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matmodlab
a Material Model Laboratory

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Abstract

The Material Model Laboratory (matmodlab) is a suite of tools whose purpose is to aid in the rapid development and testing of material models. matmodlab is made up of several components, the most notable being the Material Model Driver mmd. mmd can be thought to drive a single material point of a finite element simulation through very specific user designed paths. This permits exercising material models in ways not possible in finite element calculations, desgining verification and validation tests of the material response, among others. matmodlab is a small suite of tools at the developers disposal to aid in the design and implementation of material models in larger finite element host codes. It is also a useful tool to analysists for understanding and parameterizing a material's response to deformation.

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Chapter 1

Introduction to matmodlab

The Material Model Laboratory (matmodlab) is a suite of tools whose purpose is to aid in the rapid development and testing of material models. matmodlab is made up of several components, the most notable being the Material Model Driver mmd. mmd can be thought to drive a single material point of a finite element simulation through very specific user designed paths. This permits exercising material models in ways not possible in finite element calculations, desgining verification and validation tests of the material response, among others. matmodlab is a small suite of tools at the developers disposal to aid in the design and implementation of material models in larger finite element host codes. It is also a useful tool to analysists for understanding and parameterizing a material's response to deformation.

The core of matmodlab code base is written in Python and leverages Python's object oriented programming (OOP) design. OOP techniques are used throughout matmodlab to setup and manage simulation data. Computationally heavy portions of the code, and the material models themselves are written in Fortran for its speed and ubiquity in scientific computing. Calling Fortran procedures from Python is made possible by the f2py module, standard in Numpy, that compiles and creates Python shared object libraries from Fortran sources.

Output files from matmodlab simulations are in the EXODUSII [?] database format, devloped at Sandia National Labs for storing finite element simulation data. Since matmodlab is designed to be used by material model developers, it is expected that the typical user will want access to *all* available output from a material model, thus all simulation data is written to the output database. EXODUSII database files can be visualized via the mmv utility, in addition to other visualization packages such as PARAVIEW [?].

matmodlab is free software released under the MIT License.

Why a Single Element Driver?

Due to their complexity, it is often over kill to use a finite element code for constitutive model development. In addition, features such as artificial viscosity can mask the actual material response from constitutive model development. Single element drivers allow the constituive model developer to concentrate on model development and not the finite element response. Other advantages of `matmodlab` (or, more generally, of any stand-alone constitutive model driver) are

- `matmodlab` is a very small, special purpose, code. Thus, maintaining and adding new features to `matmodlab` is very easy.
- Simulations are not affected by irrelevant artifacts such as artificial viscosity or uncertainty in the handling of boundary conditions.
- It is straightforward to produce supplemental output for deep analysis of the results that would otherwise constitute an unnecessary overhead in a finite element code.
- Specific material benchmarks may be developed and automatically run quickly any time the model is changed.
- Specific features of a material model may be exercised easily by the model developer by prescribing strains, strain rates, stresses, stress rates, and deformation gradients as functions of time.

Why Python?

Python is an interpreted, high level object oriented language. It allows for writing programs rapidly and, because it is an interpreted language, does not require a compiling step. While this might make programs written in python slower than those written in a compiled language, modern packages and computers make the speed up difference between python and a compiled language for single element problems almost insignificant.

For numeric computations, the NumPy and SciPy modules allow programs written in Python to leverage a large set of numerical routines provided by LAPACK, BLASPACK, EIGPACK, etc. Python's APIs also allow for calling subroutines written in C or Fortran (in addition to a number of other languages), a prerequisite for model development as most legacy material models are written in Fortran. In fact, most modern material models are still written in Fortran to this day.

Python's object oriented nature allows for rapid installation of new material models.

Historical Background

When I was a graduate student at the University of Utah I had the good fortune to have as my advisor Dr. Rebecca Brannon. Prof. Brannon instilled in me the necessity to develop material models in small special purpose drivers, free from the complexities of larger finite element codes. To this end, I began developing material models in Prof. Brannon's `MED` driver (available upon request from Prof. Brannon). The `MED` driver was a special purpose driver for driving material models through predefined strain paths. After completing graduate school I began employment as a member of the Technical Staff at Sandia National Labs. Among the many projects I worked on was the development of material models for geologic applications. There, I found need to drive the material models through prescribed stress paths to match experimental records. This capability was not present in the `MED` and I sought a different solution. The solution came from the `MMD` driver, created years earlier at Sandia, by Tom Pucick. The `MMD` driver had the capability to drive material models through prescribed stress and strain paths, but also lacked many of the IO features of the `MED`. And so, for some time I used both the `MED` and `MMD` drivers in applications that suited their respective strengths. After some time using both drivers, I decided to combine the best features of each in to my own driver. Both the `MED` and `MMD` drivers were written in Fortran and I decided to write the new driver in Python so that I could leverage the large number of builtin libraries. The Numpy and Scipy Python libraries would be used for handling most number crunching. The new driver came to be known as `payette`. `payette` added many unique capabilities and became a capable piece of software used by other staff members at Sandia. But, `payette` suffered from the fact that it was my first foray in to programming with Python. After some time, the bloat and bad programming practices with `payette` caused me to spend a few weekends re-writing it in to what is now known as `matmodlab`.

Obtaining `matmodlab`

`matmodlab` is an open source project licensed under the MIT license. A copy of may be obtained from `https://github.com/tjfulle/matmodlab`

About This Guide

`matmodlab` is developed as a tool for developers and analysts who care to understand the responses of material models to specific deformation paths. The target audience is assumed to have a basic knowledge of continuum mechanics and familiarity with other finite element codes. Accordingly, concepts of continuum mechanics and finite element methods are not described in detail and programing techniques are also not described.

Chapter 2

matmodlab Quick Start Guide

This guide provides an outline for building and running `matmodlab`.

Build matmodlab See Chapter 3.

- Download `matmodlab` and setup environment
- $ `cd $MMLROOT`
- $ `python setup.py build_all`

Prepare Input Inputs are xml specification files. See Chapters 6 - 9.

- Set up the desired simulation path.
- Add material model.
- Add desired extraction requests.

Run

- $ `mmd` [options] *runid* [,*runid_1*, . . ., *runid_n*]
  runid is prefix of ".xml" file.
- Complete list of options given by
  $ `mmd -h`

Postprocess

- $ `mmv` *runid* [,*runid_1*, . . ., *runid_n*]
- PARAVIEW also reads exodus files.

Chapter 3

Building **matmodlab**

matmodlab's code base is largely written in Python and requires no additional compiling. However, several fortran procedures, the ExodusII third party library, and material models written in fortran must be built.

System and Software Requirements

matmodlab has been built and tested extensively on several versions of linux and the Apple Mac OSX operating systems. It is unknown whether or not matmodlab will run on Windows.

matmodlab requires the following software installed for your platform:

- Python 2.7
- NumPy 1.6
- SciPy 0.10
- A fortran compiler, preferably the same used to build numpy and scipy

The required software may be obtained in several ways, though all development has been made using Enthought Canopy (http://http://www.enthought.com).

Installation

Ensure that all matmodlab prerequisites are installed and working properly before proceeding.

Set Environment and Path

MMLROOT Optional, name of installation directory

PATH `$MMLROOT/toolset:$PATH`

MMLMTLS ":" separated list of paths to directories containing user defined material models. See Section 10.

Set Up

Set up and build TPLs, fortran utilities, and material models.

```
$ cd $MMLROOT
$ python setup.py build_all
```

In addition to building components, `setup.py` generates the following executable scripts

buildmtls Build material models

mml Run `matmodlab` simulations

exdump Read a `matmodlab` output and dumps requested variables to ascii columnar files

mmv 2D plots of `matmodlab` output

runtests Run the regression tests

Each script is a wrapper to another `matmodlab` Python file. In the wrapper, relevant environment variables are set (e.g., `$PYTHONPATH`) and the correct Python executable (the one used to set up) is used to interpret the `matmodlab` source file. The full set of options for each script is obtained by

```
$ scriptname -h
```

where `scriptname` is the name of the script.

The TPLs will build the first time `matmodlab` is setup. Thereafter after, they are only built if requested. Execute

```
$ python setup.py build_tpl --help
```

for options to rebuild the TPLs.

Build Materials

Material models are built during the setup stage, but can be built separately with the `buildmtls` utility.

Test the Installation

To test `matmodlab` after installation, execute

```
$ runtests $MMLROOT/tests [-j N]
```

which will run `matmodlab` regression tests.

Troubleshooting

If you experience problems when building/installing/testing `matmodlab`, you can ask help from `matmodlab` developers. Please include the following information in your message:

- Platform information OS, its distribution name and version information etc.

  ```
  $ python -c "import os,sys;print os.name,sys.platform"
  $ uname -a
  ```

- Information about C,C++,Fortran compilers/linkers as reported by the compilers when requesting their version information, e.g., the output of

  ```
  $ gcc -v
  $ gfortran --version
  ```

- Python version

  ```
  $ python -c "import sys;print sys.version"
  ```

- NumPy version

  ```
  $ python -c "import numpy;print numpy.__version__"
  ```

- SciPy version

  ```
  $ python -c "import scipy;print scipy.__version__"
  ```

- Feel free to add any other relevant information.

Chapter 4

`matmodlab` Solution Method

`matmodlab` exercises a material model directly by "driving" it through user specified paths using a designated driver. Currently installed drivers are the `solid` and `eos` drivers. For each driver type, `matmodlab` computes an increment in deformation for a given step and requires that the material model update the stress in the material to the end of that step, given the current state and an increment in deformation. Because of the similarity of the material model interface in `matmodlab` with many commercial finite element codes, transitioning material models developed and tested in `matmodlab` to full finite element codes should be an easy process. In this chapter, the role and importance of the material model in a finite element procedure is reviewed. The solution method adopted by each driver in `matmodlab` is then described and compared with that of finite elements.

The Role of the Material Model in Continuum Mechanics

Conservation Laws

Conservation of mass, momentum, and energy are the central tenets of the analysis of the response of a continuous media to deformation and/or load. Each conservation law can be summarized by the following statement

$$\boxed{\begin{array}{c}\text{Time rate of}\\ \text{change of}\\ \text{quantity}\end{array}} = \boxed{\begin{array}{c}\text{Rate of}\\ \text{production}\\ \text{in the}\\ \text{interior}\end{array}} + \boxed{\begin{array}{c}\text{Flux}\\ \text{through the}\\ \text{boundary}\end{array}}$$

Mathematically, the conservation laws for a point in the continuum are

- Conservation of mass

$$\dot{\rho} + \rho \boldsymbol{\nabla}\cdot\dot{\boldsymbol{u}} = 0$$

- Conservtion of momentum per unit volume

$$\rho \frac{d}{dt}\dot{\boldsymbol{u}} = \underset{\text{internal forces}}{\boxed{\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}}} + \underset{\text{body forces}}{\boxed{\boldsymbol{b}}}$$

- Conservation of energy per unit volume

$$\rho\frac{d}{dt}U = \underbrace{\boxed{\rho s}}_{\text{heat source}} + \underbrace{\boxed{\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}}}_{\text{strain energy}} + \underbrace{\boxed{\nabla\cdot\boldsymbol{q}}}_{\text{heat flux}}$$

where $\boldsymbol{u}$ is the displacement, ρ the mass density, $\boldsymbol{\sigma}$ the stress, $\dot{\boldsymbol{\varepsilon}}$ the rate of strain, $\boldsymbol{b}$ the body force per unit volume, $\boldsymbol{q}$ the heat flux, s the heat source, and U is the internal energy per unit mass.

In solid mechanics, mass is conserved trivially, and many problems are adiabatic or isotrhermal, so that only the momentum balance is explicitly solved

$$\rho\frac{d}{dt}\dot{\boldsymbol{u}} = \underbrace{\boxed{\nabla\cdot\boldsymbol{\sigma}}}_{\text{internal forces}} + \underbrace{\boxed{\boldsymbol{b}}}_{\text{body forces}} \tag{4.1}$$

The balance of linear momentum is the continuum mechanics generalization of Newton's second law $F = ma$.

The first term on the RHS of (4.1) represents the internal forces, which arise in the medium to resist imposed deformation. This resistance is a fundamental response of matter and is given by the divergence of the stress field.

The balance of linear momentum represents an initial boundary value problem for applications of interest in solid dynamics:

$$\begin{aligned} \rho\frac{d}{dt}\dot{\boldsymbol{u}} = \nabla\cdot\boldsymbol{\sigma} + \boldsymbol{b} \qquad & \text{in } \Omega \\ \boldsymbol{u} = \boldsymbol{u}_0 \qquad & \text{on } \Gamma_0 \\ \boldsymbol{\sigma}\cdot\boldsymbol{n} = \boldsymbol{t}^{(n)} \qquad & \text{on } \Gamma_t \\ \dot{\boldsymbol{u}}(\boldsymbol{x},0) = \dot{\boldsymbol{u}}_0(\boldsymbol{x}) \qquad & \text{on } \boldsymbol{x}\in\Omega \end{aligned} \tag{4.2}$$

The Finite Element Method

The form of the momentum equation in (4.2) is termed the **strong** form. The strong form of the initial BVP problem can also be expressed in the weak form by introducing a test function $\boldsymbol{w}$ and integrating over space

$$\begin{aligned} \int_\Omega \boldsymbol{w}\cdot\left(\nabla\cdot\boldsymbol{\sigma} + \boldsymbol{b} - \rho\frac{d}{dt}\dot{\boldsymbol{u}}\right) d\Omega \qquad & \forall \boldsymbol{w} \\ \boldsymbol{u} = \boldsymbol{u}_0 \qquad & \text{on } \Gamma_0 \\ \boldsymbol{\sigma}\cdot\boldsymbol{n} = \boldsymbol{t}^{(n)} \qquad & \text{on } \Gamma_t \\ \dot{\boldsymbol{u}}(\boldsymbol{x},0) = \dot{\boldsymbol{u}}_0(\boldsymbol{x}) \qquad & \text{on } \boldsymbol{x}\in\Omega \end{aligned} \tag{4.3}$$

Integrating (4.3) by parts allows the traction boundary conditions to be incorporated in to the governing equations

$$\begin{aligned} \int_{\Omega} \rho \boldsymbol{w} \cdot \boldsymbol{a} + \boldsymbol{\sigma} : \nabla \boldsymbol{w} \, d\Omega &= \int_{\Omega} \boldsymbol{w} \cdot \boldsymbol{b} \, d\Omega + \int_{\Gamma} \boldsymbol{w} \cdot \boldsymbol{t}^{(n)} \, d\Gamma_t \quad \forall \boldsymbol{w} \\ \boldsymbol{u} &= \boldsymbol{u}_0 \qquad \text{on } \Gamma_0 \\ \dot{\boldsymbol{u}}(\boldsymbol{x}, 0) &= \dot{\boldsymbol{u}}_0(\boldsymbol{x}) \qquad \text{on } \boldsymbol{x} \in \Omega \end{aligned} \tag{4.4}$$

This form of the IBVP is called the **weak** form. The weak form poses the IBVP as a integro-differential equation and eliminates singularities that may arise in the strong form. Traction boundary conditions are incorporated in the governing equations. The weak form forms the basis for finite element methods.

In the finite element method, forms of $\boldsymbol{w}$ are assumed in subdomains (elements) in Ω and displacements are sought such that the force imbalance R is minimized:

$$R = \int_{\Omega} \boldsymbol{w} \cdot \boldsymbol{b} \, d\Omega + \int_{\Gamma} \boldsymbol{w} \cdot \boldsymbol{t}^{(n)} \, d\Gamma_t - \int_{\Omega} \rho \boldsymbol{w} \cdot \boldsymbol{a} + \boldsymbol{\sigma} : \nabla \boldsymbol{w} \, d\Omega \tag{4.5}$$

The equations of motion as described in (4.5) are not closed, but require relationships relating $\boldsymbol{\sigma}$ to $\boldsymbol{u}$

$$\boxed{\text{Constitutive model} \longrightarrow \text{relationship between } \boldsymbol{\sigma} \text{ and } \boldsymbol{u}}$$

In the typical finite element procedure, the host finite element code passes to the constitutive routine the stress and material state at the beginning of a finite step (in time) and kinematic quantities at the end of the step. The constitutive routine is responsible for updating the stress to the end of the step. At the completion of the step, the host code then uses the updated stress to compute kinematic quantities at the end of the next step. This process is continued until the simulation is completed. The host finite element handles the allocation and management of all memory, including memory required for material variables.

Solution Procedure

In addition to providing a platform for material model developers to formulate and test constitutive routines, `matmodlab` aims to provide users of material models an independent platform to exercise, parameterize, and compare material responses against single element finite element simulations. To this end, the solution procedure in `matmodlab` is similar to that of the finite element method, in that the host code (`matmodlab`) provides to the constitutive routine a measure of deformation at the end of a finite step and expects the updated stress in return. However, rather than solve the momentum equation at the beginning of each step and advancing kinematic quantities to the step's end, `matmodlab` retrieves updated kinematic quantities from user defined tables and/or functions.

The path through which a material is exercised is defined by piecewise continuous "legs" in which components of the "control type" c_i are specified at discrete points in time, shown in Figure 4.1. The c_i are used to obtain a sequence of piecewise constant strain rates that are used to advance the kinematic state. Supported control types are strain, strain rate, stress, stress rate, deformation gradient, displacement, and velocity. "Mixed-modes" of strain and stress (and their rates) are supported. Components of displacement and velocity control are applied only to the "+" faces of a unit cube centered at the coordinate origin.

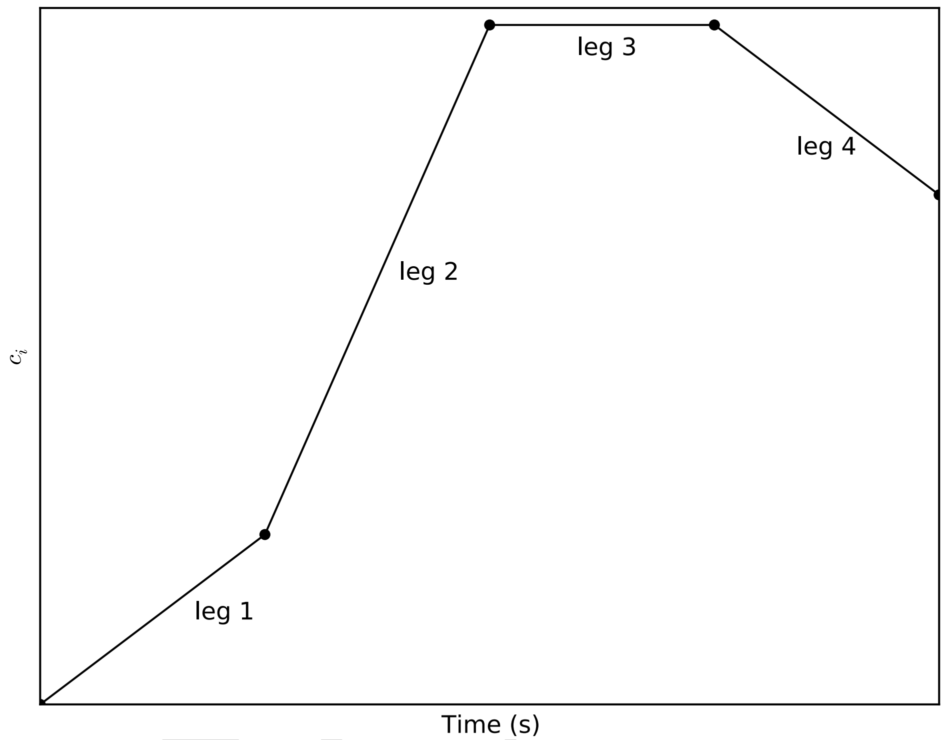

Figure 4.1. User defined path for the i^{th} component of "c". c may represent strain, strain rate, stress, stress rate, deformation gradient, displacement, or velocity.

The components of strain are defined by

$$\boldsymbol{\varepsilon} = \frac{1}{\kappa}\left(\boldsymbol{U}^{\kappa} - \boldsymbol{\delta}\right) \tag{4.6}$$

where $\boldsymbol{U}$ is the right Cauchy stretch tensor, defined by the polar decomposition of the deformation gradient $\boldsymbol{F} = \boldsymbol{R}\cdot\boldsymbol{U}$, and κ is a user specified "Seth-Hill" parameter that controls the strain definition. Choosing $\kappa = 2$ gives the Lagrange strain, which might be useful when testing models cast in a reference coordinate system. The choice $\kappa = 1$, which gives the engineering strain, is convenient when driving a problem over the same strain path as was used in an experiment. The choice $\kappa = 0$ corresponds to the logarithmic (Hencky) strain. Common values of κ and the associated names for each (there is some ambiguity in the names) are listed in Table 4.

κ	Name(s)
-2	Green
-1	True, Cauchy
0	Logarithmic, Hencky, True
1	Engineering, Swainger
2	Lagrange, Almansi

The volumetric strain ε_v is defined

$$\varepsilon_v = \begin{cases} \frac{1}{\kappa}\left(J^\kappa - 1\right) & \text{if } \kappa \neq 0 \\ \ln J & \text{if } \kappa = 0 \end{cases} \tag{4.7}$$

where the Jacobian J is the determinant of the deformation gradient.

Each leg in the control table, from time $t = 0$ to $t = t_f$ is subdivided into a user-specified number of steps and the material model evaluated at each step. When volumetric strain, deformation gradient, displacement, or velocity are specified for a leg, `matmodlab` internally determines the corresponding strain components. If a component of stress is specified, `matmodlab` determines the strain increment that minimizes the distance between the prescribed stress component and model response.

Strain Rate from Prescribed Stress

The approach to determining unknown components of the strain rate from the prescribed stress is an iterative scheme employing a multidimensional Newton's method that satisfies

$$\boldsymbol{\sigma}\left(\dot{\boldsymbol{\varepsilon}}\left[\mathrm{v}\right]\right) = \boldsymbol{\sigma}^p$$

where, v is a vector subscript array containing the components for which stresses (or stress rates) are prescribed, and $\boldsymbol{\sigma}^p$ are the components of prescribed stress.

Each iteration begins by determining the submatrix of the material stiffness

$$\mathbb{C}_\mathrm{v} = \mathbb{C}\left[\mathrm{v},\mathrm{v}\right]$$

where $\mathbb{C}$ is the full stiffness matrix $\mathbb{C} = d\boldsymbol{\sigma}/d\boldsymbol{\varepsilon}$. The value of $\dot{\boldsymbol{\varepsilon}}\left[\mathrm{v}\right]$ is then updated according to

$$\dot{\boldsymbol{\varepsilon}}\left[\mathrm{v}\right] = \dot{\boldsymbol{\varepsilon}}\left[\mathrm{v}\right] - \mathbb{C}_\mathrm{v} \colon \boldsymbol{\sigma}^*(\dot{\boldsymbol{\varepsilon}}\left[\mathrm{v}\right])/dt$$

where

$$\boldsymbol{\sigma}^*(\dot{\boldsymbol{\varepsilon}}\left[\mathrm{v}\right]) = \boldsymbol{\sigma}(\dot{\boldsymbol{\varepsilon}}\left[\mathrm{v}\right]) - \boldsymbol{\sigma}^p$$

The Newton procedure will converge for valid stress states. However, it is possible to prescribe invalid stress state, e.g. a stress state beyond the material's elastic limit. In these cases, the Newton procedure may not converge to within the acceptable tolerance and a Nelder-Mead simplex method is used as a back up procedure. A warning is logged in these cases.

Solid Driver

As the name implies, the `solid` driver is designed to exercise the type of material models encountered in solid mechanics. The solution method is similar to that of many finite element codes, so that material models developed and tested in `matmodlab` can be easily transitioned to them.

Electrical

Electric field can be prescribed for testing piezoelectric models.

Chapter 5

Running Simulations in matmodlab

The command line utility mmd is the main program with which users interact with matmodlab. To run a simulation with matmodlab, be sure that $MMLROOT is on your path and execute

```
$ mmd runid[.xml]
```

where runid is the basename of the input file. Input file formatting is covered in Chapters 6 - 9.

The following files will be produced by mmd in the current working directory

```
$ ls runid.*
runid.exo       runid.log       runid.xml
```

runid.exo is the ExodusII output database, runid.log the log file, and runid.xml the input file.

For a complete list of options, execute

```
$ mmd -h
```

Chapter 6

User Input: Overview

User input is via xml control files. In general, tags use CamelCase and attributes lower case. Attributes are described in this document as

```
attr="type[default]{choices}"
```

where `default` is the default value (if any) and {`choices`} are valid choices (if any). Any attribute not having a default value is required. Types are `str`, `int`, `real`, `list` and `boolean`. Lists are given as space separated lists (e.g., ”1 2 3”), boolean arguments should be true or false.

In the following, elements shown in red are required input.

<MMLSpec>

All input files must have as their root element `<MMLSpec>`.

```
<MMLSpec>
```

Recognized subelements of `<MMLSpec>` are

- `<Physics>`
- `<Permutation>`
- `<Optimization>`

The following elements are read from any scope in the input file

- `<Include>`
- `<Function>`

The `<Physics>`, `<Permutation>`, `<Optimization>` and input blocks are described separately in their own chapters.

Preprocessing

Preprocessing allows specifying variables in the input inside of comment tags for use in other parts of the input. Syntax mirrors that of `aprepro`. Preprocessor also evaluates (nearly) any Python expression.

The `random()` expression generates a random number. The `random_seed` variable sets the random state seed. Note, expressions are evaluated in order, therefore, if setting the `random_seed` it should occur early.

The following input stub demonstrates specifying the <Material> parameter `K` and `G`, and <Path> parameter `estar` as variables

```
<MMLSpec>
  <!-- {random_seed = 7}
       {G = 54e9 * random()}
       {K = 23e9}
       {estar = -.05}
  -->
  <Physics>
    <Material model="elastic">
      <K> {K} </K>
      <G> {G} </G>
    </Material>
    <Path type="prdef" estar="{estar}">
      ...
    </Path>
  </Physics>
</MMLSpec>
```

<Include>

Path to file to be included as if its contents were inplace in the input file

```
<Include href="str"/>
```

The following stub input demonstrates how to include a file in place

```
<Include href="/path/to/some/file.ext"/>
```

<Function>

Define functions to be used elsewhere in input. `id=0` and `id=1` are reserved for the constant 0 and 1 functions, respectively. `href` is the path to a file containing the function definition (useful when the function is a large piecewise linear table). `cols` specifies the columns in which data is located in a piecewise linear table.

```
<Function id="int"
          type="str{analytic_expression, piecewise_linear}"
          var="str[x]" href="str[]" cols="list[1 2]">
```

The following input stub demonstrates how to define an analytic expression and piecewise linear table as functions

```
<Function id="2" type="analytic_expression" var="t">
  sin(t)
</Function>
<Function id="3" type="piecewise_linear">
  1 2
  2 3
  3 5
</Function>

<!-- Read a piecewise linear table from an external file using
     columns 1 and 3
-->
<Function id="4" type="piecewise_linear" href="./file.dat"
          cols="1 3"/>
```

```
$ cat file.dat
# Column1 Column2 Column3
1 1 4
2 3 7
.
.
.
100 4.2 1.43
```

Chapter 7

User Input: <Physics>

Define the physics of the simulation.

```
<Physics driver="str[solid]{solid, eos}"
         termination_time="real[]" runid="str[filename]">
```

Recognized subelements of <Physics> are

- <Path>
- <Material>
- <Extract>

<Physics> Attributes

driver

driver specifies the driver type to use. Defaults to solid.

termination_time

If specified, termination_time defines the termination time for the simulation. If not specified, termination time is taken as final time in <Path>.

runid

If specified, runid redefines the runid from its default basename(file).

<Path>

Define deformation paths through with the material will be exercised.

```
<Path type="str{prdef, surface}"
      format="str[default]{default, table, fcnspec}"
      cols="list[1, ..., n]" cfmt="str"
      tfmt="str[time]{time,dt}"
      nfac="int[1]" kappa="real[0]" rstar="real[1]"
      tstar="real[1]" estar="real[1]" sstar="real[1]"
      amplitude="real[1]" ratfac="real[1]" href="str">
```

<Path> Attributes

type

The type of path specified. Valid types are `prdef` and `surface`.

The `prdef` type defines a prescribed deformation. The jth leg of `<Path>` is sent to the driver in form `[tf, n, cfmt, Cij]`, where `tf`, `n`, `cfmt`, and `Cij` are the termination time, number of steps, control format, and control values. Methods of inputing legs depends on the attributes of `<Path>` and will be shown in examples to follow.

The `surface` input is similar to the `prdef` specification, but leg termination time is not specified. Control parameters also differ, as shown in Table 7.2.

format

The format by which the legs of the deformation path are specified. Valid formats are `default`, `table`, and `fcnspec`. In the following subsections, the different formats are described.

format="default" The `default` format offers the most control. In this format, the termination time, number of steps, control format, and components of deformation are specified for each leg as in the following stub input

```
<Path type="default">
  <!-- tterm nsteps cfmt c1 c2 c3 ... -->
  0  0 222222 0 0 0 0 0 0
  1 10 222222 1 0 0 0 0 0
</Path>
```

See Section 7 for a full description of the control format cfmt and its relationship with the c1, c2, c3, ...

format="table" The table format allows reading in deformation paths from a columnar table of data. Control format is uniform for all legs. Specify control format as cfmt attribute of <Path>. Specify which columns to read data with the cols attribute. The first column is assumed to be the time specifier. See Section 7 for a description of the cols attribute. The tfmt attribute specifies if the time column represents the actual time (tfmt="time") or time step (tmft="dt"). The number of steps for each leg can be set by nfac. The href attribute specifies an external file to read the table.

The following input stubs demonstrate reading a table from the input file and from an external file.

```
<!-- Read entries from table. -->
<Path type="prdef" format="table" cols="1:4" cfmt="222"
      tfmt="time">
  0 0 0 0
  1 1 0 0
    ...
  n 2 0 0
</Path>
```

```
<!-- Read table from external file -->
<Path type="prdef" format="table" cols="1 3:8" cfmt="222222"
      tfmt="time" href="exmpls.tbl"/>
```

format="fcnspec" The fcnspec format allows defining a deformation path by a function. A deformation path defined by fcnspec must have only 1 leg defining the termination time and the function specifier defining the values of the components of deformation. The function specifier is of the form

```
function_id[:scale]
```

where `function_id` is the ID of the function as specified in its `<Function>` element. The optional scale is multiplied by the function.

The following input stub demonstrates uniaxial strain deformation, using a user defined function to specify the 11 component of strain through time.

```
<Path type="prdef" format="fcnspec" cfmt="222" nfac="200">
  <!-- termination time, fcn spec -->
  {2 * pi} 2:1.e-1 0 0
</Path>
```

`cfmt`

The control format `cfmt` is concatenated integer list specifying in its i^{th} component the i^{th} component of deformation, i.e., `cfmt[i]` instructs the driver as to the type of deformation represented by `Cij[i]`. Types of deformation represented by `cfmt` are shown in Table 7.1 for the `solid` driver and Table 7.2 for the `eos` driver.

For example, the following `cfmt` instructs the driver that the components of `Cij` represent [stress, strain, stress rate, strain rate, strain, strain], respectively:

```
cfmt="423122"
```

Mixed modes are allowed only for components of strain rate, strain, stress rate, and stress. Electric field components can be included with any deformation type. If only one component of stress rate, stress, strain rate, or strain is specified, the component `Cij` is taken to be either the pressure of volumetric strain.

The components `Cij` take the following order

Vectors: [X, Y, Z]

Symmetric tensors: [XX, YY, ZZ, XY, YZ, XZ]

Tensors: [XX, XY, XZ, YX, YY, YZ ZX, ZY, ZZ]

If `len(Cij)` $\neq$ `6` (or 9 for deformation gradient), the missing components are assumed to be zero strain.

`cfmt`	Deformation type
1	Strain rate
2	Strain
3	Stress rate
4	Stress
5	Deformation gradient
6	Electric field

Table 7.1. Supported deformation types and `cfmt` code for `solid prdef` paths

`cfmt`	Variable type
1	Density
2	Temperature

Table 7.2. Supported surface variable types and `cfmt` code for `eos surface` paths

`tfmt`

The `tfmt=["time"]`{`"time","dt"`} flag specifies the time format. If `tfmt="time"` (default) the first value of each leg is interpreted as the termination time for the leg. For `tfmt="dt"` the first value of each leg is interpreted as a time increment.

`cols`

The columns to be read from a table or the `fcnspec` leg are specified by the `cols` attribute. `cols` are specified as a space separated list of columns. Numbering is 1 based. Ranges can be specified using Python slice syntax.

The following input stubs demonstrate two equivalent ways to to read columns 1, 3, 4, 5, 8, 9, and 13 from a table.

```
cols="1 3 4 5 8 9 13"
```

```
cols="1 3:5 8:9 13"
```

kappa

kappa is the Seth-Hill strain definition parameter κ described in section 4.

nfac

nfac is a multiplier on the number of steps for each leg.

amplitude

amplitude is a factor multiplied to all components of deformation.

The "star" Multipliers

[rtes(ef)]star are multipliers on the components of density, time (temperature for type="surface"), strain, stress, and electric field, respectively. The [rtes(ef)]star are first multiplied by amplitude.

ratfac

ratfac is a divisor to the termination time of each leg, thereby effectively increasing the rate of deformation.

More Examples

The following examples will help clarify the <Path> input syntax

```
<!-- uniaxial strain, all six components of strain
     prescribed -->
<Path type="prdef" kappa="0" tstar="1" estar="-.5"
      amplitude="1" ratfac="1">
  <!-- termination time, number of steps, cfmt, Cij -->
  0   0 222222 0 0 0 0 0 0
  1 100 222222 1 0 0 0 0 0
  2 100 222222 2 0 0 0 0 0
  3 100 222222 1 0 0 0 0 0
  4 100 222222 0 0 0 0 0 0
</Path>
```

```
<!-- uniaxial strain, stress controlled -->
<Path type="prdef" nfac="100">
  0 0 444 0 0 0
  1 1 444 -7490645504 -3739707392 -3739707392
  2 1 444 -14981291008 -7479414784 -7479414784
  3 1 444 -7490645504 -3739707392 -3739707392
  4 1 444 0 0 0
</Path>
```

```
<!-- uniaxial stress, mixed mode -->
<Path type="prdef" nfac="100">
  0 0 222 0 0 0
  1 1 244 {epsmax} 0 0
  4 1 244 0 0 0
</Path>
```

```
<!-- volumetric strain -->
<Path type="prdef" kappa="0" estar="-.5">
  0   0 2 0
  1 100 2 1
  2 100 2 2
  3 100 2 1
  4 100 2 0
</Path>
```

Example of `type="surface"`

The following examples demonstrate the `type="surface"`

```
<Path type="surface">
  <!-- nsteps, control, Cij -->
  <!-- control goes as 1 -> density
                      2 -> temperature -->
    0 12 1 100
  100 12 5 300
</Path>
```

```
<Path type="surface" format="table" cfmt="12" nfac="100">
  <!-- Cij -->
  1 100
  5 300
</Path>
```

<Material>

Define the material model.

```
<Material model="str" constant_jacobian="boolean[false]">
```

Subelements of <Material> are

- <Matlabel>
- <ParameterArray>
- <InitialState>
- <Key>*

*<Key> is a valid material parameter name.

<Material> Attributes

model

The name of the material model.

constant_jacobian

Use the initial (constant) jacobian during inverse stress driven problems.

<Matlabel>

Insert model parameters from a database file.

```
<Matlabel href="str[F_MTL_PARAM_DB]" material="str"/>
```

<Matlabel> Attributes

href The path to the database file. Defaults to `$MMLROOT/materials/material_properties.db` if no file is given.

material Name of material as given in the database file.

The following input stub demonstrates the use of `<Matlabel>`

```
<Material model="elastic">
  <Matlabel href="./materials.xml" material="aluminum"/>
</Material>
```

<ParameterArray>

Specify the parameter array for the material as whitespace separated list of floats. The list of values must be the same length as the parameter array for the material or an error will occur.

```
<ParameterArray>
  VAL1 VAL2 ... VALN
</ParameterArray>
```

<InitialState>

Specify the initial state of the material as a whitespace separated list of floats. Six stress values must be followed by material variables (if any). The length of the material variables must be the same as the length of the `xtra` variable array for the material or an error will occur. Note, implementation is material model specific.

```
<InitialState>
  STRESS_XX STRESS_YY ... STRESS_XZ XTRA1 XTRA2 ... XTRAN
</InitialState>
```

Specify Individual Parameters

Specify individual parameters as xml text nodes

```
<Key> float </Key>
```

<Key> is replaced by specific material model parameters. The following stub inputs demonstrate the <Material> input

```
<Material model="elastic">
  <G>  54E+09 </G>
  <K> 124E+09 </K>
</Material>
```

<Extract>

Extract variables and paths from ExodusII output and (optionally) write to different formats.

```
<Extract format="str[ascii]{ascii, mathematica, ndarray}"
         step="int[1]" ffmt="str[.18f]">
```

Recognized subelements of <Extract> are

- <Path>*
- <Variables>

* eos driver only

<Extract> Attributes

format

The format to write the output. ascii format writes out columnar data as an ascii text file, mathematica writes an ascii text file that can be read by Mathematica, and ndarray writes the data to a file in the numpy .npy binary format.

`step`

Extract every `step`th timestep.

`ffmt`

The string format used write out variables.

<Variables>

Variables to extract from the ExodusII output database. Variables are specified children of the `<Variables>` element. All components of vector and tensor variables will be extracted if only the basename is specified. Time is always extracted as the first entry of the output file. Extracted variables are in `runid.out` or `runid.math` depending if the format is ascii or mathematica.

```
<Variables>
  VAR_1, ..., VAR_N
</Variables>
```

The following example demonstrates how to extract all components of stress and strain

```
<Extract format="ascii">
  <variables>
    STRESS STRAIN
  </variables>
</Extract>
```

Extract only the XX, YY, and ZZ components of stress

```
<Extract format="ascii">
  <variables>
    STRESS_XX STRESS_YY STRESS_ZZ
  </variables>
</Extract>
```

Extract all variables

```
<Extract format="ascii">
  <variables>
    ALL
  </variables>
</Extract>
```

<Path>

Extract a specified path from the equation of state surface through the specified density range starting at the initial temperature.

```
<Path type="str{isotherm, hugoniot}" increments="int[100]"
      density_range="list" initial_temperature="real">
```

The following input stub demonstrates extracting Hugoniot and Isotherm paths

```
<Extract>
  <Path type="isotherm" increments="200"
        density_range="1 3" initial_temperature="225"/>
  <Path type="hugoniot" increments="100"
        density_range="1 3" initial_temperature="100"/>
</Extract>
```

Chapter 8

User Input: <Permutation>

Permutate model input parameters, running jobs with different realization of parameters. Ideal for investigating model sensitivities.

```
<Permutation method="str[zip]{zip,combine,shotgun}"
                      seed="real[date]"
                      correlation="list[none]{plot,table,none}">
```

Recognized subelements of <Permutation> are

- <Permutate>
- <ResponseFunction>

Each <Permutation> job creates a directory runid.eval

```
$ ls runid.eval
eval_0/     eval_2/     mml-evaldb.xml
eval_1/     ...         runid.log
```

The eval_i directory holds the output of the i^{th} job, including params.in with the values of each permutated parameter for that job. mml-tabular.xml contains a summary of each job run. mmv recognizes mml-tabular.xml files.

<Permutation> Attributes

method

The method attribute describes which method to use to determine parameter combinations to run.

The zip method runs one job for each set of parameters (and, thus, the number of realizations for each parameter must be identical), the combine method runs every combination of parameters.

correlation

Create correlation table and plots of relating permutated parameters and value of response function. `correlation` is only meaningful if a `<ResponseFunction>` is specified.

seed

The seed for the random number generator. `date` is todays date in seconds.

<Permutate>

Specify the paramaters to permutate.

```
<Permutate var="str"
           values="func{range,list,weibull,uniform,
                        normal,percentage}"
```

<Permutate> Attributes

var

`var` is the name of the variabe and should occur elsewhere in the input file in preprocessing braces.

values

`values` are the specific values. The `range,list,weibull,uniform,normal,percentage` are all specified as functions with the following form

```
values="func(start,stop,N)"
```

The following input stub demonstrates how to permutate the `K` and `G` parameters

```
<Permutation method="zip" seed="12">
  <Permutate var="K" values="weibull(125.e9, 14, 3)"/>
  <Permutate var="G" values="percentage(45.e9, 10, 3)"/>
</Permutation>
```

In the `<Material>` element, the `K` and `G` parameters are specified as

```
<Material model="elastic">
  <K> {K} </K>
  <G> {G} </G>
</Material>
```

<ResponseFunction>

The `<ResponseFunction>` returns the response from permutation or optimization jobs.

```
<ResponseFunction href="str" function="mmlfcn"
                  descriptor="str[]"/>
```

One of `href` or `function` must be specified.

<ResponseFunction> Attributes

descriptor

`descriptor` is the name given to the response function in the output.

href

`href` is the path to an executable file script containing the response function. The script is called from the command line as

```
% ./scriptname runid.exo
```

An example of a response function specifying `href` is

```
<ResponseFunction href="./scriptname" descriptor="PRES"/>
```

function

`function` is the name of a builtin `matmodlab` response function. Built in response functions are

- `mml.max` maximum value of a simulation variable output
- `mml.min` minimum value of a simulation variable output
- `mml.mean` mean value of a simulation variable output
- `mml.ave` average value of a simulation variable output
- `mml.absmax` maximum absolute value of a simulation variable output
- `mml.absmin` minimum absolute value of a simulation variable output

Built in response functions operate only on variabes in the simulation output file.

An example of a response function specifying `function` is

```
<ResponseFunction function="mml.max(PRESSURE)"
                  descriptor="PRES"/>
```

Chapter 9

User Input: <Optimization>

Optimize specified parameters against user specified objective function.

```
<Optimization method="str[simplex]{simplex, powell, cobyla}"
              maxiter="int[25]" tolerance="real[1e-6]">
```

Recognized subelements of `<Optimization>` are

- <Optimize>
- <ResponseFunction>
- <AuxiliaryFile>

Like `<Permutation>` jobs, each `<Optimization>` job creates a directory `runid.eval`

```
$ ls runid.eval
eval_0/     eval_2/     mml-evaldb.xml     runid.log
eval_1/     ...         params.opt
```

The `eval_i` directory holds the output of the i^{th} job, including `params.in` with the values of each parameter for that job. `mml-tabular.xml` contains a summary of each job run. `mmv` recognizes `mml-tabular.xml` files. `params.opt` has the final, optimized, parameters.

<Optimization> Attributes

method

`method` specifies the optimization method. All optimization routines utilize the scipy.optimize module.

maxiter

`maxiter` is the maximum number of iterations.

tolerance

`tolerance` is the optimization tolerance.

<Optimize>

Specify the variable to be optimized.

```
<Optimize var="str" initial_value="real" bounds="list[]"/>
```

<Optimize> Attributes

var

`var` is the name of the variabe and should occur elsewhere in the input file in preprocessing braces.

initial_value

`initial_value` is the initial value of `var`

bounds

`bounds` specifies lower and upper bounds on `var`. Only the `cobyla` method accepts bounds.

<AuxiliaryFile>

Path to any auxiliary file needed by the optimization objective function.

```
<AuxiliaryFile href="str"/>
```

<ResponseFunction>

Same as for `<Permutation>`, except that auxiliary files are also passed to the function. The value returned from the response function is interpreted as the error to be minimized.

If the `<ResponseFunction>` is given by `href`, it is called as

```
% ./scriptname runid.exo [AuxFile1[AuxFile2[...]]]
```

Example

Optimize the `K` and `G` parameters

```
<Optimization method="simplex" maxiter="25" tolerance="1e-4">
  <ResponseFunction href="opt-sig-v-time"
                    descriptor="SIG_V_TIME"/>
  <AuxiliaryFile href="opt-baseline.dat"/>
  <Optimize var="opt_k" initial_value="129.e9"/>
  <Optimize var="opt_g" initial_value="54.e9"/>
</Optimization>
```

In the `<Material>` element, the `K` and `G` parameters are specified as

```
<Material model="elastic">
  <K> {opt_k} </K>
  <G> {opt_g} </G>
</Material>
```

Chapter 10

User Material Interface

`matmodlab` can be made to find, build, and execute user materials outside of `$MMLROOT`. User materials can be written in Python or Fortran and and `matmodlab` interacts with them through the application programming interface (API). In general, the following pattern is followed for exercising a material model with `matmodlab`:

1. create a material model interface (MMI)
2. build and link the material model to `matmodlab`
3. exercise the model

Material Model Interface

`matmodlab` interacts with materials through a material interface file. The material interface file defines the material class which must be a subclass of `$MMLROOT/materials._material.Material`. In this section, methods of the `Material` class are described.

Material Class Instantiation

The base class `Material` in `$MMLROOT/materials._material` creates new `matmodlab` materials and provides the interface with which `matmodlab` interacts. Each class must define its name (`Material.name`) and an ordered list of material parameter names (`Material.param_names`) as they should appear in the input file. Parameter aliases are supported by specifying a parameter name as a ":" separated list of allowed names. The class should not define an `__init__` method and if it does, should call the `__init__` of the base class.

Material: Interface

```
mtl = Material()
```

The following is an example of a `Material` declaration for the `Elastic` material model. Aliases for `K` are noted.

```
from materials._material import Material
class Elastic(Material)
    name = "elastic"
    param_names = ["K:BMOD:B0", "G"]
```

Setup the Material

Each material must provide the method `setup` that sets up the material model by checking and adjusting the material parameter array, requesting allocation of storage of material variables, and computing and storing the `bulk_modulus` and `shear_modulus` of the material. `setup` is called by the base class method `setup_new_material` that parses and stores the user given parameters in the `Material.params` array.

Material.setup: Interface

```
mtl.setup()
```

The following is an example of a `setup` method

```
def setup(self):
    if elastic is None:
        raise Error1("elastic model not imported")
    elastic.elastic_check(self.params, log_error, log_message)
    K, G = self.params
    self.bulk_modulus = K
    self.shear_modulus = G
```

Adjust the Initial State

The method `adjust_initial_state` adjusts the initial state after the material is setup. Method provided by base class should be adequate for most materials. A material should only overide the base method if absolutely necessary.

Material.adjust_initial_state: Interface

```
mtl.adjust_initial_state(xtra)
```

ndarray xtra

Material variables

Update the Material State

The material state is updated to the end of the step via the `update_state` method. Each material model must provide its own `update_state` method.

Material.update_state: Interface

```
stress, xtra = mtl.update_state(dt, d, sig, xtra,
                                f, ef, t, rho, tmpr, *args)
```

`real dt`

timestep size

`ndarray d`

rate of deformation

`ndarray sig`

stress at beginning of step

`ndarray xtra`

extra state variables at beginning of step

`ndarray f`

deformation gradient at end of step

`ndarray ef`

electric field

`real t`

time

`real rho`

density at end of step

`real tmpr`

temperature at end of step

`tuple args`

extra args (not used)

`dict kwargs`

extra keyword args (not used)

`ndarray stress`

stress at end of step

`ndarray xtra`

extra state variables at end of step

The following code segment is used by the driver to update the material state

```
args = []
sig, xtra = mtl.update_state(dt, d, sig, xtra,
                             f, ef, t, rho, tmpr, *args)
```


Example

The following example demonstrates the implementation of a simple elastic model.

```
import numpy as np
from materials._material import Material
from core.io import Error1, log_error, log_message
try:
    import lib.elastic as elastic
except ImportError:
    elastic = None

class Elastic(Material):
    name = "elastic"
    param_names = ["K", "G"]
    def __init__(self):
        super(Elastic, self).__init__()

    def setup(self):
        if elastic is None:
            raise Error1("elastic model not imported")
        elastic.elastic_check(self.params, log_error, log_message)
        K, G, = self.params
        self.bulk_modulus = K
        self.shear_modulus = G

    def update_state(self, dt, d, stress, xtra, *args):
        elastic.elastic_update_state(dt, self.params, d, stress,
                                     log_error, log_message)
        return stress, xtra

    def jacobian(self, dt, d, stress, xtra, v):
        return self.constant_jacobian(v)
```


Building and Linking Materials

matmodlab comes with and builds several builtin material models that are specified in $MMLROOT/materials/library/mmats.py. User materials are found by looking in directories in the $MMLMTLS environment variable for a single file umat.py. umat.conf communicates to matmodlab information needed to build the material's extension module.

Building User Materials

User materials are built [1] by `matmodlab` using `numpy`'s distutils. A material communicates to `matmodlab` information required by distutils back to `matmodlab` through the `umat.conf` function.

umat.conf: Interface

```
name, info = conf(*args)
```

`tuple args`

not currently used

`str name`

The name of the material model

`dict info`

Information dictionary

The `info` Dictionary

The `info` dict contains the following keys

`list source_files`

The list of source files to be built. If the material is a pure python module, specify as `None`

`str includ_dir`

Directory to look for includes during compile [default: dirname(interface_file)

`str interface_file`

Path to the material's interface file

`str class`

The name of the material model class

Below is an example of `umat.conf`

[1] Only pure python and fortran models have been implemented. Implementing models in other languages is possible, but would have to be sorted out.

```
D = os.path.dirname(os.path.realpath(__file__))

def conf(*args):
    name = "dsf"
    source_files = [os.path.join(D, f) for f in ("material.F", "material.pyf")]
    assert all(os.path.isfile(f) for f in source_files)
    info = {"source_files": source_files, "includ_dir": D,
            "interface_file": os.path.join(D, "material.py"),
            "class": "MaterialModel"}
    return name, info
```

Chapter 11

Regression Testing

Introduction

Regression testing is crucial to the model development process. Regression tests in matmodlab are special purpose problems that serve several purposes, most notably, component tests for the core capabilities of matmodlab, and verification and validation (V&V) of material models. In the first role, problems are fast running and exercise specific features of matmodlab in a unit-test type fashion. In the second, material models are exercised through specific paths with known, or expected outcomes.

The runtests Tool

runtests gathers, runs, and analyzes tests. To run the tests with matmodlab, be sure that $MMLROOT is on your path and execute

```
$ runtests
```

runtests will create a results directory TestResults.<platform>, where <platform> is is the machine platform (as determined by Python's sys.platform) on which the tests are being run. The following files and directories will be produced by runtests in the TestResults.<platform>, directory

```
$ ls TestsResults.darwin
completed_tests.db mmd/ summary.html
```

completed_tests.db is a database file containing information on all completed tests and summary.html is an html summary file, viewable in any web browser.

runtests Options

The full list of options to `runtests` is

```
$ runtests -h
usage: test.py [-h] [-k K] [-K K] [-j J] [-F] [--plot-failed] [--plot-all]
               [--list] [--testdirs TESTDIRS] [-D D] [-w] [--rebaseline]
               [--run-failed]
               [tests [tests ...]]

positional arguments:
  tests                 Specific tests to run [default: None]

optional arguments:
  -h, --help            show this help message and exit
  -k K                  Keywords of tests to include [default: []]
  -K K                  Keywords of tests to exclude [default: []]
  -j J                  Number of simultaneous tests [default: 1]
  -F                    Force tests previously run to rerun [default: False]
  --plot-failed         Create overlay plots for failed tests [default: False]
  --plot-all            Create overlay plots for completed tests [default:
                        False]
  --list                List matching tests and exit [default: False]
  --testdirs TESTDIRS   Additional directories to find tests [default: []]
  -D D                  Directory to run tests [default: /Users/tjfulle/Develop
                        er/Applications/matmodlab/tests/TestResults.darwin]
  -w                    Wipe test directory before running tests [default:
                        False]
  --rebaseline          Rebaseline test in current directory [default: False]
  --run-failed          Run tests that previously had failed [default: False]
```

Regression Test Specification File

`runtests` searches for test specification files in the `$MMLROOT/tests` directory and directories in the `MMLMTLS` environment variable. The tests specification files xml files with a `.rxml` extension. All test files must have as their root element `<rtest>`.

```
<rtest name="str" repeat="int[1]">
```

Recognized subelements of `<rtest>` are

- <keywords>
- <execute name=lstr">
- <link_files>

Attributes of rtest

name

`name` is the name of the test. The name should include a directory name and test name. For example `<rtest name="dir/test_name">`. The test will then run and in `TestResults.<platform>/dir/test_name/`. If no directory name is given, it will default to `orphaned`.

repeat

`repeat` tells `runtests` to repeat the test `repeat` times. Useful for tests that have random inputs and should be run for various realizations of those inputs.

An example <rtest> is

```
<rtest name="mml/super" repeat="3">
...
</rtest>
```

Name Substitution

`runtests` performs string substitution for strings in braces when processing the test specification file. The following string/substitute are currently recognized

NAME The test base name

Additionally, environment variables are expanded.

keywords

Regression test keywords

```
<keywords> kw_1 kw_2 ... kw_n </keywords>
```

One of `[long,medium,fast]` must be specified as a keyword for every test. Fast tests are defined as those that run in less than 5 seconds, medium in up to 25 seconds, and long any test that takes longer than 25 seconds to execute. When `runtests` is executed with the `-k kw` option, only tests of `kw` will be run.

An example keywords block is

```
<keywords> fast elastic super </keywords>
```

link_files

Link files to the test directory

```
<link_files> file_1 file_2 ... file_n </link_files>
```

execute

Execute the executable given by `name`

```
<execute name="exe"> args </execute>
```

`exe` is the name of an executable and must be on your `$PATH`. `args` are the arguments to `exe` as if called from the command line. When a test is run, `runtests` executes the executable in a shell as specified by each <execute> tag in the order encountered.

Examples

In the following example, a simulation is run with `mmd` and `exdif` is used to determine if the output differs from a known base result `example.base_exo`

```
<rtest name="example">
  <keywords> fast elastic </keywords>

  <link_files>
    base.exdiff {NAME}.rxml {NAME}.xml {NAME}.base_exo
  </link_files>

  <execute name="mmd"> {NAME}.xml </execute>

  <execute name="exdiff">
    -f base.exdiff {NAME}.exo {NAME}.base_exo
  </execute>
</rtest>
```

Chapter 12

Hyperelasticity

The Second Piola-Kirchhoff Stress Tensor

$$\boldsymbol{T} = 2\frac{\partial W}{\partial \mathbf{C}} \tag{12.1}$$

For isotropic hyperelasticity $W = W(\bar{I}_1, \bar{I}_2, J)$ and

$$\boldsymbol{T} = 2\left(\frac{\partial W}{\partial \bar{I}_1}\frac{\partial \bar{I}_1}{\partial \mathbf{C}} + \frac{\partial W}{\partial \bar{I}_2}\frac{\partial \bar{I}_2}{\partial \mathbf{C}} + \frac{\partial W}{\partial J}\frac{\partial J}{\partial \mathbf{C}}\right) = \boldsymbol{A}\cdot\boldsymbol{B} \tag{12.2}$$

where

$$\boldsymbol{A} = \begin{bmatrix} \dfrac{\partial W}{\partial \bar{I}_1} & \dfrac{\partial W}{\partial \bar{I}_2} & \dfrac{\partial W}{\partial J} \end{bmatrix} \tag{12.3}$$

and

$$\boldsymbol{B} = \begin{bmatrix} \dfrac{\partial \bar{I}_1}{\partial \mathbf{C}} & \dfrac{\partial \bar{I}_2}{\partial \mathbf{C}} & \dfrac{\partial J}{\partial \mathbf{C}} \end{bmatrix} \tag{12.4}$$

The *B* Term

$$\boldsymbol{B} = \begin{bmatrix} J^{-2/3}\left(\boldsymbol{\delta} - \dfrac{1}{3}I_1\mathbf{C}^{-1}\right) & J^{-4/3}\left(I_1\boldsymbol{\delta} - \mathbf{C} - \dfrac{2}{3}I_2\mathbf{C}^{-1}\right) & \dfrac{1}{2}J\mathbf{C}^{-1} \end{bmatrix} \tag{12.5}$$

Elastic Stiffness

Elastic stiffness in the material frame is given by

$$\mathbb{L} = 4\frac{\partial^2 W}{\partial \mathbf{C}\partial \mathbf{C}} = 4\frac{\partial}{\partial \mathbf{C}}(\boldsymbol{A}\cdot\boldsymbol{B}) \tag{12.6}$$

$$= 4\left(\frac{\partial \boldsymbol{A}}{\partial \mathbf{C}}\cdot\boldsymbol{B} + \boldsymbol{A}\cdot\frac{\partial \boldsymbol{B}}{\partial \mathbf{C}}\right) \tag{12.7}$$

where

$$\frac{\partial \boldsymbol{A}}{\partial \mathbf{C}} = \boldsymbol{H}^{\mathrm{T}}\cdot\boldsymbol{B}, \quad \boldsymbol{H} = \begin{bmatrix} \frac{\partial^2 W}{\partial \bar{I}_1{}^2} & \frac{\partial^2 W}{\partial \bar{I}_1 \partial \bar{I}_2} & \frac{\partial^2 W}{\partial \bar{I}_1 \partial J} \\ \frac{\partial^2 W}{\partial \bar{I}_2 \partial \bar{I}_1} & \frac{\partial^2 W}{\partial \bar{I}_2{}^2} & \frac{\partial^2 W}{\partial \bar{I}_2 \partial J} \\ \frac{\partial^2 W}{\partial J \partial \bar{I}_1} & \frac{\partial^2 W}{\partial J \partial \bar{I}_2} & \frac{\partial^2 W}{\partial J^2} \end{bmatrix} \tag{12.8}$$

and

$$\frac{\partial \boldsymbol{B}}{\partial \mathbf{C}} = \left\{ \begin{matrix} \frac{1}{3}J^{-2/3}\left[\boldsymbol{\delta}\mathbf{C}^{-1} - \mathbf{C}^{-1}\boldsymbol{\delta} - I_1\left(\mathbf{C}^{-1}\odot\mathbf{C}^{-1} - \frac{1}{3}\mathbf{C}^{-1}\mathbf{C}^{-1}\right)\right] \\ \frac{2}{3}J^{-4/3}\left[\frac{3}{2}\left(\mathbb{I}_1 - \mathbb{I}_2\right) + \left(\mathbf{C}^{-1}\mathbf{C} + \mathbf{C}\mathbf{C}^{-1}\right) - I_1\left(\mathbf{C}^{-1}\boldsymbol{\delta} + \boldsymbol{\delta}\mathbf{C}^{-1}\right) - I_2\left(\mathbf{C}^{-1}\odot\mathbf{C}^{-1} - \frac{2}{3}\mathbf{C}^{-1}\mathbf{C}^{-1}\right)\right] \\ \frac{1}{4}J\left(\mathbf{C}^{-1}\mathbf{C}^{-1} - 2\mathbf{C}^{-1}\odot\mathbf{C}^{-1}\right) \end{matrix} \right\} \tag{12.9}$$

See Appendix A for the straightforward, yet lengthy, derivation of the above quantitites.

Requirements of Objectivity

The constitutive equations for the hyperelastic material evaluate the stress directly in the reference configuration. The components of the stress are identified as the components of the Second Piola-Kirchhoff stress $\boldsymbol{T}$. The push forward of the Second Piola-Kirchhoff stress gives the Cauchy stress $\boldsymbol{\sigma}$ in the spatial configuration, as required by most finite element packages. The push forward of the corresponding material stiffness $\mathbb{L}$ does not, however, correspond to the rate of Cauchy stress but the Truesdell rate of the Cauchy stress. Furthermore, the rate of Cauchy stress is not objective, requiring finite element packages to use other so-called objective stress rates in the incremental solution of the momentum equation. In the following sections, it is demonstrated that the push forward of the material stiffness does not correspond to the rate of Cauchy stress and equations relating the stiffness corresponding to the Jaumann rate of the Kirchhoff stress to the push forward of the material stiffness are developed.

The rate of change of Cauchy stress

$$\dot{\boldsymbol{\sigma}} = \frac{d}{dt}\left(\frac{1}{J}\boldsymbol{F}\cdot\boldsymbol{T}\cdot\boldsymbol{F}^{\mathrm{T}}\right) \tag{12.10}$$

$$= \frac{1}{J}\left[-\frac{\dot{J}}{J}\boldsymbol{F}\cdot\boldsymbol{T}\cdot\boldsymbol{F}^{\mathrm{T}} + \dot{\boldsymbol{F}}\cdot\boldsymbol{T}\cdot\boldsymbol{F}^{\mathrm{T}} + \boldsymbol{F}\cdot\frac{d\boldsymbol{T}}{d\boldsymbol{E}}:\dot{\boldsymbol{E}}\cdot\boldsymbol{F}^{\mathrm{T}} + \boldsymbol{F}\cdot\boldsymbol{T}\cdot\dot{\boldsymbol{F}}^{\mathrm{T}}\right] \tag{12.11}$$

With the following identities

$$\mathrm{tr}\,\boldsymbol{d} = \frac{\dot{J}}{J} \quad \dot{\boldsymbol{F}} = \boldsymbol{l}\cdot\boldsymbol{F} \quad \dot{\boldsymbol{E}} = \boldsymbol{F}^{\mathrm{T}}\cdot\boldsymbol{d}\cdot\boldsymbol{F} \quad \dot{\boldsymbol{F}}^{\mathrm{T}} = \boldsymbol{F}^{\mathrm{T}}\cdot\boldsymbol{l}^{\mathrm{T}} \tag{12.12}$$

$$\dot{\boldsymbol{\sigma}} = \frac{1}{J}\left[-\operatorname{tr}\boldsymbol{d}\boldsymbol{\sigma} + \boldsymbol{l}\cdot\boldsymbol{F}\cdot\boldsymbol{T}\cdot\boldsymbol{F}^{\mathrm{T}} + \boldsymbol{F}\cdot\mathbb{L}\colon(\boldsymbol{F}^{\mathrm{T}}\cdot\boldsymbol{d}\cdot\boldsymbol{F})\cdot\boldsymbol{F}^{\mathrm{T}} + \boldsymbol{F}\cdot\boldsymbol{T}\cdot\boldsymbol{F}^{\mathrm{T}}\cdot\boldsymbol{l}^{\mathrm{T}}\right] \tag{12.13}$$

$$= -\operatorname{tr}\boldsymbol{d}\boldsymbol{\sigma} + \boldsymbol{l}\cdot\boldsymbol{\sigma} + \frac{1}{J}\left(\boldsymbol{F}\cdot\boldsymbol{F}\cdot\mathbb{L}\cdot\boldsymbol{F}^{\mathrm{T}}\cdot\boldsymbol{F}^{\mathrm{T}}\right)\boldsymbol{:}\boldsymbol{d} + \boldsymbol{\sigma}\cdot\boldsymbol{l}^{\mathrm{T}} \tag{12.14}$$

$$= -\operatorname{tr}\boldsymbol{d}\boldsymbol{\sigma} + \boldsymbol{l}\cdot\boldsymbol{\sigma} + \mathbb{C}\boldsymbol{:}\boldsymbol{d} + \boldsymbol{\sigma}\cdot\boldsymbol{l}^{\mathrm{T}} \tag{12.15}$$

rearranging

$$\dot{\boldsymbol{\sigma}} - \boldsymbol{\sigma}\cdot\boldsymbol{l}^{\mathrm{T}} - \boldsymbol{l}\cdot\boldsymbol{\sigma} + \operatorname{tr}\boldsymbol{d}\boldsymbol{\sigma} = \mathbb{C}\boldsymbol{:}\boldsymbol{d} \tag{12.16}$$

$$\mathring{\boldsymbol{\sigma}} = \mathbb{C}\boldsymbol{:}\boldsymbol{d} \tag{12.17}$$

where

$$\mathring{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \boldsymbol{\sigma}\cdot\boldsymbol{l}^{\mathrm{T}} - \boldsymbol{l}\cdot\boldsymbol{\sigma} + \operatorname{tr}\boldsymbol{d}\boldsymbol{\sigma} \tag{12.18}$$

is the Truesdell rate of the Cauchy stress. The Truesdell rate of the Cauchy stress is related to the Lie derivative of the Kirchhoff stress by

$$\mathring{\boldsymbol{\sigma}} = \frac{1}{J}\boldsymbol{F}\cdot\dot{\boldsymbol{T}}\boldsymbol{F}^{\mathrm{T}} = \frac{1}{J}\boldsymbol{F}\cdot\left[\frac{d}{dt}\left(J\boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma}\boldsymbol{F}^{-\mathrm{T}}\right)\right]\boldsymbol{F}^{\mathrm{T}} = \frac{1}{J}\boldsymbol{F}\cdot\left[\frac{d}{dt}\left(\boldsymbol{F}^{-1}\cdot\boldsymbol{\tau}\boldsymbol{F}^{-\mathrm{T}}\right)\right]\boldsymbol{F}^{\mathrm{T}} = \frac{1}{J}\mathscr{L}_{\varphi}[\boldsymbol{\tau}] \tag{12.19}$$

Thus, the push forward of the material time derivative of $\boldsymbol{T}$ is the Truesdell rate of the Cauchy stress. Correspondingly, the Truesdell rate of the Kirchhoff stress is related to the push forward of the material stiffness $\mathbb{L}$ by

$$\mathscr{L}_{\varphi}[\boldsymbol{\tau}] = J\mathbb{C}\boldsymbol{:}\boldsymbol{d} \tag{12.20}$$

In many finite element packages, the Jaumann rate of the Kirchhoff stress, and not the Truesdell rate, is required. Thus, the elastic stiffness must correspond to the Jaummann rate.. The Jaumann rate of the Kirchhoff stress is given by

$$\overset{\triangledown}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} + \boldsymbol{\tau}\cdot\boldsymbol{w} - \boldsymbol{w}\cdot\boldsymbol{\tau} \tag{12.21}$$

$$= \dot{J}\boldsymbol{\sigma} + J\dot{\boldsymbol{\sigma}} + J\boldsymbol{\sigma}\cdot\boldsymbol{w} - \boldsymbol{w}\cdot J\boldsymbol{\sigma} \tag{12.22}$$

$$= J\left(\operatorname{tr}\boldsymbol{d}\boldsymbol{\sigma} + \dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\cdot\boldsymbol{w} - \boldsymbol{w}\cdot J\boldsymbol{\sigma}\right) \tag{12.23}$$

$$= J\mathbb{D}\boldsymbol{:}\boldsymbol{d} \tag{12.24}$$

where $\mathbb{D}$ is the stiffness corresponding to the Jaumann rate. Subtracting $\mathring{\boldsymbol{\tau}}$ from $\overset{\triangledown}{\boldsymbol{\tau}}$, the Jaumann stiffness can be cast in terms of $\mathbb{C}$

$$(\mathbb{D} - \mathbb{C})\boldsymbol{:}\boldsymbol{d} = \left(\operatorname{tr}\boldsymbol{d}\boldsymbol{\sigma} + \dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\cdot\boldsymbol{w} - \boldsymbol{w}\cdot\boldsymbol{\sigma}\right) - \left(\dot{\boldsymbol{\sigma}} - \boldsymbol{\sigma}\cdot\boldsymbol{l}^{\mathrm{T}} - \boldsymbol{l}\cdot\boldsymbol{\sigma} + \operatorname{tr}\boldsymbol{d}\boldsymbol{\sigma}\right) \tag{12.25}$$

$$= \boldsymbol{\sigma}\cdot\boldsymbol{w} + \boldsymbol{\sigma}\cdot\boldsymbol{l}^{\mathrm{T}} + \boldsymbol{l}\cdot\boldsymbol{\sigma} - \boldsymbol{w}\cdot\boldsymbol{\sigma} \tag{12.26}$$

$$= \boldsymbol{\sigma}\cdot\boldsymbol{d} + \boldsymbol{d}\cdot\boldsymbol{\sigma} \tag{12.27}$$

$$\tag{12.28}$$

Using indicial notation, and the fact that $\boldsymbol{d}$ and $\boldsymbol{\sigma}$ are symmetric,

$$\begin{aligned} \left(D_{ijkl} - C_{ijkl}\right) d_{kl} &= \sigma_{im} d_{mj} + d_{im} \sigma_{mj} &\quad (12.29)\\ &= \left(\sigma_{ik}\delta_{jl} + \delta_{il}\sigma_{jk}\right) d_{kl} &\quad (12.30) \end{aligned}$$

from which

$$\left(D_{ijkl} - C_{ijkl} - \sigma_{ik}\delta_{jl} - \delta_{il}\sigma_{jk}\right) d_{kl} = 0_{ij} \quad (12.31)$$

Since (12.31) must hold for all d_{kl}, the stiffness corresponding to the Jaumman rate of the Kirchhoff stress is related to the stiffness corresponding to the Truesdell rate of the Kirchhoff stress as

$$D_{ijkl} = C_{ijkl} + \sigma_{ik}\delta_{jl} + \delta_{il}\sigma_{jk} \quad (12.32)$$

Note that the stiffness in (12.32) must be made minor symmetric

$$D_{ijkl} = C_{ijkl} + \frac{1}{2}\left(\sigma_{ik}\delta_{jl} + \sigma_{il}\delta_{jk} + \delta_{il}\sigma_{jk} + \delta_{ik}\sigma_{jl}\right) \quad (12.33)$$

The correct stiffness corresponding to the Jaumann rate of the Kirchhoff stress, in terms of the push forward of the material stiffness is, thus, given by (12.33). Equations relating $\mathbb{C}$ to stiffnesses corresponding to other objective rates are derived similarly.

References

DRAFT

Appendix A

Derivation of Hyperelastic Material Stiffness

Elastic stiffness in the material frame is given by

$$\mathbb{L} = 4\frac{\partial^2 W}{\partial \mathbf{C} \partial \mathbf{C}} = 4\frac{\partial}{\partial \mathbf{C}}\left(\boldsymbol{A} \cdot \boldsymbol{B}\right) \tag{A.1}$$

$$= 4\left(\frac{\partial \boldsymbol{A}}{\partial \mathbf{C}} \cdot \boldsymbol{B} + \boldsymbol{A} \cdot \frac{\partial \boldsymbol{B}}{\partial \mathbf{C}}\right) \tag{A.2}$$

A.0.1 The $\partial \boldsymbol{A}/\partial \mathrm{C}$ Term

$$\frac{\partial A_1}{\partial \mathbf{C}} = \frac{\partial A_1}{\partial \bar{I}_1}\frac{\partial \bar{I}_1}{\partial \mathbf{C}} + \frac{\partial A_1}{\partial \bar{I}_2}\frac{\partial \bar{I}_2}{\partial \mathbf{C}} + \frac{\partial A_1}{\partial J}\frac{\partial J}{\partial \mathbf{C}} \tag{A.3}$$

$$= \frac{\partial^2 W}{\partial \bar{I}_1{}^2}\frac{\partial \bar{I}_1}{\partial \mathbf{C}} + \frac{\partial^2 W}{\partial \bar{I}_2 \partial \bar{I}_1}\frac{\partial \bar{I}_2}{\partial \mathbf{C}} + \frac{\partial^2 W}{\partial J \partial \bar{I}_1}\frac{\partial J}{\partial \mathbf{C}} \tag{A.4}$$

$$\frac{\partial A_2}{\partial \mathbf{C}} = \frac{\partial A_2}{\partial \bar{I}_1}\frac{\partial \bar{I}_1}{\partial \mathbf{C}} + \frac{\partial A_2}{\partial \bar{I}_2}\frac{\partial \bar{I}_2}{\partial \mathbf{C}} + \frac{\partial A_2}{\partial J}\frac{\partial J}{\partial \mathbf{C}} \tag{A.5}$$

$$= \frac{\partial^2 W}{\partial \bar{I}_1 \partial \bar{I}_2}\frac{\partial \bar{I}_1}{\partial \mathbf{C}} + \frac{\partial^2 W}{\partial \bar{I}_2{}^2}\frac{\partial \bar{I}_2}{\partial \mathbf{C}} + \frac{\partial^2 W}{\partial J \partial \bar{I}_2}\frac{\partial J}{\partial \mathbf{C}} \tag{A.6}$$

$$\frac{\partial A_3}{\partial \mathbf{C}} = \frac{\partial A_3}{\partial \bar{I}_1}\frac{\partial \bar{I}_1}{\partial \mathbf{C}} + \frac{\partial A_3}{\partial \bar{I}_2}\frac{\partial \bar{I}_2}{\partial \mathbf{C}} + \frac{\partial A_3}{\partial J}\frac{\partial J}{\partial \mathbf{C}} \tag{A.7}$$

$$= \frac{\partial^2 W}{\partial \bar{I}_1 \partial J}\frac{\partial \bar{I}_1}{\partial \mathbf{C}} + \frac{\partial^2 W}{\partial \bar{I}_2 \partial J}\frac{\partial \bar{I}_2}{\partial \mathbf{C}} + \frac{\partial^2 W}{\partial J^2}\frac{\partial J}{\partial \mathbf{C}} \tag{A.8}$$

Combining gives

$$
\begin{aligned}
\frac{\partial \boldsymbol{A}}{\partial \mathbf{C}} &= \begin{bmatrix} \frac{\partial^2 W}{\partial \bar{I}_1{}^2} & \frac{\partial^2 W}{\partial \bar{I}_2 \partial \bar{I}_1} & \frac{\partial^2 W}{\partial J \partial \bar{I}_1} \\ \frac{\partial^2 W}{\partial \bar{I}_1 \partial \bar{I}_2} & \frac{\partial^2 W}{\partial \bar{I}_2{}^2} & \frac{\partial^2 W}{\partial J \partial \bar{I}_2} \\ \frac{\partial^2 W}{\partial \bar{I}_1 \partial J} & \frac{\partial^2 W}{\partial \bar{I}_2 \partial J} & \frac{\partial^2 W}{\partial J^2} \end{bmatrix} \begin{Bmatrix} \frac{\partial \bar{I}_1}{\partial \mathbf{C}} \\ \frac{\partial \bar{I}_2}{\partial \mathbf{C}} \\ \frac{\partial J}{\partial \mathbf{C}} \end{Bmatrix} && \text{(A.9)} \\
&= \boldsymbol{H}^{\mathrm{T}} \cdot \boldsymbol{B} && \text{(A.10)}
\end{aligned}
$$

where

$$
\boldsymbol{H} = \begin{bmatrix} \frac{\partial^2 W}{\partial \bar{I}_1{}^2} & \frac{\partial^2 W}{\partial \bar{I}_1 \partial \bar{I}_2} & \frac{\partial^2 W}{\partial \bar{I}_1 \partial J} \\ \frac{\partial^2 W}{\partial \bar{I}_2 \partial \bar{I}_1} & \frac{\partial^2 W}{\partial \bar{I}_2{}^2} & \frac{\partial^2 W}{\partial \bar{I}_2 \partial J} \\ \frac{\partial^2 W}{\partial J \partial \bar{I}_1} & \frac{\partial^2 W}{\partial J \partial \bar{I}_2} & \frac{\partial^2 W}{\partial J^2} \end{bmatrix} \tag{A.11}
$$

A.0.2 The $\partial \boldsymbol{B}/\partial \mathbf{C}$ Term

$$
\begin{aligned}
\frac{\partial B_1}{\partial \mathbf{C}} &= \frac{\partial}{\partial \mathbf{C}} \left(J^{-2/3} \left(\boldsymbol{\delta} - \frac{1}{3} I_1 \mathbf{C}^{-1} \right) \right) \\
&= \frac{\partial}{\partial \mathbf{C}} \left(J^{-2/3} \right) \left(\boldsymbol{\delta} - \frac{1}{3} I_1 \mathbf{C}^{-1} \right) + J^{-2/3} \frac{\partial}{\partial \mathbf{C}} \left(\boldsymbol{\delta} - \frac{1}{3} I_1 \mathbf{C}^{-1} \right) \\
&= -\frac{1}{3} J^{-2/3} \mathbf{C}^{-1} \left(\boldsymbol{\delta} - \frac{1}{3} I_1 \mathbf{C}^{-1} \right) - \frac{1}{3} J^{-2/3} \left(\boldsymbol{\delta} \mathbf{C}^{-1} - I_1 \mathbf{C}^{-1} \odot \mathbf{C}^{-1} \right) \\
&= \frac{1}{3} J^{-2/3} \left(-\mathbf{C}^{-1} \boldsymbol{\delta} - \boldsymbol{\delta} \mathbf{C}^{-1} + \frac{1}{3} I_1 \mathbf{C}^{-1} \mathbf{C}^{-1} + I_1 \mathbf{C}^{-1} \odot \mathbf{C}^{-1} \right) \\
&= \boxed{\frac{1}{3} J^{-2/3} \left[-\left(\mathbf{C}^{-1} \boldsymbol{\delta} + \boldsymbol{\delta} \mathbf{C}^{-1} \right) + I_1 \left(\mathbf{C}^{-1} \odot \mathbf{C}^{-1} + \frac{1}{3} \mathbf{C}^{-1} \mathbf{C}^{-1} \right) \right]} && \text{(A.12)}
\end{aligned}
$$

$$\begin{aligned}
\frac{\partial B_2}{\partial \mathbf{C}} &= \frac{\partial}{\partial \mathbf{C}}\left(J^{-4/3}\left(I_1\boldsymbol{\delta} - \mathbf{C} - \frac{2}{3}I_2\mathbf{C}^{-1}\right)\right) \\
&= \frac{\partial}{\partial \mathbf{C}}\left(J^{-4/3}\right)\left(I_1\boldsymbol{\delta} - \mathbf{C} - \frac{2}{3}I_2\mathbf{C}^{-1}\right) + J^{-4/3}\frac{\partial}{\partial \mathbf{C}}\left(I_1\boldsymbol{\delta} - \mathbf{C} - \frac{2}{3}I_2\mathbf{C}^{-1}\right) \\
&= -\frac{2}{3}J^{-4/3}\mathbf{C}^{-1}\left(I_1\boldsymbol{\delta} - \mathbf{C} - \frac{2}{3}I_2\mathbf{C}^{-1}\right) + J^{-4/3}\left(\mathbb{I}_1 - \mathbb{I}_2 - \frac{2}{3}\left(\left(I_1\boldsymbol{\delta} - \mathbf{C}\right)\mathbf{C}^{-1} - I_2\mathbf{C}^{-1}\odot\mathbf{C}^{-1}\right)\right) \\
&= J^{-4/3}\left[\frac{2}{3}\mathbf{C}^{-1}\left(-I_1\boldsymbol{\delta} + \mathbf{C} + \frac{2}{3}I_2\mathbf{C}^{-1}\right) + \left(\mathbb{I}_1 - \mathbb{I}_2 - \frac{2}{3}\left(\left(I_1\boldsymbol{\delta} - \mathbf{C}\right)\mathbf{C}^{-1} - I_2\mathbf{C}^{-1}\odot\mathbf{C}^{-1}\right)\right)\right] \\
&= J^{-4/3}\left[-\frac{2}{3}I_1\mathbf{C}^{-1}\boldsymbol{\delta} + \frac{2}{3}\mathbf{C}^{-1}\mathbf{C} + \frac{2}{3}\frac{2}{3}I_2\mathbf{C}^{-1}\mathbf{C}^{-1} + \mathbb{I}_1 - \mathbb{I}_2 - \frac{2}{3}\left(\left(I_1\boldsymbol{\delta} - \mathbf{C}\right)\mathbf{C}^{-1} - I_2\mathbf{C}^{-1}\odot\mathbf{C}^{-1}\right)\right] \\
&= J^{-4/3}\left(-\frac{2}{3}I_1\mathbf{C}^{-1}\boldsymbol{\delta} + \frac{2}{3}\mathbf{C}^{-1}\mathbf{C} + \frac{2}{3}\frac{2}{3}I_2\mathbf{C}^{-1}\mathbf{C}^{-1} + \mathbb{I}_1 - \mathbb{I}_2 - \frac{2}{3}I_1\boldsymbol{\delta}\mathbf{C}^{-1} + \frac{2}{3}\mathbf{C}\mathbf{C}^{-1} + \frac{2}{3}I_2\mathbf{C}^{-1}\odot\mathbf{C}^{-1}\right) \\
&= J^{-4/3}\left[-\frac{2}{3}I_1\left(\mathbf{C}^{-1}\boldsymbol{\delta} + \boldsymbol{\delta}\mathbf{C}^{-1}\right) + \frac{2}{3}\left(\mathbf{C}^{-1}\mathbf{C} + \mathbf{C}\mathbf{C}^{-1}\right) + \mathbb{I}_1 - \mathbb{I}_2 + \frac{2}{3}I_2\left(\mathbf{C}^{-1}\odot\mathbf{C}^{-1} + \frac{2}{3}\mathbf{C}^{-1}\mathbf{C}^{-1}\right)\right] \\
&= \boxed{\frac{2}{3}J^{-4/3}\left[\frac{3}{2}\left(\mathbb{I}_1 - \mathbb{I}_2\right) + \left(\mathbf{C}^{-1}\mathbf{C} + \mathbf{C}\mathbf{C}^{-1}\right) - I_1\left(\mathbf{C}^{-1}\boldsymbol{\delta} + \boldsymbol{\delta}\mathbf{C}^{-1}\right) + I_2\left(\mathbf{C}^{-1}\odot\mathbf{C}^{-1} + \frac{2}{3}\mathbf{C}^{-1}\mathbf{C}^{-1}\right)\right]}
\end{aligned} \tag{A.13}$$

where

$$(\mathbb{I}_1)_{ijkl} = \delta_{ij}\delta_{kl} \tag{A.14}$$

$$(\mathbb{I}_2)_{ijkl} = \delta_{ik}\delta_{jl} \tag{A.15}$$

$$\begin{aligned}
\frac{\partial B_3}{\partial \mathbf{C}} &= \frac{1}{2}\frac{\partial}{\partial \mathbf{C}}\left(J\mathbf{C}^{-1}\right) \\
&= \frac{1}{2}\left(\frac{1}{2}J\mathbf{C}^{-1}\mathbf{C}^{-1} - J\mathbf{C}^{-1}\odot\mathbf{C}^{-1}\right) \\
&= \boxed{\frac{1}{4}J\left(\mathbf{C}^{-1}\mathbf{C}^{-1} - 2\mathbf{C}^{-1}\odot\mathbf{C}^{-1}\right)}
\end{aligned} \tag{A.16}$$

Collecting

$$\frac{\partial \boldsymbol{B}}{\partial \mathbf{C}} = \left\{\begin{array}{c}
\frac{1}{3}J^{-2/3}\left[\boldsymbol{\delta}\mathbf{C}^{-1} - \mathbf{C}^{-1}\boldsymbol{\delta} - I_1\left(\mathbf{C}^{-1}\odot\mathbf{C}^{-1} - \frac{1}{3}\mathbf{C}^{-1}\mathbf{C}^{-1}\right)\right] \\
\frac{2}{3}J^{-4/3}\left[\frac{3}{2}\left(\mathbb{I}_1 - \mathbb{I}_2\right) + \left(\mathbf{C}^{-1}\mathbf{C} + \mathbf{C}\mathbf{C}^{-1}\right) - I_1\left(\mathbf{C}^{-1}\boldsymbol{\delta} + \boldsymbol{\delta}\mathbf{C}^{-1}\right) - I_2\left(\mathbf{C}^{-1}\odot\mathbf{C}^{-1} - \frac{2}{3}\mathbf{C}^{-1}\mathbf{C}^{-1}\right)\right] \\
\frac{1}{4}J\left(\mathbf{C}^{-1}\mathbf{C}^{-1} - 2\mathbf{C}^{-1}\odot\mathbf{C}^{-1}\right)
\end{array}\right\} \tag{A.17}$$

Combining, we get

$$\mathbb{L} = 4\left(\frac{\partial \boldsymbol{A}}{\partial \mathbf{C}} \cdot \boldsymbol{B} + \boldsymbol{A} \cdot \frac{\partial \boldsymbol{B}}{\partial \mathbf{C}}\right) \tag{A.18}$$

(A.19)

$I_1 = \mathbf{C}:\boldsymbol{\delta}$	$I_2 = \frac{1}{2}\left(I_1^2 - \mathbf{C}:\mathbf{C}\right)$	$J = \det \boldsymbol{F} = \sqrt{\det \mathbf{C}}$
$\frac{\partial I_1}{\partial \mathbf{C}} = \boldsymbol{\delta}$	$\frac{\partial I_2}{\partial \mathbf{C}} = I_1\boldsymbol{\delta} - \mathbf{C}$	$\frac{\partial J}{\partial \mathbf{C}} = \frac{1}{2}J\mathbf{C}^{-1}$
$\bar{I}_1 = J^{-2/3} I_1$	$\bar{I}_2 = J^{-4/3} I_2$	
$\frac{\partial \bar{I}_1}{\partial \mathbf{C}} = J^{-2/3}\left(\boldsymbol{\delta} - \frac{1}{3}I_1\mathbf{C}^{-1}\right)$	$\frac{\partial \bar{I}_2}{\partial \mathbf{C}} = J^{-4/3}\left(I_1\boldsymbol{\delta} - \mathbf{C} - \frac{2}{3}I_2\mathbf{C}^{-1}\right)$	$\frac{\partial}{\partial \mathbf{C}}\left(J^{-i/3}\right) = -\frac{i}{6}J^{-i/3}\mathbf{C}^{-1}$
$\operatorname{tr}\boldsymbol{d} = \frac{\dot{J}}{J}$	$\dot{\boldsymbol{F}} = \boldsymbol{l}\cdot\boldsymbol{F}$	$\dot{\boldsymbol{F}}^{\mathrm{T}} = \boldsymbol{F}^{\mathrm{T}}\cdot\boldsymbol{l}^{\mathrm{T}}$
$\dot{\boldsymbol{E}} = \boldsymbol{F}^{\mathrm{T}}\cdot\boldsymbol{d}\cdot\boldsymbol{F}$		

Table A.1. Identities

Sandia National Laboratories