1.4.1 → Poisson Process

→ defs → <u>Stochastic process</u> → a collection of RVs

→ <u>Counting process</u>, N or $\{N(t), t \geq 0\}$

→ This is a stochastic process whose RVs have values that are non decreasing & non-negative integers

→ for time $t \geq 0$, a counting process counts the # of events that occur after time 0 & up to (& including) time t.

→ $N(t)$ = # events that occur in interval $(0, t]$

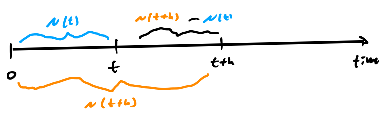
→ $N(t) \geq 0$

→ $N(t)$ is an integer

→ $N(0) = 0$

→ $N(t+h) \geq N(t)$ for $h > 0$

→ The <u>increment</u> $N(t+h) - N(t)$ represents the # of events that occur in the time interval $(t, t+h]$



→ <u>Poisson process</u> → counting process where each increment is a Poisson RV & non-overlapping increments are independent of each other

→ for Poisson process N w/ rate function $\lambda(t)$

→ the increment $N(t+h) - N(t) \sim \text{Poisson}\left(\lambda = \int_t^{t+h} \lambda(u)\,du\right)$

→ If the rate function is constant (i.e. $\lambda(t) = \lambda$), then N is a <u>homogeneous Poisson process</u>

→ $N(t+h) - N(t) \sim \text{Poisson}(h\lambda)$

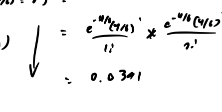
→ If the rate function $\lambda(t)$ varies w/ t, the process N is a <u>non-homogeneous Poisson process</u>

→ b/c increments $N(t)$ & $N(t+h) - N(t)$ are ⫫, the distribution of the # of events after time t is unaffected by the # of events before or at time t

$$P(N(t+h) - N(t) = x \mid N(t) = n) = \frac{P[N(t+h) - N(t) = x \cap N(t) = n]}{P(N(t) = n)}$$

$$= \frac{P[N(t+h) - N(t) = x]\,\cancel{P[N(t) = n]}}{\cancel{P(N(t) = n)}}$$

$$= P[N(t+h) - N(t) = x]$$



MUST be non-overlapping

→ Examples

→ Buses arrive at a stop at a Poisson rate of $\lambda = 4$ per hour ⇒ homogeneous Poisson Process

→ $P(N(1/3) = 2) = \frac{e^{-4/3}(4/3)^2}{2!} = 0.2343$ (20 min)

↳ $\sim \text{Poisson}(\lambda = \frac{1}{3}(4))$

→ $P(N(1/6) = 1 \cap N(1) - N(5/6) = 2) = (\ldots) * (\ldots)$ < b/c ⫫ >

$\sim \text{Poisson}(\lambda = \frac{1}{6}(4))$

$$= \frac{e^{-4/6}(4/6)^1}{1!} * \frac{e^{-4/6}(4/6)^2}{2!}$$

$$= 0.0391$$

→ Taxis arrive according to Poisson Process w/ rate function $\lambda(t) = 3t$

→ $P(N(0.5) - N(0.2) \geq 2) = 1 - P(N(0.5) - N(0.2) = 0, 1)$

↳ $\lambda = \int_{0.2}^{0.5} 3t\,dt = 0.315$

↳ $\sim \text{Poisson}(\lambda)$

$$= 1 - \left[\frac{e^{-0.315}\,0.315^0}{0!} + \frac{e^{-0.315}\,0.315^1}{1!}\right]$$

$$= 0.0403$$

→ $P(N(0.5) = 5 \mid N(0.2) = 2) = \frac{P(N(0.5) - N(0.2) = 3 \cap N(0.2) = 2)}{P(N(0.2) = 2)}$

$$= \frac{P(N(0.5) - N(0.2) = 3)\,\cancel{P(N(0.2) = 2)}}{\cancel{P(N(0.2) = 2)}}$$

$$= P(N(0.5) - N(0.2) = 3) = \frac{e^{-0.315}\,0.315^3}{3!} = 0.0038$$

↳ $\sim \text{Poisson}(\lambda = .315)$

→ Assignment

→ Q2) $N(t)$ poisson process w/ rate 3

$P(N(2) < 3) = P(N(2) = 0, 1, 2)$

↳ $\sim \text{Poisson}(2(3))$

$$= e^{-6}\left[\frac{6^0}{0!} + \frac{6^1}{1!} + \frac{6^2}{2!}\right]$$

$$= 0.0619$$

→ Q3) $N(t) \sim PP(\text{rate} = 3)$

$E[N(1) \cdot N(3)] = E[N(1) \cdot \{N(3) - N(1) + N(1)\}]$

dependent

$$= E[N(1) \cdot \{N(3) - N(1)\} + N(1)^2]$$

$$= E[N(1) \cdot \{N(3) - N(1)\}] + E[N(1)^2]$$

$$= E[N(1)]\,E[N(3) - N(1)] + [V(N(1)) + [E(N(1))]^2]$$

$\sim \text{Pois}(\lambda = 3)$; $\sim \text{Pois}(\lambda = 2(3))$

$$= 3 \cdot 6 + [3 + 3^2]$$

$$= 30$$

1.4.2 → Time between events

→ def → T_K → time until the K^{th} event occurs

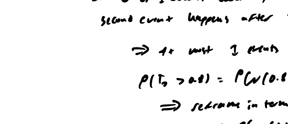
V_K → time from $(K-1)^{st}$ event to the K^{th} event (inter-arrival time)

→ $V_K = T_K - T_{K-1}$, $K = 1, 2, 3, \ldots$

→ $T_K = V_1 + V_2 + \cdots + V_K$, $K = 1, 2, 3, \ldots$

→ $T_1 = V_1$

→ $T_0 = 0$, $V_0 = 0$

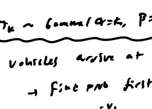


→ Poisson Process & Time to events

→ $\boxed{P(T_K > s) = P(N(s) < K)}$ → Prob of at least $t = s$ for the K^{th} event equals Prob at most $(K-1)$ events by $t = s$

→ ex) Probability second event occurs after $t = 0.8$?

$P(T_2 > 0.8)$



If 0 or 1 events occur by time 0.8, then guaranteed second event happens after $t = 0.8$

⇒ at most 1 events can occur before $t = 0.8$

$P(T_2 > 0.8) = P(N(0.8) \leq 1)$

⇒ reframe in terms of K

$= P(N(0.8) < 2)$

→ For a <u>homogeneous Poisson process</u>, the inter-arrival time & time to the K^{th} event are simplified

↳ N w/ rate λ

→ $V_K \sim \text{Exp}(\lambda)$

→ $T_K \sim \text{Gamma}(\alpha = K, \beta = \lambda)$

→ examples → vehicles arrive at PP w/ rate 3/hour

→ Find prob first vehicle in 30 min

→ $P(T_1 < 0.5) = 1 - e^{-0.5(1/3)} = 0.7769$ ($T_1 = V_1$)

↳ $\sim \text{Exp}(\lambda = 1/3)$

OR

$P(T_1 < 0.5) = P(N(0.5) \geq 1) = 1 - P(N(0.5) = 0) = 1 - e^{-1.5}$

↳ $\sim \text{Pois}(\lambda = 0.5(3))$

→ Buses arrive according to a PP w/ rate $\lambda = 4$ per hour → homogeneous

$P(T_4 > 1)$

↳ $= P(N(1) < 4) = \sum_{i=0}^{3} \frac{e^{-4}4^i}{i!} = 0.4335$

↳ $\text{Pois}(\lambda = 1(4))$

OR

↳ $1 - P(T_4 \leq 1) = 1 - F_{T_4}(1) = 1 - \text{Gamma.Dist}(x = 1, \alpha = 4, \beta = 1/4, \text{cumulative} = \text{TRUE})$ (scale)

↳ $\sim \text{Gamma}(\alpha = 4, \beta = 4)$ ✓ $= 0.4335$

→ Taxis arrive according to PP w/ rate $\lambda(t) = 3t$

$P(0.5 \leq T_3 \leq 0.9) = P(T_3 \leq 0.9) - P(T_3 \leq 0.5)$

$$= P(N(0.9) \geq 3) - P(N(0.5) \geq 3)$$

↳ $\sim \text{Pois}(\lambda = \int_0^{0.9} 3t\,dt = 1.215)$; ↳ $\sim \text{Pois}(\lambda = \int_0^{0.5} 3t\,dt = 0.375)$

$$= (1 - P(N(0.9) \leq 2)) - (1 - P(N(0.5) \leq 2))$$

$$= \left(1 - \sum_{i=0}^{2} \frac{e^{-1.215}\,1.215^i}{i!}\right) - \left(1 - \sum_{i=0}^{2} \frac{e^{-0.375}\,0.375^i}{i!}\right)$$

$$= (1 - 0.8762) - (1 - 0.9933)$$

$$= 0.1171$$

→ Assignment

→ Q2) Claims $\sim PP(\lambda = 5/\text{year})$

$P(V_3 > 0.6) = e^{-5(0.6)} = 0.0498$

↳ $\text{Exp}(\lambda = 5)$

1.4.3 → Conditional distribution of arrival times

→ overview → Let N be a homogeneous process. Given that n events happened by time t, what can we say about the distribution of the arrival times of those n events??

→ order stats → $X_{(1)} = T_1$, $X_{(2)} = T_2$, ⋮, $X_{(n)} = T_n$

(inherent order in T_K)

→ Notation → $T_K = X_{(K)}$ = ordered times of events

X_K = unordered arrival times

↳ not necessarily time to K^{th} event

→ Goal → find dist of T_K

→ Given n events for homogeneous Poisson Process N happened by time t. So for a given event, each interval of the same length within $[0, t]$ is equally likely to contain that event

⇒ → Given that n events occur <u>by time t</u>, <u>$N(t) = n$</u>, the unordered times of past events $X_1, \ldots, X_n \overset{iid}{\sim} \text{uniform}(0, t)$

→ Given <u>time to n^{th} event</u> is t, <u>$T_n = t$</u>, the unordered times of past events $X_1, \ldots, \underline{X_{n-1}} \overset{iid}{\sim} \text{uniform}(0, t)$

↳ already know when n^{th} event occurs

★ ⇒ T_K is the K^{th} order stat of iid uniform RVs

→ shortcut Expected value for order stat of iid uniform(a, b)

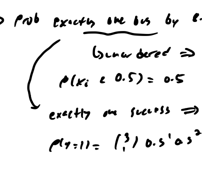
$$E(T_K) = E(X_{(K)}) = a + \frac{K(b-a)}{n+1}$$

→ Examples → Bus arrives at a PP(rate = 4 / hour)

→ Let t = hours since 8 am

★ Given $N(1) = 3$

→ Expected arrival time of 5th bus



⇒ $T_2 \sim \text{Gamma}(\alpha = 2, \beta = 4)$

$E(T_2) = \alpha/\beta = 2/4 = 0.5$

⇒ 9:30 am

→ Expected arrival time of 2nd bus

$E(T_2) = E(X_{(2)}) = a + \frac{K(b-a)}{n+1}$ < $X_{(i)} \sim \text{Uniform}(0,1)$, $i = 1, 2, 3$ > ★

$$= 0 + \frac{2(1-0)}{3+1}$$

$$= 2/4 \Rightarrow 8{:}30 \text{ am}$$

→ Prob <u>exactly one bus</u> by 8:30 am

↳ unordered ⇒ working w/ X_i's

$P(X_i < 0.5) = 0.5$ < b/c $X_i \overset{iid}{\sim} \text{uniform}(0,1)$ >

exactly one success ⇒ $Y \sim \text{Bin}(n = 3, p = 0.5)$

$P(Y = 1) = \binom{3}{1} 0.5^1\, 0.5^2 = 0.375$

★ Given $T_3 = 1$

→ $E(T_2) = E(X_2)$ < $X_i \overset{iid}{\sim} \text{Uniform}(0,1)$, $i = 1, 2$ > ★

$$= 0 + \frac{2(1)}{2+1} = 2/3 \Rightarrow 8{:}40 \text{ Am}$$

→ Assignment

→ Q1) $N(t) \sim PP(\lambda = 3)$ → homogeneous

$$P[N(1) = 5 \mid N(3) = 10] = \frac{P[N(1) = 5 \cap N(3) = 10]}{P(N(3) = 10)}$$

$$= \frac{P[N(1) = 5 \cap N(3) - N(1) = 5]}{P(N(3) = 10)}$$

$$= \frac{P(N(1) = 5) * P[N(3) - N(1) = 5]}{P(N(3) = 10)}$$ < non overlapping ⇒ ⫫ >

$\sim \text{Pois}(\lambda = 1(3))$; $\sim \text{Pois}(\lambda = 2(3))$; ↳ $\sim \text{Pois}(\lambda = 3(3))$

$$= \frac{\frac{e^{-3}3^5}{5!} * \frac{e^{-6}6^5}{5!}}{\frac{e^{-9}9^{10}}{10!}}$$

$$= 0.1366$$

<u>OR</u>

$N(3) = 10 \Rightarrow X_i \overset{iid}{\sim} \text{uniform}(0, 3)$

$P(X_i < 1) = F_{X_i}(1)$

$$= \frac{1-0}{3-0}$$

$$= 1/3$$

→ $Y \sim \text{Bin}(n = 10, p = 1/3)$

↳ # $X_i < 1$

→ $P(Y = 5) = \binom{10}{5} (1/3)^5 (2/3)^5 = 0.1366$ ✓