

1.2.1 → Policy limits

→ Policy limits
 → But → max amount the insurer will pay
 $\rightarrow X \sim u = \begin{cases} X, & X \leq u \\ u, & X > u \end{cases}$

\downarrow limited loss variable
 upper at $u \Rightarrow$ pay whichever is lower, x or u

\rightarrow Limited expected value $\rightarrow E(x \wedge u) = \int_0^u x f(x) dx + \underbrace{\int_u^\infty u f(x) dx}_{= u S(u)}$

→ Alternative calculation

$$\rightarrow E[X \wedge u] = \int_0^u S(x) dx$$

$$E[(X \wedge u)^k] = \int_0^u kx^{k-1} S(x) dx$$

$E[g(x)] = \int_0^\infty g'(x) S(x) dx$

Always zero, regardless if g' exists or not

$x \rightarrow$ must be non-negative
 $g(0) = 0$

Mixtures → Even w/ two, formula for $E[(X \wedge u)^k]$ works by taking the weighted average of the raw moments for each component

$$E(X \wedge u) = \sum_{i=1}^n w_i E[X_i \wedge u]$$

↓ mixture

→ ASSE: general

→ Q1) $f(x) = \frac{375000}{x^4}, x > 50$

$$V(X \wedge 500) = E[(X \wedge 500)^2] - (E[X \wedge 500])^2$$

$$= \left[\int_{50}^{100} x^2 f(x) dx + \int_{100}^{\infty} 500^2 f(x) dx \right] - \left[\int_{50}^{100} x f(x) dx + \int_{100}^{\infty} 500 f(x) dx \right]^2$$

$$= \dots$$

$$= (6750 + 250) - (74.25 + 0.5)^2$$

$$= 1412.4375$$

OR

$X \sim SP\text{-Pareto}(\alpha=3, \theta=50)$

$$E[(X \wedge 500)^2] = \frac{4000}{4-\alpha} - \frac{k \theta^\alpha}{(\alpha-\alpha) \cdot \theta^{\alpha-2}}$$

$$= \frac{2(10^3)}{3-2} - \frac{2(50^3)}{(3-2) \cdot 500^{3-2}}$$

$$= 7000$$

$$E(X \wedge 500) = \frac{3(10)}{3-1} - \frac{f_0^3}{(3-1) \cdot 500^{3-1}}$$

$$= 74.25$$

$$V(X \wedge 500) = 7000 - 74.25^2$$

$$= 1412.4375$$

→ Deductibles

→ Deductible is the amount the policyholder is responsible for paying before the insurer will pay anything on the claim

→ Two types → ordinary & franchise deductibles

→ The insurer then pays the remaining amount

$$(x-d)_+ = \begin{cases} 0 & x \leq d \\ x-d & x > d \end{cases}$$

→ full loss amount = sum of parts

$$(x+d)_+ + (x-d)_+ = x$$

$$\downarrow$$

$$= \min(x, d) \quad \max(x-d, 0)$$

$\Rightarrow E[(x+d)_+] + E[(x-d)_+] = E(x)$

\Rightarrow Expected insurance payment can be calculated as

$E[(x-d)_+] = E(x) - E(x+d)$

→ recommended formula bc pieces are easy to find / given in table

→ It does not hold for higher moments

$$E[(x-d)^2] \neq E(x^2) - E[(x+d)^2]$$

→ need to use integration

$$E[(x-d)_+] = \int_0^d 0 \cdot dx + \int_d^\infty (x-d) f(x) dx$$

$$= 0 + 1$$

→ can be extended to the n^{th} moment too

$$E[(x-d)_+^n] = \int_d^\infty (x-d)_+^n s(x) dx$$

using survival method = $\int_d^\infty S(x) dx$

\rightarrow Loss Elimination ratio (LER) \rightarrow measures insurer saves by imposing an order

Payment per loss vs per payment

→ ex. Losses: 2 3 7 9 14
Policy has an ordinary deductible of 5

→ Expected payment per loss = $\frac{\text{Total payment}}{\text{# losses}} = \frac{0+0+2+4+9}{5} = 3$

→ Expected payment per payment = $\frac{\text{Total payment}}{\text{# payments}} = \frac{2+4+9}{3} = 5$

$\Rightarrow E[Y^P] \geq E[Y^L]$

\downarrow payment per payment RV \downarrow payment per loss RV

$\downarrow = \frac{E[y^L]}{S(d)}$

↳ easier way to switch between expected payment per payment & expected payment per loss
 $E[y^P] = \frac{E(y^L)}{S(d)} \Leftrightarrow E(y^L) = E(y^P) S(d)$

↳ works for higher order moments too $E[(y^P)^n] : E[y^L]^n S(d) \Leftrightarrow \dots$
 ↳ with no deductible, $y^P = y^L$ (all losses are payments)

example → losses $X \sim \text{Exponential}(\lambda = 1/200)$ w/ ordinary deductible of 100
 ↳ define variables → $y^L = (X - 100)_+$
 $y^P = X - 100 / X > 100$
 ↳ goal $\rightarrow V(y^P) = E[(y^P)^2] - (E(y^P))^2 = ?$
 $E(y^P) = \frac{E(y^L)}{S(d)}$

$$\rightarrow E[y^4] = \frac{E(x) - E(x+d)}{e^{-\frac{dx}{\gamma_{y=100}}}} = 0(1 - e^{-\frac{d}{\gamma_0}})$$

$$= \frac{\frac{1}{\gamma_{y=100}} - 40.67}{0.8187}$$

$$= 500$$

$$\rightarrow E[(y^4)^2] = \int_d^\infty \gamma(x-d) f(x) dx$$

$$= \int_d^\infty \gamma(\pi-100) e^{-\frac{x-\pi}{\gamma_0}} dx$$

$$= e^{-\frac{\pi}{\gamma_0}}$$

$$= 409,365,3765$$

$$E[(y^4)^2] = \frac{E[(y^4)^2]}{S(b)}$$

$$= 500,000$$

$$\Rightarrow v(y^4) = 500,000 - 500^2$$

$$= 250,000$$

<u>Loss</u>	<u>Avgm</u>	<u>Excess Loss</u>	<u>Avgm Excess Loss</u>
$\text{Exp}(\theta)$	θ	$\text{Exp}(\theta)$	θ
uniform(a, b)	$\frac{a+b}{2}$	uniform($0, b-\theta$)	$\frac{b-\theta}{2}$
parcto(α, θ)	$\frac{\theta}{\alpha-1}$	parcto($\alpha, \theta+d$)	$\frac{\theta+d}{\alpha-1}$
Sp-parcto(α, θ)	$\frac{\alpha\theta}{\alpha-1}$	parcto(α, θ)	$\frac{\theta}{\alpha-1}$
Beta($1, \theta, \theta$)	$\frac{\theta}{1+\theta}$	Beta($1, \theta, \theta+d$)	$\frac{\theta+d}{1+2\theta+d}$

anchise deductibles

- def → pays full amount of losses over the deductible
(in contrast to ordinary deductible which only
pays amount in excess of the deductible)

$$Y^L = \begin{cases} 0 & x \leq d \\ x & x > d \end{cases}$$

→ Expected values → $E(Y^L) = \int_d^\infty x \cdot f(x) dx$

$$\left| \begin{aligned} &= \int_d^\infty x \cdot f(x) dx + \int_d^\infty d \cdot f(x) dx + \int_d^\infty d \cdot f(x) dx \\ &= \int_d^\infty (x-d) f(x) dx + d \int_d^\infty f(x) dx \\ &= E[(x-d)_+] + d S(d) \end{aligned} \right.$$

$\Rightarrow E(Y^L) = \frac{E(Y^L)}{S(d)}$

$$\left| \begin{aligned} &= \frac{E[(x-d)_+] + d S(d)}{S(d)} \end{aligned} \right.$$

\downarrow

$$= c(d) + d$$

$s(d)$

→ Example → $X \sim \text{Pareto}(\alpha = 2.5, \theta = 100)$

$\rightarrow E(Y^d) = c(d) + d$

\downarrow

$$x-d \mid x > d \sim \text{Pareto}(\alpha, \theta+d)$$

$\Rightarrow c(d) = E[x-d \mid x > d]$

\downarrow

$$= \frac{\theta+d}{\alpha-1}$$

$= \frac{100+d}{2.5-1} + 50$

\downarrow

$$\approx 150$$

part of deductibles on claim frequency

→ Recall if losses x & payments necessarily when there is a deductible d

→ If the losses x & y modelled by an (α, b, θ) class distribution, then the x & y will follow the same distribution, but w/

Dist	\sim	\sim'
Poisson	λ	$v\lambda$
Binomial	n, p	u, vp
NB	r, β	$r, v\beta$

where v is the prob of loss being greater than the deductible $E(X-d) = \text{prob of } + \text{payment}$

7 moment

- $X \sim Exp(\theta)$
 $e(\alpha) = E(X-\delta | X > d)$
 $X-d | X > d \sim Exp(\theta)$
 $\Rightarrow e(\theta) = \theta$
- $X \sim Exp(\theta=2000)$
 $d = 500$
 $\delta' = 1000$
 $E(X-d) - E(X-\delta') = ??$

$$\Rightarrow E(X-d) = \int_{100}^{\infty} (x-100) \frac{1}{2000} e^{-\frac{x-100}{200}} dx$$

↓

$$= \dots$$

$$= 1557.603$$

$$E(X-d') = \int_{100}^{\infty} (x-100) \frac{1}{2000} e^{-\frac{x-100}{200}} dx$$

↓

$$= \dots$$

$$= 1213.061$$

$$\Rightarrow |E(X-d) - E(X-d')| \approx 344.54$$

$$\frac{R}{-} = y^L$$

$$E[(X-d)_+] = E(X) - E(X-d) \quad n=6 \quad (1 - e^{-d/\sigma_0})$$

↓

$$= 2000 - 2000(1 - e^{-300/200})$$

$$= 490.799$$

↓

$$= 1557.603$$

$$E[(X-d')_+] = \dots$$

↓

$$= 1213.061$$

~~W W W W W W~~

↳ coinsurance is the portion of the loss the insurer is responsible for

→ Suppose loss $x \rightarrow$ insurer pays αx , $\alpha < 1$ + policyholder pays $(1-\alpha)x$

charge $E(x) = \alpha E(x)$

insurance + deductible → insurance comes into play after the deductible is met

$$\Rightarrow Y^* = \begin{cases} 0 & x \leq d \\ \alpha(x-d) & x > d \end{cases}$$

$$\Rightarrow E(Y^*) = \alpha [E(X) - E(X_{d+})]$$

→ If insurance is applied before the deductible, then instead of applying the entire loss x against the deductible, the insurer will only apply the coinsured portion of the loss, αx .

The policyholder will still be responsible for the remaining portion $(1-\alpha)x$ + amount needed to meet deductible

$$\Rightarrow Y^* = \begin{cases} 0 & \alpha x \leq d \\ \alpha x - d & \alpha x > d \end{cases}$$

$$\downarrow = \begin{cases} 0 & x \leq \frac{d}{\alpha} \\ \alpha(x - \frac{d}{\alpha}) & x > \frac{d}{\alpha} \end{cases}$$

$\Rightarrow \text{ex: } \alpha = 0.8, x = 3, d = 2$

After \Rightarrow payment = $0.8(3-2) = 0.8$

Before \Rightarrow payment = $0.8(3)-2 = 0.4$

\Rightarrow to solve for Expected value, make the following adjustment

$$E(Y^d) = \alpha [E(X) - E(x \wedge \frac{d}{\alpha})]$$

General formula for all coverage modifications

$$E(Y^d) = \alpha [E(X_m) - E(x-d)]$$

\star

$X = \text{loss RV}$
 $u = \text{policy limit (soft or no limit)}$
 $d = \text{deductible (0 if none)}$
 $\alpha = \text{insurance (1 if none)}$
 $m = \text{maximum covered loss}$

$$\boxed{\mathbb{1} = \frac{u}{\alpha} + d}$$

$x < d$	0	$x \leq d$
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$$\begin{aligned}
 & \rightarrow \text{biseit } X \sim \text{Beta}(a=2, b=100) \\
 & u = 250, t = ? \\
 & E(tX) = E(X \wedge u) \quad \longrightarrow \\
 & \frac{c E(tu)}{a+b} = \frac{250}{100} (0.4567 + 250(1-0.4567)) \\
 & g(100) = \frac{250}{100} \approx 163 \quad \int_0^u x t \ln x dx + u \int_0^u \ln x dx = 250t \\
 & \Rightarrow t \approx 0.815 \quad a \ln(4+1), b \in [1 - \Gamma(a, u/10)] \\
 & g(100) \Gamma(3, \frac{250}{100}) + 250 \left[1 - \Gamma(2, \frac{250}{100}) \right] \\
 & \Rightarrow X \sim \text{Beta}(3, 2.5) \quad \downarrow \\
 & \Rightarrow P(3, 2.5) = 1 - \rho(1 \sim c3) \quad \approx 0.71 \\
 & P(u, \frac{b}{a}) = \text{Beta}(a, b) \text{DIST}(x, a, b, \theta = 0, \text{TRUE}) \\
 & b = c \mapsto (250, 3, 100) \\
 & \text{choose } x \text{ so that } x = \frac{u}{b} \\
 & \rightarrow X \sim \text{pareto}(a=3, \theta=400) \\
 & B = \begin{cases} 0.15(400 - x) & x < 500 \\ 0 & x \geq 500 \end{cases} \quad \text{the survival defining RV}
 \end{aligned}$$

$$\begin{aligned}
 \downarrow &= 0.15(500 - x) + \\
 &\quad \underbrace{\text{switched compared to usual}}_{\text{switched}}
 \end{aligned}$$

$\Rightarrow E(D) = E[0.15(500 - x)_+]$
 $= 0.15 E[(500 - x)_+]$
 $= \left[E(500) - \underbrace{E(\min(X))}_{\text{min}(X, 500)} \right]$
 $\Leftrightarrow \min(500, X) = \min(X, 500)$
 $= E[X \wedge 100]$
 $= [500 - \frac{400}{3-1} \left(1 - \left(\frac{400}{500+400} \right)^{3-1} \right)]$
 $= 50.43$

additional
 $\Rightarrow Q1 \Rightarrow \text{less } x \sim \text{Exp}(0.1600)$
 $\lambda = 100, \mu = 0.80, n = 800$

$\Rightarrow Y^L = \begin{cases} 0 & x \leq d \\ \alpha(x-d) & d < x \leq m \\ \alpha_n & x \geq m \end{cases}$
 $\Leftrightarrow \frac{Y^L - d}{\alpha} \sim \text{Exp}(1)$

$$\begin{aligned}
 &= \begin{cases} 0 & x \leq 200 \\ 0.8(x-200) & 200 < x \leq 1700 \\ 1200 & x \geq 1700 \end{cases} \\
 \rightarrow f(y^*) &= \alpha [E(x_{\text{new}}) - E(x_{\text{old}})] = 0(1 - e^{-\frac{1200}{600}}) \\
 &= 0.8 [E(x_{\text{new}}) - E(x_{\text{old}})] \\
 &= 0.8 [600(1 - e^{-\frac{1200}{600}}) - 600(1 - e^{-\frac{1000}{600}})] \\
 \rightarrow E(Y^*|x_{\text{old}}) &= \frac{778.979}{600} = e^{-200/600} \\
 &= 389.3397
 \end{aligned}$$

Q8

Square root rule (the square root property) $\rightarrow x - \delta/x + \delta = x$

$$\begin{aligned}
 E(Y^*) &= E(Y^*) = \alpha E(x_{\text{new}}) \\
 &= 0.8(600)(1 - e^{-\frac{1000}{600}}) \\
 &= 389.3397
 \end{aligned}$$

Inflation

inflation & its impacts on coverage and premiums

$$\begin{aligned}
 E[(1+r)X \wedge 220] &= (1+r) E\left[X \wedge \frac{220}{1+r}\right] \\
 \downarrow \rightarrow \text{lets } X \sim \text{gep para } (\alpha=2, \theta=100) & \\
 r=0.1, n=220 & \\
 E[(1+0.1)X \wedge 220] &= 1.1 E\left[X \wedge \frac{220}{1+0.1}\right] \xrightarrow{\text{approx}} \frac{0.1 \cdot 220}{2-1} - \frac{0.1 \cdot 220^2}{(2-1) \cdot 220^{2-1}} \\
 &\quad \downarrow \\
 &\approx 1.1 \left[\frac{220}{2-1} - \frac{(1.1) 220^2}{(2-1) \cdot 220^{2-1}} \right] \\
 &= 165 \\
 \underline{\text{OR}} \\
 Y &= (1+r)X \\
 \downarrow &= 1.1X \sim \text{gep para } (\alpha=2, \theta=1.1(100)) \\
 &\quad \downarrow \quad \&= 110 \\
 \Rightarrow E[Y \wedge 220] &= \frac{2(110)}{2-1} - \frac{(1.1) 110^2}{(2-1) \cdot 220^{2-1}} \\
 &\quad \downarrow \quad \&= 165 \\
 \underline{\text{OR}} \\
 E[1.1X \wedge 220] &= \int_0^{\infty} x \cdot 1.1 \cdot f(x, 220) \cdot f(x) dx \\
 &= \int_{1.1}^{\infty} x \cdot 1.1 \cdot \frac{220}{1.1} \cdot f(x) dx
 \end{aligned}$$

10515) $X_{1k} \sim \text{Uniform}(0, 1000)$
 $r < 0.7$, $\delta = 300$

7016 $\rightarrow E(r) = 500$
 $E(F(1300)) = \int_0^{1300} S(x) dx$
 $= \int_0^{1300} \left(1 - \frac{x}{1000}\right) dx$
 $= e^{-1.3}$
 ≈ 255

7017 $\rightarrow E(1.7x) = 600$
 $E((1+x)^2 x) = (1+x)^2 E[x \cdot e^{\frac{-x}{1+x}}]$
 $= 1.7 \int_0^{200} x e^x dx$
 $= e^{-1.7}$
 ≈ 362.5

↳ $LER = \frac{E(X^{1+\alpha})}{E(X^\alpha)} = \frac{\frac{231}{500}}{\frac{202.5}{600}} = 0.51 - 0.4771$
 ≈ 0.0725

↳ Using all coverage modification formula for inflation

$E(Y) = \alpha(1+r) \left[E(X^{\alpha} \cdot \frac{m}{1+r}) - E(X^{\alpha} \cdot \frac{d}{1+r}) \right]$

$E[(Y^k)^n] = \frac{E[(Y^*)^k]}{s_n(\frac{d}{1+r})}$

↳

$E[(Y^*)^k] = E[(Y^*)^k] s_k(\frac{d}{1+r})$

↳ or use $X^* = (1+r)X + d^* = (1+r)d$

$\therefore E(Y^*) = \alpha \left[E(X^{1+\alpha m}) - E(X^{1+\alpha d}) \right]$

↳ w/ w/ scale distributions

Example $\rightarrow f(x) = 1 - \frac{x}{200}, 0 \leq x \leq 200$

$\rightarrow X \sim \text{Uniform}(0, 200)$

$$f = 20, M = 100 \Rightarrow u = 100 - 20 = 80$$

$$v(y^t) = E[(y^t)^2] - E(y^t)^2$$

$$= \frac{E[(y^t)^2]}{S(d)} - \frac{E(y^t)^2}{S(d)}$$

$$= \frac{4053.373}{0.9} - \frac{56}{0.9}$$

$$= 632.0988$$

$\rightarrow E[(y^t)^2] = \int_0^{100} (x-20)^2 \frac{1}{200} dx + 80 S(100)$

\downarrow

$$= \dots$$

$$= 4053.373$$

$\rightarrow E(y^t) = \int_{20}^{100} (x-20) \frac{1}{200} dx + 80 S(100)$

\downarrow

$$= \dots$$

$$= 56$$

$$\begin{aligned}
 E[X_{12457}] &= \sum_{n=1}^{\infty} \left[1 - \left(\frac{0}{n+0} \right)^{n-1} \right] \\
 &= \sum_{n=1}^{100} \left[1 - \left(\frac{1600}{12457+1600} \right)^{n-1} \right] \\
 &\approx 989.991 \\
 \\
 E[\underbrace{1.05X}_{\sim \text{Pareto}} \wedge 12457] &= \text{as above except w/ 1680 for 1600} \\
 \sim \text{Pareto}(\alpha=2, \theta=1.05/1680) &\rightarrow 997.766 \\
 b = 1680
 \end{aligned}$$

\rightarrow 101st X ~ $\text{Pareto}(\alpha=4, \theta=1500)$, $\beta=0.90$

$$\begin{aligned}
 E[(1+r)X - 100]_+ &= E[(X^4 - 100)_+] \\
 &\sim \text{Pareto}(\alpha=4, \theta=1.2(1500)) \\
 &= E[X^4] - E[X^4 \wedge 100] \\
 &= \frac{0^4}{0-1} - \frac{0^4}{0-1} \left[1 - \left(\frac{0^4}{1+0} \right)^{4-1} \right] \\
 &= \frac{1600}{4-1} - \frac{1600}{4-1} \left[1 - \left(\frac{1600}{100+1600} \right)^{4-1} \right] \\
 &\approx 510.1818
 \end{aligned}$$