1.6.6 → Time reversibility

→ overview → Recall that an ergodic Markov chain is irreducible, positive recurrent + aperiodic.
Suppose we have a stationary ergodic Markov chain $(X_n, n \geq 0)$, + we want to trace the sequence of states going back in time, i.e. $X_n, X_{n-1}, X_{n-2}, \ldots$ This is called the reversed process of a Markov chain

→ Then, it turns out that the reversed process of an ergodic Markov chain is itself a Markov chain w/ the following transition probabilities

$$\boxed{R_{i,j} = \frac{\pi_j P_{j,i}}{\pi_i}}$$

→ If $R_{i,j} = P_{i,j}$ for every i + j, then the Markov chain is time reversible.
Thus, for a time reversible Markov chain, the following must be true:

$$\boxed{\pi_i P_{i,j} = \pi_j P_{j,i}}$$

→ This means that for a time reversible Markov chain starting in state i, any path back to state i has the same probability as the reversed path.
For example $i \to j \to k \to i$ has the same prob as $i \to k \to j \to i$ + therefore

$$P_{i,j} P_{j,k} P_{k,i} = P_{i,k} P_{k,j} P_{j,i}$$

→ Example → weather example again

$$P = \begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.1 & 0.2 & 0.7 \\ 0.6 & 0 & 0.4 \end{bmatrix}$$

→ Calculate the transition probabilities for the reversed process

$$\pi = \begin{bmatrix} 0.3467 \\ 0.2479 \\ 0.3554 \end{bmatrix} \quad \text{(found earlier)}$$

$$R_{1,2} = \frac{\pi_2 P_{2,1}}{\pi_1}$$

$$= \frac{0.2479}{0.3467} (0.1)$$

$$= 0.0623$$

(repeat for $R_{1,3}, R_{2,1}, R_{2,3}, R_{3,1} + R_{3,2}$)

$$\Rightarrow R = \begin{bmatrix} 0.4 & 0.0625 & 0.5375 \\ 0.8 & 0.2 & 0 \\ 0.116 & 0.4884 & 0.4 \end{bmatrix}$$

→ notice main diagonal of R is the same as that of P

$$\text{diag}(R) = \text{diag}(P)$$

(this is just a natural consequence of the formula)

→ Note → An ergodic Markov chain is time reversible if + only if $P_{i,j} = 0$ whenever $P_{j,i} = 0$ + any path back to state i starting in state i has the same probability as the reverse path.
From above example, can immediately tell it is not time reversible b/c $P_{3,2} = 0$ but $P_{2,3} \neq 0$

→ Examples

→ Given incomplete transition matrix for a time reversible 3-state Markov chain

$$P = \begin{bmatrix} 0.4 & 0.3 & P_{1,3} \\ P_{2,1} & 0.3 & 0.6 \\ 0.06 & P_{3,2} & P_{3,3} \end{bmatrix}$$

→ rows Σ = 1

$$\Rightarrow P = \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.1 & 0.3 & 0.6 \\ 0.06 & P_{3,2} & P_{3,3} \end{bmatrix}$$

$P_{3,3} = ??$

→ $1 = 0.06 + \underbrace{P_{3,2}}_{= 0.36} + P_{3,3}$

⇒ $P_{3,3} = 0.58$

→ time reversible

(can set up system of equations using + $\pi_i P_{i,j} = \pi_j P_{j,i}$)

OR

we know → any path back to itself has the same probability

$$\Rightarrow 1 \to 2 \to 3 \to 1 = 1 \to 3 \to 2 \to 1$$

$$P_{1,2} P_{2,3} P_{3,1} = P_{1,3} P_{3,2} P_{2,1}$$

$$0.3(0.6)(0.06) = 0.3 P_{3,2} (0.1)$$

$$\Rightarrow P_{3,2} = 0.36$$

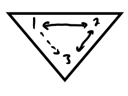
$$\Rightarrow P_{3,3} = 0.58$$

→ Given the following 3-state Markov chain, determine if it is
1) irreducible, 2) positive recurrent, 3) aperiodic, 4) time reversible

$$P = \begin{bmatrix} 0.3 & 0.4 & 0.3 \\ 0.4 & 0 & 0.6 \\ 0 & 1 & 0 \end{bmatrix}$$

1) Irreducible ⇒ one class ✓

$1 \to 2, 3$
$2 \to 1, 3$
$3 \to 2$



2) Positive recurrent ⇒ way 1) all states in an irreducible, finite Markov chain are automatically positive recurrent

$m_j < \infty$
↳ expected time to return

⇒ all states communicate ⇒ will eventually get back

way 2 → unique solution to $\pi = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} = (\ldots) = \begin{bmatrix} 0.2434 \\ 0.4268 \\ 0.3293 \end{bmatrix}$

3) Aperiodic ⇒ limiting probabilities ✓ → if aperiodic, limiting probs =

$$P^{(2)} = P \cdot P = \begin{bmatrix} 0.25 & 0.42 & 0.33 \\ 0.12 & 0.76 & 0.12 \\ 0.40 & 0 & 0.60 \end{bmatrix}$$

$$P^{(4)} = P^{(2)} \cdot P^{(2)} = \begin{bmatrix} 0.2449 & 0.4242 & 0.3309 \\ 0.1692 & 0.678 & 0.2028 \\ 0.34 & 0.168 & 0.492 \end{bmatrix}$$

$$P^{(8)} = P^{(4)} \cdot P^{(4)} = \begin{bmatrix} 0.2483 & 0.4259 & 0.3259 \\ 0.2166 & 0.5002 & 0.2831 \\ 0.2790 & 0.3324 & 0.3886 \end{bmatrix}$$

⇒ columns appear to be converging

4) Time reversible
→ b/c this Markov chain is irreducible, positive recurrent + aperiodic ⇒ ergodic
→ for ergodic Markov chain to be time reversible, $P_{i,j}$ must = 0 whenever $P_{j,i} = 0$ (+ vice versa)
→ But $P_{3,1} = 0 \neq 0.3 = P_{1,3}$ ⇒ not time reversible

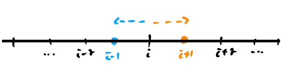
1.6.7 → Applications of Markov chains

→ Random walk model

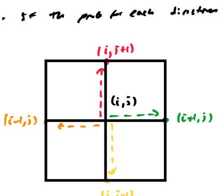
→ A one-dimensional random walk is a Markov chain that can only transition to state i+1 or i-1 at every step, given that the chain is currently at state i.
The transition probability to state i+1 given the current state is p, where 0<p<1

$$\boxed{\begin{aligned} P_{i,i+1} &= p \\ P_{i,i-1} &= 1-p \end{aligned}}$$

→ A one-dimensional random walk where p=0.5 is symmetric.



→ now imagine a 2D space, where each state is represented by a pair of integers (i,j).
A two-dimensional random walk is a Markov chain that can transition to state (i+1,j), (i-1,j), (i,j+1), or (i,j-1) at every step, given the chain is currently at state i ⇒ can think of this as a process that can move one step up, down, left or right at each transition. If the prob for each direction is equal, then it is symmetric



→ Transience vs recurrence
→ 1D + 2D symmetric random walks are recurrent, while all higher-dimensional symmetric random walks are transient. In addition, all non-symmetric random walks are transient

→ Gambler's ruin problem

→ Suppose that in each round of a game, a gambler wins a chip w/ a probability of p + loses a chip w/ prob 1-p. We might want to know the probability that a gambler who starts w/ i chips will end up w/ j chips. This is known as the gambler's ruin problem.

→ A gambling model is used to model the gambler's ruin problem. It is similar to a random walk. However instead of having an infinite # of states, it is a finite Markov chain. A gambling model has the following properties:

→ $P_{0,0} = P_{j,j} = 1$ → This is b/c the gambler stops when they have either 0 or j chips
⇒ 0 + j are absorbing states

→ $P_{i,i+1} = p = 1 - P_{i,i-1}$, where $i = 1, 2, \ldots, j-1$
→ prob of winning a chip is = complement of prob of losing a chip, obviously

→ There are 3 classes → $\{0\}, \{1, \ldots, j-1\}, \{j\}$

→ $\{0\} + \{j\}$ are recurrent + $\{1, \ldots, j-1\}$ is transient

→ Let P_i be the probability of starting w/ i chips + ending w/ j chips, + let q = 1-p.
So the complement, $1 - P_i$, is the probability of starting w/ i chips + ending w/ 0 chips (only two possible outcomes)

$$\boxed{P_i = \begin{cases} \dfrac{1 - \left(\frac{q}{p}\right)^i}{1 - \left(\frac{q}{p}\right)^j} & p \neq 1/2 \\ \dfrac{i}{j} & p = 1/2 \end{cases}}$$

→ Let X be a RV representing the # of chips the gambler has at the end.
Since the gambler will play until they have 0 or j chips, X ~ Bern w/ 2 possible values, j + 0, which have probabilities P_i + $1-P_i$, respectively

→ Examples

→ Bet flipping coins + earning candy
L starts w/ 25 candies + J has 10, L wins w/ Heads + J wins w/ tails

→ Calculate variance of final count for L, assuming fair coin

gambler's ruin → $X_L = \begin{cases} 0 & p = 10/35 \\ 35 & p = 25/35 \end{cases}$

↳ $p = 0.5 \Rightarrow P_i = i/j$
$P_{25} = 25/35$
↳ start w/ 25 + end w/ 35

$$\rightarrow V(X_L) = \overbrace{(35-0)^2 \left(\tfrac{25}{35}\right)\left(\tfrac{10}{35}\right)}^{\text{Bernoulli shortcut}} = 250$$

$$= V(X_L) = V(35 Y_L) = 35^2 V(Y_L) = 35^2 pq \overset{\checkmark}{=} 250$$

↳ ~ Bern(p=0.5) ↳ $\frac{25}{35}\left(\frac{10}{35}\right)$

→ Calculate the average final count for J assuming biased coin p=0.55 for tails

$$P_{10} = \frac{1 - \left(\frac{0.45}{0.55}\right)^{10}}{1 - \left(\frac{0.45}{0.55}\right)^{35}} = 0.8163$$

$$\Rightarrow E(X_J) = 0(1 - P_{10}) + 35 P_{10} = 28.5719$$

↓
X_{10}
(notation??)

→ Assignment

→ Q2) Gambler's ruin → p = 0.2, i = 5 + j = 10

$$P_5 = \frac{1 - \left(\frac{0.8}{0.2}\right)^5}{1 - \left(\frac{0.8}{0.2}\right)^{10}} = 0.00097$$

1.6.8 → Branching Processes

→ overview → Suppose we have a population of individuals, where each individual produces j new offspring by the end of their lifetime w/ a probability of P_j. Then, μ is the average # of new offspring an individual produces.

$$\boxed{\mu = \sum_{j=0}^{\infty} j P_j}$$

→ Let σ^2 be the variance of the # of new offspring an individual produces. Then

$$\boxed{\sigma^2 = \sum_{j=0}^{\infty} (j-\mu)^2 P_j}$$

→ Let X_n be the size of the n^{th} generation. Then X_0 is the size of the zeroth generation. When $X_0 = 1$, the mean + variance of the size of the n^{th} generation are

$$\boxed{\begin{aligned} E(X_n) &= \mu^n \\ V(X_n) &= \begin{cases} \sigma^2 \mu^{n-1} \left(\dfrac{1-\mu^n}{1-\mu}\right) & \mu \neq 1 \\ n\sigma^2 & \mu = 1 \end{cases} \end{aligned}}$$

☆ → If $X_0 = k$ ⇒ $E(X_n) = \underbrace{k E(X_n)}$ + $V(X_n) = \underbrace{k V(X_n)}$
↳ adding RVs

→ Let π_0 be the probability that the population will eventually die out when $X_0 = 1$

$$\boxed{\pi_0 = \begin{cases} 1 & \mu \leq 1 \text{ (always will eventually die out)} \\ \sum_{j=0}^{\infty} \pi_0^j \underbrace{P_j} & \mu > 1 \end{cases}}$$

↳ probability of j offspring
↳ probability of j offspring dying out $(\underbrace{\pi_0 \cdot \pi_0 \cdot \pi_0} = \pi_0^3)$
3 individuals → P(D.1 ∩ D.2 ∩ D.3)

→ Note that for μ>1, π_0 is dependent on π_0^j. Since we have to solve a polynomial equation to determine π_0, there may be multiple solutions for π_0. Therefore, the correct solution is the smallest positive value of π_0 that satisfies the equation $\pi_0 = \sum_{j=0}^{\infty} \pi_0^j P_j$

→ When $X_0 = k$, the probability that the population will eventually die out is π_0^k.

→ In cases where individuals are guaranteed to produce at least one offspring (i.e. $P_0 = 0$), the above equation for π_0 doesn't apply + $\pi_0 = 0$

→ Example → Island has population of 10 individuals w/ the following table

# of offspring	Prob
0	0.1
1	0.5
2	0.4

$$\rightarrow E(X_{20}) = \mu^{20} = 1.3^{20} = 190.0496$$

↳ $= \sum x f(x) = 1.3$

⇒ w/ k = 10 → $10 E(X_{20}) = 1900.49$
= 1900 (round down)

→ $\mu = 1.3 > 1 \Rightarrow \pi_0 = \sum_{j=0}^{2} \pi_0^j P_j = \pi_0^0 P_0 + \pi_0^1 P_1 + \pi_0^2 P_2$

$$= 0.1 + 0.5\pi_0 + 0.4\pi_0^2$$

$$0 = 0.4\pi^2 - 0.5\pi_0 + 0.1$$

$$= 4\pi^2 - 5\pi_0 + 1$$

$$\pi_0 = \frac{5 - \sqrt{5^2 - 4(4)(1)}}{2(4)}$$

$$= 0.25 \vee 1$$

⇒ $\pi_0 = 0.25$ (smallest positive value)

→ k = 10 ⇒ $\pi_0 = 0.25^{10} \approx 0$

→ Assignment

→ Q1) $X_0 = 2$
offspring ~ Geo(β=2)

$$K V(X_4) = 2 V(X_4)$$

$$= 2\left[\sigma^2 \mu^{n-1} \left(\frac{1-\mu^n}{1-\mu}\right)\right]$$

$$= 2\left[\sigma^2 \mu^{4-1} \left(\frac{1-\mu^4}{1-\mu}\right)\right]$$

$$= 2\left[6(2^3)\left(\frac{1-2^4}{1-2}\right)\right]$$

$$= 1440$$

$\mu = \sum_{j=0}^{\infty} j P_j$ (geometric)
⇒ μ = β = 2

$\sigma^2 = \sum_{j=0}^{\infty} (j-\mu)^2 P_j$
→ (geometric)
⇒ $\sigma^2 = \beta(1+\beta)$
= 2(1+2)
= 6