3.1 → Introduction

→ link function → $g(\mu_i) = x_i^T \beta$

3.2 → Exponential family of distributions

→ Let Y be a variable whose probability distribution depends on a single parameter θ. The distribution $\in$ exponential family if it can be written in the form

$$f(y|\theta) = s(y)\, t(\theta)\, e^{a(y)\, b(\theta)}$$

↳ known functions

$$= \exp\{\ln[s(y)\, t(\theta)\, e^{a(y)\, b(\theta)}]\}$$

$$\boxed{= \exp\{a(y)\, b(\theta) + c(\theta) + d(y)\}}$$

$s(y) = e^{d(y)} \Rightarrow d(y) = \ln(s(y))$

$t(\theta) = e^{c(\theta)} \Rightarrow c(\theta) = \ln(t(\theta))$

→ terms
- → if $a(y) = y \Rightarrow$ canonical form (standard)
- → $b(\theta)$ = natural parameter
- → other parameter are nuissance parameters ⇒ treated as known

→ Examples →

Dist	Natural parameter	c	d
Poisson	$\ln(\theta)$	$-\theta$	$-\ln(y!)$
Normal	μ/σ^2	$-\frac{\mu^2}{2\sigma^2} - \frac{1}{2}\ln(2\pi\sigma^2)$	$-\frac{y^2}{2\sigma^2}$
Binomial	$\ln(\pi/1-\pi)$	$n\ln(1-\pi)$	$\ln\binom{n}{y}$

→ Poisson dist

→ $f(y|\theta) = \frac{e^{-\theta}\theta^y}{y!} = \exp\{\ln[e^{-\theta}\theta^y/y!]\}$

$$= \exp\{\underbrace{-\theta}_{c(\theta)} + \underbrace{y}_{a(y)}\underbrace{\ln(\theta)}_{b(\theta)} - \underbrace{\ln(y!)}_{d(y)}\}$$

⇒ canonical form

→ Poisson models count data
- → Real data often has higher variance ⇒ overdispersed (see ch 8)

→ Normal dist

→ $f(y|\mu) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y-\mu)^2}$

$= \exp\{\ln[\ \langle\ldots\rangle\ ]\}$

$= \exp\{-\frac{1}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}(y-\mu)^2\}$

$= \exp\{-\frac{1}{2}\ln(2\pi\sigma^2) \underbrace{- \frac{1}{2\sigma^2}y^2}_{d(y)} + \underbrace{\frac{1}{\sigma^2}\mu}_{b(\mu)}\underbrace{y}_{a(y)} - \frac{1}{2\sigma^2}\mu^2\}$ ⇒ canonical form

$c(\mu)$ (could be included differently)

- → Normal models continuous, symmetric data
  - → CLT ⇒ even if skewed, average is ≈ Normal
  - → lots of theoretical properties ⇒ want to transform to approx Normal if possible → e.g. $y' = \sqrt{y}$ or $y' = \ln(y)$

→ Binomial dist

→ $f(y|\pi) = \binom{n}{y}\pi^y(1-\pi)^{n-y}$

$= \exp\{\ln[\ \langle\ldots\rangle\ ]\}$

$= \exp\{\ln\binom{n}{y} + y\ln(\pi) + (n-y)\ln(1-\pi)$

$= \exp\{\ln\binom{n}{y} + y\ln(\pi) + n\ln(1-\pi) - y\ln(1-\pi)\}$

$= \exp\{\underbrace{\ln\binom{n}{y}}_{d(y)} + \underbrace{y}_{a(y)}\underbrace{[\ln(\pi) - \ln(1-\pi)]}_{b(\pi) = \ln(\frac{\pi}{1-\pi})} \underbrace{- n\ln(1-\pi)}_{c(\pi)}\}$ → canonical form

→ Binomial dist models processes w/ binary outcomes

→ etc.

3.3 → Properties of distributions in the exponential family

→ Need expressions for $E[a(y)]$ & $V[a(y)]$

→ needed results (assuming order of integration & derivation can be interchanged) (true for any dist)

→ $\int f(y|\theta)\, dy = 1$

→ differentiate both sides

$$\frac{d}{d\theta}\int f(y|\theta)\, dy = \frac{d}{d\theta}1 = 0$$

→ reverse order of derivative & integral

$$\int \frac{d}{d\theta} f(y|\theta)\, dy = 0$$

→ differentiate twice

$$\int \frac{d^2}{d\theta^2} f(y|\theta)\, dy = 0$$

→ These results can now be used for distributions in the exponential family

→ $f(y|\theta) = \exp\{a(y)\, b(\theta) + c(\theta) + d(y)\}$

⇒ $\frac{d f(y|\theta)}{d\theta} = \{a(y)\, b'(\theta) + c'(\theta)\}\, \underbrace{f(y|\theta)}_{= e^{a(y)b(\theta)+c(\theta)+d(y)}}$ → $\left(\frac{d}{dx} e^{g(x)} = g'(x) e^{g(x)}\right)$

⇒ $0 = \int \frac{d}{d\theta} f(y|\theta)\, dy = \int \{a(y)\, b'(\theta) + c'(\theta)\}\, f(y|\theta)\, dy$

$= \int (a(y)\, b'(\theta)\, f(y|\theta) + c'(\theta)\, f(y|\theta))\, dy$

$= b'(\theta) \underbrace{\int a(y)\, f(y|\theta)\, dy}_{= E(a(y))} + c'(\theta)\underbrace{\int f(y|\theta)\, dy}_{=1}$

$= b'(\theta)\, E[a(y)] + c'(\theta)$

$$\Rightarrow \boxed{E[a(y)] = \frac{-c'(\theta)}{b'(\theta)}}$$

→ Similarly, $0 = \int \frac{d^2}{d\theta^2} f(y|\theta)\, dy$ → product rule $\frac{d}{dx} g(x)f(x) = g'(x)f(x) + f'(x)g(x)$

$= \int [(a(y)\, b''(\theta) + c''(\theta))\, f(y|\theta) + [\underbrace{a(y)\, b'(\theta) + c'(\theta)}]^2 f(y|\theta)]\, dy$

$= b''(\theta)\int a(y) f(y|\theta) + c''(\theta)\int f(y|\theta) + \ldots$ $\quad a(y)\, b'(\theta) - \overbrace{b'(\theta)\, E(a(y))}^{= c'(\theta)}$

$= \ldots + \int [(b'(\theta))^2 (a(y) - E(a(y)))^2]\, f(y|\theta)\, dy$

$= \ldots\ (b'(\theta))^2 \int (a(y) - E(a(y)))^2 f(y|\theta)\, dy$

$= b''(\theta)\, E[a(y)] + c''(\theta) + (b'(\theta))^2\, V[a(y)]$

⇒ $V(a(y)) = \frac{-b''(\theta)\, E[a(y)] - c''(\theta)}{(b'(\theta))^2}$

$= \frac{b''(\theta)\frac{c'(\theta)}{b'(\theta)} - c''(\theta)}{(b'(\theta))^2}$

$$\boxed{= \frac{b''(\theta)\, c'(\theta) - c''(\theta)\, b'(\theta)}{(b'(\theta))^3}}$$

→ Can use $\boxed{\text{equations}}$ to verify formulas for normal, poisson, binomial, etc.

→ Expressions are also needed for expected value & variance of derivatives of the log-likelihood function

for a distribution in the exponential family, the log likelihood is

→ $\ell(\theta|y) = \ln[\exp\{a(y)\, b(\theta) + c(\theta) + d(y)\}] = a(y)\, b(\theta) + c(\theta) + d(y)$

→ derivative of $\ell(\theta|y)$ w/ respect to θ

$$\boxed{U = \frac{d\ell(\theta|y)}{d\theta} = a(y)\, b'(\theta) + c'(\theta)}$$

↳ score statistic

→ depends on y ⇒ it is a RV

$U = a(y)\, b'(\theta) + c'(\theta)$

→ $E(U) = E[a(y)\, b'(\theta) + c'(\theta)]$

$= b'(\theta)\, E(a(y)) + E(c'(\theta))$

$= -b'(\theta)\frac{c'(\theta)}{b'(\theta)} + c'(\theta)$

$\boxed{= 0}$

→ $V(U) = V[a(y)\, b'(\theta) + c'(\theta)]$

$= (b'(\theta))^2\, V(a(y))$

$= (b'(\theta))^2\left[\frac{b''(\theta)\, c'(\theta) - c''(\theta)\, b'(\theta)}{(b'(\theta))^3}\right]$

$$\boxed{= \frac{b''(\theta)\, c'(\theta)}{b'(\theta)} - c''(\theta) = \mathcal{J} = \text{Information}}$$

→ The score statistic is used for inference about parameter values in glms

→ Property of score statistic → $V(U) \overset{①}{=} E(U^2) \overset{②}{=} -E(U')$

① $E(U^2) - \underbrace{(E(U))^2}_{=0}$

② $U' = \frac{dU}{d\theta} = \frac{d}{d\theta}[a(y)\, b'(\theta) + c'(\theta)]$

$= a(y)\, b''(\theta) + c''(\theta)$

$E(U') = E(\langle\ldots\rangle)$

$= b''(\theta)\, E[a(y)] + c''(\theta)$

$= -b''(\theta)\frac{b'(\theta)}{c'(\theta)} + c''(\theta)$

$= -V(U) = -\mathcal{J}$

3.4 → GLMs

→ GLM definition → $Y_1, \ldots, Y_N$ are ⫫ ~ Exponential family

→ 1) each Y_i has canonical form & depends on single parameter θ_i (don't have to be the same)

⇒ $f(y_i|\theta_i) = \exp[y_i\, b_i(\theta_i) + c_i(\theta_i) + d_i(y_i)]$

→ 2) distributions of all Y_i's are of the same form (e.g.) all normal, or Binomial)

⇒ subscripts on a, b, c & d are not needed

→ Joint pdf

$f(y_1, \ldots, y_N|\theta_1, \ldots, \theta_N) = \prod_{i=1}^{N} \exp[y_i\, b(\theta_i) + c(\theta_i) + d(y_i)]$

$$\boxed{= \exp\left[\sum_{i=1}^{N} y_i\, b(\theta_i) + \sum_{i=1}^{N} c(\theta_i) + \sum_{i=1}^{N} d(y_i)\right]}$$

→ parameters θ_i are usually not of direct interest (could be one for each obs).
For model specification, usually interested in a smaller set of parameters $\beta_1, \ldots, \beta_p$ ($p < N$)
Let $E(Y_i) = \mu_i$ where μ_i is some function of θ_i

→ for a GLM, there is a transformation of μ_i such that

$$g(\mu_i) = x_i^T\beta$$

→ ☆ this equation ⇒ 3 parts of glm model

→ 1) Response variable $Y_i \overset{iid}{\sim} \underbrace{f(\theta)}$ ↳ exponential family

2) Set of parameters & explanatory variables $\beta = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_p \end{bmatrix}$ & $x_i = \begin{bmatrix} x_{i1} \\ \vdots \\ x_{ip} \end{bmatrix}$

$X = \begin{bmatrix} x_1^T \\ \vdots \\ x_N^T \end{bmatrix} = \begin{bmatrix} x_{11} & \cdots & x_{1p} \\ \vdots & & \vdots \\ x_{N1} & \cdots & x_{Np} \end{bmatrix}$

↳ design matrix

→ link function g is monotone & differentiable such that

$g(\mu_i) = x_i^T\beta$
↳ $= E(Y_i)$

3.5 → Examples

→ Normal linear model → $E(Y_i) = \mu_i = x_i^T\beta$, $Y_i \overset{⫫}{\sim} N(\mu_i, \sigma^2)$

↳ $g(\mu_i) = \mu_i$ (identity)

⇒ $Y = \underbrace{X\beta}_{\text{signal}} + \underbrace{\varepsilon}_{\text{noise}}$ ↳ $\overset{iid}{\sim} N(0, \sigma^2)$

→ MLR, ANOVA, & ANCOVA are all this form

→ Bernoulli model

→ $Y_i = \begin{cases} 1 \\ 0 \end{cases}$

$P(Y_i = 1) = e^{-\theta t} = \pi$ (parameter)

⇒ $P(Y_i = 0) = 1 - e^{-\theta t} = 1 - \pi$

→ Interested in θ rather than π

⇒ $g(\pi) = \ln(\pi) = \ln(e^{-\theta t}) = -\theta t$
↳ $= E(Y_i)$

⇒ $g(\pi)$ is linear in parameter θ & $x_i = [-t]$, $\beta = [\theta]$

→ Poisson model

→ $f(y|\mu) = \frac{e^{-\mu}\mu^y}{y!}$, $y = 0, 1, 2, \ldots$

→ $E(y) = \mu$

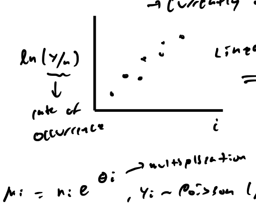
→ parameter μ depends on various characteristics of the population & the population size

⇒ $E(Y) = \mu = n\,\underbrace{\lambda(x^T\beta)}$

↳ function depends on

→ currently unknown ☆

Figured it out



Linear on log scale

⇒ exponential growth ☆

$Y = e^{f(x)}$

→ $E(Y_i) = \mu_i = n_i e^{\theta_i}$ → multiplication, $Y_i \sim \text{Poisson}(\mu_i)$

→ $g(\mu_i) = \ln(\mu_i) = \ln[n_i e^{\theta_i}] = \ln n_i + \theta_i$

⇒ linear component $x_i^T\beta \to = \begin{bmatrix} 1 \\ \theta \end{bmatrix}$
↳ $= [\ln n_i\ \ i]$

Same process as Bernoulli model

⇒ $E(Y) = \mu$ = some function of θ

$g(\mu)$ = linear comb. of θ