Arie, A. Verhoeven
Error analysis of BDF Compound-Fast multirate method for differential-algebraic equations

HG 8 47 (Technische Universiteit Eindhoven)
Den Dolech 2
5600 MB Eindhoven
The Netherlands
averhoev@win.tue.nl
Jan, E.J.W. ter Maten
Bob, R.M.M. Mattheij
Beelen Theo, T.G.J. El Guennouni Ahmed, A. Tasić Bratislav B.

Analogue electrical circuits are usually modeled by differential-algebraic equations (DAE) of type:

$$\frac{d}{dt}\left[\mathbf{q}(t, \mathbf{x})\right] + \mathbf{j}(t, \mathbf{x}) = \mathbf{0}, \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^d$ represents the state of the circuit. A common analysis is the transient analysis, which computes the solution $\mathbf{x}(t)$ of this non-linear DAE along the time interval $[0, T]$ for a given initial state. Often, parts of electrical circuits have latency or multirate behaviour.

For a multirate method it is necessary to partition the variables and equations into an active (A) and a latent (L) part. The active and latent parts can be expressed by $\mathbf{x}_A = \mathbf{B}_A\mathbf{x}, \mathbf{x}_L = \mathbf{B}_L\mathbf{x}$ where $\mathbf{B}_A, \mathbf{B}_L$ are permutation matrices. Then equation (1) is written as the following partitioned system:

$$\begin{aligned} \tfrac{d}{dt}\left[\mathbf{q}_A(t, \mathbf{x}_A, \mathbf{x}_L)\right] + \mathbf{j}_A(t, \mathbf{x}_A, \mathbf{x}_L) = \mathbf{0}, \\ \tfrac{d}{dt}\left[\mathbf{q}_L(t, \mathbf{x}_A, \mathbf{x}_L)\right] + \mathbf{j}_L(t, \mathbf{x}_A, \mathbf{x}_L) = \mathbf{0}. \end{aligned} \tag{2}$$

In contradiction to classical integration methods, multirate methods integrate both parts with different stepsizes. Besides the coarse time-grid $\{T_n, 0 \leq n \leq N\}$ with stepsizes $H_n = T_n - T_{n-1}$, also a refined time-grid $\{t_{n-1,m}, 1 \leq n \leq N, 0 \leq m \leq q_n\}$ is used with stepsizes $h_{n,m} = t_{n,m} - t_{n,m-1}$ and multirate factors q_n. If the two time-grids are synchronized, $T_n = t_{n,0} = t_{n-1,q_n}$ holds for all n. There are a lot of multirate approaches for partitioned systems but we will consider the Compound-Fast version of the BDF methods. This method performs the following four steps:

1. The complete system is integrated at the coarse time-grid.
2. The latent interface variables are interpolated at the refined time-grid.

3. The active part is integrated at the refined time-grid, using the interpolated values at the latent interface.

4. The active solution at the coarse time-grid is updated.

The above methods can be shown to be stable under reasonable conditions. In this paper we will concentrate on error control.

The local discretization error δ^n of the compound phase still has the same behaviour $\delta^n = O(H_n^{K+1})$. Let $\bar{\mathbf{P}}^n, \bar{\mathbf{Q}}^n$ be the Nordsieck vectors which correspond to the predictor and corrector polynomials of $\mathbf{q}$. Then th error δ^n can be estimated by $\hat{\delta}^n$:

$$\hat{\delta}^n = \frac{-H_n}{T_n - T_{n-K-1}} \left[\bar{\mathbf{Q}}_1^n - \bar{\mathbf{P}}_1^n\right]. \tag{3}$$

Now $\hat{r}_C^n = \|\mathbf{B}_L \hat{\delta}^n\| + \tau \|\mathbf{B}_A \hat{\delta}^n\|$ is the used weighted error norm, which must satisfy $\hat{r}_C^n < \mathrm{TOL}_C$.
At the refined time-grid the DAE has been perturbed by the interpolated latent variables. The local discretization error $\delta^{n,m}$ is defined as the residue after inserting the exact solution in the BDF scheme of the refinement phase. However during the refinement instead of $\delta^{n,m}$ the perturbed local error $\tilde{\delta}^{n,m}$ is estimated. During the refinement each step $\mathbf{x}_A^{n-1,m}$ is computed from the following scheme:

$$\alpha_{n-1,m}\mathbf{q}_A(t_{n-1,m}, \mathbf{x}_A^{n-1,m}, \hat{\mathbf{x}}_L^{n-1,m}) + h_{n-1,m}\mathbf{j}_A(t_{n-1,m}, \mathbf{x}_A^{n-1,m}, \hat{\mathbf{x}}_L^{n-1,m}) + \tilde{\beta}_{n-1,m} = \mathbf{0}. \tag{4}$$

Here $\tilde{\beta}_{n-1,m}$ is a constant which depends on the previous values of $\mathbf{x}_A$ and $\hat{\mathbf{x}}_L$.
A tedious analysis yields the following asymptotic behaviour:

$$\mathbf{B}_A \delta^{n-1,m} \doteq \mathbf{B}_A \tilde{\delta}^{n-1,m} + \frac{1}{4} h \mathbf{K}_{n-1,m} \mathbf{B}_L \rho^{n-1,m}. \tag{5}$$

Here $\rho^{n-1,m}$ is the interpolation error at the refined grid and $\mathbf{K}_{n-1,m}$ is the coupling matrix. The perturbed local discretization error $\mathbf{B}_A \tilde{\delta}^{n,m}$ behaves as $O(h_{n-1,m}^{k+1})$ and can be estimated in a similar way as δ^n. Thus the active error estimate $\mathbf{B}_A \hat{\delta}^{n-1,m}$ satisfies $\mathbf{B}_A \hat{\delta}^{n-1,m} \doteq \mathbf{B}_A \hat{\tilde{\delta}}^{n-1,m} + \frac{1}{4} h \hat{\mathbf{K}}_{n-1,m} \mathbf{B}_L \hat{\rho}^{n-1,m}$. Let L be the interpolation order, then it can be shown that $\frac{1}{4}\|\hat{\mathbf{K}}_n \mathbf{B}_L \rho^{n-1,m}\|$ is less than

$$\hat{r}_I^n = \frac{1}{4} \frac{H_n}{T_n - T_{n-L-1}} \|\hat{\mathbf{K}}_n \mathbf{B}_L \left[\bar{\mathbf{X}}_1^n - \bar{\mathbf{Y}}_1^n\right] \|. \tag{6}$$

Here $\bar{\mathbf{Y}}^n, \bar{\mathbf{X}}^n$ are the Nordsieck vectors which correspond to the predictor and corrector polynomials of $\mathbf{x}$. This error estimate $\hat{r}_I^n$ has the asymptotic behaviour $\hat{r}_I^n = O(H_n^{L+1})$. It follows that $\|\mathbf{B}_A \hat{\delta}^{n,m}\|$ satisfies:

$$\|\mathbf{B}_A \hat{\delta}^{n-1,m}\| \quad \leq \quad \hat{\tilde{r}}_A^{n-1,m} + h \hat{r}_I^n =: \hat{r}_A^{n-1,m}. \tag{7}$$

If $\hat{r}_I^n \leq \mathrm{TOL}_I = \sigma \mathrm{TOL}_A$ and $\hat{\tilde{r}}_A^{n-1,m} \leq \tilde{\mathrm{TOL}}_A = (1-\sigma h)\mathrm{TOL}_A$ then $\hat{r}_A^{n-1,m} \leq \tilde{\mathrm{TOL}}_A + h\mathrm{TOL}_I = \mathrm{TOL}_A$.

We tested a circuit with 5×10 inverters. The location of the active part is controlled by the connecting elements and the voltage sources. The connecting elements were chosen such that the active part consists of 3 inverters. We did an Euler Backward Compound-Fast multirate simulation on $[0, 10^{-8}]$ with $\sigma = 0.5, \tau = 0$. We get accurate results combined with a speedup factor 13.