# 1.5.4 → Reliability of systems

→ overview → reliability of a component is the probability that the component is functioning

- → Reliability of a system is the prob that the system is functioning
- → Reliability of a function can be expressed as a function of the reliability of its components ⟹ called the reliability function

→ Let $X_i \sim$ Bernoulli RV representing the state of component $i$ & let $p_i$ be the prob that component $i$ functions (i.e. $P(X_i=1)=p_i$). Then define two vectors

- → $X=(X_1, X_2, \dots, X_n)$ is the vector of RVs for the components (similar to state vector $x$, just w/ RVs now)
- → $p=(p_1, p_2, \dots, p_n)$ is the vector of reliabilities of the components in a system

→ $r(p)$ = reliability function of a system = $\sum P(\text{path vectors}) = 1-\sum P(\text{cut vectors})$

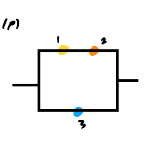
$$\boxed{r(p)=P[\phi(X)=1]=1-P[\phi(X)=0]}$$

→ If we know $\phi(x)$, we can determine $r(p)$ w/

$r(p)=E[\phi(X)]$ (b/c $\phi(X)$ is a Bernoulli RV)

→ Example

→ determine $r(p)$



⟹ 5 path vectors
$\{(0,0,1), (0,1,1), (1,0,1), (1,1,0), (1,1,1)\}$
& 3 cut vectors
$\{(0,0,0), (0,1,0), (1,0,0)\}$

→ $r(p)=P(\phi(X)=1)$ (paths)

$$\begin{aligned}&=P[X=(0,0,1)]+P[X=(0,1,1)]+\dots+P[X=(1,1,1)]\\&=(1-p_1)(1-p_2)p_3+(1-p_1)p_2p_3+\dots+p_1p_2p_3\\&=(\dots)\\&=p_1p_2+p_3-p_1p_2p_3\end{aligned}$$

$$\begin{aligned}&=1-P(\phi(X)=0) \quad \text{(cuts)}\\&=1-\{P(X=(0,0,0))+P[X=(0,1,0)]+P[X=(1,0,0)]\}\\&=1-[(1-p_1)(1-p_2)(1-p_3)+\dots+p_1(1-p_2)(1-p_3)]\\&\overset{\checkmark}{=}p_1p_2+p_3-p_1p_2p_3\end{aligned}$$

→ we also know $\phi(x)=x_1x_2+x_3-x_1x_2x_3$ from earlier example

⟹ $r(p)=E[\phi(X)]$

$$\begin{aligned}&=E[X_1X_2+X_3-X_1X_2X_3]\\&=E(X_1)E(X_2)+E(X_3)-E(X_1)E(X_2)E(X_3) \quad \text{(b/c } \perp\text{)}\\&\overset{\checkmark}{=}p_1p_2+p_3-p_1p_2p_3\end{aligned}$$

→ lastly if know $p_i$ ⟹ plug in & calculate

→ suppose $p_i=0.7$ ⟹ $r(p)=0.7(0.7)+0.7-0.7(0.7)(0.7)=0.847$

→ If $r(p)$ is a reliability function of a system of $\perp$ components, then $r(p)$ is an increasing function of $p$ ⟹ system is monotone

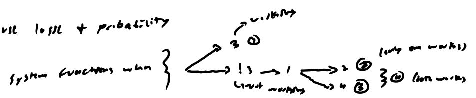
→ Examples

→ Given information → minimal path sets are $A_1=\{3\}$, $A_2=\{1,2\}$, $A_3=\{1,4\}$
→ all components are $\perp$
→ $p_i=0.8$ for all $i$

→ calculate reliability

→ Basic probability → $2^4=16$ possible state vectors
⟹ Tedious to list all of them & $\sum P(\phi(x)=1)$

⟹ use logic + probability



$$r(p)=\overset{①}{0.8}+0.2(0.8)[\overset{②}{0.8(0.2)}+\overset{③}{0.2(0.8)}+\overset{④}{0.8(0.8)}]$$

$$=0.9536$$

→ Structure function

→ $\phi(x)=x_3+x_1x_2+x_1x_4-x_1x_2x_3-x_1x_3x_4-x_1x_2x_4+x_1x_2x_3x_4$
(from earlier example)

⟹ $r(p)=E[\phi(X)]$

$$\begin{aligned}&=E[X_3+X_1X_2+X_1X_4-X_1X_2X_3-X_1X_3X_4-X_1X_2X_4+X_1X_2X_3X_4]\\&=p_3+p_1p_2+\dots+p_1p_2p_3p_4\\&=0.8+0.8^2+\dots+0.8^4\\&\overset{\checkmark}{=}0.9536\end{aligned}$$

★ can only substitute $p_i$ for $X_i$ when $\phi(x)$ is expanded enough

$E[(1-X_3)(1-X_1X_2)(1-X_1X_4)] \neq (1-E(X_3))(1-E(X_1)E(X_2))(1-E(X_1)E(X_4))$

multiplying same term is the clue need to keep expanding

→ Given info about 3-out-of-4 system

→ All components are $\perp$
→ $p_i=0.8$, $i=1,4$
$p_j=0.9$, $j=2,3$

→ calculate $r(p)$

→ Tedious to find & expand $\phi(x)$ ⟹ use basic probability + logic

working scenarios (where $\geq 3$ $X_i=1$)

$p_1 - p_2 - p_3 - !p_4$ ①
$p_1 - p_2 - !p_3 - p_4$ ②
$p_1 - !p_2 - p_3 - p_4$ ③
$!p_1 - p_2 - p_3 - p_4$ ④
$p_1 - p_2 - p_3 - p_4$ ⑤

| Scenario | P(scenario) |
|---|---|
| ① | $p_1p_2p_3(1-p_4)=0.1296$ |
| ② | $p_1p_2(1-p_3)p_4=0.0576$ |
| ③ | $p_1(1-p_2)p_3p_4=0.0576$ |
| ④ | $(1-p_1)p_2p_3p_4=0.1296$ |
| ⑤ | $p_1p_2p_3p_4=0.5184$ |
| $\sum(\cdot)$ | $r(p)=0.8928$ |

↳ = one form of $r(p)$ ⟹ can derive structure function from this

$\phi(x)=x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4+x_1x_2x_3x_4$

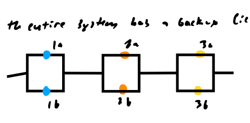
(x's that are "on")

→ Assume you want to build a series system w/ 3 types of components, with $p_i=0.6$. You have 2 of each component, & all are $\perp$

→ calculate $P(\phi(X)=1)$ if:

1) each component type has a backup (i.e. the series is a series of 3 parallel systems)
2) the entire system has a backup (i.e. the system is a parallel of 2 series systems)

→ 1)



⟹ system fails if both of the same type fail

⟹ minimal cut sets are $\{1a,1b\}$, $\{2a,2b\}$, $\{3a,3b\}$
minimal cut sets approach

$$\begin{aligned}\phi(x)&=\min\{\max(x_{1a},x_{1b}),\max(2a,2b),\max(3a,3b)\}\\&=\min\{(1-(1-x_{1a})(1-x_{1b})),(1-(1-x_{2a})(1-x_{2b})),(1-(1-x_{3a})(1-x_{3b}))\}\\&=(1-(1-x_{1a})(1-x_{1b}))\cdot(1-(1-x_{2a})(1-x_{2b}))\cdot(1-(1-x_{3a})(1-x_{3b}))\\&=(x_{1a}+x_{1b}-x_{1a}x_{1b})(x_{2a}+x_{2b}-x_{2a}x_{2b})(x_{3a}+x_{3b}-x_{3a}x_{3b})\end{aligned}$$

$$\begin{aligned}r(p)&=E(\phi(X))\\&=E[(\dots)]\\&=(p+p-p^2)^3\\&=(0.6+0.6-0.6^2)^3\\&=0.5927\end{aligned}$$

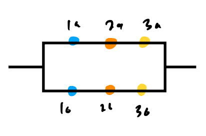
(always multiplying different x terms ⟹ can sub $p_i$ for $X_i$)

OR

★ → logic → $P(\phi(X)=1)=P(\text{all 3 parallel systems function})$

$$\begin{aligned}&=[1-P(\text{parallel system fails})]^3\\&=(1-(0.4)^2)^3\\&=0.5927\end{aligned}$$

→ 2)



★ logic → $P(\phi(X)=1)=P(\geq 1 \text{ series system functions})$

$$\begin{aligned}&=1-P(\text{Both series systems fail})\\&=1-\{P(\geq 1 \text{ component fails})\}^2\\&=1-[1-P(\text{all components function})]^2\\&=1-[1-0.6^3]^2\\&=0.3853\end{aligned}$$

OR

→ minimal path sets = $\{1a,2a,3a\}$ & $\{1b,2b,3b\}$

$$\begin{aligned}\Rightarrow \phi(x)&=\max[\min(x_{1a},x_{2a},x_{3a}),\min(x_{1b},x_{2b},x_{3b})]\\&=\max[(x_{1a}\cdot x_{2a}\cdot x_{3a}),(x_{1b}\cdot x_{2b}\cdot x_{3b})]\\&=1-(1-x_{1a}\cdot x_{2a}\cdot x_{3a})(1-x_{1b}x_{2b}x_{3b})\end{aligned}$$

$$\begin{aligned}\Rightarrow E[\phi(X)]&=E[(\dots)]\\&=(\dots)\\&=1-(1-p^3)^2\\&\overset{\checkmark}{=}0.3853\end{aligned}$$

→ Key takeaway → reliability is higher in the system where each component has a backup, rather than where the entire system has a backup. This is generally true for a system w/ n components

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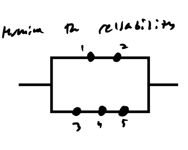
→ Assignment

→ Q1 → Determine the reliability



$p_i=0.72$, $i=1,2,\dots,5$

→ logic → $P(\phi(X)=1)=P(\geq 1 \text{ parallel system functions})$

$$\begin{aligned}&=1-P(\text{Both parallel systems fail})\\&=1-P(\geq 1 \text{ component fails in each parallel system})\\&=1-(1-P(\text{All components function in each parallel systems}))\\&=1-(1-p_1p_2)(1-p_3p_4p_5)\\&=1-(1-0.72(1)0.72(2))(1-0.72(3)0.72(4)0.72(5))\\&=0.5716\end{aligned}$$

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# 1.5.5 → Bounds of reliability function

→ overview → previously learned how to derive exact reliability function of a system. However this is often impractical / unnecessary. Rather we can approximate w/ lower & upper bounds

→ There are two methods for this

→ Method of inclusion & exclusion

→ Recall union probabilities

$P(E_1\cup E_2)=P(E_1)+P(E_2)-P(E_1\cap E_2)$

$$\begin{aligned}P(E_1\cup E_2\cup E_3)=&P(E_1)+P(E_2)+P(E_3)\\&-[P(E_1\cap E_2)+P(E_1\cap E_3)+P(E_2\cap E_3)]\\&+P(E_1\cap E_2\cap E_3)\end{aligned}$$

⟹ generalized to union of n events

$$\boxed{\begin{aligned}P\Big(\bigcup_{i=1}^{n}E_i\Big)=&\sum_{i=1}^{n}P(E_i) \quad ①\\&-\sum_{j=1}^{n}\sum_{k>j}P(E_j\cap E_k) \quad ②\\&+\sum_{j=1}^{n}\sum_{k>j}\sum_{l>k}P(E_j\cap E_k\cap E_l) \quad ③\\&\vdots\\&\dots+(-1)^{n+1}P(E_1\cap E_2\cap\dots\cap E_n) \quad ⓝ\end{aligned}}$$

★ this is the main formula we will use next

→ Notice if only include terms ①, we are overestimating the probability of the union. Similarly, if we include terms ①+②, we are underestimating the prob of the union

⟹ can use these to create bounds

→ Minimal path sets

→ Let $A_1,\dots,A_s$ be the minimal path sets & $p_i$ be the reliability of component $i$. The bounds of the system's reliability are:

$$\boxed{\begin{aligned}&① \; r(p)\leq\sum_{j=1}^{s}\Big(\prod_{i\in A_j}p_i\Big) && (\textstyle\sum \text{(all on)})\\&② \; r(p)\geq(\text{------}) - \sum_{j=1}^{s}\sum_{k>j}\Big(\prod_{i\in A_j\cup A_k}p_i\Big) && (-\textstyle\sum \text{(all on in either)})\\&③ \; r(p)\leq(\text{----------------}) + \sum_{j=1}^{s}\sum_{k>j}\sum_{l>k}\Big(\prod_{i\in A_j\cup A_k\cup A_l}p_i\Big) && (+(\dots))\\&\vdots\end{aligned}}$$

→ The bounds of the reliability will converge to the exact reliability as more terms are included

⟹ line ③ upper bound is more precise than line ① upper bound

⟹ including all possible terms in the formula will result in the exact reliability function of the system, so this is another method

→ This approximation using minimal path sets is most effective when the reliabilities of the components are low. If the reliabilities are high, the reliability interval will widen & become less useful

→ Example → Minimal path sets of a 3-out-of-4 system are:

$A_1=\{1,2,3\}$ $A_2=\{1,2,4\}$, $A_3=\{1,3,4\}$, $A_4=\{2,3,4\}$

→ Terms of the bounds formulas are:

$$\begin{aligned}① \sum_{j=1}^{4}\Big(\prod_{i\in A_j}p_i\Big)&=\prod_{\substack{i\in A_1\\ i\in\{1,2,3\}}}p_i+\prod_{i\in A_2}p_i+\prod_{i\in A_3}p_i+\prod_{i\in A_4}p_i\\&=p_1p_2p_3+p_1p_2p_4+p_1p_3p_4+p_2p_3p_4\\&=4p^3 \quad \text{(all } p_i=p\text{)}\end{aligned}$$

$$\begin{aligned}② \sum_{j=1}^{4}\sum_{k>j}\Big(\prod_{i\in A_j\cup A_k}p_i\Big)&=\prod_{\substack{i\in A_1\cup A_2\\ i\in\{1,2,3\}\cup\{1,2,4\}}}p_i+\prod_{i\in A_1\cup A_3}p_i+\dots+\prod_{i\in A_3\cup A_4}p_i \quad \Big(\tbinom{4}{2}\text{ total terms in }\textstyle\sum\sum\Big)\\&=p_1p_2p_3p_4+p_1p_2p_3p_4+\dots+p_1p_2p_3p_4\\&=6p_1p_2p_3p_4\\&=6p^4\end{aligned}$$

⟹ Bounds $① - ② \leq r(p) \leq ①$

$4p^3-6p^4\leq r(p)\leq 4p^3$

⟹ if $p=0.2$ ⟹ $0.0224\leq r(p)\leq 0.032$ → narrow ✓

→ exact reliability $=p_1p_2p_3+p_1p_2p_4+p_1p_3p_4+p_2p_3p_4-3p_1p_2p_3p_4$

$$\begin{aligned}&=4p^3-3p^4\\&=0.0272\end{aligned}$$

⟹ if $p=0.6$ ⟹ $0.0864\leq r(p)\leq 0.864$ → wide ✗

→ can narrow interval by including more terms

→ Minimal cut sets

→ Let $C_1,\dots,C_m$ be the minimal cut sets & $p_i$ be the reliability of component $i$. The bounds of the system's reliability are

$$\boxed{\begin{aligned}&① \; 1-r(p)\leq\sum_{j=1}^{m}\Big(\prod_{i\in C_j}(1-p_i)\Big) && (\textstyle\sum \text{(all off)})\\&② \; 1-r(p)\geq(\text{------}) - \sum_{j=1}^{m}\sum_{k>j}\Big(\prod_{i\in C_j\cup C_k}(1-p_i)\Big) && (-\textstyle\sum \text{(all off in either)})\\&③ \; 1-r(p)\leq(\text{----------------}) + \sum_{j=1}^{m}\sum_{k>j}\sum_{l>k}\Big(\prod_{i\in C_j\cup C_k\cup C_l}(1-p_i)\Big) && (+(\dots))\\&\vdots\end{aligned}}$$

→ Again converges w/ more terms. Most effective w/ minimal cut sets when reliabilities are very high

→ Example → minimal cut sets of a 3-out-of-4 system are

$C_1=\{1,2\}$, $C_2=\{1,3\}$, $C_3=\{1,4\}$
$C_4=\{2,3\}$, $C_5=\{2,4\}$, $C_6=\{3,4\}$

→ Terms of bounds are:

$$\begin{aligned}① \sum_{j=1}^{6}\Big(\prod_{i\in C_j}(1-p_i)\Big)&=\prod_{\substack{i\in C_1\\ i\in\{1,2\}}}(1-p_i)+\prod_{i\in C_2}(1-p_i)+\dots+\prod_{i\in C_6}(1-p_i)\\&=(1-p_1)(1-p_2)+(1-p_1)(1-p_3)+\dots+(1-p_3)(1-p_4)\\&=6(1-p)^2 \quad \text{(all } p_i=p\text{)}\end{aligned}$$

$$\begin{aligned}② \sum_{j=1}^{6}\sum_{k>j}\Big(\prod_{i\in C_j\cup C_k}(1-p_i)\Big)&=\prod_{\substack{i\in C_1\cup C_2\\ i\in\{1,2\}\cup\{1,3\}}}(1-p_i)+\dots+\prod_{\substack{i\in C_1\cup C_6\\ i\in\{1,2\}\cup\{3,4\}}}(1-p_i)+\dots+\prod_{\substack{i\in C_5\cup C_6\\ i\in\{2,4\}\cup\{3,4\}}}(1-p_i)\\&=(1-p_1)(1-p_2)(1-p_3)+\dots+(1-p_1)(1-p_2)(1-p_3)(1-p_4)+\dots+(1-p_2)(1-p_3)(1-p_4)\\&=12(1-p)^3+3(1-p)^4 \quad \Big(\text{all } p_i=p \text{ \& } \tbinom{6}{2}=15 \text{ total terms in series}\Big)\end{aligned}$$

⟹ bounds

$① - ② \leq 1-r(p)\leq ①$

$6(1-p)^2-[12(1-p)^3+3(1-p)^4]\leq 1-r(p)\leq 6(1-p)^2$

⟹ if $p=0.9$ ⟹ $0.0477\leq 1-r(0.9)\leq 0.06$

⟹ $0.0477-1\leq -r(0.9)\leq 0.06-1$

$0.94\leq r(0.9)\leq 0.9523$ → narrow ✓

→ exact reliability → $r(0.9)=0.9477$

⟹ if $p=0.6$ ⟹ $(\dots)$

⟹ $0.04\leq r(0.6)\leq 0.8848$ → wide ✗

→ Method of intersection

→ Instead of union of events, this method is based on intersection of events

→ Let $A_1,\dots,A_s$ be the minimal path sets & $C_1,\dots,C_m$ be the minimal cut sets & $p_i$ be the reliability of component $i$. Bounds using method of intersection are:

$$\boxed{\underbrace{\prod_{j=1}^{m}\Big[1-\prod_{i\in C_j}(1-p_i)\Big]}_{P\left(\text{at least one component in each minimal cut set is functioning}\right)}\leq r(p)\leq 1-\underbrace{\prod_{j=1}^{s}\Big[1-\prod_{i\in A_j}p_i\Big]}_{P\left(\text{all components in at least one minimal path are functioning}\right)}}$$

?

⟹ remove prob of complete failure of components in all minimal cut sets (b/c as long as one component is alive in each minimal cut set, the system will function)

⟹ remove probability of complete failure of all components possible in the minimal path sets (so ≥1 minimal path set can survive)

→ method of intersection works best when the values of p are all very large or very small

→ Example → Consider 3-out-of-4 system where all components have reliability p. Determine bounds using the method of intersection

→ Minimal path sets:

$A_1=\{1,2,3\}$ $A_2=\{1,2,4\}$, $A_3=\{1,3,4\}$, $A_4=\{2,3,4\}$

→ upper bound is:

$$1-\prod_{j=1}^{4}\Big[1-\prod_{i\in A_j}p_i\Big]=1-\Big[1-\prod_{\substack{i\in A_1\\ i\in\{1,2,3\}}}p_i\Big]\Big[1-\prod_{i\in A_2}p_i\Big]\Big[1-\prod_{i\in A_3}p_i\Big]\Big[1-\prod_{i\in A_4}p_i\Big]$$

$$=1-(1-p^3)^4$$

→ minimal cut sets

$C_1=\{1,2\}$, $C_2=\{1,3\}$, $C_3=\{1,4\}$
$C_4=\{2,3\}$, $C_5=\{2,4\}$, $C_6=\{3,4\}$

→ lower bound

$$\prod_{j}\Big[1-\prod_{i\in C_j}(1-p_i)\Big]=\Big[1-\prod_{i\in C_1}(1-p_i)\Big]\cdots\Big[1-\prod_{i\in C_6}(1-p_i)\Big]$$

$$=(1-(1-p)^2)^6$$

⟹ combined → $(1-(1-p)^2)^6\leq r(p)\leq 1-(1-p^3)^4$

(bounds are tight w/ r(0.2) & r(0.9) ⟹ good approximation)

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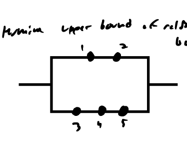
→ Assignment

→ Q1 → Determine upper bound of reliability of the system using first two terms of inclusion-exclusion bounds based on minimal path sets



$p_i=0.72$, $i=1,2,\dots,5$

→ $A_1=\{1,2\}$, $A_2=\{3,4,5\}$

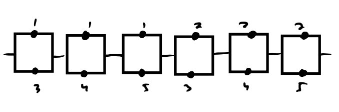
→ upper bound → ① $\sum_{j=1}^{2}\Big(\prod_{i\in A_j}p_i\Big)=p_1p_2+p_3p_4p_5$

② $\sum_{j=1}^{2}\sum_{k>j}\prod_{i\in A_j\cup A_k}p_i=p_1p_2p_3p_4p_5$ → not needed for this problem

$r(p)\leq ①$

$$\begin{aligned}&=0.72^2(1)(2)+0.72^3(3)(4)(5)\\&=0.56\end{aligned}$$

→ Q2)



$p_i=0.72$, $i=1,2,\dots,5$

→ Determine lower bound of reliability of the system using first two terms of inclusion-exclusion bounds based on minimal cut sets

$C_1=\{1,3\}$, $C_2=\{1,4\}$, $C_3=\{1,5\}$
$C_4=\{2,3\}$, $C_5=\{2,4\}$, $C_6=\{2,5\}$

$$\begin{aligned}① \sum_{j=1}^{6}\Big(\prod_{i\in C_j}(1-p_i)\Big)&=(1-p_1)(1-p_3)+(1-p_1)(1-p_4)+(1-p_1)(1-p_5)+(1-p_2)(1-p_3)+(1-p_2)(1-p_4)+(1-p_2)(1-p_5)\\&=(1-p_1)\Big[3-\sum_{i=3}^{5}p_i\Big]+(1-p_2)\Big[3-\sum_{i=3}^{5}p_i\Big]\\&=0.84\end{aligned}$$

$$\begin{aligned}\Rightarrow 1-r(p)&\leq 0.84\\-r(p)&\leq 0.84-1\\0.16&\leq r(p)\end{aligned}$$

★ bounds flip! ⟹ ① is actually lower bound & ② is upper bound

② $\sum_{j=1}\sum_{k>j}\Big(\prod_{i\in C_j\cup C_k}(1-p_i)\Big)=(1-p_1)(1-p_3)(1-p_4)+\dots+(1-p_2)(1-p_4)(1-p_5)$ $\Big(\tbinom{6}{2}=15 \text{ terms}\Big)$ (not worth my time...)

$=??$ → not needed