1.1) ✓

$\rightarrow Y_1 \sim N(1,3)$ $\quad W_1 = Y_1 + 2Y_2$

$Y_2 \sim N(2,5)$ $\quad W_2 = 4Y_1 - Y_2$

$Y_1 \perp\!\!\!\perp Y_2$ $\quad (W_1, W_2) = ??$

$\rightarrow W_1 \sim N(\mu_{Y_1} + 2\mu_{Y_2},\ \sigma^2_{Y_1} + 4\sigma^2_{Y_2})$
$= 1 + 2(2),\ 3 + 4(5)$
$= 5,\ 23$

$W_2 \sim N(4\mu_{Y_1} - \mu_{Y_2},\ 16\sigma^2_{Y_1} + \sigma^2_{Y_2})$
$= 4(1) - 2,\ 16(3) + 5$
$= 2,\ 53$

$\rightarrow \boxed{(W_1, W_2) \sim MVN\left[\begin{pmatrix}5\\2\end{pmatrix}, \begin{pmatrix}23 & 2\\ 2 & 53\end{pmatrix}\right]}$

$Cov(W_1, W_2) = cov(Y_1, 4Y_1) - Cov(Y_1, Y_2) + Cov(2Y_2, 4Y_1) - Cov(2Y_2, Y_2)$
$= 4(3) \quad = 0 \quad = 0 \quad = 2(5)$
$= 2$

↳ Σ Cov(pairs)

1.2) ✓

$\rightarrow Y_1 \sim N(0,1)$

$Y_2 \sim N(3,4)$

$Y_1 \perp\!\!\!\perp Y_2$

a) $Y_1^2 = Z^2 \sim \chi^2_1$

b) $\rightarrow \underline{y} = \begin{bmatrix} Y_1 \\ (Y_2 - 3)/2 \end{bmatrix}$

$\rightarrow y^T y = \underset{1\times 2}{[Z_1\ \ Z_2]} \underset{2\times 1}{\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}} = \chi^2_1 + \chi^2_1 = \chi^2_2$

c) $\rightarrow \underline{y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \sim MVN\left(\begin{bmatrix} 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\right)$

$\rightarrow y^T V^{-1} y = \frac{1}{4}[Y_1\ \ Y_2] \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \frac{1}{4}[4Y_1\ \ Y_2] \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \frac{1}{4}[4Y_1^2 + Y_2^2] = Y_1^2 + \frac{1}{4}Y_2^2 \sim \chi^2_{(2,\lambda)}$

non-central χ^2

↳ $\lambda = \mu^T V^{-1} \mu$
$= [0\ \ 3]\ \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 3 \end{bmatrix}$
$= \frac{1}{4}[0\ \ 3]\begin{bmatrix} 0 \\ 3 \end{bmatrix}$
$= 9/4$

$V = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$V^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$V^{-1} = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$

if $y \sim MVN(\mu, V) \Rightarrow y^T V^{-1} y \sim \chi^2_{(n,\lambda)}$
↳ $\lambda = \mu^T V^{-1} \mu$

1.3 ✓

$\rightarrow (Y_1, Y_2) \sim MVN\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 & 1 \\ 1 & 9 \end{bmatrix}\right)$

$\rightarrow$ a) $(y - \mu)^T V^{-1} (y - \mu) = [(Y_1 - 2)\ \ (Y_2 - 3)]\ \frac{1}{35}\begin{bmatrix} 9 & -1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} Y_1 - 2 \\ Y_2 - 3 \end{bmatrix}$

$= \frac{1}{35}[(9(Y_1 - 2) - (Y_2 - 3))\ \ -(Y_1 - 2) + 4(Y_2 - 3)]\begin{bmatrix} Y_1 - 2 \\ Y_2 - 3 \end{bmatrix}$

$= \frac{1}{35}[9(Y_1 - 2)^2 - (Y_2 - 3)(Y_1 - 2) - (Y_1 - 2)(Y_2 - 3) + 4(Y_2 - 3)^2]$

$$\boxed{\begin{aligned} &= \tfrac{1}{35}[9(Y_1 - 2)^2 - 2(Y_1 - 2)(Y_2 - 3) + 4(Y_2 - 3)^2] \\ &\sim \chi^2_2 \text{ by theorem} \end{aligned}}$$

$\rightarrow$ b) $y^T V^{-1} y = [Y_1\ \ Y_2]\ \frac{1}{35}\begin{bmatrix} 9 & -1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$

$= \frac{1}{35}[(9Y_1 - Y_2)\ \ (-Y_1 + 4Y_2)]\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$

$= \frac{1}{35}[9Y_1^2 - Y_1Y_2 - Y_1^2 + 4Y_2^2]$

$$\boxed{\begin{aligned} &= \tfrac{1}{35}[8Y_1^2 - Y_1Y_2 + 4Y_2^2] \\ &\sim \chi^2_{(2,\lambda)} \end{aligned}}$$

↳ $\lambda = \mu^T V^{-1} \mu$
$= [2\ \ 3]\ \frac{1}{35}\begin{bmatrix} 9 & -1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$
$= \frac{1}{35}[15\ \ 10]\begin{bmatrix} 2 \\ 3 \end{bmatrix}$
$= \frac{1}{35}[60]$
$= \frac{12}{7}$

1.4 ✗

$\rightarrow Y_i \overset{iid}{\sim} N(\mu, \sigma^2) \quad i = 1, \dots, n$

$\bar{Y} = \frac{1}{n}\sum Y_i \qquad S^2 = \frac{1}{n-1}\sum (Y_i - \bar{Y})^2$

a) $\bar{Y} \sim N(\mu, \sigma^2/n)$

b) show $S^2 = \frac{1}{n-1}[\underbrace{\sum (Y_i - \mu)^2 - n(\bar{Y} - \mu)^2}]$

↳ $= \sum (Y_i - \bar{Y})^2$

$\sum (Y_i - \bar{Y})^2 = \sum[(Y_i - \mu) - (\bar{Y} + \mu)]^2$

$= \sum[(Y_i - \mu)^2 - 2(Y_i - \mu)(\bar{Y} + \mu) + (\bar{Y} + \mu)^2]$

$= \sum[(Y_i - \mu)^2 - 2Y_i\bar{Y} - 2Y_i\mu + 2\bar{Y}\mu + 2\mu^2 + (\bar{Y} + \mu)^2]$

$\sum (Y_i - \mu)^2 - 2\bar{Y}\sum Y_i - 2\mu\sum Y_i + 2n\bar{Y}\mu + 2n\mu^2 + n(\bar{Y} + \mu)^2$

$-2\bar{Y}n\bar{Y} - \cancel{2\mu n\bar{Y}} + \cancel{2n\bar{Y}\mu} + 2n\mu^2 + \cancel{n\bar{Y}^2} + 2n\bar{Y}\mu + \cancel{n\mu^2}$

$-2\bar{Y}^2 n$

$\sum (Y_i - \mu)^2 - n\bar{Y}^2 + 3n\mu^2 + 2n\bar{Y}\mu$

$n\bar{Y}^2 - 2n\bar{Y}\mu + n\mu^2$

?

c)

d) $\frac{(n-1)}{\sigma^2}S^2 \sim \chi^2_{n-1}$

e) $\frac{\bar{Y} - \mu}{S/\sqrt{n}} \sim t_{n-1}$

1.5

✓ a.) $\rightarrow Y_1, \dots, Y_n \overset{iid}{\sim} \text{Poisson}(\theta)$

$\rightarrow$ show $E(Y_i) = \theta \qquad M_{Y_i}(t) = e^{\theta(e^t - 1)}$

$E(Y_i) = M'_{Y_i}(t)\Big|_{t=0} = \theta e^t e^{\theta(e^t - 1)}\Big)_{t=0} \overset{\checkmark}{=} \theta$

b) let $\theta = e^\beta \rightarrow$ find MLE β

$\rightarrow \beta = \ln(\theta) \longrightarrow \hat{\theta}_{MLE} = \bar{Y} \Rightarrow \hat{\beta}_{MLE} = \tau(\hat{\theta}_{MLE}) = \boxed{\ln(\bar{Y})}$ by invariance property

$\vartheta = \tau(\theta)$

c) Minimize $\sum (Y_i - e^\beta)^2$ to get $\hat{\beta}_{LSE}$

$\rightarrow \frac{d}{d\beta}(\dots) = -\sum 2e^\beta(Y_i - e^\beta)$

$0 = -2e^\beta \sum (Y_i - e^\beta)$

$\downarrow = \sum Y_i - ne^\beta$

$ne^\beta = \sum Y_i$

$\Rightarrow \boxed{\hat{\beta}_{LSE} = \ln(\bar{Y})}$

1.6 ✓

a) (calculator work)

b) $Y_i \overset{\perp\!\!\!\perp}{\sim} Bin(n_i, \theta)$, $i = 1, \dots, n$

$\rightarrow$ MLE $\theta \rightarrow L(\theta) = f(Y_1, \dots, Y_n | \theta) = \prod_{i=1}^{n} f(Y_i | \theta)$

$= \prod_{i=1}^{n} \binom{n_i}{Y_i} \theta^{Y_i}(1 - \theta)^{n_i - Y_i}$

$= \theta^{\sum Y_i}(1 - \theta)^{\sum(n_i - Y_i)} \prod \binom{n_i}{Y_i}$

$\rightarrow \ell(\theta) = \ln[L \dots]$

$\downarrow = \sum Y_i \ln(\theta) + \sum(n_i - Y_i)\ln(1 - \theta) + \sum \ln\binom{n_i}{Y_i}$

$\rightarrow \ell'(\theta) = \frac{d}{d\theta}[\ell \dots]$

$\downarrow = \frac{\sum Y_i}{\theta} - \frac{\sum(n_i - Y_i)}{1 - \theta}$

$\rightarrow 0 = \frac{\sum Y_i}{\theta} - \frac{\sum(n_i - Y_i)}{1 - \theta}$

$= \frac{(1 - \theta)\sum Y_i - \theta\sum(n_i - Y_i)}{\cancel{\theta(1 - \theta)}}$

$= \sum Y_i - \cancel{\theta\sum Y_i} - \theta\sum n_i + \cancel{\theta\sum Y_i}$

$\boxed{\hat{\theta}_{MLE} = \frac{\sum Y_i}{\sum n_i}}$

↳ using data $\rightarrow \hat{\theta}_{MLE} = \frac{363}{734} = 0.4946$

c) (on computer)