## 1.7.3 → Life Annuities

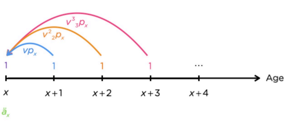
→ overview → In order to purchase different types of life insurance covered in the previous subsections, insureds typically pay annual premiums. These annual premiums paid by policyholders have the payment structure of life annuities.

- → A life annuity is similar to a regular annuity, just w/ an additional mortality component. So payments are made as long as the person is alive.

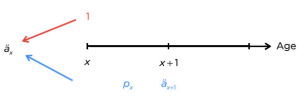
- → An important symbol for life annuities is $\ddot{a}_x$, which is referred to as a life-annuity-due. This is the APV of a life annuity w/ payments of 1 made at the beginning of each year as long as $(x)$ is alive.
  - → The first payment will be made immediately at age $x$. The APV of this payment is simply 1.
  - → The next payment of 1 will only be made if the life survives to age $x+1$. The APV of this payment is $vp_x$
  - → Then another payment will be made if the life survives two years to age $x+2$, which makes the APV $v^2{}_2p_x$

- → This payment structure continues until death

$$\boxed{\begin{aligned}\ddot{a}_x &= 1 + vp_x + v^2{}_2p_x + \cdots\\ &= 1 + vp_x\left[vp_{x+1} + v^2{}_2p_{x+1} + \cdots\right]\\ &= 1 + vp_x\,\ddot{a}_{x+1}\end{aligned}}$$



- → Notice $\ddot{a}_x$ can be expressed as a recursive formula. $\ddot{a}_x$ is equivalent to a payment of 1 now + a life annuity on life age $x+1$ in one year if $(x)$ survives until age $x+1$.



- → If the payments are instead made at the end of the year, assuming $(x)$ is alive, this is referred to as a life-annuity-immediate. In this case, the APV is denoted as $a_x$

- → There are different types of life annuities that correspond to the different types of life insurance covered in the previous subsection. For example, similar to term life insurance, a term life annuity is an annuity w/ payments made at the beginning of each year for $t$ years as long as $(x)$ is alive + w/ no payments made after $t$ years.

  (→ not covering all of them cause similar application to last section, instead an example)

→ Example → Given the following info

- → (50) has a 5-year term life insurance policy
- → She pays 200 at the beginning of each year she is alive during the term of the insurance
- → The death benefit, $Y$, is paid at the end of the year of death
- → < standard table >

→ Calculate APV of the premiums

→ APV = 200 [ < term life > ]

$$\begin{aligned}&= \left[1 + vp_{50} + v^2{}_2p_{50} + v^3{}_3p_{50} + v^4{}_4p_{50}\right]\\ &= \left[1 + 1.06^{-1}\frac{\ell_{51}}{\ell_{50}} + 1.06^{-2}\frac{\ell_{52}}{\ell_{50}} + \cdots + 1.06^{-4}\frac{\ell_{54}}{\ell_{50}}\right]\\ &= [\,\ldots\,]\\ &= [4.4114]\\ &= 882.2743\end{aligned}$$

$$= 200\left[\ddot{a}_{50} - \underbrace{v^5{}_5p_{50}}_{{}_5E_{50}\ \to\ \text{discount + survive} \Rightarrow \text{deferral}}\ddot{a}_{55}\right] \quad \text{< subtraction of life annuity dues >}$$

$$\begin{aligned}&= \left[13.2668 - 0.73137\,(12.1758)\right]\\ &= [4.4114]\\ &= 882.2812 \quad \text{< slight rounding differences >}\end{aligned}$$

→ Life annuities + life insurance

- → There is a valuable relationship between the APVs of a life annuity + life insurance that will allow us to convert between the two easily. Before we get to that, let's review a few concepts from interest theory related to an annuity-certain, which is an annuity where the # of payments is fixed (i.e. there is no mortality component).

- → Consider an annuity-certain which requires an investment of 1 now + pays interest of $d$ at the beginning of every period for $n$ periods. At the end of $n$ periods, you will receive your investment of 1 back. So, the present value (at an interest rate consistent w/ $d$) of the interest payments + the original investment will equal 1. From this, we get the formula for an annuity due.

$$\boxed{\begin{aligned}d\,\ddot{a}_{\overline{n}|} + v^n &= 1\\ \ddot{a}_{\overline{n}|} &= \frac{1-v^n}{d}\end{aligned}}$$

- → The interest paid at the beginning of each year, $d$, is called discount + is equal to $\frac{i}{1+i}$. Notice this is different than $d_x$ used in the life table.

- → This is analogous to the fact that a bond w/ a coupon rate equal to the yield to maturity has a price that is equal to its face amount.

- → We can extend this concept to a life annuity + life insurance. Instead of an annuity certain over $n$ periods, we have a life annuity that will pay $d$ at the beginning of every year until death. And instead of getting 1 back at the end of $n$ periods, we get a death benefit of 1 at the end of the year of death.

- → Therefore, investing 1 now is equivalent to receiving $d$ as interest at the beginning of each year when $(x)$ is alive + receiving a payment of 1 at the end of the year of death of $(x)$.

$$\boxed{\begin{aligned}1 &= d\ddot{a}_x + A_x\\ \ddot{a}_x &= \frac{1-A_x}{d}\end{aligned}} \Rightarrow \$1 \text{ today} = \text{< life annuity w/ payments of } d \text{ >} + \text{< life insurance policy w/ death benefit of 1 >}$$

→ Example → Given the following info

- → A whole life policy on (59) has a death benefit of 1 paid at the end of the year of death
- → APV = 0.35
- → $i = 0.05$ + mortality follows this table:

| $x$ | $q_x$ |
|---|---|
| 58 | 0.01 |
| 59 | 0.02 |
| 60 | 0.03 |

→ Calculate the APV of a whole life annuity due w/ annual payments of 1 on (60)

→ Goal is to calculate $\ddot{a}_{60} = \frac{1-A_{60}}{d}$

$$\begin{aligned}A_{60}:\ A_{59} &= vq_{59} + vp_{59}A_{60}\\ 0.35 &= 1.05^{-1}(0.02) + 1.05^{-1}(1-0.02)A_{60}\\ &\Rightarrow A_{60} = 0.3546\end{aligned}$$

$$\Rightarrow \ddot{a}_{60} = \frac{1-0.3546}{0.05/(1+0.05)} = 13.5536$$

→ Joint lives

- → We can also have annuities that are based on the survival of multiple lives. The most common type are for two lives, $(x)$ + $(y)$, which are usually a married couple.

- → We will briefly introduce two types of annuities for joint lives. The first is called a joint life annuity, + its APV is denoted as $\ddot{a}_{xy}$. This annuity makes payments until the first death of $(x)$ or $(y)$. The second is called last survivor annuity, which makes payments until the second death of $(x)$ + $(y)$. The APV of this type of annuity is denoted as $\ddot{a}_{\overline{xy}}$. We can relate the annuities on joint lives to the annuities on individual lives:

$$\boxed{\ddot{a}_x + \ddot{a}_y = \ddot{a}_{xy} + \ddot{a}_{\overline{xy}}}$$

- → If $(x)$ dies first, then $\ddot{a}_{xy} = \ddot{a}_x$ + $\ddot{a}_{\overline{xy}} = \ddot{a}_y$.
  If $(y)$ dies first, then $\ddot{a}_{xy} = \ddot{a}_y$ + $\ddot{a}_{\overline{xy}} = \ddot{a}_x$
  ⇒ regardless of order of death, the value of $\ddot{a}_{xy}$ matches $\ddot{a}_x$ or $\ddot{a}_y$ + $\ddot{a}_{\overline{xy}}$ matches the other.

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→ Assignment

→ Q1) → Given the following info for a 10-year term life annuity-due of 1 on $(x)$ that is payable annually:

$$\begin{aligned}&\ddot{a}_x = 28.1758\\ &\ddot{a}_{x+10} = 26.6914\\ &v = (0.9)^{1/5}\\ &{}_{10}q_x = \tfrac{10}{100}\end{aligned}$$

→ $APV = \ddot{a}_x - {}_{10}E_x\,\ddot{a}_{x+10}$

$$\begin{aligned}&= \downarrow - (v^{10}\,\underbrace{{}_{10}p_x}_{1-{}_{10}q_x})\,\ddot{a}_{x+10}\\ &= 28.1758 - (0.9)^{10/5}\left(1-\tfrac{10}{100}\right)26.6914\\ &= 8.7177\end{aligned}$$

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## 1.7.4 → Equivalence principle

→ overview → The equivalence principle is one of the most important concepts in life contingencies. It equates the APV of the benefits from an insurance policy w/ the APV of the premiums so that insureds receive a "fair" contract. From an insurance company's standpoint, expenses + profit need to be included when pricing products. This means the APV of the premiums received by the company need to cover benefits as well as expenses + profit.

- → For this exam, we are only going to focus on benefits + net premium. Net premium is the amount of premium needed to cover the cost of the benefits paid by the insurance company, ignoring expenses + profit.

$$\boxed{APV_{Premium} = APV_{Benefits}}$$

→ Examples

→ Given the following info

- → A 3-year term life insurance policy on (55) provides a level death benefit of 3000.
- → The premium $P$ is paid at the beginning of each year during the term as long as the insured is alive.
- → The death benefit is paid at the end of the year of death
- → $i = 0.06$, with mortality table

| $x$ | $q_x$ |
|---|---|
| 55 | 0.01 |
| 56 | 0.02 |
| 57 | 0.04 |

→ Calculate $P$ using the equivalence principle.

→ $P = ??$

$$\begin{aligned}\to APV_{Benefit} &= 3000\left[vq_{55} + v^2p_{55}q_{56} + v^3{}_2p_{55}q_{57}\right]\\ &= \left[1.06^{-1}(0.01) + (1.06)^{-2}(1-0.01)(0.02) + 1.06^{-3}(1-0.01)(1-0.02)(0.04)\right]\\ &= [0.0596]\\ &= 178.9195\end{aligned}$$

$$\begin{aligned}\to APV_{premium} &= P\left[1 + vp_{55} + v^2{}_2p_{55}\right]\\ &= \left[1 + 1.06^{-1}(1-0.01) + 1.06^{-2}(1-0.01)(1-0.02)\right]\\ &= P[2.7474]\end{aligned}$$

$$\begin{aligned}\Rightarrow APV_{Benefit} &= APV_{Premium}\\ 178.9195 &= 2.7174P\\ &\Rightarrow P = 65.9584\end{aligned}$$

→ Given the following information

- → A life insurance company issues a special 3-year term life insurance policy on $(x)$
- → If the insured dies, there will be a random draw at the end of the year of death. The death benefit will either be 100,000 w/ prob = 0.75 or 0 w/ prob = 0.25
- → At the beginning of each year the policy is in effect, if $(x)$ is alive, there will be a random draw. The insured will either pay $P$, w/ prob = 0.2 or 0 w/ prob = 0.8
- → Random draws are ⫫
- → The prob of dying between age age $x+t$ + age $x+t+1$ is $q_{t+x} = 1 - 0.98^{t+1}$ for $t = 0, 1, 2, \ldots$
- → $i = 0.05$

→ Calculate $P$ using the equivalence principle.

$$\begin{aligned}\to APV_{Benefit} = {}& vq_x\left[100000(0.75) + 0(0.25)\right] + v^2p_xq_{x+1}\left[100000(0.75) + 0(0.25)\right] +\\ & v^3{}_2p_xq_{x+2}\left[100000(0.75) + 0(0.25)\right]\end{aligned}$$

$$\begin{aligned}&= 1.05^{-1}(0.02)[\ldots] + 1.05^{-2}(0.98)(1-0.98^2)[\ldots] + 1.05^{-3}(0.98)(0.98^2)(1-0.98^3)[\ldots]\\ &= 7654.55\end{aligned}$$

$$\begin{aligned}\to APV_{Premium} &= 1\left[P(0.2) + 0(0.8)\right] + vp_x\left[P(0.2) + 0(0.8)\right] + v^2{}_2p_x\left[P(0.2) + 0(0.8)\right]\\ &= 0.2P + 1.05^{-1}(0.98)[0.2P] + 1.05^{-2}(0.98)(0.98^2)[P(0.2)]\\ &= P[\,\ldots\,]\\ &= \downarrow[0.5574]\end{aligned}$$

$$\begin{aligned}\to APV_{Benefit} &= APV_{premium}\\ 7654.5526 &= 0.5574P\\ \Rightarrow P &= 13{,}732.6024\end{aligned}$$

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→ Assignment

→ Q1) For a 3-year term insurance of 1000 on (40), you are given:

- → The death benefit is payable at the end of the year of death
- → Level net premiums are payable at the beginning of each year for 3 years
- → Mortality rates are as follows:

| $x$ | $q_x$ |
|---|---|
| 40 | 0.0019 |
| 41 | 0.0024 |
| 42 | 0.0027 |

- → Interest rates are as follows:

| Year | $i$ |
|---|---|
| 1 | 0.02 |
| 2 | 0.03 |
| 3 | 0.04 |

→ Calculate annual net premium

$$\begin{aligned}\to APV_{premium} &= P\left[1 + v_1p_{40} + v_1v_2{}_2p_{40}\right]\\ &= \left[1 + 1.02^{-1}(1-0.0019) + (1.02 \times 1.03)^{-1}(1-0.0019)(1-0.0024)\right]\\ &= [2.9263]\end{aligned}$$

$$\begin{aligned}\to APV_{Benefit} &= 1000\left[v_1q_{40} + v_1v_2p_{40}q_{41} + v_1v_2v_3{}_2p_{40}q_{42}\right]\\ &= \left[1.02^{-1}(0.0019) + (1.02 \times 1.03)^{-1}(1-0.0019)(0.0024) + (1.02 \times 1.03 \times 1.04)^{-1}(1-0.0019)(1-0.0024)(0.0027)\right]\\ &= [0.006603]\\ &= 6.6033\end{aligned}$$

$$\begin{aligned}\Rightarrow APV_{Benefit} &= APV_{premium}\\ 6.6033 &= 2.9263P\\ &\Rightarrow P = 2.2565\end{aligned}$$