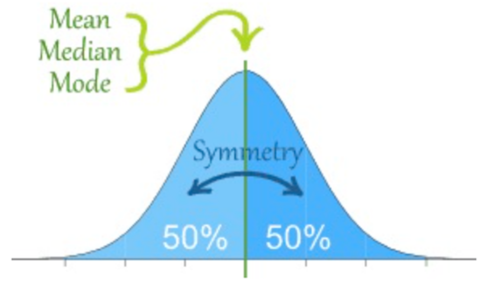
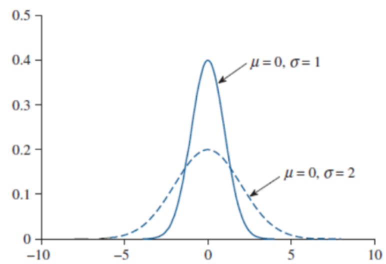
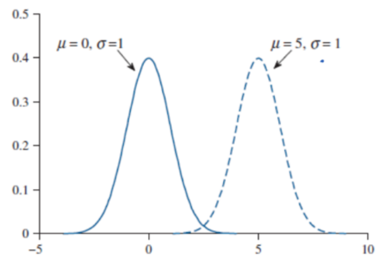
11.4 The Normal Distribution – Overview

Normal Distribution Properties



- It's a symmetric, unimodal and bell-shaped distribution
  $\Rightarrow$ which implies mean = median = mode.
- Total area under curve (probability) is equal to 1 = 100%.
- Completely described by its mean μ (location) and standard deviation $\boldsymbol{\sigma}$ (spread).





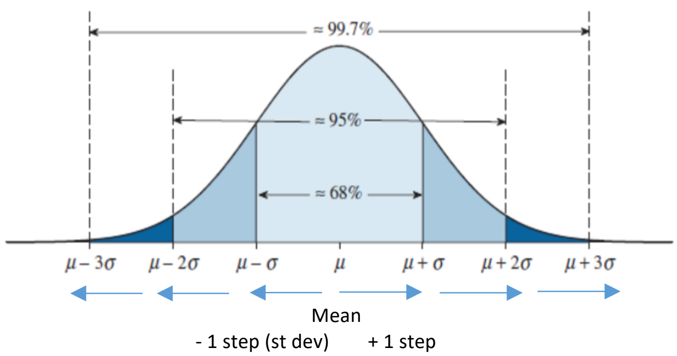
Empirical Rule (68 – 95 – 99.7 Rule)

- When data is approximately bell shaped, the standard deviation allows us to make fairly accurate approximations about the locations of our data values.

  68 % of the data lies within 1 standard deviation (Step) of the mean.

  95 % of the data lies within 2 standard deviations of the mean.

  99.7 % of the data lies within 3 standard deviations of the mean.



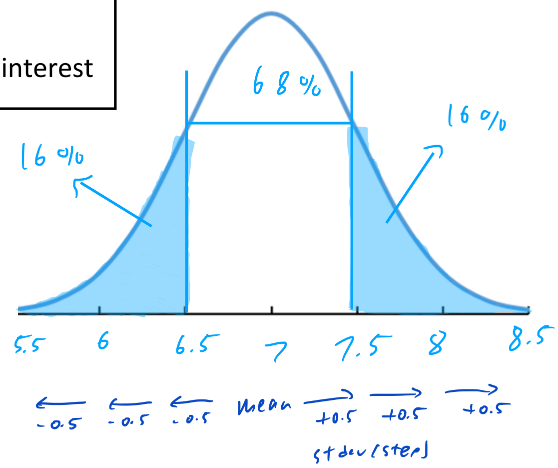
- We can use these breakdowns to find probabilities within certain intervals.

Example 1: Suppose that diameters of a new species of apple have a bell-shaped distribution with a mean of 7 cm and a standard deviation of 0.5 cm. Using the empirical rule, find the following percentages of apples with diameters that are:

<u>Step 1</u> Draw and label curve <u>Step 2</u> Shade area of interest

a) Between 5.5 cm and 8.5 cm

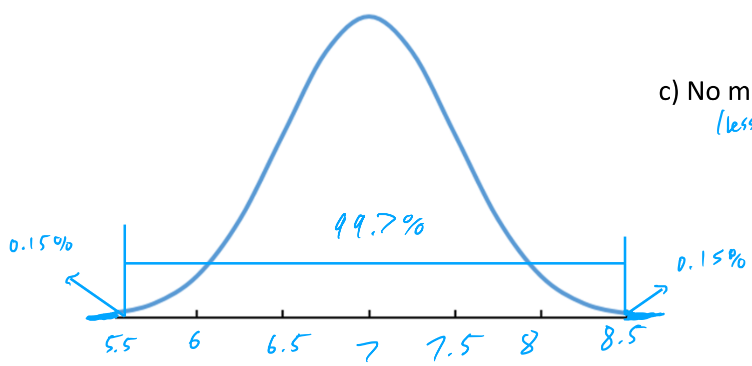
Between 3 steps ⇒ $\boxed{99.7\%}$



b) More than 7.5 cm. → one step

Outside → Total 100% − Between (Inside) 68% = 32%

Only Right → $\frac{32\%}{2}$ = $\boxed{16\%}$



c) No more than 5.5 cm. → 3 steps
(less)

Outside → Total 100% − Between (Inside) 99.7% = 0.3%

Only Left → $\frac{0.3\%}{2}$ = $\boxed{0.15\%}$

<u>Normal Distribution – Again</u>

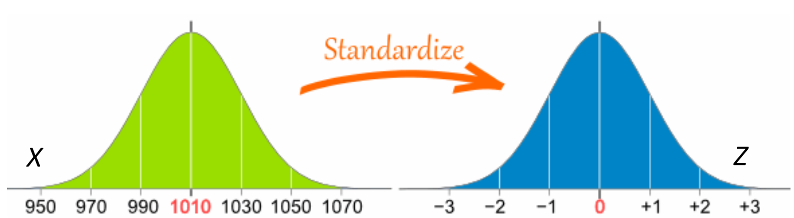
- The normal distribution allows us to find any probability, not just for points that lie exactly 1, 2, or 3 standard deviations ("steps") away from the mean like with the empirical rule!

Z-scores ("Standard" scores in Hawkes Certify)

- Definition: A **z-score** <u>standardizes</u> observations based on the <u>mean</u> (center) and <u>standard deviation</u> (spread) of the distribution.
  - Allows for comparisons on different scales.
  - Ex) ACT vs SAT

Formula: $z = \frac{x-\mu}{\sigma} = \frac{x-\bar{x}}{s} = \frac{obs - mean}{st\ dev}$

- Interpretation:
  - A **z-score** tells us how many standard deviations an observation is away from the mean.
  - The unit of a **z-score** is standard deviations.



below − ← Mean → + above

Example 2: For each data set with the stated μ and σ, find the standard score (z score) corresponding to the given observation, x.

$z = \frac{x-\mu}{\sigma}$

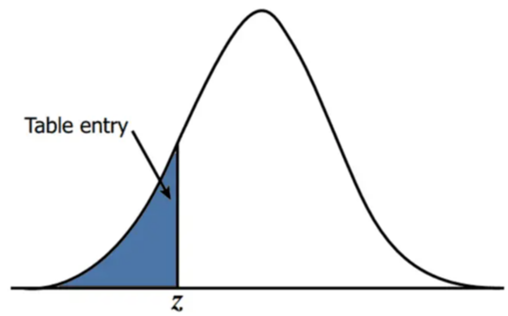
a) $\mu = 8,\ \sigma = 3,\ x = 17$ $z = \frac{17-8}{3} = 3$, above mean

b) $\mu = 100,\ \sigma = 16,\ x = 80$ $z = \frac{80-100}{16} = -1.25$, below mean

c) Which observation is further from the mean relatively?

(a) $z=3$ because "Larger" value

Finding probabilities based on the Normal Distribution

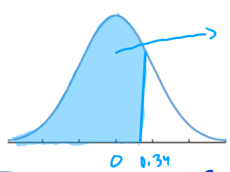


- Handout: Normal Distribution Table
  - Use the handout to convert z-scores to percentiles ("left probabilities").
  - ALWAYS gives probability LESS THAN Z: P(Z < z).
    "Left"

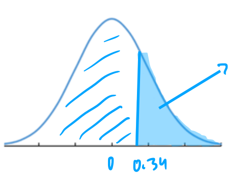
- Different types of probabilities
  - Left probability = Table (Directly)

Draw, Label and Shade curve

  - Example: Find the total area under the standard normal curve (probability or percentage) to the left of z = 0.34.

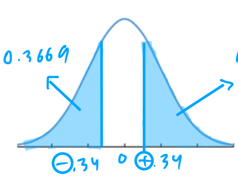


  - Right probability = 1 − Left (table)
    - Examples: Find the probability to the right of z = 0.34.



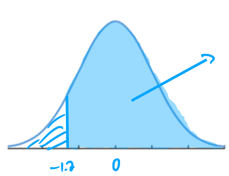
Right = 1 − Left
= 1 − 0.6331
= 0.3669

Symmetric



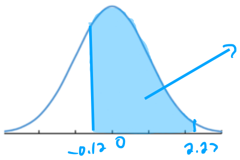
left −z = Right +z

Find the probability to the right of z = -1.2.



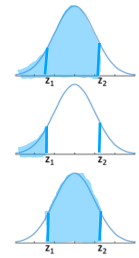
Right = 1 − Left
= 1 − 0.1151
= 0.8849

- Between probability = Left z_2 − Left z_1
  - Example: Find the probability between $z_1 = -0.12$ and $z_2 = 2.27$.



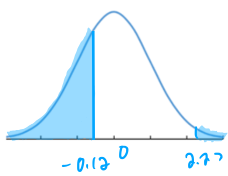
Between = left z_2 − left z_1
= 0.9884 − 0.4522
= 0.5362

Why this works

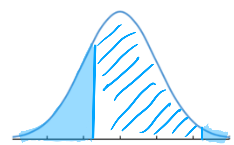


Left z_2
(minus) −
left z_1
=
Between

- Outside probability = Left z_1 + Right z_2
  - Example: Find the probability to the left of $z_1 = -0.12$ and to the right of $z_2 = 2.27$.



Outside = Left z_1 + Right z_2
= 0.4552 + 0.0116
= 0.4638



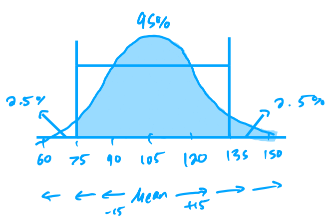
Outside = 1 − Between
= 1 − 0.5362
= 0.4638

Examples

Example 3: Suppose that IQ scores have a bell-shaped distribution with a mean of 105 and a standard deviation of 15. Using the empirical rule answer the following questions:

Steps
① Draw + label
② shade

a) What percentage of IQ scores are greater than 75? → two steps



Outside → Total 100% − Between (Inside) 95% = 5%

Only Right → $\frac{5\%}{2}$ = 2.5%

Final Answer = Between 95% + Right 2.5% = 97.5% OR Total 100% − Left 2.5% = $\boxed{97.5\%}$

b) Between which two values do the middle 68% of IQ scores fall between?

Middle 68% ⟹ ± 1 step ⟹ $\boxed{(90, 120)}$

→ Just want the two values on the curve

Example 4: Suppose there is a new breed of giant cats, whose weights are normally distributed with an average of 100 pounds and a standard deviation of 15 lbs. You would like to own a smaller version of this type of cat, specifically between [redacted] lbs and 69 lbs. What is the probability you can find a cat between these two weights.
59

$P(59 \le x \le 69) = P(-2.73 \le z \le -2.07)$ = Between = Left z_2 − Left z_1 = 0.0192 − 0.0032 = $\boxed{0.0016}$

Convert to z-scores