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Multilevel Preconditioners for Nonselfadjoint or Indefinite Orthogonal Spline Collocation Problems

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We develop and study symmetric multilevel preconditioners forthe computation of the orthogonal spline collocation (OSC) solution of aDirichlet boundary value problem (BVP) with a nonselfadjoint or anindefinite operator. The OSC solution is sought in the space ofpiecewise Hermite bicubic spline functions defined on a uniformpartition. We consider an additive and a multiplicative multilevel-preconditioners that are used with the preconditioned conjugategradient (PCG) method. Results and algorithms presented in this paperare closely related to those in [1], [2], [3], and [4].Let Ω be a unit square $(0,1) \times (0,1)$ with the boundary$\partial\Omega$, and let $x = (x_1, x_2)$. We consider a BVP

$$Lu \equiv \sum_{i,j=1}^{2} a_{ij}(x) u_{x_i x_j} + \sum_{i=1}^{2} b_i(x) u_{x_i} + c(x) u = f(x), \quad x \in \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{1}$$

Operator L could be non-selfadjoint or indefinite in L^2 innerproduct. We assume that the principal part of L satisfies theuniform ellipticity condition and that BVP (1) has aunique solution in $H^2(\Omega)$.Let π_0 be a uniform coarsest rectangular partition of Ω.We obtain a set of partitions $\{\pi_k\}_{k=0}^{K}$ by standardcoarsening, and let $V_0 \subset V_1 \subset \ldots \subset V_K \equiv V_h$be the set of corresponding nested spaces of piecewise Hermitebicubics that vanish on $\partial\Omega$. Let $\sum$ denote thetwo-dimensional composite Gauss quadrature corresponding to partitionπ_h with 4 nodes in each element. Let $\mathcal{G}_h$ denote thecorresponding set of Gauss points. The OSC discretization of BVP(1) is defined by

$$u_h \in V_h, \quad Lu_h(\xi) = f(\xi), \quad \xi \in \mathcal{G}_h, \tag{2}$$

and it can be written as the operator equation $L_h u_h = f_h$ in theHilbert space V_h with the inner product $(v, w)_h = \sum vw$.Let $\{\psi_{k,j}\}_{j=1}^{N_k}$ be the standard finite element basisof V_k consisting of products of one-dimensional value and slopebasis functions. Using space decomposition

$$V_h = V_0 + \sum_{k=1}^{J} \sum_{j=1}^{N_k} V_{k,j}, \quad V_{k,j} = \text{span}(\psi_{k,j}),$$

we define and study multilevel additive $B_{\mathrm{a}}$and multiplicative B_{m} preconditioners forsolving the normal equation $L_h^* L_h u_h = L_h^* f_h$, where L_h^*isthe adjoint to L_h. The implementation of B and B_{m} is based on relationshipsbetween basis functions for two consecutive partitions and theimplementation of B_{m} is similar to thatforV(1,1)-cycle with the Gauss-Seidel smoothing. A problem on thecoarsest partition is assumed sufficiently small, and it is solvedexactly. The computational cost of the preconditioning algorithmsisO(N_K). The following is our main result.**Theorem.** *There are positive independent of h and K constants*α_{a}*,* β_{a}*,*α_{m}*, and* β_{m}*,such that*

$$\begin{aligned}
&\alpha_{\mathrm{a}}\,(B_{\mathrm{a}}v,v)_h \le (L_h^* L_h v,v)_h \le \beta_{\mathrm{a}}\,(B_{\mathrm{a}}v,v)_h, \quad v\in V_h, \\
&\alpha_{\mathrm{m}}\,(B_{\mathrm{m}}v,v)_h \le (L_h^* L_h v,v)_h \le \beta_{\mathrm{m}}\,(B_{\mathrm{m}}v,v)_h, \quad v\in V_h.
\end{aligned} \tag{3}$$

To obtain these results, we prove the key assumptions in the generaltheory of Schwarz methods presented in [4], and use the inequalities

$$C^{-1}\|v\|^2_{H^2(\Omega)} \le a_h(v,v) \le C\|\Delta v\|^2_{L^2(\Omega)}, \quad v\in V_h,$$

obtained in [2]. We present numerical results that demonstratethe efficiency of our preconditioning algorithms.**References**

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