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A polynomial multigrid smoother for the iterative solution of the heterogeneous Helmholtz problem

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This contribution is focused on the numerical solution of the indefinite Helmholtz equation

$$Hu \equiv -(\Delta + k^2(x,y))u = f \text{ in } \Omega \qquad (1)$$

with a space a space dependent wave number k and exterior complex stretching as absorbing boundary conditions. The solution method is based on a preconditioned Krylov subspace, however, the preconditioner operator is the Helmholtz equation discretized on a grid with a complex grid distance, which is approximately inverted with one multigrid cycle.

The failure of multigrid on the Helmholtz problem have been widely documented and is caused by the indefinite spectrum of the operator H. Discretized on a grid with grid distance h, the negative Laplacian, $-\Delta$, leads to a spectrum that is spread between 0 and $\mathcal{O}(1/h^2)$ on the real axis. However, the wave number will shift this spectrum in the negative direction. This leads to an indefinite matrix for wave numbers k larger than a certain threshold. This means that the spectrum is spread over both the positive real part as the negative real part of the complex plane, not necessarily excluding zero as an eigenvalue. Realistic problems have absorbing boundary conditions, that move the eigenvalues slightly below the real axis.

Two difficulties emerge when multigrid is applied. First, typical smoothers like weighted Jacobi or Gauss-Seidel become unstable for indefinite problems. However, the unstable modes are the smoothest modes and should be stabilized by the coarse grid correction, for problems that are slightly indefinite. The second problem is the appearance of diverging coarse grid corrections for certain wave numbers. This has been analyzed in detail by [1]. The peaks in the convergence plots can be interpreted as resonances.

These difficulties are avoided when multigrid is applied to a complex shifted Laplacian [2,3]. Then the wave number is complex valued which cannot cause

resonances in the coarse grid correction. In [4], we show that the complex shifting is equivalent with discretizing the Laplacian with a complex valued grid distance.

It is important to stress that introducing the complex shift suppresses and broadens the resonances during the coarse grid correction, but it leaves the possibility that the smoother is unstable. In this contribution we report on our analysis of replacing the standard smoother such as weighted Jacobi or Gauss-Seidel with a polynomial smoother.

In particular we have analysed the performance of polynomial smoothers [5] on Helmholtz problems discretized with finite difference and an exterior complex scaling as an absorbing boundary conditions. The spectrum of the operator is bounded by a triangle [4] where the first corner lies in $-k^2$ and the second in $-k^2 + 4/h^2$ and a third corner lies in down in the complex plane. This triangle lies entirely in the lower half of the complex plane. The eigenvalues of the smooth modes lie near $-k^2$ while in the other corners the modes are rapidly oscillating.

The smoothers we have studied have the form

$$f(t) = (1 - \omega_1 t)(1 - \omega_2 t)(1 - \omega_3 t), \qquad (2)$$

where the weights are choosen such that the error iteration, $e^{(k+1)} = f(A)e^{(k)}$, fits the following requirements. It satisfies the fixed point property since $f(0) = 1$. The smoothest modes near $-k^2$ are mapped close to the unit cirlcle since f is such that $f(-k^2) = e^{\iota\varphi}$ with $\varphi \in [0, 2\pi[$. The two other corners are mapped to zero such that the oscillatory modes are smoothed. In [5] we show that the coefficients ω_1, ω_2 en ω_3 can be choosen such that the f is stable and behaves as a smoother given certain conditions on the complex grid of the preconditioner operator.

In practice, however, this hand tuned particular polynomial can be replaced by GMRES(3), which optimizes the coefficients ω_i each iteration. Since the particular polynomial provides a upper bound the GMRES(3) convergence, the GMRES(3) smoother inherits all the properties from the hand tuned polynomials. Note that since we use a complex grid, we only need a restart of three, while [1] requires a complicated smoothing strategy.

We have extensively tested this solution strategy and it gives a satisfactory multigrid convergence that is independent of the wave number. However, the krylov convergence is increases linearly in the wave number.

References

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