

**Law of total probability**

Motivation

- Example: A company has three assembly lines. The first line, the second line and the third produce 30%, 50% and 20% of productions, respectively.

Additionally, 1 of 100 productions is defective in Line 1; 2 of 100 productions are defective in Line 2; 3 of 100 productions are defective in Line 3.

We want to know the probability of defectives produced in the company.

- *STRATEGY*:
  - (a) Define all (unconditional) events given in the problem and find their probabilities.  
If conditional events are given, define them using unconditional events.
  - (b) Find the event of interest. Try to express the event of interest as a composition (union) of the given events.  
Often it is desirable to form compositions mutually exclusive or independent events.
  - (c) Use the general multiplication rule to find the probability for the event of interest.

- Visualizing scenario:

(a) Using a Venn Diagram:

(b) Using a Tree Diagram:

## Law of total probability

- The **law of total probability** says that a partition  $(A_1, \dots, A_n)$  of the sample space will lead to a partition of any event  $B$  into mutually exclusive pieces.

Then we can write  $P(B)$  as the sum of the probabilities of those pieces. Note that an event  $A$  and its complement  $\sim A$  always partition  $S$ .

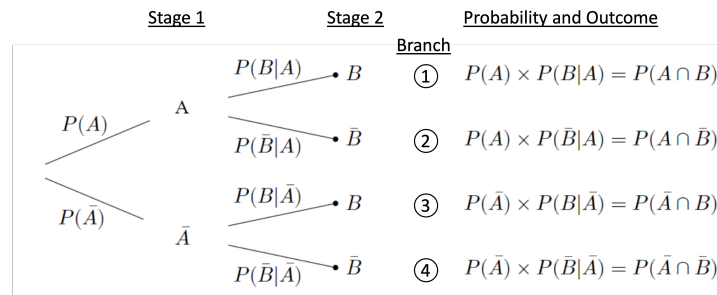
- Definition: **Law of total probability**

Let  $B$  be an event. If  $A_1, \dots, A_n$  partition the sample space, then

$$P(B) =$$

- Using the general tree diagram below, we can summarize

Law of total probability =  $P(\text{Second stage event}) = \sum \text{Branches of interest}$

Bayes' Theorem

## Bayes' Theorem

- Using the same terminology, we can summarize

$$\text{Bayes' Theorem} = P(\text{First stage event} \mid \text{Second stage event}) = \frac{\text{Main branch of interest}}{\sum \text{All branches of interest}}$$

- In essence, Bayes' Theorem reverses the natural order of the tree for the conditional probability of interest.

- Continuing example:

Find the probability that a defective product was made in Line 1.

- Definition: **Bayes' Theorem**

Let  $B$  be an event. If  $A_1, \dots, A_n$  partition the sample space, then

$$P(A_i | B) =$$

- Example: At the beginning of a certain study of a group of persons, 15% were classified as heavy smokers, 30% as light smokers, and 55% as nonsmokers. In the five year study, it was determined that the death rates of the heavy and light smokers were five and three times that of the nonsmokers, respectively.

A randomly selected participant died over the five-year period; calculate the probability that the participant was a nonsmoker.

Bayes' Theorem from another perspective

- Bayes' Theorem is all about changing probabilities based on new evidence.
- In a previous example, we drew a tree diagram about testing for the presence of a disease and the result of the test. We used the following events:

$D$  = the person tested has the disease

$\sim D$  = the person tested does not have the disease

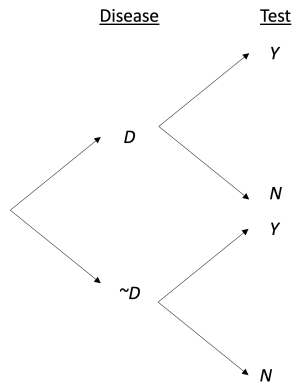
$Y$  = the test is positive

$N$  = the test is negative

Lets now consider a disease test that is “95% accurate”, which can be defined as follows:

- a. If you have the disease, 95% chance of a positive test.
- b. If you do not have the disease, 95% chance of a negative test.

Further, suppose only 1% of the population actually have this disease (aka prevalence).



- Terminology:
  - Prior probability: Original unconditional probabilities
  - Evidence: Conditional probability given the prior information
  - Posterior probability: Prior probability conditioned on the new evidence
- Some calculations in context: The good and the bad of Bayes' Theorem
  1. Find the probability of testing positive (total probability).
  2. Lets' solve for the probability of having the disease *given that you test positive*.
    - (a) For a randomly selected person from the population, we had our original prior probability of having the disease,  $P(D) = 0.01$ .  
(We don't know if they do or don't have the disease, it remains unknown).
    - (b) Then this person got tested, and tested positive; this is our evidence.  
  
Intuitively, this likelihood of the person having the disease should \_\_\_\_\_; we are adjusting the prior probability \_\_\_\_\_ based on the new evidence.

(c) Now we can calculate this new posterior probability.

3. Suppose you know that someone has tested positive for this disease. What is the probability that the person does not actually have the disease?

- The practical information here is interesting.
  1. The "95% accurate" test will classify \_\_\_\_\_ of the population as positives, compared to the true prevalence of  $P(D) = 0.01$ .
  2. By updating our prior probability with the new evidence, we drastically increased our information about this person having the disease. This is the good side of Bayes' Theorem!
  3. \_\_\_\_\_ of the individuals who tested positive will actually \_\_\_\_\_ have the disease. This percentage depends heavily on the prevalence, for example if  $P(D) = 0.1 \rightarrow P(\sim D | Y) = \underline{\hspace{1cm}}$ ; and if  $P(D) = 0.001 \rightarrow P(\sim D | Y) = \underline{\hspace{1cm}}$ .

Final example

- Alice writes to Bob and does not receive an answer. Assuming that one letter in  $n$  is lost in the mail, find the probability that Bob received the letter. It is to be assumed that Bob would have answered the letter if he had received it.
- Let  $A$  = Alice receives letter from Bob and  $B$  = Bob receives letter from Alice.