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Structured Matrices, Multigrid Methods, and the Helmholtz Equation

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We wish to present a different approach to the multigrid solution of the Helmholtz equation with constant coefficients. It is primarily based on certain classes of structured matrices and their strong correspondence to generating functions. Discretization of the Helmholtz equation with certain boundary conditions results in structured linear systems which are associated with generating functions. Depending on the kind of boundary conditions, the discretized Helmholtz equation is a linear system of Toeplitz, tau, circulant, or DCT-III type. By solving these systems with normal equations, we have the advantage that the corresponding generating functions are nonnegative, although they have a whole curve of zeros.

The multigrid methods we develop are especially designed for structured matrix classes, making heavy use of the associated generating functions. Over the last ten years, a specific theory of multigrid methods has been developed for structured matrices whose generating functions have isolated zeros. It is based on the AMG approach and the convergence theory of Ruge and Stüben. In this work, we extend some of these theoretical results to the case of generating functions with whole zero curves, and apply these modified multigrid methods to the solution of the Helmholtz equation. We propose two different strategies how this can be done.

- The first strategy is based on the idea of representing the whole zero curve on all grids. For a multigrid method based on the Galerkin approach, we can prove optimal two-grid convergence, but such a method is computationally too expensive. Therefore we propose a rediscretization technique where the zero curve is approximated on each grid. This results in fast convergence and in coarse grid matrices with the same banded structure as the given matrix. The only disadvantage of this approach is that zero curves become larger on coarser levels, and therefore the number of grids is limited.
- The second strategy consists of splitting the original problem into a fixed number of coarse grid problems. Corresponding to a generating function

with isolated zeros, each of these problems locally represents one part of the zero curve. Each coarse grid problem is solved with a standard multigrid method. We combine this splitting technique with the first strategy to construct a faster and more robust multigrid solver.

We wish to discuss the use of our multigrid strategies not only as a solver, but also as a preconditioner for Krylov subspace methods. Moreover, the methods can also be applied to anisotropic linear systems whose generating functions have a whole zero curve.