MULTIGRID FOR A WEAKLY OVER-PENALIZED INTERIOR PENALTY METHOD

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Abstract. We study a W-cycle multigrid algorithm for a weakly over-penalized interior penalty method for the two-dimensional Poisson problem on convex polygonal domains.

1. Introduction

Let Ω be a bounded convex polygonal domain in $\mathbb{R}^2$, $f \in L_2(\Omega)$ and $u \in H_0^1(\Omega)$ satisfy

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall \, v \in H_0^1(\Omega). \tag{1.1}$$

In this paper we investigate a W-cycle algorithm for a weakly-overpenalized interior penalty method for the model problem (1.1).

Let $\mathcal{T}_h$ be a simplicial triangulation of Ω and V_h be the discontinuous P_1 finite element space, i.e.,

$$V_h = \{ v \in L_2(\Omega) : \, v_T = v\big|_T \in P_1(T) \quad \forall \, T \in \mathcal{T}_h \}.$$

We define the jumps $[\![v]\!]$ and means $\{\!\{\nabla v\}\!\}$ of a function v in the usual way [2]. Let e be an interior edge shared by the triangles $T_1, T_2 \in \mathcal{T}_h$. Then we define on e,

$$[\![v]\!] = v_1 \boldsymbol{n}_1 + v_2 \boldsymbol{n}_2, \tag{1.2}$$

$$\{\!\{\nabla v\}\!\} = \frac{1}{2}\big(\nabla v_1 + \nabla v_2\big), \tag{1.3}$$

where $v_1 = v\big|_{T_1}$, $v_2 = v\big|_{T_2}$ and $\boldsymbol{n}_1$ (resp. $\boldsymbol{n}_2$) is the unit normal of e pointing towards the outside of T_1 (resp. T_2). On an edge e along $\partial\Omega$, we define

$$[\![v]\!] = (v\big|_e)\boldsymbol{n}, \tag{1.4}$$

$$\{\!\{\nabla v\}\!\} = (\nabla v)\big|_e, \tag{1.5}$$

where $\boldsymbol{n}$ is the unit normal of e pointing outside Ω.

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The weakly over-penalized nonsymmetric interior penalty (WOPNIP) method is: Find $u_h \in V_h$ such that

$$a_h(u_h, v) = \int_\Omega f v \, dx \qquad \forall\, v \in V_h, \tag{1.6}$$

where

$$\begin{aligned} a_h(w, v) = \sum_{T \in \mathcal{T}_h} \int_T \nabla w \cdot \nabla v \, dx - \sum_{e \in \mathcal{E}_h} \int_e \{\!\{\nabla w\}\!\} \cdot [\![v]\!] \, ds \\ + \sum_{e \in \mathcal{E}_h} \int_e \{\!\{\nabla v\}\!\} \cdot [\![w]\!] \, ds + \sum_{e \in \mathcal{E}_h} \frac{\eta}{|e|^3} \int_e \Pi_e^0 [\![w]\!] \cdot \Pi_e^0 [\![v]\!] \, ds, \end{aligned} \tag{1.7}$$

Π_e^0 is the orthogonal projection operator from $L_2(e)$ onto $P_0(e)$, the space of constant functions on e, and

$$\eta \geq \eta_0 > 0 \tag{1.8}$$

is a penalty parameter (η_0 is arbitrary but fixed).

Remark 1.1. Note that

$$\Pi_e^0 v = \frac{1}{|e|} \int_e v \, ds \qquad \forall\, v \in L_2(e).$$

Furthermore, it follows from the midpoint rule that

$$\Pi_e^0 v = v(m_e) \qquad \forall\, v \in P_1(e), \tag{1.9}$$

where m_e is the midpoint of e. Therefore, for the WOPNIP method, it is convenient to use the nodal values of the discontinuous P_1 finite element functions at the *midpoints* of the edges for the purpose of programming.

The symmetric and antisymmetric parts of a_h are defined by

$$\begin{aligned} s_h(w, v) &= \frac{1}{2}\big(a_h(v, w) + a_h(w, v)\big) \\ &= \sum_{T \in \mathcal{T}_h} \int_T \nabla w \cdot \nabla v \, dx + \sum_{e \in \mathcal{E}_h} \frac{\eta}{|e|^3} \int_e \Pi_e^0 [\![w]\!] \cdot \Pi_e^0 [\![v]\!] \, ds, \end{aligned} \tag{1.10}$$

$$\begin{aligned} n_h(w, v) &= \frac{1}{2}\big(a_h(v, w) - a_h(w, v)\big) \\ &= \sum_{e \in \mathcal{E}_h} \int_e \{\!\{\nabla w\}\!\} \cdot [\![v]\!] \, ds - \sum_{e \in \mathcal{E}_h} \int_e \{\!\{\nabla v\}\!\} \cdot [\![w]\!] \, ds. \end{aligned} \tag{1.11}$$

Let the mesh-dependent norm $\|\!|\cdot|\!\|_h$ be defined by

$$\|\!|v|\!\|_h^2 = \sum_{T \in \mathcal{T}_h} \|\nabla v\|_{L_2(T)}^2 + \sum_{e \in \mathcal{E}_h} \frac{\eta}{|e|^3} \|\Pi_e^0 [\![v]\!]\|_{L_2(e)}^2 = s_h(v, v) = a_h(v, v). \tag{1.12}$$

Then we have (cf. [8])

$$a_h(w,v) \le C\|\!|w\|\!|_h \|\!|v\|\!|_h \qquad \forall\, v,w \in V_h, \tag{1.13}$$

and the following error estimate is valid:

$$\|u-u_h\|_{L_2(\Omega)} + h\|\!|u-u_h\|\!|_h \le Ch^2\|f\|_{L_2(\Omega)}. \tag{1.14}$$

From here on we will use C (with or without subscripts) to denote a positive constant independent of η (under condition (1.8)), h and k (the grid level in multigrid) that can take different values at different occurrences.

Remark 1.2. The WOPNIP method is convergent for any penalty parameter and its formulation can be easily extended to nonconforming meshes. It is therefore an attractive alternative for adaptive algorithms. The study of the standard W-cycle algorithm in this paper is a first step towards a multilevel adaptive WOPNIP method.

Remark 1.3. There are rigorous multigrid convergence results [11, 10] for the symmetric interior penalty method [19, 1]. Because of the lack of a quasi-optimal error estimate in the L_2 norm, this rigorous analysis is missing from previous multigrid work [13, 14] for some of the nonsymmetric interior penalty methods described in [18].

Remark 1.4. The over-penalization increases the condition number of the resulting discrete system from $O(h^{-2})$ to $O(h^{-4})$. However there is a simple block diagonal preconditioner (cf. Remark 2.2 and the estimate (3.1) below) that reduces the condition number back to $O(h^{-2})$. This block diagonal preconditioner also plays an important role in the smoothing steps (cf. (2.12) and (2.14) below) of multigrid algorithms for (1.6).

Remark 1.5. The WOPNIP method differs from the over-penalized NIPG method in [18] because of the presence of the projection operator Π_e^0 in the penalty term, which forces $\Pi_e^0[\![u_h]\!] \approx 0$ (i.e., u_h is almost weakly continuous) whereas the over-penalized method in [18] forces $[\![u_h]\!] \approx 0$ (i.e., u_h is almost continuous). For the Poisson problem these methods have similar behavior. However, it is well-known [16] that for Maxwell's equations the continuous P_1 finite element method does not work for nonconvex domains. But there are methods [6, 7] based on the weakly continuous P_1 finite element that do work on nonconvex domains. Therefore the work in this paper is also relevant for multigrid methods for Maxwell's equations.

The rest of the paper is organized as follows. The W-cycle algorithm is defined in Section 2. Convergence results and numerical results are presented in Section 3 and Section 4, followed by some concluding remarks in Section 5.

2. A W-Cycle Multigrid Algorithm

Let $\mathcal{T}_1, \mathcal{T}_2, \ldots$ be a sequence of triangulations of Ω obtained by regular subdivisions, h_k be the mesh size of $\mathcal{T}_k$, and V_k be the corresponding discontinuous P_1 finite element space associated with $\mathcal{T}_k$. The k-th level discrete problem corresponding to (1.6) is: Find $u_k \in V_k$ such that

$$a_k(u_k, v) = \int_\Omega f v\, dx \qquad \forall\, v \in V_k. \tag{2.1}$$

Here $a_k(\cdot,\cdot)$ is the analog of (1.7) for V_k, i.e.,

$$\begin{aligned} a_k(w, v) = \sum_{T\in\mathcal{T}_k} \int_T \nabla w \cdot \nabla v\, dx - \sum_{e\in\mathcal{E}_k} \int_e \{\!\{\nabla w\}\!\} \cdot [\![v]\!]\, ds \\ + \sum_{e\in\mathcal{E}_k} \int_e \{\!\{\nabla v\}\!\} \cdot [\![w]\!]\, ds + \sum_{e\in\mathcal{E}_k} \frac{\eta}{|e|^3} \int_e \Pi_e^0 [\![w]\!] \cdot \Pi_e^0 [\![v]\!]\, ds. \end{aligned} \tag{2.2}$$

Similarly, $s_k(\cdot,\cdot)$ and $n_k(\cdot,\cdot)$ are the analogs of (1.10) and (1.11) for V_k.

Let the operators $A_k, S_k, N_k : V_k \longrightarrow V_k'$ be defined by

$$\langle A_k w, v\rangle = a_k(w, v) \qquad \forall\, w, v \in V_k, \tag{2.3}$$

$$\langle S_k w, v\rangle = s_k(w, v) = \frac{1}{2}\big(a_k(v, w) + a_k(w, v)\big) \qquad \forall\, w, v \in V_k, \tag{2.4}$$

$$\langle N_k w, v\rangle = n_k(w, v) = \frac{1}{2}\big(a_k(v, w) - a_k(w, v)\big) \qquad \forall\, w, v \in V_k, \tag{2.5}$$

where $\langle\cdot,\cdot\rangle$ is the canonical bilinear form on $V_k' \times V_k$. These operators are related by

$$A_k = S_k - N_k. \tag{2.6}$$

In terms of A_k, the discrete problem (2.1) can be written as

$$A_k u_k = \phi_k,$$

where $\phi_k \in V_k'$ is defined by

$$\langle \phi_k, v\rangle = \int_\Omega f v\, dx \qquad \forall\, v \in V_k.$$

In order to construct a preconditioned relaxation scheme as the smoother in the multigrid algorithms, we define an operator $B_k : V_k \longrightarrow V_k'$ by

$$\langle B_k w, v\rangle = \sum_{T\in\mathcal{T}_k} \int_T w v\, dx + \sum_{e\in\mathcal{E}_k} \frac{\eta}{|e|} \int_e \Pi_e^0 [\![w]\!] \cdot \Pi_e^0 [\![v]\!]\, ds \qquad \forall\, w, v \in V_k. \tag{2.7}$$

Remark 2.1. It follows from a standard quadrature rule for quadratic functions that

$$\int_T w v\, dx = \frac{|T|}{3} \sum_{m\in\mathcal{M}_T} w(m) v(m) \qquad \forall\, w, v \in P_1(T), \tag{2.8}$$

where $\mathcal{M}_T$ is the set of the three midpoints of T.

Remark 2.2. Let $\mathbf{B}_k$ be the matrix representing B_k with respect to the nodal basis associated with the midpoints of $\mathcal{T}_k$ and its canonical dual basis, i.e.,

$$\mathbf{v}^T\mathbf{B}_k\mathbf{w} = \langle B_k w, v\rangle,$$

where $\mathbf{w}$ (resp. $\mathbf{v}$) is the coordinate vector for w (resp. v) in V_k. Because of (1.9) and (2.8), the matrix $\mathbf{B}_k$ is block diagonal with 2×2 blocks (corresponding to the midpoints of interior edges) and 1×1 blocks (corresponding to the midpoints of boundary edges). Therefore it is trivial to compute $\mathbf{B}_k^{-1}$, which is needed in the smoothing steps (2.12) and (2.14).

We define the coarse-to-fine operator $I_{k-1}^k : V_{k-1} \longrightarrow V_k$ by averaging. More precisely, let m_e be a midpoint of an interior edge e in $\mathcal{T}_k$, then we define

$$(I_{k-1}^k v)(m_e) = \frac{1}{2}\big(v_1(m_e) + v_2(m_e)\big), \tag{2.9}$$

where $v_i = v\big|_{T_i}$ and T_1, T_2 are the triangles in $\mathcal{T}_k$ that share e as a common edge. If $e \subset \partial\Omega$, then we define $(I_{k-1}^k v)(m_e) = 0$.

The fine-to-coarse operator $I_k^{k-1} : V_k' \longrightarrow V_{k-1}'$ is the transpose of I_{k-1}^k with respect to the canonical bilinear form, i.e.,

$$\langle I_k^{k-1}\alpha, v\rangle = \langle \alpha, I_{k-1}^k v\rangle \qquad \forall\, \alpha \in V_k',\ v \in V_{k-1}. \tag{2.10}$$

We are now ready to define a W-cycle algorithm for the equation

$$A_k z = \psi \tag{2.11}$$

where $\psi \in V_k'$.

Algorithm 2.3. The output of the algorithm will be denoted by $MG_W(k, \psi, z_0, m_1, m_2)$, where $z_0 \in V_k$ is the initial guess and m_1 (resp. m_2) is the number of pre-smoothing (resp. post-smoothing) steps.

For $k = 1$, we take the output to be $A_1^{-1}\psi$.

For $k > 1$, we proceed in three steps.

Pre-Smoothing Compute $z_k \in V_k$ for $1 \le k \le m_1$ recursively by

$$z_k = z_{k-1} + \omega_k B_k^{-1}(\psi - A_k z_{k-1}), \tag{2.12}$$

where ω_k is a damping factor (to be chosen in (3.3) below).

Coarse-Grid Correction Compute $q \in V_{k-1}$ by

$$\begin{aligned} \rho &= I_k^{k-1}(\psi - A_k z_{m_1}) \\ q_* &= MG_W(k-1, \rho, 0, m_1, m_2) \\ q &= MG_W(k-1, \rho, q_*, m_1, m_2) \end{aligned} \tag{2.13}$$

and take

$$z_{m_1+1} = z_{m_1} + I_{k-1}^k q.$$

Post-Smoothing Compute $z_k \in V_k$ for $m_1 + 2 \le k \le m_1 + m_2 + 1$ recursively by

$$z_k = z_{k-1} + \omega_k B_k^{-1}(\psi - A_k z_{k-1}). \tag{2.14}$$

The final output is

$$MG_W(k, \psi, z_0, m_1, m_2) = z_{m_1+m_2+1}.$$

We need the following operators for the analysis of Algorithm 2.3. The operator $R_k : V_k \longrightarrow V_k$ that measures the effect of one smoothing step is defined by

$$R_k = Id_k - \omega_k B_k^{-1} A_k, \tag{2.15}$$

where $Id_k : V_k \longrightarrow V_k$ is the identity operator. The operator $P_k^{k-1} : V_k \longrightarrow V_{k-1}$ is the transpose of the coarse-to-fine operator I_{k-1}^k with respect to the bilinear forms that define the WOPNIP method, i.e.,

$$a_{k-1}(P_k^{k-1} w, v) = a_k(w, I_{k-1}^k v) \qquad \forall\, w \in V_k,\ v \in V_{k-1}. \tag{2.16}$$

We denote the error propagation operator for Algorithm 2.3 by $\mathbb{E}_k$, i.e.,

$$\mathbb{E}_k(z - z_0) = z - MG_W(k, \psi, z_0, m_1, m_2). \tag{2.17}$$

The following recursive relation [12, 15, 9] is well-known:

$$\mathbb{E}_k = R_k^{m_2}\big(Id_k - I_{k-1}^k P_k^{k-1} + I_{k-1}^k \mathbb{E}_{k-1}^2 P_k^{k-1}\big) R_k^{m_1}. \tag{2.18}$$

3. Convergence Results

The convergence analysis is based on the facts [8] that

$$0 < C_1 \le \lambda_{\min}(B_k^{-1} S_k) \le \lambda_{\max}(B_k^{-1} S_k) \le C_2 h_k^{-2} \tag{3.1}$$

and

$$\langle N_k w, v\rangle \le C \eta^{-1/2} h_k \|w\|_k \|v\|_k \qquad \forall\, w, v \in V_k, \tag{3.2}$$

where

$$\|v\|_k = \sqrt{s_k(v,v)} = \sqrt{a_k(v,v)}$$

is the analog of the norm $\|\cdot\|_h$ defined in (1.12).

The estimates (3.1) and (3.2) show that $B_k^{-1} S_k$ behaves like a second order differential operator and that the antisymmetric part is a *compact* perturbation of the symmetric part. This is crucial for proving multigrid convergence results for nonsymmetric methods [3, 5, 4].

In view of (3.1), we will take

$$\omega_k = c_* h_k^2 \tag{3.3}$$

in the smoothing steps (2.12) and (2.14), where c_* is chosen so that $\omega_k \lambda_{\max}(\mathcal{S}_k) \le 1$.

Using (2.6), (2.15) and (3.1)–(3.3), we can prove the following smoothing property of Algorithm 2.3.

Lemma 3.1. *There exist positive constants C_1 and C_2 such that*

$$\|R_k^m v\|_{1,k} \le C_1 h_k^{-1} m^{-1/2} (1 + C_2 \eta^{-1/2} h_k)^m \langle B_k v, v\rangle^{1/2} \qquad \forall\, v \in V_k,\ k, m \ge 1. \tag{3.4}$$

Furthermore, we can prove the following approximation property using a duality argument and the quasi-optimal L_2 error estimate in (1.14).

Lemma 3.2. *There exists a positive constant C such that*

$$\langle B_k(Id_k - I_{k-1}^k P_k^{k-1})v, v\rangle^{1/2} \le Ch_k \|\!|v|\!\|_{1,k} \qquad \forall\, v \in V_k,\ k \ge 2. \tag{3.5}$$

Combining Lemma 3.1, Lemma 3.2 and (2.18), we have the following convergence result for the W-cycle algorithm with only post-smoothing steps.

Theorem 3.3. *For any $0 < \gamma < 1$, there exist a positive integer m_γ and a positive number δ_γ, such that when the Algorithm 2.3 is applied with m_γ post-smoothing steps and without pre-smoothing to the equation* (2.11), *we have*

$$\|\!|z - MG_W(k, \psi, z_0, 0, m_\gamma)|\!\|_k \le \gamma \|\!|z - z_0|\!\|_k, \tag{3.6}$$

provided $\eta^{-1/2} h_1 \le \delta_\gamma$.

4. Numerical Results

In the first set of experiments, we computed the k-th level contraction number

$$\gamma_k = \sup_{v \in V_k \setminus \{0\}} \frac{\|\!|\mathbb{E}_k v|\!\|_k}{\|\!|v|\!\|_k} \tag{4.1}$$

for Algorithm 2.3 on the unit square $(0,1)\times(0,1)$, and we took $\eta = 1$. The W-cycle algorithm with only post-smoothing, which is analyzed in this paper, is a contraction for $m \ge 3$. The values of γ_k for $k = 2, \dots, 7$ and $m = 3, \dots 10$ are presented in Table 1.

k	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$	$m=9$	$m=10$
2	0.905	0.879	0.853	0.829	0.804	0.779	0.756	0.734
3	0.916	0.894	0.871	0.849	0.827	0.805	0.784	0.763
4	0.920	0.898	0.876	0.854	0.833	0.812	0.792	0.772
5	0.920	0.899	0.878	0.856	0.835	0.814	0.794	0.774
6	0.921	0.899	0.878	0.857	0.835	0.815	0.795	0.775
7	0.921	0.899	0.878	0.857	0.835	0.815	0.795	0.775

Table 1. γ_k for the unit square with $m_1 = 0$ and $m_2 = m$.

We also computed γ_k for the W-cycle algorithm with only pre-smoothing, and for the symmetric W-cycle algorithm with both pre-smoothing and post-smoothing. The results for the W-cycle algorithm with only pre-smoothing are practically identical with those in Table 1. The results for the symmetric W-cycle algorithm are presented in Table 2.

In the second set of experiments we computed the contraction numbers of Algorithm 2.3 on the L-shaped domain with vertices $(0,0)$, $(0,1)$, $(-1,1)$, $(-1,-1)$, $(1,-1)$ and $(1,0)$, and we took $\eta = 1$. The results for the W-cycle algorithm with only post-smoothing and the

k	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$	$m=9$	$m=10$
2	0.828	0.780	0.735	0.691	0.648	0.609	0.574	0.539
3	0.848	0.805	0.764	0.724	0.686	0.650	0.616	0.584
4	0.854	0.813	0.772	0.734	0.697	0.662	0.629	0.598
5	0.856	0.814	0.774	0.736	0.700	0.665	0.632	0.601
6	0.857	0.815	0.775	0.737	0.701	0.666	0.633	0.602
7	0.857	0.815	0.775	0.737	0.701	0.666	0.634	0.602

TABLE 2. γ_k for the unit square with $m_1 = m_2 = m$.

k	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$	$m=9$	$m=10$
2	0.815	0.855	0.819	0.787	0.763	0.734	0.709	0.674
3	0.862	0.879	0.853	0.828	0.804	0.780	0.756	0.733
4	0.916	0.894	0.871	0.849	0.827	0.805	0.784	0.763
5	0.920	0.898	0.876	0.855	0.834	0.812	0.792	0.772
6	0.920	0.899	0.878	0.856	0.835	0.814	0.794	0.774
7	0.921	0.899	0.878	0.857	0.835	0.815	0.795	0.775

TABLE 3. γ_k for the L-shaped domain with $m_1 = 0$ and $m_2 = m$.

k	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$	$m=9$	$m=10$
2	0.788	0.730	0.678	0.630	0.577	0.544	0.499	0.469
3	0.739	0.779	0.733	0.689	0.648	0.609	0.572	0.538
4	0.848	0.805	0.763	0.724	0.686	0.650	0.616	0.584
5	0.855	0.812	0.772	0.734	0.697	0.662	0.629	0.598
6	0.856	0.814	0.774	0.736	0.700	0.665	0.632	0.601
7	0.857	0.815	0.775	0.737	0.701	0.666	0.633	0.602

TABLE 4. γ_k for the L-shaped domain with $m_1 = m_2 = m$.

symmetric W-cycle algorithm with both pre-smoothing and post-smoothing are presented in Table 3 and Table 4. They are similar to the results for the unit square.

In the last experiment we computed the contraction numbers of the W-cycle algorithm with only post-smoothing on the unit square at level 5 using penalty parameters 0.1, 1, 10 and 100. The results in Table 5 show that the behavior of the W-cycle algorithm is independent of η.

5. Concluding Remarks

We have analyzed a W-cycle algorithm with only post-smoothing for a weakly over-penalized interior penalty method for the Poisson problem on convex polygonal domains and

η	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$	$m=9$	$m=10$
0.1	0.918	0.896	0.874	0.852	0.830	0.808	0.788	0.767
1	0.920	0.899	0.878	0.856	0.835	0.814	0.794	0.774
10	0.921	0.900	0.878	0.857	0.836	0.815	0.795	0.775
100	0.921	0.900	0.878	0.857	0.835	0.815	0.795	0.775

TABLE 5. γ_5 for the unit square with $m_1 = 0$, $m_2 = m$ and various penalty parameters.

obtained the first rigorous multigrid convergence result for nonsymmetric interior penalty methods. The result in this paper can be easily generalized to three dimensional convex polyhedral domains.

Our numerical results indicate that the convergence results can be extended to the general W-cycle algorithm with both pre-smoothing and post-smoothing steps, and also to nonconvex domains. Classical multigrid convergence results in [3] showed the adverse effect of increasing the number of smoothing steps in the case of nonsymmetric problems. However, this adverse effect has not been observed in our numerical experiments. We believe our convergence result can be refined to reflect this phenomenon.

These and other issues (such as V-cycle algorithms) will be further investigated in [17].

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