1.3.1 → Conditional Tail Expectation

→ Quantile → $\pi_p = \underbrace{F_X^{-1}(p)}$
↳ quantile function

→ Example) → $X \sim \text{Weibull}(\theta = 5000, \tau = 3)$

$\pi_{0.95} \rightarrow 0.95 = F_X(\pi_{0.95})$

$\downarrow = 1 - e^{-(\pi/5000)^3}$

$\left(\frac{\pi}{5000}\right)^3 = -\ln(0.05)$

$\pi = 5000\left(-\ln(0.05)\right)^{1/3}$

$\downarrow = 7207.83$

$\Rightarrow P(X > 7207.83) = 0.05$

→ Smoker: $X_s \sim \text{Exp}(\theta = 300)$ → $P(S) = 0.3$

Non-smoker: $X_N \sim \text{Exp}(\theta = 150)$ → $P(N) = 0.7$

$\pi_{0.95,N} \rightarrow 0.95 = F_{X_N}^{-1}(\pi_{0.95,N})$

$\downarrow = 1 - e^{-\pi/150}$

$\pi = 150(-\ln(0.05))$

$\downarrow = 449.36$

$\pi_{0.95} \rightarrow 0.90 = F_Y^{-1}(\pi_{0.90})$

↓ randomly selected individual

↳ $Y = 0.3X_s + 0.7X_{ns}$ (mixture)

$= 0.3F_{X_s}^{-1}(\pi_{0.95}) + 0.7F_{X_N}^{-1}(\pi_{0.95})$

$= 0.3(1 - e^{-\pi/300}) + 0.7(1 - e^{-\pi/150})$

$0.3\underbrace{e^{-\pi/300}} + 0.7\left(\underbrace{e^{-\pi/300}}\right)^2 = 0.1$

$= u$

$0.3u + 0.7u^2 = 0$

$0.7u^2 + 0.3u - 0.1 = 0$

$\Rightarrow u = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-0.3 \pm \sqrt{0.3^2 - 4(0.7)(-0.1)}}{2(0.7)}$

$= 0.2202 \vee -0.6487$

$\Rightarrow 0.2202 = e^{-\pi/300}$

$\Rightarrow \pi_{0.90} = 453.469$

★ Have to find quantiles of the mixture, not mix the quantiles

$\pi_{0.9,Y} \neq 0.3\pi_{0.9,s} + 0.7\pi_{0.9,X_N}$

→ Conditional tail expectation (CTE)

→ def → for a cont'n's RV X, the CTE with tolerance probability $1-p$ is X's conditional expectation given X exceeds the quantile π_p

→ $CTE_p(X) = E[X \mid X > \pi_p]$

$= \frac{\int_{\pi_p}^{\infty} x f(x)\,dx}{P(X > \pi_p)}$

$\boxed{= \frac{\int_{\pi_p}^{\infty} x f(x)\,dx}{1-p}}$

$\boxed{= \pi_p + E[X - \pi_p \mid X > \pi_p]}$ ⟨avoiding integration⟩

⟨starting point⟩ + ⟨extra⟩

$= \pi_p + e(\pi_p)$

$\boxed{= \pi_p + \left[\frac{E(X) - E[X \wedge \pi_p]}{1-p}\right]}$

→ This gives us info about the expected size of losses above a certain quantile

→ Examples → $f(x) = \frac{2}{3}x$, $1 < x < 2$

→ $F(x) = \int_1^x \frac{2}{3}t\,dt = \frac{1}{3}x^2 - \frac{1}{3} = \frac{1}{3}(x^2 - 1)$, $1 < x < 2$

→ $\pi_{0.75} \rightarrow 0.75 = F_X^{-1}(\pi_{0.75})$

$\downarrow = \frac{1}{3}(\pi^2 - 1)$

$\Rightarrow \pi = \sqrt{3.25} \approx 1.8028$

→ $CTE_{0.75} = E(X \mid X > \pi_p)$

$= \frac{\int_{\pi_p}^{\infty} x f(x)\,dx}{1-p}$

$= \frac{\int_{\sqrt{3.25}}^{2} x\left(\frac{2}{3}x\right)dx}{1 - 0.75}$

$= \frac{\frac{2}{9}x^3 \Big|_{\sqrt{3.25}}^{2}}{0.25}$

$= \frac{\frac{2}{9}\left(2^3 - 3.25^{3/2}\right)}{0.25}$

$= 1.903$

→ Continuing smoker example → $Y = 0.3X_s + 0.7X_N$

→ $CTE_{0.9} = \pi_{0.9} + e_Y(\pi_{0.9})$

$= \downarrow \quad \frac{E(Y) - E[Y \wedge \pi_{0.9}]}{1 - 0.9}$

$= 453.47 + \frac{195 - 170.091}{0.1}$

$= 703.058$

$E(Y) = 0.3E(X_s) + 0.7E(X_N)$

$= 0.3(300) + 0.7(150)$

$= 195$

$E(Y \wedge \pi_{0.9}) = 0.3E[X_s \wedge \pi_{0.9}] + 0.7E[X_N \wedge \pi_{0.9}]$ → $\theta(1 - e^{-x/\theta})$

$= 0.3\left[300\left(1 - e^{-\frac{453.47}{300}}\right)\right] + 0.7\left[150\left(1 - e^{-\frac{453.47}{150}}\right)\right]$

$= 170.091$

★ Have to find the CTE of the mixture, not mix the CTEs

$CTE_{0.9}(Y) \neq 0.3\,CTE_{0.9}(X_s) + 0.7\,CTE_{0.9}(X_N)$

→ Shortcut formulas

→ Specifically for lognormal RVs, doesn't require calculation of π_p

$CTE_p(X) = E(X) \cdot \left[\frac{\Phi(\sigma - z_p)}{1-p}\right]$ (→ CDF)

→ Specifically for normal RVs

$CTE_p(X) = \mu + \sigma\left[\frac{\phi(z_p)}{1-p}\right]$ (→ PDF)

→ Assignment

→ Q1) → Losses

x	$F(x)$	$E(X \wedge x)$
1000	0.10	800
4000	0.20	2500
22,500	0.80	9500
72,500	0.90	1100
∞	1.00	20,000

→ $CTE_{0.8} = \pi_{0.8} + \frac{E(X) - E(X \wedge \pi_{0.8})}{1 - 0.8}$

$= 22{,}500 + \frac{20{,}000 - 9500}{0.2}$

$= 75{,}000$

→ Q2) $X \sim \text{Exp}(\lambda)$

$E(X \mid X > \pi_p) = CTE_{1-p} = \frac{\int_{\pi_p}^{\infty} x f(x)\,dx}{1 - (1-p)}$

$= \pi_p + \frac{E(X) - E(X \wedge \pi_p)}{1 - (1-p)}$

$p = F_X^{-1}(\pi_p)$

$= 1 - e^{-\lambda\pi}$

$\pi_p = \frac{-\ln(1-p)}{\lambda}$

$= \pi_p + \cancel{E[X \mid X > \pi_p]}$ memoryless

→ $E(X) = 1/\lambda$

$= \pi_p + E(X - \pi_p \mid X > \pi_p)$ memoryless $\sim E(X)$

$1 - p = F_X(\pi_p)$

$= 1 - e^{-\lambda\pi}$

$\frac{-\ln(p)}{\lambda} = \pi$

$= \frac{-\ln(p)}{\lambda} + 1/\lambda$

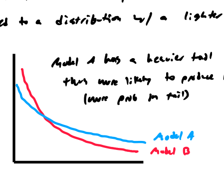
$= \frac{1}{\lambda}\left[1 - \ln(p)\right]$

1.3.2 → Tail weight

→ define tail weight as how distributions describe the likelihood of extreme losses occurring ("tail" = right tail)

⇒ A heavier tail weight implies that large claims are more likely to occur compared to a distribution w/ a lighter tail

→ Ex) Model A has a heavier tail & thus more likely to produce large values (even past an infl)



→ Comparing tails → can't compare every pdf by hand during exam ⇒ 4 ways to compare tails

→ 1) # of positive raw moments

→ for greater tail weight, it becomes less likely that the integral $\int x^k f(x)$ over all possible values will converge

? → If the integral doesn't converge for higher values of k, then these moments don't exist

★ ⇒ fewer positive moments that exist ⇒ greater the tail weight

→ 2) Ratio of the survival functions (or densities)

→ The faster the survival function approaches zero, the thinner the tail

⇒ if $\lim_{x \to \infty} \frac{S_1(x)}{S_2(x)} = \infty \Rightarrow X_1$ has thicker tail

$= 0 \Rightarrow X_2$ — — — —

or $\lim_{x \to \infty} \frac{-f_1(x)}{-f_2(x)} = \langle \cdots \rangle \Rightarrow \langle \cdots \rangle$

↳ L'Hospital's rule

→ 3) Hazard rate function

→ If the hazard rate function decreases as x increases, this indicates there is a higher prob of extreme losses ⇒ heavier tail

→ 4) CTEs (or quantiles)

→ The larger a given CTE (or quantile) is for a distribution, the larger the extreme values are

⇒ distribution with a larger CTE (or quantile) for a given value of p has greater tail weight

→ Examples

→ Consider the following models → A: $\text{Pareto}(\alpha = 3, \theta = 60)$

B: $\text{Exp}(\theta = 30)$

Determine the model w/ the heavier tails using each method

→ 1) # positive raw moments

→ From tables, Pareto only has positive moments for $-1 < k < 3$ (b/c $\alpha = 3$)

→ Exp has infinite moments

⇒ $A > B$
↳ heavier tail

→ 2) ratio of survival functions

$A \rightarrow S_A(x) = \left(\frac{60}{x+60}\right)^3$

$B \rightarrow S_B(x) = e^{-x/30}$

$\rightarrow \lim_{x \to \infty} \frac{S_A(x)}{S_B(x)} = \frac{\left(\frac{60}{x+60}\right)^3}{e^{-x/30}}$

$= \frac{60^3 e^{x/30}}{(x+60)^3}$ → increases faster

⇒ ∞

⇒ $A > B$

→ 3) Hazard rate test

$A \rightarrow h_A(x) = \frac{f_A(x)}{S_A(x)} = \langle \cdots \rangle = \frac{3}{x+60}$ → decreases w/ x

$B \rightarrow h_B(x) = \frac{f_B(x)}{S_B(x)} = \langle \cdots \rangle = 1/30$ → constant

⇒ $A > B$

→ 4) CTEs

⟨… find general CTE + plug in a few p's …⟩

Compare values & find $CTE_p(A) > CTE_p(B)$ consistently + increases at a faster rate

⇒ $A > B$