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Two grid spectral preconditioning for general sparse linear systems

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Multigrid methods are among the fastest techniques to solve a linear system $Ax = b$ arising from the discretization of a partial differential equation. The core of the multigrid algorithms is a two-grid procedure that is applied recursively. The basic idea of the two-grid solver is :

1. given x_0, perform a few (μ_1) steps of a basic stationary method of the form $x^{(k+1)} = (I - MA)x^{(k)} + g$ to compute x^{μ_1}. This step is referred to as the pre-smoothing.
2. project the residual $r = b - Ax^{\mu_1}$ on a coarse space using a restriction operator R and solve the linear system $RAPe = Rr$, where P is the prolongation operator.
3. prolongate the error in the fine space and update $x = x^{\mu_1} + Pe$.
4. perform few (μ_2) steps of a basic stationary method of the form $x^{(k+1)} = (I - MA)x^{(k)} + g$ to compute x^{μ_2}. This step is referred to as the post-smoothing.
5. If x_2^{μ} is accurate enough stop, else $x_0 = x^{\mu_2}$, go to Step 1.

The smoother iterations aim at reducing the high frequencies of the error (i.e. the components of the error in the space spanned by the vectors associated with the largest eigenvectors of A). The restriction operator and consequently the coarse space is chosen so that this space contains the low frequency of the error (i.e. the components associated with the smallest eigenvalues). In classical multigrid, the coarse space is not defined explicitly through the knowledge of the eigencomponents but by the selection of a space that is expected to capture them. The scheme presented above is a multiplicative algorithm [1] but additive variants [2] also exist.

In this work, we exploit the idea of the two-grid method to design additive and multiplicative preconditioners for general linear systems. We explicitly

define the coarse space by computing the eigenvectors V associated with the smallest eigenvalues of MA (that is, the components of the error that are not damped efficiently by the smoother). In that context, the prolongation operator is $P = V$. We show that our preconditioners shift the smallest eigenvalues of MA to one and tend to cluster those that were already in the neighbourhood of one closer to one. We illustrate the performance of our method through numerical experiments on a set of general linear systems, both symmetric and positive definite and unsymmetric. Finally, we consider a case study of a non-overlapping domain decomposition method of semiconductor device modelling for the solution of the drift-diffusion equations.

[1] W. Hackbusch. Multigrid methods and applications. Springer-Verlag, 1985.

[2] R. S. Tuminaro. A highly parallel multigrid-like method for the solution of the Euler equations. *SIAM J. Scientific and Statistical Computing* vol. 13, pages 88-100, 1992.