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Multilevel Preconditioners for Nonselfadjoint or Indefinite Orthogonal Spline Collocation Problems

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TheoremWe develop and study symmetric multilevel preconditioners for the computation ofthe orthogonal spline collocation (OSC) solution of a Dirichlet boundary value problem(BVP) with a nonselfadjoint or an indefinite operator.The OSC solution is sought in the space of piecewise Hermite bicubic spline functionsdefined on a uniform partition.We consider an additive and a multiplicative multilevel preconditionersthat are used with the preconditioned conjugate gradient (PCG) method.Our results and algorithms are closely related to those in[1],[2],[3],and[4].Let Ω be a unit square $(0,1)\times(0,1)$ with the boundary $\partial\Omega$, andlet $x=(x_1,x_2)$. We consider a BVP

$$Lu \equiv \sum_{i,j=1}^{2} a_{ij}(x)u_{x_ix_j} + \sum_{i=1}^{2} b_i(x)u_{x_i} + c(x)u = f(x), \quad x\in\Omega, \quad u=0 \text{ on } \partial\Omega. \tag{1}$$

Operator L could be non-selfadjoint or indefinite in L^2 inner product.We assume that the principal part of L satisfies the uniform ellipticity conditionand that BVP (1) has a unique solution in $H^2(\Omega)$.Let π_0 be a uniform coarsest rectangular partition of Ω. We obtain a set of partitions$\{\pi_k\}_{k=0}^K$ by standard coarsening, andlet $V_0 \subset V_1 \subset \ldots \subset V_K \equiv V_h$ be the set of corresponding nested spaces ofpiecewise Hermite bicubics that vanish on $\partial\Omega$.Let $\sum$ denote the 2-D composite Gauss quadrature corresponding to partition π_h with 4 nodesin each element. Let $\mathcal{G}_h$ denote the corresponding set of Gauss points.The OSC discretization of BVP (1) is defined by

$$u_h \in V_h, \quad Lu_h(\xi) = f(\xi), \quad \xi \in \mathcal{G}_h, \tag{2}$$

and it can be written as the operator equation $L_h u_h = f_h$ in the Hilbert space V_h with theinner product $(v,w)_h = \sum vw$.Let $\{\psi_{k,j}\}_{j=1}^{N_k}$ be the standard finite element basis of V_k consistingof products of one-dimensional value and slope basis functions.Using the space decomposition

$$V_h = V_0 + \sum_{k=1}^{J}\sum_{j=1}^{N_k} V_{k,j}, \quad V_{k,j} = \text{span}(\psi_{k,j}),$$

we define and study multilevel additiveB_a and multiplicative B_m preconditionersfor solving the normal equation $L_h^* L_h u_h = L_h^* f_h$, where L_h^* is the adjoint to L_h.The implementation of B_a and $B_\mathrm{m}$is based on relationships between basis functions for two consecutive partitionsand the implementation of B_m is similar to that forV(1,1)-cycle with the Gauss-Seidel smoothing.A problem on the coarsest partition is assumed sufficiently small, and it is solved exactly.The computational cost of the preconditioning algorithms is $\mathrm{O}(N_K)$.The following is our main result.There are positive independent of h and K constantsα_a,β_a,α_m,andβ_m,such that

$$\begin{aligned} &\alpha_\mathrm{a}\,(B_\mathrm{a} v, v)_h \le (L_h^* L_h v, v)_h \le \beta_\mathrm{a}\,(B_\mathrm{a} v, v)_h, \quad v \in V_h, \\ &\alpha_\mathrm{m}\,(B_\mathrm{m} v, v)_h \le (L_h^* L_h v, v)_h \le \beta_\mathrm{m}\,(B_\mathrm{m} v, v)_h, \quad v \in V_h. \end{aligned} \tag{3}$$

To obtain this result, we prove the key assumptions in the general theory of Schwarz methodsformulated in [4] and use the inequalities(see [2])

$$C^{-1}\|v\|^2_{H^2(\Omega)} \le a_h(v,v) \le C\|\Delta v\|^2_{L^2(\Omega)}, \quad v \in V_h.$$

In the following table, we present results of our numerical computations; that is,the ratios of spectral constants in (3),the convergence factor $\bar{\rho}$, which is the geometric mean of consecutiveresidual ratios, and the CPU time.Under *Additive* and *Multiplicative*, we list results for Poisson's equation,and under *General* – results fora PDE with a general nonselfadjoint and indefinite operator L with the coefficients

$$\begin{aligned} &a_{11}(x) = e^{x_1 x_2}, \quad a_{12}(x) = 0.5/(1 + x_1 + x_2), \quad a_{22}(x) = e^{-x_1 x_2}, \\ &b_1(x) = x_2 e^{x_1 x_2} + 10\cos[\pi(x_1 + x_2)],_2(x) = -x_1 e^{-x_1 x_2} + 50\sin(2\pi x_1 x_2), \\ &c(x) = 50[1 + 1/(1 + x_1 + x_2)]. \end{aligned}$$

	Additive			*Multiplicative*			*General*		
J	$\beta_\mathrm{a}/\alpha_\mathrm{a}$	$\bar{\rho}$	$t(s)$	$\beta_\mathrm{m}/\alpha_\mathrm{m}$	$\bar{\rho}$	$t(s)$	$\beta_\mathrm{m}/\alpha_\mathrm{m}$	$\bar{\rho}$	$t(s)$
3	3.883	0.072	0.19	1.367	0.005	0.18	925.2	0.094	0.33
4	4.490	0.101	0.59	1.435	0.007	0.90	515.2	0.096	1.80
5	5.016	0.125	2.13	1.476	0.008	3.87	457.5	0.121	8.44
6	5.488	0.142	9.13	1.500	0.009	16.45	402.4	0.166	42.67
7	5.845	0.156	49.93	1.515	0.009	73.43	381.4	0.202	199.40
8	6.162	0.168	278.10	1.524	0.009	334.60	377.3	0.224	995.60

problems are solved by PCG with the multilevel additiveand the multiplicative preconditioners for Poisson's equation andwith the multiplicative preconditioner for the general equation.We set $\pi_0 = \Omega$ and reduce the relative residual to less than 10^{-12}.The numerical results demonstrate the efficiency of our preconditioning algorithms.

Bibliography

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