
T. K. Nilssen
**On the Reuse of Standard Preconditioners for Higher
Order Time Discretizations of Parabolic PDEs**

Simula Research Laboratory
Martin Linges vei 17
Fornebu
P.O. Box 134
N-1325 Lysaker
Norway
trygvekn@simula.no
K.-A. Mardal

In this abstract we will describe a preconditioner for some higher order time discretizations of parabolic problems. The preconditioner is optimal with respect to the spatial discretization parameters, that typically are the characteristic mesh size parameter h and the polynomial degree p . The preconditioner is also order optimal with respect to Δt . The only assumption is that there exists a preconditioner for the low order time discretization schemes such as Crank-Nicholson or implicit Euler. Such preconditioners are standard, c.f. e.g., [1], [2] and [3].

We study the model problem

$$\begin{aligned}\frac{\partial u}{\partial t} &= \Delta u, & \text{in } \Omega, t > 0 \\ u &= 0, & \text{on } \partial\Omega, t > 0 \\ u &= u_0, & \text{in } \Omega, t = 0.\end{aligned}$$

This equation is discretized in space and time to give the following linear system to be solved for each time level

$$Q_{kj}(\Delta t A) u^n = P_{kj}(\Delta t A) u^{n-1},$$

where Δt is the time stepping parameter, the two polynomials Q_{kj} and P_{kj} are the (k, j) – Padé approximation to the exponential function and A is a discrete Laplacian. The polynomials are given by (c.f. [3]):

$$\begin{aligned}P_{kj}(\Delta t A) &= \sum_{i=0}^k \binom{k}{i} \frac{(k+j-i)!}{(k+j)!} (\Delta t A)^i \\ Q_{kj}(\Delta t A) &= P_{jk}(-\Delta t A).\end{aligned}$$

The proposed (exact) preconditioner is on the form,

$$R_c^{-j} = (I - c\Delta t A)^{-j},$$

$j \backslash k$	j	$j - 1$	$j - 2$
2	1.07	1.10	1.17
4	1.26	1.31	1.40
6	1.49	1.56	1.66
8	1.76	1.85	1.97
10	2.08	2.20	2.34

Table 1: Upper bound on the condition number for various values of j and k .

where c is determined such that the highest order term of $R_c^{-j}(\Delta t A)$ equals the highest order term of $Q_{kj}(\Delta t A)$. This can be done by choosing

$$R_{kj}(\Delta t A) = \left(I - \sqrt[j]{\frac{j!}{(j+k)!}} \Delta t A \right)^j.$$

Hence, R_{kj} is a standard preconditioner for a low order time discretization of a parabolic PDE, used j times.

In Table 1 we show an upper bound on the condition number of the preconditioned system, using an exact preconditioner for R_{kj} . Further we prove the following lemma.

Lemma 1 *The polynomials $R_{kj}(\Delta t A)$ and $Q_{kj}(\Delta t A)$ are spectrally equivalent independent of A and Δt , and the condition number of the preconditioned system is bounded by*

$$\kappa \left((R_{kj}(\Delta t A))^{-1} Q_{kj}(\Delta t A) \right) < 1.8 \cdot 1.09^j.$$

Bibliography

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- [3] V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems*, Springer-Verlag, 2nd edition, 1997.