(skipping to assignments >

3.8.1 → Exponential family

→ Q1) show which are exponential family

$$f(x|\theta) = \frac{\theta^x}{(1+\theta)^{x+1}} \quad , x > 0$$

$= \exp\{\ln(\ldots)\}$

$= \exp\{x\ln(\theta) - \underline{(x+1)\ln(1+\theta)}\}$

$x\ln(1+\theta) - \ln(1+\theta)$

$= \exp\{x\ln(\frac{\theta}{1-\theta}) - \ln(1+\theta)\}$

$a(x)$ $b(\theta)$ $c(\theta)$ $d(x)=0$

↳ canonical

⇒ ✓

$$g(y|\beta) = \frac{\beta e^{-\beta/y}}{y^2} \quad , y > 0$$

$= \exp\{\ln(\ldots)\}$

$= \exp\{\ln(\beta) - \beta y^{-1} - 2\ln(y)\}$

$c(\beta)$ $b(\beta)$ $a(y)$ $d(y)$

↳ not canonical

⇒ ✓

3.8.2 → model framework

★ range of linear component $(x^T\beta)$ must match range of link function $g(\mu)$

→ ex) if Y~ gamma ⇒ $g(\mu) = \mu$ (identity) is not possible b/c $x^T\beta$ could be < 0, but $\mu > 0$

mismatch

3.8.3 → Parameter estimation

→ Q1) Given GLM model w/ Poisson response & link function

$$g(\lambda) = \ln(\lambda) = \beta_0 + \beta_1 x$$

→ method of scoring is used to estimate β coefficients

→ given

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 5 \\ 1 & 7 \\ 1 & 9 \end{bmatrix} \quad Y = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \\ 4 \end{bmatrix}$$

$$\hat\beta^{(1)} = \begin{bmatrix} 0.1 \\ 0.4 \end{bmatrix} \quad u^{(1)} = \begin{bmatrix} -61.106 \\ -481.732 \end{bmatrix} \quad J^{(1)} = \begin{bmatrix} 72.106 & 544.732 \\ 544.732 & 4{,}405.591 \end{bmatrix}$$

→ calculate estimate for λ_2, the mean response for the second observation, on the second iteration

→ $\hat\beta^{(2)} = \hat\beta^{(1)} + (J^{(1)})^{-1}u^{(1)} = \langle$ excel $\rangle = \begin{bmatrix} -0.2244 \\ 0.3308 \end{bmatrix}$

→ $\ln(\lambda) = x\hat\beta^{(2)} = 0.769$

⇒ $\lambda = e^{0.768} = 2.155$

→ Q2) Given GLM model w/ Poisson response & link function

$$g(\lambda) = \ln(\lambda) = \beta_0 + \beta_1 x$$

→ Estimate $\hat\beta$ w/ IWLS

→ given

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 5 \\ 1 & 7 \\ 1 & 9 \end{bmatrix} \quad Y = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \\ 4 \end{bmatrix}$$

$$(X'W^{(1)}X)^{-1} = \begin{bmatrix} 0.4750 & -0.0583 \\ -0.0583 & 0.0102 \end{bmatrix}$$

$$X'W^{(1)}z^{(1)} = \begin{bmatrix} 10.2273 \\ 67.7721 \end{bmatrix}$$

→ Estimate λ_3, the mean response for the third observation, on the second iteration

→ $b^{(2)} = [(X'WX)^{(1)}]^{-1} X'Wz^{(1)}$

$= \langle$ excel $\rangle$

$= \begin{bmatrix} 0.2455 \\ 0.0450 \end{bmatrix}$

→ $\ln(\lambda) = x\beta^{(2)} = 0.8705$

⇒ $\lambda = e^{0.8705} = 2.388$

→ Q3) → Sample of losses: 650, 400, 800, 1000

X~ Exp(θ)

→ method of scoring starting w/ $\theta^{(0)} = 1200$

→ $\theta^{(1)} = \theta^{(0)} + (J^{(0)})^{-1}u^{(0)}$

$= 1200 + \frac{1200^2}{4} \times \left[\frac{\Sigma x_i}{1200^2} - \frac{4}{1200}\right]$

$= 712.5$

→ $f(x) = \frac{1}{\theta}e^{-x/\theta}$

$= \exp\{\ln(\frac{1}{\theta}) - x/\theta\}$

→ $l(\theta) = \Sigma(\ln(1) - \ln(\theta) - x/\theta)$

→ $U = l'(\theta)$

$= \Sigma(\frac{x_i}{\theta^2} - \frac{1}{\theta})$

→ $J = -E(U')$

$= -E(\underset{\sim}{l''(\theta)})$

↳ $= \Sigma(-\frac{2x_i}{\theta^3} + \frac{1}{\theta^2})$

$= -E[-\frac{2\Sigma x_i}{\theta^3} + \frac{n}{\theta^2}]$

$= -(-\frac{2n}{\theta^2} + \frac{n}{\theta^2})$

$= \frac{n}{\theta^2}$

3.8.4 → Numerical results

→ Q1) ~ Y ~ Exp(λ)

→ Find deviance of model

Y	$1/\hat\lambda = \hat\mu$
5.7	6.2
8.4	8.9
6.1	7.3
4.9	4.5

$\hat\theta_{max} = \hat\theta_{saturated\ model}$

→ $D = 2[l_n(\hat\lambda_{max}) - l_n(\hat\lambda)]$

$= 2[\{\Sigma\ln(\hat\lambda_{max}) - \Sigma\hat\lambda_{max}y_i\} - [\Sigma\ln(\hat\lambda_i) - \Sigma\hat\lambda_i y_i]]$

$= 2[\underbrace{\Sigma\ln(\frac{1}{y_i}) - \Sigma\frac{1}{y_i}y_i}_{Ⓐ} - (\underbrace{\Sigma\ln(\frac{1}{\hat\mu_i}) - \Sigma\frac{1}{\hat\mu_i}y_i}_{Ⓑ})]$

$= 2[Ⓐ - Ⓑ]$

↳ $= -11.290$

↳ $= -11.266$

$= 0.048$

$f(y) = \lambda_i e^{-\lambda_i y_i}$

$= \exp\{\ln(\lambda_i) - \lambda_i y_i\}$

$l(\lambda) = \Sigma(\ln(\lambda_i) - \lambda_i y_i)$

$= \Sigma\ln(\lambda_i) - \Sigma\lambda_i y_i$

know $\hat\lambda_{MLE} = \frac{1}{y_i}$

→ Q2) deviance residual $= \pm\sqrt{D_i}$

↳ individual term of $2[l_n(\hat\theta_{max}) - l_n(\hat\theta)]$

sign to match sign of raw residual

→ Q3) Pearson residual $= \frac{e_i}{\sqrt{\hat v(y_i)}} = \frac{\text{Actual} - \text{predicted}}{\sqrt{\hat v(y_i)}}$

↳ if $y_i \sim Exp(\lambda)$

⇒ $v(Y_i) = \frac{1}{\lambda_i^2}$

3.8.5 → Inference

Q1) LRT →

~~$\lambda = \frac{L(\hat\lambda_{max})}{L(\hat\lambda)}$~~

~~$= \frac{\Pi L_i(1/y_i)}{\Pi_i L_i(\frac{1}{\hat\mu_i})}$~~

~~$= \frac{\Pi y_i^{-1} e^{-\Sigma y_i^{-1}}}{\Pi\frac{1}{\hat\mu_i}e^{-\Sigma\frac{1}{\hat\mu_i}y_i}}$~~

~~$= \frac{0.000623}{0.00787}$~~

~~$= 0.161$~~

$L(\lambda) = \Pi\lambda e^{-\lambda y_i}$

$= \Pi\lambda_i e^{-\Sigma\lambda_i y_i}$

LRT ⇒ Deviance statistic, but for Full & reduced models

more complicated ⇏ saturated

↳ "Deviance" deviance is always saturated vs fitted model

→ $D = 2[l_n(\hat\lambda_{full}) - l_n(\hat\lambda_{red})]$

using $\hat\mu$

$= \frac{1}{\bar y}$

$= 2[\Sigma\ln(\hat\lambda_i) - \Sigma\hat\lambda_i y_i - [\Sigma\ln(\hat\lambda_{MLE}) - \hat\lambda_{MLE}\Sigma y_i]]$

$= 2[\underbrace{\Sigma\ln(\frac{1}{\hat\mu_i}) - \Sigma\frac{1}{\hat\mu_i}y_i}_{Ⓐ} - (\underbrace{4\ln(\frac{1}{\bar y}) - \frac{1}{\bar y}\Sigma y_i}_{Ⓑ})]$

$= 2[Ⓐ - Ⓑ]$

↳ $= -11.346$

↳ $= -11.290$

$= 0.1121$

→ Q2) Normal glm w/ identity link = MLR

⇒ scaled deviance = SSE