1) → $N|\Lambda \sim Poisson(\Lambda)$

$\Lambda|\theta \sim Gamma(\alpha=2, \beta=\theta)$

$\theta \sim Beta(a=2, b=3, \theta=1)$

→ $SD(N) = \sqrt{V(N)} = \sqrt{E_\theta[\underbrace{E_{\Lambda|\theta}[V_{N|\Lambda}(N|\Lambda,\theta)]}_{①}] + E_\theta[\underbrace{V_{\Lambda|\theta}[E_{N|\Lambda}(N|\Lambda,\theta)]}_{②}]}$

if was just $N|\Lambda \sim Poisson(\Lambda)$ & $\Lambda \sim Gamma(\alpha,\beta)$

$\Rightarrow E(N) = E_\Lambda[E_N(N|\Lambda)] = E_\Lambda[\Lambda] = \alpha/\beta$

→ ① $= E_\theta[E_{\Lambda|\theta}[\underbrace{V_{N|\Lambda}(N|\Lambda,\theta)}_{u=\Lambda}]] = E_\theta[E_{\Lambda|\theta}[\Lambda]] = E_\theta[\underbrace{\alpha\theta}_{u=\alpha\theta}] = \alpha E_\theta[\theta] = \alpha \underbrace{E[\theta]}_{u=\frac{a}{a+b}} = 2[4] = 8$

→ ② $= E_\theta[V_\Lambda[\underbrace{E_{N|\Lambda}(N|\Lambda,\theta)}_{u=\Lambda}]] = E_\theta[\underbrace{V_{\Lambda|\theta}(\Lambda)}_{u=\alpha\theta^2}] = E_\theta[\alpha\theta^2] = \alpha E_\theta[\theta^2] = \alpha \underbrace{E_\theta[\theta^2]}_{u=\frac{a^2(a+1)}{b(a)}} = 2\left[\frac{24}{5}\right] = 256/5$

$\Rightarrow SD(N) = \sqrt{① + ②} = \sqrt{8 + 256/5} = \boxed{6.6267}$

☆ Problems aren't as hard as make them to be!

$N|\theta$ is a Poisson-Gamma Mixture $\Rightarrow N|\theta \sim NB(r=\alpha=2, p=\theta)$

Poisson this term above?

$\Rightarrow V(N|\theta) = V_\theta[\underbrace{E_N(N|\theta)}_{\alpha\theta}] + E_\theta[\underbrace{V_N(N|\theta)}_{=\alpha\theta(1+\theta)}]$

$= V_\theta(\alpha\theta) + E[\alpha\theta(1+\alpha\theta)]$

$= \alpha^2 V(\theta) + \alpha[E(\theta) + E(\theta^2)]$

$= \alpha^2[E(\theta^2) - (E(\theta))^2] + [\alpha E(\theta) + \alpha E(\theta^2)]$

$= 2^2\left[\frac{120}{7} - 4^2\right] + \left[8 + \frac{256}{7}\right]$

$= \frac{376}{7} \Rightarrow \boxed{SD(N) = \sqrt{\frac{376}{7}} \simeq 7.3}$

---

2) $X \sim Lognormal(\mu=7, \sigma^2=7)$

$V(1+\alpha)^2 X) = (1+\alpha)^2 \underbrace{V(X)}_{\alpha=0.05}$ $\quad u = E(X^2) - E(X)^2$

$\boxed{\approx 22655}$

$= \exp[2\mu + 2\sigma^2/2] - (\exp[\mu + \sigma^2/2])^2$

$= \exp[7(2) + 7(7)/2] - (\exp[7 + 7/2])^2$

$= 20929.83$

☆ Memorize variances of distributions

☆ double check inequalities #s

---

3)

	Dist		# moments
	$V \sim Gamma(\alpha=6, \theta)$	$\Rightarrow$	∞
✓	$W \sim Pareto(\alpha=3, \theta)$	$\Rightarrow$	$-1 < k < \infty \Rightarrow 2$
	$X \sim Inverse\ Burr(\tau=1, \theta, \gamma=5)$	$\Rightarrow$	$-5 < k < 5 \Rightarrow 4$
	$Y \sim Weibull(\theta, \tau=6)$	$\Rightarrow$	∞
	$Z \sim Inverse\ gamma(\alpha=2, \theta)$	$\Rightarrow$	$k < 2 \Rightarrow 1$

---

4) $X \sim SP\text{-}Pareto(\alpha=2, \theta=500)$

$Y \sim Gamma(\alpha=2, \theta=500)$

→ 1 ! mean residual life function

→ 2 → $e(x) = \frac{E(x)}{S(x)} = \frac{\mu(x)}{1 - F(x;\alpha,\theta)}$! increasing or decreasing

→ 3 → $h(x) = \frac{f(x)}{S(x)} = \frac{\frac{\alpha\theta^\alpha x^{\alpha-1}}{(\theta+x)^{\alpha+1}}}{(\theta/\theta+x)^\alpha} = \frac{\alpha x}{\theta} \Rightarrow$ decreasing

$e(x) = \frac{E(X) - E(X \wedge x)}{S(x)} = E[X - x \mid X > x]$

= exp pareto, there is a shortcut

$= \frac{x}{\alpha-1} \Rightarrow$ linear increasing

---

5) $T \sim Exp(\lambda = 1/4)$

→ replace w/ spliced model → uniform $\in [0,3]$

→ $kT \in [3, \infty)$

→ continuous $\Rightarrow f_1(3) = f_2(3)$

→ $F_{Y_{0.6}}(y) = 0.6$

→ $F_Y(3) = \int_0^3 c_1/3\, dy = \frac{c_1}{3} x \Big|_0^3 = 0.4 = 2(?)$

$F_Y(y_{0.6}) = c_1 + \int_3^y c_2 \frac{1}{4} e^{-1/4 x} dx = c_1 + c_2[E(y) - F_T(3)] = c_1 + c_2[e^{-3/4} - e^{-y/4}]$

$0.6 = c_1 + \ldots$

$\Rightarrow y = -4\ln\left(-\left[\frac{0.6 - c_1}{c_2} - e^{-3/4}\right]\right) = \boxed{4.4787}$

→ $f_Y(y) = \begin{cases} c_1/3 & 0 \le y \le 3 \\ c_2 \frac{1}{4} e^{-1/4 y} & y > 3 \end{cases}$

$1 = \int_0^3 c_1/3\, dy + \int_3^\infty \frac{c_2}{4} e^{-1/4 y} dy = c_1 + c_2 S_T(3) = c_1 + c_2 e^{-1/4(3)}$

$\Rightarrow c_1 = 1 - c_2 e^{-3/4}$

$f_1(3) = f_2(3)$

$\frac{1}{3} c_1 = c_2 \frac{1}{4} e^{-1/4(3)}$

$\Rightarrow \frac{1}{3}(1 - c_2 e^{-3/4}) = c_2 \frac{1}{4} e^{-3/4}$

$\frac{1}{3} - \frac{1}{3} c_2 e^{-3/4} = \frac{1}{3} = \frac{7}{12} c_2 e^{-3/4}$

$\Rightarrow c_2 = 1.2097$

$\Rightarrow c_1 = 0.47857$

For exp dist → $pr(a \le X \le b)$: $F(b) - F(a) = e^{-a\lambda} - e^{-b\lambda}$

---

6) → $X \sim Exp(\lambda = 1/10)$

$Y \sim Exp(\lambda = 1/15)$

$D \sim Pareto(\alpha=3, \theta=5000)$ → on average, every 3rd transaction

if $D > 7500 \Rightarrow$ handled by manager ~~(not needed info)~~

→ let N = # transactions by teller (combined both X + Y) per day

$\sim PP(\lambda = \lambda_x + \lambda_y)$ → per minute

$= \frac{1}{10} + \frac{1}{15} = 1/6$

☆ Thinning PP

(per day $\lambda = \underbrace{6}_{\text{hrs}} \cdot \underbrace{(60)}_{\text{hour}} \cdot \underbrace{(1/6)}_{\text{per minute}} = 60$ per day

→ $P(D \le 7500) = F_D(7500) = 1 - \left[\frac{\theta}{x+\theta}\right]^\alpha = 1 - \left(\frac{5000}{7500+5000}\right)^3 = 0.936$

→ let N' be PP for transactions less than 7500

$E(N') = 60 \times \left(\frac{1}{3}\right)(0.936) = 18.72$

transactions

* prop of success

* prop ≤ 7500 (?)

→ $E(D \mid D \le 7500) = \frac{\int_0^{7500} d f(d)\, dd}{F_D(7500)} = \frac{E(X \wedge 7500) - 7500 S(7500)}{F_D(7500)}$

policy limit ⇒ take out losses above limit that get cut 7500

$= \frac{\frac{5000}{2}\left[1 - \left(\frac{5000}{7500+5000}\right)^2\right] - 7500\left(\frac{5000}{7500+5000}\right)^3}{1 - \left(\frac{5000}{7500+5000}\right)^3} = 1730.7692$

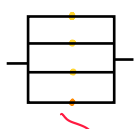
→ Total Deposit $= E(N')\, E(D \mid D > 7500)$

$= (18.72)(1730.7692)$

$= \boxed{32400}$

---

7)



$X_i \overset{iid}{\sim} Exp(\lambda = 1/4)$, $i = 1,2,3$

$X_4 \sim Exp(\lambda = 1/8)$ } all are ⊥

→ $\cancel{\phi(x) = \max(x_1, x_2, x_3, x_4)}$

$\cancel{= 1 - \prod_{i=1}^4 (1 - x_i)}$

$\cancel{= 1 - (1-x_1)^3(1-x_4)}$ $\quad$ ($x_1, x_2, x_3 \sim iid$)

→ let $Y = \max(x_1, x_2, x_3, x_4)$

→ $P(Y > 2 \mid Y > 1) = P(\max(x_1,x_2,x_3,x_4) > 2 \mid \max(x_1,x_2,x_3,x_4) > 1)$

$= 1 - \cancel{P(\text{ALL } x_i < 2 \mid \max(x_1,x_2,x_3,x_4) > 1)}$ applied definition wrong

$= 1 - \cancel{\frac{P(\text{ALL } x_i < 2 \wedge \max(x_1,x_2,x_3,x_4) > 1)}{P(\max(x_1,x_2,x_3,x_4) > 1)}}$

$= \frac{P(\max(x_1,x_2,x_3,x_4) > 2)}{P(\max(x_1,x_2,x_3,x_4) > 1)}$ $\quad$ ($\max(X) > 2$ & $\max(X) > 1$) $\Rightarrow P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}$, $A \subseteq B$

$= \frac{1 - P(\text{ALL } x_i \le 2)}{1 - P(\text{ALL } x_i \le 1)}$

$= \frac{1 - (1 - e^{-2/4})^3(1 - e^{-2/8})}{1 - (1 - e^{-1/4})^3(1 - e^{-1/8})} = \boxed{0.8925}$

---

8) $V_X(t) = \ln 1.06 \Rightarrow X \sim Exp(\lambda = \ln(1.06))$

$r_Y(t) = \frac{1}{20 - t} \Rightarrow Y \sim Uniform(0, 20)$

$\phi(x,y) = \min(x,y)$

↓ = xy → joint pdf density ⇒ $z_1 = xy$ & $z_2 = y$

⇒ $x = h(z_1, z_2) = z_1/z_2$ & $y = h(z_1, z_2) = z_2$

$f_{Z_1,Z_2}(z_1,z_2) = f_{X,Y}(z_1/z_2, z_2^{-1}) \left|J\right|$

$= \frac{1}{\ln(1.06)} e^{-\ln(1.06) z_1/z_2} \left(\frac{1}{20}\right) \begin{vmatrix} \frac{\partial x}{\partial z_1} & \frac{\partial y}{\partial z_1} \\ \frac{\partial x}{\partial z_2} & \frac{\partial y}{\partial z_2} \end{vmatrix}$

$= (\ldots) \begin{vmatrix} \frac{1}{z_2} & 0 \\ -\frac{z_1}{z_2^2} & 1 \end{vmatrix}$

$= \ldots$

$f(x,y) = f(x) f(y) = \frac{1}{\ln(1.06)} e^{-\ln(1.06) x} \left(\frac{1}{20}\right)$

$v(x,y) = E[\min(x,y)] = E(xy) = E(x)E(y) = \frac{10}{\ln(1.06)}$

$P\left(\begin{matrix} 5 \le X \le 9 \\ 3 \le Y \le 9 \end{matrix} \middle| \begin{matrix} X > 3, \\ Y > 3 \end{matrix}\right)$

☆ use logic → system functions only if both machines function

⇒ fails if at least one machine fails

⇒ $P(5 \le T \le 9 \mid T > 3) = P(5 \le X \le 9 \mid X > 3) * P(5 \le Y \le 9 \mid Y > 3)$ → only 2 scenarios

$= \frac{P(5 \le X \le 9)}{P(X > 3)} * \frac{P(5 \le Y \le 9)}{P(Y > 3)}$ $\quad + P(X > 9 \mid X > 3) P(5 \le Y \le 9 \mid Y > 3) + P(5 \le X \le 9 \mid X > 3) P(Y > 9 \mid Y > 3)$

$= \frac{e^{-5\ln(1.06)} - e^{-9\ln(1.06)}}{e^{-3\ln(1.06)}} * \frac{4/20}{17/20}$

$= 0.0425$ ① $\quad +$ ② $\quad +$ ③

$= 0.3791$

---

9) → ≥1 recurrent classes in markov chain

(defs) → recurrence is class property

---

(10) → All states in finite irreducible markov chain are recurrent

(defs) → all states recurrent ⇏ irreducible (one class)

→ 2-D symmetric Random walk has all recurrent classes

---

11) $X_i \overset{iid}{\sim} Geometric(p = 0.7)$

$P(X_1 \le 4, X_2 \le 4 \mid \max(X_1, X_2) \ge 3) = \frac{P(X_1 \le 4 \wedge X_2 \le 4 \wedge \max(X_1,X_2) \ge 3)}{P(\max(X_1, X_2) \ge 3)}$

Scenarios: ① $X_1 \ge 3$ $X_2 \le 3$
② $X_1 \le 3$ $X_2 \ge 3$
③ $X_1 \ge 3$ $X_2 \ge 3$
always ≤ 4

$= \frac{\sum_{i=1}^{3} ①}{1 - P(\text{ALL } X_i \le 3)}$

① $= P(3 \le X_1 \le 4) P(X_2 \le 3)$

$= [0.8^3(0.3) + 0.8^2(0.3)](1 - 0.8^3)$

$= ①$

② $= P(3 \le X_1 \le 4) P(3 \le X_2 \le 4)$

$= [0.8^3(0.3) + 0.8^2(0.3)]^2$

$= \frac{① + ② + ③}{1 - (1-0.8^3)(1-0.8^3)} = \boxed{0.7517}$

$1 - F_G(x) = 1 - q^x$

☆ memorize geometric shortcut formulas

---

(12) $e_{x+1} > e_x$ → recursive formula: $e_x = p_x(1 + e_{x+1})$

$\sum_{k=1}^{\infty} k p_{x+1} > \sum_{k=1}^{\infty} k p_x$

Chance one year + survive one year & now expected life formula

0.00

skipping

Q13) $Z \sim Geometric(p = 1/c)$

$= 1/2.5$

$= 0.4$

$P(Z = 1, 3, 5, \ldots) = \sum_{i=1,3,5,\ldots} p(z)$

$= \sum_{i=1,3,5,\ldots} q^{z-1} p$

→ Excel approximation up to $z = 1, 3, 5, \ldots, 71$

⇒ $f = 0.625$

Q14) $X \sim Pareto(\alpha = 2, \theta)$

$X \wedge 20000$

$\hat{\theta}_{MME}$: $E(X \wedge 20000) = \frac{\theta}{\alpha - 1}\left[1 - \frac{\theta}{20000 + \theta}\right]$

$= \theta\left[\frac{20000 + \theta - \theta}{20000 + \theta}\right]$

$= \frac{20{,}000\theta}{20000 + \theta}$

→ $\bar{x} = \frac{20000\theta}{20000 + \theta}$

$20000\bar{x} + \bar{x}\theta = 20000\theta$

$20000\bar{x} = \theta(20000 - \bar{x})$

⇒ $\hat{\theta} = \frac{20000\bar{x}}{20000 - \bar{x}}$, $\bar{x} = 9200$

⇒ $\boxed{\hat{\theta} = 17283.117}$

(Rest are stats/sim problems)