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Multilevel Preconditioners for Nonselfadjoint or Indefinite Orthogonal Spline Collocation Problems

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We develop and study symmetric multilevel preconditioners for the computationof the orthogonal spline collocation (OSC) solution of a Dirichlet boundary-value problem (BVP) with a nonselfadjoint or an indefinite operator.The OSC solution is sought in the space of piecewise Hermite bicubic splinefunctions defined on a uniform partition.We consider an additive and a multiplicative multilevel preconditionersthat are used with the preconditioned conjugate gradient (PCG) method.Let Ω be a unit square $(0,1) \times (0,1)$ with the boundary$\partial\Omega$, and let $x = (x_1, x_2)$. We consider a BVP

$$Lu \equiv \sum_{i,j=1}^{2} a_{ij}(x)u_{x_i x_j} + \sum_{i=1}^{2} b_i(x)u_{x_i} + c(x)u = f(x), \quad x \in \Omega, \quad u = 0 \ \text{ on } \ \partial\Omega.$$
(1)

Operator L could be non-selfadjoint or indefinite in L^2 inner product.We assume that the principal part of L satisfies the uniform ellipticitycondition and that BVP (1) has a unique solution in $H^2(\Omega)$.Let π_0 be a uniform coarsest rectangular partition of Ω. We obtaina set of partitions $\{\pi_k\}_{k=0}^{K}$ by standard coarsening, andlet $V_0 \subset V_1 \subset \ldots \subset V_K \equiv V_h$ be the set of correspondingnested spaces of piecewise Hermite bicubics that vanish on $\partial\Omega$.Let $\sum$ denote the two-dimensional composite Gauss quadrature correspondingto partition π_h with 4 nodes in each element. Let $\mathcal{G}_h$ denote thecorresponding set of Gauss points. The OSC discretization of BVP (1)is defined by

$$u_h \in V_h, \quad Lu_h(\xi) = f(\xi), \quad \xi \in \mathcal{G}_h,$$
(2)

and it can be written as the operator equation $L_h u_h = f_h$ in the Hilbertspace V_h with the inner product $(v, w)_h = \sum vw$.Let $\{\psi_{k,j}\}_{j=1}^{N_k}$ be the standard finite element basis ofV_k consisting of products of one-dimensional value and slope basis functions.Using space decomposition

$$V_h = V_0 + \sum_{k=1}^{J} \sum_{j=1}^{N_k} V_{k,j}, \quad V_{k,j} = \text{span}(\psi_{k,j}),$$

we define and study multilevel additive B_a and multiplicative B_m preconditioners for solving the normal equation $L_h^* L_h u_h = L_h^* f_h$, where L_h^* is the adjoint to L_h. The implementation of B_a and B_m is based on relationships between basis functions for two consecutive partitions and the implementation of B_m is similar to that for V(1,1)-cycle with the Gauss-Seidel smoothing. A problem on the coarsest partition is assumed sufficiently small, and it is solved exactly. The computational cost of the preconditioning algorithms is $\mathrm{O}(N_K)$. The following is our main result.

Theorem 1 *There are positive independent of h and K constants $\alpha_\mathrm{a}, \beta_\mathrm{a}, \alpha_\mathrm{m}, and \beta_\mathrm{m}, such that*

$$\alpha_\mathrm{a} (B_\mathrm{a} v, v)_h \leq (L_h^* L_h v, v)_h \leq \beta_\mathrm{a} (B_\mathrm{a} v, v)_h, \quad v \in V_h, \qquad (3)$$
$$\alpha_\mathrm{m} (B_\mathrm{m} v, v)_h \leq (L_h^* L_h v, v)_h \leq \beta_\mathrm{m} (B_\mathrm{m} v, v)_h, \quad v \in V_h.$$

We present numerical results that demonstrate the efficiency of our preconditioning algorithms.