Problem 1

$\to Y_i \overset{iid}{\sim} Exp(\lambda)$ or $f(y_i|\lambda) = \lambda e^{-\lambda y_i}$, $i = 1, \dots, 30$

→ part a → informative prior with Exp value = 2.5 & variance = 0.2

⟹ use conjugate prior & find parameters to match info

$\Rightarrow \lambda \sim Gamma(\alpha, \beta) \Rightarrow \pi(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}$

$E(\lambda) = \frac{\alpha}{\beta} = 2.5 \Rightarrow \alpha = 2.5\beta$

$V(\lambda) = \frac{\alpha}{\beta^2} = 0.2 \qquad \frac{2.5\beta}{\beta^2} = 0.2$

$\frac{2.5}{\beta} = 0.2$

$\Rightarrow \beta = 12.5 \;\&\; \alpha = 31.25$

$\Rightarrow \boxed{\lambda \sim Gamma(\alpha = 31.25, \beta = 12.5)}$

→ Part b → Posterior distribution

$\to (\lambda|\underset{\sim}{y}) \propto f(\underset{\sim}{y}|\lambda)\,\pi(\lambda)$

$= \left[\prod_{i=1}^{n} \lambda e^{-\lambda y_i}\right] \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}$

$\propto \lambda^{(n+\alpha)-1} e^{-\lambda(\beta + \Sigma y_i)}$

$\boxed{\propto Gamma(n+\alpha, \beta + \Sigma y_i)}$

$\Rightarrow$ using α, β, n & $\underset{\sim}{y}$ $\to \lambda|\underset{\sim}{y} \sim Gamma(61.25, 90.6)$

Problem 2

$\to y_i \overset{iid}{\sim} N(\mu, \sigma^2 = 1.5^2) \Rightarrow f(y_i|\mu) = \frac{1}{1.5\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y_i - \mu}{1.5}\right)^2}$

→ part a → Informative prior with average of 8 & ranges from 4 to 12

$\Rightarrow \boxed{\cancel{\mu \sim Uniform(4, 12) \Rightarrow \pi(\mu) = \frac{1}{8}}}$

~~→ part b → Posterior distribution~~

$\to p(\mu|\underset{\sim}{y}) \propto f(\underset{\sim}{y}|\mu)\,\pi(\mu)$

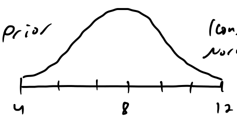
$= \left[\prod_{i=1}^{n} \frac{1}{1.5\sqrt{2\pi}} e^{-\frac{1}{2(1.5^2)}(y_i - \mu)^2}\right] \frac{1}{8}$

$\propto \exp\left\{-\frac{1}{2(1.5^2)}(n\mu^2 - 2\mu\Sigma y_i)\right\}$

$\propto$??

~~Not conjugate!!~~ ~~→ Not general form of a normal model~~

→ part a again → Prior (conjugate) Normal



range = $\mu_0 \pm 3\tau_0$ ⟹ nearly all values in range

$\Rightarrow \tau_0 = 4/3$

$\to \boxed{\mu \sim Normal(\mu_0 = 8, \tau_0 = 4/3) \Rightarrow \pi(\mu) = \frac{1}{\sqrt{2\pi \frac{4}{3}}} e^{-\frac{1}{2}\left(\frac{\mu - 8}{4/3}\right)^2}}$

→ part b again → Posterior

$p(\mu|\underset{\sim}{y}) \propto f(\underset{\sim}{y}|\mu)\,\pi(\mu)$

$= \left[\prod_{i=1}^{n} \frac{1}{1.5\sqrt{2\pi}} e^{-\frac{1}{2(1.5^2)}(y_i-\mu)^2}\right]\left[\frac{1}{\sqrt{2\pi\frac{4}{3}}} e^{-\frac{1}{2(4/3)^2}(\mu - 8)^2}\right]$

$\propto \exp\left\{-\frac{1}{2(1.5^2)}(n\mu^2 - 2\mu\Sigma y_i) - \frac{1}{2(4/3)^2}(\mu^2 - 2(8)\mu)\right]$

$= \exp\left\{-\left(\frac{n}{2(1.5)^2} + \frac{1}{2(4/3)^2}\right)\mu^2 + \left(\frac{\Sigma y_i}{(1.5)^2} + \frac{8}{(4/3)^2}\right)\mu\right\}$

→ Normal dist (rearrangement) → $f(x) \propto \exp\left\{-\frac{1}{2\sigma^2}x + \frac{\mu}{\sigma^2}x\right\}$

$= \exp\left\{-\frac{1}{2\tau_n^2}\mu^2 + \frac{\mu_n}{\tau_n^2}\mu\right\}$

$\to \begin{cases} \frac{1}{\tau_n^2} = \frac{n}{1.5^2} + \frac{1}{(4/3)^2} \\ \frac{\mu_n}{\tau_n^2} = \frac{\Sigma y_i}{1.5^2} + \frac{8}{(4/3)^2} \end{cases} \Rightarrow \begin{cases} \tau_n^2 = \left(\frac{n}{\sigma^2} + \frac{1}{\tau_0^2}\right)^{-1} = \left(\frac{n}{1.5^2} + \frac{1}{(4/3)^2}\right)^{-1} \\ \mu_n = \tau_n^2\left(\frac{\Sigma y_i}{\sigma^2} + \frac{\mu_0}{\tau_0^2}\right) = \tau_n^2\left(\frac{\Sigma y_i}{1.5^2} + \frac{8}{(4/3)^2}\right) \end{cases}$

→ or $\Sigma y_i = n\bar{y}$ like in notes

$\boxed{\propto Normal(\mu_n, \tau_n^2)}$

$\Rightarrow$ using $\underset{\sim}{y}$ $\to \mu|\underset{\sim}{y} \sim Normal(\underbrace{5.544}_{mean}, \underbrace{0.072}_{var})$

⟹ Part d → Posterior Predictive distribution

$\to p(\tilde{y}|\underset{\sim}{y}) = \int_\theta p(\tilde{y}|\theta)\,p(\theta|\underset{\sim}{y})\,d\theta$

→ Simulate for $m = 1, \dots, M$ → 1) $\mu^{(m)} \sim p(\mu|\underset{\sim}{y}) = Normal(\mu_n, \tau_n^2)$ → < using result of part b >

2) $\tilde{y} \sim p(\underset{\sim}{y}|\mu^{(m)}) = Normal(\mu^{(m)}, \sigma^2)$

↳ data dist, except w/ generated $\mu^{(m)}$

→ Approximate → $E(\tilde{y}|\underset{\sim}{y}) \approx \frac{1}{M}\Sigma\tilde{y}$

$SD(\tilde{y}|\underset{\sim}{y}) \approx \sqrt{\frac{1}{M-1}\Sigma(\tilde{y} - \,)^2}$

Problem 3

→ Data dists → Female BMI → $X_i \overset{iid}{\sim} N(\mu_x, \sigma^2 = 1)$, $i = 1, \dots, 32$

→ Male BMI → $Y_j \overset{iid}{\sim} N(\mu_y, \sigma^2 = 1)$, $j = 1, \dots, 28$

→ Priors → $\mu_x \sim N(\mu_{x,0} = 22, \tau_{x,0}^2 = 4)$

→ $\mu_y \sim N(\mu_{y,0} = 20.5, \tau_{y,0}^2 = 4)$

→ Posteriors → using results from problem 2 ...

$\to \boxed{\mu_x|\underset{\sim}{x} \sim N(\mu_{x,n}, \tau_{n,x}^2)} \Rightarrow \tau_{n,x}^2 = \left(\frac{n}{\sigma_x^2} + \frac{1}{\tau_{0,x}^2}\right)^{-1} = \left(\frac{32}{1} + \frac{1}{4}\right)^{-1}$

$\mu_{n,x} = \tau_{n,x}^2\left(\frac{\Sigma x_i}{\sigma_x^2} + \frac{\mu_{0,x}}{\tau_{0,x}}\right) = \tau_{n,x}^2\left(\frac{\Sigma x_i}{1} + \frac{22}{4}\right)$

→ using $\underset{\sim}{x}$ ⟹ $\mu_x|\underset{\sim}{x} \sim N(\text{mean} \approx 21.978, \text{var} \approx 0.031)$

$\to \boxed{\mu_y|\underset{\sim}{y} \sim N(\mu_{n,y}, \tau_{n,y}^2)} \Rightarrow \tau_{n,y}^2 = \left(\frac{n}{\sigma_y^2} + \frac{1}{\tau_{0,y}^2}\right)^{-1} = \left(\frac{28}{1} + \frac{1}{4}\right)^{-1}$

$\mu_{n,y}^2 = \tau_{n,y}^2\left(\frac{\Sigma y_i}{\sigma_y^2} + \frac{\mu_{0,y}}{\tau_{0,y}^2}\right) = \tau_{n,y}^2\left(\frac{\Sigma y_i}{1} + \frac{20.5}{4}\right)$

→ using $\underset{\sim}{y}$ ⟹ $\mu_y|\underset{\sim}{y} \sim N(\text{mean} \approx 20.372, \text{var} \approx 0.035)$

→ Equal tails interval for difference in means ⟹ is just simply an interval based on the resulting distribution from subtracting the two posteriors

$\to \mu_x|\underset{\sim}{x} - \mu_y|\underset{\sim}{y} \sim N(\text{mean} = \mu_{n,x} - \mu_{n,y}, \text{var} = \tau_{n,x}^2 + \tau_{n,y}^2)$

↳ = < ... algebra ... > , ↳ = < ... algebra ... >

$\approx 21.978 - 20.372$, $= 0.031 + 0.035$

$= 1.606$, $= 0.066$

⟹ 95% equal tails interval → LB = qnorm(0.025, mean = 1.606, sd = $\sqrt{0.066}$)

UB = qnorm(0.975, – – – – – – –)

Problem 4

→ Let $Y|\theta \sim Poisson(\theta) \Rightarrow f(Y|\theta) = \frac{e^{-\theta}\theta^Y}{Y!} = L(\theta|Y)$

→ Fisher's information = $I(\theta)$

$= -E\left[\frac{\partial^2 \ln(\theta)}{\partial\theta^2}\right]$

$= -E[-Y/\theta^2]$

$= \frac{1}{\theta^2}\underbrace{E(Y)}_{=\theta}$

$= \frac{1}{\theta}$

→ Jeffrey's prior $p(\theta) \propto \sqrt{I(\theta)}$

$= \sqrt{1/\theta}$

$\boxed{= \theta^{-1/2}}$

$\to l(\theta) = \ln[\;]$

$= -\theta + Y\ln(\theta) - Y!$

$\to \frac{\partial l(\theta)}{\partial\theta} = \frac{\partial}{\partial\theta}[\;]$

$= -1 + \frac{Y}{\theta}$

$\to \frac{\partial^2 l(\theta)}{\partial\theta^2} = \frac{\partial}{\partial\theta}[\;]$

$= \frac{-Y}{\theta^2}$