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**On the Reuse of Standard Preconditioners for Higher
Order Time Discretizations of Parabolic PDEs**

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In this abstract we will describe a preconditioner for some higher order time discretizations of parabolic problems. The preconditioner is optimal with respect to the spatial discretization parameters, that typically are the characteristic mesh size parameter h and the polynomial degree p . The preconditioner is also order optimal with respect to Δt . The only assumption is that there exists a preconditioner for the low order time discretization schemes such as Crank-Nicholson or implicit Euler. Such preconditioners are standard, c.f. e.g., [16], [17] and [18].

We study the model problem

$$\begin{aligned}\frac{\partial u}{\partial t} &= \Delta u, & \text{in } \Omega, t > 0 \\ u &= 0, & \text{on } \partial\Omega, t > 0 \\ u &= u_0, & \text{in } \Omega, t = 0.\end{aligned}$$

This equation is discretized in space and time to give the following linear system to be solved for each time level

$$Q_{kj}(\Delta t A)u^n = P_{kj}(\Delta t A)u^{n-1},$$

where Δt is the time stepping parameter, the two polynomials Q_{kj} and P_{kj} are the (k, j) -Padé approximation to the exponential function and A is a discrete Laplacian. The polynomials are given by (c.f. [18]):

$$\begin{aligned}P_{kj}(\Delta t A) &= \sum_{i=0}^k \binom{k}{i} \frac{(k+j-i)!}{(k+j)!} (\Delta t A)^i \\ Q_{kj}(\Delta t A) &= P_{jk}(-\Delta t A).\end{aligned}$$

$j \backslash k$	j	$j - 1$	$j - 2$
2	1.07	1.10	1.17
6	1.49	1.56	1.66
10	2.08	2.20	2.34

Table 1: Upper bound on the condition number for various values of j and k .

The proposed (exact) preconditioner is

$$R_{kj}(\Delta t A) = \left(I - \sqrt[j]{\frac{j!}{(j+k)!}} \Delta t A \right)^j.$$

Hence, R_{kj} is a standard preconditioner for a low order time discretization of a parabolic PDE, used j times. The coefficient before $\Delta t A$ is chosen such that the highest order term of $R_{kj}(\Delta t A)$ and $Q_{kj}(\Delta t A)$ are equal.

In Table 1 we show an upper bound on the condition number of the preconditioned system, using an exact preconditioner for R_{kj} . Further we prove the following lemma.

Lemma 1 *The condition number of the preconditioned system is bounded by*

$$\kappa \left((R_{kj}(\Delta t A))^{-1} Q_{kj}(\Delta t A) \right) < 1.8 \cdot 1.09^j.$$

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