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Multilevel Preconditioners for Nonselfadjoint or Indefinite Orthogonal Spline Collocation Problems

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TheoremWe develop and study symmetric multilevel preconditioners for the computation ofthe orthogonal spline collocation (OSC) solution of a Dirichlet boundary value problem(BVP) with a nonselfadjoint or an indefinite operator.The OSC solution is sought in the space of piecewise Hermite bicubic spline functionsdefined on a uniform partition.We consider an additive and a multiplicative multilevel preconditionersthat are used with the preconditioned conjugate gradient (PCG) method.Our results and algorithms are closely related to those in[1], [2], [3],and [4].Let Ω be a unit square $(0,1)\times(0,1)$ with the boundary $\partial\Omega$, andlet $x=(x_1,x_2)$. We consider a BVP

$$Lu \equiv \sum_{i,j=1}^{2} a_{ij}(x)u_{x_ix_j} + \sum_{i=1}^{2} b_i(x)u_{x_i} + c(x)u = f(x), \quad x\in\Omega, \quad u=0 \text{ on } \partial\Omega. \tag{1}$$

Operator L could be non-selfadjoint or indefinite in L^2 inner product.We assume that the principal part of L satisfies the uniform ellipticity conditionand that BVP (1) has a unique solution in $H^2(\Omega)$.Let π_0 be a uniform coarsest rectangular partition of Ω. We obtain a set of partitions$\{\pi_k\}_{k=0}^{K}$ by standard coarsening, andlet $V_0\subset V_1\subset\ldots\subset V_K\equiv V_h$ be the set of corresponding nested spaces ofpiecewise Hermite bicubics that vanish on $\partial\Omega$.Let $\sum$ denote the 2-D composite Gauss quadrature corresponding to partition π_h with 4 nodesin each element. Let $\mathcal{G}_h$ denote the corresponding set of Gauss points.The OSC discretization of BVP (1) is defined by

$$u_h\in V_h, \quad Lu_h(\xi)=f(\xi), \quad \xi\in\mathcal{G}_h, \tag{2}$$

and it can be written as the operator equation $L_hu_h=f_h$ in the Hilbert space V_h with theinner product $(v,w)_h=\sum vw$.Let $\{\psi_{k,j}\}_{j=1}^{N_k}$ be the standard finite element basis of V_k consistingof products of one-dimensional value and slope basis functions.Using the space decomposition

$$V_h = V_0 + \sum_{k=1}^{J}\sum_{j=1}^{N_k} V_{k,j}, \quad V_{k,j}=\operatorname{span}(\psi_{k,j}),$$

we define and study multilevel additiveB_a and multiplicative B_m preconditionersfor solving the normal equation $L_h^* L_h u_h = L_h^* f_h$, where L_h^* is the adjoint to L_h.The implementation of B_a and $B_\mathrm{m}$is based on relationships between basis functions for two consecutive partitionsand the implementation of B_m is similar to that forV(1,1)-cycle with the Gauss-Seidel smoothing.A problem on the coarsest partition is assumed sufficiently small, and it is solved exactly.The computational cost of the preconditioning algorithms is $\mathrm{O}(N_K)$.The following is our main result.There are positive independent of h and K constantsα_a,β_a,α_m,andβ_m,such that

$$\begin{aligned}
&\alpha_\mathrm{a}\,(B_\mathrm{a}v,v)_h \le (L_h^* L_h v,v)_h \le \beta_\mathrm{a}\,(B_\mathrm{a}v,v)_h, \quad v\in V_h, \\
&\alpha_\mathrm{m}\,(B_\mathrm{m}v,v)_h \le (L_h^* L_h v,v)_h \le \beta_\mathrm{m}\,(B_\mathrm{m}v,v)_h, \quad v\in V_h.
\end{aligned} \tag{3}$$

To obtain this result, we prove the key assumptions in the general theory of Schwarz methodsformulated in [4] and use the inequalities(see [2])

$$C^{-1}\|v\|^2_{H^2(\Omega)} \le a_h(v,v) \le C\|\Delta v\|^2_{L^2(\Omega)}, \quad v\in V_h.$$

In the following table, we present results of our numerical computations; that is,the ratios of spectral constants in (3),the convergence factor $\bar{\rho}$, which is the geometric mean of consecutiveresidual ratios, and the CPU time.

	Additive			*Multiplicative*			*General*		
J	$\beta_\mathrm{a}/\alpha_\mathrm{a}$	$\bar{\rho}$	$t(s)$	$\beta_\mathrm{m}/\alpha_\mathrm{m}$	$\bar{\rho}$	$t(s)$	$\beta_\mathrm{m}/\alpha_\mathrm{m}$	$\bar{\rho}$	$t(s)$
3	3.883	0.072	0.19	1.367	0.005	0.18	925.2	0.094	0.33
4	4.490	0.101	0.59	1.435	0.007	0.90	515.2	0.096	1.80
5	5.016	0.125	2.13	1.476	0.008	3.87	457.5	0.121	8.44
6	5.488	0.142	9.13	1.500	0.009	16.45	402.4	0.166	42.67
7	5.845	0.156	49.93	1.515	0.009	73.43	381.4	0.202	199.40
8	6.162	0.168	278.10	1.524	0.009	334.60	377.3	0.224	995.60

Under *Additive* and *Multiplicative*, we list results for Poisson's equation,and under *General* – results fora PDE with a general nonselfadjoint and indefinite operator L with the coefficients

$$\begin{aligned}
&a_{11}(x) = e^{x_1x_2}, \quad a_{12}(x) = 0.5/(1+x_1+x_2), \quad a_{22}(x) = e^{-x_1x_2}, \\
&b_1(x) = x_2e^{x_1x_2} + 10\cos[\pi(x_1+x_2)],_2(x) = -x_1e^{-x_1x_2} + 50\sin(2\pi x_1x_2), \\
&c(x) = 50[1+1/(1+x_1+x_2)].
\end{aligned}$$

The problem with the general equation is solved using the multilevel multiplicative preconditioner.We set $\pi_0 = \Omega$ and reduce the relative residual to less than 10^{-12}.The numerical results demonstrate the efficiency of our preconditioning algorithms.

Bibliography

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