

Probability by counting equally likely outcomes

- Now we can update our original definition of probability using the counting concepts.
- Definition: Let  $A$  be an event from a sample space in which all outcomes are equally likely. The **probability of  $A$** , denoted  $P(A)$ , is defined by:

$$\text{Probability of an event} = \frac{\text{Number of outcomes in the event}}{\text{Total number of possible outcomes}}$$

$$P(A) =$$

- Examples: A standard 52 card deck is shuffled and one card is picked at random. Find the probabilities of the following events:
  - (a) Club and King:
  - (b) Club or King:
  - (c) Not Club nor King:
  - (d) Club or Hearts:
- We will formalize these probability ideas soon.

More counting probability problems

- Now we can use all of the counting tools we've learned and our new probability knowledge to look at more interesting problems.
- *COUNTING STRATEGY*: Solve the numerator and denominator separately.
  - Numerator: “IS a condition” (selecting from restricted sample space).
  - Denominator: “NO condition” (selecting from unrestricted sample space).
- Examples:
  1. A box of jerseys for a pick-up game of basketball contains 8 extra-large jerseys, 7 large jerseys, and 5 medium jerseys. If you are first to the box and grab 3 jerseys, what is the probability that you randomly grab 3 extra-large jerseys.
  2. Suppose we have a shuffled deck and deal seven cards. What is the probability that we draw no queens?
  3. Suppose we have a shuffled deck and deal three cards. What is the probability that we draw exactly one queen?

- Remember in order to use counting tools when finding probabilities, all outcomes need to be equally likely.

This means the just the smallest possible results of the experiment (e.g. drawing a single card), not events (e.g. king or heart).

### Generalizing probability

#### Motivation

- In real data studies, outcomes are rarely equally likely. So we need methods to work with probabilities in these scenarios as well.
- Example: A research study into the percentage of births which involve more than one child leads to the following probability table:

Number of children	1	2	3
Probability	0.9670	0.0311	0.0019

Intuitively we can easily find the probability of an event, such as a randomly selected birth involving more than one child, based on this table.

Let's break down what we implicitly did.

1. Assigned probabilities to each of the \_\_\_\_\_ in the sample space (i.e. the table).
  2. Wrote the \_\_\_\_\_ of interest in terms of the outcomes of interest.
  3. \_\_\_\_\_ probabilities of the mutually exclusive outcomes.
- This previous example illustrates a natural method for assigning probabilities to events of certain types of experiments.

#### Sample point method for calculated probabilities

- Theorem: Let  $S = \{O_1, \dots, O_n\}$  be a finite set, where all  $O_i$  are individual outcomes each with probability  $P(O_i) \geq 0$  and  $\sum P(O_i) = 1$ . For any  $A \in S$ ,

$$P(A) = \sum_{O_i \in A} P(O_i)$$

- Using this theorem, we have steps / techniques to find probability of any event.

- Example: When players  $A$  and  $B$  play tennis, the probability that  $A$  wins is  $2/3$ . Suppose that  $A$  and  $B$  play two matches. What is the probability that  $A$  wins at least one match?

Let  $AB$  denote the outcome that player  $A$  wins the first game and player  $B$  wins the second.

1. What are the sample space and individual outcomes?
2. Find the probability of each individual outcomes.
3. Find the event of interest and express it as a union of individual outcomes.

#### General definition of probability

- Not all sample spaces are finite or easy to handle. So there are axioms that give general properties that an assignment of probabilities to events must have.
- **Axioms of Probability:** If you define a way to assign a probability  $P(A)$  to any event  $A$ , the following axioms must be true:

1.  $P(A) \geq 0$  for any event  $A$ .
2.  $P(S) = 1$ .
3. Suppose  $A_1, \dots, A_n$  is a (possibly infinite) sequence of pairwise mutually exclusive events. Then

$$P(A_1 \cup \dots \cup A_n) = P(A_1) + \dots + P(A_n)$$

- Using this new definition, we have two important properties:
1. Any \_\_\_\_\_ can be expressed as a union of mutually exclusive outcomes.
  2. The probability of an event is the \_\_\_\_\_ of probabilities of the mutually exclusive outcomes.

#### Theorems and their proofs

- Theorem: For probability assignment  $P(\cdot)$  and any event  $A$  in the sample space  $S$ ,

$$(a) P(\emptyset) = 0 \quad (b) P(A) \leq 1 \quad (c) P(\sim A) = 1 - P(A)$$

- Proofs (easier to prove out of order):

(c)  $P(\sim A) = 1 - P(A)$

*PROOF STRATEGY:* Rewrite events we know and use axioms to simplify their probabilities.

(b)  $P(A) \leq 1$

*HINT: Use part (c) of theorem*

Combined with Axiom 1, this theorem gives us bounds on the probability of any event  $A$ :

(a)  $P(\emptyset) = 0$

*HINT: Use part (c) of theorem*

- Theorem: For probability assignment  $P(\cdot)$  and any events  $A$  and  $B$  in the sample space  $S$ ,

(a)  $P(A \cap \sim B) =$

(b)  $P(A \cup B) =$

(c) If  $B \subset A$ , then  $P(B) \leq P(A)$

- Proofs:

(a)  $P(A \cap \sim B) = P(A) - P(A \cap B)$

*HINT: Start with  $A$  and use identity  $A = (A \cap B) \cup (A \cap \sim B)$*

(b)  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

*HINT: Use identity  $(B \cup A) = (B \cup A) \cap (B \cup \sim B)$*

(c) If  $B \subset A$ , then  $P(B) \leq P(A)$

*HINT: Start with Axiom 1 and  $P(A \cap \sim B)$*

More examples and concepts / theorems

1. A fair coin is flipped successively until the same face is observed on successive flips. Find the probability that it will take three or more flips of the coin to observe the same face on two consecutive flips.

2. Find the probability that in a room of 20 people, there are at least two people sharing the same birthday.

- Note on order: When solving these types of problems, the numerator and the denominator must ALWAYS MATCH (be consistent) in terms of order.

So should never have

3. A cryptocurrency exchange sells Bitcoin, Litecoin and Ethereum.

Let  $B = \{\text{buys Bitcoin}\}$ ,  $L = \{\text{buys Litecoin}\}$  and  $E = \{\text{buys Ethereum}\}$ .

Based on past sales the exchange determines that for any new customer

$$\begin{aligned} P(B) &= 0.50, P(L) = 0.22, P(E) = 0.20, \\ P(B \cap L) &= 0.10, P(B \cap E) = 0.15, P(L \cap E) = 0.09, \\ P(B \cap L \cap E) &= 0.06 \end{aligned}$$

Find the probability that a new customer purchases at least one of these three currencies.

- Theorem: For any events  $A$ ,  $B$ , and  $C$ ,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

The analogous counting rule for the union of three events is:

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

#### Quick review

- Everything we are studying begins from \_\_\_\_\_.  
From these, we obtain \_\_\_\_\_, which together make up the \_\_\_\_\_.
- We want to know \_\_\_\_\_ of events (subsets of  $S$ ).  
To calculate this, we defined how to assign \_\_\_\_\_ in general via the \_\_\_\_\_.
- Then there are two methods to compute \_\_\_\_\_:  
1.  
2.

#### Odds

- Odds of an event  $A$  are generally written as a \_\_\_\_\_ of two integers, such as \_\_\_\_\_. The odds against  $A$  are given by the \_\_\_\_\_.  
Formally, odds are another way to represent probability (but the terms are not interchangeable).
- Definition: The **odds** for an event  $A$  are defined as the ratio  $P(A)$  to  $P(\sim A)$ .

Odds of  $A$  =

- Converting between odds and probability:  
Example: Suppose a soccer team is in a playoff game.  
(a) If the probability winning is 0.20, what are the odds of winning?  
  
(b) If the odds of losing are 7:9, find the probability of winning.