AN ALGEBRAIC MULTILEVEL METHOD FOR ANISOTROPIC ELLIPTIC EQUATIONS BASED ON SUBGRAPH MATCHING

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Abstract. This paper presents a strength of connection measure for algebraic multilevel algorithms for a class of linear systems corresponding to the graph Laplacian on a general graph. The coarsening in the multilevel algorithm is based on matching in the underlying graph. Our main new idea is to define a local measure of the quality of the matching whose maximum gives an upper bound on the stability (energy norm) of the projection on the coarse space. As an application, we focus on utilizing this measure as a tool for constructing coarse spaces for anisotropic diffusion problems. Specifically, we consider the diffusion equation with grid aligned as well as non-grid aligned anisotropies in the diffusion coefficient and show that the strength of connection measure is able to appropriately capture the correct anisotropic behavior in both cases. We then study a coarsening algorithm that uses this measure in a greedy strategy to find the subgraph matching (set of aggregates). The process forms an initial set of subgraphs, each consisting of a single vertex, and then adds vertices to these subgraphs corresponding to the local direction of the anisotropy as determined by the proposed measure.

1. Introduction. The main task in setting up a variational algebraic multigrid method (AMG) is to define an interpolation operator to map corrections from the coarse to fine grids. Interpolation must by definition be accurate for the so-called "smooth" error that cannot be eliminated by Jacobi or Gauss-Seidel iterations. To construct an interpolation, the classical AMG methods use the idea that such smooth error varies little along the strongly connected nodes [1, 2, 9, 11].

From this point of view, a key issue to consider when designing an algorithm for constructing AMG interpolation is to find appropriate weights that describe the ability of a coarse grid node to distribute smooth errors to its neighbors. The basic approach used in classical AMG is based on heuristics derived using properties of M-matrices [11]. Roughly speaking, each off-diagonal entry in a given row of the system matrix, say the i-th row, is compared to the maximum value of the off-diagonal entries in this row (times a parameter) and if these values are proportional, then the given variable is said to be strongly coupled to the i-th one. It is well known that this measure can be sensitive to the parameter used in defining it for anisotropic diffusion problems [4].

An alternative way of defining AMG interpolation is to find a set of aggregates such that each aggregate represents exactly one coarse grid node and contains a collection of strongly connected fine grid neighbors, defined in terms of the graph of the matrix. Then, given this set of aggregates, interpolation is chosen as the basis of the space of piecewise constant vectors over the aggregates. Such approaches and their smoothed aggregation variants have been successfully analyzed for and applied to elliptic-value problems [8, 10, 12–14]. A multilevel method using matching in graphs is found in [7], where such method is also applied to convection-diffusion problems. A main step in such a construction of coarse spaces is to select the aggregates and, as in classical AMG, the basic approach relies on a strength of connection measure that is difficult to apply to anisotropic problems.

Aggregation based methods are particularly attractive from a theoretical point of view in that their simplicity allows for direct calculation of a variety of key quantities used in estimating their convergence and complexity properties. Two-level and multilevel methods based on such interpolation can be proven to be uniformly convergent with respect to the size of the finest grid, under certain assumptions on the aggregates. We mention some recent theoretical results on convergence of plain aggregation using an Adaptive Multilevel Iteration [3, 15] and on Smoothed Aggregation using polynomial approximations to smooth the tentative prolongation operator [5]. In this paper, we focus on linear systems whose matrices are weighted graph Laplacians.

The remainder of the paper is organized as follows. First, in Section 2 we review some properties of the graph Laplacians as well as the multilevel matching (MLM) algorithm. Further, we state some results from the convergence theory of the MLM algorithm applied to such systems. Such results are based on estimates of the stability properties of the interpolation defined via the matching. The latter stability estimates are obtained by constructing an operator as part of a commutative diagram which in addition to this operator involves the l_2 projection on a subset of vertices and the discrete gradient. In Section 3, we introduce a local measure of the convergence and prove that the convergence rate of the two-level matching method can be bounded by an increasing function of this measure and an additional constant related to the performance of the smoother restricted to a subspace. Then, we introduce a black box algorithm for selecting a matching

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minimizing this measure. Finally, in Section 4, we show the promise of the algorithm by testing it for three examples of anisotropic elliptic problems. Specifically, we present bounds on the stability of the matching interpolation for a family of graph Laplacians whose matrices correspond to finite element (finite difference) discretizations of anisotropic diffusion problems on structured grids.

2. Notation and Preliminary Results. In this section, we introduce some essential notations related to linear systems corresponding to graph laplacians and also give an outline of our multilevel matching algorithm for solving this system. We conclude the section with a review of some of the AMG theories found in [6] and references therein related to our algorithm. For specifics on the two-level and multilevel convergence analysis of the approach we refer the reader to our work in [3].

2.1. Graphs and Graph Laplacians. Consider a undirected weighted graph $G = \{V, E, W\}$, which is a triplet of sets of vertices, edges, and weights. For each edge $k = (i, j) \in E$ we have a weight w_k. We assume throughout the paper that all weights are positive. The graph Laplacian A corresponding to the graph G is a matrix defined via the following relation:

$$(Au, v) = \sum_{k=(i,j)\in E} w_k(u_i - u_j)(v_i - v_j), \quad \forall u, v \in \mathbb{R}^{|V|},$$

where $|V|$ is the number of vertices in the set V. Equivalently, we write the linear system

$$Au = f, \tag{2.1}$$

where A is the weighted graph Laplacian.

For a graph, G, we define the discrete gradient operator $B : \mathbb{R}^{|V|} \to \mathbb{R}^{|E|}$ as

$$B^T e_k = e_i - e_j, \quad k = (i, j) \in E,$$

where e_k is the k-th standard basis in $\mathbb{R}^{|E|}$. Define $D : \mathbb{R}^{|E|} \to \mathbb{R}^{|E|}$ as a diagonal matrix and $D_{kk} = w_k$, then we can rewrite A as

$$A = B^T DB.$$

Then, equation (2.1) can be written in variational form:

$$\text{Find } u \text{ such that } (DBu, Bv) = (f, v), \quad \forall v \in \mathbb{R}^{|V|}. \tag{2.2}$$

Our goal is to define a multilevel matching algorithm to solve for u in (2.2).

2.2. Graph matching. A *matching* in a graph is a subset of the set of edges, such that no two edges from the matching have a common vertex. A matching is called maximal if no more edges can be added to it. Clearly any maximal matching corresponds to a partitioning of the graph in subgraphs. Each of the subgraphs in such partitioning is at most of size 2 and has at most two vertices and one edge connecting them. Such a partitioning in subgraphs is often referred to as a 1-1 matching.

A matching is called *perfect* if the endpoints of the edges in the matching cover all the vertices of the original graph. Any perfect 1-1 matching is equivalent to a partition of the graph such that all subgraphs contain *exactly* two vertices and a single edge.

Here we use a more general definition of matching, as a set of subgraphs $\{G_i\}$. Each subgraph is connected and is given by $G_i = \{V_i, E_i, W_i\}$. The set of subgraphs satisfy that $\cup_i V_i = V$ and

$$k = (i, j) \in E_l \text{ if and only if } i \in V_l, j \in V_l, \text{ and } k \in E.$$

The weights of the edges in the subgraphs are the weights of these edges in the original graph G.

REMARK 2.1. *There are a variety of methods to form a matching of G. Graph partitioning methods can generate matchings consisting of any number of subgraphs. In the AMG setting, the aim is to form a matching in a way that the number of vertices in any subgraph is bounded by a constant independent of the size of the graph. Such condition allows for control of the fill in of the coarse level matrix, $A_c = P^T AP$. For example if G is planar, one can easily prove that a partitioning in subgraphs of a planar graph and constructing piecewise constant interpolation P (constant on each of the subgraphs) results in A_c which corresponds to a planar graph as well. Such property is not present in constructions of P that use "smoothing of the prolongation".*

2.3. Space of Piecewise Constant Vectors. On a given fine graph, G, we assign values to all the vertices; the ordered collection of these values are thus vectors in $\mathbb{R}^{|V|}$. We refer to this space as U and view it as the fine-level space. Given this fine-level space, we aim to construct a coarse space consisting of a collection of subgraphs, each representing a coarse vertex on the coarse graph.

Given a set of subgraphs $\{G_i\}$, we define a subspace

$$U_c = \{u \in U | u \text{ is constant on any } V_i\},$$

where U_c is the coarse space for the two-level method mentioned later. Define Q as the l_2-projection onto the space U_c, such that the i-th component of Qu is

$$(Qu)_i = \frac{1}{|V_j|} \sum_{k \in V_j} u_k, \quad \text{for } i \in V_j.$$

Algebraic multigrid theory indicates that the coarse space should represent all smooth errors that cannot not be eliminated by relaxations, i.e., errors with small A-norm. Noticing that

$$|Qu|_A \leq |Q|_A |u|_A,$$

we can measure the A-norm of the operator Q to check if the Q-projection of a smooth error is still a smooth error. However, evaluating $|Q|_A$ is not straightforward. As such, we introduce an alternative way to estimate $|Q|_A$ that is relatively easier to compute.

2.4. The Commutative Diagram. Here, we introduce a commutative diagram and then show how the corresponding commutative relations can be used to estimate the stability of a projection to the coarse space and also to analyze the convergence properties of the matching algorithm for the graph Laplacian system.

LEMMA 2.2. *Let $\{G_i = (V_i, E_i, W_i)\}$ be a splitting of G in subgraphs with non-overlapping sets of vertices. Let B be the discrete gradient on G and Q be the l_2 projection onto the space of piecewise constant (with respect to the splitting of the vertices) vectors. Then there exists a $\Pi : \mathbb{R}^{|E|} \to \mathbb{R}^{|E|}$ such that*

$$\Pi B = BQ. \tag{2.3}$$

We note that the lemma does not assume anything on the size of the subgraphs. It holds for any splitting of a graph G into subgraphs.

The commutative diagram corresponding to (2.3) is provided in Figure 2.1 The proof of the lemma follows

$$\begin{array}{ccc} \mathbb{R}^{|V|} & \xrightarrow{B} & \mathbb{R}^{|E|} \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle \Pi} \\ \mathbb{R}^{|V|} & \xrightarrow[B]{} & \mathbb{R}^{|E|} \end{array}$$

Fig. 2.1: Commutative diagram relating the l_2 projection onto the space of piecewise constant vectors of a matching and the projection $\Pi : \mathbb{R}^{|E|} \to \mathbb{R}^{|E|}$ given in Lemma 2.2.

by a row-wise construction of a special form of Π such that $\Pi Bu = BQu$ for any u. Furthermore, it is easy to show that Π is not unique.

The main result leading to our local stability measure, π_k, follows from the lemma:

$$\begin{aligned} |Q|_A^2 &= \sup_u \frac{u^T Q^T B^T D B Q u}{u^T B^T D B u} \\ &= \sup_u \frac{u^T B^T \Pi^T D \Pi B u}{u^T B^T D B u} \\ &\leq \sup_v \frac{v^T D^{-1/2} \Pi^T D \Pi D^{-1/2} v}{v^T v} \\ &= \|D^{1/2} \Pi D^{-1/2}\|^2. \end{aligned}$$

Our main goal in this paper is to describe the matrix $D^{1/2}\Pi D^{-1/2}$ and then to define a local constant π_k to control the norm $\|D^{1/2}\Pi D^{-1/2}\|$.

3. The Construction of Subgraphs and the Local Constant π_k. Assuming that the graph, G, is partitioned into a set of subgraphs, we define a constant, π_k, on each of the edges. The values of π_k are determined by the subgraphs connected to its associated edge (or the subgraph that contains this edge). The precise definition of π_k and an algorithm for its computation are given next.

3.1. The Local Constant π_k. The identity $\Pi B = BQ$ means that Π maps the discrete gradient of any vector in $\mathbb{R}^{|V|}$ to the the discrete gradient of its average value on each of the subgraphs. Since the gradient of a constant is equal to zero, we can assign zeros to a row of Π corresponding to edges that are contained completely in a subgraph. For the edges that connect two subgraphs, we can use a discrete Taylor expansion to compute the corresponding row of Π.

We refer to the two types of edges as "internal edges" and "external edges". Using the above results, it is straightforward to compute one such matrix Π. Specifically, the k-th row of Π can be computed as follows:

$$(\Pi)_k = \begin{cases} \mathbf{0}^T & \text{if } k=(i,j) \text{ is an internal edge,} \\ e_k^T + C_{i,k}^T - C_{j,k}^T & \text{if } k=(i,j) \text{ is an external edge,} \end{cases}$$

where

$$C_{i,k} = \frac{1}{|V_i|} B_i (A_i + e_{i[k]} e_{i[k]}^t)^{-1} \mathbf{1}_i. \tag{3.1}$$

Here A_i, B_i, V_i and $\mathbf{1}_i$ are the graph Laplacian, discrete gradient, set of vertices, and constant vector on the i-th subgraph and $e_{i[k]}$ stands for a standard basis in $\mathbb{R}^{|V_i|}$ with the component associated with the endpoint of the edge k having a value of 1. A simple calculation shows that the nonzero row of $D^{1/2}\Pi D^{-1/2}$ is

$$(D^{1/2}\Pi D^{-1/2})_k = e_k^T + D_{kk}^{1/2} C_{i,k}^T D_i^{-1/2} - D_{kk}^{1/2} C_{j,k}^T D_j^{-1/2},$$

where D_{kk} is the k-th diagonal entry in D and D_i and D_j are the weight matrices for the i-th and j-th subgraph.

Noticing that a certain row of $D^{1/2}\Pi D^{-1/2}$ is determined by at most two subgraphs, both of which are assumed to be local, we compute the relation between the norm $|Q|_A$ and this row and then derive an appropriate scheme to compute the local constant π_k for each row. Let

$$X = D^{1/2}\Pi D^{-1}\Pi^T D^{1/2}.$$

Then

$$\begin{aligned} |Q|_A^2 &\le \|D^{1/2}\Pi D^{-1/2}\|^2 \\ &= \|D^{-1/2}\Pi^T D^{1/2}\|^2 \\ &= \lambda_{max}(D^{1/2}\Pi D^{-1}\Pi^T D^{1/2}) \\ &= \lambda_{max}(X). \end{aligned}$$

By definition, the nonzero diagonal entries X_{kk} are all larger than 1. Now using the Cauchy-Schwarz inequality on a given nonzero off-diagonal entry we have

$$X_{kl}^2 \le (X_{kk}-1)(X_{ll}-1),$$

which implies $|X_{kl}| < X_{kk}$ or $|X_{kl}| < X_{ll}$. By the Gershgorin Theorem, the largest eigenvalue of X can be estimated by

$$\lambda_{max}(X) \le c \max_i (X_{ii}),$$

where c is the the maximal number of nonzeros in a row of X. Assuming that in the i-th row there are c nonzeros and using the definition of an entry of X, as

$$X_{ij} = (D^{1/2}\Pi D^{-1/2})_i (D^{1/2}\Pi D^{-1/2})_j^T,$$

we can show that X_{ij} is a nonzero entry, only if $(\Pi)_i(\Pi_j)^T$ is not zero. By our construction, $(\Pi)_i$ is a sparse vector and can have nonzero entries only on the edges that correspond to the subgraphs that the i-th edge

connects. Therefore, $(\Pi)_i(\Pi_j)^T$ is nonzero only if the i-th and j-th edges connect to the same subgraph. For a fixed edge, i, the number of nonzeros in $(X)_i$ is bounded by the number of vertices in any two connected subgraphs times the maximum degree of the whole graph. Therefore, for graph Laplacians describing the finite-element method discretization of a scalar elliptic equation, the degree of the graph is bounded. This then implies that we can restrict the size of the subgraph to control the bound on c.

Here, it is clear that the norm $|Q|_A$ can be bounded by the maximal diagonal entry of X multiplied by a constant that depends on the structure of X as induced by the matching. We define the local constant, π_k, as

$$\pi_k = D_{kk}C_{i,k}^T D_i^{-1} C_{i,k} + D_{kk}C_{j,k}^T D_j^{-1} C_{j,k},$$

For positive definite matrix A that has negative weights, we define

$$\pi_k = |D_{kk}|C_{i,k}^*(D_i^{-1/2})^* D_i^{-1/2} C_{i,k} + |D_{kk}|C_{j,k}^*(D_j^{-1/2})^* D_j^{-1/2} C_{j,k}, \tag{3.2}$$

which depends on the two subgraphs and the weight of the edge that connects them. By using the Gershgorin theorem, we have

$$|Q|_A^2 \le \max_k \sum_l |X_{kl}| \le c \max_k(X_{kk}) = c \max_k(1+\pi_k). \tag{3.3}$$

This estimation is not the sharpest. However, it is very practical to evaluate or optimize one or a few constants π_k when compared with the costs of computing $|Q|_A$.

The locality of π_k provides some idea on how to construct a matching without using the global information. One extreme case is to begin the matching procedure with no subgraphs specified. At this point, we cannot evaluate $|Q|_A$ directly since Q has not been formed. However, we can construct two subgraphs locally such that the π_k on all edges that connect them are minimized. Further, since the values of π_k do not change when other subgraphs are formed later, locally optimizing π_k can lead to an effective and efficient scheme when compared to the global optimization problem.

Another extreme case is given by assuming that a set of subgraphs has been formed and there is a vertex which does not belong to any of the subgraphs. Assuming the degree of the graph is bounded by a small number, we can test all possibilities, i.e., group the vertex with any of the nearby subgraphs. Ideally, we would compute the norm $|Q|_A$ in each case, but this is a global operation and thus not very efficient if the graph is large. Instead, we enumerate all cases of grouping the last vertex and then compute the associated values of π_k. These are local operations and the cost is proportional to a polynomial of the size of the surrounding subgraphs, which we can set a limit on in the algorithm. Our actual algorithm for selecting the subgraph matching is given next.

3.2. A Subgraph Matching Algorithm. We call the procedure to form the set of subgraphs a "matching of subgraphs." To form such a matching that locally minimizes π_k on each proceeding step, we first construct subgraphs that each contain only a single vertex. Specifically, the subgraphs are distributed "densely" on the graph using a standard maximal independent set algorithm. Then, given this initial matching, we "grow" the subgraphs by adding ungrouped vertices v together to form larger subgraphs. During this step, a quasi-optimal choice of the subgraph to append the vertex with is made based on the constant, π_k, defined on the edges that connect v to its neighboring subgraphs.

We use the following algorithm that combines these ideas for the numerical tests provided in a later section.

Subgraph Matching

1. Find a maximal independent set $\{v_i\}$, and initialize a set of subgraphs $\{S_i\}$, defining them as $S_i = \{v_i\}$.
2. Find a vertex v which is not in any of the subgraphs, for all its neighboring subgraphs S_i, do
   (a) $S_i \leftarrow S_i \cup \{v\}$.
   (b) Compute $d_i = \max_k \pi_k$ where π_k are computed on the edges that connect v to a subgraph S_k.
   (c) $S_i \leftarrow S_i \backslash \{v\}$.
3. Find $k = \arg\min d_i$, and let $S_k \leftarrow S_k \cup \{v\}$.
4. Goto step (2) if there are vertices that do not belong to any subgraph.

This is a greedy algorithm in which we do not impose any restrictions on the size of the subgraphs. The idea is that starting with a maximal independent set, i.e., a dense and irregular initial set of subgraphs gives the freedom to grow the subgraphs in all directions as needed for general anisotropic diffusion problems. We observe in our tests for generic anisotropic elliptic problems on structured grids that by using a randomly distributed maximal independent set as the initial matching, this procedure generates a matching of subgraphs that generally follows the direction of the specified anisotropy.

4. Applications to Anisotropic Elliptic Equations. We analyze and test our approach for the second order, elliptic boundary value problem with Neumann boundary conditions discretized by finite elements (FE space is piece-wise linear functions S_h): Find $u \in S_h$, for a given polygonal (polyhedral) domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) and a source term $f \in L_2(\Omega)$, $\int_\Omega f = 0$, such that

$$\int_\Omega a(\boldsymbol{x})\, \nabla u \cdot \nabla v \;=\; \int_\Omega f(\boldsymbol{x}) v(\boldsymbol{x})\,, \quad \text{for all} \quad v \in S_h. \tag{4.1}$$

Our discussion focuses on discretizations on structured grids and the matrix valued function $a(\boldsymbol{x})$ is constant or variable.

4.1. Constant Coefficient Anisotropic Diffusion. Here we consider the case of a constant $a(\boldsymbol{x})$ corresponding to grid-aligned anisotropy: $0 < \epsilon < 1$ and

$$a(\boldsymbol{x}) = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}. \tag{4.2}$$

This is a problem for which the performance of AMG is well understood. Assuming that we use a finite difference method to discretize this equation on an $n \times n$ uniform grid, the stiffness matrix A corresponds to a weighted-graph Laplacian on which the weights are ϵ on the vertical edges along the x_2 axis and 1 on the horizontal edges along the x_1 axis.

To simplify our analysis we assume a Richardson iteration as a smoother and consider the two-level case, such that the coarse-level system, $P^T AP$, is solved directly. Given a matching of subgraphs, the coarse space is constructed from a piecewise constant basis – with each subgraph we associate a basis vector which equals one for each index in the subgraph, and equals zero otherwise. The prolongation matrix P then has columns consisting of these basis vectors.

Our analysis of this problem uses a result in [6] showing that the norm of the two-level error propagation operator can be estimated as

$$\rho(E_{TL}) \leq 1 - \frac{1}{\kappa_s |Q|_A^2}\,, \tag{4.3}$$

where κ_s is the spectral equivalence constant for the symmetrized smoother, $\tilde{M}$, and the matrix, $S^T AS$, with S chosen to satisfy $S^T P = 0$. For the Richardson iteration, it is then easy to show that $\kappa_s = \kappa(S^T AS)$, which is the condition number of $S^T AS$.

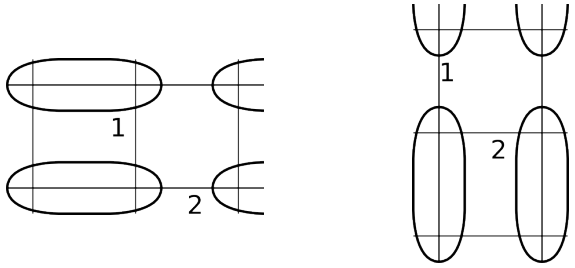
We note that for our model problem, since $\epsilon < 1$, the direction of the anisotropy is aligned with the horizontal or the x_1-axis. A smooth error is thus geometrically smooth along this direction of strong anisotropy. Of course, if a matching is chosen such that all subgraphs are aligned along the x_1-axis, then we expect the two level method to converge fast, since in this case smooth errors are well approximated by the coarse space. If we instead match vertices vertically, then our bound of the convergence rate for the two-level method should degrade. To illustrate these cases analytically and numerically, we compare the local constant, π_k, and κ_s for these two cases.

To compute π_1 for the horizontal matching shown in Figure 4.1a, which stands for the local measure π_k on the edge marked 1, we form the graph Laplacians, A_i and A_j, and the discrete gradients, B_i and B_j, for the i-th and j-th subgraphs that are connected by edge 1. Then,

$$A_i = A_j = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad B_i = B_j = \begin{pmatrix} 1 & -1 \end{pmatrix}.$$

By using (3.1), we can compute C_{1i} and C_{1j}. Then, defining the weight matrix

$$D_i = D_j = \begin{pmatrix} 1 \end{pmatrix}, \quad D_{11} = \epsilon,$$



(a) Horizontal (b) Vertical

Fig. 4.1: Horizontal (left) and vertical (right) matching of subgraphs.

and applying this result, we can evaluate $\pi_k, k = 1, 2$ using (3.2). This yields

$$\pi_1 = \frac{\epsilon}{2} \quad \text{and} \quad \pi_2 = \frac{1}{2},$$

where π_1 and π_2 are defined according to the matching used in Figure 4.1a. It is also straight forward to show that $\kappa_s \leq \frac{1}{2}$ in this case. This and the fact that $\pi_k \leq \frac{1}{2}$ on all edges independent of the choice of ϵ imply that $|Q|_A$ is bounded. By the estimate in (4.3), we conclude that the convergence rate of the two-level method for horizontal matching is bounded uniformly with respect to the grid size and the constant ϵ. Further, we mention that our local to global estimate in Equation (3.3) gives $|Q|_A \leq c \cdot \frac{3}{2}$ where $c = 11$ for this problem and the matching aligned with the x_1-axis.

If we instead match the subgraphs vertically, as in Figure 4.1b, then

$$\pi_1 = \frac{1}{2} \quad \text{and} \quad \pi_2 = \frac{1}{2\epsilon},$$

and κ_s grows proportionally to $1/\epsilon$. The resulting two-level method thus fails in two ways according to our theory: the Richardson iteration is not a suitable fine-point relaxation, and the coarse-level correction does not efficiently attenuate smooth errors. Of course, a lower bound on the convergence rate of the method similar to the one derived in [16] for anisotropic problems is needed to make this statement rigorous. Nonetheless, the results of our analysis show that our local constant, π_k, is able to detect a suitable coarsening for this anisotropic model problem.

We now estimate the condition numbers of this system preconditioned by a two-level matching algorithm obtained by choosing subgraphs aligned with the anisotropy. We test the system for different grid sizes and different values for ϵ. The results reported in Table 4.1 correspond to the condition numbers of the preconditioned system $B_{TL}Ax = B_{TL}b$, where B_{TL} is the action of the two-level method. We observe that the condition numbers remain bounded even when the grid size is increased and ϵ is decreased. We mention that if we align the subgraphs in the vertical direction to define the matching, then our numerical results show that the condition number of the two-level preconditioned system grows.

	32^2 grid	64^2 grid	128^2 grid
$\epsilon = 10^{-2}$	2.18	2.22	2.23
$\epsilon = 10^{-4}$	2.19	2.23	2.23
$\epsilon = 10^{-6}$	2.20	2.23	2.24

Table 4.1: The condition number estimates of the two-level preconditioned system for Problem (4.1) with $a(\boldsymbol{x})$ defined by (4.2) for various problem sizes and choices of ϵ.

Next, we consider the case when the direction of the anisotropy is not aligned with the grid, that is, the differential equation (4.1) where now

$$a(\boldsymbol{x}) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{4.4}$$

By using linear elements and a uniform triangulation, coupled with Neumann boundary conditions, we obtain a positive semi-definite stiffness matrix A. We label the weights on the horizontal, vertical, and "backslash" grid lines w_h, w_v, w_d, respectively, as illustrated in Figure 4.2. Direct computation then gives

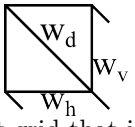


Fig. 4.2: Part of the finite element grid that is considered as a graph Laplacian

$$w_v = (1-\epsilon)\cos\theta\sin\theta + \sin^2\theta + \epsilon\cos^2\theta,$$
$$w_h = (1-\epsilon)\cos\theta\sin\theta + \cos^2\theta + \epsilon\sin^2\theta,$$
$$w_d = (\epsilon-1)\cos\theta\sin\theta.$$

We note if we assume that the problem (4.1) is defined on a rectangular domain, then the weights w_v, and w_h take on different forms for edges on the boundary. Further, the corresponding graph Laplacian is always positive semi definite, but it may have negative weights on some of the edges, depending on the choice of the parameters θ and ϵ. In this case, we can prove that if we form matchings of subgraphs containing exactly two vertices such that all subgraphs are aligned in the same direction and the angle between the edges in the subgraphs and the direction of the anisotropy is less than $\pi/8$, then the two level method converges uniformly with respect to ϵ and the problem size, for $\theta \in [-\pi/2, 0]$.

In the extreme case, when $a(\boldsymbol{x})$ is defined as (4.4), and $\theta = \pi/4$, it is difficult to determine an appropriate matching of subgraphs, even using the underlying geometry. For this system, none of the grid lines form an angle of less than $\pi/4$ with the direction of the anisotropy.

We test the performance of our matching algorithm applied to this problem to see if our algebraic approach can select a suitable matching in this setting. The plot in Figure 4.3 contains the matching our algorithm produces on a 16^2 grid. Here, the independent set algorithm chooses a initial set of subgraphs that corresponds to standard coarsening and then the algorithm to minimize π_k produces the given matching. We observe that the algorithm selects subgraphs along the direction of anisotropy.

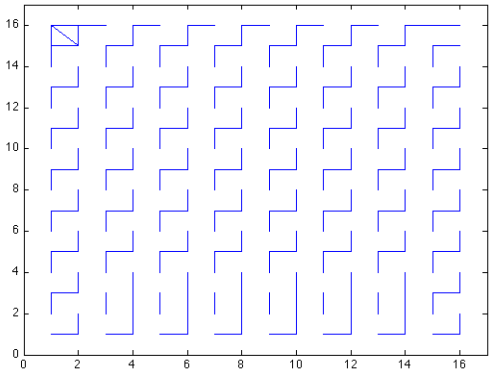


Fig. 4.3: Matching obtained on a 16^2 grid with $\theta = \pi/4$ and $\epsilon = 0.1$.

Generally, our numerical tests suggest that this strategy yields a two-level method that converges uniformly for this problem. In Table 4.2, we report the condition numbers of the two-level preconditioned systems for a variety of choices of ϵ and several grid sizes.

	16^2 grid	32^2 grid	64^2 grid	128^2 grid
$\epsilon = 10^{-2}$	11.16	13.15	13.92	14.38
$\epsilon = 10^{-4}$	11.16	13.16	13.70	14.20
$\epsilon = 10^{-6}$	11.16	13.15	13.93	14.43

Table 4.2: The condition number estimates of the two-level preconditioned system for Problem (4.1) with $a(\boldsymbol{x})$ defined by (4.4) for $\theta = \pi/4$.

We note that our definition of π_k uses the concept of "weak connections", similar to the notion of "strong connections" used in classical AMG methods. The aim in our approach is to find a set of subgraphs such that the inter-connections between any two subgraphs are "weak" according to the local measure π_k, when compared to the overall connections within the subgraphs.

4.2. Variable Coefficient Anisotropic Diffusion. Having studied the matching algorithm and local measure in some detail for the case of constant $a(\boldsymbol{x})$, we proceed to test the approach for (4.1) when $a(\boldsymbol{x})$ varies with $\boldsymbol{x} = (x_1, x_2)$. Specifically, we set

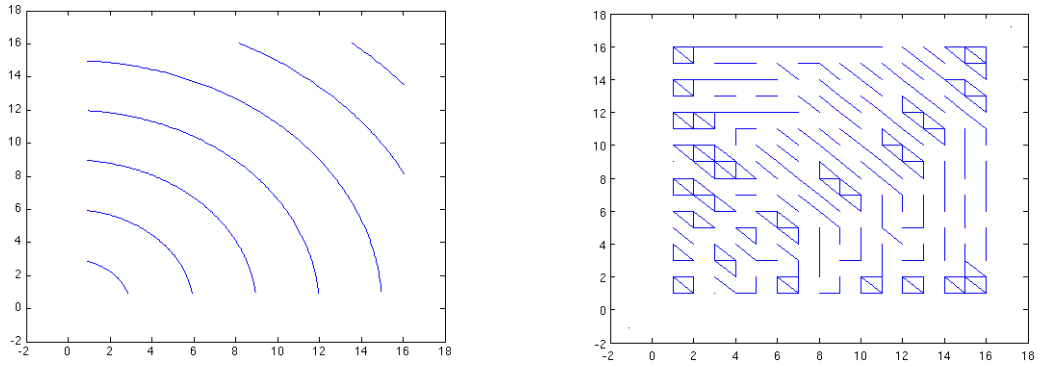
$$a(\boldsymbol{x}) = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \begin{pmatrix} c & -s \\ s & c \end{pmatrix}, \tag{4.5}$$

where

$$\epsilon = 0.2, \quad c = \frac{x_2}{\sqrt{x_1^2 + x_2^2}}, \quad s = \frac{-x_1}{\sqrt{x_1^2 + x_2^2}}, \tag{4.6}$$

and then discretize the problem using linear finite elements on a uniform triangulation. The directions of the strong anisotropy form circles centered at the origin, see Figure 4.4a. Noticing that discretizing this problem at the origin ($x = 0$, $y = 0$) is difficult, we instead discretize this equation on the two dimensional domain $[h, 1] \times [h, 1]$, where h is the size of the grid.

A plot of the direction of the anisotropy for this problem on a 16^2 grid is provided in Figure 4.4a and the matching our algorithm selects for this problem is given in Figure 4.4b.



(a) Direction of anisotropy. (b) Chosen matching.

Fig. 4.4: Direction of the anisotropy (left) and matching obtained on a 16^2 grid (right).

Here, we observe that the algorithm selects subgraphs that are aligned with the direction of the anisotropy, even when it is curved. Next, we report the condition numbers of the system preconditioned by the two-level method generated by our algorithm for different problem sizes. As the results show, the condition numbers

Grid size	16^2	24^2	32^2	48^2	64^2
$\kappa(B_{TL}A)$	7.05	12.53	19.16	35.61	59.83

Table 4.3: The condition number estimates of the system preconditioned by the two-level method for Problem (4.1) with $a(\boldsymbol{x})$ defined by (4.5), with various problem sizes.

grow with the problem size. Considering the matching given in Figure 4.4b that our algorithm selects for this problem, we notice that the subgraphs near the boundary contain many vertices. We note that in our numerical tests we observed similar behavior for larger graphs. The estimate in (3.3) suggests that $|Q|_A$ will grow in proportion to the maximal number of vertices in these subgraphs, even when π_k is bounded independent of the problem size.

A simple modification we make to our algorithm to overcome this issue is to introduce a post-processing step, in which we disaggregate the subgraphs. The idea is to fix a threshold, m, and then cut such subgraphs into several smaller ones. Estimates of the condition numbers of the preconditioned system obtained using this post-processing technique to adjust the initial matching are reported in Table 4.4. Here, we notice a marked improvement in the results, suggesting that the growth in condition numbers obtained for the matchings generated by our scheme without the post processing step are indeed due to the large cardinalities of some of the subgraphs selected by the initial matching step.

We conclude this section with a brief discussion on the computational complexity associated with our setup algorithm as well as the properties of the matching it generates. Assuming that the graph is partitioned into a set of subgraphs, the constant, π_k, is defined on all edges, and its value is determined by the subgraphs connecting to the ends of this edge (or the subgraph that contains the edge). If all subgraphs contain a

Grid size	16^2	24^2	32^2	48^2	64^2
$\kappa(B_{TL}A)$	2.97	3.54	3.84	4.14	4.32

Table 4.4: The condition number estimates of the Problem (4.1) with $a(\boldsymbol{x})$ defined by (4.5) preconditioned by the two-level method, for various problem sizes and choices of ϵ. Here, the post processing step is used to disaggregate the subgraphs.

limited number of vertices, then evaluating the constant π_k is of complexity $O(1)$. Therefore, optimizing the matching locally to minimize π_k can be done in a computationally optimal way.

Although we do not provide implementation details of our matching algorithm for sake of brevity, we mention that it is designed such that the number of subgraphs in the final matching is the same as the number of the vertices in a maximal independent set U. We thus expect that any vertex that is not in U has two neighbors in U on average. This in turn implies that the average size of a subgraph is $d/2+1$, where d is the average degree of the graph. As an example, consider a uniform triangulation corresponding to a two dimensional planar region. Then, we have $d=6$ and the matching algorithm will generate subgraphs with $d/2+1=4$ vertices on average.

5. Conclusions. We introduced a measure for choosing the matching used in forming AMG interpolation. It is also not difficult to generalize the commutative diagram to a matrix A that corresponds to a graph Laplacian with negative weights (but still globally positive semi-definite) and the two-level convergence theory of the matching algorithm still holds. We construct a black box algorithm based on local optimization, that can construct a two-level method that solves a large family of non M-matrix positive definitive systems.

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