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On a complex Shifted-Laplace preconditioner for high wavenumber heterogeneous Helmholtz problems

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In this presentation, we are concerned with the numerical solution of the boundary value problem

$$\mathcal{L}u \equiv \left(\partial_{xx} + \partial_{yy} + k^2(x,y)\right) u = f \text{ in } \Omega \in \mathbb{R}^2, \quad (1)$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial n} - iku\right) = 0 \text{ on } \Gamma = \partial\Omega. \quad (2)$$

This Helmholtz problem arises in many applications, e.g. acoustics, electromagnetics and geophysics. Here, we consider problems which mimic geophysical applications with inhomogeneous material properties and high wavenumbers k. Furthermore, we develop solution methods based on the iterants in the Krylov subspace.

For high wavenumbers, the matrix A obtained from any discretization of (1) and (2) has the property of indefiniteness, with the spectrum is distributed in the left and right complex half plane. Furthermore, the condition number is extremely large. These two properties are not favorable for Krylov subspace iterative methods. Applying Krylov methods on the normal equations representation, A^*A, does not provide a practical remedy since, even though the linear system can be made definite, the condition number is even worse. Another remedy is efficient preconditioning.

In *[Erlangga et. al., TU Delft-AMA Report 03-01]*, we propose a class of preconditioners, called the Shifted-Laplace preconditioners, for the Helmholtz problem. The preconditioners are constructed based on an operator

$$\mathcal{M} \equiv \partial_{xx} + \partial_{yy} - \alpha k^2(x,y), \quad (3)$$

where $\alpha \in \mathbb{C}$. We have previously found that $\alpha = i$, ($i^2 = -1$) gives very satisfactorily convergence within this class of preconditioners. We call the preconditioner (3) with $\alpha = i$ the Complex Shifted-Laplace preconditioner (CSL).

We can show also that (i) the spectrum of $M^{-1}A$ is bounded above by one, and (ii) the lower bound of the spectrum of $M^{-1}A$ is of $\mathcal{O}(1/k^2)$. From the latter insight, we may expect that the convergence rate for increasing k is mainly determined by the smallest eigenvalue. We have used the preconditioner in combination with BiCGSTAB. A significant reduction in the number of iterations is observed.

In this talk, we discuss another issue in relation with the preconditioner solves. (Previously, we solved the preconditioner exactly using direct methods. This process is very costly.) Since the preconditioning matrix M is complex, symmetric positive definite (CSPD), several efficient iteration methods can be implemented to approximate M^{-1}. We study the use of an incomplete LU decomposition of M and multigrid as the preconditioner solver. For ILU factorizations for M, one level of fill-in is allowed. For constructing the LU factors, an algorithm based on regular stencil is used. The algorithm is not only fast in computing the LU factors (so it reduces initialization cost) but also requires a small amount of storage.

For multigrid, we implement a geometric multigrid algorithm as discussed in *[Oosterlee et. al., SIAM J. Sci. Comput. 19(1) (1998), pp.87–110]*. It is originally developed for real-valued matrices for structured grid applications. For our complex-valued applications, the method needs not be modified.

We compare the number of iterations and the time to convergence for various wavenumbers k. The conclusions are as follows:

(1) ILU of M results in better performance than ILU directly applied to A. The improvement becomes more significant as k increases.

(2) One V(1,1) multigrid iteration is sufficient to further improve the computational performance. In comparison with the unpreconditioned and ILU-preconditioned case, the computational time is reduced almost by factor 10 and 3 for sufficiently large k, respectively. Furthermore, the number of iterations are reduced by factor of 50 and 8, respectively.

(3) Preconditioner solves using multigrid seem to be less sensitive to the inhomogeneity of the media. In comparison with constant wavenumbers, only less than 10 % increase in the number of iterations is observed with multigrid.