# Multiple Mittag-Leffler stability of fractional-order competitive neural networks with Gaussian activation functions

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# Multiple Mittag-Leffler stability of fractional-order competitive neural networks with Gaussian activation functions

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## HIGHLIGHTS

- FOCNNs with Gaussian activation functions are introduced.
- Coexistence and local stability of multiple equilibrium points are investigated.
- $n$-neuron networks have exactly $3^k$ equilibrium points with $0 \leq k \leq n$.
- $2^k$ equilibrium points are locally Mittag-Leffler stable.
- The obtained results cover both multistability and mono-stability.

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## ABSTRACT

In this paper, we explore the coexistence and dynamical behaviors of multiple equilibrium points for fractional-order competitive neural networks with Gaussian activation functions. By virtue of the geometrical properties of activation functions, the fixed point theorem and the theory of fractional-order differential equation, some sufficient conditions are established to guarantee that such $n$-neuron neural networks have exactly $3^k$ equilibrium points with $0 \leq k \leq n$, among which $2^k$ equilibrium points are locally Mittag-Leffler stable. The obtained results cover both multistability and mono-stability of fractional-order neural networks and integer-order neural networks. Two illustrative examples with their computer simulations are presented to verify the theoretical analysis.



## 1. Introduction

Neural Networks are said to be multistable if they possess multiple equilibrium points or periodic orbits which are locally stable. Multistability is necessary whenever neural networks are applied in pattern recognition, associative memories and image processing. For this reason, multistability analysis of neural networks has been an active area of research in the past decades and some valuable and interesting results on this topic have been obtained. For example, Cheng, Lin, and Shih (2007) investigated the multistability of Hopfield neural networks with sigmoidal and nondecreasing saturated activation functions, and some parameter conditions were derived to ascertain the coexistence of $3^n$ equilibrium points and local stability of $2^n$ equilibrium points for such $n$-neuron delayed neural networks. Based on the approach of sequential contracting, the above results were further extended in Cheng, Lin, Shih, and Tseng (2015), and it was shown therein that under some conditions, the $n$-neuron Hopfield neural networks have exactly $3^k$ equilibrium points with $0 \leq k \leq n$, among which $2^k$ equilibrium points are attracting and locally stable. For various piecewise linear activation functions, including nondecreasing, continuous nonmonotonic and discontinuous nonmonotonic functions, the multistability and complete stability of recurrent neural networks were discussed in Liu, Zeng, and Wang (2017a), Di Marco, Forti, Grazzini, and Pancioni (2012), Nie, Cao, and Fei (2019), Nie and Zheng (2015a, 2015b, 2016), Nie, Zheng, and Cao (2016), Wang and Chen (2012), Wang, Lu, and Chen (2010) and Zeng, Huang, and Zheng (2010). In addition, Chen, Zhao, Song, and Hu (2017), Liang, Gong, and Huang (2016) and Rakkiyappan, Velmurugan, and Cao (2015) focused on the multistability analysis of complex-valued neural networks. Many other relevant works can be found in Di Marco, Forti, and Pancioni (2017), Huang, Raffoul, and Cheng (2014), Huang, Song, and Feng (2010) and Kaslik and Sivasundaram (2011)and references therein.

It is well known that the type of activation functions plays a crucial role in the multistability analysis of neural networks. It should be mentioned here that most of the activation functions employed in the existing literature (including the abovementioned works)

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are restricted to be either nondecreasing or piecewise linear. However, as reported in Morita (1993, 1996), the neural networks with nonmonotonic activation functions exhibit good performance in terms of associative memory, including memory capacity and retrieval ability. Actually, there are still several widely used neural networks utilizing activation functions that are nonmonotonic and nonlinear. Among which is the known radial basis function (RBF) neural networks, where the activation functions are radially symmetric and produce a localized response to input stimulus (Moody & Darken, 1989). In traditional RBF neural networks, Gaussian function is the most frequently used activation function, and a tunable Gaussian computation circuit for analog neural networks was implemented in Lin, Huang, and Chiueh (1998). In the past decades, RBF neural networks with Gaussian activation function have been successfully applied to classification (Er, Wu, Lu, & Toh, 2002; Savitha, Suresh, & Sundararajan, 2012), function approximation (Hartman, Keeler, & Kowalski, 1990; Leonard, Kramer, & Ungar, 1992; Saha, Wu, & Tang, 1993) and prediction (Lin, Ko, Chuang, Su, & Lin, 2012). Besides, as pointed out in Gundogdu, Egrioglu, Aladag, and Yolcu (2016) and Kamimura (2006), neural networks with Gaussian activation function can have faster learning capacity and better forecasting performance because of locally tuned neurons, compared with other neural networks with sigmoidal or piecewise linear activation functions. Furthermore, Gaussian function has important biological background. It was claimed in O'Keefe and Burgess (1996) that hippocampal place cell formation in both rats and humans plays an important role in spatial navigation. A model of hippocampal place cell formation was proposed in O'Keefe and Burgess (1996) and the essence of this model is that each place field comprises the summation of two or more Gaussian functions (O'Keefe, Burgess, Donnett, Jeffery, & Maguire, 1998). Hence, it is valuable to explore the multistability of neural networks with Gaussian activation function. Very recently, Liu, Zeng, and Wang (2017b) studied the issue of complete stability and multistability for classical integer-order delayed Hopfield neural networks with Gaussian activation function. Based on the geometrical properties and algebraic properties of nonsingular $M$-matrix, some sufficient conditions were obtained to ensure the addressed neural networks have exactly $3^k$ equilibrium points, among which $2^k$ equilibrium points are locally exponentially stable and the others are unstable.

It is worth pointing out that all the aforementioned multistability results are based on classical integer-order neural networks. Fractional-order calculus is a generalization of integer-order differentiation and integration to arbitrary non-integer order. Compared with integer-order calculus, fractional-order derivative is nonlocal and has weakly singular kernels, and has proved to be an excellent tool for the description of memory and hereditary properties of various materials and processes. Taking into account of these benefits, some researchers incorporate fractional-order derivative into neural networks model, leading to fractional-order neural networks (FONNs). As pointed out in Chen, Wu, Cao, and Liu (2015), the main advantages of FONNs lie in two aspects, one is its infinite-memory property, the other is that fractional-order parameter enriches the system performance by increasing one degree of freedom, and integer-order neural networks can be viewed as the special case of fractional-order ones. As a result, the dynamical analysis of FONNs became an important topic of recent research, and some valuable results have been achieved in this new field. However, to the best of our knowledge, most results on dynamical analysis of FONNs concentrated upon mono-stability and stabilization (Li, Cao, Alsaedi, & Alsaadi, 2017; Wu & Zeng, 2017), synchronization (Chen, Cao et al., 2017; Wu, Wang, Niu, & Wang, 2017), bifurcation control (Huang, Cao, Xiao, Alsaedi, & Alsaadi, 2017; Kaslik & Radulescu, 2017), and there are very few results on the multistability of FONNs. The incorporation of fractional-order derivative is beneficial for the accurate modeling of neural networks, but brings some additional challenges for their multistability analysis. The most frequently used multistability method for integer-order neural networks cannot be extended easily to FONNs, especially for the estimation of positively invariant sets and the analysis of local stability of multiple equilibrium points. Very recently, Liu, Zeng, and Wang (2017c) considered the multistability of fractional-order Hopfield neural networks with a general class of activation functions. Based on the algebraic properties of nonsingular $M$-matrix and the theory of fractional-order differential equation, it was shown in Liu et al. (2017c) that the $n$-neuron fractional-order Hopfield neural networks without delay can have $\prod_{i=1}^{n}(2K_i+1)$ equilibrium points, and $\prod_{i=1}^{n}(K_i+1)$ equilibrium points are locally Mittag-Leffler stable.

Competitive neural networks model the dynamics of both neural activity level (i.e., the short-term memory) and unsupervised synaptic modifications (i.e., the long-term memory), which were firstly proposed in Meyer-Baese, Ohl, and Scheich (1996). The networks combine an additive activation dynamics of Hopfield neural networks with a competitive learning law for the synaptic modifications. If no learning occurs, then competitive neural networks reduce to the known Hopfield neural networks. The multistability analysis of integer-order competitive neural networks was investigated in Nie and Cao (2009), Nie, Cao, and Fei (2013), Nie and Zheng (2016) and Nie et al. (2019). We remark that the activation functions considered in these papers include sigmoidal and nondecreasing saturated functions, nondecreasing piecewise linear functions and nonmonotonic piecewise linear functions, which are all either nondecreasing or piecewise linear.

With the motivations illustrated as above, our main objective in this paper is to explore the multistability of fractional-order competitive networks (FOCNNs) with Gaussian activation function. Both the model and the activation function considered in this paper are totally different from those in the existing multistability results of competitive neural networks. On the one hand, the model in this paper is based on fractional-order derivative and is more general and accurate, thus the multistability analysis of FOCNNs is much challenging. On the other hand, the activation function employed in this paper is Gaussian function, which is nonmonotonic and nonlinear. More precisely, the main contributions of this paper can be summarized in the following aspects. Firstly, we establish sufficient conditions under which the $n$-neuron FOCNNs with Gaussian activation functions have exactly $3^k$ equilibrium points with $0 \leq k \leq n$, by means of the geometrical properties of Gaussian functions, the fixed point theorem and the contraction mapping theorem. Secondly, the theory of fractional-order differential equation is further utilized to estimate the invariant sets and analyze the dynamical behaviors of multiple equilibrium points for the corresponding neural networks. It is shown that the addressed FOCNNs have exactly $3^k$ equilibrium points, among which $2^k$ equilibrium points are locally Mittag-Leffler stable. Thirdly, the obtained results cover both the mono-stability and multistability of fractional-order and integer-order competitive neural networks. Besides, as a byproduct, the corresponding results of fractional-order and integer-order Hopfield neural networks can be obtained, respectively. Finally, two illustrative examples with computer simulations are presented to demonstrate the effectiveness of the theoretical results.

## 2. Model formulation and preliminaries

There are three common definitions of fractional-order differential operators: Grunwald–Letnikov, Riemann–Liouville and Caputo definitions. In this paper, the Caputo fractional-order derivative will be employed due to its initial condition is the same with the one of integer-order derivation, which is well-understood in physical situations and more applicable to practical problems.

**Definition 1.** The fractional-order integral with order $q \in \Re^+$ for function $x(t)$ is defined as follows:

$$
{}_{t_0}D_t^{-q}x(t) = \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-\tau)^{q-1}x(\tau)d\tau, \tag{1}
$$

where $\Gamma(\cdot)$ is the Gamma function, and $\Gamma(\tau) = \int_0^{+\infty} t^{\tau-1}e^{-t}dt$, $t_0$ is the initial time.

**Definition 2.** The Caputo fractional-order derivative with order $q \in \Re^+$ for function $x(t)$ is defined as follows:

$$
{}_{t_0}^{C}D_t^{q}x(t) = \frac{1}{\Gamma(n-q)}\int_{t_0}^{t}(t-\tau)^{n-q-1}x^{(n)}(\tau)d\tau, \tag{2}
$$

where $n-1 < q < n$, $n$ is a positive integer.

For the reason of convenience, in the rest of this paper, we use the notation $D^q x(t)$ to denote Caputo fractional-order derivative ${}_{t_0}^{C}D_t^{q}x(t)$.

**Definition 3.** $\{\xi,\infty\}$-norm: for $x = (x_1, x_2, \ldots, x_n)^T \in \Re^n$, define $\|x\|_{\{\xi,\infty\}} = \max_{1\le i\le n}\{\xi_i^{-1}|x_i|\}$, where $\xi_i > 0,\ \ i = 1, 2, \ldots, n$.

Next, we give the definition of Mittag-Leffler function which has the similar effect with exponential function used in integer-order systems. The Mittag-Leffler function with one parameter is given as follows:

$$
E_\alpha(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+1)}, \tag{3}
$$

where $\alpha > 0$, $z \in \mathbb{C}$. The Mittag-Leffler function with two parameters is given as follows:

$$
E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\beta)},
$$

where $\alpha > 0$, $\beta > 0$ and $z \in \mathbb{C}$. For $\beta = 1$, we have $E_{\alpha,1}(z) = E_\alpha(z)$ and $E_{1,1}(z) = e^z$ when $\alpha = \beta = 1$.

In this paper, based on the integer-order competitive neural networks considered in Meyer-Baese et al. (1996), Nie and Cao (2009), Nie et al. (2013) and Nie and Zheng (2016), we introduce the following FOCNNs:

$$
\begin{cases}
D^q x_i(t) = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + B_i \sum_{j=1}^{P} m_{ij}(t) y_j + I_i, \\
D^q m_{ij}(t) = -\alpha_i m_{ij}(t) + y_j \beta_i f_i(x_i(t)),
\end{cases} \tag{4}
$$

where $i = 1, \ldots, n, j = 1, \ldots, P$, $0 < q < 1$, $x_i(t)$ is the neural current activity level, $f_j(x_j(t))$ is the output of neurons, $m_{ij}(t)$ is the synaptic efficiency, $y_j$ is the constant external stimulus, $a_{ij}$ represents the connection weight between the $i$th neuron and the $j$th neuron, $B_i$ is the strength of the external stimulus, $I_i$ is the constant input, $\alpha_i > 0$, $\beta_i \ge 0$ denote disposable scaling constants.

After setting $S_i(t) = \sum_{j=1}^{P} m_{ij}(t)y_j = \mathbf{y}^T\mathbf{m}_i(t)$, where $\mathbf{y} = (y_1, \ldots, y_P)^T$, $\mathbf{m}_i(t) = (m_{i1}(t), \ldots, m_{iP}(t))^T$, and assume the input stimulus $\mathbf{y}$ to be normalized with unit magnitude $|\mathbf{y}|^2 = y_1^2 + \cdots + y_P^2 = 1$, then the above networks are equivalent to

$$
\begin{cases}
D^q x_i(t) = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + B_i S_i(t) + I_i, \\
D^q S_i(t) = -\alpha_i S_i(t) + \beta_i f_i(x_i(t)), \quad i = 1, \ldots, n.
\end{cases} \tag{5}
$$

In this paper, we consider a class of both non-monotonic and nonlinear activation functions, which are called Gaussian functions

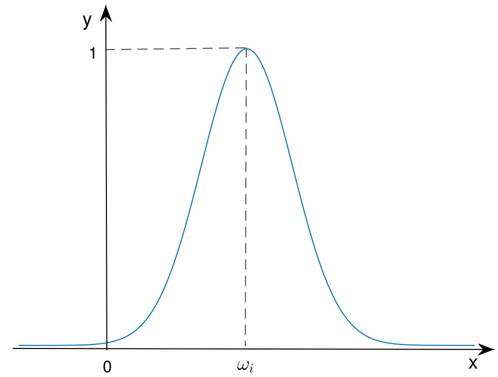



**Fig. 1.** Graph of Gaussian activation function.

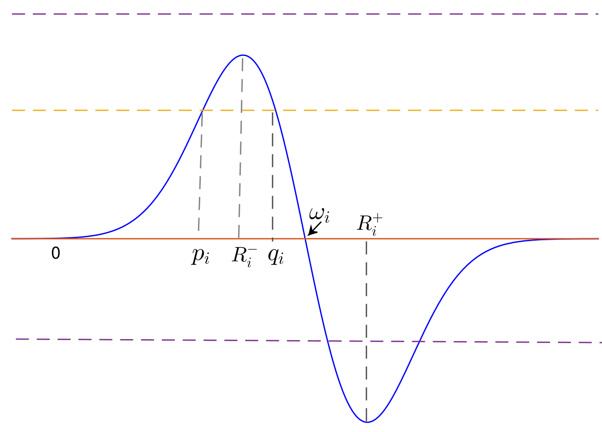



**Fig. 2.** Graph of the derivative of Gaussian activation function.

defined by (see Fig. 1):

$$
f_i(r) = \exp\left(-\frac{(r-\omega_i)^2}{\rho_i^2}\right), \tag{6}
$$

where $\omega_i$, $\rho_i$ represent the center and the width, respectively. It is easy to see that the activation function satisfies $0 < f_i(r) \le 1$. Moreover, by simple computations, we get

$$
f_i'(r) = -\frac{2(r-\omega_i)}{\rho_i^2}\exp\left(-\frac{(r-w_i)^2}{\rho_i^2}\right),
$$

$$
f_i''(r) = \frac{2(2r^2 - 4r\omega_i + 2\omega_i^2 - \rho_i^2)}{\rho_i^4}\exp\left(-\frac{(r-w_i)^2}{\rho_i^2}\right).
$$

By solving the equation $f_i'(r) = 0$, we can find it just has one root $R_i = \omega_i$. Noting that $f_i'(r) > 0\,(r < \omega_i)$, $f_i'(r) < 0\,(r > \omega_i)$ and $\lim_{r\to-\infty}f_i(r) = \lim_{r\to+\infty}f_i(r) = 0$, it follows that $\omega_i$ is the maximum point of $f_i(r)$ with $f_i(\omega_i) = 1$ and $\inf_{r\in\Re}f_i(r) = 0$. Similarly, from the equation $f_i''(r) = 0$, two roots can be found as follows:

$$
R_i^- = \omega_i - \frac{\sqrt{2}}{2}\rho_i,\ \ R_i^+ = \omega_i + \frac{\sqrt{2}}{2}\rho_i.
$$

It is easy to see that $\lim_{r\to-\infty}f_i'(r) = \lim_{r\to+\infty}f_i'(r) = 0$, $f_i''(r) > 0\,(r < R_i^-)$, $f_i''(r) < 0\,(R_i^- < r < R_i^+)$ and $f_i''(r) > 0\,(r > R_i^+)$, hence $R_i^-$ and $R_i^+$ are the maximum point and minimum point of $f_i'(r)$, respectively (see Fig. 2). And the maximum value and the minimum value of $f_i'(r)$ are as follows:

$$
\max_{r\in\Re} f_i'(r) = f_i'(R_i^-) = \sqrt{2}\exp(-1/2)/\rho_i,
$$

$$
\min_{r\in\Re} f_i'(r) = f_i'(R_i^+) = -\sqrt{2}\exp(-1/2)/\rho_i.
$$

**Definition 4.** An equilibrium point of FOCNNs (5) is a vector $(x^{*T}, S^{*T})^T \in \Re^{2n}$ that satisfies

$$\begin{cases} -d_i x_i^* + (a_{ii} + \alpha_i^{-1} B_i \beta_i) f_i(x_i^*) + \sum_{j\neq i, j=1}^{n} a_{ij} f_j(x_j^*) + I_i = 0, \\ S_i^* = \alpha_i^{-1} \beta_i f_i(x_i^*), \ i = 1, \dots, n. \end{cases} \tag{7}$$

**Definition 5.** Suppose that $D \subset \Re^{2n}$ is a positively invariant set and $(x^{*T}, S^{*T})^T \in D$ is an equilibrium point of FOCNNs (5). The equilibrium point $(x^{*T}, S^{*T})^T$ is said to be locally Mittag-Leffler stable in $D$, if for any initial value $(x^T(t_0), S^T(t_0))^T \in D$, the corresponding state $(x^T(t), S^T(t))^T \in D$ of FOCNNs (5) satisfies that

$$\begin{aligned} &\left\| \left(x^T(t), S^T(t)\right)^T - \left(x^{*T}, S^{*T}\right)^T \right\| \\ &\le M \left\| \left(x^T(t_0), S^T(t_0)\right)^T - \left(x^{*T}, S^{*T}\right)^T \right\| E_q(-\sigma t^q), \ t \ge t_0, \end{aligned}$$

where $M$ and $\sigma$ are two positive constants.

In order to derive the main results in this paper, the following lemmas are needed.

**Lemma 1** (Liu et al., 2017c). *Assume that $0 < q < 1$, $t_0 \ge 0$ and $h(t) \in C^1([t_0, +\infty), \Re)$, if there exists $t_1 > t_0$, such that $h(t_1) = 0$ and $h(t) < 0 (\le 0)$ for $t_0 \le t < t_1$, then $D^q h(t_1) > 0 (\ge 0)$.*

**Proof.** From the definition of Caputo derivative and $0 < q < 1$, we get

$$\begin{aligned} D^q h(t_1) &= \frac{1}{\Gamma(1-q)} \int_{t_0}^{t_1} (t_1 - \tau)^{-q} h'(\tau) d\tau \\ &= \frac{1}{\Gamma(1-q)} \int_{t_0}^{t_1} (t_1 - \tau)^{-q} dh(\tau) \\ &= \frac{1}{\Gamma(1-q)} \left[ \left. \frac{h(\tau)}{(t_1-\tau)^q} \right|_{t_0}^{t_1} - q \int_{t_0}^{t_1} \frac{h(\tau)}{(t_1-\tau)^{q+1}} d\tau \right]. \end{aligned}$$

Since $h(t)$ is continuously differential, we have

$$\lim_{\tau \to t_1} \frac{h(\tau)}{(t_1-\tau)^q} = -\frac{1}{q} \lim_{\tau \to t_1} h'(\tau)(t_1-\tau)^{1-q} = 0,$$

and

$$-q \int_{t_0}^{t_1} \frac{h(\tau)}{(t_1-\tau)^{q+1}} d\tau = -q \lim_{\varepsilon \to 0^+} \int_{t_0}^{t_1-\varepsilon} \frac{h(\tau)}{(t_1-\tau)^{q+1}} d\tau \ge 0.$$

Then it follows from $h(t_0) < 0 (\le 0)$ that

$$D^q h(t_1) \ge -\frac{1}{\Gamma(1-q)} \frac{h(t_0)}{(t_1-t_0)^q} > 0 (\ge 0). \quad \blacksquare$$

**Lemma 2.** *If $x(t) \in C^1([t_0, +\infty), \Re)$, $t_0 \ge 0$, and the following inequality holds:*

$$-a x(t) + b_1 \le D^q x(t) \le -a x(t) + b_2,$$

*where $a > 0$ and $b_1 < b_2$, then $x(t_0) \in (a^{-1} b_1, a^{-1} b_2)$ implies that $x(t) \in (a^{-1} b_1, a^{-1} b_2)$ for all $t \ge t_0$.*

**Proof.** If $x(t_0) \in (a^{-1} b_1, a^{-1} b_2)$, we claim that the solution $x(t) \in (a^{-1} b_1, a^{-1} b_2)$ for all $t \ge t_0$. If this is not true, then there exists $t_1 > 0$ such that $x(t_1) = a^{-1} b_1$ or $x(t_1) = a^{-1} b_2$. Without loss of generality, assume that $x(t_1) = a^{-1} b_2$. Define $y(t) = x(t) - a^{-1} b_2$, then

$$\begin{cases} y(t_1) = 0, \\ y(t) < 0, \ t_0 \le t < t_1. \end{cases}$$

According to Lemma 1, $D^q y(t_1) > 0$, which is a contradiction with $D^q y(t_1) = D^q x(t_1) \le -a x(t_1) + b_2 = 0$. Hence, if $x(t_0) \in (a^{-1} b_1, a^{-1} b_2)$, then the corresponding solution $x(t)$ will never escape from $(a^{-1} b_1, a^{-1} b_2)$ and stay inside it for all $t \ge t_0$. $\blacksquare$

**Lemma 3** (Zhang, Yu, & Wang, 2015). *If $h(t) \in C^1([t_0, +\infty), \Re)$ denotes a continuously differentiable function, then the following inequality holds almost everywhere:*

$$D^q |h(t)| \le sign(h(t)) D^q h(t), \ 0 < q < 1.$$

**Lemma 4** (Chen, Zeng, & Jiang, 2014). *Let $V(t)$ be a continuously differentiable function defined on $[t_0, +\infty)$, and satisfies $D^q V(t) \le c V(t)$, where $0 < q < 1$, $t_0 \ge 0$, $c$ is a constant. Then $V(t) \le V(t_0) E_q(c t^q)$ holds for $t \ge t_0$.*

## 3. Main results

In this section, the dynamical behaviors of multiple equilibrium points for FOCNNs (5) with Gaussian activation functions (6) are investigated. First of all, based on the geometrical properties of Gaussian functions, the fixed point theorem and the contraction mapping theorem, several sufficient criteria are established to ascertain the exact existence of $3^k$ equilibrium points. Then, the $2^k$ equilibrium points located in positively invariant sets are proved to be locally Mittag-Leffler stable, by applying the theory of fractional-order differential equation. The main results are stated one by one as follows.

### 3.1. Coexistence of multiple equilibrium points

First, we define the upper and lower functions as follows:

$$\begin{cases} \check{g}_i(r) = -d_i r + a_{ii} f_i(r) + \check{s}_i, \\ \hat{g}_i(r) = -d_i r + a_{ii} f_i(r) + \hat{s}_i, \end{cases} \tag{8}$$

where

$$\check{s}_i = \sum_{j\neq i, j=1}^{n} \min\{a_{ij}, 0\} + \min\{0, \alpha_i^{-1} B_i \beta_i\} + I_i,$$

$$\hat{s}_i = \sum_{j\neq i, j=1}^{n} \max\{a_{ij}, 0\} + \max\{0, \alpha_i^{-1} B_i \beta_i\} + I_i.$$

And we also define

$$g_i(r) = -d_i r + a_{ii} f_i(r) + s_i,$$

where $\check{s}_i \le s_i \le \hat{s}_i$, then $\check{g}_i(r), g_i(r), \hat{g}_i(r)$ are vertical shifts of one another.

Let $\mathcal{N} = \{1, 2, \dots, n\}$ and $L_i = \sqrt{2} \exp(-1/2)/\rho_i$, for the given parameters $d_i$ and $a_{ii}$, denote

$$\mathcal{M}_1 = \left\{ i \in \mathcal{N} \middle| 0 < \frac{d_i}{a_{ii}} < L_i \right\}, \qquad \mathcal{M}_2 = \left\{ i \in \mathcal{N} \middle| \frac{d_i}{a_{ii}} > L_i \right\},$$

$$\mathcal{M}_3 = \left\{ i \in \mathcal{N} \middle| -L_i < \frac{d_i}{a_{ii}} < 0 \right\}, \qquad \mathcal{M}_4 = \left\{ i \in \mathcal{N} \middle| \frac{d_i}{a_{ii}} < -L_i \right\}.$$

**Remark 1.** If $i \in \mathcal{M}_1 \cup \mathcal{M}_2$, then $a_{ii} > 0$; If $i \in \mathcal{M}_3 \cup \mathcal{M}_4$, then $a_{ii} < 0$.

**Lemma 5.** *If $i \in \mathcal{M}_1$ (or $\mathcal{M}_3$), then there exist $p_i$ and $q_i$ with $p_i < R_i^- < q_i < \omega_i$ (or $\omega_i < p_i < R_i^+ < q_i$) such that $g_i'(p_i) = g_i'(q_i) = 0$. If $i \in \mathcal{M}_2 \cup \mathcal{M}_4$, then $g_i'(r) < 0$ for any $r \in \Re$.*

**Proof.** The equation $g_i'(r) = -d_i + a_{ii} f_i'(r) = 0$ is equivalent to $f_i'(r) = d_i / a_{ii}$.

As shown in Fig. 2, if $i \in \mathcal{M}_1$, there exist two points $p_i, q_i$ with $p_i < R_i^- < q_i < \omega_i$ such that $f_i'(p_i) = f_i'(q_i) = d_i/a_{ii}$, which follows that $g_i'(p_i) = g_i'(q_i) = 0$. Similarly, if $i \in \mathcal{M}_3$, there also exist two points $p_i, q_i$ with $\omega_i < p_i < R_i^+ < q_i$ such that $f_i'(p_i) = f_i'(q_i) = d_i/a_{ii}$, which also follows that $g_i'(p_i) = g_i'(q_i) = 0$.

If $i \in \mathcal{M}_2$, recalling that $a_{ii} > 0$ (see Remark 1), it follows from the definition of subset $\mathcal{M}_2$ that $-d_i + a_{ii} L_i < 0$. Then we obtain

$$g_i'(r) = -d_i + a_{ii} f_i'(r) \le -d_i + a_{ii} L_i < 0.$$

If $i \in \mathcal{M}_4$, noting that $a_{ii} < 0$ (see Remark 1), it follows from the definition of subset $\mathcal{M}_4$ that $-d_i - a_{ii} L_i < 0$. Then we get

$$g_i'(r) = -d_i + a_{ii} f_i'(r) \le -d_i - a_{ii} L_i < 0. \quad \blacksquare$$

**Remark 2.** A consequence of Lemma 5 is that if $i \in \mathcal{M}_1 \cup \mathcal{M}_3$, then $g_i(r)$ is strictly decreasing on $(-\infty, p_i) \cup (q_i, +\infty)$, and strictly increasing on $[p_i, q_i]$.

In this paper, we just consider the following disjoint subsets of $\mathcal{M}_1$ and $\mathcal{M}_3$:

$$\mathcal{M}_1^1 = \{i \in \mathcal{M}_1 | \hat{g}_i(p_i) < 0, \breve{g}_i(q_i) > 0\},$$
$$\mathcal{M}_1^2 = \{i \in \mathcal{M}_1 | \hat{g}_i(q_i) < 0\},$$
$$\mathcal{M}_1^3 = \{i \in \mathcal{M}_1 | \breve{g}_i(p_i) > 0\},$$
$$\mathcal{M}_3^1 = \{i \in \mathcal{M}_3 | \hat{g}_i(p_i) < 0, \breve{g}_i(q_i) > 0\},$$
$$\mathcal{M}_3^2 = \{i \in \mathcal{M}_3 | \hat{g}_i(q_i) < 0\},$$
$$\mathcal{M}_3^3 = \{i \in \mathcal{M}_3 | \breve{g}_i(p_i) > 0\}.$$

Then we have the following lemma:

**Lemma 6.** *If $i \in \mathcal{M}_1^1 \cup \mathcal{M}_3^1$, then $\breve{g}_i(r)$ has three zeros $\breve{a}_i, \breve{b}_i, \breve{c}_i$ and $\hat{g}_i(r)$ also has three zeros $\hat{a}_i, \hat{b}_i, \hat{c}_i$ with $\breve{a}_i \le \hat{a}_i < p_i < \hat{b}_i \le \breve{b}_i < q_i < \breve{c}_i \le \hat{c}_i$.*

*If $i \in \mathcal{M}_1^2 \cup \mathcal{M}_3^2$, then $\breve{g}_i(r)$ just has one zero $\check{m}_i$, $\hat{g}_i(r)$ also just has one zero $\hat{m}_i$ with $\check{m}_i \le \hat{m}_i < p_i$.*

*If $i \in \mathcal{M}_1^3 \cup \mathcal{M}_3^3$, then $\breve{g}_i(r)$ just has one zero $\check{m}_i$, $\hat{g}_i(r)$ also just has one zero $\hat{m}_i$ with $q_i < \check{m}_i \le \hat{m}_i$.*

*If $i \in \mathcal{M}_2 \cup \mathcal{M}_4$, then $\breve{g}_i(r)$ just has one zero $\check{m}_i$, $\hat{g}_i(r)$ also just has one zero $\hat{m}_i$ with $\check{m}_i \le \hat{m}_i$.*

**Proof.** According to the definitions of $\breve{g}_i(r)$ and $\breve{g}_i(r)$, we have $\lim_{r\to-\infty}\breve{g}_i(r) = \lim_{r\to-\infty}\hat{g}_i(r) = +\infty$, $\lim_{r\to+\infty}\breve{g}_i(r) = \lim_{r\to+\infty}\hat{g}_i(r) = -\infty$.

If $i \in \mathcal{M}_1^1 \cup \mathcal{M}_3^1$, then $\breve{g}_i(p_i) \le \hat{g}_i(p_i) < 0, \hat{g}_i(q_i) \ge \breve{g}_i(q_i) > 0$. According to the continuity of function $\breve{g}_i(r)$, $\breve{g}_i(r)$ has three zeros $\breve{a}_i, \breve{b}_i, \breve{c}_i$. Similarly, $\hat{g}_i(r)$ also has three zeros $\hat{a}_i, \hat{b}_i, \hat{c}_i$. Due to $\breve{g}_i(r)$, $\hat{g}_i(r)$ are vertical shifts of one another, we get $\breve{a}_i \le \hat{a}_i < p_i < \hat{b}_i \le \breve{b}_i < q_i < \breve{c}_i \le \hat{c}_i$.

If $i \in \mathcal{M}_1^2 \cup \mathcal{M}_3^2$, we have $\breve{g}_i(p_i) \le \breve{g}_i(q_i) < 0$ and $\hat{g}_i(p_i) \le \hat{g}_i(q_i) < 0$ , then there exist $\check{m}_i, \hat{m}_i$ with $-\infty < \check{m}_i \le \hat{m}_i < p_i$, such that $\breve{g}_i(\check{m}_i) = 0$ and $\hat{g}_i(\hat{m}_i) = 0$.

If $i \in \mathcal{M}_1^3 \cup \mathcal{M}_3^3$, we have $0 < \breve{g}_i(p_i) \le \breve{g}_i(q_i)$ and $0 < \hat{g}_i(p_i) \le \hat{g}_i(q_i)$, then there exist $\check{m}_i, \hat{m}_i$ with $q_i < \check{m}_i \le \hat{m}_i < +\infty$, such that $\breve{g}_i(\check{m}_i) = 0$ and $\hat{g}_i(\hat{m}_i) = 0$.

If $i \in \mathcal{M}_2 \cup \mathcal{M}_4$, then $\breve{g}_i'(r) < 0, \hat{g}_i'(r) < 0$ and $\lim_{r\to-\infty}\breve{g}_i(r) = \lim_{r\to-\infty}\hat{g}_i(r) = +\infty$, $\lim_{r\to+\infty}\breve{g}_i(r) = \lim_{r\to+\infty}\hat{g}_i(r) = -\infty$. Then there exist $\check{m}_i, \hat{m}_i$ with $\check{m}_i \le \hat{m}_i$ such that $\breve{g}_i(\check{m}_i) = 0$ and $\hat{g}_i(\hat{m}_i) = 0$. $\blacksquare$

In the following, we present the first theorem on the coexistence of multiple equilibrium points for FOCNNs (5) with Gaussian activation functions (6). We show that for each $1 \le k \le n$, there exist parameters with which model (5) admits $3^k$ equilibrium points. Let card(•) denote the cardinality of set •.

**Table 1**
$\bar{L}_i$ for different $i \in \mathcal{N}$.

| $i$ | $\bar{L}_i$ |
|---|---|
| $i \in \mathcal{M}_1^1$ | $\min\left\{a_{ii}\min\left(f_i'(\hat{b}_i), f_i'(\breve{b}_i)\right) - d_i, d_i - a_{ii}\max\{f_i'(\hat{a}_i), f_i'(\breve{c}_i), f_i'(\hat{c}_i)\}\right\}$ |
| $i \in \mathcal{M}_1^2$ | $d_i - a_{ii} f_i'(\hat{m}_i)$ |
| $i \in \mathcal{M}_1^3$ | $d_i - a_{ii}\max\{f_i'(\check{m}_i), f_i'(\hat{m}_i)\}$ |
| $i \in \mathcal{M}_2$ | $d_i - a_{ii} L_i$ |
| $i \in \mathcal{M}_3^1$ | $\min\left\{a_{ii}\max\{f_i'(\hat{b}_i), f_i'(\breve{b}_i)\} - d_i, d_i - a_{ii}\min\{f_i'(\hat{a}_i), f_i'(\breve{a}_i), f_i'(\breve{c}_i)\}\right\}$ |
| $i \in \mathcal{M}_3^2$ | $d_i - a_{ii}\min\{f_i'(\hat{m}_i), f_i'(\check{m}_i)\}$ |
| $i \in \mathcal{M}_3^3$ | $d_i - a_{ii} f_i'(\check{m}_i)$ |
| $i \in \mathcal{M}_4$ | $d_i + a_{ii} L_i$ |

**Theorem 1.** *If $\mathcal{M}_1^1 \cup \mathcal{M}_1^2 \cup \mathcal{M}_1^3 \cup \mathcal{M}_3^1 \cup \mathcal{M}_3^2 \cup \mathcal{M}_3^3 \cup \mathcal{M}_2 \cup \mathcal{M}_4 = \mathcal{N}$, and $k = card(\mathcal{M}_1^1 \cup \mathcal{M}_3^1)$, then there exist $3^k$ equilibrium points for FOCNNs (5) with Gaussian activation functions (6).*

**Proof.** We consider $3^k$ disjoint closed regions in $\Re^n$. Let $\mathbf{w} = (w_1, \ldots, w_n)$, with $w_i =$ "$l$", "$m$", "$r$" if $i \in \mathcal{M}_1^1 \cup \mathcal{M}_3^1$, $w_i =$ "$s$" if $i \in \mathcal{M}_1^2 \cup \mathcal{M}_1^3 \cup \mathcal{M}_3^2 \cup \mathcal{M}_3^3 \cup \mathcal{M}_2 \cup \mathcal{M}_4$. Set $\Omega^{\mathbf{w}} = \{(x_1, \ldots, x_n)^T \in \Re^n | x_i \in \Omega_i^{w_i}\}$, where $\Omega_i^l = [\breve{a}_i, \hat{a}_i]$, $\Omega_i^m = [\hat{b}_i, \breve{b}_i]$, $\Omega_i^r = [\breve{c}_i, \hat{c}_i]$ and $\Omega_i^s = [\check{m}_i, \hat{m}_i]$ are compact intervals, as shown in Fig. 3. Let us take $\widetilde{\Omega}^{\mathbf{w}}$ as any one of the $3^k$ regions. For any given $(x_1, \ldots, x_n)^T \in \widetilde{\Omega}^{\mathbf{w}}$, define

$$F_i(r) = -d_i r + a_{ii} f_i(r) + \sum_{j=1, j\neq i}^{n} a_{ij} f_j(x_j) + \alpha_i^{-1} B_i \beta_i f_i(x_i) + I_i.$$

Note that $F_i(r)$ is a vertical shift of $g_i(r)$ and lies between $\breve{g}_i(r)$ and $\hat{g}_i(r)$. Therefore, if $i \in \mathcal{M}_1^1 \cup \mathcal{M}_3^1$, we can always find three zeros of $F_i(r)$, which lie in $[\breve{a}_i, \hat{a}_i]$, $[\hat{b}_i, \breve{b}_i]$ and $[\breve{c}_i, \hat{c}_i]$, respectively. If $i \in \mathcal{M}_1^2 \cup \mathcal{M}_1^3 \cup \mathcal{M}_3^2 \cup \mathcal{M}_3^3 \cup \mathcal{M}_2 \cup \mathcal{M}_4$, there exists one zero of $F_i(r)$ which lies in $[\check{m}_i, \hat{m}_i]$.

Define a mapping $\Phi : \widetilde{\Omega}^{\mathbf{w}} \to \widetilde{\Omega}^{\mathbf{w}}$, $\Phi(x_1, \ldots, x_n) = (\bar{x}_1, \ldots, \bar{x}_n)$, where $\bar{x}_i$ is the zero solution of $F_i(r)$ lying in $\widetilde{\Omega}_i^{w_i}$. The mapping $\Phi$ is continuous, since each $f_i(\cdot)$ is continuous. Based on Brouwer fixed point theorem, there exists one fixed point $x^* = (x_1^*, \ldots, x_n^*)^T$ of $\Phi$ in $\widetilde{\Omega}^{\mathbf{w}}$. Thus, we obtain an equilibrium point $(x^{*T}, S^{*T})^T = (x_1^*, \ldots, x_n^*, \alpha_1^{-1}\beta_1 f_1(x_1^*), \ldots, \alpha_n^{-1}\beta_n f_n(x_n^*))^T$ of FOCNNs (5) with Gaussian activation functions (6) in $\widetilde{\Omega}^{\mathbf{w}} \times \prod_{i=1}^n [0, \alpha_i^{-1}\beta_i]$. By arbitrariness of $\widetilde{\Omega}^{\mathbf{w}}$, we conclude that FOCNNs (5) with Gaussian activation functions (6) can have $3^k$ equilibrium points located in $\Omega^{\mathbf{w}} \times \prod_{i=1}^n [0, \alpha_i^{-1}\beta_i]$. $\blacksquare$

Next, we further apply the contraction mapping theorem to assert the exact existence of $3^k$ equilibrium points for FOCNNs (5), under additional conditions.

**Theorem 2.** *Assume that the conditions in Theorem 1 hold. Furthermore, if the following conditions hold for all $i \in \mathcal{N}$:*

$$\sum_{j=1, j\neq i}^{n} |a_{ij}| L_j + \alpha_i^{-1} |B_i| \beta_i L_i < \bar{L}_i, \tag{9}$$

*where $\bar{L}_i$ is shown in Table 1, then there exist exactly $3^k$ equilibrium points in $\Re^n \times \prod_{i=1}^n [0, \alpha_i^{-1}\beta_i]$ for FOCNNs (5) with Gaussian activation functions (6).*

**Proof.** According to Theorem 1, there exist $3^k$ equilibrium points for FOCNNs (5) with Gaussian activation functions (6). For each $\widetilde{\Omega}^w$, we will show the mapping $\Phi$ in Theorem 1 is a contracting mapping, hence there exists exactly one equilibrium point in $\widetilde{\Omega}^{\mathbf{w}} \times \prod_{i=1}^n [0, \alpha_i^{-1}\beta_i]$. Assume that $\Phi(x) = \bar{x}, \Phi(y) = \bar{y}, \forall x, y \in \widetilde{\Omega}^{\mathbf{w}}$. That

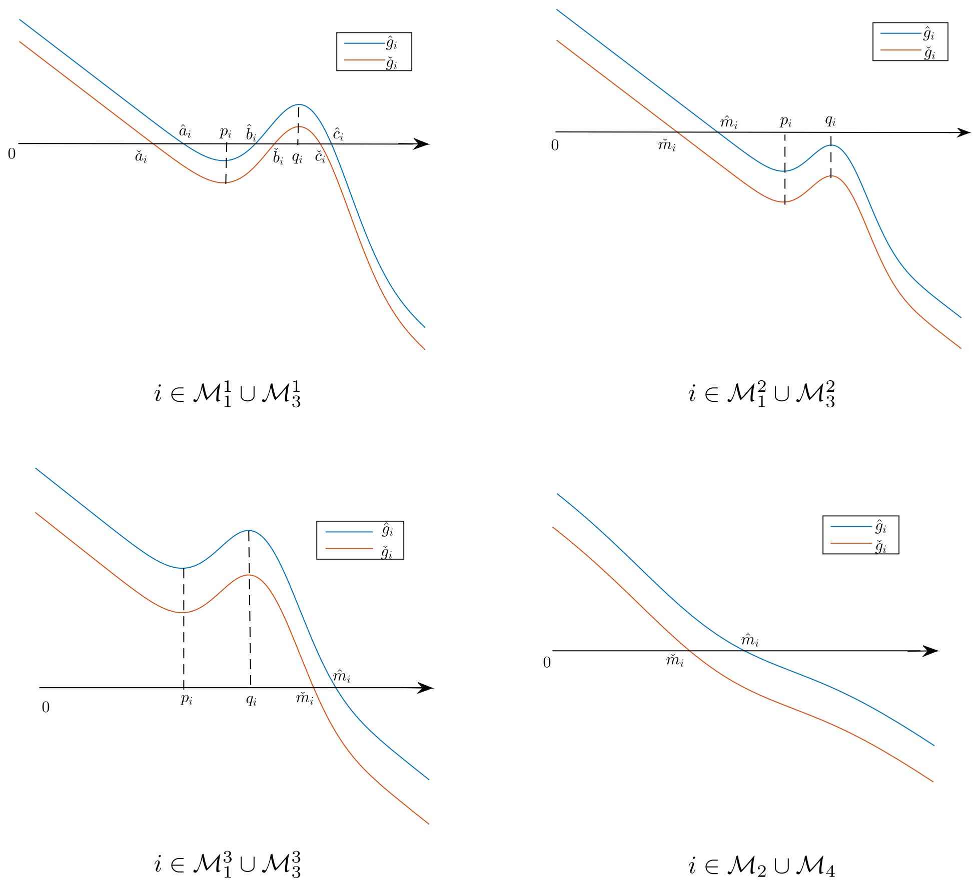



**Fig. 3.** Graphs of $\check{g}_i$ and $\hat{g}_i$ for different subsets of $\mathcal{N}$.

is, for each $i \in \mathcal{N}$, we get

$$\begin{cases} -d_i\,\bar{x}_i + a_{ii}\,f_i(\bar{x}_i) + \sum_{j=1,j\neq i}^{n} a_{ij}f_j(x_j) + \alpha_i^{-1}\,B_i\,\beta_i\,f_i(x_i) + I_i = 0, \\ -d_i\,\bar{y}_i + a_{ii}\,f_i(\bar{y}_i) + \sum_{j=1,j\neq i}^{n} a_{ij}f_j(y_j) + \alpha_i^{-1}\,B_i\,\beta_i\,f_i(y_i) + I_i = 0. \end{cases}$$

Subtracting one equation from the other, we have

$$\begin{aligned} &|-d_i(\bar{x}_i - \bar{y}_i) + a_{ii}\,(f_i(\bar{x}_i) - f_i(\bar{y}_i))| \\ &= \left|d_i - a_{ii}\,f_i'(\xi_i^*)\right|\,|\bar{x}_i - \bar{y}_i| \\ &= \left|\sum_{j=1,j\neq i}^{n} a_{ij}(f_j(x_j) - f_j(y_j)) + \alpha_i^{-1}\,B_i\,\beta_i(f_i(x_i) - f_i(y_i))\right| \\ &\leq \sum_{j=1,j\neq i}^{n} \left|a_{ij}\right|\,L_j\,\left|x_j - y_j\right| + \alpha_i^{-1}\,|B_i|\,\beta_i\,L_i\,\left|x_i - y_i\right|, \end{aligned}$$

where $\xi_i^*$ is some number lying between $\bar{x}_i$ and $\bar{y}_i$. Let us divide the discussion into eight cases.

Case 1: If $i \in \mathcal{M}_1^1$ and $\widetilde{\Omega}_i^{w_i} = [\hat{b}_i, \check{b}_i]$, then $f_i'(\xi_i^*) > d_i/a_{ii}$. Noting $a_{ii} > 0$ and $\max_{x\in\Re} f_i'(x) = L_i \geq f_i'(\xi_i^*) \geq \min\left(f_i'(\hat{b}_i), f_i'(\check{b}_i)\right)$, we get from the definition of $\bar{L}_i$

$$\left|d_i - a_{ii}\,f_i'(\xi_i^*)\right| = a_{ii}\,f_i'(\xi_i^*) - d_i \geq a_{ii}\min\left\{f_i'(\hat{b}_i), f_i'(\check{b}_i)\right\} - d_i \geq \bar{L}_i.$$

Similarly, if $i \in \mathcal{M}_1^1$ and $\widetilde{\Omega}_i^{w_i} = [\check{a}_i, \hat{a}_i]$ or $[\check{c}_i, \hat{c}_i]$, then $f_i'(\xi_i^*) < d_i/a_{ii}$ and $a_{ii} > 0$. It follows from $-L_i \leq f_i'(\xi_i^*) \leq \max\left\{f_i'(\hat{a}_i), f_i'(\check{c}_i), f_i'(\hat{c}_i)\right\}$ and the definition of $\bar{L}_i$ that

$$\begin{aligned} &|d_i - a_{ii}\,f_i'(\xi_i^*)| = d_i - a_{ii}\,f_i'(\xi_i^*) \\ &\geq d_i - a_{ii}\max\left\{f_i'(\hat{a}_i), f_i'(\check{c}_i), f_i'(\hat{c}_i)\right\} \geq \bar{L}_i. \end{aligned}$$

Case 2: If $i \in \mathcal{M}_1^2$, then $0 < f_i'(\xi_i^*) < d_i/a_{ii}$. Combining $a_{ii} > 0$ and $f_i'(\check{m}_i) \leq f_i'(\xi_i^*) \leq f_i'(\hat{m}_i)$ together leads to

$$|d_i - a_{ii}\,f_i'(\xi_i^*)| = d_i - a_{ii}\,f_i'(\xi_i^*) \geq d_i - a_{ii}\,f_i'(\hat{m}_i) = \bar{L}_i.$$

Case 3: If $i \in \mathcal{M}_1^3$, then $f_i'(\xi_i^*) < d_i/a_{ii}$ and $a_{ii} > 0$. By virtue of $-L_i \leq f_i'(\xi_i^*) \leq \max\{f_i'(\check{m}_i), f_i'(\hat{m}_i)\}$, we get

$$|d_i - a_{ii}\,f_i'(\xi_i^*)| = d_i - a_{ii}\,f_i'(\xi_i^*) \geq d_i - a_{ii}\,\max\left\{f_i'(\check{m}_i), f_i'(\hat{m}_i)\right\} = \bar{L}_i.$$

Case 4: If $i \in \mathcal{M}_2$, $d_i/a_{ii} > L_i$ and $a_{ii} > 0$. Then we have

$$|d_i - a_{ii}\,f_i'(\xi_i^*)| = d_i - a_{ii}\,f_i'(\xi_i^*) \geq d_i - a_{ii}\,L_i = \bar{L}_i.$$

Case 5: If $i \in \mathcal{M}_3^1$ and $\widetilde{\Omega}_i^{w_i} = [\hat{b}_i, \check{b}_i]$, then $f_i'(\xi_i^*) < d_i/a_{ii}$. Considering that $a_{ii} < 0$ and $-L_i \leq f_i'(\xi_i^*) \leq \max\{f_i'(\hat{b}_i), f_i'(\check{b}_i)\}$, we have

$$\left|d_i - a_{ii}\,f_i'(\xi_i^*)\right| = a_{ii}\,f_i'(\xi_i^*) - d_i \geq a_{ii}\,\max\left\{f_i'(\hat{b}_i), f_i'(\check{b}_i)\right\} - d_i \geq \bar{L}_i.$$

Similarly, if $i \in \mathcal{M}_3^1$ and $\widetilde{\Omega}_i^{w_i} = [\check{a}_i, \hat{a}_i]$ or $[\check{c}_i, \hat{c}_i]$, then $f_i'(\xi_i^*) > d_i/a_{ii}$ and $a_{ii} < 0$. It follows from $L_i \geq f_i'(\xi_i^*) \geq \min\{f_i'(\check{a}_i), f_i'(\hat{a}_i), f_i'(\check{c}_i)\}$ that

$$\begin{aligned} &\left|d_i - a_{ii}\,f_i'(\xi_i^*)\right| = d_i - a_{ii}\,f_i'(\xi_i^*) \\ &\geq d_i - a_{ii}\,\min\left\{f_i'(\check{a}_i), f_i'(\hat{a}_i), f_i'(\check{c}_i)\right\} \geq \bar{L}_i. \end{aligned}$$

Case 6: If $i \in \mathcal{M}_3^2$, then $f_i'(\xi_i^*) > d_i/a_{ii}$. From $a_{ii} < 0$ and $L_i \geq f_i'(\xi_i^*) \geq \min\{f_i'(\check{m}_i), f_i'(\hat{m}_i)\}$, we obtain

$$\left|d_i - a_{ii}\,f_i'(\xi_i^*)\right| = d_i - a_{ii}\,f_i'(\xi_i^*) \geq d_i - a_{ii}\,\min\left\{f_i'(\check{m}_i), f_i'(\hat{m}_i)\right\} = \bar{L}_i.$$

**Table 2**
$\Phi_i$ for different $i \in \mathcal{N}$.

| $i$ | $\Phi_i$ |
|---|---|
| $i \in \mathcal{M}_1^1 \cup \mathcal{M}_3^1$ | $\{(\check{a}_i - \epsilon, \hat{a}_i + \epsilon),\ (\check{c}_i - \epsilon, \hat{c}_i + \epsilon)\}$ |
| $i \in \mathcal{M}_1^2 \cup \mathcal{M}_1^3 \cup \mathcal{M}_3^2 \cup \mathcal{M}_3^3 \cup \mathcal{M}_2 \cup \mathcal{M}_4$ | $\{(\check{m}_i - \epsilon, \hat{m}_i + \epsilon)\}$ |

Case 7: If $i \in \mathcal{M}_3^3$ , then $f_i'(\xi_i^*) > d_i/a_{ii}$. It follows from $a_{ii} < 0$ and $f_i'(\hat{m}_i) \geq f_i'(\xi_i^*) \geq f_i'(\check{m}_i)$ that

$$\left|d_i - a_{ii} f_i'(\xi_i^*)\right| = d_i - a_{ii} f_i'(\xi_i^*) \geq d_i - a_{ii} f_i'(\check{m}_i) = \bar{L}_i.$$

Case 8: If $i \in \mathcal{M}_4$ , $d_i/a_{ii} < -L_i$ and $a_{ii} < 0$. Then we get

$$\left|d_i - a_{ii} f_i'(\xi_i^*)\right| = d_i - a_{ii} f_i'(\xi_i^*) \geq d_i + a_{ii} L_i = \bar{L}_i.$$

Summarizing Cases 1–8, we obtain

$$\begin{aligned}|\Phi_i(x) - \Phi_i(y)| &= |\bar{x}_i - \bar{y}_i| \\ &\leq \frac{\sum_{j=1,j\neq i}^n |a_{ij}|L_j + \alpha_i^{-1}\,|B_i|\,\beta_i\,L_i}{\bar{L}_i} \|x-y\|_\infty,\ \ i \in \mathcal{N},\end{aligned}$$

where $\|x-y\|_\infty = \max_{1\leq i\leq n}|x_i - y_i|$.

Let

$$\alpha = \max_{1\leq i\leq n}\left\{\bar{L}_i^{-1}\left(\sum_{j=1,j\neq i}^{n} |a_{ij}|L_j + \alpha_i^{-1}\,|B_i|\,\beta_i\,L_i\right)\right\},$$

it is clear that $\alpha < 1$, due to the condition (9). Then we get

$$\|\Phi(x) - \Phi(y)\|_\infty \leq \alpha\,\|x-y\|_\infty. \tag{10}$$

That is, $\Phi$ defined in each $\widetilde{\Omega}^{\mathbf{w}}$ is a contracting mapping. Consequently, there exists exactly one equilibrium point in $\widetilde{\Omega}^{\mathbf{w}} \times \prod_{i=1}^n[0, \alpha_i^{-1}\beta_i]$. On the other hand, if $(x^{*T}, S^{*T})^T$ is an equilibrium point of system (5), then its components satisfy (7), and thus must lie in some $\widetilde{\Omega}^{\mathbf{w}} \times \prod_{i=1}^n[0, \alpha_i^{-1}\beta_i]$. Therefore, FOCNNs (5) with Gaussian activation functions (6) admit exactly $3^k$ equilibrium points located in $\Re^n \times \prod_{i=1}^n[0, \alpha_i^{-1}\beta_i]$. ■

### 3.2. Multiple Mittag-Leffler stability

In this subsection, we will investigate the dynamical behaviors of $3^k$ equilibrium points for FOCNNs (5) with Gaussian activation functions (6). For this purpose, we need to give some positively invariant sets for system (5), which will establish a foundation of multiple Mittag-Leffler stability.

Denote

$$\epsilon_i = \begin{cases}\min\{p_i - \hat{a}_i, \check{c}_i - q_i\}, & i \in \mathcal{M}_1^1 \cup \mathcal{M}_3^1;\\ p_i - \hat{m}_i, & i \in \mathcal{M}_1^2 \cup \mathcal{M}_3^2;\\ \check{m}_i - q_i, & i \in \mathcal{M}_1^3 \cup \mathcal{M}_3^3;\\ \dfrac{1}{2}, & i \in \mathcal{M}_2 \cup \mathcal{M}_4.\end{cases}$$

Then let $\epsilon$ be a small positive number such that $0 < \epsilon < \min_{1\leq i\leq n}\{\epsilon_i\}$. Denote

$$\Phi = \left\{\prod_{i=1}^n (\lambda_i, \mu_i) | (\lambda_i, \mu_i) \in \Phi_i\right\}, \tag{11}$$

where $\Phi_i$ is shown in Table 2. It is easy to check that for any element $(\lambda_i, \mu_i) \in \Phi_i, \breve{g}_i(\lambda_i) > 0,\ \hat{g}_i(\mu_i) < 0$.

**Theorem 3.** *Assume that $\mathcal{M}_1^1 \cup \mathcal{M}_1^2 \cup \mathcal{M}_1^3 \cup \mathcal{M}_3^1 \cup \mathcal{M}_3^2 \cup \mathcal{M}_3^3 \cup \mathcal{M}_2 \cup \mathcal{M}_4 = \mathcal{N}$. Then $\Phi \times \prod_{i=1}^n(0, \alpha_i^{-1}\beta_i)$ are positively invariant sets of FOCNNs (5) with Gaussian activation functions (6).*

**Proof.** First of all, note that $0 < f_i \leq 1$, it follows from the second equation of FOCNNs (5) that

$$-\alpha_i\,S_i(t) \leq D^q S_i(t) \leq -\alpha_i\,S_i(t) + \beta_i.$$

According to Lemma 2, $S_i(0) \in (0, \alpha_i^{-1}\beta_i)$ always implies that $S_i(t) \in (0, \alpha_i^{-1}\beta_i)$ for $t \geq 0$. That is, if $S(0) \in \prod_{i=1}^n(0, \alpha_i^{-1}\beta_i)$, then the solution $S(t)$ will always stay in $\prod_{i=1}^n(0, \alpha_i^{-1}\beta_i)$ for all $t \geq 0$.

Select a region arbitrarily from $\Phi$ as $\widetilde{\Phi} = \prod_{i=1}^n(\lambda_i, \mu_i)$. We will show that if $x(0) \in \widetilde{\Phi}$, then the corresponding solution $x(t) \in \widetilde{\Phi}$ for all $t \geq 0$. If it is not true, then there exist some index $i \in \mathcal{N}$ and $t_1 > 0$, such that

$$\begin{cases}x_i(t_1) = \mu_i,\\ x_i(t) \leq \mu_i,\ \ 0 \leq t < t_1.\end{cases}$$

or

$$\begin{cases}x_i(t_1) = \lambda_i,\\ x_i(t) \geq \lambda_i,\ \ 0 \leq t < t_1.\end{cases}$$

For the first case, let $y_i(t) = x_i(t) - \mu_i$, then

$$\begin{cases}y_i(t_1) = 0,\\ y_i(t) \leq 0,\ \ 0 \leq t < t_1.\end{cases}$$

According to Lemma 1, $D^q y_i(t_1) \geq 0$. One the other hand, $D^q y_i(t_1) = D^q x_i(t_1) \leq \hat{g}_i(\mu_i) < 0$, which is a contradiction. For the second case, let $\tilde{y}_i(t) = \lambda_i - x_i(t)$, then

$$\begin{cases}\tilde{y}_i(t_1) = 0,\\ \tilde{y}_i(t) \leq 0,\ \ 0 \leq t < t_1.\end{cases}$$

Thus, we get that $D^q\tilde{y}_i(t_1) \geq 0$, owing to Lemma 1. However, $D^q\tilde{y}_i(t_1) = -D^q x_i(t_1) \leq -\breve{g}_i(\lambda_i) < 0$, which is also a contradiction. Consequently, $x_i(0) \in (\lambda_i, \mu_i)$ implies $x_i(t) \in (\lambda_i, \mu_i)$ for all $t \geq 0$, $i = 1, \ldots, n$.

From the analysis of the above two cases, we know that if the initial condition $(x^T(0), S^T(0))^T \in \widetilde{\Phi} \times \prod_{i=1}^n(0, \alpha_i^{-1}\beta_i)$, the corresponding solution $(x^T(t), S^T(t))^T \in \widetilde{\Phi} \times \prod_{i=1}^n(0, \alpha_i^{-1}\beta_i)$ for all $t \geq 0$. Hence, $\widetilde{\Phi} \times \prod_{i=1}^n(0, \alpha_i^{-1}\beta_i)$ is an invariant set of model (5), and the proof is completed. ■

Below, we are ready to prove that the equilibrium points located in the positively invariant sets are locally Mittag-Leffler stable. Let $(x^T(t), S^T(t))^T = (x_1(t), \ldots, x_n(t), S_1(t), \ldots, S_n(t))^T$ be a solution of FOCNNs (5) with initial condition $(x^T(0), S^T(0))^T = (x_1(0), \ldots, x_n(0), S_1(0), \ldots, S_n(0))^T$ located in $\widetilde{\Phi} \times \prod_{i=1}^n(0, \alpha_i^{-1}\beta_i)$, where $\widetilde{\Phi}$ is a region selected arbitrarily from the set $\Phi$. And $(x^{*T}, S^{*T})^T = (x_1^*, \ldots, x_n^*, S_1^*, \ldots, S_n^*)^T$ is the equilibrium point located in the subset $\widetilde{\Phi} \times \prod_{i=1}^n(0, \alpha_i^{-1}\beta_i)$. It follows from model (5) that

$$\begin{cases}D^q(x_i(t) - x_i^*) = -d_i(x_i(t) - x_i^*) + a_{ii}(f_i(x_i(t)) - f_i(x_i^*))\\ \qquad\qquad + \displaystyle\sum_{j=1,j\neq i}^n a_{ij}(f_j(x_j(t)) - f_j(x_j^*)) + B_i(S_i(t) - S_i^*),\\ D^q(S_i(t) - S_i^*) = -\alpha_i(S_i(t) - S_i^*) + \beta_i(f_i(x_i(t)) - f_i(x_i^*)).\end{cases} \tag{12}$$

According to Lemma 3 and (12), we obtain

$$\begin{aligned}D^q|x_i(t) - x_i^*| &\leq \mathrm{sign}(x_i(t) - x_i^*)D^q(x_i(t) - x_i^*)\\ &= \mathrm{sign}(x_i(t) - x_i^*)\Big[-d_i(x_i(t) - x_i^*) + a_{ii}(f_i(x_i(t)) - f_i(x_i^*))\\ &\quad + \sum_{j=1,j\neq i}^n a_{ij}(f_j(x_j(t)) - f_j(x_j^*)) + B_i(S_i(t) - S_i^*)\Big].\end{aligned} \tag{13}$$

**Table 3**
$\bar{\eta}_i$ for different $i \in \mathcal{N}$.

| $i$ | $\bar{\eta}_i$ |
|---|---|
| $i \in \mathcal{M}_1^1$ | $\max\left\{f_i'(\hat{a}_i+\epsilon), f_i'(\check{c}_i-\epsilon), f_i'(\hat{c}_i+\epsilon)\right\}$ |
| $i \in \mathcal{M}_1^2$ | $f_i'(\hat{m}_i+\epsilon)$ |
| $i \in \mathcal{M}_1^3$ | $\max\left\{f_i'(\check{m}_i-\epsilon), f_i'(\hat{m}_i+\epsilon)\right\}$ |
| $i \in \mathcal{M}_2$ | $\sqrt{2}\exp(-\frac{1}{2})/\rho_i$ |
| $i \in \mathcal{M}_3^1$ | $\min\left\{f_i'(\check{a}_i-\epsilon), f_i'(\hat{a}_i+\epsilon), f_i'(\check{c}_i-\epsilon)\right\}$ |
| $i \in \mathcal{M}_3^2$ | $\min\left\{f_i'(\check{m}_i-\epsilon), f_i'(\hat{m}_i+\epsilon)\right\}$ |
| $i \in \mathcal{M}_3^3$ | $f_i'(\check{m}_i-\epsilon)$ |
| $i \in \mathcal{M}_4$ | $-\sqrt{2}\exp(-\frac{1}{2})/\rho_i$ |

Note that

$$\begin{aligned} &\operatorname{sign}(x_i(t)-x_i^*)\, a_{ii}\,(f_i(x_i(t))-f_i(x_i^*)) \\ &= \operatorname{sign}(x_i(t)-x_i^*)\, a_{ii}\,\frac{f_i(x_i(t))-f_i(x_i^*)}{x_i(t)-x_i^*}(x_i(t)-x_i^*) \\ &= a_{ii}\,|x_i(t)-x_i^*|\,\frac{f_i(x_i(t))-f_i(x_i^*)}{x_i(t)-x_i^*} \\ &\le a_{ii}\,\bar{\eta}_i\,|x_i(t)-x_i^*|, \end{aligned} \tag{14}$$

where $\bar{\eta}_i$ is shown in Table 3. Consequently, we get

$$\begin{aligned} D^q|x_i(t)-x_i^*| &\le (-d_i+a_{ii}\bar{\eta}_i)|x_i(t)-x_i^*| \\ &\quad + \sum_{j=1,j\neq i}^{n} |a_{ij}|L_j|x_j(t)-x_j^*| + |B_i|\,|S_i(t)-S_i^*|. \end{aligned} \tag{15}$$

Similarly, we have

$$\begin{aligned} D^q|S_i(t)-S_i^*| &\le \operatorname{sign}(S_i(t)-S_i^*)D^q(S_i(t)-S_i^*) \\ &\le -\alpha_i|S_i(t)-S_i^*| + \beta_i\,L_i|x_i(t)-x_i^*|. \end{aligned} \tag{16}$$

**Theorem 4.** *Assume that the conditions in Theorems 1 and 2 hold. Furthermore, if there exist positive constants $\xi_1, \ \xi_2, \dots, \xi_{2n}$ such that*

$$\begin{cases} \xi_i(-d_i+a_{ii}\,\bar{\eta}_i) + \sum_{j=1,j\neq i}^{n} \xi_j|a_{ij}|L_j + \xi_{n+i}|B_i| < 0, \\ -\alpha_i\,\xi_{n+i} + \xi_i\,\beta_i\,L_i < 0 \end{cases} \tag{17}$$

*hold for all $i \in \mathcal{N}$, then FOCNNs (5) with Gaussian activation functions (6) have exactly $3^k$ equilibrium points and $2^k$ of them are locally Mittag-Leffler stable, where $k = card(\mathcal{M}_1^1 \cup \mathcal{M}_3^1)$.*

**Proof.** First of all, according to Theorems 1 and 2, there exist exactly $3^k$ equilibrium points for FOCNNs (5) with Gaussian activation functions (6). In the following, we shall prove that the $2^k$ equilibrium points located in positively invariant sets are locally Mittag-Leffler stable.

Let $\varphi_i(t) = \xi_i^{-1}|x_i(t)-x_i^*|, \ \ \psi_i(t) = \xi_{n+i}^{-1}|S_i(t)-S_i^*|$ and define

$$V(t) = \max_{1\le i\le n}\{\varphi_i(t), \psi_i(t)\} = \|(x^T(t), S^T(t))^T - (x^{*T}, S^{*T})^T\|_{\{\xi,\infty\}}, \tag{18}$$

where $\xi = (\xi_1, \dots, \xi_n, \xi_{n+1}, \dots, \xi_{2n})^T \in \Re^{2n}$, $\xi_i > 0$, $i = 1, 2, \dots, 2n$.

Let $c$ be as in Box I. In the following, we will prove that

$$D^qV(t) \le -c\,V(t) \tag{19}$$

holds almost everywhere. In fact, for any $t > 0$, there exists some index $k \in \{1, \dots, n\}$, such that

$$\begin{cases} \varphi_k(t) = V(t), \\ \varphi_j(s) \le V(s), \quad s \in [0,t), j = 1, \dots, n. \\ \psi_j(s) \le V(s), \quad s \in [0,t), j = 1, \dots, n. \end{cases} \tag{20}$$

or

$$\begin{cases} \psi_k(t) = V(t), \\ \psi_j(s) \le V(s), \quad s \in [0,t), j = 1, \dots, n. \\ \varphi_j(s) \le V(s), \quad s \in [0,t), j = 1, \dots, n. \end{cases} \tag{21}$$

For the first case, let $H(t) = \varphi_k(t) - V(t)$. Then $H(t) = 0, H(s) \le 0$ for $s \in [0,t)$. According to Lemma 1, we have $D^qH(t) = D^q\varphi_k(t) - D^qV(t) \ge 0$, which implies $D^qV(t) \le D^q\varphi_k(t)$. Hence, it follows from (15) that

$$\begin{aligned} D^qV(t) &\le \xi_k^{-1}D^q|x_k(t)-x_k^*| \\ &\le (-d_k+a_{kk}\bar{\eta}_k)\,\varphi_k(t) + \sum_{j=1,j\neq k}^{n} \xi_k^{-1}\,\xi_j\,|a_{kj}|\,L_j\,\varphi_j(t) \\ &\quad + \xi_k^{-1}\,\xi_{n+k}\,|B_k|\,\psi_k(t) \\ &\le \xi_k^{-1}\left[(-d_k+a_{kk}\,\bar{\eta}_k)\,\xi_k + \sum_{j=1,j\neq k}^{n} \xi_j\,|a_{kj}|\,L_j + \xi_{n+k}\,|B_k|\right]V(t). \end{aligned} \tag{22}$$

Similarly, for the second case, let $\widetilde{H}(t) = \psi_k(t) - V(t)$. Then $\widetilde{H}(t) = 0$, $\widetilde{H}(s) \le 0$ for $s \in [0,t)$. By virtue of Lemma 1, we derive that $D^q\widetilde{H}(t) = D^q\psi_k(t) - D^qV(t) \ge 0$, i.e., $D^qV(t) \le D^q\psi_k(t)$. From (16), we obtain that

$$\begin{aligned} D^qV(t) &\le \ \xi_{n+k}^{-1}\,D^q|S_k(t)-S_k^*| \\ &\le \ -\alpha_k\,\psi_k(t) + \xi_{n+k}^{-1}\,\xi_k\,\beta_k\,L_k\,\varphi_k(t) \\ &\le \ \xi_{n+k}^{-1}(-\alpha_k\,\xi_{n+k} + \xi_k\,\beta_k\,L_k)V(t). \end{aligned} \tag{23}$$

From (22) and (23), we can arrive at $D^qV(t) \le -c\,V(t)$, which follows that $V(t) \le V(0)E_q(-c\,t^q)$, $t > 0$, due to Lemma 4. That is,

$$\begin{aligned} &\left\|\left(x^T(t), S^T(t)\right)^T - \left(x^{*T}, S^{*T}\right)^T\right\|_{\{\xi,\infty\}} \\ &\le \left\|\left(x^T(0), S^T(0)\right)^T - \left(x^{*T}, S^{*T}\right)^T\right\|_{\{\xi,\infty\}} E_q(-c\,t^q). \end{aligned}$$

Therefore, the equilibrium point $(x^{*T}, S^{*T})^T$ located in the positively invariant set $\widetilde{\Phi} \times \prod_{i=1}^n (0, \alpha_i^{-1}\beta_i)$ is locally Mittag-Leffler stable.

By arbitrariness of $\widetilde{\Phi} \times \prod_{i=1}^n (0, \alpha_i^{-1}\beta_i)$, we conclude that each equilibrium point located in positively invariant set is locally Mittag-Leffler stable. Hence, FOCCNs (5) with Gaussian activation functions (6) have $2^k$ locally Mittag-Leffler stable equilibrium points. ■

Applying the preceding Theorem 4, we can easily obtain the next corollary.

**Corollary 1.** *Assume that $card(\mathcal{M}_1^1 \cup \mathcal{M}_3^1) = n$. Furthermore, if (9) and (17) hold for all $i \in \mathcal{N}$, then FOCCNs (5) with Gaussian activation functions (6) have exactly $3^n$ equilibrium points and $2^n$ of them are locally Mittag-Leffler stable, where*

$$\bar{L}_i = \begin{cases} \min\left\{a_{ii}\min\left(f_i'(\hat{b}_i), f_i'(\check{b}_i)\right) - d_i, \right. \\ \left. \quad d_i - a_{ii}\max\{f_i'(\hat{a}_i), f_i'(\check{c}_i), f_i'(\hat{c}_i)\}\right\}, & i \in \mathcal{M}_1^1, \\ \min\left\{a_{ii}\max\{f_i'(\hat{b}_i), f_i'(\check{b}_i)\} - d_i, \right. \\ \left. \quad d_i - a_{ii}\min\{f_i'(\hat{a}_i), f_i'(\check{a}_i), f_i'(\check{c}_i)\}\right\}, & i \in \mathcal{M}_3^1 \end{cases}$$

$$c=\frac{\min_{1\le i\le n}\left\{(d_i-a_{ii}\,\bar{\eta}_i)\,\xi_i-\sum_{j=1,j\neq i}^{n}\xi_j|a_{ij}|L_j-\xi_{n+i}|B_i|,\ \ \alpha_i\,\xi_{n+i}-\xi_i\,\beta_i\,L_i\right\}}{\max_{1\le i\le 2n}\{\xi_i\}}.$$

**Box I.**

*and*

$$\bar{\eta}_i=\begin{cases}\max\left\{f_i'(\hat{a}_i+\epsilon),f_i'(\check{c}_i-\epsilon),f_i'(\hat{c}_i+\epsilon)\right\}, & i\in\mathcal{M}_1^1,\\ \min\left\{f_i'(\check{a}_i-\epsilon),f_i'(\hat{a}_i+\epsilon),f_i'(\check{c}_i-\epsilon)\right\}, & i\in\mathcal{M}_3^1.\end{cases}$$

**Remark 3.** If $card(\mathcal{M}_1^1\cup\mathcal{M}_3^1)=0$, then it can be seen from Theorem 4 that under the corresponding conditions, the FOCCNs (5) with Gaussian activation functions (6) have a unique equilibrium point which is globally Mittag-Leffler stable. That is, Theorem 4 in this paper covers the results of mono-stability and multistability.

If $q=1$, then system (5) turns into the following classical integer-order competitive neural networks:

$$\begin{cases}\dot{x}_i(t)=-d_i\,x_i(t)+\sum_{j=1}^{n}a_{ij}\,f_j(x_j(t))+B_i\,S_i(t)+I_i,\\ \dot{S}_i(t)=-\alpha_i\,S_i(t)+\beta_i\,f_i(x_i(t)),\ \ i=1,\ldots,n.\end{cases}\tag{24}$$

For system (24), the results of Theorem 4 still hold.

**Corollary 2.** *Under the conditions of Theorem 4, the classical integer-order competitive neural networks (24) with Gaussian activation functions (6) have exactly $3^k$ equilibrium points, among which $2^k$ equilibrium points are locally exponentially stable.*

**Proof.** First of all, note that system (5) and (24) share the same equilibrium points. Moreover, by applying the arguments similar to Nie et al. (2013), Nie and Zheng (2016), we can show that the sets $\Phi\times\prod_{i=1}^{n}(0,\alpha_i^{-1}\beta_i)$ in Theorem 3 are also the invariant sets of (24). Define the same Lyapunov function $V(t)$ in (18), we can get from the proof of Theorem 4 that $D^+V(t)\le -c\,V(t)$, where $D^+V(t)=\limsup_{h\to 0^+}(V(t+h)-V(t))/h$. This follows that $V(t)\le V(0)\,e^{-c\,t}$, i.e.,

$$\begin{aligned}&\left\|\left(x^T(t),S^T(t)\right)^T-\left(x^{*T},S^{*T}\right)^T\right\|_{\{\xi,\infty\}}\\ \le&\left\|\left(x^T(0),S^T(0)\right)^T-\left(x^{*T},S^{*T}\right)^T\right\|_{\{\xi,\infty\}}e^{-c\,t}.\end{aligned}$$

Therefore, the equilibrium point $(x^{*T},S^{*T})^T$ located in the each positively invariant set is locally exponentially stable. ■

**Remark 4.** Corollary 2 implies that Theorem 4 in this paper include multistability of integer-order competitive neural networks as a special case.

**Remark 5.** The multistability analysis of integer-order competitive neural networks has been investigated in Nie and Cao (2009), Nie et al. (2013), Nie and Zheng (2016), Nie et al. (2019). Compared with the existing works, the main contributions in this paper lie in the following. (1) The model in this paper is based on fractional-order derivative, which is more general and accurate. (2) The activation functions considered in Nie and Cao (2009), Nie et al. (2013), Nie and Zheng (2016), Nie et al. (2019) include sigmoidal and nondecreasing saturated functions, nondecreasing piecewise linear functions and nonmonotonic piecewise linear functions, which are all either nondecreasing or piecewise linear. Different from these activation functions, the Gaussian functions employed in this paper are both nonmonotonic and nonlinear. (3) The number of equilibrium points in this paper is in terms of $k$-power with $0\le k\le n$, whereas the number of equilibrium points in Nie and Cao (2009), Nie et al. (2013), Nie and Zheng (2016) and Nie et al. (2019) is in terms of $n$-power. Hence, the main results in this paper include the mono-stability and multistability of both fractional-order and integer-order competitive neural networks as special cases, which improve and generalize the existing results.

**Remark 6.** Let $B_i=\beta_i=S_i(t)=0\,(i=1,2,\ldots,n)$, then systems (5) and (24) are transformed respectively into the following fractional-order Hopfield neural networks

$$D^q x_i(t)=-d_i\,x_i(t)+\sum_{j=1}^{n}a_{ij}f_j(x_j(t))+I_i\tag{25}$$

and integer-order Hopfield neural networks

$$\dot{x}_i(t)=-d_i\,x_i(t)+\sum_{j=1}^{n}a_{ij}f_j(x_j(t))+I_i.\tag{26}$$

Thus, from Theorem 4 and Corollary 2, we can obtain the corresponding multistability and mono-stability results of (25) and (26), respectively.

**Remark 7.** In Liu et al. (2017b), the multistability and complete stability were studied for integer-order Hopfield neural networks (26) with Gaussian activation functions, under the condition $a_{ii}>0$. In this paper, we investigate the multistability of FOCNNs. Moreover, we consider the cases of both $a_{ii}>0$ and $a_{ii}<0$, and the restrictive condition $a_{ii}>0$ in Liu et al. (2017b) is removed in this paper.

**Remark 8.** Besides Gaussian activation functions, Mexican hat functions are another class of both nonmonotonic and nonlinear activation functions, which have been employed in Liu et al. (2017a) for multistability analysis of integer-order delayed Hopfield neural networks. In fact, the theory and analysis in this paper can be applied to study the multistability of FOCNNs with Mexican hat activation functions.

**Remark 9.** It is well known that RBF neural network with Gaussian activation function is a good candidate for approximation problems because of its faster learning capability. However, as pointed out in Lee, Chung, Tsai, and Chang (1999), it is difficult to utilize a Gaussian function to approximate a constant valued function unless its bandwidth is very large. In order to cope with this problem, a new RBF network is proposed in Lee et al. (1999) which is based on sequences of sigmoidal functions. This new activation function is still a nonmonotonic and nonlinear function. The multistability analysis of FOCNNs with this new type of activation function will be investigated in the future.

## 4. Numerical simulations

In order to demonstrate the validity of our theoretical results, two examples of FOCNNs are presented in this section.

**Example 1.** Consider the following 3-D FOCNNs:

$$\begin{cases} D^q x_1(t) = -x_1(t) + 2.25 f_1(x_1(t)) + 0.2 f_2(x_2(t)) \\ \qquad -0.1 f_3(x_3(t)) - 0.1 S_1(t) - 1.95, \\ D^q x_2(t) = -x_2(t) - 0.2 f_1(x_1(t)) - 2.2 f_2(x_2(t)) \\ \qquad +0.1 f_3(x_3(t)) + 0.1 S_2(t) + 1.95, \\ D^q x_3(t) = -x_3(t) - 0.2 f_1(x_1(t)) + 0.1 f_2(x_2(t)) \\ \qquad -2.3 f_3(x_3(t)) + 0.05 S_3(t) + 1.95, \\ D^q S_i(t) = -S_i(t) + f_i(x_i(t)),\ i = 1, 2, 3, \end{cases} \tag{27}$$

where $q = 0.96, f(r) = f_i(r) = exp(-r^2)(i = 1, 2, 3)$. For this activation function, we have $L_i = \sqrt{2/e} \approx 0.8578$ $(i = 1, 2, 3)$. Note that $d_1/a_{11} = 4/9 < L_1, -L_2 < d_2/a_{22} = -5/11 < 0, -L_3 < d_3/a_{33} = -10/23 < 0$ , which imply that $1 \in \mathcal{M}_1, 2 \in \mathcal{M}_3$, $3 \in \mathcal{M}_3$. Then the upper and lower functions are as follows:

$$\begin{cases} \check{g}_1(r) = -r + 2.25 f(r) - 2.15, & \hat{g}_1(r) = -r + 2.25 f(r) - 1.75, \\ \check{g}_2(r) = -r - 2.2 f(r) + 1.75, & \hat{g}_2(r) = -r - 2.2 f(r) + 2.15, \\ \check{g}_3(r) = -r - 2.3 f(r) + 1.75, & \hat{g}_3(r) = -r - 2.3 f(r) + 2.1. \end{cases}$$

By some simple computations, we can easily obtain that $p_1 \approx -1.3406, q_1 \approx -0.2348, p_2 \approx 0.2408, q_2 \approx 1.3289, p_3 \approx 0.2291, q_3 \approx 1.3519, \check{a}_1 \approx -2.1254, \check{b}_1 \approx -0.6221, \check{c}_1 \approx 0.0841, \hat{a}_1 \approx -1.5400, \hat{b}_1 \approx -1.1488, \hat{c}_1 \approx 0.3028, \check{a}_2 \approx -0.2819, \check{b}_2 \approx 1.1140, \check{c}_2 \approx 1.5524, \hat{a}_2 \approx -0.0455, \hat{b}_2 \approx 0.5819, \hat{c}_2 \approx 2.1260, \check{a}_3 \approx -0.3226, \check{b}_3 \approx 1.1844, \check{c}_3 \approx 1.5258, \hat{a}_3 \approx -0.1493$ , $\hat{b}_3 \approx 0.7093$ and $\hat{c}_3 \approx 2.0681$. Moreover, it is easy to verify that $\hat{g}_1(p_1) \approx -0.0364 < 0, \check{g}_1(q_1) \approx 0.2141 > 0, \hat{g}_2(p_2) \approx -0.1669 < 0, \check{g}_2(q_2) \approx 0.0449 > 0, \hat{g}_3(p_3) \approx -0.3115 < 0$, and $\check{g}_3(q_3) \approx 0.0283 > 0$, i.e., $1 \in \mathcal{M}_1^1, 2 \in \mathcal{M}_3^1, 3 \in \mathcal{M}_3^1$, $card(\mathcal{M}_1^1 \cup \mathcal{M}_3^1) = 3$.

In addition, we can verify the conditions (9) are satisfied:

$$\bar{L}_1 = \min\left\{a_{11} \min\{f'(\hat{b}_1), f'(\check{b}_1)\} - d_1,\right.$$
$$\left. d_1 - a_{11} \max\{f'(\hat{a}_1), f'(\check{c}_1) f'(\hat{c}_1)\}\right\}$$
$$\approx 0.3531 > |a_{12}|L_2 + |a_{13}|L_3 + |B_1|L_1 \approx 0.3431,$$
$$\bar{L}_2 = \min\left\{a_{22} \max\{f'(\hat{b}_2), f'(\check{b}_2)\} - d_2,\right.$$
$$\left. d_2 - a_{22} \min\{f'(\hat{a}_2), f'(\check{a}_2), f'(\check{c}_2)\}\right\}$$
$$\approx 0.3864 > |a_{21}|L_1 + |a_{23}|L_3 + |B_2|L_2 \approx 0.3431,$$
$$\bar{L}_3 = \min\left\{a_{33} \max\{f'(\hat{b}_3), f'(\check{b}_3)\} - d_3,\right.$$
$$\left. d_3 - a_{33} \min\{f'(\hat{a}_3), f'(\check{a}_3), f'(\check{c}_3)\}\right\}$$
$$\approx 0.3158 > |a_{31}|L_1 + |a_{32}|L_2 + |B_3|L_3 \approx 0.3002.$$

Finally, we are ready to verify the conditions (17) are satisfied. We can easily get that $\epsilon_1 = \min\{p_1 - \hat{a}_1, \check{c}_1 - q_1\} \approx 0.1994$, $\epsilon_2 = \min\{p_2 - \hat{a}_2, \check{c}_2 - q_2\} \approx 0.2235, \epsilon_3 = \min\{p_3 - \hat{a}_3, \check{c}_3 - q_3\} \approx 0.1739$. Choose $\epsilon = 0.0001$, and $\xi = (1, 1, 1, 9/10, 1, 1)^T$, and then we have

$$\begin{cases} \xi_1(-d_1 + a_{11}\bar{\eta}_1) + \xi_2|a_{12}|L_2 + \xi_3|a_{13}|L_3 + \xi_4|B_1| \\ \quad \approx -0.0058 < 0, -\xi_4\alpha_1 + \xi_1\beta_1 L_1 \approx -0.0422 < 0, \\ \xi_2(-d_2 + a_{22}\bar{\eta}_2) + \xi_1|a_{21}|L_1 + \xi_3|a_{23}|L_3 + \xi_5|B_2| \\ \quad \approx -0.0291 < 0, -\xi_5\alpha_2 + \xi_2\beta_2 L_2 \approx -0.1422 < 0, \\ \xi_3(-d_3 + a_{33}\bar{\eta}_3) + \xi_1|a_{31}|L_1 + \xi_2|a_{32}|L_2 + \xi_6|B_3| \\ \quad \approx -0.0082 < 0, \\ -\xi_6\alpha_3 + \xi_3\beta_3 L_3 \approx -0.1422 < 0, \end{cases}$$

where $\bar{\eta}_1 = \max\{f'(\hat{a}_1 + \epsilon), f'(\check{c}_1 - \epsilon), f'(\hat{c}_1 + \epsilon)\} \approx 0.2875$, $\bar{\eta}_2 = \min\{f'(\check{a}_2 - \epsilon), f'(\hat{a}_2 + \epsilon), f'(\check{c}_2 - \epsilon)\} \approx -0.2789$, $\bar{\eta}_3 = \min\{f'(\check{a}_3 - \epsilon), f'(\hat{a}_3 + \epsilon), f'(\check{c}_3 - \epsilon)\} \approx -0.2976$. According to Corollary 1, FOCNNs (27) have exactly $3^3 = 27$ equilibrium points, among which $2^3 = 8$ equilibrium points are locally Mittag-Leffler stable. In fact, by virtue of Matlab, we can obtain all the 27 equilibrium points as follows:

$^{[1]}(x_1^*, x_2^*, x_3^*, S_1^*, S_2^*, S_3^*)^T \approx (-1.7464, -0.0554, -0.1558, 0.0474, 0.9969, 0.9760)^T$,

$^{[2]}(x_1^*, x_2^*, x_3^*, S_1^*, S_2^*, S_3^*)^T \approx (-1.5757, -0.1297, 1.9885, 0.0835, 0.9833, 0.019)^T$,

$^{[3]}(x_1^*, x_2^*, x_3^*, S_1^*, S_2^*, S_3^*)^T \approx (-2.0033, 2.0044, -0.2072, 0.0181, 0.0180, 0.9580)^T$,

$^{[4]}(x_1^*, x_2^*, x_3^*, S_1^*, S_2^*, S_3^*)^T \approx (-1.8859, 1.8877, 1.8820, 0.0285, 0.0283, 0.0290)^T$,

$^{[5]}(x_1^*, x_2^*, x_3^*, S_1^*, S_2^*, S_3^*)^T \approx (0.2084, -0.1805, -0.2541, 0.9575, 0.9679, 0.9375)^T$,

$^{[6]}(x_1^*, x_2^*, x_3^*, S_1^*, S_2^*, S_3^*)^T \approx (0.2525, -0.2273, 1.7533, 0.9382, 0.9496, 0.0462)^T$,

$^{[7]}(x_1^*, x_2^*, x_3^*, S_1^*, S_2^*, S_3^*)^T \approx (0.2525, -0.2273, 1.7533, 0.9382, 0.9496, 0.0462)^T$,

$^{[8]}(x_1^*, x_2^*, x_3^*, S_1^*, S_2^*, S_3^*)^T \approx (0.1556, 1.6018, 1.5729, 0.9761, 0.0769, 0.0842)^T$,

$^{[9]}(x_1^*, x_2^*, x_3^*, S_1^*, S_2^*, S_3^*)^T \approx (-1.6821, -0.0881, 0.7441, 0.0590, 0.9923, 0.5748)^T$,

$^{[10]}(x_1^*, x_2^*, x_3^*, S_1^*, S_2^*, S_3^*)^T \approx (-1.8402, 0.6323, -0.1732, 0.0338, 0.6705, 0.9704)^T$,

$^{[11]}(x_1^*, x_2^*, x_3^*, S_1^*, S_2^*, S_3^*)^T \approx (-1.7936, 0.6877, 0.7802, 0.0401, 0.6232, 0.5441)^T$,

$^{[12]}(x_1^*, x_2^*, x_3^*, S_1^*, S_2^*, S_3^*)^T \approx (-1.7323, 0.7553, 1.9455, 0.0497, 0.5653, 0.0227)^T$,

$^{[13]}(x_1^*, x_2^*, x_3^*, S_1^*, S_2^*, S_3^*)^T \approx (-1.9458, 1.9471, 0.8439, 0.0227, 0.0226, 0.4906)^T$,

$^{[14]}(x_1^*, x_2^*, x_3^*, S_1^*, S_2^*, S_3^*)^T \approx (-0.9109, -0.1140, -0.2002, 0.4362, 0.9871, 0.9607)^T$,

$^{[15]}(x_1^*, x_2^*, x_3^*, S_1^*, S_2^*, S_3^*)^T \approx (-0.9766, -0.1368, 0.8172, 0.3853, 0.9815, 0.5128)^T$,

$^{[16]}(x_1^*, x_2^*, x_3^*, S_1^*, S_2^*, S_3^*)^T \approx (-1.0632, -0.1594, 1.9283, 0.3229, 0.9749, 0.0243)^T$,

$^{[17]}(x_1^*, x_2^*, x_3^*, S_1^*, S_2^*, S_3^*)^T \approx (-0.8017, 0.7587, -0.2315, 0.5259, 0.5624, 0.9478)^T$,

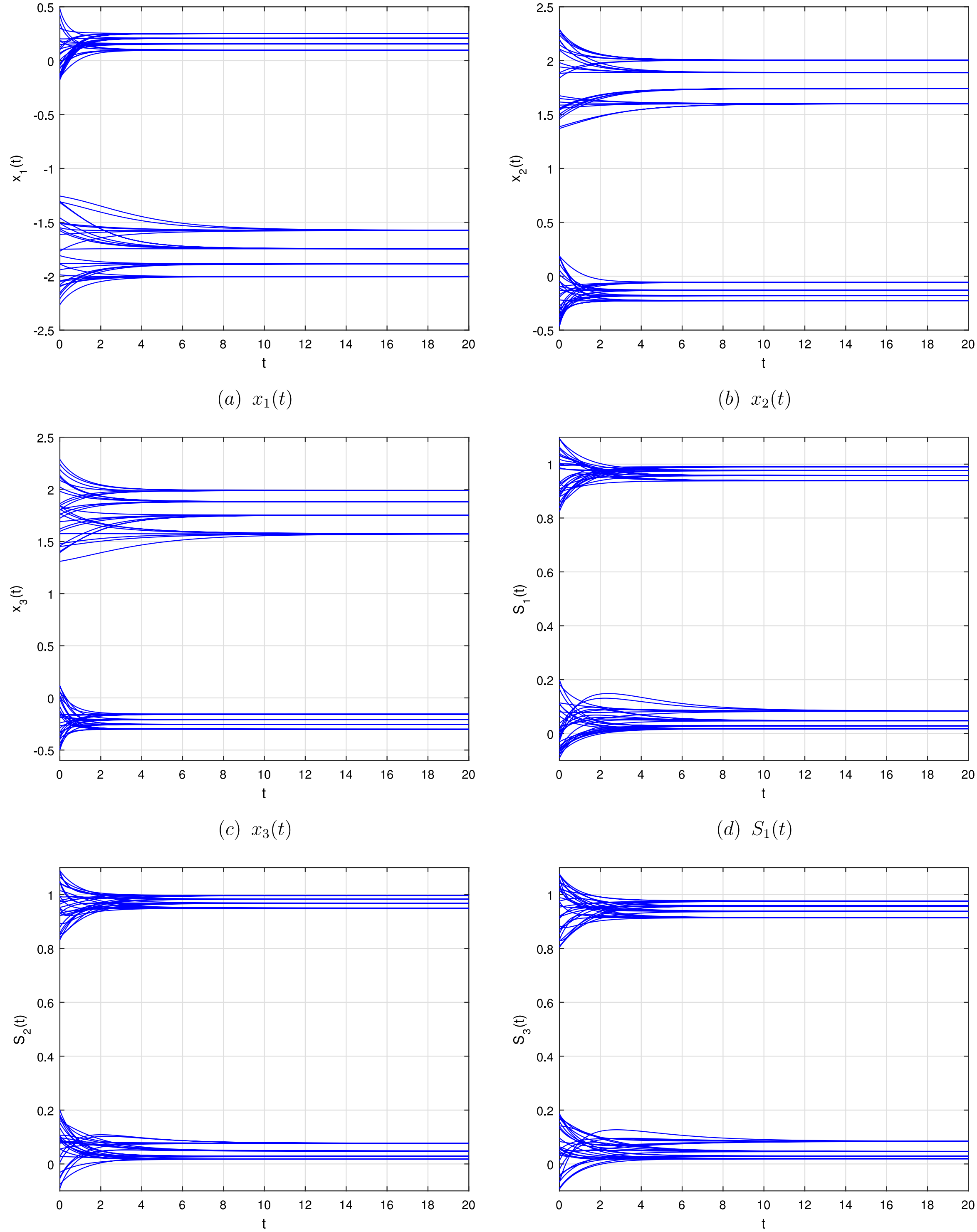
$^{[18]}(x_1^*, x_2^*, x_3^*, S_1^*, S_2^*, S_3^*)^T \approx (-0.8532, 0.8117, 0.8947, 0.48290.5174, 0.4491)^T$,

$^{[19]}(x_1^*, x_2^*, x_3^*, S_1^*, S_2^*, S_3^*)^T \approx (-0.8984, 0.8573, 1.8296, 0.4461, 0.4795, 0.0352)^T$,

$^{[20]}(x_1^*, x_2^*, x_3^*, S_1^*, S_2^*, S_3^*)^T \approx (-0.6770, 1.8475, -0.2676, 0.6323, 0.0329, 0.9309)^T$,

$^{[21]}(x_1^*, x_2^*, x_3^*, S_1^*, S_2^*, S_3^*)^T \approx (-0.7443, 1.7869, 0.9829, 0.5747, 0.0410, 0.3806)^T$,

$^{[22]}(x_1^*, x_2^*, x_3^*, S_1^*, S_2^*, S_3^*)^T \approx (-0.78561.7481, 1.7365, 0.5395, 0.0471, 0.0490)^T$,




(a) $x_1(t)$ (b) $x_2(t)$ (c) $x_3(t)$ (d) $S_1(t)$ (e) $S_2(t)$ (f) $S_3(t)$

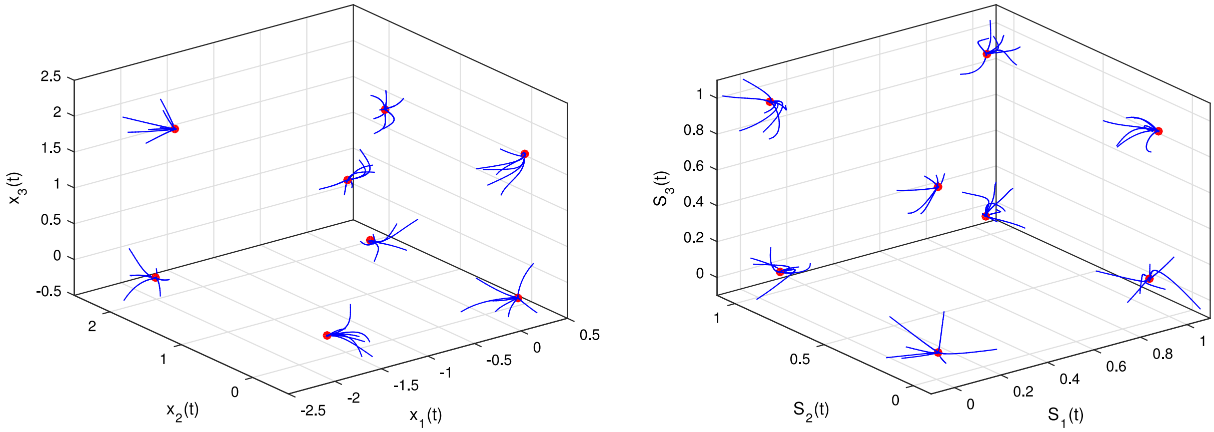
**Fig. 4.** Transient responses of state variables $x_1, x_2, x_3, S_1, S_2$ and $S_3$ in Example 1.

$$^{[23]}(x_1^*, x_2^*, x_3^*, S_1^*, S_2^*, S_3^*)^T \approx (0.2355, -0.2093, 0.9582, 0.9460, 0.9571, 0.3993)^T,$$

$$^{[24]}(x_1^*, x_2^*, x_3^*, S_1^*, S_2^*, S_3^*)^T \approx (0.15100.8815, -0.2799, 0.9775, 0.4598, 0.9246)^T,$$

$$^{[25]}(x_1^*, x_2^*, x_3^*, S_1^*, S_2^*, S_3^*)^T \approx (0.1780, 0.9706, 1.0533, 0.9688, 0.3898, 0.3297)^T,$$

$$^{[26]}(x_1^*, x_2^*, x_3^*, S_1^*, S_2^*, S_3^*)^T \approx (0.1890, 1.0164, 1.6396, 0.9649, 0.3559, 0.0680)^T,$$




$(a)\ (x_1, x_2, x_3)^T$ $(b)\ (S_1, S_2, S_3)^T$

**Fig. 5.** Phase plots of state variables $(x_1, x_2, x_3)^T$ and $(S_1, S_2, S_3)^T$ in Example 1.

$$^{[27]}(x_1^*, x_2^*, x_3^*, S_1^*, S_2^*, S_3^*)^T \approx (0.1422, 1.6396, 1.1223, 0.9800, 0.0680, 0.2838)^T.$$

Among them, $^{[i]}(x_1^*, x_2^*, x_3^*, S_1^*, S_2^*, S_3^*)^T (i = 1, 2, \ldots, 8)$ are locally Mittag-Leffler stable. Transient behaviors of FOCNNs (27) with 56 random initial values are depicted in Figs. 4–5, which show that the above eight equilibrium points $^{[i]}(x_1^*, x_2^*, x_3^*, S_1^*, S_2^*, S_3^*)^T (i = 1, 2, \ldots, 8)$ are locally Mittag-Leffler stable.

**Example 2.** Consider the following 2-D FOCNNs:

$$\begin{cases} D^q x_1(t) = -x_1(t) + 2f_1(x_1(t)) - 0.1f_2(x_2(t)) + 0.12\,S_1(t) - 1.8, \\ D^q x_2(t) = -1.5\,x_2(t) + f_2(x_2(t)) + 0.2f_1(x_1(t)) - 0.2\,S_2(t) + 1.2, \\ D^q S_1(t) = -S_1(t) + f_1(x_1(t)), \\ D^q S_2(t) = -S_2(t) + f_2(x_2(t)). \end{cases} \tag{28}$$

where $q = 0.9$, $f_i(r) = f(r) = \exp(-r^2)$ for $i = 1, 2$. For this activation function, we have $L_i = \sqrt{2/e} \approx 0.8578\,(i = 1, 2)$. It is easy to get that $d_1/a_{11} = 0.5 < L_1$, $d_2/a_{22} = 1.5 > L_2$, which imply that $1 \in \mathcal{M}_1$ and $2 \in \mathcal{M}_2$.

Then the upper and lower functions are as follows:

$$\begin{cases} \check{g}_1(r) = -r + 2f_1(r) - 1.9, & \hat{g}_1(r) = -r + 2f_1(r) - 1.68, \\ \check{g}_2(r) = -1.5\,r + f_2(r) + 1, & \hat{g}_2(r) = -1.5\,r + f_2(r) + 1.4. \end{cases}$$

By some simple computations, we can easily obtain that $p_1 \approx -1.2770$, $q_1 \approx -0.2687$, $\check{a}_1 \approx -1.8297$, $\check{b}_1 \approx -0.7353$, $\check{c}_1 \approx 0.0854$, $\hat{a}_1 \approx -1.3916$, $\hat{b}_1 \approx -1.1637$, $\hat{c}_1 \approx 0.2230$, $\check{m}_2 \approx 0.9414$, $\hat{m}_2 \approx 1.1225$. Moreover, it is easy to verify that $\hat{g}_1(p_1) < 0$ and $\check{g}_1(q_1) > 0$, i.e., $1 \in \mathcal{M}_1^1$. Hence, $card(\mathcal{M}_1^1 \cup \mathcal{M}_3^1) = 1$.

In addition, we can verify the conditions (9) are satisfied:

$$\begin{aligned} \bar{L}_1 &= \min\Big\{a_{11}\min\Big(f'(\hat{b}_1), f'(\check{b}_1)\Big) - d_1, \\ &\quad d_1 - a_{11}\max\big\{f'(\hat{a}_1), f'(\hat{c}_1), f'(\check{c}_1)\big\}\Big\} \\ &\approx 0.1974 > |a_{12}|\,L_2 + |B_1|\,L_1 \approx 0.1887, \\ \bar{L}_2 &= d_2 - a_{22}\,L_2 \approx 0.6422 > |a_{21}|\,L_1 + \alpha_2^{-1}\,|B_2|\,\beta_2\,L_2 \approx 0.3432. \end{aligned}$$

Finally, we examine the rest of conditions of Theorem 4. It is easy to get that $\epsilon_1 = \min\{p_1 - \hat{a}_1, \check{c}_1 - q_1\} \approx 0.1146$, $\epsilon_2 = 0.5$. Choose $\epsilon = 0.001$ and $\xi = (1/2, 9/20, 43/100, 9/20)^T$, then we obtain

$$\begin{cases} \xi_1(-d_1 + a_{11}\,\bar{\eta}_1) + \xi_2|a_{12}|L_2 + \xi_3|B_1| \approx -0.0076 < 0, \\ -\alpha_1\,\xi_3 + \xi_1\,\beta_1\,L_1 \approx -0.0011 < 0, \\ \xi_2(-d_2 + a_{22}\,\bar{\eta}_2) + \xi_1|a_{21}|L_1 + \xi_4|B_2| \approx -0.1132 < 0, \\ -\alpha_2\,\xi_4 + \xi_2\,\beta_2\,L_2 \approx -0.0640 < 0, \end{cases}$$

where $\bar{\eta}_1 = \max\{f'(\hat{a}_1 + \epsilon), f'(\check{c}_1 - \epsilon), f'(\hat{c}_1 + \epsilon)\} \approx 0.4022$, $\bar{\eta}_2 = \sqrt{2}\exp(-0.5) \approx 0.8578$. According to Theorem 4, FOCNNs (28) have exactly $3^1 = 3$ equilibrium points, among which $2^1 = 2$ equilibrium points are locally Mittag-Leffler stable. In fact, by using Matlab, we can obtain all the three equilibrium points as follows:

$$\begin{aligned} &^{[1]}(x_1^*, x_2^*, S_1^*, S_2^*)^T \approx (-1.7305, 1.0021, 0.0501, 0.3663)^T, \\ &^{[2]}(x_1^*, x_2^*, S_1^*, S_2^*)^T \approx (0.2035, 1.0904, 0.9594, 0.3045)^T, \\ &^{[3]}(x_1^*, x_2^*, S_1^*, S_2^*)^T \approx (-0.9132, 1.0391, 0.4343, 0.3397)^T. \end{aligned}$$

We generate 20 random initial values to track the solution trajectories by simulations. Figs. 6–7 confirm that the two equilibrium points $^{[1]}(x_1^*, x_2^*, S_1^*, S_2^*)^T$ and $^{[2]}(x_1^*, x_2^*, S_1^*, S_2^*)^T$ are locally Mittag-Leffler stable.

## 5. Conclusions

In this paper, the multistability issue has been investigated for fractional-order competitive neural networks with Gaussian activation functions. Several sufficient conditions have been derived to ascertain the exact existence of $3^k$ equilibrium points and local Mittag-Leffler stability of $2^k$ equilibrium points. Two numerical examples have been presented to validate the theoretical findings. Further study will focus on multistability and complete stability of fractional-order neural networks with time delays, where the activation functions are modeled by both non-monotonic and nonlinear functions.

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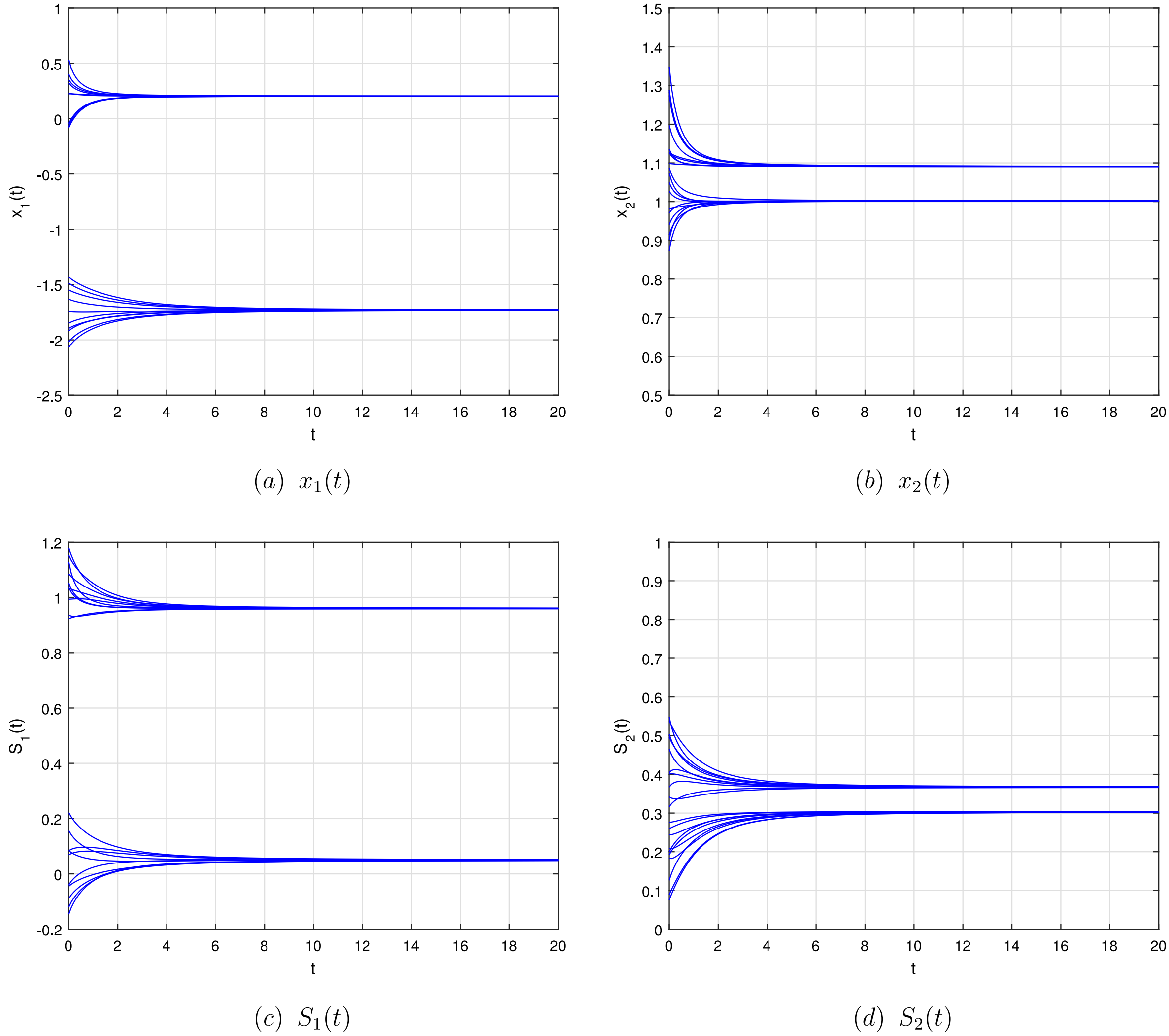


**Fig. 6.** Transient responses of state variables $x_1$, $x_2$, $S_1$ and $S_2$ in Example 2.

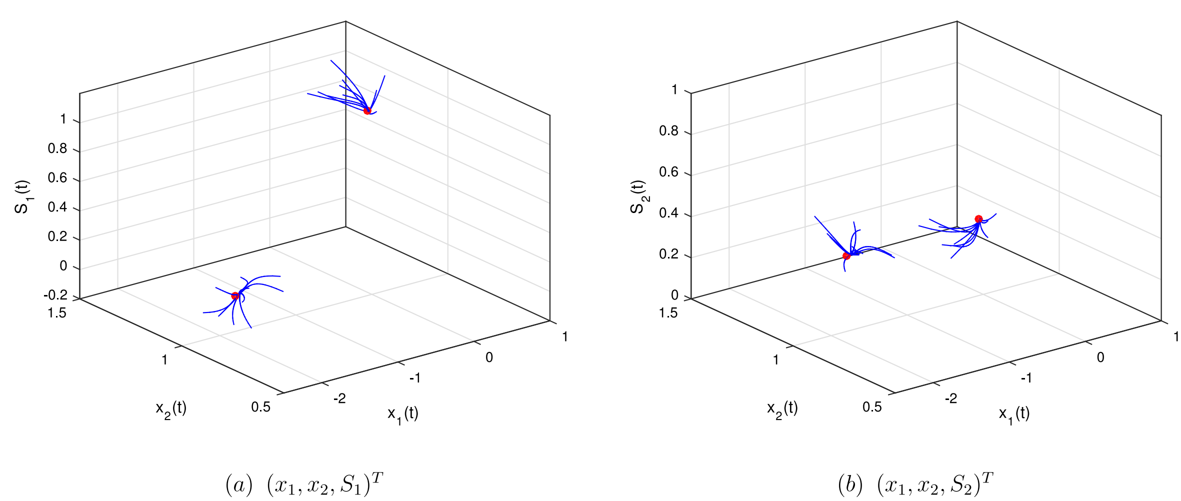


**Fig. 7.** Phase plots of state variables $(x_1, x_2, S_1)^T$ and $(x_1, x_2, S_2)^T$ in Example 2.

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