→ $Y \sim$ Binomial $(100, \theta)$

⇒ Data distribution $f(y|\theta) = \binom{100}{y} \theta^y (1-\theta)^{100-y}$, $y = 0, 1, \ldots, 100$
$0 \le \theta \le 1$

→ Derive posteriors for different priors

1) Prior → $\theta \sim$ Uniform$(0,1)$ = Beta$(1,1)$

$\pi(\theta) = 1 \quad , 0 \le \theta \le 1$

Posterior → $p(\theta|y) \propto f(y|\theta)\,\pi(\theta)$

$$\propto \theta^y (1-\theta)^{100-y} [1]$$

$$\propto \theta^{(y+1)-1} (1-\theta)^{(101-y)-1}$$

$$\sim \text{Beta}(\alpha = y+1, \beta = 101-y)$$

Posterior mean → $E(\theta|y) = \frac{\alpha}{\alpha+\beta} = \frac{y+1}{(y+1)+(101-y)} = \frac{y+1}{102}$

2) Prior → $\theta \sim$ Beta $(1, 10)$

$\pi(\theta) = \frac{\Gamma(1+10)}{\Gamma(1)\Gamma(10)} \theta^{1-1} (1-\theta)^{10-1} \quad , 0 \le \theta \le 1$

Posterior → $p(\theta|\underset{\sim}{y}) \propto f(y|\theta)\,\pi(\theta)$

$$\propto \theta^y (1-\theta)^{100-y} [\theta^0 (1-\theta)^9]$$

$$\propto \theta^{(y+1)-1} (1-\theta)^{(110-y)-1}$$

$$\sim \text{Beta}(y+1, 110-y)$$

Posterior mean → $E(\theta|y) = \frac{y+1}{(y+1)+(110-y)} = \frac{y+1}{111}$