1.1) ✓

$\rightarrow Y_1 \sim N(1,3)$ $\qquad W_1 = Y_1 + 2Y_2$

$Y_2 \sim N(2,5)$ $\qquad W_2 = 4Y_1 - Y_2$

$Y_1 \perp\!\!\!\perp Y_2$ $\qquad (W_1, W_2) = ??$

$\rightarrow W_1 \sim N(\mu_{Y_1} + 2\mu_{Y_2},\ \sigma^2_{Y_1} + 4\sigma^2_{Y_2})$
$= 1 + 2(2),\quad 3 + 4(5)$
$= 5,\quad 23$

$W_2 \sim N(4\mu_{Y_1} - \mu_{Y_2},\ 16\sigma^2_{Y_1} + \sigma^2_{Y_2})$
$= 4(1) - 2,\quad 16(3) + 5$
$= 2,\quad 53$

$\rightarrow \boxed{(W_1, W_2) \sim MVN\left[\begin{pmatrix}5\\2\end{pmatrix}, \begin{pmatrix}23 & 2\\ 2 & 53\end{pmatrix}\right]}$

$Cov(W_1,W_2) = cov(Y_1, 4Y_1) - Cov(Y_1,Y_2) + Cov(2Y_2, 4Y_1) - Cov(2Y_2, Y_2)$
$= 4(3) \qquad = 0 \qquad = 0 \qquad = 2(5)$
$= 2$

↳ $\sum Cov(pairs)$

1.2) ✓

$\rightarrow Y_1 \sim N(0,1)$

$Y_2 \sim N(3,4)$

$Y_1 \perp\!\!\!\perp Y_2$

a) $Y_1^2 = Z^2 \sim \chi^2_1$

b) $\rightarrow \underline{y} = \begin{bmatrix} Y_1 \\ (Y_2-3)/2 \end{bmatrix}$

$\rightarrow y^T y = \underset{1\times 2}{[Z_1\ \ Z_2]} \underset{2\times 1}{\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}} = \chi^2_1 + \chi^2_1 = \chi^2_2$

c) $\rightarrow \underline{y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \sim MVN\left(\begin{bmatrix} 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\right)$

$\rightarrow y^T V^{-1} y = \frac{1}{4}[Y_1\ \ Y_2]\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \frac{1}{4}[4Y_1\ \ Y_2]\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \frac{1}{4}[4Y_1^2 + Y_2^2] = Y_1^2 + \frac{1}{4}Y_2^2 \sim \chi^2_{(2,\lambda)}$ → non-central χ^2

↳ $\lambda = \mu^T V^{-1} \mu$
$= [0\ \ 3]\ \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 3 \end{bmatrix}$
$= \frac{1}{4}[0\ \ 3]\begin{bmatrix} 0 \\ 3 \end{bmatrix}$
$= 9/4$

$V = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$V^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$V^{-1} = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$

if $y \sim MVN(\mu, V) \Rightarrow y^T V^{-1} y \sim \chi^2_{(n,\lambda)}$
↳ $\lambda = \mu^T V^{-1} \mu$

1.3 ✓

$\rightarrow (Y_1, Y_2) \sim MVN\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 & 1 \\ 1 & 9 \end{bmatrix}\right)$

$\rightarrow$ a) $(y-\mu)^T V^{-1} (y-\mu) = [(Y_1-2)\ \ (Y_2-3)]\ \frac{1}{35}\begin{bmatrix} 9 & -1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} Y_1-2 \\ Y_2-3 \end{bmatrix}$

$= \frac{1}{35}[(9(Y_1-2) - (Y_2-3))\quad -(Y_1-2) + 4(Y_2-3)]\begin{bmatrix} Y_1-2 \\ Y_2-3 \end{bmatrix}$

$= \frac{1}{35}[9(Y_1-2)^2 - (Y_2-3)(Y_1-2) - (Y_1-2)(Y_2-3) + 4(Y_2-3)^2]$

$\boxed{= \frac{1}{35}[9(Y_1-2)^2 - 2(Y_1-2)(Y_2-3) + 4(Y_2-3)^2] \quad \sim \chi^2_2 \text{ by theorem}}$

$\rightarrow$ b) $y^T V^{-1} y = [Y_1\ \ Y_2]\ \frac{1}{35}\begin{bmatrix} 9 & -1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$

$= \frac{1}{35}[(9Y_1 - Y_2)\ \ (-Y_1 + 4Y_2)]\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$

$= \frac{1}{35}[9Y_1^2 - Y_1Y_2 - Y_1^2 + 4Y_2^2]$

$\boxed{= \frac{1}{35}[8Y_1^2 - Y_1Y_2 + 4Y_2^2] \quad \sim \chi^2_{(2,\lambda)}}$

↳ $\lambda = \mu^T V^{-1} \mu$
$= [2\ \ 3]\ \frac{1}{35}\begin{bmatrix} 9 & -1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$
$= \frac{1}{35}[15\ \ 10]\begin{bmatrix} 2 \\ 3 \end{bmatrix}$
$= \frac{1}{35}[60]$
$= \frac{12}{7}$

1.4 ✗

$\rightarrow Y_i \overset{iid}{\sim} N(\mu, \sigma^2) \quad i = 1, \ldots, n$

$\bar{y} = \frac{1}{n}\sum Y_i \qquad S^2 = \frac{1}{n-1}\sum (Y_i - \bar{y})^2$

a) $\bar{y} \sim N(\mu, \sigma^2/n)$

b) show $S^2 = \frac{1}{n-1}[\underbrace{\sum (Y_i-\mu)^2 - n(\bar{y}-\mu)^2}]$
↳ $= \sum (Y_i - \bar{y})^2$

$\sum (Y_i - \bar{Y})^2 = \sum[(Y_i - \mu) - (\bar{Y}+\mu)]^2$

$= \sum[(Y_i-\mu)^2 - 2(Y_i-\mu)(\bar{Y}+\mu) + (\bar{Y}+\mu)^2]$

$= \sum[(Y_i-\mu)^2 - 2Y_i\bar{Y} - 2Y_i\mu + 2\bar{Y}\mu + 2\mu^2 + (\bar{Y}+\mu)^2]$

$\sum (Y_i-\mu)^2 - 2\bar{Y}\sum Y_i - 2\mu\sum Y_i + 2n\bar{Y}\mu + 2n\mu^2 + n(\bar{Y}+\mu)^2$

$-2\bar{Y}n\bar{Y} - \cancel{2\mu n\bar{Y}} + \cancel{2n\bar{Y}\mu} + 2n\mu^2 + n\bar{Y}^2 + 2n\bar{Y}\mu + n\mu^2$

$-2\bar{Y}^2 n$

?

$\sum (Y_i-\mu)^2 - n\bar{Y}^2 + 3n\mu^2 + 2n\bar{Y}\mu$

$n\bar{Y}^2 - 2n\bar{Y}\mu + n\mu^2$

c)

d) $\frac{(n-1)}{\sigma^2}S^2 \sim \chi^2_{n-1}$

e) $\frac{\bar{Y}-\mu}{S/\sqrt{n}} \sim t_{n-1}$

1.5

a.) $\rightarrow Y_1, \ldots, Y_n \overset{iid}{\sim} Poisson(\theta)$

$\rightarrow$ show $E(Y_i) = \theta \qquad M_{Y_i}(t) = e^{\theta(e^t - 1)}$

$E(Y_i) = M'_{Y_i}(t)\Big|_{t=0} = \qquad \Big|_{t=0} \overset{\checkmark}{=} \theta$

$e^{\theta(e^t-1)}$

$f(g(x))$