1.5.6 → Random Graphs

→ overview → many of the previous topics can be applied to random graphs, but need to understand graphs first

→ defs → A graph consists of a set of <u>nodes</u> & a set of arcs connecting the nodes

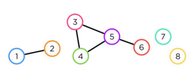
Let N be the set of nodes & A be the set of arcs.

<u>A graph has a maximum of $\binom{n}{2}$ arcs</u> (b/c $\binom{n}{2}$ different pairs of nodes).

A graph can be subdivided into subgraphs, called <u>components</u>, where each component consists of non-overlapping connected nodes

→ A graph is <u>connected</u> if there is only one component. In other words, given any two nodes, there must be a path from one node to the other for the graph to be connected

→ Example → list all nodes, arcs, & components of this graph



→ 8 nodes → $N = \{1,2,3,4,5,6,7,8\}$

5 arcs → $A = \{\{1,2\},\{3,4\},\{3,5\},\{4,5\},\{5,6\}\}$

4 components → $\{\{1,2\},\{3,4,5,6\},\{7\},\{8\}\}$

$>1 \Rightarrow$ not connected

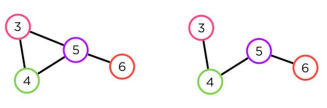
→ Random graphs

→ Now consider a graph w/ n nodes, where any two nodes (node i & node j) are not connected with certainty, but w/ probability of $P_{i,j}$. This graph is known as a <u>random graph</u>. Let $X_{i,j}$ be a RV that represents the existence of an arc between node i & node j

$$X_{i,j} = \begin{cases} 1 & \text{if } \{i,j\} \text{ is an arc} \\ 0 & \text{otherwise} \end{cases}$$

→ $P(X_{i,j}=1) = P_{i,j}$ & $P(X_{i,j}=0) = 1-P_{i,j}$

→ Suppose we are interested in the probability that a random graph is connected. Notice that a graph doesn't need to have every node connected to every other node to be considered connected

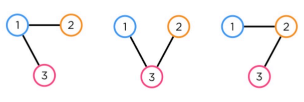


⟹ both are connected

→ the connectedness of a graph is similar to the reliability of a system. The reliability of a system depends on the reliability of its components. A system will function if & only if all components of at least one minimal path set function

→ Like a system, a graph has minimal path/cut sets. A random graph is connected as long as all arcs of at least one minimal path set exist. A random graph w/ n nodes has <u>n^{n-2} minimal path sets</u> & <u>$2^{n-1}-1$ minimal cut sets</u>.

→ e.g.) A random graph w/ 3 nodes has $3^{3-2}=3$ minimal path sets



⟹ starting/ending point doesn't matter

→ Let's assume each RV is iid (all $P_{i,j}=p$). Let $P_n = P$(random graph w/ n nodes is connected). Then

$$\boxed{P_n = 1 - \sum_{k=1}^{n-1}\binom{n-1}{k-1} q^{k(n-k)} P_k \quad , n = 2,3,4,\ldots}$$

where $P_1 = 1$, $P_2 = p$ & $q = 1-p$. Note that this is a <u>recursive formula</u> (the value of P_n depends on the values of all P_i from $i = 1,\ldots,n-1$).

This formula gets complicated quickly

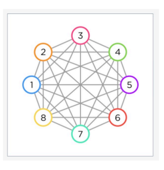
→ To avoid the exact calculation, we get two bounds

$$\boxed{nq^{n-1} - \binom{n}{2}q^{2n-3} \le 1 - P_n \le (n+1)q^{n-1}}$$

→ For large n, it can be shown

$$\boxed{P_n \approx 1 - nq^{n-1}}$$

→ random graph w/ all $\binom{8}{2} = 28$ arcs (for reference)



→ calculating bounds is just plug & chug ...

→ Example → given info on graph

→ 4 nodes
→ prob of arc between any two nodes is 0.6
→ Every arc is ⫫

→ Find the prob the graph is connected

→ need P_4 ⟹ know $P_1 = 1$, $P_2 = p = 0.6$

→ first find P_3

$$P_3 = 1 - \sum_{k=1}^{3-1}\binom{3-1}{k-1} q^{k(3-k)} P_k$$
$$= 1 - \left[\binom{2}{0}q^2 P_1 + \binom{2}{1}q^2 P_2\right]$$
$$= 1 - q^2 - 2q^2p$$
$$= 1 - 0.4^2 - 2(0.4^2)0.6$$
$$= 0.648$$

→ $P_4 = 1 - \sum_{k=1}^{4-1}\binom{4-1}{k-1} q^{k(4-k)} P_k$

$$= 1 - \left[\binom{3}{0}q^3P_1 + \binom{3}{1}q^4P_2 + \binom{3}{2}q^3P_3\right]$$
$$= 1 - \left[q^3 + 3q^4p + 3q^3P_3\right]$$
$$= 1 - 0.4^3 + 3(0.4^4)0.6 + 3(0.4^3)(0.648)$$
$$= 0.7655$$

→ Assignment

→ Q1) Given info → graph has 4 nodes
prob of arc between any 2 nodes is 0.5
every arc is ⫫

→ we know
→ max arcs: $\binom{n}{2} = \binom{4}{2} = 6$
→ minimal path sets: $n^{n-2} = 4^{4-2} = 16$
→ minimal cut sets: $2^{n-1}-1 = 2^{4-1}-1 = 7$
→ max # of graphs: $2^{\text{max \# arcs}} = 2^6 = 64$

→ Q2) Given info → graph has 3 nodes
prob of arc between any 2 nodes is 0.4
every arc is ⫫

→ Find prob graph is not connected

→ $P_1 = 1$
→ $P_2 = p = 0.4$
→ $P_3 = 1 - \sum_{k=1}^{3-1}\binom{3-1}{k-1} q^{k(3-k)} P_k$

$$= 1 - \left[\binom{2}{0}q^2P_1 + \binom{2}{1}q^2P_2\right]$$
$$= 1 - \left[q^2 + 2q^2p\right]$$
$$= 1 - \left[0.6^2 + 2(0.6^2)(0.4)\right]$$
$$= 0.352$$

⟹ $1 - P_3 = 1 - 0.352 = 0.648$

☆ set up whole problem on left, solve parts on right
⟹ so don't get wrong answer by stopping problem too early or forgetting a final step

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1.5.7 → Lifetime of Systems

→ overview → Let T_i represent the lifetime of component i. Then

→ $S_i(t)$ is the survival function of the lifetime of component i

→ $s(t) = (S_1(t), S_2(t), \ldots, S_n(t))$ is the vector of survival functions of the n components

→ If we know the reliability function of a system, we can easily calculate the moments & probabilities for the lifetime of a system. Let T be the lifetime of a system. Then, the probability that the system lifetime exceeds t is:

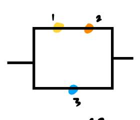
$$\boxed{P(T>t) = r[s(t)]}$$

↳ reliability function in terms of the components survival functions (rather than their probabilities), we set $p = s(t)$ or equivalently $p_i = S_i(t)$ for all i

→ The expected value of the system lifetime is

$$\boxed{E(T) = \int_0^\infty r[s(t)]\,dt}$$

→ Example → Assume the lifetimes of all components ~ Uniform(0,2)



→ $E(T) = \int_0^2 r[s(t)]$

$$= \int_0^2 \left(\frac{t^3-4t^2+8}{8}\right)dt$$
$$= \ldots$$
$$= 1.167$$

$r(p) = p_1p_2 + p_3 - p_1p_2p_3$ (from earlier examples)

$S_i(t) = \frac{2-t}{2}$ (b/c uniform)

$s(t) = \left(\frac{2-t}{2}\right)^2 + \frac{2-t}{2} + \left(\frac{2-t}{2}\right)^3$

$= \frac{t^3-4t^2+8}{8}$

→ Structure functions, Reliability functions, + survival functions

→ There is a relationship between all these

→ derive $r(p)$ from $\phi(x)$
→ then derive $s(t)$ from $r(p)$

⟹ follow steps below to calc probs & expected values involving the lifetime of systems

> → 1) <u>Determine the structure function</u> using minimal path sets or minimal cut sets (min/max stuff)
> 2) <u>Derive the reliability function</u> from the structure function or basic probability (substitute p)
> 3) <u>Develop the survival function</u> from the reliability function (substitute $s_i(t)$)
> 4) <u>calculate the desired probability or expected value</u>

→ Examples

→ Given info about system of 4 components

→ minimal path sets = $\{3\}$, $\{1,2\}$, $\{1,4\}$
All components are ⫫
lifetimes $T_i \overset{iid}{\sim}$ uniform(0,3)

→ $P(T<1.5) = 1 - P(T \ge 1.5)$

$$= 1 - s(1.5)$$
$$= 1 - 0.6875$$
$$= 0.3125$$

$r(p) = p_3 + p_1p_2 + p_1p_4 - p_1p_2p_3 - p_1p_3p_4 - p_1p_2p_4 + p_1p_2p_3p_4$ (from earlier ex)

$S_i(t) = \frac{3-t}{3}$

$s(t) = \frac{3-t}{3} + 2\left(\frac{3-t}{3}\right)^2 - 3\left(\frac{3-t}{3}\right)^3 + \left(\frac{3-t}{3}\right)^4$

→ Given info about 3-out-of-4 system

→ All components are ⫫
→ Lifetime $T_i \overset{iid}{\sim}$ Exp($\lambda=1$)

→ $E(T) = \int_0^\infty r[s(t)]\,dt$

$$= \int_0^\infty \left(4e^{-3t} - 3e^{-4t}\right)dt$$
$$= \ldots$$
$$= 0.5833$$

$r(p) = p_1p_2p_3 + p_1p_2p_4 + p_1p_3p_4 + p_2p_3p_4 - 3p_1p_2p_3p_4$ (from earlier ex)

$S_i(t) = e^{-t}$

$s(t) = 4(e^{-t})^3 - 3(e^{-t})^4$

$= 4e^{-3t} - 3e^{-4t}$

→ shortcut → For k-out-of-n systems of iid exponential components, each w/ mean θ, the mean lifetime of the system is

$$\boxed{E(T) = \theta\sum_{i=k}^{n}\frac{1}{i}}$$

→ Given info about a series system w/ two ⫫ machines

→ The hazard rate for each machine (w/ x in months) is

$$\mu_x = \frac{1}{50-x}, \quad 0 \le x < 50 \Rightarrow T_i \sim \text{uniform}(0,50)$$

One machine has worked for 15 months & the other for 20 months

→ calculate the probability the system will function for another 10 months

→ $P(T_1 > 25 \cap T_2 > 30 \mid T_1 > 15 \cap T_2 > 20)$

$$= \frac{P(T_1>25 \cap T_2>30)}{P(T_1>15 \cap T_2>20)}$$
$$= \frac{P(T_1>25) * P(T_2>30)}{P(T_1>15) * P(T_2>20)} \quad \text{(b/c ⫫)}$$
$$= \frac{S_1(25) * S_2(30)}{S_1(15) * S_2(20)}$$
$$= \frac{\left(\frac{50-25}{50}\right)\left(\frac{50-30}{50}\right)}{\left(\frac{50-15}{50}\right)\left(\frac{50-20}{50}\right)} \quad \text{(given } \tfrac{50-t}{50}\text{)}$$
$$= 0.4762$$

→ If don't recognize hazard function, can always derive survival function from it

$$\boxed{S(x) = \exp\left(-\int_{-\infty}^{x} h(t)\,dt\right)}$$
$$= \exp\left(-\int_0^x \frac{1}{50-t}\,dt\right)$$
$$= \exp\left(\ln(50-t)\Big|_0^x\right)$$
$$= \exp\left(\ln(50-x) - \ln(50)\right)$$
$$= \exp\left(\ln\left(\frac{50-x}{50}\right)\right)$$
$$✓ = \frac{50-x}{50} \quad \text{(for this example)}$$

→ Assignment

→ Q1) Given info about a <u>parallel</u> system w/ two components

→ $T_1 \sim$ Exp($\lambda=1$) + $T_2 \sim$ Exp($\lambda=1/2$)

☆ still do good notation on exam & all steps so can follow work when checking & don't skip steps

→ $P(T>3) = P(T_1>3 \cup T_2>3)$ (b/c parallel)

$$= P(T_1>3) + P(T_2>3) - P(T_1>3 \cap T_2>3) \quad \text{(} P(A\cup B) = P(A)+P(B)-P(A\cap B) \text{)}$$
$$= \text{— — — — —} \quad P(T_1>3) * P(T_2>3) \quad \text{(b/c ⫫)}$$
$$= S_1(3) + S_2(3) - S_1(3) * S_2(3)$$
$$= e^{-3} + e^{-3/2} - e^{-3}e^{-3/2}$$
$$= 0.2618$$

→ Q2) Given info about a <u>series</u> system w/ 2 components

→ $T_1 \sim$ uniform(0,1)
$T_2 \sim$ Exp($\lambda=1$)

→ $E(T) = \int_0^\infty r[s(t)]\,dt$

$$= \int_0^\infty (1-t)e^{-t}\,dt$$

$r(p) = p_1p_2$

$s(t) = S_1(t)S_2(t)$

$= \frac{1-t}{1}e^{-t}$