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Boundary Summation Equations

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The boundary element method (BEM) relies on the fact that a linear partial differential equation (PDE) with constant coefficients can be reduced to a boundary integral equation (BIE). In this presentation, it is shown that a similar reduction is possible for linear difference equations with constant coefficients. The reduced equations involve a summation operator and are called boundary summation equations (BSE).

There are two key properties that BSE and discretized BIE have in common; the number of unknowns is low (compared to methods where the interior domain is discretized) and the resulting linear system is full. The latter property implies that special techniques are required in an iterative solver, especially in higher dimensions. One example for BIE is the multipole method, which can be used to apply an integral operator. For the BSE, it is possible to apply the summation operator efficiently by FFT, provided that the domain is simple enough.

The concepts of BSE are illustrated by a two dimensional example. Let

$$Pu = f,$$

be the linear system that represents the usual five point Laplacian with Dirichlet boundary conditions on a rectangular domain $\bar{\Omega} = (0, \dots, n_1) \times (0, \dots, n_2)$. Let also E be a discrete fundamental solution, i.e. a function that satisfies

$$\hat{P}E_i = \begin{cases} 1, & i = 0 \\ 0, & i \neq 0 \end{cases},$$

where $\hat{P}$ represents the discrete Laplacian. Define the summation operator K according to

$$Kv_i = \sum_{j \in \bar{\Omega}} E_{i-j} v_j.$$

Now, P can be reordered such that

$$\begin{pmatrix} P_\Gamma & P_{\Gamma\Omega} \\ P_{\Omega\Gamma} & P_\Omega \end{pmatrix} \begin{pmatrix} u_\Gamma \\ u_\Omega \end{pmatrix} = \begin{pmatrix} f_\Gamma \\ f_\Omega \end{pmatrix}, \tag{1}$$

where subscript Γ denoted boundary points and Ω denotes interior points. Mixed subscripts denote the couplings. If the same ordering is applied when K is represented by a matrix, one can show that

$$\begin{pmatrix} P_\Gamma & P_{\Gamma\Omega} \\ P_{\Omega\Gamma} & P_\Omega \end{pmatrix} \begin{pmatrix} K_\Gamma & K_{\Gamma\Omega} \\ K_{\Omega\Gamma} & K_\Omega \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & I \end{pmatrix}, \tag{2}$$

where

$$A = P_\Gamma K_\Gamma + P_{\Gamma\Omega} K_{\Omega\Gamma},$$

and

$$B = P_\Gamma K_{\Gamma\Omega} + P_{\Gamma\Omega} K_\Omega.$$

A new vector v of unknowns is introduced, such that

$$\begin{pmatrix} u_\Gamma \\ u_\Omega \end{pmatrix} = \begin{pmatrix} K_\Gamma & K_{\Gamma\Omega} \\ K_{\Omega\Gamma} & K_\Omega \end{pmatrix} \begin{pmatrix} v_\Gamma \\ v_\Omega \end{pmatrix}.$$

By inserting the new vector into (1) and using (2), it is easy to see that $v_\Omega = f_\Omega$ and then, by elimination, that

$$Av_\Gamma = f_\Gamma - Bf_\Omega. \tag{3}$$

The original problem, with $(n_1 + 1)(n_2 + 1)$ unknowns, is hence reduced to a system of equations with $2(n_1 + n_2)$ unknowns. The additional cost for the reduction process lies in the construction of a fundamental solution. An algorithm requiring $\mathcal{O}(n_1 n_2 \log n_1 n_2)$ arithmetic operations is described in [1].

In this case, where the domain is rectangular, it is possible to apply the matrix A by an embedding technique, related to the way in which Toeplitz matrices can be applied by embedding them in circulants. The complexity for this operation is also $\mathcal{O}(n_1 n_2 \log n_1 n_2)$. The same holds for computing the original vector of unknowns from the solution of the reduced system. The over all cost for solving the original problem is hence $\mathcal{O}(k n_1 n_2 \log n_1 n_2)$, where k is the number of iterations needed for the reduced system to converge to a desired level. One detail that is worth pointing out is that there is no additional cost in determining the residual of the original system when performing iterations on the reduced system.

The reduction process generalizes to several dimensions and is not restricted to any special kind of difference operator, as long as it is linear and has constant coefficients. The fast application technique requires that the domain is a hyper cuboid. Fast application techniques for more complicated domains is a subject for future research. Numerical experiments for the solution of (3) will be presented in the talk. Results for various difference equations and various boundary conditions will be shown and preliminary results indicate grid independent GMRES iteration counts for discretizations of first order PDEs.

Bibliography

[1] H. Brandén and P. Sundqvist *An Algorithm for Computing Fundamental Solutions of Difference Operators* Tech. Rep. 2003-006 Dept. of Information Technology, Uppsala Univ., Uppsala, Sweden, 2003