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Nullspace preserving multigrid for saddle-point problems

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Consider the saddle point problem

$$A \begin{bmatrix} \mathbf{u} \\ \mathbf{x} \end{bmatrix} \equiv \begin{bmatrix} \mathcal{A} & \mathcal{B}^T \\ \mathcal{B} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ 0 \end{bmatrix}.$$

Assuming that \mathcal{A} is s.p.d., this problem is transformed, by a (computable) projection π ($\pi^2 = \pi$) such that $\mathcal{B}\pi = \mathcal{B}$, to an equivalent s.p.d. problem for \mathbf{u} ,

$$[(I - \pi^T)\mathcal{A}(I - \pi) + \pi^T\mathcal{A}\pi] \mathbf{u} = (I - \pi^T)\mathbf{f}.$$

We present a set of conditions for a smoother \mathcal{M}^{-1} and an interpolation matrix Π , such that if a current iterate in the resulting two-grid method belongs to the subspace $\text{Null}(\mathcal{B})$, then after smoothing the iterate stays in $\text{Null}(\mathcal{B})$, and finally, the (interpolated) coarse-grid correction also stays in the subspace $\text{Null}(\mathcal{B})$. Thus a multigrid method can be devised without explicit knowledge of a computable basis of $\text{Null}(\mathcal{B})$. The tools needed are: computable projections π_k , such that π_k^T are also computable, interpolation matrices \mathcal{P}_k for the \mathbf{u} -variable and interpolation matrices \mathcal{Q}_k for the second unknown \mathbf{x} , at all levels $k \geq 0$. Let $\mathcal{B}_0 = \mathcal{B}$ and $\mathcal{A}_0 = \mathcal{A}$ (i.e., $k = 0$ stands for the finest level). The projections π_k have the form $\mathcal{R}_k\mathcal{B}_k$ and satisfy $\mathcal{B}_k\pi_k = \mathcal{B}_k$.

There is a common “*null-space preserving*” assumption on \mathcal{Q}_k , \mathcal{P}_k , and $\mathcal{B}_{k+1} \equiv \mathcal{Q}_k^T\mathcal{B}_k\mathcal{P}_k$:

$$\mathcal{B}_{k+1}\mathbf{v}_c = 0 \text{ implies } \mathcal{B}_k\mathcal{P}_k\mathbf{v}_c = 0.$$

Define a standard multigrid method based on the s.p.d. matrices

$$(I - \pi_k^T)\mathcal{A}_k(I - \pi_k) + \pi_k^T\mathcal{A}_k\pi_k, \quad \mathcal{A}_k = \mathcal{P}_{k-1}^T\mathcal{A}_{k-1}\mathcal{P}_{k-1}, \quad \mathcal{A}_0 = \mathcal{A},$$

smoothers (for given s.p.d. matrices $\overline{\mathcal{M}}_k^{-1}$),

$$\mathcal{M}_k^{-1} = (I - \pi_k)\overline{\mathcal{M}}_k^{-1}(I - \pi_k^T) + \pi_k\overline{\mathcal{M}}_k^{-1}\pi_k^T,$$

and (modified) interpolation matrices

$$\Pi_k = \mathcal{P}_k(I - \pi_{k+1}).$$

The resulting multigrid method (with zero initial iterate) keeps all iterates in $\text{Null}(\mathcal{B})$ since the initial residual is $(I - \pi^T)\mathbf{f}$ and all successive residuals also have the form $(I - \pi^T)\mathbf{r}$.

We provide a specific construction of the (computable) projections π as well as alternative choices of the null-space preserving smoothers \mathcal{M}^{-1} for some mixed finite element saddle-point matrices A . This work was performed under

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