Sixth Order Finite Difference Computation with Multigrid Method and Extrapolation for 2D Poisson Equation

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Abstract

We develop a sixth order finite difference discretization strategy to solve the two dimensional Poisson equation, which is based on the multigrid method, Richardson extrapolation and an operator interpolation scheme. Our modified multigrid algorithm, which is similar to the full multigrid method (FMG), can yield the fourth order approximate solution on both the fine grid and the coarse grid. Then we apply the Richardson extrapolation technique to compute a sixth order accurate coarse grid solution. The sixth order accurate fine grid solution can be computed by using an operator interpolation scheme. Numerical experiments are conducted to show the accuracy and efficiency of our new method, compared to the sixth order Richardson extrapolation compact (REC) discretization strategy using Alternating Direction Implicit (ADI) method and the standard fourth order compact difference (FOC) scheme using multigrid method.

Keywords: Poisson equation, compact difference scheme, multigrid method, Richardson extrapolation.

1 Introduction

Poisson equation is a partial differential equation with broad application in electrostatics, mechanical engineering, theoretical physics and other fields. The 2D Poisson equation can be written in the form of

$$u_{xx}(x,y) + u_{yy}(x,y) = f(x,y), \qquad (x,y) \in \Omega, \tag{1}$$

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where Ω is a rectangular domain, with suitable boundary conditions defined on $\partial\Omega$. The solution $u(x, y)$ and the forcing function $f(x, y)$ are assumed to be sufficiently smooth and have the required continuous partial derivatives.

By applying the standard second order central difference operator to $u_{xx}(x, y)$ and $u_{yy}(x, y)$ in Eq. (1), we can compute an approximate solution with second order accuracy. High order (more than two) accuracy discretization methods need more complex procedure than the second order discretization method to compute the coefficient matrix, but they usually generate linear systems of much smaller size, compared with that from the lower order discretization methods [1, 8, 10]. There has been growing interest in developing high order discretization methods, especially the high order compact difference schemes to solve the partial differential equations [11, 15, 18, 23, 25, 26]. We call them "compact" because these schemes only use the minimum three grid points in one dimension in the approximation formulas.

Previously, Chu and Fan [5, 6] proposed a new three point combined compact difference (CCD) scheme for solving two dimensional Stommel Ocean model, which is a special two dimensional convection diffusion equation. They use Hermitian polynomial approximation to achieve sixth order accuracy for the inner grid points and fifth order accuracy for the boundary grid points. The advantage of the CCD scheme is that they can be used to solve many types of PDEs without major modifications. And the Alternating Direction Implicit (ADI) [14] method can be used to reduce the higher dimensional problems to a series of lower dimensional problems. So, their scheme is referred to as implicit high order compact scheme because they do not compute the solution of the variables directly.

In contrary, the explicit fourth order compact schemes [7, 10, 12, 13, 18], compute the solution of the variables directly. Some accelerating iterative methods like multigrid method and preconditioned iterative method have been used to solve the resulting sparse linear systems [22, 24, 25]. But the higher order explicit compact schemes are more complicated to develop in higher dimensions [9, 27], compared with the implicit compact scheme. As far as we know, there is no existing explicit compact scheme on a single scale grid that is higher than the fourth order accuracy.

Since a sixth order explicit scheme may be impossible to develop on a single scale grid, the multiscale grid method has been considered to achieve the sixth order accuracy for the explicit scheme. Sun and Zhang [20] proposed a sixth order explicit finite difference discretization strategy for solving 2D convection diffusion equation. Their scheme is different from the CCD scheme. They used two scale grid method to compute the fourth order solution on the fine and the coarse grids first, then apply the Richardson extrapolation technique and the operator based interpolation to achieve the sixth order solution on the fine grid. They chose the ADI iteration to solve the fine and the coarse grid solutions separately, and the extrapolation and operator based interpolation procedures are used during each ADI iteration. Since the ADI iteration is not scalable with respect to the meshsize, when the mesh becomes finer, the number of iterations increases quickly.

We intend to develop a new explicit sixth order compact schemes to solve the 2D Poisson equation, which can efficiently solve the resulting linear system and is scalable to the meshsize. The idea is similar to Sun-Zhang's sixth order compact scheme, but instead of using the ADI method, we choose multigrid method as our convergence acceleration method combined with Gauss-Seidel relaxation method to directly solve the resulting sparse linear system to get the fourth order accuracy solution on both the fine and the coarse grids. Then we apply the Richardson extrapolation technique combined with our operator based interpolation method to get the sixth order accuracy solution on the fine grid.

This paper is arranged as follows. In Section 2, we present the sixth order compact difference discretization strategy for 2D Poisson equation. In Section 3, we develop our modified multigrid method to solve the fourth order accuracy solution on the fine and the coarse grid. Section 4 contains the numerical experiments to demonstrate high accuracy of the sixth order compact difference scheme, as well as efficiency of our modified multigrid method. Concluding remarks are given in Section 5.

2 Sixth Order Compact Approximations

In this section, we first introduce the fourth order compact difference scheme for 2D Poisson equation. The basic idea of the fourth order compact difference scheme is from Zhang's previous papers [20, 25, 28]. More detailed discussions about the fourth order compact difference schemes are in [7, 17].

In order to discretize the Eq. (1), let us consider a rectangular domain $\Omega = [0, L_x] \times [0, L_y]$. We discretize Ω with uniform meshsizes $\Delta x = L_x/N_x$ and $\Delta y = L_y/N_y$ in the x and y coordinate directions, respectively. Here N_x and N_y are the number of uniform intervals in the x and y coordinate directions. The mesh points are (x_i, y_j) with $x_i = i\Delta x$ and $y_j = j\Delta y$, $0 \le i \le N_x$, $0 \le j \le N_y$. We write the standard second order central difference operators as

$$\delta_x^2 u_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2}, \qquad \delta_y^2 u_{i,j} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2}.$$

Using Taylor series expansions, at the grid point (x_i, y_j), we have

$$\delta_x^2 u_{i,j} = u_{xx} + \frac{\Delta x^2}{12} u_x^4 + \frac{\Delta x^4}{360} u_x^6 + O(\Delta_x^6), \tag{2}$$

and

$$\delta_y^2 u_{i,j} = u_{yy} + \frac{\Delta y^2}{12} u_y^4 + \frac{\Delta y^4}{360} u_y^6 + O(\Delta_y^6). \tag{3}$$

With Eqs. (2) and (3), we then apply the symbolic fourth order compact approximation operator to the second derivatives u_{xx} and u_{yy} in Eq. (1), respectively. The discrete 2D

Poisson equation will be formulated symbolically as [19]

$$(1+\frac{\Delta x^2}{12}\delta_x^2)^{-1}\delta_x^2 u+(1+\frac{\Delta y^2}{12}\delta_y^2)^{-1}\delta_y^2 u=f+\tau_1\Delta x^4+\tau_2\Delta y^4+O(\Delta^6), \tag{4}$$

where τ_1 and τ_2 are used to denote some complex representations which will be cancelled in the Richardson extrapolation procedure, Δ^6 denotes the truncated terms in the order of $O(\Delta x^6+\Delta y^6)$. By applying the symbolic operators, setting τ_1 and τ_2 both equal to zero and dropping the Δ^6, the Eq. (4) can be rewritten as

$$\begin{aligned}&(1+\tfrac{\Delta y^2}{12}\delta_y^2)\delta_x^2 u+(1+\tfrac{\Delta x^2}{12}\delta_x^2)\delta_y^2 u\\&=(1+\tfrac{\Delta x^2}{12}\delta_x^2)(1+\tfrac{\Delta y^2}{12}\delta_y^2)f+O(\Delta^4)\\&=[1+\tfrac{1}{12}(\Delta x^2\delta_x^2+\Delta y^2\delta_y^2)]f+O(\Delta^4).\end{aligned} \tag{5}$$

By absorbing the $O(\Delta x^2.\Delta y^2)$ into $O(\Delta^4)$, the general fourth order compact approximation scheme for Eq. (1) is given by

$$(\delta_x^2+\delta_y^2)u+\frac{1}{12}(\Delta x^2+\Delta y^2)\delta_x^2\delta_y^2 u=f+\frac{1}{12}(\Delta x^2\delta_x^2+\Delta y^2\delta_y^2)f. \tag{6}$$

If we denote the mesh aspect ratio $\gamma=\Delta x/\Delta y$, we can rewrite the Eq. (6) into the following form [25]

$$\begin{aligned}&au_{i,j}+b(u_{i+1,j}+u_{i-1,j})+c(u_{i,j+1}+u_{i,j-1})\\&+d(u_{i+1,j+1}+u_{i+1,j-1}+u_{i-1,j+1}+u_{i-1,j-1})\\&=\tfrac{\Delta x^2}{2}(8f_{i,j}+f_{i+1,j}+f_{i-1,j}+f_{i,j+1}+f_{i,j-1}),\end{aligned} \tag{7}$$

which has a nine point computational stencil. Here the coefficients are

$$a=-10(1+\gamma^2),\qquad b=5-\gamma^2,\qquad c=5\gamma^2-1,\qquad d=(1+\gamma^2)/2.$$

With Eq. (7) we can compute the fourth order solutions $u_{i,j}^{2h}$ and $u_{i,j}^{h}$ on the Ω_{2h} grid and Ω_h grid, respectively. Since the approximate solutions are of fourth order accuracy, we can use the Richardson extrapolation technique [4], to compute a sixth order solution $\tilde{u}_{i,j}^{2h}$ on Ω_{2h} as

$$\tilde{u}_{i,j}^{2h}=\frac{(16u_{2i,2j}^{h}-u_{i,j}^{2h})}{15}. \tag{8}$$

By direct interpolation, $\tilde{u}_{2i,2j}^{h}=\tilde{u}_{i,j}^{2h}$ is a sixth order approximate solution at the $(even, even)$ indexed grid points on Ω_h. For other grid points, we need an operator interpolation scheme to get the sixth order accuracy.

In Sun-Zhang's sixth order method, they apply the Richardson extrapolation combined with the operator based interpolation in each ADI iteration. But our new operator based interpolation procedure is carried out only after we get the converged fine and coarse grid fourth order solutions. So, the CPU cost of the interpolation procedure of our method should be much less than that in Sun-Zhang's method.

Algorithm 1 Operator based interpolation iteration combined with the sixth order Richardson extrapolation technique.

1: Let $u^h_{old} = \tilde{u}^{h,k}$.
2: Update every $(even, even)$ grid point on Ω_h.
From $\tilde{u}^{2h,k}_{i,j} \in \Omega^4_{2h}$ and $\tilde{u}^{h,k}_{2i,2j} \in \Omega^4_h$, we compute $\tilde{u}^{2h,k+1}_{i,j} \in \Omega^6_{2h}$ and $\tilde{u}^{h,k+1}_{2i,2j} \in \Omega^6_h$ by Eq. (8);
3: Update every (odd, odd) grid point on Ω_h.
From Eq. (7), for (odd, odd) point (i, j), the updated solution is

$$\begin{aligned}\tilde{u}^{h,k+1}_{i,j} = \tfrac{1}{a}[&F_{i,j} - b(u_{i+1,j} + u_{i-1,j}) - c(u_{i,j+1} + u_{i,j-1}) \\ &- d(u_{i+1,j+1} + u_{i+1,j-1} + u_{i-1,j+1} + u_{i-1,j-1})]\end{aligned}$$

Here, $F_{i,j}$ represents the right hand side part of Eq. (7).
4: Update every $(odd, even)$ grid point on Ω_h.
From Eq. (7), the idea is similar to that for the (odd, odd) grid point.
5: Update every $(even, odd)$ grid point on Ω_h.
6: Compute the norm $R = ||\tilde{u}^{h,k+1} - u^h_{old}||_2$.

Assuming that N_x and N_y are both even numbers, our iterative operator interpolation algorithm (from step k to $k+1$) based on the Richardson extrapolation technique is outlined in Algorithm 1.

In the above algorithm, Ω^4_h and Ω^4_{2h} denote the fourth order solution space, Ω^6_h and Ω^6_{2h} mean the sixth order solution space. $\tilde{u}^{h,k}$ is the approximate solution after k iterations. The operator based interpolation iteration will continue until the 2-norm R of the correction vector is reduced to below a certain tolerance.

3 Modified Multigrid Method

The convergence rate of the multigrid method is independent of the grid size [2, 3, 21]. It is a very efficient method to solve large sparse linear systems arising from PDEs. Various multigrid implementation strategies with the fourth order compact schemes to solve the 2D or 3D Poisson Equations or other PDEs like convection diffusion equations are discussed in [8, 10, 16]. In this paper, we use a geometric multiscale multigrid method [3, 16], similar to the full multigrid method, to compute the fourth order solution on both the fine and the coarse grids.

We use the notations u_{lh}, f_{lh} and L_{lh} to represent the approximate solution, right hand side vector and the finite difference operator for grid Ω_{lh}, respectively. $I^{lh}_{(l-1)h}$ is the restriction operator from grid $\Omega_{(l-1)h}$ to Ω_{lh} and $I^{(l-1)h}_{lh}$ is the interpolation operator from grid Ω_{lh} to grid $\Omega_{(l-1)h}$.

Below we describe a multigrid V-cycle based algorithm to solve the 2D Poisson equation.

Algorithm 2 Multiscale Multigrid Method

1: Run multigrid V-Cycle algorithm $MG(u_{4h}, f_{4h})$ on coarser grid Ω_{4h} for one or two cycles to get an approximate solution u_{4h}.

2: Use some high order interpolation schemes like bicubic interpolation or operator based interpolation to interpolate u_{4h} to the coarse grid Ω_{2h}, $u_{2h} = I_{4h}^{2h} u_{4h}$.

3: Relax ν_1 times on $L_{2h}u_{2h} = f_{2h}$.

4: Use u_{2h} from the previous step as the initial guess to run multigrid V-Cycle algorithm $MG(u_{2h}, f_{2h})$ on coarse grid Ω_{2h} until it converges. We can get the converged fourth order solution u_{2h}.

5: Use high order interpolation to interpolate u_{2h} to the fine grid Ω_h like $u_h = I_{2h}^{h} u_{2h}$.

6: Relax ν_1 times on $L_h u_h = f_h$.

7: Use u_h from the previous step as the initial guess to run multigrid V-Cycle algorithm $MG(u_h, f_h)$ on fine grid Ω_h until it converges. We can get the converged fourth order solution u_h.

The Algorithm 2 is similar to the full multigrid method, but we do not start from the coarsest grid. Since we use the interpolated coarse grid solution as the initial guess for the fine grid V-Cycle, this algorithm will need fewer number of multigrid cycles than we run the V-Cycle on Ω_h and Ω_{2h} separately to get the fourth order solutions u_h and u_{2h} [3, 16].

4 Numerical Results

In this section, we compare our new sixth order multigrid method with Richardson extrapolation (MG-Six) strategy with Sun-Zhang's sixth order REC method [20] and with the standard fourth order compact difference scheme using multigrid (MG-FOC). The codes are written in Fortran 77 programming language and run on one processor of an IMB HS21 blade cluster at the University of Kentucky. The processor has 2GB of memory and runs at 2.0GHZ.

The multigrid V-Cycle in our code use the Gauss-Seidel relaxation. The initial guess for the V-Cycle on Ω_{4h} is the zero vector. The V-Cycle for the Ω_{2h} and Ω_h grid will stop when the 2-norm of the residual vector is reduced by 10^{-12}. For the interpolation iteration combined with the Richardson extrapolation, the iteration will stop when the 2-norm of the correction vector of the approximate solution is less than 10^{-12}.

The 2D Poisson equation we tested is one of the test cases in Sun-Zhang's paper [20]. Sun-Zhang used a 2D convection diffusion equation, we set the convection coefficient as zero, then the equation is equal to a 2D Poisson equation. The test case is as follows:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\alpha \sin(\frac{\pi}{b} y), \qquad (x, y) \in \Omega = [0, \lambda] \times [0, b], \tag{9}$$

where the boundary conditions are

$$u(0, y) = u(\lambda, y) = u(x, 0) = u(x, b) = 0.$$

In this equation, the parameter α is chosen as

$$\alpha = \frac{F\pi}{Rb}.$$

The analytic solution of Eq. (9) is:

$$u = -\alpha(\frac{b}{\pi})^2 \sin(\frac{\pi y}{b})(e^{\frac{\pi x}{b}} - 1).$$

The other parameters are chosen as

$$\lambda = 10^7 m, b = 2\pi \times 10^6 m, F = 0.3 \times 10^{-7} m^2 s^{-2}, R = 0.6 \times 10^{-3} m s^{-1}.$$

In the following, we define $N_x = N_y = n$. The mesh sizes Δ_x and Δ_y are equal to λ/n and b/n, respectively. Table 1 to Table 3 show the comparison results of different strategies indexed by the number of multigrid cycles or iterations, CPU time and the maximum absolute errors.

Table 1: Comparison of iteration counts for different methods with different discretization schemes.

n	REC-ADI	MG-Six	MG-FOC
16	234	(10,10), **40**	12
32	894	(10,11), **39**	13
64	3438	(11,12), **36**	14
128	>5000	(12,12), **33**	14
256	>5000	(12,12), **30**	14

Table 1 shows the number of iterations for different strategies that we compared. We can find that when the mesh becomes finer, the number of ADI iterations increases quickly. When $n > 64$, the number of ADI iteration is greater than 5000, which means it could not converge within the maximum number of iterations we set. For the MG-Six method, the number of iterations contains three parts. These are the number of V-Cycles for the Ω_{2h}, the number of V-Cycles for the Ω_h, and the number of iterations for the iterative interpolation combined with Richardson extrapolation. These three numbers are listed in the column of MG-Six of Table 1, respectively. We can see that, by using our new sixth order compact scheme, the number of V-Cycles for the Ω_h and the Ω_{2h} have been reduced compared to the traditional multigrid V-Cycle with FOC scheme.

The data in Table 2 and Table 3 indicate that the accuracy of the solution computed by our new sixth order method and the Sun-Zhang's method is comparable. And when the

Table 2: Comparison of CPU times in seconds for different methods with different discretization schemes.

n	REC-ADI	MG-Six	MG-FOC
16	0.017	0.002	0.001
32	0.295	0.005	0.004
64	4.618	0.028	0.016
128	not converged	0.121	0.074
256	not converged	0.790	0.382

Table 3: Comparison of maximum absolute errors of the computed approximate solutions for different methods with different discretization schemes.

n	REC-ADI	MG-Six	MG-FOC
16	1.32e-6	1.32e-6	1.63e-5
32	2.27e-8	2.27e-8	1.02e-6
64	3.68e-10	3.67e-10	6.37e-8
128	not converged	5.10e-12	3.94e-9
256	not converged	2.27e-12	1.87e-10

grid size goes greater than 64, our method can also get the high accuracy solution. The CPU time for our method is much less than the time needed by the ADI iteration, and is better than running the MG-FOC separately twice to get u_h and u_{2h}.

5 Concluding Remarks

We designed a new sixth order compact computation scheme with multigrid method and Richardson extrapolation to solve 2D Poisson equation. This new idea is based on designing a geometric multiscale multigrid method, similar to the full multigrid method, to solve the approximate solution using the fourth order compact scheme in both the fine and the coarse grids. And we also present a new iterative interpolation scheme, which is combined with Richardson extrapolation to achieve the sixth order accuracy on the fine grid.

Numerical results show that the new method can solve the 2D Poisson equation with high accuracy solution compared with other sixth order compact schemes, and also keep the low CPU cost. This idea can also be extended to solve other PDE equations like the 3D Poisson equation, 2D, and 3D convection diffusion equations.

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