1.5.1 → Introduction to systems

→ overview → systems have multiple components, each w/ its own lifetime (a RV)

$$x_i = \begin{cases} 1 & \text{if component } i \text{ is functioning} \\ 0 & \text{— — — has failed} \end{cases}$$

→ states of all components in a system can be written as a vector

<u>state vector</u> : $x = (x_1, x_2, \dots, x_n)$

→ has 2^n distinct state vectors

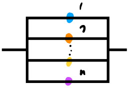
→ state of system depends on states of components

$$\phi(x) = \begin{cases} 1 & \text{if system is functioning} \\ 0 & \text{— — has failed} \end{cases}$$

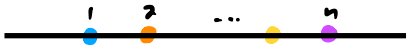
↳ structure function

→ Different types of systems

→ parallel system → functions as long as one of the components is functioning



→ Series system → functions only when all components are functioning
if one component fails, the whole system fails



→ k-out-of-n system → functions if & only if at least k out of n components are functioning

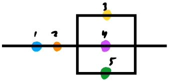
→ parallel system = 1-out-of-n system
series system = n-out-of-n system

→ Bridge system → Two branches that are "bridged" somewhere in the middle

→ other systems → in reality, there is an unlimited # of systems.
they can be constructed as any combo of the 4 given above

→ example → Draw a 2-out-of-2 system placed w/ a 1-out-of-3 system

$\underbrace{\text{2-out-of-2}}_{\text{↳ series}}$ $\underbrace{\text{1-out-of-3}}_{\text{↳ parallel}}$



Combo of states resulting in a functioning system

x_1	x_2	x_3	x_4	x_5
1	1	1	0	0
1	1	0	1	0
1	1	0	0	1
1	1	1	1	0
1	1	0	1	1

1.5.2 → Minimal path sets & Minimal cut sets

→ def → A state vector is a <u>path vector</u> if the system is functioning, i.e $\phi(x) = 1$

→ A path vector is a <u>minimal path vector</u> if the system would fail if any of the functioning components in x were to fail, i.e there are no "backup" components ⇒ turn any off & system fails

→ x is a minimal path vector if $\phi(y) = 0$ for every $y < x$

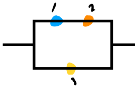
→ $y < x$ = each y_i must be $\leq x_i$, with at least one $y_i < x_i$

→ Each minimal path vector can be represented as a <u>minimal path set</u>, or a minimal set of components whose functioning guarantees the functioning of the system

→ Let A be a minimal path set, then given a minimal path vector x, the minimal path set is $\{i : x_i = 1\}$

★ ⇒ A system will function if & only if all components of at least one minimal path set function

→ Example → Determine minimal path sets for this system



→ All possible states of components & resulting state of system

x_1	x_2	x_3	$\phi(x)$	
0	0	0	0	
0	0	1	1	✓
0	1	0	0	
0	1	1	1	✓
1	0	0	0	
1	0	1	1	✓
1	1	0	1	✓
1	1	1	1	✓

} 5 path vectors (i.e system is functioning)

→ To determine which of these are minimal path vectors, first list all vectors y that satisfy $y < x$ (for each x)

path vector x	All y
(0, 0, 1)	(0, 0, 0)
(0, 1, 1)	(0.0.0), (0.0.1), (0.1.0)
⋮	⋮

→ $y \overset{\checkmark}{<} x$ b/c $y_1 \leq x_1$, $y_2 \leq x_2$ & $\underline{y_3 < x_3}$

↳ need a strictly less than comparison

→ After listing all vectors y, check if $\phi(y) = 0$

→ A path vector x can only be a minimal path vector if **all** of its corresponding y vectors result in $\phi = 0$

→ only (0,0,0), (0,1,0), & (1,0,0) meet this

⇒ (0,0,1) & (1,1,0) are minimal path vectors

⇒ {3}, {1,2} are the minimal path sets for the system

⇒ As long as component 3 or components 1 & 2 are functioning, the system will function

→ could also logic instead of the formal process

→ we know a minimal path set is a minimal set of components whose functioning guarantees the functioning of the system

⇒ then look for sets of non-overlapping components for which no component can be removed w/o causing the system to fail

⇒ obviously it is {3} & {1,2}

→ minimal cut sets

→ defs → In contrast to a path vector, a state vector x is a cut vector if the system is not functioning (i.e $\phi(x) = 0$).

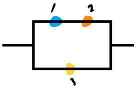
→ A cut vector is a minimal cut vector if repairing any of the non-functioning components in x would result in the whole system functioning ⇒ turn any on & system works

→ So x is a minimal cut vector if $\phi(y) = 1$ for every $y > x$ (again all $y_i \geq x_i$ & at least one $y_i > x_i$)

→ Each minimal cut vector can be represented as a minimal cut set, or a minimal set of components whose failure guarantees the failure of the system. Let C be a minimal cut set, then given a minimal cut vector x, the minimal cut set C is $\{i : x_i = 0\}$

☆ ⇒ A system will fail if & only if all components of at least one minimal cut set fail

→ Example → Determine minimal cut sets for this system



→ formal process → All possible states of components & resulting state of system

x_1	x_2	x_3	$\phi(x)$	
0	0	0	0	×
0	0	1	1	
0	1	0	0	×
0	1	1	1	
1	0	0	0	×
1	0	1	1	
1	1	0	1	
1	1	1	1	

} 3 path vectors (i.e system is not functioning)

→ list all $y > x$ for

cut vector x	all y
(0,0,0)	(0,0,1), (0,1,0), ...
(0,1,0)	⋮
(1,0,0)	

→ A cut vector can only be a minimal cut vector if <u>all</u> of its corresponding y vectors result in $\phi(y)=1$

⇒ only works for (0,1,0) & (1,0,0)

⇒ $\{1,3\}$ & $\{2,3\}$ are the minimal cut sets

→ if components 1 & 3 or components 2 & 3 stop functioning, the system is gaurenteed to fail

→ OR use <u>logic</u> again

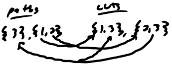
→ obviously system fails if all 3 components fail, but all failing isn't necessary (want <u>minimal</u>)

→ component 3 must fail, but only one of the other two must fail

⇒ $\{1,3\}, \{2,3\}$

☆ → notes → all minimal path sets must have at least one component from each minimal cut set

→



→ Counting

→ We can then calculate the # of minimal path sets & minimal cut sets for the common systems shown earlier

System	# minimal path sets	# minimal cut sets
parallel	n	1
series	1	n
K-out-of-n	$\binom{n}{k}$	$\binom{n}{n-k+1}$
Bridge	4	4

→ If we have multiple systems & they are placed in...

→ parallel → total minimal path sets = $\sum$ individual amounts
total minimal cut sets = $\prod$ individual amounts

→ series → total minimal path sets = $\prod$ individual amounts
total minimal cut sets = $\sum$ individual amounts

→ Assignment

→ Q1) structure function → $\phi(x) = x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4 - 2 x_1 x_2 x_3 x_4$

a path vector $x = (1,0,1,1) \Rightarrow \phi(x) \overset{\checkmark}{=} 1$

1.5.3 → Structure functions

→ overview → structure functions $\phi(x)$ are functions that represent the state of a system. Can be derived using either minimal path sets or minimal cut sets

→ for parallel system → functions as long as at least one component functions

$$\boxed{\Rightarrow \phi(x) = \max(x_1, \dots, x_n)}$$

→ for series system → functions if & only if all of its components function

$$\boxed{\Rightarrow \phi(x) = \min(x_1, \dots, x_n)}$$

→ Since x is binary, the following relationships are true

$$\boxed{\begin{aligned} &\rightarrow \max(x_1, \dots, x_n) = 1 - \prod_{i=1}^{n} (1 - x_i) \\ &\rightarrow \min(x_1, \dots, x_n) = \prod_{i=1}^{n} x_i \\ &\rightarrow x_i^{k} = x_i \,,\ k > 0 \end{aligned}}$$

→ minimal path sets approach

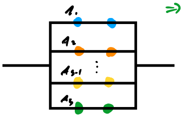
→ A system will function if & only if all components of at least one minimal path set function

→ A parallel system will function if & only if at least one component functions

★ ⇒ can treat any system as a parallel system of its minimal path sets.

Therefore, given the minimal path sets of a system, we can express the structure function of the system the same way we express the structure function of a parallel system, just treat minimal path sets as individual components

→ Let $A_1, \dots, A_s$ be minimal path sets of a system. This can be expressed as (not what the actual system looks like, just how we can think about it)

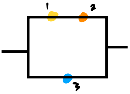


⇒ there is another representation using minimal cut sets approach

$$\Rightarrow \phi(x) = \max\left(\min_{i \in A_1} x_i \,,\ \min_{i \in A_2} x_i \,,\ \dots ,\ \min_{i \in A_s} x_i \right)$$

$$\downarrow \quad = \max_{i} \prod_{i \in A_i} x_i$$

→ Example → determine $\phi(x)$ using minimal path sets approach



↓ $\{3\}, \{1,2\}$

$$\begin{aligned} \rightarrow \phi(x) &= \max(\min(x_1, x_2), \min(x_3)) \\ &= \max(x_1 x_2 \,,\ x_3) \\ &= 1 - (1 - x_1 x_2)(1 - x_3) \\ &= 1 - (1 - x_1 x_2 - x_3 + x_1 x_2 x_3) \\ &= x_1 x_2 + x_3 - x_1 x_2 x_3 \end{aligned}$$

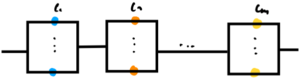
→ minimal cut sets approach

→ Relationship exist just like above

→ A system will fail if & only if all components of at least one minimal cut set fail

→ A series system will fail if & only if at least one component fails

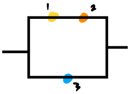
⇒ any system can be thought of as a series system of its minimal cut sets.
If know minimal cut sets, treat as individual components of the system & draw same way as a series system



$$\Rightarrow \quad \phi(x) = \min\left(\max_{i\in C_1} x_i, \max_{i\in C_2} x_i, \ldots, \max_{i\in C_m} x_i\right)$$

$$\downarrow \quad = \prod_{j=1}^{m} \max_{i\in C_j} x_i$$

→ Example → determine $\phi(x)$ using minimal cut sets approach



↓ $\{1,3\}, \{2,3\}$

$$\rightarrow \quad \phi(x) = \min(\max(x_1,x_3), \max(x_2,x_3))$$

$$= \min[1-(1-x_1)(1-x_3), 1-(1-x_2)(1-x_3)]$$

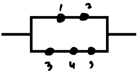
$$= (1-(1-x_1)(1-x_3))(1-(1-x_2)(1-x_3))$$

$$= (\ldots)$$

$$\checkmark = x_1x_2 + x_3 - x_1x_2x_3 \quad \text{(same as before)}$$

→ Assignment

→ Q1) Determine structure function



⇒ currently shown as minimal paths sets approach

$$\Rightarrow \quad \phi(x) = \max(\min(x_1,x_2), \min(x_3,x_4,x_5))$$

$$= \max(x_1x_2, x_3x_4x_5)$$

$$= 1-(1-x_1x_2)(1-x_3x_4x_5)$$

$$= 1-[1 - x_3x_4x_5 - x_1x_2 + x_1x_2x_3x_4x_5]$$

$$= x_3x_4x_5 + x_1x_2 - x_1x_2x_3x_4x_5$$