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Nonlinear monotone finite volume method for anisotropic diffusion equation

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Predictive numerical simulations of subsurface processes require not only more sophisticated physical models but also more accurate and reliable discretization methods for these models. We study a new monotone finite volume method for diffusion problems with a heterogeneous anisotropic material tensor. Examples of anisotropic diffusion includes diffusion in geological formations, head conduction in structured materials and crystals, image processing, biological systems, and plasma physics. Development of a new discretization scheme should be based on the requirements motivated by both practical implementation and physical background. This scheme must

- be locally conservative;

- be monotone, i.e. preserve positivity of the differential solution;

- be applicable to unstructured, anisotropic, and severely distorted meshes;

- allow arbitrary diffusion tensors;

- result in sparse systems with a minimal number of non-zero entries;

- have higher than the first-order accuracy for smooth solutions.

The discretization methods used in existing simulations, such as Mixed Finite Element (MFE) method, Finite Volume (FV) method, Mimetic Finite Difference (MFD) method, Multi Point Flux Approximation (MPFA) method, satisfy most of these requirements except the **monotonicity**. They fail to preserve positivity of a continuum solution when the diffusion tensor is heterogeneous and anisotropic or the computational mesh is strongly perturbed. Monotonicity is a very important as well as a difficult requirement to satisfy.

The mixed form of the diffusion equation includes the mass conservation equation and the constitutive equation:

$$\operatorname{div} \mathbf{q} = Q, \qquad \mathbf{q} = -D \operatorname{grad} C,$$

where D is the diffusion tensor, Q is the source term, and $\mathbf{q}$ is the flux of concentration C.

All the methods mentioned above use the same discretization of the mass conservation equation and differ by their approximation of the flux (constitutive) equation. In the nonlinear finite volume scheme a *reference* point $\mathbf{x}_T$ is defined for each mesh cell T to approximate the concentration C. The position of the reference point depends on the geometry of T and value of the diffusion tensor. For isotropic diffusion tensors and triangular cell T, the center of the inscribed circle is the acceptable position for the reference point.

The flux $\mathbf{q}$ is approximated at the middle of each mesh edge using a weighted difference of concentrations in two neighboring cells. Nonlinearity comes from the fact that these weights depend on a solution. Therefore the nonlinear finite volume method results in a nonlinear algebraic system. This system is very sparse and the dimension is equal to the number of mesh cells T. For triangular meshes, the matrix of this system has at most four non-zero elements in each row. To solve the nonlinear algebraic problem we use the Picard iterative method which guarantees monotonicity of the discrete solution for all iterative steps. The computational results demonstrate the flexibility and accuracy of the scheme. For sufficiently smooth solutions, we achieve the second-order convergence for concentration C and at least the first-order for flux $\mathbf{q}$ in a mesh-dependent L_2-norm. For non-smooth, highly anisotropic solutions we observe at least the first-order convergence for both unknowns. The solution remains **non-negative** on different types of meshes and for different directions of anisotropy. Both RT_0 and P_1 methods produce **negative** values.