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Block Preconditioners for PDEs Arising from the Bidomain Model of the Electrical Activity in the Heart

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Introduction. In this abstract we will study three different (block) preconditioners for two alternative formulations of the Bidomain model. The Bidomain model describes the electrical activity in the heart and is of importance to understand cardiac diseases, which are a major problem in particular in the western world. Here, we will study preconditioners for the following linearized version of the Bidomain model,

$$\begin{aligned} v_t &= \nabla \cdot (M_i \nabla v) + \nabla \cdot (M_i \nabla u), && (1) \\ 0 &= \nabla \cdot (M_i \nabla v) + \nabla \cdot ((M_i + M_e) \nabla u), && (2) \end{aligned}$$

where u and v are the extracellular potential and the transmembrane potential, respectively, and M_i and M_e are conductivity tensors in the heart. An alternative formulation can be obtained by expressing the equations with the intracellular potential, $w = u + v$ instead of v. These equations are,

$$\begin{aligned} w_t &= \nabla \cdot (M_i \nabla w) + u_t, && (3) \\ u_t &= w_t + \nabla \cdot (M_e \nabla u). && (4) \end{aligned}$$

The problems (1)-(2) and (3)-(4) must be equipped with suitable boundary and initial conditions. More details on these equations can be found in, e.g. [2] and [3].

The two alternative formulations have the same solution, but give rise to two different linear systems to be solved. The first one reads,

$$\begin{pmatrix} I + \frac{\Delta t}{2} A_i & \frac{\Delta t}{2} A_i \\ \frac{\Delta t}{2} A_i & \frac{\Delta t}{2} A_{i+e} \end{pmatrix} \begin{pmatrix} v^n \\ u^n \end{pmatrix} = \begin{pmatrix} b^{n-1} \\ c^{n-1} \end{pmatrix}. \qquad (5)$$

And the second linear system is,

$$\begin{pmatrix} I + \frac{\Delta t}{2} A_i & -I \\ -I & I + \frac{\Delta t}{2} A_e \end{pmatrix} \begin{pmatrix} w^n \\ u^n \end{pmatrix} = \begin{pmatrix} b^{n-1} \\ c^{n-1} \end{pmatrix}. \qquad (6)$$

We have used a Crank-Nicholson scheme in time and a finite element discretization in space. Hence, I is the mass matrix, while the various versions of A_α, where $\alpha = i, e, i{+}e$, are "similar" to a discrete Laplacian, because the M tensors are positive definite and bounded. Notice further that preconditioners for I, A_α and $I + \frac{\Delta t}{2} A_\alpha$ are off-the-shelves preconditioners that are order optimal (with respect to both h and Δt). These preconditioner can be made by e.g., multigrid or domain decomposition.

We will investigate the efficiency of the three following (block) preconditioners for both formulations of the Bidomain model,

$$
\begin{aligned}
\mathcal{B}_J^{-1} &= \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad \text{block Jacobi}, \\
\mathcal{B}_{GS}^{-1} &= \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}, \quad \text{block Gauss-Seidel (GS)}, \\
\mathcal{B}_{SGS}^{-1} &= \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \ \text{symmetric block GS}.
\end{aligned}
$$

To check the efficiency of the preconditioner we measure the condition number of the preconditioned matrix, $\mathcal{B}\mathcal{A}$. The condition number is computed via singular value decomposition.

Numerical Experiments. In all the experiments we have partioned the unit square into bilinear elements with size h and used the standard Galerkin finite element method to generate the matrices.

In Table 1 and 2 the three different exact preconditioners are checked for the two alternative formulations. It seems that the preconditioners for the second formulation (6) are the best. However, the condition numbers seem to increase as Δt decreases. In contrast, the condition numbers for the first formulation (5) seem to be bounded independent of both h and Δt. In Table 3 and 4 we investigate this behavior further, but we use a multigrid preconditioner instead of the exact preconditioner, due to the large number of unknowns. The condition number is estimated as a bi-product of the Conjugate-Gradient method as described in [where-ever-this-was]. The Conjugate-Gradient method is stopped when the relative residual is less than 10^{-18}. The condition number for the preconditioned matrix using the second formulation is clearly dependent on Δt. Therefore, it seems that the first formulation should be applied if Δt is small, while the second formulation is better when Δt is large.

This study is an extension of the work in [2] and [3]. The above block algorithms are described in e.g., [1].

Prec	$\mathcal{B}_J$				$\mathcal{B}_{GS}A$				$\mathcal{B}_{SGS}A$			
$h\backslash\Delta t$	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-2}	2^{-3}	2^{-4}	2^{-4}	2^{-2}	2^{-3}	2^{-4}	2^{-5}
2^{-1}	5.0	4.3	3.4	2.7	2.3	2.2	2.0	1.9	1.9	1.7	1.5	1.3
2^{-2}	6.2	6.0	5.7	5.3	2.6	2.6	2.6	2.5	2.1	2.1	2.0	1.9
2^{-3}	6.3	6.3	6.2	6.1	2.6	2.7	2.7	2.7	2.2	2.1	2.1	2.1
2^{-4}	6.3	6.3	6.3	6.3	2.6	2.7	2.7	2.7	2.2	2.2	2.2	2.2

Table 1: Condition numbers for the exact preconditioners, using the first formulation.

Prec	$\mathcal{B}_J$				$\mathcal{B}_{GS}A$				$\mathcal{B}_{SGS}A$			
$h\backslash\Delta t$	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-2}	2^{-3}	2^{-4}	2^{-4}	2^{-2}	2^{-3}	2^{-4}	2^{-5}
2^{-1}	1.3	1.7	2.3	3.7	1.2	1.3	1.6	2.2	1.0	1.1	1.2	1.6
2^{-2}	1.4	1.8	2.5	4.1	1.2	1.3	1.7	2.4	1.0	1.1	1.3	1.7
2^{-3}	1.4	1.8	2.6	4.2	1.2	1.4	1.7	2.4	1.0	1.1	1.3	1.7
2^{-4}	1.4	1.8	2.6	4.2	1.2	1.4	1.7	2.4	1.0	1.1	1.3	1.8

Table 2: Condition numbers for exact preconditioners, using the second formulation.

$h\backslash\Delta t$	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}
2^{-2}	5.7	5.6	5.4	5.0	4.4	3.7
2^{-3}	5.8	5.8	5.7	5.6	5.5	5.2
2^{-4}	5.9	5.9	5.8	5.8	5.7	5.6
2^{-5}	6.1	6.0	5.9	5.9	5.8	5.8
2^{-6}	6.2	6.1	6.1	6.0	5.9	5.9

Table 3: Condition numbers for Jacobi preconditioner with multigrid, first version.

$h\backslash\Delta t$	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}
2^{-2}	1.4	1.8	2.5	4.1	7.2	13.3
2^{-3}	1.4	1.8	2.6	4.2	7.4	13.8
2^{-4}	1.4	1.8	2.6	4.2	7.5	13.9
2^{-5}	1.4	1.8	2.6	4.2	7.5	14.0
2^{-6}	1.4	1.8	2.6	4.2	7.5	14.0

Table 4: Condition numbers for Jacobi preconditioner with multigrid, second version.

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