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Numerical simulation of glacial rebound using preconditioned iterative solution methods

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We consider the problem to compute the stress (σ) and displacement ($\mathbf{u}$) fields in a (visco)elastic inhomogeneous layered media, in response to a surface load. The underlying physical phenomenon, which is modeled, is glacial advance and recession, and the post-glacial rebound caused by the latter, which reflects the viscoelastic properties of the mantle. The nowadays observed residual rebound implies that the lithosphere might not completely reach equilibrium within itself before another glacial period. Therefore post-glacial rebound has to be taken into account in the context of other problems, such as for example predicting safety of nuclear waste repositories.

Recently, this problem has attracted much attention and simulations are performed using available commercial finite element packages, the consequences of this being that

(a) only direct solution methods are used which entail in general high demands on the computer resources and
(b) almost all well-tested finite element packages are engineering-oriented and are designed to solve the stiffness equation $\nabla\underline{\sigma} + \mathbf{f} = 0$, with $\mathbf{f}$ being the acting forces, which turns out to be overly simplified for geophysical applications and does not include some very important phenomena, such as the so-called advection of pre-stress, for instance.

The material incremental momentum equation for quasi-static infinitesimal perturbations of a stratified, compressible fluid Earth, initially in hydrostatic equilibrium, subject to gravitational forces but neglecting internal forces (cf. [5]) is

$$\underbrace{\nabla\cdot\sigma}_{(A)} + \underbrace{\nabla(\mathbf{u}\cdot\nabla\mathbf{p^{(0)}})}_{(B)} + \underbrace{\rho^{(\Delta)}\mathbf{g^{(0)}}}_{(C)} + \underbrace{\rho^{(0)}\nabla\mathbf{g}^{(\Delta)}}_{(D)} = \mathbf{0}.$$

Here, term (A) describes the force from spatial gradients in stress. If a large elastic solid is put in a gravitational field, it becomes gravitationally pre-stressed with pressure $p^{(0)}$. This pressure can be regarded as an initial condition imposed on the problem and does not cause deformations. Term (B) represents the advection of this pre-stress and describes how it is carried by the moving material. Terms (C) and (D) describe perturbations of the gravitational force and gravitational acceleration due to changes of density.

In the present study, an incompressible non-selfgravitating (flat) Earth model is used, which implies constant gravity field and constant density, so that these two terms vanish. Term (B) is further simplified assuming that the advection term describes the advection in the direction of the gravity field only.

Incorporating the above simplifications with respect to terms (B), (C) and (D), we obtain the following form of the governing equilibrium equation

$$\nabla \cdot \sigma + \rho^{(0)} g^{(0)} \nabla(u_d) = \mathbf{0} \quad \mathbf{x} \in \Omega \subset R^d, d = 2, 3 \tag{1}$$

with suitable boundary conditions.

In the linear case small deformations are assumed, i.e. strain ε and displacements are related as $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$, $1 \leq i, j \leq n$. In its full complexity, the model includes viscoelastic constitutive relations. In this work we discuss a purely elastic material behavior only, as is analyzed in [5], for instance.

Problem (1) is discretized using stable mixed finite element pairs or a suitable stabilized formulation, which lead to a system of linear equations with a non-symmetric two-by-two block matrix of a saddle point form.

Results from numerical experiments solving the so-arising algebraic system with preconditioned iterative solution methods are presented. Several preconditioning strategies are tested, based on the techniques and experience described in [4], [3], [1], [2] and other authors. The performance of the tested preconditioned iterative solution methods is compared with that of a commercial FEM package solver.

Bibliography

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