# 1.7 → Life contingencies

## 1.7.1 → Probabilities

→ notation → $_tp_x$ or $_nP_x$ = $P(x)$

   → $_tp_x$ = B lives at least to age $x$

   → $_tq_x$ = B lives or reaches to before age $x+t$

     $\downarrow$ = $l_x - l_{x+t}$

   → $_tq_x$ : prob of $(x)$ dying before age $x+t$

     $\downarrow$ = $\dfrac{d_x}{l_x}$

   → complement $_tp_x$ = prob of $(x)$ surviving to age $x+t$

     $\downarrow$ = $1 - _tq_x = 1 - \dfrac{d_x}{l_x} = \dfrac{l_{x+t}}{l_x}$

   → $_{t}p_x$ = all $x$ survive to age $x+t$

     additional to $\downarrow$ = $_tp_x \cdot _sp_{x+t} = _{t+s}p_x$

        = $\dfrac{l_{x+t}}{l_x}$

   → $_t|q_x = 1 - _tp_x$ , prob of $(x)$ die before age $x+t$

     $\downarrow$ = $\dfrac{l_x - l_{x+t}}{l_x}$

   → Before to an illustration of this, who survives for $t$ years, but dies with the next $u$ years. Probability of that occurring is $_tp_x \cdot _uq_{x+t}$



     → $D$ = $\#$ or deaths in the next $t$ years from a group of $n$ insureds, all $(x)$

     $\downarrow$ = Binomial $(t|q_x)$

     $E(D) = n \cdot _tq_x$     $\}$ mean CT → $E(x) = 2 \cdot a_x \cdot \sqrt{v(x)}$

     $v(D) = n \cdot _tq_x \cdot _tp_x$

     $\downarrow = (1 - _tq_x)$

→ Curtate life expectancy → avg $\#$ of full years expected to live

     $\downarrow$ = $\displaystyle\sum_{k=1}^{\infty} k \cdot _kp_x$

        = $e_x$

→ complete expectation of life → curtate + 0.5
     $\downarrow$ assume deaths occur uniformly throughout year

## 1.7.2 → Life Insurance

→ Actuarial present value → discounts payments for interest & probability

   $APV = v \cdot q_x + v^2 \cdot _1p_x \cdot q_{x+1} + v^3 \cdot _2p_x \cdot q_{x+2} + \cdots$

discount factor
= $\dfrac{1}{1+i}$



   → discount payment of $1$ at end of year not death

→ whole life insurance → $n$ that product

   → $A_x$ = APV of whole life insurance that pays $1$ for $(x)$

      = $\displaystyle\sum_{k=0}^{\infty} v^{k+1} \cdot _kp_x \cdot q_{x+k}$

      = $v \cdot q_x + v^2 \cdot _1p_x \cdot q_{x+1} + \cdots$

      = $v \cdot q_x + v p_x \cdot A_{x+1}$

     $\underbrace{\qquad\qquad}_{\text{recursive formula}}$



   → $^2A_x = \displaystyle\sum_{k=0}^{\infty} v^{2(k+1)} \cdot _kp_x \cdot q_{x+k}$ = 2nd raw moment (can use if to calculate variance)

     $\downarrow$ pays twice the force of interest $[v (1+i)^2 - 1]$

→ Deferred life insurance → only pays if death occurs after $t$ years (0 payment if before)

   → $APV = \dfrac{v^t \cdot _tp_x \cdot A_{x+t}}{}$

     $\underbrace{\qquad}_{tE_x}$ = represents present value of $1$ paid $t$ years from now, conditional on surviving $t$ years ("discount & survival")

     $\downarrow$ can be broken down into smaller components e.g. $_tE_x = _1E_x \cdot _1E_{x+t}$, e.g. $_tp_x = _np_x \cdot _{t-n}p_{x+n}$

→ Term life insurance → pays if death occurs in first $t$ years (= whole life insurance w/ expiration date)

   → $APV = A^1_{x:\overline{t}|} = \displaystyle\sum_{k=0}^{t-1} v^{k+1} \cdot _kp_x \cdot q_{x+k}$

     $\downarrow$ = $v q_x + v^2 _1p_x \cdot q_{x+1} + \cdots + v^t \cdot _{t-1}p_x \cdot q_{x+t-1}$

→ Relationship between all our above types of insurance

   → $A_x = A^1_{x:\overline{t}|} + _tE_x \cdot A_{x+t}$

      whole life       $\downarrow$       deferred whole life

     ⟹ from this: $A_x = _tp_x \cdot A_{x+t}$



→ Pure Endowment → pays at end of year if survives until then = nothing otherwise (survival benefit)

   → $APV = _tE_x$

→ Endowment insurance → pays either death or survival benefit. Death benefit at end of year of death w/in $t$ years + survival benefit at end of year $t$ if survive $t$ years

   → $APV = A_{x:\overline{t}|} = \left[ \displaystyle\sum_{k=0}^{t-1} v^{k+1} \cdot _kp_x \cdot q_{x+k} \right] + v^t _tp_x$

      = $v q_x + v^2 _1p_x q_{x+1} + \cdots + v^t _{t-1}p_x q_{x+t-1} + _tE_x$

   → Endowment, term life + pure endowment

     $A_{x:\overline{t}|} = A^1_{x:\overline{t}|} + _tE_x$

     endowment  =  term life  +  pure endowment
     insurance     insurance

## 1.7.3 → Life annuities

→ $\ddot{a}_x$ : life annuity due
   = APV of a life annuity w/ payments of $1$ made at the beginning of each year as long as $(x)$ is alive

   → this payment structure (continues until death)

     $\ddot{a}_x = 1 + v p_x + v^2 _2p_x + \cdots$

        = $1 + v p_x [v p_x + v^2 _2p_{x+1} + \cdots]$

         $\downarrow$ = $1 + v p_x \ddot{a}_{x+1}$
         $\underbrace{\qquad}_{\text{recursion formula}}$



→ $a_x$ : life annuity immediate in when payments are made at the end of year

→ different types of life annuities, just like types of insurance

   → annuity certain → fixed # of payments

     → $\ddot{a}_{\overline{n}|} \cdot d + v^n = 1 \implies \ddot{a}_{\overline{n}|} = \dfrac{1 - v^n}{d}$

       $\downarrow$ annuity due
       discount: $d = \dfrac{i}{1+i}$



     ⟹ from above, but now of life annuities + $v^n$ insurance

       $1 = d \ddot{a}_x + A_x$

       $\ddot{a}_x = \dfrac{1 - A_x}{d}$

→ Joint lives → $\ddot{a}_{x+y} = \ddot{a}_x + \ddot{a}_y + \ddot{a}_{\overline{xy}}$
     $\downarrow$ payments until first death
     payments until second death

## 1.7.4 → Equivalence principle

→ fair insurance → APV premium = APV benefits

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# 1.8 → Simulation

## 1.8.1 → Uniform # generator

→ algorithm → choose $x_0$ (seed)
     ⟹ $x_{n+1} = (a \cdot x_n + c) \bmod m$ , $m, u$
     ⟹ $\dfrac{x_{n+1}}{m}$ → pseudo between $[0, 1]$

## 1.8.2 → Inversion method

→ process → 1) Generate uniform random $U$
   → 2) let $F(x) = U \iff$ solve for $X$ ⟹ $X = F^{-1}(U)$

## 1.8.3 → Acceptance rejection method



→ process ⟹ find a c such that $\dfrac{f(x)}{g(x)} \le c$ i.e. find $\dfrac{d}{dx}\left[\dfrac{f(x)/g(x)}{}\right]$ c $= \dfrac{f(x)}{g(x)}$
     (maximize)       plug in

   → 1) simulate $y$ from $g$ (will be your)
      + $U$ (will be yours)

   → 2) compare $u \le \dfrac{f(y)}{c g(y)}$ ⟹ if TRUE ⟹ accept $y$ as sample point from $f$