★ only writing new stuff
(maybe quick summaries of old stuff)

1.1.2 → Pareto

→ overview → originally used to study income distributions
→ can be derived as a mixture of exponential RVs
★ → "pareto" on the exam refers to the two-parameter Pareto unless otherwise specified

→ Def → $\boxed{\text{If } X \sim \text{Pareto}(\alpha, \theta) \rightarrow f(x|\alpha,\theta) = \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}} \quad x \geq 0}$
↳ recognize constant $*(x+\theta)^{-\alpha}$

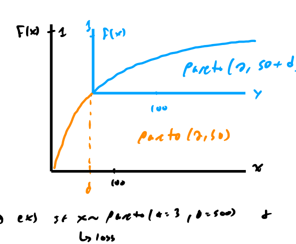
→ trivia → hazard function for Pareto decreases w/ x for any values of α + θ

→ No MGF
→ Expected value + variance

$$\boxed{E(X) = \frac{\theta}{\alpha-1} \qquad V(X) = (E(X))^2\frac{\alpha}{(\alpha-2)} = \frac{\theta^2}{(\alpha-1)^2(\alpha-2)}}$$

→ note only has $k < \alpha$ moments

→ $\boxed{\text{If } X\sim \text{Pareto}(\alpha,\theta) \text{ + } Y = X-d \mid X>d \Rightarrow Y \sim \text{Pareto}(\alpha, \theta+d)}$
→ useful for deductibles



→ ex) If $X \sim \text{Pareto}(\alpha=3, \theta=500)$ + insurance only covers losses above 200,
↳ loss
the payment is $Y = X - 200$. Find $E(Y)$

→ $Y = X - 200$
↓ $\sim \text{Pareto}(\alpha=3, \theta = 500+200)$
$= 700$

$\Rightarrow E(Y) = \frac{\theta}{\alpha-1} = \frac{700}{2} = 350$

→ Inverse Pareto distribution
→ If $X \sim \text{Pareto}(\alpha, \theta^*)$ + $Y = X^{-1}$

$\Rightarrow Y \sim \text{Inv Pareto}(\tau = \alpha, \theta = \theta^{*-1})$

$f(y) = \frac{\tau\theta y^{\tau-1}}{(y+\theta)^{\tau+1}}, \ y \geq 0$

→ resembles original pareto dist, except has an $x^{\tau-1}$ term in numerator

$f(x) = \frac{c \cdot x^{\tau-1}}{(x+\theta)^{\tau+1}}$

★ → Note this inversion methodology applies to all distributions on the exam table that have an inverse counterpart
→ θ is parameterized such that $X^{-1} \sim$ Inv c dist (same parameters, θ^{-1})
ex) Gamma(α, β) + InvGamma(α, 1/β)

→ Proof → let $X \sim \text{Pareto}(\alpha,\theta)$ + $Y = \frac{1}{X}$
↳ $x = h(y) = \frac{1}{y}$
$h'(y) = -\frac{1}{y^2}$

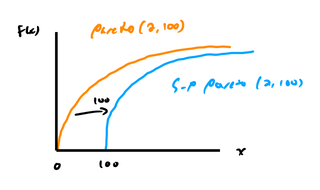
$$f_Y(y) = f_X(h(y))\,|h'(y)|$$
$$= \frac{\alpha\theta^\alpha}{(\frac{1}{y}+\theta)^{\alpha+1}}\left|\frac{-1}{y^2}\right|$$
$$= \frac{\alpha\theta^\alpha}{\left(\frac{1+\theta y}{y}\right)^{\alpha+1} y^2}$$
$$= \frac{\alpha\theta^\alpha y^{\alpha+1}}{(y+\theta)^{\alpha+1}y^2}$$
$$= \frac{\alpha\theta^\alpha y^{\alpha-1}}{(y+\theta)^{\alpha+1}}$$
$\sim \text{Inv Pareto}(\alpha, \theta^{-1})$

→ Single parameter pareto distribution

$\boxed{\text{If } X \sim \text{S-P Pareto}(\alpha, \theta) \Rightarrow f(x|\alpha,\theta) = \frac{\alpha\theta^\alpha}{x^{\alpha+1}}, \ x > \theta}$

amount the pareto is shifted & must be determined in advance (not estimated)
⇒ α is the only the parameter
→ hence the name

→ A single-parameter Pareto dist is a pareto dist shifted by θ
⇒ if X Pareto(α, θ) + $Y = X + \theta$
⇒ $Y \sim$ S-P Pareto(α, θ)



→ Expected value → $E(Y) = E(X+\theta)$
$= E(X) + \theta$
$= \frac{\theta}{\alpha-1} + \theta$
$= \boxed{\frac{\alpha\theta}{\alpha-1}}$

variance → $\boxed{V(Y) = V(X) = (E(X))^2\left(\frac{\alpha}{\alpha-2}\right) = \left(\frac{\theta}{\alpha-1}\right)^2\left(\frac{\alpha}{\alpha-2}\right)}$

→ Assignment

→ Q1 → $P(X > 25) = \int_{25}^{\infty} 200x^{-3}\,dx$
↳ ~ S-P Pareto (α = 2, θ = 10)

$\Rightarrow F_X(x) = 1 - (\theta/x)^\alpha$
$\Rightarrow S_X(x) = (\theta/x)^\alpha \rightarrow S(25) = \left(\frac{10}{25}\right)^2 = 4/25$

→ Q2 → $X \sim \text{Pareto}(\alpha, \theta=100)$, given $CV = 3$, find $V(X)$

→ $V(X) = (E(X))^2\left(\frac{\alpha}{\alpha-2}\right)$
↓ $= \left(\frac{\theta}{\alpha-1}\right)^2\left(\frac{\alpha}{\alpha-2}\right)$

→ $CV = \frac{\sigma}{\mu} \Rightarrow 3 = \frac{\sqrt{V(X)}}{E(X)}$
↓ $= \frac{\cancel{\left(\frac{\theta}{\alpha-1}\right)}\sqrt{\frac{\alpha}{\alpha-2}}}{\cancel{\theta/\alpha-1}}$

$9 = \frac{\alpha}{\alpha-2}$
$9\alpha - 18 = \alpha$
$8\alpha = 18$
$\alpha = 9/4$

$\Rightarrow V(X) = \left(\frac{100}{9/4-1}\right)^2\left(\frac{9/4}{9/4-2}\right)$
↓ $= 57{,}600$

→ Q3 → 2012 → $X \sim \text{Pareto}(\alpha=2, \theta)$
2013 → $X \sim \text{Pareto}(\alpha=2, \theta = k\theta)$, $k > 1$

$P(X_{12} > E(X_{13})) = 0.16$

S-P pareto

$\Rightarrow S(E(X_{13})) = (\theta/\cancel{E(X_{13})})^2$

$0.16 = (\theta/k\theta)^2$
$0.16 = ((k)^{-2})^{-1/2}$
$(0.16)^{-1/2} = k$
$2.5 = k$

$E(X_{13}) = \frac{k\theta}{2-1}$
↓ $= k\theta$

$\left(\frac{\theta}{E(X_{13})+\theta}\right)^2$
$= \left(\frac{\theta}{\theta k+\theta}\right)^2$
$(0.16)^{-1/2} = k+1 \leftarrow = \left(\frac{\theta}{\theta(k+1)}\right)^2$
$k = 1.5$

1.1.3 → Beta & Uniform

→ Beta dist

$$\boxed{\text{→ More general Beta pdf → If } X \sim \text{Beta}(\alpha,\beta,\theta) \rightarrow f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\left(\frac{x}{\theta}\right)^\alpha\left(1-\frac{x}{\theta}\right)^{\beta-1}\frac{1}{x}, \quad 0 \leq x \leq \theta}$$

→ if θ = 1 ⇒ beta dist I am used to

→ Mean + variance
→ $E(X) = \frac{\alpha}{\alpha+\beta}\cdot\theta$
→ $V(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}\cdot\theta^2$

→ uniform dist
→ special case of Beta
→ $X \sim \text{Beta}(\alpha=1, \beta=1, \theta) = \text{uniform}(0,\theta)$

→ Assignment
→ Q1) → from pdfs recognize
→ $f(x) = 3(1-x)^2$, $0<x<1 \Rightarrow X \sim \text{Beta}(\alpha=1, \beta=3, \theta=1)$
$f(y) = \frac{3y^2}{1000}$, $0<y<1000 \Rightarrow Y \sim \text{Beta}(\alpha^*=3, \beta^*=1, \theta^*=10)$
→ $\alpha\beta\theta + \alpha^*\beta^*\theta^* = 1(3)(1) \ 3(1)(10) = 33$

1.1.4 → Normal & Lognormal

→ lognormal dist

$\boxed{\begin{array}{l}\text{→ log-transformed normal dist} \\ \text{if } X \sim N(\mu,\sigma^2) + Y = e^X \\ \Rightarrow Y \sim \text{lognormal}(\mu,\sigma^2) \\ f_Y(y) = \frac{1}{y\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{1}{2\sigma^2}(\ln(y)-\mu)^2\right\}, \ y>0\end{array}}$

$\boxed{\begin{array}{l}\text{→ Mean + variance} \\ E(Y) = e^{\mu+\frac{1}{2}\sigma^2} \\ V(Y) = (E(X))^2(e^{\sigma^2}-1)\end{array}}$

→ cdf → found by log transforming the normal RV

→ $Y = e^X \Rightarrow X = \ln(y)$
$P(Y \leq y) = P(e^X \leq y)$
lognormal RV
$= P(X \leq \ln(y))$
$= P\left(\frac{X-\mu}{\sigma^2} \leq \frac{\ln(y)-\mu}{\sigma^2}\right)$
$= \boxed{P\left(Z \leq \frac{\ln(y)-\mu}{\sigma^2}\right)}$

→ Product of lognormal RVs
→ $Y_i \sim \text{lognormal}(\mu_i, \sigma_i^2)$, $i = 1,2$

$\boxed{\begin{array}{l}\text{→ if } Y = Y_1 Y_2 \\ = e^{X_1}e^{X_2} \\ = e^{X_1+X_2} \\ = e^X \\ \Rightarrow Y \sim \text{lognormal}(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)\end{array}}$

→ Assignment
→ Q1) → $X_1 \sim N(\mu=5, \sigma^2=4)$ $X_1 \perp X_2$
$X_2 \sim N(\mu=6, \sigma^2=3)$
→ Find $E(e^{X_1+X_2}) = E(e^Y)$

$Y \sim \text{lognormal}(\mu = 5+6, \sigma^2 = 4+3)$

$\Rightarrow E(Y) = e^{\mu+\frac{1}{2}\sigma^2}$
$= e^{11+\frac{1}{2}7}$
$= 1{,}982{,}759.264$

→ Q2) ⇒ $X \sim N(\mu=12, \sigma^2) \rightarrow x_{0.95} = 17$
$Y = e^X$
↓ ~ lognormal (μ = 12, σ²)
→ find |mode(X) − mode(Y)|
μ
$= 12 - \exp(\mu - \sigma^2)$
$= 12 - e^{12-9.1827}$
$= \cancel{4.7316}$

→ 17 = invNorm (0.95, 12, $\sqrt{\sigma^2}$)
⇒ $z_{0.95} = \frac{x_{0.95}-\mu}{\sigma}$
$1.65 = \frac{17-12}{\sigma}$
⇒ σ = 3.03
σ² = 9.1827

Correct logic → use 3 decimals ⇒ use critical value knowledge

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1.1.5 → Gamma, Exponential, & Weibull

→ Gamma dist
→ cdf → $F(x) = \Gamma(\alpha, x\beta)$ → incomplete gamma function
rate
$= \frac{x}{\theta}$
↳ scale

→ $\Gamma(\alpha, x) = \frac{1}{\Gamma(\alpha)}\int_0^x t^{\alpha-1}e^{-t}\,dt, \quad \alpha>0 + t>0$

→ need Excel if $\alpha \notin \mathbb{Z}^+$

$\boxed{\begin{array}{l}\text{→ But if } \alpha \in \mathbb{Z}^+ \Rightarrow \text{ where } N \sim \text{Poisson}(\lambda = x) \\ \Gamma(\alpha, x) = 1 - P(N < \alpha) \\ \rightarrow f(n) = \frac{e^{-x}x^n}{n!}\end{array}}$

→ example → let $X \sim \text{Gamma}(\alpha=2, \beta=1/5)$

→ $F(20) = \Gamma(\alpha, x\beta)$
$= \Gamma(2, 20(\frac{1}{5}))$
$= 4$
$= 1 - P(N < 2)$, $N \sim \text{Poisson}(\lambda = 4)$
$= 1 - [f_N(0) + f_N(1)]$
$= 1 - \left[\frac{e^{-4}4^0}{0!} + \frac{e^{-4}4^1}{1!}\right]$
$= 1 - 0.09157$
$= 0.9084$

→ Weibull dist
→ overview → Weibull dist is a transformed exponential dist

$\boxed{\begin{array}{l}\text{→ If } X \sim \text{Exp}(\mu) + Y = X^{1/\tau} \\ \Rightarrow Y \sim \text{Weibull}(\theta = \mu^{1/\tau}, \tau) \\ f_Y(y) = \frac{\tau(x/\theta)^\tau e^{-(x/\theta)^\tau}}{x} \quad x>0\end{array}}$

similar to gamma pdf, except
$f(x) = c \cdot x^{\tau-1}e^{-(x/\theta)^{\tau}}$

→ when τ = 1 ⇒ $Y \sim \text{Exp}(\mu)$
when θ = 1, it is the standard Weibull

→ hazard function of X increases with x if τ > 1 & decreases w/ x if τ < 1

$\boxed{\text{→ mean + variance → } E(X) = \theta\,\Gamma(1+\tfrac{1}{\tau}) \quad \rightarrow V(X) = \theta^2\,\Gamma(1+\tfrac{2}{\tau}) - [\theta\cdot\Gamma(1+\tfrac{1}{\tau})]^2}$

→ Inverse counterparts

→ If $X \sim \text{Gamma}(\alpha, \theta)$ + $Y = X^{-1}$
⇒ $Y \sim \text{Inv Gamma}(\alpha, \theta^{-1})$

→ If $X \sim \text{Exp}(\theta)$ + $Y = X^{-1}$
⇒ $Y \sim \text{Inv Exponential}(\theta^{-1})$

→ If $X \sim$ Weibull + $Y = X^{-1}$
⇒ $Y \sim \text{Inv Weibull}(\theta^{-1}, \tau)$

→ Assignment
→ Q1) $f(x) = c \cdot x^2 e^{-5x}$, $x>0$

→ Find c → $X \sim \text{Gamma}(\alpha=3, \beta=5)$

$c = \frac{\beta^\alpha}{\Gamma(\alpha)} = \frac{5^3}{2!} = 62.5$

→ Q2) → $X \sim \text{Weibull}(\tau, \theta)$, $x_{0.25} = 1000$, $x_{0.75} = 100{,}000$
→ Find τ
→ $F(x) = 1 - e^{-(x/\theta)^\tau}$

$0.25 = 1 - e^{-\left(\frac{1000}{\theta}\right)^\tau}$
$-\ln(0.75) = \left(\frac{1000}{\theta}\right)^\tau$
(a) $\theta = 1000(-\ln(0.75))^{-1/\tau}$

$0.75 = 1 - e^{-\left(\frac{100{,}000}{\theta}\right)^\tau}$
$-\ln(0.25) = \left(\frac{100{,}000}{\theta}\right)^\tau$
(b) $\theta = 100{,}000(-\ln(0.25))^{-1/\tau}$

$\frac{(a)}{(b)} = 1 = \frac{1}{100}\left(\frac{\ln(.75)}{\ln(.25)}\right)^{-1/\tau}$
$\ln(100^{-\tau}) = \ln\left(\frac{\ln(.75)}{\ln(.25)}\right)$
$-\tau\ln(100) = \ln\left[\frac{\ln(0.75)}{\ln(0.25)}\right]$
$\tau = \frac{d}{-\ln(100)}$
$= 0.3415$

1.1.6 → Properties of the exponential dist

→ hazard rate → $h(x) = \frac{f(x)}{S(x)} = \frac{f(x)}{1-F(x)}$

→ exponential hazard rate is constant (only continuous dist w/ this property)
→ $\boxed{h(x) = \frac{\lambda e^{-\lambda x}}{e^{-\lambda x}} = \lambda}$

→ Memoryless property
$\boxed{\begin{array}{l}\rightarrow P(X > t+s \mid X>t) = P(X>s) \\ \rightarrow X-d \mid X>d \sim \text{Exp}(\lambda)\end{array}}$

→ $E(X-d \mid X>d) = E(X) = 1/\lambda$
→ $V(X-d \mid X>d) = V(X) = 1/\lambda^2$
→ $P(X-d \leq a \mid X>d) = P(X \leq a)$

→ minimum of exponential RVs

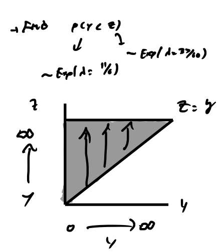
$\boxed{\begin{array}{l}\text{→ let } X_i \overset{\perp\!\!\!\perp}{\sim} \text{Exp}(\lambda_i) + Y = X_{(1)} \\ \rightarrow F_Y(y) = P(Y \leq y) \\ = P(\text{At least one } X_i \leq y) \\ = 1 - P(\text{All } X_i > y) \\ = 1 - e^{-y\sum\lambda_i} \text{ add rates} \\ \sim \text{Exp}(\lambda = \textstyle\sum\lambda_i)\end{array}}$

→ Assignment
→ Q1) → $X_i \overset{iid}{\sim} \text{Exp}\left(\frac{1}{\theta}\right)$, let $Y = \sum X_i$ + $Z = \min(X_1, \ldots, X_n)$
rate

find $E(Y-Z) = E(Y) - E(Z)$
$= E(\sum X_i) - E(X_{(1)})$
$\sim \text{Gamma}(\alpha = n, \theta)$ $\sim \text{Exp}\left(\frac{n}{\theta}\right)$ ↳ rate
$= n\theta - \frac{\theta}{n}$
$= \theta\left(n - \frac{1}{n}\right)$

→ Q2) → $X_i \overset{\perp\!\!\!\perp}{\sim} \text{Exp}(\lambda_i)$, $\lambda_i = \frac{1}{i}$, $i = 1, \ldots, 6$
→ $Y = \min(X_1, X_2, X_3)$ + $Z = \min(X_4, X_5, X_6)$

→ Find $P(Y < Z)$
$\sim \text{Exp}(\lambda = 11/6)$ $\sim \text{Exp}(\lambda = 37/60)$



$$P(Y<Z) = \int_0^\infty\int_y^\infty \frac{37}{60}\left(\frac{11}{6}\right)e^{-\left(\frac{37}{60}\right)z}e^{-\left(\frac{11}{6}\right)y}\,dz\,dy$$
$$= \int_0^\infty \frac{11}{6}e^{-(11/6)y}\left[-e^{-(37/60)z}\right]_y^\infty dy$$
$$= \int_0^\infty \frac{11}{6}e^{-(11/6)y}\left[0 + e^{-(37/60)y}\right]dy$$
$$= -\frac{20}{49}\left(\frac{11}{6}\right)e^{-49/20\,y}\Big|_0^\infty$$
$$= 0.7483$$