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A comparative study of iterative solvers exploiting spectral information for SPD systems

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When solving the Symmetric Positive Definite (SPD) linear system

$$\mathbf{Ax} = \mathbf{b}$$

with the Conjugate Gradient (CG) method, the smallest eigenvalues of the matrix A often slow down the convergence. This is highlighted by the bound on the rate of convergence of CG given by

$$||\mathbf{e}^{(k)}||_{\mathbf{A}} \leq \mathbf{2}\, ||\mathbf{e}^{(0)}||_{\mathbf{A}} \left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^{k},$$

where $\mathbf{e}^{(n)} = \mathbf{x}^* - \mathbf{x}^{(n)}$ denotes the forward error associated with the iterate at step k, $\kappa = \kappa(\mathbf{A}) = \lambda_{max}/\lambda_{min}$ denotes the condition number of $\mathbf{A}$ and $||\mathbf{x}||_{\mathbf{A}} = (\mathbf{x}^T\mathbf{A}\mathbf{x})^{1/2}$ denotes the $\mathbf{A}-$norm of $\mathbf{x}$.
From this bound, it can be seen that increasing the size of the smallest eigenvalues will improve the convergence rate of CG. Consequently if the smallest eigenvalues of $\mathbf{A}$ could be somehow “removed” the convergence will be improved. This observation is still of course true when a preconditioner is used and some extra techniques might be investigated to improve the convergence rate of CG on the preconditioned system. Several techniques have been proposed in the literature that either consists of updating the preconditioner or enforcing CG to work in the orthogonal of invariant subspace associated with small eigenvalues. The aim of this work it to compare several of these techniques in terms of numerical efficiency and computational complexity. Among these approaches we consider first the two-phase algorithm recently proposed in [1]. In a first stage this algorithm computes a partial spectral decomposition simply using matrix-vector products. More precisely it combines Chebyshev iterations with a block Lanczos procedure to accurately compute an orthogonal basis of the invariant subspace associated with the smallest eigenvalues of $\mathbf{A}$. Then, the solution on this small subspace is computed using a direct solver while the solution in the orthogonal complement is obtained with Chebyshev iterations that benefits

from the reduced condition number.
For sake of comparison this eigen-information is used in combination with other techniques. In particular we consider the deflated version of conjugate gradient described in [5]. This algorithm exploits the link between the Lanczos algorithm and the standard conjugate gradient algorithm, and is mathematically equivalent to Nicolaide's algorithm which is derived from a "deflated" Lanczos procedure [4].
As representative of techniques exploiting the spectral information to update the preconditioner we consider the approaches proposed in [2] and [3] that attempts to shift the smallest eigenvalues close to one where most of the eigenvalues of the preconditioned matrix should be located. In this talk, we will describe theses various variants as well as the observed numerical behaviour on a set of model problems from Matrix Market or arising from the discretization via finite element technique of some heterogeneous diffusion PDEs. We will discuss their numerical efficiency, computational complexity and sensitivity to the accuracy of the eigencalculation. Perspectives for future work will also be debated.

References

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