1.6.6 → Time reversibility

→ overview → Recall that an ergodic Markov chain is irreducible, positive recurrent & aperiodic.
Suppose we have a stationary ergodic Markov chain (X_n, $n \geq 0$), & we want to trace the sequence of states going back in time, i.e. $X_n, X_{n-1}, X_{n-2}, \ldots$ This is called the reversed process of a Markov chain.

→ Then, it turns out that the reversed process of an ergodic Markov chain is itself a Markov chain w/ the following transition probabilities:

$$\boxed{Q_{i,j} = \frac{\pi_j P_{j,i}}{\pi_i}}$$

→ If $Q_{i,j} = P_{i,j}$ for every i & j, then the Markov chain is time reversible. Then, for a time reversible Markov chain, the following must be true:

$$\boxed{\pi_i P_{i,j} = \pi_j P_{j,i}}$$

→ This means that for a time reversible Markov chain starting in state i, any path back to state i has the same probability as the reversed path.
For example $i \to j \to k \to i$ has the same prob as $i \to k \to j \to i$
therefore

$$P_{i,j} P_{j,k} P_{k,i} = P_{i,k} P_{k,j} P_{j,i}$$

→ Example → Another example again

$P = $ [illegible]

→ Calculate the transition probabilities for the reversed process

$\pi = $ [illegible]

$Q_{1,2} = \frac{\pi_2 P_{2,1}}{\pi_1} = $ [illegible]

$Q = $ [illegible]

→ Note → An ergodic Markov chain is time reversible only if $P_{i,j} > 0$ whenever $P_{j,i} > 0$. [illegible]

→ Examples

→ Given example transition matrix for a time reversible 3-state Markov chain

$P = $ [illegible]

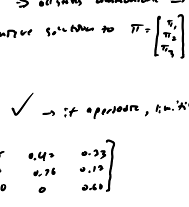
$P_{3,2} = ??$

→ $P_{3,2} = 0.38$ [illegible]

→ Given the following transition prob. matrix, determine if the MC is 1) irreducible, 2) positive recurrent, 3) aperiodic, 4) time reversible

$P = $ [illegible]

1) irreducible → one class



2) positive recurrent → every state in an irreducible finite Markov chain is automatically positive recurrent

3) aperiodic → calculate probabilities

$P^{(2)} = P \cdot P = $ [illegible]

$$P^{(4)} = P^{(2)} \cdot P^{(2)} = \begin{bmatrix} 0.2449 & 0.4247 & 0.3304 \\ 0.1697 & 0.628 & 0.2028 \\ 0.34 & 0.168 & 0.492 \end{bmatrix}$$

$$P^{(8)} = P^{(4)} \cdot P^{(4)} = \begin{bmatrix} 0.2483 & 0.4259 & 0.3259 \\ 0.2166 & 0.5002 & 0.2831 \\ 0.2790 & 0.3324 & 0.3886 \end{bmatrix}$$

⇒ Columns appear to be converging

4) Time reversible

→ b/c this Markov chain is irreducible, positive recurrent & aperiodic ⇒ ergodic

→ for ergodic Markov chain to be time reversible, $P_{i,j}$ must $= 0$ whenever $P_{j,i} = 0$ (& vice versa)

→ But $P_{3,1} = 0 \neq 0.3 = P_{1,3}$ ⇒ not time reversible

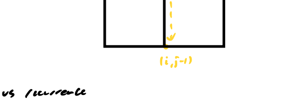
1.6.7 → Applications of Markov chains

→ Random walk model

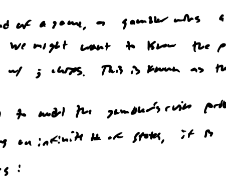
→ A one-dimensional **random walk** is a Markov chain that can only transition to state $i+1$ or $i-1$ at every step, given that the chain is currently at state i. The transition probability to state $i+1$ given the current state is p, where $0 < p < 1$

$$\boxed{\begin{aligned} P_{i,i+1} &= p \\ P_{i,i-1} &= 1-p \end{aligned}}$$

→ A one-dimensional random walk where $p = 0.5$ is symmetric.



→ now imagine a 2D space, where each state is represented by a pair of integers (i, j). A two-dimensional random walk is a Markov chain that can transition to state $(i+1, j)$, $(i-1, j)$, $(i, j+1)$, or $(i, j-1)$ at every step, given the chain is currently at state i ⇒ can think of this as a process that can move one step up, down, left or right at each transition. If the prob for each direction is equal, then it is symmetric



→ Transience vs recurrence

→ 1D & 2D symmetric random walks are recurrent, while all higher-dimensional symmetric random walks are transient. In addition, all non-symmetric random walks are transient

→ Gambler's ruin problem

→ Suppose that in each round of a game, a gambler wins a chip w/ a probability of p & loses a chip w/ prob 1-p. We might want to know the probability that a gambler who starts w/ i chips will end up w/ j chips. This is known as the gambler's ruin problem.

→ A **gambling model** is used to model the gambler's ruin problem. It is similar to a random walk. However instead of having an infinite # of states, it is a finite Markov chain. A gambling model has the following properties:

→ $P_{0,0} = P_{j,j} = 1$ → This is b/c the gambler stops when they have either 0 or j chips
⇒ 0 & j are absorbing states

→ $P_{i,i+1} = p = 1 - P_{i,i-1}$, where $i = 1, 2, \ldots, j-1$
→ prob of winning a chip is = complement of prob of losing a chip, obviously

→ There are 3 classes → $\{0\}$, $\{1, \ldots, j-1\}$, $\{j\}$

→ $\{0\}$ & $\{j\}$ are recurrent & $\{1, \ldots, j-1\}$ is transient

→ Let p_i be the probability of starting w/ i chips & ending w/ j chips, & let $q = 1-p$.
So the complement, $1 - p_i$, is the probability of starting w/ i chips & ending w/ 0 chips (only two possible outcomes)

$$\boxed{p_i = \begin{cases} \dfrac{1 - \left(\frac{q}{p}\right)^i}{1 - \left(\frac{q}{p}\right)^j} & p \neq 1/2 \\ \dfrac{i}{j} & p = 1/2 \end{cases}}$$

→ Let X be a RV representing the # of chips the gambler has at the end. Since the gambler will play until they have 0 or j chips, X ~ f(x) w/ 2 possible values, j & 0, which have probabilities p_i & $1 - p_i$, respectively

→ Examples

→ Bet flipping coins & earning candy
L starts w/ 25 candies & J has 10, L wins w/ Heads & J wins w/ tails

→ calculate variance of final count for L, assuming fair coin

gambler's ruin → $X_L = \begin{cases} 0 & p = 10/35 \\ 35 & p = 25/35 \end{cases}$

↳ $p = 0.5$ ⇒ $p_i = i/j$
$p_{25} = 25/35$
↳ start w/ 25 & end w/ 35

→ $V(X_L) = \underbrace{(35-0)^2 \left(\frac{25}{35}\right)\left(\frac{10}{35}\right)}_{\text{Bernoulli shortcut}} = 250$

$= V(X_L) = V(35 Y_L) = 35^2 V(Y_L) = 35^2 pq = 250$ ✓
↳ ~ Bern(p = 0.5)
↳ $\frac{25}{35}\left(\frac{10}{35}\right)$

→ Calculate the average final count for J assuming biased coin p = 0.55 for tails

$$p_{10} = \frac{1 - \left(\frac{0.45}{0.55}\right)^{10}}{1 - \left(\frac{0.45}{0.55}\right)^{35}} = 0.8163$$

⇒ $E(X_J) = 0(1 - p_{10}) + 35 p_{10} = 28.5714$
↓
X_{10}
(notation??)

→ Assignment

→ Q2) Gambler's ruin → $p = 0.2$, $i = 5$ & $j = 10$

$$p_5 = \frac{1 - \left(\frac{0.8}{0.2}\right)^5}{1 - \left(\frac{0.8}{0.2}\right)^{10}} = 0.00097$$

1.6.8 → Branching Processes

→ overview → Suppose we have a population of individuals, where each individual produces j new offspring by the end of their lifetime w/ a probability of P_j. Then, μ is the average # of new offspring an individual produces.

$$\boxed{\mu = \sum_{j=0}^{\infty} j P_j}$$

→ Let σ^2 be the variance of the # of new offspring an individual produces. Then

$$\boxed{\sigma^2 = \sum_{j=0}^{\infty} (j - \mu)^2 P_j}$$

→ Let X_n be the size of the n^{th} generation. Then X_0 is the size of the zeroth generation. When $X_0 = 1$, the mean & variance of the size of the n^{th} generation are

$$\boxed{\begin{aligned} E(X_n) &= \mu^n \\ V(X_n) &= \begin{cases} \sigma^2 \mu^{n-1} \left(\dfrac{1 - \mu^n}{1 - \mu}\right) & \mu \neq 1 \\ n\sigma^2 & \mu = 1 \end{cases} \end{aligned}}$$

☆ → If $X_0 = k$ ⇒ $E(X_n) = \underbrace{k E(X_n)}$ & $V(X_n) = \underbrace{k V(X_n)}$
↳ adding RVs

→ Let π_0 be the probability that the population will eventually die out when $X_0 = 1$

$$\boxed{\pi_0 = \begin{cases} 1 & \mu \leq 1 \quad \text{(always will eventually die out)} \\ \sum_{j=0}^{\infty} \pi_0^j P_j & \mu > 1 \end{cases}}$$

↳ probability of j offspring
↳ probability of j offspring dying out $(\underbrace{\pi_0 \cdot \pi_0 \cdot \pi_0}_{3 \text{ individuals} \to P(\text{d.o} \cap \text{d.o} \cap \text{d.o})} = \pi_0^3)$

→ Note that for $\mu > 1$, π_0 is dependent on π_0^j. Since we have to solve a polynomial equation to determine π_0, there may be multiple solutions for π_0. Therefore, the correct solution is the **smallest positive value** of π_0 that satisfies the equation $\pi_0 = \sum_{j=0}^{\infty} \pi_0^j P_j$

→ When $X_0 = k$, the probability that the population will eventually die out is π_0^k.

→ In cases where individuals are guaranteed to produce at least one offspring (i.e. $P_0 = 0$), the above equation for π_0 doesn't apply & $\pi_0 = 0$

→ Example → Island has population of 10 individuals w/ the following table

# of offspring	Prob
0	0.1
1	0.5
2	0.4

→ $E(X_{20}) = \mu^{20} = 1.3^{20} = 190.0496$
↳ $= \sum x f(x) = 1.3$

⇒ w/ k = 10 → $10 E(X_{20}) = 1900.49$
≈ 1900 (round down)

→ $\mu = 1.3 > 1$ ⇒ $\pi_0 = \sum_{j=0}^{2} \pi_0^j P_j = \pi_0^0 P_0 + \pi_0^1 P_1 + \pi_0^2 P_2$
$= 0.1 + 0.5\pi_0 + 0.4\pi_0^2$

$0 = 0.4\pi^2 - 0.5\pi_0 + 0.1$
$= 4\pi^2 - 5\pi_0 + 1$

$$\pi_0 = \frac{5 - \sqrt{5^2 - 4(4)(1)}}{2(4)}$$

$= 0.25 \vee 1$

⇒ $\pi_0 = 0.25$ (smallest positive value)

→ $k = 10$ ⇒ $\pi_0 = 0.25^{10} \approx 0$

→ Assignment

→ B1) $X_0 = 2$
offspring ~ Geo($\beta = 2$)

$KV(X_4) = 2V(X_4)$
$= 2\left[\sigma^2 \mu^{n-1} \left(\dfrac{1 - \mu^n}{1 - \mu}\right)\right]$
$= 2\left[\sigma^2 \mu^{4-1} \left(\dfrac{1 - \mu^4}{1 - \mu}\right)\right]$
$= 2\left[6(2^3)\left(\dfrac{1 - 2^4}{1 - 2}\right)\right]$
$= 1440$

$\mu = \sum_{j=0}^{\infty} j P_j$ (geometric)
⇒ $\mu = \beta = 2$

$\sigma^2 = \sum_{j=0}^{\infty} (j - \mu)^2 P_j$
→ (geometric)
⇒ $\sigma^2 = \beta(1 + \beta)$
$= 2(1+2)$
$= 6$