✗ **3.1**

a.) weight $= Y_i \sim N(\mu_i, \sigma^2)$

$E(Y_i) = \mu_i = \beta_0 + \beta_1 x_{age} + \beta_2 x_{sex} + \beta_3 x_{height} + \beta_4 x_{food} + \beta_5 x_{energy}$

b) $Y_j \sim \text{Binomial}(n=20, \pi_j)$

$E(\frac{Y_j}{n}) = \pi_j = \underbrace{\frac{e^{x\beta}}{1+e^{x\beta}}}$ → logistic function b/c $0<\pi<1$ $\Rightarrow \ln(\pi/(1-\pi)) = x^T\beta$

↳ model comes first ⇒ then decipher link

~~$Y_i \sim \text{Binomial}(n, \pi_i)$~~

~~$E(Y_i) = n\pi_i$~~

~~$g(E(Y_i)) = g(\pi_i) = \frac{1}{n}\pi_i = \beta_0 + \beta_1 x_2 + \beta_2 x_3 + \beta_3 x_4 + \beta_4 x_5$~~

c) # trips $= Y_i \sim \text{Poisson}(\mu_i)$

→ need positive response ⇒ put linear component in exponent

$\mu = e^{\beta_0 + \beta_1 x_1 + \dots + \beta_3 x_3} \Rightarrow \ln(\mu) = x_i^T\beta$

✓ **3.2**

$\rightarrow f(y|\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y}$

$= \exp\{\ln(\dots)\}$

$= \exp\{\underbrace{\alpha\ln(\beta)}_{c(\beta)} \underbrace{- \ln(\Gamma(\alpha)) + (\alpha-1)\ln(y)}_{d(y)} \underbrace{-\beta}_{b(\beta)} \underbrace{y}_{a(y)}\}$

$\rightarrow E(Y) = \frac{-c'(\beta)}{b'(\beta)} = \frac{-\frac{d}{d\beta}[\alpha\ln(\beta)]}{\frac{d}{d\beta}-\beta} = \frac{-\alpha/\beta}{-1} \overset{\checkmark}{=} \alpha/\beta$

$\rightarrow V(Y) = \frac{b''(\theta)c'(\theta) - c''(\theta)b'(\theta)}{(b'(\theta))^3} = \frac{0 + \frac{\alpha}{\beta^2}(-1)}{(-1)^3} \overset{\checkmark}{=} \frac{\alpha}{\beta^2}$

✓ **3.3**

a) Pareto dist → $f(y|\theta) = \frac{\theta}{y^{\theta+1}} = \theta y^{-(\theta+1)}$

$= \exp\{\ln[\theta y^{-(\theta+1)}]\}$

$= \exp\{\underbrace{\ln(\theta)}_{c(\theta)} - \underbrace{(\theta+1)}_{b(\theta)}\underbrace{\ln(y)}_{a(y)}\}$

$d(y)=0$

b) Exponential dist → $f(y|\theta) = \theta e^{-\theta y}$

$= \exp\{\ln[\theta e^{-\theta y}]\}$

$= \exp\{\underbrace{\ln(\theta)}_{c(\theta)} - \underbrace{\theta}_{b(\theta)}\underbrace{y}_{a(y)}\}$

$d(y)=0$

c) negative Binomial dist → $f(y|\theta) = \binom{y+r-1}{r-1}\theta^r(1-\theta)^y$

$= \exp\{\ln[\dots]\}$

$= \exp\{\underbrace{\ln\binom{y+r-1}{r-1}}_{d(y)} + \underbrace{r\ln(\theta)}_{c(\theta)} + \underbrace{y}_{a(y)}\underbrace{\ln(1-\theta)}_{b(\theta)}\}$

✓ **3.4**

a) Poisson dist → $E(Y) = \frac{-c'(\theta)}{b'(\theta)} = \frac{-(-\theta)'}{(\ln(\theta))'} = \frac{1}{1/\theta} \overset{\checkmark}{=} \theta$

$V(\theta) = \frac{b''(\theta)c'(\theta) - c''(\theta)b'(\theta)}{(b'(\theta))^3} = \frac{1/\theta^2 - 0}{(1/\theta)^3} \overset{\checkmark}{=} \theta$

b) normal dist → $E(Y) = \langle\dots\rangle = \frac{-(-\frac{\mu^2}{2\sigma^2} - \frac{1}{2}\ln(2\pi\sigma^2))'}{(\mu/\sigma^2)'} = \frac{\mu/\sigma^2}{1/\sigma^2} \overset{\checkmark}{=} \mu$

$V(y) = \langle\dots\rangle = \frac{0 + (1/\sigma^2)^2}{(1/\sigma^2)^3} \overset{\checkmark}{=} \sigma^2$

c) Binomial dist → $E(Y) = \langle\dots\rangle = \frac{-(n\ln(1-\pi))'}{(\ln(\frac{\pi}{1-\pi}))'} = \frac{\frac{n}{1-\pi}}{\frac{1}{\pi} + \frac{1}{1-\pi}} = \frac{\frac{n}{1-\pi}}{\frac{\cancel{1-\pi}+\pi}{\pi(1-\pi)}} \overset{\checkmark}{=} n\pi$

$= \ln(\pi) - \ln(1-\pi)$

$V(Y) = \langle\dots\rangle = \frac{(-\pi^{-2} + (1-\pi)^{-2})\frac{n}{1-\pi} + n(1-\pi)^{-2}(\frac{1}{\pi} + \frac{1}{1-\pi})}{(\frac{1}{\pi} + \frac{1}{1-\pi})^3}$

$= \frac{(\frac{1}{\pi^2} + \frac{1}{(1-\pi)^2})(\frac{n}{1-\pi}) + \frac{n}{(1-\pi)^2}(\frac{1}{\pi} + \frac{1}{1-\pi})}{(\frac{1}{\pi} + \frac{1}{1-\pi})^3}$

$= \frac{\frac{(\pi^2 + (1-\pi)^2)n}{\pi^2(1-\pi)^3} + \frac{n\cancel{(\pi + (1-\pi))}\pi}{\pi^2(1-\pi)^3}}{(\frac{\pi + (1-\pi)}{\pi(1-\pi)})^3}$

? algebra somewhere (not important)

✓ **3.5**

a) Negative Binomial dist → $E(Y) = \frac{-c'(\theta)}{b'(\theta)} = \frac{-(r\ln(\theta))'}{(\ln(1-\theta))'} = \frac{-r/\theta}{-\frac{1}{1-\theta}} \overset{\checkmark}{=} r\frac{(1-\theta)}{\theta}$

$V(Y) = \frac{b''(\theta)c'(\theta) - c''(\theta)b'(\theta)}{(b'(\theta))^3} = \frac{-\frac{1}{(1-\theta)^2}(\frac{r}{\theta}) + \frac{r}{\theta^2}(-\frac{1}{1-\theta})}{(-\frac{1}{1-\theta})^3}$

$= \frac{\frac{-r(1-\theta)}{(1-\theta)^2\theta^2} - \frac{r\theta}{\theta^2(1-\theta)^2}}{-\frac{1}{(1-\theta)^3}}$

$= \frac{\frac{-r\cancel{+r\theta}\cancel{-r\theta}}{(1-\theta)^2\theta^2}}{-\frac{1}{(1-\theta)^3}} \overset{\checkmark}{=} \frac{r(1-\theta)}{\theta^2}$

b) Poisson → $E(Y) = V(Y)$
Binomial → $E(Y) > V(Y)$
NB → $E(Y) < V(Y)$
} based on relationship between $\hat{\mu}$ + $\hat{\sigma}^2$, can pick the corresponding dist

✓ **3.6**

< computer >

✓ **3.7**

→ Bernoulli model → $P(Y_i = 1) = \pi_i$ + $P(Y_i = 0) = 1-\pi_i$

$\Rightarrow P(Y=y) = \pi^{y_i}(1-\pi)^{1-\pi_i}$, $y = 0,1$

a) → b) → $f(y|\pi) = \exp\{\ln(\pi^{y_i}(1-\pi)^{1-y})\}$

$= \exp\{y_i\ln(\pi) + (1-y_i)\ln(1-\pi)\}$

$= \exp\{\underbrace{y_i}_{a(y)}\underbrace{\ln(\frac{\pi}{1-\pi})}_{b(\pi)} + \underbrace{\ln(1-\pi)}_{c(\pi)}\}$ → logit function

$d(y)=0$

c) $E(Y_i) = \pi_i$ → $E(Y_i) = \frac{-c'(\pi)}{b'(\pi)} = \frac{-(\ln(1-\pi))'}{\ln((\frac{\pi}{1-\pi}))'} = \frac{\frac{1}{1-\pi}}{\frac{1-\pi}{\pi}(\frac{1}{1-\pi} + \frac{\pi}{(1-\pi)^2})}$

$= \frac{\frac{1}{1-\pi}}{\frac{1}{\pi} + \frac{1}{1-\pi}} = \frac{\frac{1}{1-\pi}}{\frac{\cancel{1-\pi}+\pi}{\pi(1-\pi)}} \overset{\checkmark}{=} \pi$

d) link function → $g(\pi) = \ln(\frac{\pi}{1-\pi}) = x^T\beta$

$\rightarrow e^{\ln(\frac{\pi}{1-\pi})} = e^{x^T\beta}$

$\frac{\pi}{1-\pi} = e^{x^T\beta}$

$\pi = (1-\pi)e^{x^T\beta}$

$= e^{x^T\beta} - \pi e^{x^T\beta}$

$\pi + \pi e^{x^T\beta} = e^{x^T\beta}$

$\pi(1 + e^{x^T\beta}) = e^{x^T\beta}$

$\boxed{\pi = \frac{e^{x^T\beta}}{1+e^{x^T\beta}}}$ → logistic function

e) < on computer >

3.8

→ extreme value (Gumbel) distribution

$f(y|\theta) = \frac{1}{\phi}\exp\{\frac{(y-\theta)}{\phi} - \exp\{\frac{y-\theta}{\phi}\}\}$

ϕ > 0 nuissance parameter

$= \exp\{\ln \langle\dots\rangle\}$

$= \exp\{\ln(\frac{1}{\phi}) + \frac{(y-\theta)}{\phi} - \exp\{\frac{y-\theta}{\phi}\}\}$

$= \exp\{\underbrace{\ln(\frac{1}{\phi}) + y\frac{1}{\phi}}_{d(y)} \underbrace{- \theta\frac{1}{\phi}}_{c(\theta)} - \underbrace{e^{\frac{y}{\phi}}}_{a(y)}\underbrace{e^{-\frac{\theta}{\phi}}}_{b(\theta)}\}$

⇒ Yes!

3.9

→ $Y_i \overset{iid}{\sim} \text{Pareto}(\theta)$ → $f(y|\theta) = \frac{\theta}{y^{\theta+1}}$, $y>0$

→ is $\mu_i = E(Y_i) = (\beta_0 + \beta_1 x_i)^2$ a GLM??

~~$= (x_i^T\beta)^2$~~

~~$g(\mu_i) = \sqrt{\mu_i} = x_i^T\beta$ → Yes!~~

→ Pareto ∈ exponential family, but $a(y) = \ln(y) \neq y$ ⇒ not GLM

3.10

✓ → $Y_i \overset{iid}{\sim} N(\mu, \sigma^2)$

→ is $E(Y_i) = \mu_i = \beta_0 + \ln(\beta_1 + \beta_2 x_i)$ a GLM??

→ Normal ∈ Exponential family w/ canonical form ✓

→ But ~~model cannot be transformed to linear combo of~~ X ⇒ ✗

Logic → Can I transform model to be linear in X??

→ If so, the way to get there is the link function

→ try $g(\mu_i) = \exp(\mu_i) = \exp\{\beta_0 + \ln(\beta_1 + \beta_2 x_i)\}$

$= e^{\beta_0}(\beta_1 + \beta_2 x_i)$

$= e^{\beta_0}\beta_1 + e^{\beta_0}\beta_2 x_i$

$= \beta_0^* + \beta_1^* x_i$ ⇒ linear component ✓

✓ **3.11**

→ Pareto dist → $f(y|\theta) = \frac{\theta}{y^{\theta+1}} = \exp\{\underbrace{\ln(\theta)}_{c(\theta)} - \underbrace{(\theta+1)}_{b(\theta)}\underbrace{\ln(y)}_{a(y)}\}$

$d(y)=0$

→ $U = b'(\theta)a(y) + c'(\theta)$

$= (\theta+1)'\ln(y) + (\ln(\theta))'$

$= \ln(y) + \frac{1}{\theta}$

→ $E(U) = E[\ln(y) + \frac{1}{\theta}]$

$= E[\ln(y)] + \frac{1}{\theta}$

↳ $-\frac{c'(\theta)}{b'(\theta)} = \frac{-(\ln(\theta))'}{(\theta+1)'}$

$= -\frac{1}{\theta} + \frac{1}{\theta}$

$\overset{\checkmark}{=} 0$

→ $V(U) = \mathfrak{J} = \frac{b''(\theta)c'(\theta)}{b'(\theta)} - c''(\theta)$

$= \frac{(\theta+1)''(\ln(\theta))'}{(\theta+1)'} - (\ln(\theta))''$

$= 0 + \frac{1}{\theta^2}$

$= 1/\theta^2$

3.12

✓ a) Gamma = Σ exp

pdf → $\frac{\beta^\alpha}{\Gamma(\alpha)}y^{\alpha-1}e^{-\beta y}$ $\beta e^{-\beta y} \sim \text{Exp}(\beta)$

↳ if $\alpha = 1$

mgf → if $Y = \sum_{i=1}^{\alpha} X_i$

$M_Y(t) = M_{X_1+\dots+X_\alpha}(t) = \prod_{i=1}^{\alpha} M_{X_i}(t)$

$= \prod_{i=1}^{\alpha}\frac{\beta}{\beta-t}$

$= (\frac{\beta}{\beta-t})^\alpha \sim \text{Gamma}(\alpha, \beta)$

b) if $X \sim \text{Uniform}(0,1)$, show $Y = -\theta\ln(X)$ $\sim \text{Exp}(\theta)$

→ pdf technique → $Y = -\theta\ln(X)$ → $f_Y(y) = f_X(h(y))\,|h'(y)|$

$e^{\frac{y}{-\theta}} = e^{\ln(x)}$

$x = h(y) = \exp(y/-\theta)$

$= \frac{1}{\theta}e^{-y/\theta}$

$\sim \text{Exp}(1/\theta)$

$0 < x < 1$

$-\theta\ln(0) < -\theta\ln(x) < -\theta\ln(1)$

$\ln(1) < y$

c) i) show $\text{Bin}(n, \pi) \to P(\lambda)$ as $n \to \infty$

? Convergence of MGFs ⇒ Convergence in dist

ii) show $NB(r, \theta) \to P(r(1-\theta))$ as $r \to \infty$

} applications using limit def of e

d) use CLT to show

i) $P(\lambda) \to N(\mu, \mu)$ for large μ

$P(\lambda) = \sum_{i=1}^{\lambda} P(1)$, for large $\lambda \approx N(\underset{\mu}{n\lambda}, \underset{\mu}{n\lambda})$

ii) $\text{Bin}(n, \pi) \to N(n\pi, n\pi(1-\pi))$

$\text{Bin}(n, \pi) = \sum_{i=1}^{n}\text{Bern}(\pi)$, for large $n \approx N(n\pi, n\pi(1-\pi))$

iii) $G(\alpha, \beta) \to N(\alpha/\beta, \alpha/\beta^2)$

$G(\alpha, \beta) = \sum_{i=1}^{\alpha}\text{Exp}(\beta)$, for large $\alpha \approx N(\alpha/\beta, \alpha/\beta^2)$