5.1 → Introduction

→ Hypothesis tests are a statistical modelling framework performed by comparing how well two related models fit the data

→ for GLMs, models must have the same probability distribution & the same link function

→ But the linear components can differ, specifically the simple model is a subset of the more general model

→ Summary stats are used to compare how well models fit the data. Goodness of fit statistics may be based on max value of the (log)-likelihood function, minimum value of SSE or a composite statistic based on residuals

→ Summary of process & logic of hypothesis tests

1) Specify model M_0 (corresponding to H_0)
   — — — M_1 — — — — H_1
2) Fit M_0 & calculate goodness of fit statistic G_0
   — M_1 — — — — — — — G_1
3) Calculate the improvement in fit, usually $G_1 - G_0$ or G_1/G_0
4) Use sampling dist of $G_1 - G_0$ (or whatever from 3) to test hypothesis $G_0 = G_1$ vs $G_0 \neq G_1$
5) If $G_0 = G_1$ is not rejected ⇒ fail to reject H_0 + M_0 is preferred model
   $G_0 \neq G_1$ is rejected ⇒ reject H_0 + M_1 — — — — — —

→ Sampling distributions

→ For both forms of inference (CI + HT), the sampling distributions are required
(CI: need samp dist of estimator; HT: need samp dist of goodness of fit statistic)

→ If response ~ normal dist, then samp dist can be determined exactly. For others, need asymptotics based on CLT (ignoring attention to regularity conditions ⇒ if Y_i are ⫫ & from exp family ⇒ conditions are met)

→ Basic idea is that under appropriate conditions if S is a statistic of interest then approximately

$$\frac{S - E(S)}{\sqrt{V(S)}} \sim N(0,1)$$

or equivalently

$$\frac{(S-E(S))^2}{V(S)} \sim \chi^2_1$$

→ If there is a vector of statistics $S = \begin{bmatrix} S_1 \\ \vdots \\ S_p \end{bmatrix}$ w/ asymptotic expectation $E(S)$ + var-cov matrix V

$$[S - E(S)]^T V^{-1} [S - E(S)] \sim \chi^2_p$$

↳ nonsingular & inverse exists

5.2 → Sampling dist for score statistics

→ setup → Suppose $Y_1, \dots, Y_N$ are ⫫ RVs in a GLM w/ parameters β, where $E(Y_i) = \mu_i$, $g(\mu_i) = x_i^T\beta = \eta_i$

→ Score statistic → $U_j = \frac{\partial \ell}{\partial \beta_j} = \sum_{i=1}^{N} \left[\frac{Y_i - \mu_i}{V(Y_i)} x_{ij} \left(\frac{\partial \mu_i}{\partial \eta_i} \right) \right]$ for $j = 1, \dots, p$

→ As $E(Y_i) = \mu_i$ for all i ⇒ $E(U_j) = 0$ for $j = 1, \dots, p$

→ Var-cov matrix of U_j is the information matrix $\mathcal{J}$ w/ elements $\mathcal{J}_{jk} = E[U_j U_k]$

→ If only one parameter β, score stat has asymptotic dist

$$\frac{U}{\sqrt{\mathcal{J}}} \sim N(0,1) \quad \text{or equivalently} \quad \frac{U^2}{\mathcal{J}} \sim \chi^2(1)$$

$(E(U) = 0 \ \& \ V(U) = \mathcal{J})$

→ If there is a vector of parameters $\beta = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_p \end{bmatrix}$ + score vector $U = \begin{bmatrix} U_1 \\ \vdots \\ U_p \end{bmatrix} \sim MVN(0, \mathcal{J})$ (asymptotically)

⇒ $U^T \mathcal{J}^{-1} U \sim \chi^2(p)$ for large samples

→ Example → $Y_i \overset{iid}{\sim} N(\mu, \sigma^2)$ (↳ known constant)

→ log likelihood → $\ell = \frac{-1}{2\sigma^2} \sum (Y_i - \mu)^2 - N \ln(\sqrt{2\pi}\sigma)$

→ Score statistic → $U = \frac{d\ell}{d\mu} = \frac{1}{\sigma^2} \sum (Y_i - \mu) = \frac{N}{\sigma^2}(\bar{Y} - \mu)$ ① ②

→ MLE → obtained by solving $U = 0$ ⇒ $\hat{\mu} = \bar{Y}$

→ Expected value + variance → $E(U)$ = <using ①> $= \frac{1}{\sigma^2} \sum [E(Y_i) - \mu] = 0$ (↳ $= \mu$)

→ $\mathcal{J} = V(U)$ = <using ①> $= \frac{1}{\sigma^4} \sum V(Y_i) = \frac{N}{\sigma^2}$

⇒ $\frac{U}{\sqrt{\mathcal{J}}}$ = <using ②> $= \frac{(\bar{Y} - \mu)}{\sigma / \sqrt{N}} \sim N(0,1)$ asymptotically (exactly in this scenario b/c $\bar{Y} \sim N(\mu, \sigma^2/N)$)

Similarly $U^T \mathcal{J}^{-1} U = \frac{U^2}{\mathcal{J}} = \frac{(\bar{Y} - \mu)^2}{\sigma^2/N} \sim \chi^2_1$

⇒ use this sampling dist of U to make inferences about μ
e.g.) 95% CI = $\bar{y} \pm 1.96\, \sigma/\sqrt{n}$

→ Example → If $Y \sim Bin(n, \pi)$

→ log-likelihood → $\ell(\pi|y) = y \ln(\pi) + (n-y)\ln(1-\pi) + \ln \binom{n}{y}$

→ Score statistic → $U = \frac{d\ell}{d\pi} = \frac{Y}{\pi} - \frac{n-Y}{1-\pi} = \frac{Y - n\pi}{\pi(1-\pi)}$

→ Mean + variance → $E(Y) = n\pi$ ⇒ $E(U) = 0$ as expected

→ $V(Y) = n\pi(1-\pi)$ ⇒ $V(U) = \frac{1}{\pi^2(1-\pi)^2} V(Y) = \frac{n}{\pi(1-\pi)}$

⇒ $\frac{U}{\sqrt{\mathcal{J}}} = \frac{Y - n\pi}{\pi(1-\pi)} \Big/ \sqrt{\frac{n}{\pi(1-\pi)}} = \frac{Y - n\pi}{\sqrt{n\pi(1-\pi)}} \sim N(0,1)$

↳ normal approximation to the binomial used for CI + tests about π

5.3 → Taylor series approximations

→ Taylor series approximations
→ used to obtain asymptotic sampling distributions for various other statistics
→ $f(x) = f(t) + (x-t)\left[\frac{df}{dx}\right]_{x=t} + \frac{1}{2}(x-t)^2 \left[\frac{d^2f}{dx^2}\right]_{x=t} + \dots$
provided x is near t

→ For log-likelihood function of single parameter β, the first 3 terms become

$\ell(\beta) = \ell(b) + (\beta - b) U(b) + \frac{1}{2}(\beta - b)^2 U'(b)$
↳ score function evaluated at β
$U' = \frac{d^2\ell}{d\beta^2} \approx E(U') = -\mathcal{J}$ <from earlier formulas>

⇒ approximation

$\ell(\beta) = \ell(b) + (\beta - b) U(b) - \frac{1}{2}(\beta - b)^2 \mathcal{J}(b)$
↳ information evaluated at $\beta = b$

→ For vector β, this becomes

$\ell(\beta) = \ell(b) + (\beta - b)^T U(b) - \frac{1}{2}(\beta - b)^T \mathcal{J}(b)(\beta - b)$

→ For score function of a single parameter β, using first two terms of series, we get

$U(\beta) = U(b) + (\beta - b) U'(b)$ (↳ $\approx E(U') = -\mathcal{J}$)
$= U(b) - (\beta - b)\mathcal{J}(b)$

for vector ⇒ $U(\beta) = U(b) - \mathcal{J}(b)(\beta - b)$ ☆

5.4 → Sampling distributions for MLEs

→ ☆ equation directly above can be used to find sampling dist of MLE $b = \hat{\beta}$. By def, b is the estimator which maximizes $\ell(b)$ & so $U(b) = 0$ (first derivative)

⇒ $U(\beta) = -\mathcal{J}(b)(\beta - b)$ or equivalently
$(b - \beta) = \mathcal{J}^{-1} U$ <left multiply $\mathcal{J}^{-1}$ & use negative to switch b & β>

→ If $\mathcal{J}$ is treated as constant
$E(b - \beta) = 0$ b/c $E(U) = 0$
⇒ $E(b) = \beta$ ⇒ b is a consistent estimator of β (at least asymptotically)

→ The var-cov matrix for b is

$E[(b-\beta)(b-\beta)^T] = E[(\mathcal{J}^{-1}U)(\mathcal{J}^{-1}U)^T]$ <like $\mathrm{Cov}(X,Y)$: $E((X - \mu_X)(Y - \mu_Y)^T)$>
$= \mathcal{J}^{-1} E(UU^T) \mathcal{J}^{-1}$ (constant + $(\mathcal{J}^{-1})^T = \mathcal{J}^{-1}$ b/c symmetric; $E(UU^T) = \mathcal{J}$)
$= \mathcal{J}^{-1}$

⇒ Asymptotic sampling dist for b is

$(b - \beta)^T (\mathcal{J}(b)^{-1})^{-1} (b - \beta) =$
$(b - \beta)^T \mathcal{J}(b)(b - \beta) \sim \chi^2(p)$

→ This is the Wald statistic

→ for one parameter → more commonly used form → $b \sim N(\beta, \mathcal{J}^{-1})$
→ if response ~ normal, is exact result

→ Example → Consider model $E(Y_i) = \mu_i = x_i^T\beta$, $Y_i \overset{ii}{\sim} N(\mu_i, \sigma^2)$
⇒ glm w/ identity link function

→ Information matrix → $\mathcal{J}_{jk} = \sum_{i=1}^{N} \frac{x_{ij} x_{ik}}{V(Y_i)} \left(\frac{\partial \mu_i}{\partial \eta_i} \right)^2$ (↳ $= 1$)
$= \sum \frac{x_{ij} x_{ik}}{\sigma^2}$
$= \frac{1}{\sigma^2} X^T X$

→ $z_i = \sum_{k=1}^{p} x_{ik} b_k^{(m-1)} + (y_i - \mu_i) = y_i$ (evaluated @ $b^{(m-1)}$; ↳ $= x_i^T\beta$, ↳ $= x_i^T\beta$)

⇒ $X^T W X b^{(m)} = X^T W z$ is equal to
$\frac{1}{\sigma^2} X^T X b = \frac{1}{\sigma^2} X^T y$ (acting like W)
⇒ MLE is $b = (X^T X)^{-1} X^T y$

→ writing model in matrix form → $Y \sim MVN(X\beta, \sigma^2 I)$

→ expected value & variance of b
→ $E(b) = E[(X^TX)^{-1}X^Ty]$
$= (X^TX)^{-1}X^T E(y)$ (↳ $= X\beta$)
$= (X^TX)^{-1}X^TX\beta$
$= \beta$ ⇒ unbiased

→ var-cov matrix of b
$b - \beta = (X^TX)^{-1}X^Ty - \beta$
$= (X^TX)^{-1}X^Ty - (X^TX)^{-1}(X^TX)\beta$ (= 1)
$= (X^TX)^{-1}X^T(y - X\beta)$

⇒ $E[(b-\beta)(b-\beta)^T] = (X^TX)^{-1}X^T E[(y - X\beta)(y - X\beta)^T] X (X^TX)^{-1}$ ($(AB)^T = B^TA^T$)
$= (X^TX)^{-1}X^T V(y) X (X^TX)^{-1}$
$= \sigma^2 (X^TX)^{-1}$
$= \mathcal{J}^{-1}$

→ In end → MLE b is a linear combo of Y_i & b/c $Y_i \sim$ Normal
$b \sim$ normal ⇒ exact sampling dists
$b \sim N(\beta, \mathcal{J}^{-1})$, or equivalently
$(b - \beta)^T \mathcal{J}(b - \beta) \sim \chi^2_{(p)}$

5.5 → Log-likelihood ratio statistic

→ One way of assessing the adequacy of a model is to compare it with a more general model w/ the max # of parameters that can be estimated. This is called a saturated (maximal or full) model

→ If there are N obs $Y_1, \dots, Y_N$ all w/ potentially different values of the linear component $x_i^T\beta$, then a saturated model can be specified w/ N parameters

→ If there are replicates (obs w/ same combos of X), then # parameters in full model may be $< N$

→ setup → let m be the max # of parameters that can be estimated & let β_{max} denote the parameter vector for the saturated model & $b_{max} = \hat{\beta}_{max\,MLE}$

→ The likelihood + log-likelihood functions evaluated at b_{max} will be larger than any other likelihood function (using same dist + link function) b/c it provides the most complete description of the data

→ Statistic → Let $L(b|y)$ be the likelihood function for the model of interest. Then the likelihood ratio

$$\lambda = \frac{L(b_{max}|y)}{L(b|y)} \Rightarrow \ln(\lambda) = \ell(b_{max}|y) - \ell(b|y)$$

provides a way of assessing the goodness of fit for the model

→ large values of λ or $\ln(\lambda)$ ⇒ poor description for model of interest relative to saturated model

→ Need sampling dist of $\ln(\lambda)$ to determine critical values
→ But $2\ln(\lambda) \sim \chi^2$ ⇒ use this statistic instead
↳ = Deviance

5.6 → Sampling dist of deviance

→ Definition → Deviance (log-likelihood (ratio) statistic)

$$D = 2[\ell(b_{max}|y) - \ell(b|y)]$$

→ Distribution → If b is MLE of β (so that $U(b) = 0$)

(earlier equation, Taylor series approximation) $\ell(\beta) = \ell(b) + (\beta - b)^T U(b) - \frac{1}{2}(\beta - b)^T \mathcal{J}(b)(\beta - b)$ (↳ = 0)
$\ell(\beta) - \ell(b) = -\frac{1}{2}(\beta - b)^T \mathcal{J}(b)(\beta - b)$
approximately

⇒ $2[\ell(b) - \ell(\beta)] = (\beta - b)^T \mathcal{J}(b)(\beta - b) \sim \chi^2_{(p)}$ ↳ # of parameters
<order switched from earlier derivation, but essentially squaring ⇒ not important>

→ Sampling dist of D
→ $D = 2[\ell(b_{max}) - \ell(b)]$
$= 2[\ell(b_{max}) - \ell(\beta_{max})] - 2[\ell(b) - \ell(\beta)] + 2[\ell(\beta_{max}) - \ell(\beta)]$
$\sim \chi^2_m$ (↳ # params in saturated model); $\sim \chi^2_p$ (↳ # params in model of interest); $= \nu$ → positive constant near zero if model of interest fits data almost as well as the saturated model fits

⇒ $D \sim \chi^2(m - p, \nu)$ ↳ ncp

→ Use in tests
→ D is the TS in most HT for GLMs
→ If Y_i's ~ Normal ⇒ $D \sim \chi^2$ exactly
→ however it depends on $V(Y_i) = \sigma^2$ which is usually unknown
⇒ D cannot be used directly as a GoF stat
→ for other dists → $D \overset{approx}{\sim} \chi^2$
→ however for Binomial + Poisson, D can be calculated + used directly as a GoF stat

→ Example → Let $Y_i \overset{ii}{\sim} Bin(n_i, \pi_i)$

→ log-likelihood → $\ell(\beta) = \sum_{i=1}^{N} \left[y_i \ln(\pi_i) - y_i \ln(1 - \pi_i) + n_i \ln(1 - \pi_i) + \ln \binom{n_i}{y_i} \right]$

→ full model → MLE → for a saturated model, all π_i's are different, so $\beta = [\pi_1, \dots, \pi_N]^T$
→ MLE for each is $\hat{\pi}_i = y_i / n_i$
→ likelihood → max value of the log-likelihood function is

$$\ell(b_{max}|y) = \sum \left[y_i \ln\left(\frac{y_i}{n_i}\right) - y_i \ln\left(\frac{n_i - y_i}{n_i}\right) + n_i \ln\left(\frac{n_i - y_i}{n_i}\right) + \ln \binom{n_i}{y_i} \right]$$

→ Model of interest → MLE → for any other model w/ $p < N$ parameters, let $\hat{\pi}_i$ be the MLE for probabilities & let $\hat{y}_i = n_i\hat{\pi}_i$ be the fitted values
→ likelihood → the log likelihood evaluated at these values is

$$\ell(b|y) = \sum \left[y_i \ln\left(\frac{\hat{y}_i}{n_i}\right) - y_i \ln\left(\frac{n_i - \hat{y}_i}{n_i}\right) + n_i \ln\left(\frac{n_i - \hat{y}_i}{n_i}\right) + \ln \binom{n_i}{y_i} \right]$$

→ Final deviance → $D = 2[\ell(b_{max}) - \ell(b)]$
$= 2 \sum_{i=1}^{N} \left[y_i \ln\left(\frac{y_i}{\hat{y}_i}\right) + (n_i - y_i) \ln\left(\frac{n_i - y_i}{n_i - \hat{y}_i}\right) \right]$

→ Example → normal model → $E(Y_i) = \mu_i = x_i^T\beta$, $Y_i \overset{ii}{\sim} N(\mu_i, \sigma^2)$, $i = 1, \dots, N$

→ likelihood → $\ell(\beta|y) = -\frac{1}{2\sigma^2} \sum (y_i - \mu_i)^2 - \frac{1}{2} N \ln(2\pi\sigma^2)$

→ full model → all μ_i's can be different ⇒ $\dim(\beta) = N \times 1$
→ maximizing above likelihood gives $\hat{\mu}_i = y_i$
⇒ max value of likelihood is
$\ell(b_{max}|y) = -\frac{1}{2} N \ln(2\pi\sigma^2)$

→ Model of interest → Let $b = (X^TX)^{-1}X^Ty$ be the MLE
→ The corresponding max value of the likelihood is
$\ell(b|y) = -\frac{1}{2\sigma^2} \sum (y_i - x_i^Tb)^2 - \frac{1}{2} N \ln(2\pi\sigma^2)$

→ final deviance → $D = 2[\ell(b_{max}) - \ell(b)]$
$= \frac{1}{\sigma^2} \sum (y_i - x_i^Tb)^2$
$= \frac{1}{\sigma^2} \sum (y_i - \hat{\mu}_i)^2$

→ If only one parameter → $E(Y_i) = \mu$ for all i, $X = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$, $b = \hat{\mu} = \bar{y}$ & $\hat{\mu}_i = \bar{y}$ for all i

→ $D = \frac{1}{\sigma^2} \sum (y_i - \bar{y})^2$
→ This is related to the sample variance S^2
$S^2 = \frac{1}{N-1} \sum (y_i - \bar{y})^2 = \frac{\sigma^2 D}{N-1}$
→ we know $\frac{N-1}{\sigma^2} S^2 \sim \chi^2_{N-1}$ ⇒ $D \sim \chi^2_{N-1}$ exactly

→ More generally → $D = \frac{1}{\sigma^2} \sum (y_i - x_i^Tb)^2$
$= \frac{1}{\sigma^2} (y - Xb)^T (y - Xb)$
$y - Xb = y - X(X^TX)^{-1}X^Ty$
$= [I - X(X^TX)^{-1}X^T]y$
$= (I - H)y$
↳ hat matrix $= X(X^TX)^{-1}X^T$

⇒ quadratic form → $(y - Xb)^T(y - Xb) = \{(I - H)y\}^T\{(I - H)y\}$
$= y^T(I - H)y$ <b/c H is idempotent → $HH = H$ & $H^T = H$>
(rank = N; rank = p; rank$(I - H) = N - p$)

⇒ $D \sim \chi^2(N - p, \lambda)$
↳ $\lambda = (X\beta)^T(I - H)(X\beta)/\sigma^2$
→ but $(I - H)X = 0$ ⇒ $D \sim \chi^2_{N-p}$ exactly

→ Scaled deviance → $\sigma^2 D = \sum (y_i - \hat{\mu}_i)^2$

→ If model fits the data well, then $D \sim \chi^2_{N-p}$. The expected value for a RV with χ^2 dist is $N - p$
⇒ $E(D) = N - p$
→ This provides an estimate for σ^2 b/c
$\tilde{\sigma}^2 = \frac{\sum (y_i - \hat{\mu}_i)^2}{N - p}$
↳ for normal linear models, some softwares output scaled deviance & call $\tilde{\sigma}^2$ called the scale parameter

→ Deviance is also related to sum of squares of standardized residuals
$\sum r_i^2 = \frac{1}{\hat{\sigma}^2} \sum (y_i - \hat{\mu}_i)^2$
→ provides rough rule of thumb for the overall magnitude of the standardized residuals
→ If the model fits the data well so that $D \sim \chi^2_{N-p}$, you could expect $\sum r_i^2 \approx N - p$

→ Example → Poisson model → $Y_i \overset{ii}{\sim} Poisson(\lambda_i)$

→ likelihood → $\ell(\beta|y) = \sum y_i \ln(\lambda_i) - \sum \lambda_i - \sum \ln(y_i!)$

→ full model → all λ_i's are different ⇒ $\beta = [\lambda_1, \dots, \lambda_N]^T$
→ MLE → $\hat{\lambda}_i = y_i$
→ max value → $\ell(b_{max}) = \sum y_i \ln(y_i) - \sum y_i - \sum \ln(y_i!)$

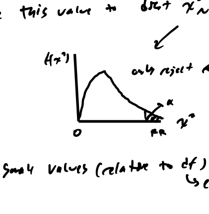
→ Model of interest → $p < N$ parameters
→ MLE → b can be used to calculate estimates $\hat{\lambda}_i$
⇒ fitted values $\hat{y}_i = \hat{\lambda}_i$ b/c $E(Y_i) = \lambda_i$
→ max value → $\ell(b) = \sum y_i \ln(\hat{y}_i) - \sum \hat{y}_i - \sum \ln(y_i!)$

→ final deviance → $D = 2[\ell(b_{max}) - \ell(b)]$
$= 2\left[\sum y_i \ln(y_i / \hat{y}_i) - \sum (y_i - \hat{y}_i)\right]$

→ For most models, it can be shown $\sum y_i = \sum \hat{y}_i$
⇒ $D = 2 \sum o_i \ln(o_i / e_i)$
(observed value y_i; estimated expected value $\hat{y}_i$)

→ This value can be calculated from the data in this case (unlike for the normal dist where D depends on unknown constant σ^2)

→ Can compare this value to dist χ^2_{N-p}



⇒ Small values (relative to df) mean good fit
↳ expected value

5.7 → Hypothesis testing

→ previous approaches → Hypotheses about the parameter vector β of length p can be tested using the sampling dist of the wald statistic $(\hat{\beta} - \beta)^T \mathcal{J}(\hat{\beta} - \beta) \sim \chi^2_p$
Occasionally the score statistic is used $U^T \mathcal{J}^{-1} U \sim \chi^2_p$

→ An alternative approach is to compare the GoF of two nested or hierarchical models (same prob dist + link function, linear component M_0 is a subset of linear component M_1)

→ new approach → H_0: $\beta = \beta_0 = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_q \end{bmatrix}$ → M_0

H_1: $\beta = \beta_1 = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_p \end{bmatrix}$ → M_1 $\quad q < p < N$

→ We can test H_0 vs H_1 using the difference of deviance stats

$\Delta D = D_0 - D_1 = 2[\ell(b_{max}) - \ell(b_0)] - 2[\ell(b_{max}) - \ell(b_1)]$
$= 2[\ell(b_1) - \ell(b_0)]$

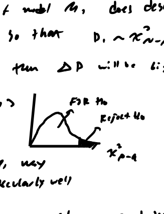
→ If both models describe the data well, then $D_0 \sim \chi^2(N - q)$ & $D_1 \sim \chi^2(N - p)$ ⇒ $\Delta D \sim \chi^2_{(p-q)}$ (provided some ⫫ conditions hold)

→ If ΔD is consistent w/ this dist, choose M_0 b/c simpler (although keep in mind practical importance of predictors & not just the statistical significance)

→ If model M_0 doesn't describe the data well, then D_0 tend to be $>$ $E(D_0)$ (↳ χ^2_{N-q}; non-central χ^2 $E(\cdot) >$)

→ If model M_1 does describe the data well so that $D_1 \sim \chi^2_{N-p}$ ⇒ if M_0 doesn't describe data well, then ΔD will be bigger than expected from $\chi^2_{(p-q)}$

→ <Usual testing conclusion>



→ Keep in mind model M_1 may not fit the data particularly well

→ Provided ΔD can be calculated, it is a good method for HT & sampling dist of ΔD (multiple deviances) is usually better approximated by χ^2 dist rather than using only a single deviance

→ For normal & some other dists, Deviance isn't totally determined by the data b/c of nuisance parameters that aren't estimated
But this can be overcome ↺

→ Example → HT normal model
$E(Y_i) = \mu_i = x_i^T\beta$ $\quad Y_i \overset{ii}{\sim} N(\mu_i, \sigma^2)$
$D = \frac{1}{\sigma^2} \sum (y_i - \hat{\mu}_i)^2$

→ Have $\hat{\mu}_i(0)$ + $\hat{\mu}_i(1)$ be from M_0 + M_1, respectively
Then $D_0 = \frac{1}{\sigma^2} \sum (y_i - \hat{\mu}_i(0))^2$ &
$D_1 = \frac{1}{\sigma^2} \sum (y_i - \hat{\mu}_i(1))^2$

→ It's usual to assume that M_1 fits the data well (i.e. H_1 is correct)
⇒ $D_1 \sim \chi^2_{N-p}$. If M_0 also fits the data well, then $D_0 \sim \chi^2(N - q)$
⇒ $\Delta D \sim D_0 - D_1$

→ If M_0 does not fit the data well (i.e. H_0 is not correct) then $\Delta D \sim$ non-central χ^2
To eliminate σ^2, the following ratio is used

$$F = \frac{D_0 - D_1}{p - q} \Big/ \frac{D_1}{N - p}$$

$$= \frac{\left[\sum (y_i - \hat{\mu}_i(0))^2 - \sum (y_i - \hat{\mu}_i(1))^2\right] / (p - q)}{\sum (y_i - \hat{\mu}_i(1))^2 / (N - p)}$$

→ Now F can be directly calculated from the fitted values
→ If H_0 is correct ⇒ $F \sim$ Central $F(p - q, N - p)$
→ If H_0 is not correct ⇒ $F \sim$ non-central $F(\dots)$