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**On the Reuse of Standard Preconditioners for Higher  
Order Time Discretizations of Parabolic PDEs**

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In this abstract we will describe a preconditioner for some higher order time discretizations of parabolic problems. The preconditioner is optimal with respect to the spatial discretization parameters, that typically are the characteristic mesh size parameter  $h$  and the polynomial degree  $p$ . The preconditioner is also order optimal with respect to  $\Delta t$ . The only assumption is that there exists a preconditioner for the low order time discretization schemes such as Crank-Nicholson or implicit Euler. Such preconditioners are standard, c.f. e.g., [1], [2] and [3].

We study the model problem

$$\begin{aligned}\frac{\partial u}{\partial t} &= \Delta u, & \text{in } \Omega, t > 0 \\ u &= 0, & \text{on } \partial\Omega, t > 0 \\ u &= u_0, & \text{in } \Omega, t = 0.\end{aligned}$$

This equation is discretized in space and time to give the following linear system to be solved for each time level

$$Q_{kj}(\Delta t A) u^n = P_{kj}(\Delta t A) u^{n-1},$$

where  $\Delta t$  is the time stepping parameter, the two polynomials  $Q_{kj}$  and  $P_{kj}$  are the  $(k, j)$ -Padé approximation to the exponential function and  $A$  is a discrete Laplacian. The polynomials are given by (c.f. [3]):

$$\begin{aligned}P_{kj}(\Delta t A) &= \sum_{i=0}^k \binom{k}{i} \frac{(k+j-i)!}{(k+j)!} (\Delta t A)^i \\ Q_{kj}(\Delta t A) &= P_{jk}(-\Delta t A).\end{aligned}$$

As mentioned earlier, we will reuse preconditioners for

$$R_c = (I - c\Delta t A).$$

$j \backslash k$	$j$	$j-1$	$j-2$
2	1.07	1.10	1.17
4	1.26	1.31	1.40
6	1.49	1.56	1.66
8	1.76	1.85	1.97
10	2.08	2.20	2.34

Table 1: Upper bound on the condition number for various values of  $j$  and  $k$ .

In fact, the proposed (exact) preconditioner is on the form,

$$R_c^{-j} = (I - c\Delta t A)^{-j},$$

where  $c$  is determined such that the highest order term of  $R_c^{-j}(\Delta t A)$  equals the highest order term of  $Q_{kj}(\Delta t A)$ . This can be done by choosing

$$R_{kj}(\Delta t A) = \left( I - \sqrt[j]{\frac{j!}{(j+k)!}} \Delta t A \right)^j.$$

Hence,  $R_{kj}$  is a standard preconditioner for a low order time discretization of a parabolic PDE, used  $j$  times.

In Table 1 we show an upper bound on the condition number of the preconditioned system, using an exact preconditioner for  $R_{kj}$ . More details about the preconditioner and the estimation of the condition number can be found in the full paper that is also submitted. Further we prove the following lemma.

**Lemma 1** *The polynomials  $R_{kj}(\Delta t A)$  and  $Q_{kj}(\Delta t A)$  are spectrally equivalent independent of  $A$  and  $\Delta t$ , and the condition number of the preconditioned system is bounded by*

$$\kappa \left( (R_{kj}(\Delta t A))^{-1} Q_{kj}(\Delta t A) \right) < 1.8 \cdot 1.09^j.$$

# Bibliography

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- [3] V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems*, Springer-Verlag, 2nd edition, 1997.