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Error analysis of BDF Compound-Fast multirate method for differential-algebraic equations

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Analogue electrical circuits are usually modeled by differential-algebraic equations (DAE) of type:

$$\frac{d}{dt}\left[\mathbf{q}(t,\mathbf{x})\right]+\mathbf{j}(t,\mathbf{x})=\mathbf{0}, \tag{1}$$

where $\mathbf{x}\in\mathbb{R}^d$ represents the state of the circuit. A common analysis is the transient analysis, which computes the solution $\mathbf{x}(t)$ of this non-linear DAE along the time interval $[0,T]$ for a given initial state. Often, parts of electrical circuits have latency or multirate behaviour.

For a multirate method it is necessary to partition the variables and equations into an active (A) and a latent (L) part. The active and latent parts can be expressed by $\mathbf{x}_A=\mathbf{B}_A\mathbf{x}, \mathbf{x}_L=\mathbf{B}_L\mathbf{x}$ where $\mathbf{B}_A, \mathbf{B}_L$ are permutation matrices. Then equation (1) is written as the following partitioned system:

$$\begin{array}{l}\frac{d}{dt}\left[\mathbf{q}_A(t,\mathbf{x}_A,\mathbf{x}_L)\right]+\mathbf{j}_A(t,\mathbf{x}_A,\mathbf{x}_L)=\mathbf{0},\\ \frac{d}{dt}\left[\mathbf{q}_L(t,\mathbf{x}_A,\mathbf{x}_L)\right]+\mathbf{j}_L(t,\mathbf{x}_A,\mathbf{x}_L)=\mathbf{0}.\end{array} \tag{2}$$

In contradiction to classical integration methods, multirate methods integrate both parts with different stepsizes or even with different schemes. Besides the coarse time-grid $\{T_n, 0\le n\le N\}$ with stepsizes $H_n=T_n-T_{n-1}$, also a refined time-grid $\{t_{n-1,m}, 1\le n\le N, 0\le m\le q_n\}$ is used with stepsizes $h_{n,m}=t_{n,m}-t_{n,m-1}$ and multirate factors q_n. If the two time-grids are synchronized, $T_n=t_{n,0}=t_{n-1,q_n}$ holds for all n. There are a lot of multirate approaches for partitioned systems but we will consider the Compound-Fast version of the BDF methods. This method performs the following four steps:

1. The complete system is integrated at the coarse time-grid.
2. The latent interface variables are interpolated at the refined time-grid.

3. The active part is integrated at the refined time-grid, using the interpolated values at the latent interface.

4. The active solution at the coarse time-grid is updated.

The above methods can be shown to be stable under reasonable conditions. In this paper we will concentrate on error control.

The local discretization error δ^n of the compound phase still has the same behaviour $\delta^n = O(H_n^{K+1})$. Let $\bar{\mathbf{P}}^n, \bar{\mathbf{Q}}^n$ be the Nordsieck vectors which correspond to the predictor and corrector polynomials of $\mathbf{q}$. Then th error δ^n can be estimated by $\hat{\delta}^n$:

$$\hat{\delta}^n = \frac{-H_n}{T_n - T_{n-K-1}} \left[\bar{\mathbf{Q}}_1^n - \bar{\mathbf{P}}_1^n\right]. \tag{3}$$

Now $\hat{r}_C^n = \|\mathbf{B}_L\hat{\delta}^n\| + \tau\|\mathbf{B}_A\hat{\delta}^n\|$ is the used weighted error norm, which must satisfy $\hat{r}_C^n < \mathrm{TOL}_C$.
At the refined time-grid the DAE has been perturbed by the interpolated latent variables. The local discretization error $\delta^{n,m}$ is defined as the residue after inserting the exact solution in the BDF scheme of the refinement phase. However during the refinement instead of $\delta^{n,m}$ the perturbed local error $\tilde{\delta}^{n,m}$ is estimated. During the refinement each step $\mathbf{x}_A^{n-1,m}$ is computed from the following scheme:

$$\alpha_{n-1,m}\mathbf{q}_A(t_{n-1,m}, \mathbf{x}_A^{n-1,m}, \hat{\mathbf{x}}_L^{n-1,m}) + h_{n-1,m}\mathbf{j}_A(t_{n-1,m}, \mathbf{x}_A^{n-1,m}, \hat{\mathbf{x}}_L^{n-1,m}) + \tilde{\beta}_{n-1,m} = \mathbf{0}. \tag{4}$$

Here $\tilde{\beta}_{n-1,m}$ is a constant which depends on the previous values of $\mathbf{x}_A$ and $\hat{\mathbf{x}}_L$. A tedious analysis yields the following asymptotic behaviour:

$$\mathbf{B}_A\delta^{n-1,m} \doteq \mathbf{B}_A\tilde{\delta}^{n-1,m} + \frac{1}{4}h\mathbf{K}_{n-1,m}\mathbf{B}_L\rho^{n-1,m}. \tag{5}$$

Here $\rho^{n-1,m}$ is the interpolation error at the refined grid and $\mathbf{K}_{n-1,m}$ is the coupling matrix. The perturbed local discretization error $\mathbf{B}_A\tilde{\delta}^{n,m}$ behaves as $O(h_{n-1,m}^{k+1})$ and can be estimated in a similar way as δ^n. Thus the active error estimate $\mathbf{B}_A\hat{\delta}^{n-1,m}$ satisfies $\mathbf{B}_A\hat{\delta}^{n-1,m} \doteq \mathbf{B}_A\hat{\tilde{\delta}}^{n-1,m} + \frac{1}{4}h\hat{\mathbf{K}}_{n-1,m}\mathbf{B}_L\hat{\rho}^{n-1,m}$. Let L be the interpolation order, then it can be shown that $\frac{1}{4}\|\hat{\mathbf{K}}_n\mathbf{B}_L\rho^{n-1,m}\|$ is less than

$$\hat{r}_I^n = \frac{1}{4}\frac{H_n}{T_n - T_{n-L-1}}\|\hat{\mathbf{K}}_n\mathbf{B}_L\left[\bar{\mathbf{X}}_1^n - \bar{\mathbf{Y}}_1^n\right]\|. \tag{6}$$

Here $\bar{\mathbf{Y}}^n, \bar{\mathbf{X}}^n$ are the Nordsieck vectors which correspond to the predictor and corrector polynomials of $\mathbf{x}$. This error estimate $\hat{r}_I^n$ has the asymptotic behaviour $\hat{r}_I^n = O(H_n^{L+1})$. It follows that $\|\mathbf{B}_A\hat{\delta}^{n,m}\|$ satisfies:

$$\|\mathbf{B}_A\hat{\delta}^{n-1,m}\| \quad \leq \quad \hat{\tilde{r}}_A^{n-1,m} + h\hat{r}_I^n =: \hat{r}_A^{n-1,m}. \tag{7}$$

If $\hat{r}_I^n \leq \mathrm{TOL}_I = \sigma \mathrm{TOL}_A$ and $\hat{\tilde{r}}_A^{n-1,m} \leq \tilde{\mathrm{TOL}}_A = (1-\sigma h)\mathrm{TOL}_A$ then $\hat{r}_A^{n-1,m} \leq \tilde{\mathrm{TOL}}_A + h\mathrm{TOL}_I = \mathrm{TOL}_A$.

We tested a circuit with 5×10 inverters. The circuit is driven by 5 voltage sources which can have different frequencies. The location of the active part is controlled by the connecting elements and the voltage sources. The connecting elements were chosen such that the active part consists of 3 inverters. We did a Euler Backward Compound-Fast multirate simulation on $[0, 10^{-8}]$ with $\sigma = 0.5, \tau = 0$. It appears that the multirate method is able to produce accurate results while the computational time is decreased with factor 13.