A Multigrid approach for Poisson equation with non-eliminated mixed boundary conditions in arbitrary domain

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January 10, 2011

Abstract

In this paper we present a Multigrid approach to accelerate the convergence of the iterative method proposed in [8] to solve the Poisson equation in arbitrary domain Ω and mixed boundary conditions over a Cartesian grid. Such multigrid technique is treated in detail in [9]. The domain is identified by a level set function φ in such a way that $\Omega = \{x \in \mathbb{R}^d : \varphi(x) < 0\}$. This Multigrid strategy can be applied also to more general problems where a non-eliminated boundary condition approach is used. Arbitrary domain makes the definition of the restriction operator for boundary conditions hard to find. A suitable restriction operator is provided in this work, together with a proper treatment of the boundary smoothing, in order to avoid degradation of the convergence factor of the Multigrid due to boundary effects. Numerical tests show evidence of the convergence factor improvement in several arbitrary domains. The method proposed is second order accurate. A brief description of the case of discontinuous coefficients is also provided.

Introduction

Multigrid technique is one of the most efficient strategy to solve a class of partial differential equations, using a hierarchy of discretizations, especially for elliptic equations.
Elliptic equation in arbitrary domain (possibly with moving boundary) is central to many applications, such as diffusion phenomena, fluid dynamics, charge transport in semiconductors, crystal growth, electromagnetism and many others. The wide range of applications may require different kinds of boundary conditions. An application we have in mind is to fluid dynamics: the aim is to model the motion of an incompressible fluid contained in a tank of arbitrary shape. The problem is modeled by incompressible Navier-Stokes equations, which are solved by Chorin's projection method [7]. This leads to an elliptic equation for the pressure, obtained enforcing the incompressibility condition. This pressure equation requires Dirichlet condition on the free surface of the fluid and Neumann condition on the rigid walls. The pressure equation is the bottleneck of the whole method and therefore requires an efficient solver.
Several techniques have been developed to solve elliptic equations on arbitrary domains. Finite Element Methods use a mesh triangulation to better fit the boundary [18]. However, in presence of moving boundary, a grid re-generation is needed at each time step, which makes the method expensive. For this reason a Cartesian grid method is preferred together with a level-set approach to keep track of the boundary at each time step. Level-set methods have been introduced to implicitly define a domain and its boundary, in order to simple handle complex topological changes of moving boundaries such as merging and breaking [22, 20].
Since the boundary may be not aligned with the grid, a special treatment is needed. The simplest method makes use of

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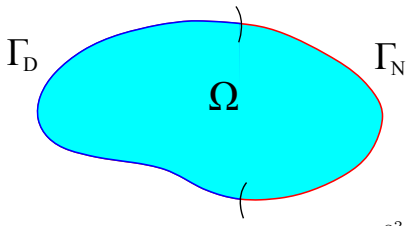
the Shortley-Weller discretization [21], that discretizes the Laplacian operator with usual central difference away from the boundary and makes use of a non symmetric stencil in the interior points of the domain close to the boundary. While this discretization provides a simple second order method for Dirichlet conditions, it cannot be immediately applied in presence of Neumann conditions. A simple approach is adopted by Hackbusch [12] to first order accuracy, and by other authors (see [1] and the references therein) to second order accuracy. However, the method proposed by Bramble in [1] for second order accuracy is quite involved and not recommendable for practical purposes.
Another class of methods is based on cut-cell techniques, obtained by a Finite Volume discretization which embeds the domain in a regular Cartesian grid [13]. Cells that are cut by the boundary require a special treatment, such as cell-merging and rotated-cell, in order to avoid a too strict restriction of the time step dictated by the CFL condition. Other methods for Dirichlet condition are Immersed Boundary Methods, first proposed by Peskin in [17] and later improved by LeVeque and Li [14], and penalization methods [5].
In our method [8] we will use a rather simple finite-difference ghost-cell technique, that adds extra grid points (ghost points) outside the domain in order to keep unchanged the symmetry of the stencil even for inside points close to the boundary. A detailed description of the method can be found in [8]. In ghost points, boundary conditions are enforced in order to close the discrete system. The ghost-cell method was first developed by Fedkiw in [10], where a two-phase contact discontinuity was discretized (Ghost Fluid Method).
A second-order accurate method for Dirichlet conditions on regular Cartesian grid is proposed by Gibou *et al.* in [11]. Other methods use a non-regular Cartesian grid, such as in [6], where Gibou *et al.* present finite difference schemes for solving the variable coefficient Poisson equation and heat equation on irregular domains with Dirichlet boundary conditions, using adaptive Cartesian grids. One efficient discretization based on cut-cell method to solve more general Robin conditions is proposed by Gibou *et al.* in [16], which provides second order accuracy for the Poisson and heat equation and first order accuracy for Stefan-type problems. Simple efficient methods based on symmetric image of ghost points to solve mixed boundary condition problems provided with a Multigrid algorithm have been recently developed in [4] and by Ma *et al.* in [15].
In our method [8], boundary conditions are neither eliminated from the discrete system (they are strongly coupled and their elimination is too hard in more than one dimension) nor directly enforced (which leads to a non-convergent iterative method): they are *relaxed* together with the interior equations. This leads us to an iterative scheme for the set of all unknowns (internal and ghost points), which is proved to converge, at least for first order accurate discretization. In [9] we provide a general Multigrid technique to solve the discrete system coming from continuous elliptic problem in case of non-eliminated boundary conditions. The smoothing procedure of the multigrid approach in the interior is Gauss-Seidel-like, while the iterations on the boundary are performed in order to provide smooth errors.
Multigrid techniques for non-eliminated boundary conditions are well-studied in literature in the case of rectangular domain (as we can see, for example, in [23]), where a restriction operator is defined separately for the interior of the domain and on the boundary, and the restriction of the boundary is performed using a one dimensional restriction operator, since ghost points are aligned with the Cartesian axis. In the case of arbitrary domain, ghost points have an irregular structure and we provide a reasonable definition of the restriction operator for general boundary conditions. We also show that a proper treatment of the boundary iterations can improve the rate of convergence of the multigrid, making it closer to the convergence factor predicted by the Local Fourier Analysis for inside equations, as suggested by Brandt in [2]. The cost of this extra computational work is negligible, i.e. tends to zero as the dimension of the problem increases.

1 Description of the model problem

Let $D = [-1, 1]^2$ be the computational domain, $\Omega \subset D$ a domain such that $\partial\Omega \cap \partial D = \emptyset$. Let Γ_D, Γ_N a partition of $\partial\Omega$ (i.e. $\Gamma_D \cup \Gamma_N = \partial\Omega, \Gamma_D \cap \Gamma_N = \emptyset$). Consider the model problem



$$-\Delta u = f \quad \text{in } \Omega \tag{1}$$
$$u = g_D \quad \text{on } \Gamma_D \tag{2}$$
$$\frac{\partial u}{\partial n} = g_N \quad \text{on } \Gamma_N \tag{3}$$

where $\mathbf{n}$ is the outward unit normal, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian operator, $f\colon \Omega \to \mathbb{R}$, $g_D\colon \Gamma_D \to \mathbb{R}$, $g_N\colon \Gamma_N \to \mathbb{R}$ are assigned functions.
In order to solve the elliptic problem (1)-(3), we can transform it in an evolutive problem (with a fictitious time) that we call the associate *time-dependent* problem (see [8]):

$$\frac{\partial \tilde{u}}{\partial t} = \Delta \tilde{u} + f \quad \text{in } \Omega \tag{4}$$
$$\frac{\partial \tilde{u}}{\partial t} = \mu_D (g_D - \tilde{u}) \quad \text{on } \Gamma_D \tag{5}$$
$$\frac{\partial \tilde{u}}{\partial t} = \mu_N \left(g_N - \frac{\partial \tilde{u}}{\partial n} \right) \quad \text{on } \Gamma_N \tag{6}$$
$$\tilde{u} = \tilde{u}_0 \quad \text{in } \Omega, \text{ when } t = 0 \tag{7}$$

where μ_D and μ_N are two positive constants. Then, we will look for the steady state solution. An iterative scheme can therefore be obtained discretizing the associate *time-dependent* problem and considering the time just as an iterative parameter.
In order to keep track of the boundary Γ, we introduce the level set function $\varphi_0\colon D \to \mathbb{R}$, in such a way:

$$(x,y) \in \mathring{\Omega} \Longleftrightarrow \varphi_0(x,y) < 0, \quad (x,y) \in \partial\Omega \Longleftrightarrow \varphi_0(x,y) = 0$$

The outward unit normal to the boundary is $\mathbf{n} = \nabla\varphi_0 / |\nabla\varphi_0|$. From the level set function, we can obtain the signed distance function φ by reinitialization, i.e. by computing the stationary solution of $\partial\varphi/\partial t = \operatorname{sgn}(\varphi_0)\,(1 - |\nabla\varphi|)$, as we can see, for instance, in [22, 20]. Let us suppose we know the signed distance function in all grid nodes.

1.1 Relaxation operator

Let us introduce some notation. Let $d \in \mathbb{N}$ be the dimension of the problem, $N \geq 1$ be an integer and $h = 2/N$ be the spatial step. Let $D_h = \mathbf{j}h, \mathbf{j} = (j_1, \ldots, j_d) \in \{-N, N\}^d$ and $\Omega_h = \Omega \cap D_h$ be the discrete versions of D and Ω respectively. D_h is the set of *grid points*. We say that a grid point $\mathbf{x}' \in D_h$ is *near* to a grid point $\mathbf{x}'' \in D_h$ if $\|\mathbf{x}' - \mathbf{x}''\|_1 = h$, i.e. $\sum_{l=1}^{d} |x'_l - y'_l| = h$. We call *ghost point* any grid point that is both outside Ω and near a grid point inside Ω. We call Γ_h the set of all ghost points.
From now on, we shall consider $d = 2$, but the results are valid also for $d > 2$.
Then, we write the basic iterative scheme (*relaxation scheme*) discretizing the *time-dependent* problem (4)-(7). For any grid point (jh, ih) of Ω_h, we write an equation obtained from the discretization of (4) in such point, using forward Euler in time and central difference in space and taking the maximum time step consented by the CFL condition (i.e. $\Delta t = h^2/4$):

$$u_{i,j}^{(n+1)} = 1/4 \left(h^2 f_{i,j} + u_{i-1,j}^{(n)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n)} + u_{i,j+1}^{(n)} \right). \tag{8}$$

Eq. (8) is equivalent to discretizing directly (1) using central difference in space and applying Jacobi iteration scheme. Since we have used the standard $5-$point stencil even for grid points close to the boundary, we have added new

unknowns to the linear system (*ghost points*).
To close the system of equations (8), we must write an equation for each ghost point. This can be done in three simple steps. Let $G \equiv (x_G, y_G)$ be a ghost point.

1. Making use of the signed distance function φ, we can compute the closest boundary point to G, that we call B (see Figure 1), by:
$$B \equiv (x_B, y_B) = G - \mathbf{n}_G \cdot \varphi(G) = G - \left(\frac{\nabla \varphi}{|\nabla \varphi|} \right) \Big|_G \varphi(G) \tag{9}$$
using a second order accurate discretization for $\nabla \varphi$, such as central difference in G.

2. Compute the nine-point stencil (depicted in Fig. 1) in Upwind direction, i.e.:
$$St_9 = \left\{ (x_G + s_x\, k_1\, h, y_G + s_y\, k_2\, h) \colon (k_1, k_2) \in \{0, 1, 2\}^2 \right\}$$
where $s_x = \mathrm{sgn}(x_B - x_G)$ and $s_y = \mathrm{sgn}(y_B - y_G)$.

3. Let $\mathcal{L}_{St_9}[u]$ be the biquadratic interpolant of the numerical solution u in the nine-point stencil. If $B \in \Gamma_D$, the iteration for the ghost point G will be obtained from the discretization of (5):
$$u_G^{(n+1)} = u_G^{(n)} + \mu_D \Delta t \left(g_D(B) - \mathcal{L}_{St_9}[u](B) \right) \tag{10}$$
while if $B \in \Gamma_N$, the iteration for the ghost point G will be obtained from the discretization of (6):
$$u_G^{(n+1)} = u_G^{(n)} + \mu_N \Delta t \left(g_N(B) - \left(\nabla \mathcal{L}_{St_9}[u] \cdot \frac{\nabla \mathcal{L}_{St_9}[\varphi]}{|\nabla \mathcal{L}_{St_9}[\varphi]|} \right) \Big|_B \right) \tag{11}$$

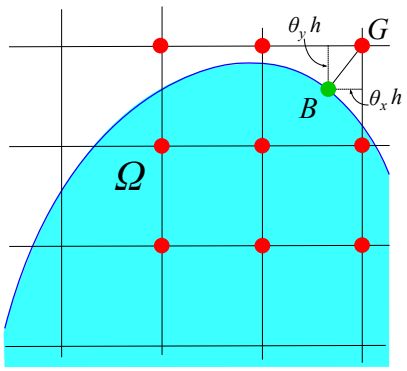


Fig. 1: B is the boundary closest point to G, while the red points are the nine-point stencil in Upwind direction referred to the ghost point G.

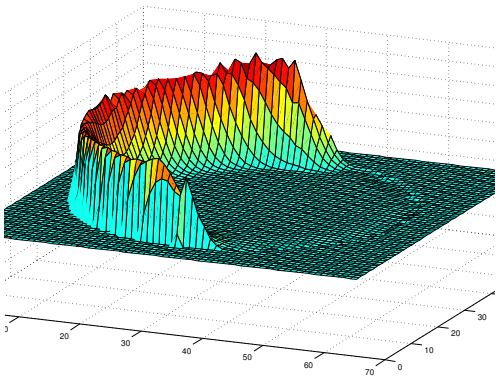


Fig. 2: After few relaxations, the defect shows a sharp gradient crossing the boundary, especially for the Neumann boundary. Here the domain is a circle.

The constants μ_D and μ_N are chosen in order to satisfy a stability condition, i.e. $\mu_D \Delta t < 1$ and $\mu_N \Delta t < h$ [8].

2 Multigrid approach

We just modify the relaxation operator, considering the Gauss-Seidel version of (8), (10), (11), in order to deal with a proper *smoother*, and we have to order all points of $\Omega_h \cup \Gamma_h$ in some way. Let us choose the lexicographic ordering (GS-LEX). In order to explain the multigrid approach, we have just to explain the steps concerning grid migration. All the other steps are straightforward (see [3, 23]).

2.1 Transfer grid operators

Since we are dealing with non-eliminated boundary condition and the operators for inner equations and for boundary conditions scale with different powers of h (see Fig. 2), the restriction of the defect to a coarser grid has to be performed separately for Ω_h and Γ_h. In addition, since $\partial\Omega$ is not aligned with the grid lines, the restriction for Γ_h cannot be performed as a $1D-$restriction on the boundary. In other words, Γ_h has a very irregular structure and is not easy to find a reasonable restriction operator.

2.1.1 Restriction of the defect on Ω_h

We want to transfer the defect $\mathbf{r}_h^\Omega$ of inner equations to a coarser grid by a suitable operator: $\mathbf{r}_{2h}^\Omega = \left(I_{2h}^h\right)^\Omega \mathbf{r}_h^\Omega$. Let $(x,y) \in \Omega_{2h}$. The value $\mathbf{r}_{2h}^\Omega(x,y)$ has to be found by weighting the values of the defect in near grid points, but always inside Ω. Let $\mathcal{N}(x,y) = \{(x+jh, y+ih) \colon j,i = -1,0,1\}$ be the neighborhood of (x,y). Consider the maximum full rectangle $\mathcal{T}$ with vertices belonging to $\mathcal{N}(x,y)$ and such that $\mathcal{T} \cap D_h \subseteq \mathcal{N}(x,y) \cap \Omega_h$ (see Fig. 3). Therefore, the stencil we use in (x,y) to transfer $\mathbf{r}_h^\Omega$ to a coarse grid depends on the size of $\mathcal{T}$. In fact, let $\mathcal{T} \cap D_h$ be a 3×3-point stencil (i.e. $\mathcal{N}(x,y) \subseteq \Omega_h$). In this case we can use the standard full-weighting operator:

$$\left(I_{2h}^h\right)^\Omega = \frac{1}{16} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}_{2h}^h \tag{12}$$

Now let $\mathcal{T} \cap D_h$ be a 3×2-point stencil. Without loss of generality, we can suppose the vertices of $\mathcal{T}$ are $(x+jh, y+ih)$, with $j \in \{-1,0\}$, $i \in \{-1,1\}$. In this case, the operator we will use is:

$$\left(I_{2h}^h\right)^\Omega = \frac{1}{16} \begin{bmatrix} 2 & 2 & 0 \\ 4 & 4 & 0 \\ 2 & 2 & 0 \end{bmatrix}_{2h}^h, \tag{13}$$

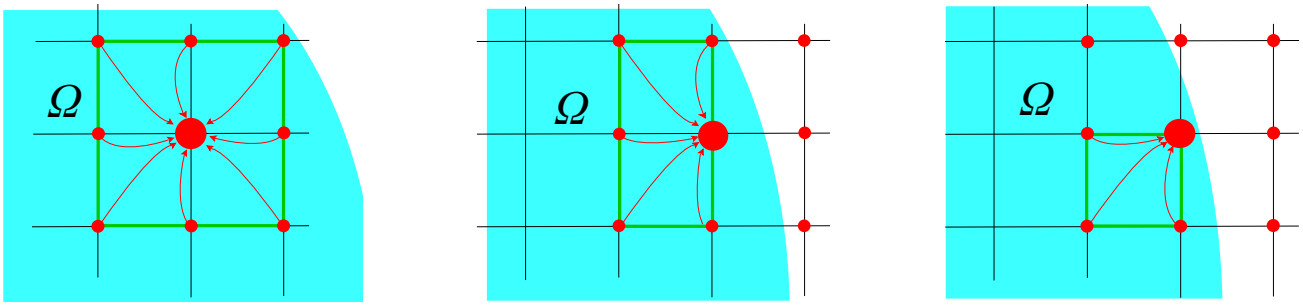
while, if $\mathcal{T}$ is a 2×2-point stencil, with vertices $(x+jh, y+ih)$, $j,i \in \{-1,0\}$, the operator will be:

$$\left(I_{2h}^h\right)^\Omega = \frac{1}{16} \begin{bmatrix} 0 & 0 & 0 \\ 4 & 4 & 0 \\ 4 & 4 & 0 \end{bmatrix}_{2h}^h. \tag{14}$$

These three cases are summarized in Fig. 3 (a description of the stencil notation can be found in [23, pag. 10]).

2.1.2 Restriction of the defect on Γ_h

The main idea is to extrapolate the values $\mathbf{r}_h^\Gamma$ of the defect on Γ_h in all $D_h - \Omega_h$ and then perform the restriction just using values outside Ω_h, by the same technique described in the previous section.



$$\frac{1}{16}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}_{2h}^{h} \qquad \frac{1}{16}\begin{bmatrix} 2 & 2 & 0 \\ 4 & 4 & 0 \\ 2 & 2 & 0 \end{bmatrix}_{2h}^{h} \qquad \frac{1}{16}\begin{bmatrix} 0 & 0 & 0 \\ 4 & 4 & 0 \\ 4 & 4 & 0 \end{bmatrix}_{2h}^{h}$$

Fig. 3: Upper, the nine points of $\mathcal{N}(x,y)$ and the green boundary of the rectangle $\mathcal{T}$. The bold red point is on the coarser and finer grids, while the little red points are on the finer grid. The arrows represent the action of the restriction operators. Below, the respective stencil to be used.

The extrapolation will be done by solving the transport equation $\partial \mathbf{r}_h^\Gamma/\partial\tau + \nabla\mathbf{r}_h^\Gamma \cdot \mathbf{n} = 0$ where $\mathbf{n} \equiv (n_x, n_y) = \nabla\varphi/|\nabla\varphi|$ is the unit normal vector to the level-set [22]. The effect of such equation is to propagate the value of $\mathbf{r}_h^\Gamma$ along the normal. In practice, few steps will be needed to define $\mathbf{r}_h^\Gamma$ on a narrow band just outside of Ω.

2.1.3 Interpolation

Since the interpolation operator will act on the error, we just use the standard bilinear interpolation operator:

$$I_h^{2h} = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}_{h}^{2h}.$$

3 Numerical experiments

In all the following numerical tests we always choose $\Gamma_D = \{(x,y) \in \Gamma \colon x \leq 0\}$ and $\Gamma_N = \{(x,y) \in \Gamma \colon x > 0\}$.

3.1 Convergence factor

The Local Fourier Analysis (LFA) is a powerful tool to obtain the theoretically convergence factor by analyzing separately the action of different parts of the Multigrid algorithm to high and low frequency components of the error. For a detailed explanation of the LFA, we refer to [23, Ch. 4].
Now, we perform a numerical test in order to check if the convergence factor is close to the one predicted by LFA, which is obtained for rectangular domain with periodic boundary conditions, then without taking into account boundary effects. For brevity, we just consider the case $\nu_1 = 1$, $\nu_2 = 2$ (ν_1 and ν_2 are respectively the number of pre- and post-relaxations). The convergence factor predicted by LFA is 0.119. In all the numerical tests we will perform, the convergence factor is estimated as the ratio of consequent defects, i.e. $q = q^{(m)} = \left\|\mathbf{r}_h^{(m)}\right\|_\infty / \left\|\mathbf{r}_h^{(m-1)}\right\|_\infty$. for m very large.

3.2 An initial test

We start testing the Multigrid algorithm on a circular domain Ω with center $(0.05, 0.05)$ and radius $r = 0.563$ (Top-Left of Fig. 4). We list in Table 1 the measured convergence factors for TGCS, V-cycle and W-cycle.

Table 1: Measured convergence factor q with $\nu_1 = 1$, $\nu_2 = 2$

N	TGCS	V-cycle	W-cycle
64	0.56	0.57	0.60
128	0.57	0.62	0.57
256	0.59	0.60	0.59

As we can see, the measured convergence factor is far from the predicted one by LFA. Then, some boundary effects degrade the convergence factor. To overcome this difficulties, a possible improvement is applying, after a single relaxation and at each grid level, λ extra relaxation sweeps on all ghost points Γ_h and on all inside grid points of Ω_h within $\delta > 0$ distance from the boundary. We experimented that a good choice of these parameters is $\lambda = 5, \quad \delta = 3\,h$. The extra computational work is $O(N)$, than negligible as $N \to \infty$.

3.3 Numerical tests

In this section we confirm that the convergence factor is improved by adding some extra-relaxations near the boundary. We provide four tests on domains with different shape, as depicted in Fig. 4: circle, ellipse, saddle-shaped and flower-shaped. The four level-set functions are:

$$\textsc{Top-Left}{:}\varphi_0 = \sqrt{(x-0.05)^2 + (y-0.05)^2} - 0.563, \quad \textsc{Top-Right}{:}\varphi_0 = \frac{(x-0.05)^2}{0.563^2} + \frac{(y-0.05)^2}{0.263^2} - 1$$

$$\textsc{Bottom-Left}{:}\varphi_0 = 9\left(\frac{1}{2}x - \frac{\sqrt{3}}{2}y\right)^2 + \left(\frac{3\sqrt{3}}{2}x + \frac{3}{2}y - 1\right)^2 \sin\left(\frac{3\sqrt{3}}{2}x + \frac{3}{2}y - 1\right) - 1$$

$$\textsc{Bottom-Right}{:}\varphi_0 = r - 0.5 - \frac{y^5 + 5x^4y - 10x^2y^3}{5r^5}, \quad r = \sqrt{x^2 + y^2}$$

In Fig. 5 we list the measured convergence factors for each test.

4 Discontinuous coefficients

The whole method proposed (discretization and Multigrid approach) can be also extended to the case of discontinuous coefficients, which models, for example, a system composed by different materials separated by an interface. We want to study the equation

$$-\nabla \cdot (\gamma\, \nabla u) = f \text{ in } \Omega$$

where the coefficient function γ is discontinuous across a lower dimensional interface Γ, which split the domain Ω into two subdomains Ω_1 and Ω_2. The idea is to split the problem into two subproblems, enforcing homogeneous transmission conditions on Γ, i.e. $[u] = 0$ and $[\gamma\, \nabla u] \cdot \mathbf{n} = 0$, where $\mathbf{n}$ is normal to Γ, and $[\cdot]$ denotes the jump operator.

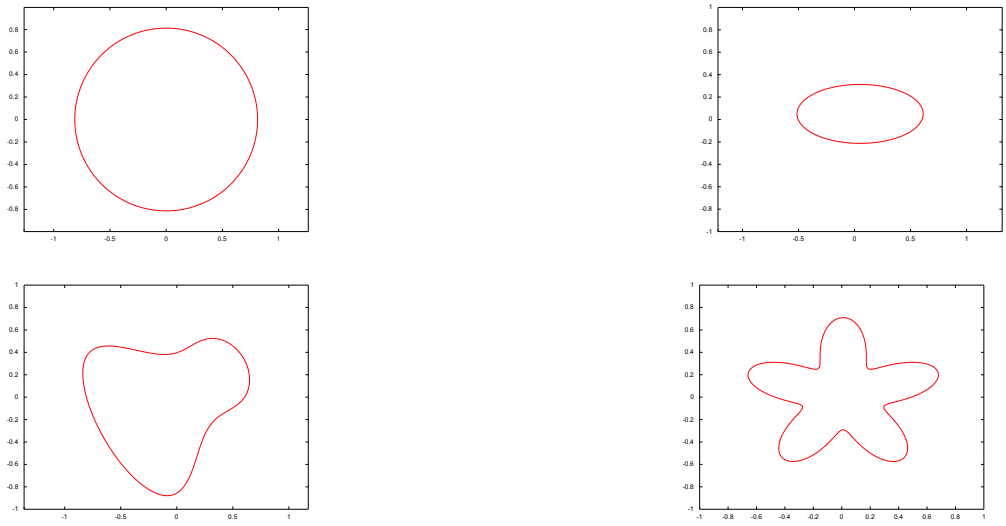


Fig. 4: Different domains used in the numerical tests of Sec. 3.3.

N / Nc	16	32	64	128	256
8	0.06	0.03	0.09	0.08	0.08
16		0.04	0.09	0.08	0.08
32			0.09	0.08	0.08
64				0.09	0.08
128					0.09

N / Nc	16	32	64	128	256
8	0.44	0.06	0.12	0.11	0.09
16		0.55	0.30	0.09	0.09
32			0.13	0.10	0.09
64				0.12	0.08
128					0.09

N / Nc	16	32	64	128	256
8	0.36	0.08	0.09	0.12	0.09
16		0.12	0.09	0.12	0.09
32			0.09	0.12	0.09
64				0.13	0.09
128					0.09

N / Nc	16	32	64	128	256
8					
16		0.89	0.75	0.50	0.25
32			0.49	0.25	0.12
64				0.24	0.11
128					0.09

Fig. 5: Convergence factor of the numerical tests of Sec. 3.3. We use $N \times N$ grid points in the finest grid against $Nc \times Nc$ grid points in the coarsest grid. In these test, $\nu = \nu_1 + \nu_2 = 3$.

The two subproblems then read:

$$\left\{ \begin{array}{rcll} -\nabla \cdot (\gamma_1 \nabla u_1) & = & f & \text{in } \Omega_1 \\ u_1^{(m+1)} & = & g & \text{on } \partial\Omega_1 - \Gamma \\ u_1^{(m+1)} & = & u_2^{(m)} & \text{on } \Gamma \end{array} \right. , \quad \left\{ \begin{array}{rcll} -\nabla \cdot (\gamma_2 \nabla u_2) & = & f & \text{in } \Omega_2 \\ u_2^{(m+1)} & = & g & \text{on } \partial\Omega_2 - \Gamma \\ \gamma_2 \dfrac{\partial u_2^{(m+1)}}{\partial n} & = & \gamma_1 \dfrac{\partial u_1^{(m+1)}}{\partial n} & \text{on } \Gamma \end{array} \right.$$

The relaxation operator can be performed just doing one iteration for the subproblem on Ω_1, and using the guess u_1 to make the right-hand side for the subproblem on Ω_2, then vice versa, and so on. This technique is very similar to the one of Domain Decomposition Method (DDM): the only difference is that in DDM one solves exactly each subproblem instead of making just a relaxation sweep. The advantage of our method is that convergence is always guaranteed, unlike in DDM, where association between the Dirichlet/Neumann transmission condition and the sub-domain cannot be arbitrary (see [19]).

Several numerical tests have been performed in $1D$, always showing a second order accuracy and a good convergence factor, even for highly oscillant coefficients and high jump on the interface. For brevity, we just show that the convergence factor does not depend on the magnitude of the jump in the coefficient (Table 2). The $2D$ case is presently under investigation.

Table 2: Measured $V(1,1)$ asymptotic convergence factors for a problem with a jumping coefficient of the order 10^p

p	0	1	2	3	4	5
ρ	0.11	0.10	0.11	0.11	0.11	0.10

Conclusion and Work in progress

A Multigrid technique for Poisson equation on an arbitrary domain and mixed boundary conditions is presented. This Multigrid strategy can also be applied in a general framework of ghost-point method in a regular Cartesian grid, in case of non-eliminated boundary conditions. A suitable transfer restriction operator for inside equations and boundary conditions is provided. The convergence factor is improved by adding some extra-relaxations on the ghost points and in a narrow band of inside grid points close to the boundary. The method is extended also to the case of discontinuous coefficients, which models, for example, a system composed by different materials separated by an interface.
Several extension of the discretization technique and Multigrid approach are presently under investigation. We mention the convection-diffusion equation in a moving domain, in order to study applications modeled by a Stefan-Type problem.
All this extensions will be coupled with the use of Adaptive Mesh Refinement (AMR) to obtain accurate solution in the case of domain with complex boundary. The AMR implementation is currently underway and will be ready in a few weeks.

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