Iterative weighted least squares Algorithm

$$b^{(m)} = (X^TWX)^{-1}X^TWz$$

$$W_{ii} = \frac{1}{v(y_i)}\left(\frac{\partial \mu_i}{\partial \eta_i}\right)^2 \qquad z_i = \sum_{k=1}^{p} x_{ik} b_k^{(m-1)} + (y_i-\mu_i)\left(\frac{\partial \eta_i}{\partial \mu_i}\right)$$

→ solve link function for μ_i

evaluated @ $b^{(m-1)}$

4.1)

a & b) <on computer>

c) Fit GLM w/ $y_i \sim \text{Poisson}(\lambda_i)$ & log-link &

$$g(\mu_i) = g(\lambda_i) = \ln(\lambda_i) = \beta_1 + \beta_2 x_i = \eta_i$$

$\hookrightarrow = \mu_i$

$$\rightarrow \underset{(n\times 2)}{X} = \begin{bmatrix} 1 & \ln(1) \\ 1 & \ln(2) \\ \vdots & \vdots \\ 1 & \ln(20) \end{bmatrix} \qquad b^{(m)} = \begin{bmatrix} b_1^{(m)} \\ b_2^{(m)} \end{bmatrix}$$

$$\rightarrow W_{ii} = \frac{1}{v(y_i)}\left(\frac{\partial \mu_i}{\partial \eta_i}\right)^2$$

$\rightarrow \ln(\mu_i) = \beta_1 + \beta_2 x_i = \eta_i$

$e^{\ln(\mu_i)} = e^{x^T\beta} = \eta_i$

$\mu_i = e^{\eta_i}$

$\Rightarrow \frac{\partial \mu_i}{\partial \eta_i} = e^{\eta_i}$ $\quad \langle \frac{d}{dx}e^x = e^x \rangle$

$= e^{-x^T\beta}\left\{\left(e^{\eta_i}\right)'\right\}^2$ $\quad \langle v(y_i) = \lambda_i = e^{x^T\beta}$

$= e^{-x^T\beta} e^{2\eta_i}$

$\hookrightarrow = e^{2(x^T\beta)}$

$= e^{x^T\beta}$

$$z_i = x^T\beta + (y_i-\mu_i)\left(\frac{\partial \eta_i}{\partial \mu_i}\right) \qquad \hookrightarrow = 1\Big/\left(\frac{\partial \mu_i}{\partial \eta_i}\right)$$

$= x^T\beta + \frac{y_i - e^{x^T\beta}}{e^{x^T\beta}}$

$= x_i^T\beta + \frac{y_i}{e^{x_i^T\beta}} - 1$

→ set $b_1^{(1)} = b_2^{(1)} = 1 \Rightarrow$ <algorithm on computer>

4.2

a) <plot>

b) → possible model → $E(y_i) = \exp(\beta_0 + \beta_1 x_i)$

(to ensure $E(Y)$ is nonnegative)

→ possible link function → $\ln(E(y_i)) = \ln(\exp(\beta_0+\beta_1 x_i))$

log-link $\qquad = \beta_0 + \beta_1 x_i$

c) $\rightarrow f(y|\theta) = \theta e^{-\theta y}$

$= \exp\{\ln(\langle\ldots\rangle)\}$

$= \exp\{\underset{c(\theta)}{\ln(\theta)} - \underset{b(\theta)a(y)}{\theta y}\}$

$$\rightarrow E(y) = \frac{-c'(\theta)}{b'(\theta)} = \frac{-(\ln(\theta))'}{(-\theta)'} \checkmark = \frac{1}{\theta}$$

$$\rightarrow v(y) = \frac{b''(\theta)c'(\theta) - c''(\theta)b'(\theta)}{(b'(\theta))^3}$$

$$= \frac{\frac{1}{\theta^2}(-\theta)}{-\frac{1}{\theta}}$$

$\overset{\checkmark}{=} 1/\theta^2$

d) <on computer>

4.3

→ Setup → $Y_i \overset{iid}{\sim} N(\ln(\beta), \sigma^2)$

$\hookrightarrow$ known

→ find MLE

→ Algorithm → Model statement → $E(y_i) = \mu_i = \ln(\beta)$

$$g(\mu_i) = \underbrace{e^{\mu_i}}_{\text{Link function}} = e^{\ln(\beta)} = \beta = \eta_i$$

$\Rightarrow b^{(m)} = (X^TWX)^{-1}X^TWz$

$\rightarrow X = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

$$\rightarrow W_{ii} = \frac{1}{v(y_i)}\left(\frac{\partial \mu_i}{\partial \eta_i}\right)^2 \qquad \mu_i = \ln(\eta_i)$$

$\hookrightarrow = \beta$

$= \frac{1}{\sigma^2}\left(\frac{1}{\beta}\right)^2$

$\rightarrow z_i = x_i\beta + (y_i-\mu_i)\left(\frac{\partial \eta_i}{\partial \mu_i}\right)$

$= x_i\beta + (y_i - \ln(\beta))\beta$

$= \beta(x_i + y_i - \ln(\beta))$

→ Iteratively solve $b^{(m)} = (X^TWX)^{-1}X^TWz$

→ usual way

$$\rightarrow f(y|\beta) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{1}{2\sigma^2}(y_i - \ln(\beta))^2\right\}$$

$$l(\beta) = \exp\{\ln(\langle\ldots\rangle)\}$$

$$= \exp\left\{-\frac{1}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}(y_i - \ln(\beta))^2\right\}$$

$$= \exp\Big\{\underbrace{-\frac{1}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}y_i^2}_{d(y)} + \underset{b(\beta)}{\frac{\ln(\beta)}{\sigma^2}}\underset{a(y)}{y_i} - \underset{c(\beta)}{\frac{1}{2\sigma^2}(\ln(\beta))^2}\Big\}$$

$$U = \frac{dl}{d\theta} = \sum_{i=1}^{n} a(y_i)b'(\beta) + c'(\beta)$$

$$= \sum_{i=1}^{n}\left(\frac{y_i}{\sigma^2\beta} - \frac{\ln(\beta)}{\sigma^2\beta}\right)$$

$$= \frac{1}{\sigma^2\beta}\sum_{i=1}^{n}(y_i - \ln(\beta))$$

$\rightarrow 0 = \langle\ldots\rangle$

$= \sum(y_i - \ln(\beta))$

$n\ln(\beta) = \sum y_i$

$\widehat{\ln(\beta)}_{MLE} = \bar{y}$

$\Rightarrow \hat{\beta}_{MLE} = e^{\bar{y}}$

$$\rightarrow \text{verify} \quad U_j = \sum_{i=1}^{n}\left[\frac{(y_i-\mu_i)}{v(y_i)}x_{ij}\left(\frac{\partial \mu_i}{\partial \eta_i}\right)\right] \rightarrow \quad g(\mu_i) = e^{\mu_i} = \beta = \eta_i$$

$\mu_i = \ln(\eta_i)$

$$\Rightarrow U = \sum\left[\frac{(y_i - \ln(\beta))}{\sigma^2}x_i\left(\frac{1}{\beta}\right)\right]$$

$\hookrightarrow = 1$ (no independent variable)

$$\overset{\checkmark}{=} \frac{1}{\sigma^2\beta}\sum_{i=1}^{n}(y_i - \ln(\beta))$$