1.5.1 → Introduction to systems

→ overview → systems have multiple components, each w/ its own lifetime (a RV)

$$x_i = \begin{cases} 1 & \text{if component } i \text{ is functioning} \\ 0 & \text{-------- has failed} \end{cases}$$

→ states of all components in the system can be written as a vector

state vector : $x = (x_1, x_2, \dots, x_n)$

→ has 2^n distinct state vectors

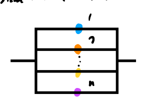
→ state of system depends on states of components

$$\phi(x) = \begin{cases} 1 & \text{if system is functioning} \\ 0 & \text{---- has failed} \end{cases}$$

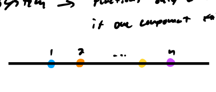
↳ structure function

→ Different types of systems

→ parallel system → functions as long as one of the components is functioning



→ Series system → functions only when all components are functioning
if one component fails, the whole system fails



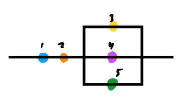
→ k-out-of-n system → functions if & only if at least k out of n components are functioning

→ parallel system = 1-out-of-n system
series system = n-out-of-n system

→ Bridge system → two branches that are "bridged" somewhere in the middle

→ other systems → in reality, there is an unlimited # of systems.
they can be constructed as any combo of the 4 given above

→ example → Draw a 2-out-of-2 system placed w/ a 1-out-of-3 system
↳ series ↳ parallel



Combo of states resulting in a functioning system

x_1	x_2	x_3	x_4	x_5
1	1	1	0	0
1	1	0	1	0
1	1	0	0	1
1	1	1	1	0
1	1	0	1	1
1	1	1	0	1
1	1	1	1	1

1.5.2 → Minimal path sets & Minimal cut sets

→ def → A state vector is a path vector if the system is functioning, i.e. $\phi(x) = 1$

→ A path vector is a minimal path vector if the system would fail if any of the functioning components in x were to fail, i.e. there are no "backup" components ⇒ turn any off & system fails

→ x is a minimal path vector if $\phi(y) = 0$ for every $y < x$

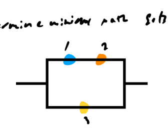
→ $y < x$ = each y_i must be $\leq x_i$, with at least one $y_i < x_i$

→ Each minimal path vector can be represented as a minimal path set, or a minimal set of components whose functioning guarantees the functioning of the system

→ Let A be a minimal path set, then given a minimal path vector x, the minimal path set is $\{i : x_i = 1\}$

★ ⇒ A system will function if & only if all components of at least one minimal path set function

→ Example → Determine minimal path sets for this system



→ All possible states of components & resulting state of system

x_1	x_2	x_3	$\phi(x)$
0	0	0	0
0	0	1	1 ✓
0	1	0	0
0	1	1	1 ✓
1	0	0	0
1	0	1	1 ✓
1	1	0	1 ✓
1	1	1	1 ✓

} 5 path vectors (i.e. system is functioning)

→ To determine which of these are minimal path vectors, first list all vectors y that satisfy $y < x$ (for each x)

path vector x	All y
(0,0,1)	(0,0,0)
(0,1,1)	(0,0,0), (0,0,1), (0,1,0)
⋮	⋮

→ $y < x$ b/c $y_1 \leq x_1$, $y_2 \leq x_2$ & $\underbrace{y_3 < x_3}_{\text{need a strictly less than comparison}}$

→ After listing all vectors y, check if $\phi(y) = 0$

→ A path vector x can only be a minimal path vector if all of its corresponding y vectors result in $\phi = 0$

→ only (0,0,0), (0,1,0), & (1,0,0) meet this

⇒ (0,0,1) & (1,1,0) are minimal path vectors

⇒ {3}, {1,2} are the minimal path sets for this system

⇒ As long as component 3 or components 1 & 2 are functioning, the system will function

→ could also logic instead of the formal process

→ we know a minimal path set is a minimal set of components whose functioning guarantees the functioning of the system

⇒ thus look for sets of non-overlapping components for which no component can be removed w/o causing the system to fail

⇒ obviously it is {3} & {1,2}

→ minimal cut sets

→ def → In contrast to a path vector, a state vector x is a cut vector if the system is not functioning (i.e. $\phi(x) = 0$).

→ A cut vector is a minimal cut vector if repairing any of the non-functioning components in x would result in the whole system functioning ⇒ turn any on & system works

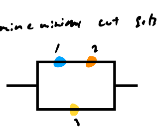
→ So x is a minimal cut vector if $\phi(y) = 1$ for every $y > x$ (means all $y_i \geq x_i$ & at least one $y_i > x_i$)

→ Each minimal cut vector can be represented as a minimal cut set, or a minimal set of components whose failure guarantees the failure of the system.

Let C be a minimal cut set, then given a minimal cut vector x, the minimal cut set C is $\{i : x_i = 0\}$

★ ⇒ A system will fail if & only if all components of at least one minimal cut set fail

→ Example → Determine minimal cut sets for this system



→ formal process → All possible states of components & resulting state of system

x_1	x_2	x_3	$\phi(x)$
0	0	0	0 ✗
0	0	1	1
0	1	0	0 ✗
0	1	1	1
1	0	0	0 ✗
1	0	1	1
1	1	0	1
1	1	1	1

} 3 path vectors (i.e. system is not functioning)

→ list all $y > x$ for

cut vector x	All y
(0,0,0)	(0,0,1), (0,1,0), …
(0,1,0)	⋮
(1,0,0)	⋮

→ A cut vector can only be a minimal cut vector if all of its corresponding y vectors result in $\phi(y) = 1$

⇒ only works for (0,1,0) & (1,0,0)

⇒ {1,3} & {2,3} are the minimal cut sets

→ if components 1 & 3 or components 2 & 3 stop functioning, the system is guaranteed to fail

→ OR use logic again

→ obviously system fails if all 3 components fail, but all failing isn't necessary (want minimal)

→ component 3 must fail, but only one of the other two must fail

⇒ {1,3}, {2,3}

★ → notes → all minimal path sets must have at least one component from each minimal cut set

→ paths: {3}, {1,2} cuts: {1,3}, {2,3}

→ Counting

→ we can then calculate the # of minimal path sets & minimal cut sets for the common systems shown earlier

system	# minimal path sets	# minimal cut sets
parallel	n	1
series	1	n
k-out-of-n	$\binom{n}{k}$	$\binom{n}{n-k+1}$
Bridge	4	4

→ If we have multiple systems & they are placed in…

→ parallel → total minimal path sets = Σ individual amounts
total minimal cut sets = Π individual amounts

→ series → total minimal path sets = Π individual amounts
total minimal cut sets = Σ individual amounts

→ Assignment

→ 01) structure function → $\phi(x) = x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 - 2x_1x_2x_3x_4$

a path vector $x = (1,0,1,1) \Rightarrow \phi(x) = 1$ ✓

1.5.3 → Structure functions

→ overview → structure functions $\phi(x)$ are functions that represent the state of a system.
can be derived using either minimal path sets or minimal cut sets

→ for parallel system → functions as long as at least one component functions

$$\boxed{\Rightarrow \phi(x) = \max(x_1, \dots, x_n)}$$

→ for series system → functions if & only if all of the components function

$$\boxed{\Rightarrow \phi(x) = \min(x_1, \dots, x_n)}$$

→ Since x is binary, the following relationships are true

$$\boxed{\begin{aligned} &\max(x_1, \dots, x_n) = 1 - \prod_{i=1}^{n}(1 - x_i) \\ &\min(x_1, \dots, x_n) = \prod_{i=1}^{n} x_i \\ &x_i^k = x_i,\ k>0 \end{aligned}}$$

→ minimal path sets approach

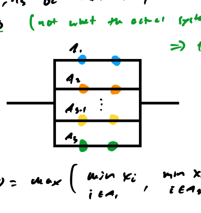
→ A system will function if & only if all components of at least one minimal path set function

→ A parallel system will function if & only if at least one component functions

★ ⇒ can treat any system as a parallel system of its minimal path sets. Therefore, given the minimal path sets of a system, we can express the structure function of the system the same way we express the structure function of a parallel system, just treat minimal path sets as individual components

→ Let $A_1, \dots, A_s$ be minimal path sets of a system. This can be expressed as (not what the actual system looks like, just how we can think about it)

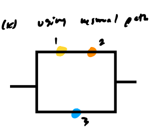
⇒ there is another representation using minimal cut sets approach



$$\Rightarrow \phi(x) = \max\left(\min_{i \in A_1} x_i, \min_{i \in A_2} x_i, \dots, \min_{i \in A_s} x_i\right)$$

$$= \max_j \prod_{i \in A_j} x_i$$

→ Example → determine $\phi(x)$ using minimal path sets approach



↓ {3}, {1,2}

$$\begin{aligned} \rightarrow \phi(x) &= \max(\min(x_1, x_2), \min(x_3)) \\ &= \max(x_1x_2, x_3) \\ &= 1 - (1 - x_1x_2)(1 - x_3) \\ &= 1 - (1 - x_1x_2 - x_3 + x_1x_2x_3) \\ &= x_1x_2 + x_3 - x_1x_2x_3 \end{aligned}$$

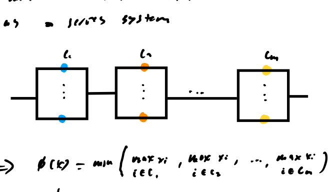
→ minimal cut sets approach

→ Relationship exist just like above

→ A system will fail if & only if all components of at least one minimal cut set fail

→ A series system will fail if & only if at least one component fails

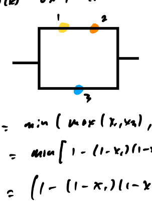
⇒ any system can be thought of as a series system of its minimal cut sets.
If know minimal cut sets, treat as individual components of the system & draw same way as a series system



$$\Rightarrow \phi(x) = \min\left(\max_{i \in C_1} x_i, \max_{i \in C_2} x_i, \dots, \max_{i \in C_m} x_i\right)$$

$$= \prod_{j=1}^{m} \max_{i \in C_j} x_i$$

→ Example → determine $\phi(x)$ using minimal cut sets approach

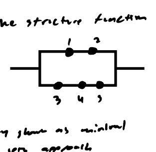


↓ {1,3}, {2,3}

$$\begin{aligned} \rightarrow \phi(x) &= \min(\max(x_1, x_3), \max(x_2, x_3)) \\ &= \min[1 - (1-x_1)(1-x_3), 1 - (1-x_2)(1-x_3)] \\ &= (1 - (1-x_1)(1-x_3))(1 - (1-x_2)(1-x_3)) \\ &= (\dots) \\ &= x_1x_2 + x_3 - x_1x_2x_3 \quad \text{(same as before)} \end{aligned}$$

→ Assignment

→ 01) Determine structure function



⇒ currently shown as minimal path sets approach

$$\begin{aligned} \Rightarrow \phi(x) &= \max(\min(x_1, x_2), \min(x_3, x_4, x_5)) \\ &= \max(x_1x_2, x_3x_4x_5) \\ &= 1 - (1 - x_1x_2)(1 - x_3x_4x_5) \\ &= 1 - [1 - x_3x_4x_5 - x_1x_2 + x_1x_2x_3x_4x_5] \\ &= x_3x_4x_5 + x_1x_2 - x_1x_2x_3x_4x_5 \end{aligned}$$