the Whitney form

$$\phi_{12}^{LO} = \lambda_{e_1} \nabla \lambda_{e_2} - \lambda_{e_2} \nabla \lambda_{e_1}$$

where λ_{e_1} and λ_{e_2} are the barycentric coordinates for verticies e_1 and e_2 respectively. These functions have constant tangential components on edge (e_1, e_2) and zero tangential components on the other two edges.

In the following, $l_n(x)$ is the nth Legendre polynomial of order p and $L_i^S(x,t)$ is the scaled integrated Legendre polynomial defined by

$$L_n^S(x,t) = t^n \int_{-1}^{x/t} l_{n-1}(s)ds$$

For each edge (e_1, e_2) and $i \in [0, p-1]$, we define the high order edge functions as

$$\phi_{12,i}^{HO} = \nabla L_{i+2}^S(\lambda_{e_1} - \lambda_{e_2}, \lambda_{e_2} + \lambda_{e_1}).$$

here L_i^S is the scaled integrated Legendre polynomial. The tangential components of these high order edge functions span the space of polynomials of order p on the interior of edge (e_1, e_2) and vanishes on all other edges. These are also gradients of the high order edge based H^1 functions.

On the interior of each triangle, the following three functions are defined, for $i + j \in [0, p-2], i, j \geq 0$,

$$u_i = L_{i+2}^S(\lambda_2 - \lambda_1, \lambda_1 + \lambda_2)$$
$$v_j = \lambda_3 l_j(2\lambda_3 - 1)$$

1. $\phi_{ij}^{int,1} = \nabla(u_i v_j)$
2. $\phi_{ij}^{int,2} = \nabla u_i v_j - u_i \nabla v_j$
3. $\phi_j^{int,3} = (\nabla\lambda_1\lambda_2 - \lambda_1\nabla\lambda_2)v_j$

These polynomials span the space of polynomials of order p on the interior of the triangle and have vanishing tangential components on the boundary of the triangle. The functions 1. are gradients of the high order face based H^1 functions.

The functions defined above are linearly independent and there are $(p+1)(p+2)$ functions which is the same as the Nedelec space of order p and hence define a basis for the space. For a more detailed explanation see [11] and [16].

4. AMG algorithm. We propose a fully algebraic multigrid method. [12] give a two level multigrid algorithm for solving $H(\text{curl})$ problems discretized by Webb's hierarchical elements. Since the basis is hierarchical, a high order discretization explicitly contains low order components as degrees of freedom. On the first level, they restrict the high order problem to the lowest order components and solve the second level directly. A similar approach is done in [16] where they use a two level additive Schwarz method in which the space is split into low order and high order components. They propose that the second level be solved directly or using the smoother of [1].

Instead of a two level method where the low order coarse space is solve directly, we propose a fully multilevel algebraic method. For our AMG method, we use V cycling. To define an AMG algorithm, we specify, for each level, the smoother and the interpolation. In the following let $W_{HO,p}^1$ and $W_{LO,0}^1$ be the high order space of order p and and lowest order finite element spaces respectively.

Assuming the discretization is of order p, then due to the hierarchical nature of the basis, there are degrees of freedom corresponding to orders $0, ..., p$. On the finest level, we smooth once using pointwise Gauss-Seidel. The coarse space is defined as those degrees of freedom corresponding to any order less than p. We remark that coarsening strategies can range from very aggressive ($W_{HO,p}^1$ to $W_{LO,0}^1$) to very mild ($W_{HO,p}^1$ to $W_{HO,p-1}^1$). The strategy used affects the convergence and cycle complexity of the algorithm. Different strategies are tested in the following section. Once we have chosen the order of the coarse space say, $W^1_{HO,k}$ where $0 < k < p$, we define the restriction operator as $R : W^1_{HO,p} \to W^1_{HO,k}$ where

$$R^e_{ij} = \begin{cases} 1, \text{ if } j \text{ is the } i\text{th dof of order } \leq k, \\ 0, \text{ otherwise.} \end{cases} \tag{4.1}$$

The interpolation operator is defined as the transpose of the above restriction and we can now define the coarse grid operator using Galerkin's method as

$$K^e_{k+1} = R^e_k K^e_k P^e_k.$$

This process is continued recursively until we reach the lowest order space, $W^1_{LO,0}$, where we complete the V cycle using the smoothed aggregation edge algorithm given in [4]. That is, we now define P^e as described in section 2 and we use the hybrid smoother of Hiptmair [7]. The recursion terminates when the size of the matrix meets a certain tolerance.

We summarize the above with the following algorithm. Suppose we are given $P_1,...,P_j,P_{j+1},...,P_l$, where $P_0,...,P_j$ are the high order interpolation operators defined by (4.1) and $P_{j+1},...,P_l$ are the smoothed interpolation operators obtained by [4]

$V\_cycle(A_k,u_k,b_k,k)$
if $k > j$ **then**
  $u_k \leftarrow Hiptmair\_hybrid(A_k,b_k,u_k)$
else
  $u_k \leftarrow Gauss\_Seidel(A_k,b_k,u_k)$
end if
if $k \leq l$ **then**
  $r_k \leftarrow b_k - A_k u_k$
  $A_{k+1} \leftarrow P_k^T A_k P_k$
  $u_{k+1} \leftarrow 0$
  $V\_cycle(A_{k+1},P_k^T r,u_{k+1},k+1)$
  $u_k \leftarrow u_k + P_k u_{k+1}$
  if $k > j$ **then**
    $u_k \leftarrow Hiptmair\_hybrid(A_k,b_k,u_k)$
  else
    $u_k \leftarrow Gauss\_Seidel(A_k,b_k,u_k)$
  end if
else
  $u_k \leftarrow A_k^{-1} b_k$
end if

5. Numerical Results. We implemented the hierarchical finite element basis defined in section 3 in Python and used PyAMG libraries [3] for the AMG solver. In our tests, we take Ω to be the unit square $[0,1]\times[0,1]$ with homogeneous Dirichlet boundary conditions triangulated by structured and unstructured meshes. We use AMG as a preconditioner to conjugate gradient for solving (1.2). For simplicity, we assume μ and σ are constants in which case the resulting system becomes

$$K^e = \mu S + \sigma M$$

where S and M are the $H(\text{curl})$ stiffness and mass matricies respectively.

We are interested in how CG preconditioned by our proposed AMG method performs on

$$K^e u = f$$

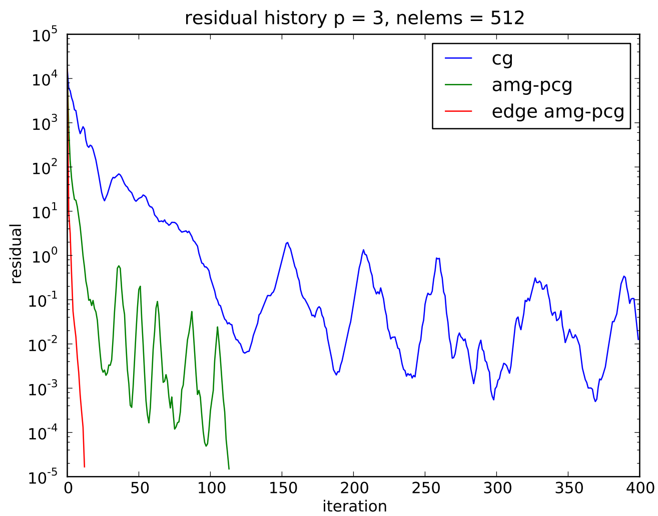

FIG. 5.1. *Residual history of CG, standard AMG-PCG, edge AMG-PCG with $p = 3$, nelems = 512*

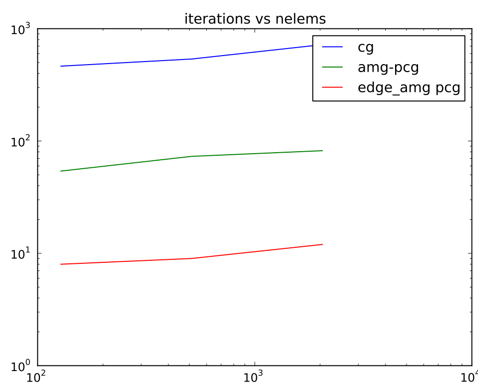

FIG. 5.2. *Dependence on h for $p = 2$, (iterations vs number of elements)*

where we obtain f by taking a random u and multiplying $K^e u = f$. In each case, we iterate until the residual is less than 10^{-8}. In the following tests, we used $V(1,1)$ cycling for the AMG preconditioner and we took $\mu = 1$ and $\sigma = 10^{-4}$.

Figure 5.1 compares the residual history of CG, standard AMG preconditioned CG, and our proposed edge AMG preconditioned CG using the most aggresive high order coarsening scheme, using two levels, with the interpolation operator being $P : W^1_{LO,0} \rightarrow W^1_{HO,p}$. The test was done for $p = 3$ and 512 elements.

We further tested the $p \rightarrow 0$ coarsening for dependence on h which can be seen in 5.2. Since we used smoothed aggregation on the lowest order space, the results were relatively h-independent.

We tested the algorithm on unstructured meshes, in the case of $p = 2$ we obtain the following iteration counts

	edge AMG-PCG	AMG-PCG	CG
num elems = 110	8	58	932
440	9	65	763
1760	11	93	998
7040	15	144	1363

We checked for effects of σ on our algorithm. For $p = 4$, the iteration counts for different σ are collected in the following table

	edge AMG-PCG	AMG-PCG	CG
$\sigma = 10$	16	80	3039
10^{-1}	15	144	6189
10^{-3}	13	113	2566

The convergence of the algorithm does not decrease as σ decreases.

In the following, we consider the different coarsening strategies and how each affects convergence and AMG cycle complexity. AMG cycle complexity gives an estimate of the total work for one V cycle relative to one smoothing iteration on the finest level. We use three different coarsening strategies. Assuming the highest order matrix is of order p, then the following describes the maximum order of each level

1. $p-1, p-2, ..., p-i, ..., 1, 0$
2. $p/2, p/4, ..., p/2^i, ..., 1, 0$
3. 0

We obtain the following PCG iterations / cycle complexities for the above three strategies on a mesh with 512 elements.

	1. $(p \to p-1)$	2. $(p \to p/2)$	3. $(p \to 0)$
$p = 1$	13 / 2.68	13 / 2.67	13 / 2.67
2	13 / 2.61	13 / 2.61	13 / 2.15
3	12 / 2.89	13 / 2.21	13 / 2.05
4	13 / 3.25	14 / 2.38	15 / 2.02
5	14 / 3.63	16 / 2.19	18 / 2.02
6	17 / 4.02	19 / 2.26	23 / 2.01
7	20 / 4.41	25 / 2.15	31 / 2.00
8	23 / 4.81	32 / 2.25	34 / 2.00
9	39 / 5.2	54 / 2.16	58 / 2.00

Although strategy 1 converges in the least number of iterations, the cycle complexity grows with p. For strategies 2 and 3, the cycle complexities do no grow with p, in particular, for strategy 2, the iteration counts are lower than those for strategy 3. For this reason, we choose 2. as our coarsening strategy.

6. Conclusion. We propose a fully algebraic multigrid method for high order hierarchical elements by extending the edge AMG algorithm of Reitzinger and Schöberl [10]. In our numerical tests we test different coarsening strategies and we demonstrate that the proposed method works well for orders up to $p = 7$. However, for each the three different coarsening strategies proposed, the the convergence begins to deteriorate after $p = 7$. In our research, we looked into the convergence of the two grid AMG method with the coarse grid being $p-1$. As we can see in Figure 6.1, the convergence begins stagnating for $p \geq 7$. We also looked into hybrid smoothers where we first perform one iteration of Gauss-Seidel on the whole system, then one iteration of block Gauss-Seidel on the gradient functions. From Figure 6.2, we see that the convergence improves, but still stagnates for $p \geq 8$. We are currently looking into different smoothers to obtain more p-independent results.

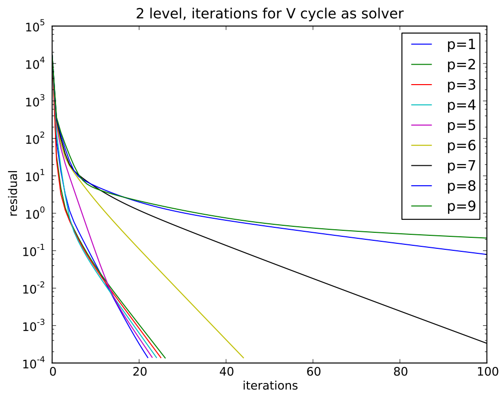

FIG. 6.1. *2 grid convergence*

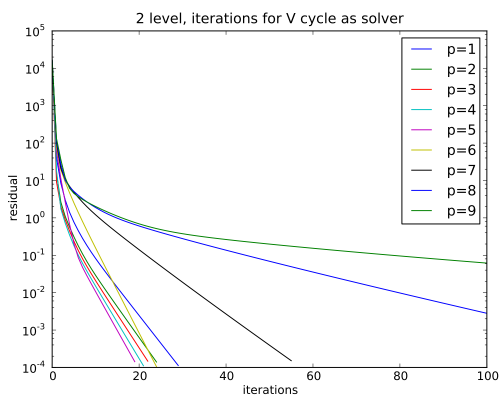

FIG. 6.2. *2 grid convergence using hybrid smoother*

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