## 1.1.2 → Pareto

★ only writing new stuff (maybe quick summaries of old stuff)

→ overview → originally used to study income distributions
→ can be derived as a mixture of exponential RVs
★ → "pareto" on the exam refers to the two-parameter Pareto unless otherwise specified

→ Def → $\boxed{\text{If } X \sim \text{Pareto}(\alpha, \theta) \rightarrow f(x|\alpha,\theta) = \frac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}} \quad x \geq 0}$

↳ recognize constant $*(x+\theta)^{-\alpha-1}$

→ trivia → hazard function for pareto decreases w/ x for any values of α + θ

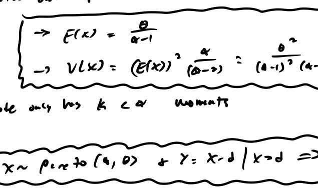
→ No MGF

→ Expected value + variance

$$E(X) = \frac{\theta}{\alpha - 1}$$
$$V(X) = (E(X))^2 \frac{\alpha}{(\alpha-2)} = \frac{\theta^2}{(\alpha-1)^2(\alpha-2)}$$

→ note only has $k < \alpha$ moments

→ $\boxed{\text{If } X \sim \text{Pareto}(\alpha,\theta) + Y = X - d \mid X > d \Rightarrow Y \sim \text{Pareto}(\alpha, \theta + d)}$
→ useful for deductibles




→ ex) If $X \sim \text{Pareto}(\alpha = 3, \theta = 500)$ + insurance only covers losses above 200,
↳ loss
the payment is $Y = X - 200$. Find $E(Y)$

→ $Y = X - 200$
↓ ~ Pareto ($\alpha = 3$, $\theta = 500 + 200 = 700$)

$\Rightarrow E(Y) = \frac{\theta}{\alpha - 1} = \frac{700}{2} = 350$

→ Inverse Pareto distribution

→ If $X \sim \text{Pareto}(\alpha, \theta^*)$ + $Y = X^{-1}$

$\Rightarrow Y \sim \text{Inv Pareto}(\tau = \alpha, \theta = \theta^{*-1})$

$$f(y) = \frac{\tau\theta y^{\tau-1}}{(y+\theta)^{\tau+1}}, \quad y \geq 0$$

→ resembles original pareto dist, except has an $x^{\tau-1}$ term in numerator

$$f(x) = \frac{c \cdot x^{\tau-1}}{(x+\theta)^{\tau+1}}$$

★ → Note this inversion methodology applies to all distributions on the exam table that have an inverse counterpart
→ θ is parameterized such that $X^{-1} \sim$ Inv <dist> (same parameters, $\theta^{-1}$)
ex) Gamma $(\alpha, \beta)$ + Inv Gamma $(\alpha, 1/\beta)$

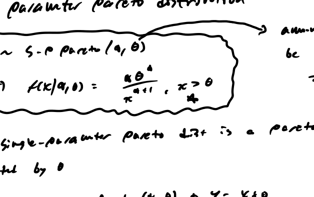
→ Proof → let $X \sim \text{Pareto}(\alpha, \theta)$ + $Y = \frac{1}{X}$
↳ $x = h(y) = \frac{1}{y}$
$h'(y) = -\frac{1}{y^2}$

$$f_Y(y) = f_X(h(y))\,|h'(y)|$$
$$= \frac{\alpha\theta^{\alpha}}{(\frac{1}{y}+\theta)^{\alpha+1}} \left|\frac{-1}{y^2}\right|$$
$$= \frac{\alpha\theta^{\alpha}}{\left(\frac{1+\theta y}{y}\right)^{\alpha+1} y^2}$$
$$= \frac{\alpha\theta^{\alpha} y^{\alpha+1}}{(y\theta)^{\alpha+1} y^2}$$
$$= \frac{\alpha\theta^{\alpha} y^{\alpha-1}}{(y+\theta)^{\alpha+1}}$$
$$\sim \text{Inv Pareto}(\alpha, \theta^{-1})$$

→ Single parameter pareto distribution

$\boxed{\text{If } X \sim \text{S-P Pareto}(\alpha, \theta) \Rightarrow f(x|\alpha,\theta) = \frac{\alpha\theta^{\alpha}}{x^{\alpha+1}}, \ x > \theta}$ → amount to pareto is shifted & must be determined in advance (not estimated)
⇒ α is the only the parameter
→ hence the name

→ A single-parameter pareto dist is a pareto dist shifted by θ
⇒ if X Pareto $(\alpha, \theta)$ + $Y = X + \theta$
⇒ $Y \sim$ S-P Pareto $(\alpha, \theta)$




→ Expected value → $E(Y) = E(X + \theta)$
$= E(X) + \theta$
$= \frac{\theta}{\alpha - 1} + \theta$
$= \boxed{\frac{\alpha\theta}{\alpha - 1}}$

variance → $\boxed{V(Y) = V(X) = (E(X))^2\left(\frac{\alpha}{\alpha-2}\right) = \left(\frac{\theta}{\alpha-1}\right)^2\left(\frac{\alpha}{\alpha-2}\right)}$

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→ Assignment

→ Q1 → $P(X > 25) = \int_{25}^{\infty} 200x^{-3}\,dx$
↳ ~ S-P Pareto ($\alpha = 2$, $\theta = 10$)

$\Rightarrow F_X(x) = 1 - (\theta/x)^{\alpha}$
$\Rightarrow S_X(x) = (\theta/x)^{\alpha} \rightarrow S(25) = \left(\frac{10}{25}\right)^2 = 4/25$

→ Q2 → $X \sim \text{Pareto}(\alpha, \theta = 100)$, given $CV = 3$, find $V(X)$

→ $V(X) = (E(X))^2\left(\frac{\alpha}{\alpha-2}\right)$
↓ $= \left(\frac{\theta}{\alpha-1}\right)^2\left(\frac{\alpha}{\alpha-2}\right)$

## 1.1.3 → Beta & Uniform

→ Beta dist

→ Uniform dist

→ Assignment

## 1.1.4 → Normal & Lognormal

→ lognormal dist

$\boxed{\begin{array}{l}\text{→ log-transformed normal dist} \\ \text{if } X \sim N(\mu, \sigma^2) + Y = e^X \\ \Rightarrow Y \sim \text{lognormal}(\mu, \sigma^2) \\ f_Y(y) = \frac{1}{y\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(\ln(y) - \mu)^2\right\}, \quad y > 0\end{array}}$

→ Mean + Variance
$E(Y) = e^{\mu + \frac{1}{2}\sigma^2}$
$V(Y) = (E(X))^2(e^{\sigma^2} - 1)$

→ CDF → found by log transforming the normal RV

→ $Y = e^X \Rightarrow X = \ln(y)$

$P(Y \leq y) = P(e^X \leq y)$
lognormal RV
$= P(X \leq \ln(y))$
$= P\left(\frac{X - \mu}{\sigma} \leq \frac{\ln(y) - \mu}{\sigma}\right)$
$= \boxed{P\left(Z \leq \frac{\ln(y) - \mu}{\sigma}\right)}$

→ Product of lognormal RVs
→ $Y_i \sim \text{lognormal}(\mu_i, \sigma_i^2)$, $i = 1, 2$

→ if $Y = Y_1 \cdot Y_2$
$= e^{X_1} e^{X_2}$
$= e^{X_1 + X_2}$
$= e^X$
$\Rightarrow Y \sim \text{lognormal}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

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→ Assignment

→ Q1) → $X_1 \sim N(\mu = 5, \sigma^2 = 4)$   $X_1 \perp X_2$
$X_2 \sim N(\mu = 6, \sigma^2 = 3)$
→ Find $E(e^{X_1 + X_2}) = E(e^Y)$

$Y \sim \text{lognormal}(\mu = 5 + 6, \sigma^2 = 4 + 3)$

$\Rightarrow E(Y) = e^{\mu + \frac{1}{2}\sigma^2}$
$= e^{11 + \frac{1}{2}7}$
$\approx 1,982,759.264$

→ Q2) → $X \sim N(\mu = 12, \sigma^2) \rightarrow x_{0.95} = 17$
$Y = e^X$
↓ ~ lognormal $(\mu = 12, \sigma^2)$
→ Find $|\text{mode}(X) - \text{mode}(Y)|$
$= 12 - \exp(\mu - \sigma^2)$
$= 12 - e^{12 - 9.1827}$
$= \cancel{4.7316}$

→ $17 = \text{invnorm}(0.95, 12, \sqrt{\sigma^2})$
$\Rightarrow z_{0.95} = \frac{x_{0.95} - \mu}{\sigma}$
$1.65 = \frac{17 - 12}{\sigma}$
$\Rightarrow \sigma = 3.03$
$\sigma^2 = 9.1827$

Correct logic → use 3 decimals ⇒ use critical value knowledge

## 1.1.5 → Gamma, Exponential, & Weibull

→ Gamma dist

→ CDF → $F(x) = \Gamma(\alpha, x\beta)$ → incomplete gamma function
(β: rate; $= \frac{x}{\theta}$ ↳ scale)

→ $\Gamma(\alpha, x) = \frac{1}{\Gamma(\alpha)}\int_0^x t^{\alpha-1}e^{-t}\,dt$, $\alpha > 0 + t > 0$

→ need Excel if $\alpha \notin \mathbb{Z}^+$

$\boxed{\begin{array}{l}\text{→ But if } \alpha \in \mathbb{Z}^+ \Rightarrow \text{where } N \sim \text{Poisson}(\lambda = x) \\ \Gamma(\alpha, x) = 1 - P(N < \alpha) \\ \rightarrow f(n) = \frac{e^{-x}x^n}{n!}\end{array}}$

→ example → let $X \sim \text{Gamma}(\alpha = 2, \beta = 1/5)$

→ $F(20) = \Gamma(\alpha, x\beta)$
$= \Gamma(2, \underbrace{20(\tfrac{1}{5})}_{=4})$
$= 1 - P(N < 2)$, $N \sim \text{Poisson}(\lambda = 4)$
$= 1 - [f_N(0) + f_N(1)]$
$= 1 - \left[\frac{e^{-4}4^0}{0!} + \frac{e^{-4}4^1}{1!}\right]$
$= 1 - 0.09157$
$= 0.9084$

→ Weibull dist

→ Overview → Weibull dist is a transformed exponential dist

$\boxed{\begin{array}{l}\text{→ If } X \sim \text{Exp}(\mu) + Y = X^{1/\tau} \\ \Rightarrow Y \sim \text{Weibull}(\theta = \mu^{1/\tau}, \tau) \\ f_Y(y) = \frac{\tau(y/\theta)^{\tau}e^{-(y/\theta)^{\tau}}}{y} \quad x > 0\end{array}}$ → Similar to gamma pdf, except $f(x) = c \cdot x^{\tau-1}e^{-(x/\theta)^{\tau}}$

→ when $\tau = 1 \Rightarrow Y \sim \text{Exp}(\mu)$
when $\theta = 1$, it is the standard weibull

→ hazard function of X increases with x if τ > 1 & decreases w/ x if τ < 1

$\boxed{\begin{array}{l}\text{→ mean + variance → } E(X) = \theta\,\Gamma(1 + \frac{1}{\tau}) \\ \rightarrow V(X) = \theta^2\,\Gamma(1 + \frac{2}{\tau}) - [\theta \cdot \Gamma(1 + \frac{1}{\tau})]^2\end{array}}$

→ Inverse counterparts

→ If $X \sim \text{Gamma}(\alpha, \theta)$ + $Y = X^{-1}$
⇒ $Y \sim \text{Inv Gamma}(\alpha, \theta^{-1})$

→ If $X \sim \text{Exp}(\theta)$ + $Y = X^{-1}$
⇒ $Y \sim \text{Inv Exponential}(\theta^{-1})$

→ If $X \sim$ Weibull + $Y = X^{-1}$
⇒ $Y \sim \text{Inv Weibull}(\theta^{-1}, \tau)$

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→ Assignment

→ Q1) $f(x) = c \cdot x^2 e^{-5x}$, $x > 0$

→ Find c → $X \sim \text{Gamma}(\alpha = 3, \beta = 5)$

$c = \frac{\beta^{\alpha}}{\Gamma(\alpha)} = \frac{5^3}{2!} = 62.5$

→ Q2) → $X \sim \text{Weibull}(\tau, \theta)$, $x_{0.25} = 1000$, $x_{0.75} = 100{,}000$

→ Find τ

→ $F(x) = 1 - e^{-(x/\theta)^{\tau}}$

$0.25 = 1 - e^{-\left(\frac{1000}{\theta}\right)^{\tau}}$
$-\ln(0.75) = \left(\frac{1000}{\theta}\right)^{\tau}$
(a) $\theta = 1000(-\ln(0.75))^{-1/\tau}$

$0.75 = 1 - e^{-\left(\frac{100{,}000}{\theta}\right)^{\tau}}$
$-\ln(0.25) = \left(\frac{100{,}000}{\theta}\right)^{\tau}$
(b) $\theta = 100{,}000(-\ln(0.25))^{-1/\tau}$

$\frac{(a)}{(b)} = 1 = \frac{1}{100}\left(\frac{\ln(.75)}{\ln(.25)}\right)^{-1/\tau}$

$\ln(100^{-\tau}) = \ln\left(\frac{\ln(.75)}{\ln(.25)}\right)$

$-\tau\ln(100) = \underbrace{\ln\left[\frac{\ln(0.75)}{\ln(0.25)}\right]}_{A}$

$\tau = \frac{A}{-\ln(100)}$
$= 0.3415$

## 1.1.6 → Properties of the exponential dist

→ hazard rate → $h(x) = \frac{f(x)}{S(x)} = \frac{f(x)}{1 - F(x)}$

→ exponential hazard rate is constant (only continuous dist w/ this property)

→ $\boxed{h(x) = \frac{\lambda e^{-\lambda x}}{e^{-\lambda x}} = \lambda}$

→ Memoryless property

$\boxed{\begin{array}{l}\rightarrow P(X > t + s \mid X > t) = P(X > s) \\ \rightarrow X - d \mid X > d \sim \text{Exp}(\lambda)\end{array}}$

→ $E(X - d \mid X > d) = E(X) = 1/\lambda$
→ $V(X - d \mid X > d) = V(X) = 1/\lambda^2$
→ $P(X - d \leq a \mid X > d) = P(X \leq a)$

→ Joint probabilities
→ $\boxed{X_i \overset{\perp}{\sim} \text{Exp}(\lambda_i) \rightarrow P(X_1 < X_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}}$

→ Generalizing to $X_1, \dots, X_n$ → $P(X_i = \min(X_1, \dots, X_n)) = \frac{\lambda_i}{\sum_{j=1}^n \lambda_j}$

→ Memoryless can also be applied to find probs involving 2 exp RVs

→ $P(X_1 - X_2 > a \mid X_1 > X_2) = P(X_1 > a)$

→ minimum of exponential RVs

$\boxed{\begin{array}{l}\text{→ let } X_i \overset{\perp}{\sim} \text{Exp}(\lambda_i) + Y = X_{(1)} \\ \rightarrow F_Y(y) = P(Y \leq y) \\ = P(\text{At least one } X_i \leq y) \\ = 1 - P(\text{All } X_i > y) \\ = 1 - e^{-y\sum\lambda_i} \text{ ↩ add rates} \\ \sim \text{Exp}(\lambda = \sum\lambda_i)\end{array}}$

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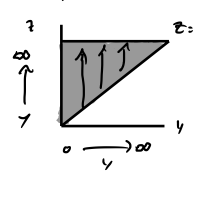
→ Assignment

→ Q1) → $X_i \overset{iid}{\sim} \text{Exp}(\frac{1}{\theta})$ (→ rate), let $Y = \sum X_i$ + $Z = \min(X_1, \dots, X_n)$

find $E(Y - Z) = E(Y) - E(Z)$
$= E(\sum X_i) - E(X_{(1)})$
~ Gamma $(\alpha = n, \theta)$;   $X_{(1)} \sim \text{Exp}(\frac{n}{\theta})$ ↳ rate
$= n\theta - \frac{\theta}{n}$
$= \theta(n - \frac{1}{n})$

→ Q2) → $X_i \overset{\perp}{\sim} \text{Exp}(\lambda_i)$, $\lambda_i = \frac{1}{i}$, $i = 1, \dots, 6$
→ $Y = \min(X_1, X_2, X_3)$ + $Z = \min(X_4, X_5, X_6)$

→ Find $P(Y < Z)$
$Y \sim \text{Exp}(\lambda = 11/6)$;   $Z \sim \text{Exp}(\lambda = 37/60)$




$$P(Y < Z) = \int_0^{\infty}\int_y^{\infty} \frac{37}{60}\left(\frac{11}{6}\right) e^{-\left(\frac{37}{60}\right)z} e^{-\left(\frac{11}{6}\right)y}\,dz\,dy$$
$$= \int_0^{\infty} \frac{11}{6} e^{-\left(\frac{11}{6}\right)y}\left[-e^{-\left(\frac{37}{60}\right)z}\right]_y^{\infty} dy$$
$$= \int_0^{\infty} \frac{11}{6} e^{-\left(\frac{11}{6}\right)y}\left[0 + e^{-\left(\frac{37}{60}\right)y}\right] dy$$
$$= -\frac{20}{49}\left(\frac{11}{6}\right) e^{-\frac{49}{20}y}\Big|_0^{\infty}$$
$$= 0.7483$$