MATH 320: Test 3 Study Guide

Lecture 10 – Discrete Distributions (2.1, 2.3, 2.4, 2.5, 2.6, 2.7)

See distribution table

Summary of four Bernoulli-based experiments

- Distributions: Binomial, Geometric, Negative Binomial, and Hypergeometric
- Throughout all of these, there were three important aspects:

  (1) Number of successes (2) Number of trials (3) Probability of success
- Organization of the four distributions based on what we are interested in (the random variable) and what we are given (as parameters).
  - Distributions counting the number of successes: Binomial and Hypergeometric
    - Interested in: (1)
    - (2) and (3) are given as parameters.
    - Only difference is with vs without replacement
  - Distributions counting the number of trials: Geometric and Negative Binomial
    - Interested in: (2)
    - (1) and (3) are given as parameters.
    - Only difference is the number of successes

Poisson distribution

- The random variable X counts the number of events in a given unit.

Lecture 11 – Continuous Distributions (3.1, 3.2, 3.3)

See distribution table

Survival function

- $S(t) = P(T > t) = 1 - F(t)$.

Linear transformation of normal random variables

- Theorem: If $X \sim \text{Normal}\,(\mu, \sigma^2)$ and $Y = aX + b \to Y \sim \text{Normal}\,(a\mu + b, a^2\sigma^2)$.
- Standardizing: If $X \sim \text{Normal}\,(\mu, \sigma^2)$ and $Z = \frac{X-\mu}{\sigma} \to Z \sim \text{Normal}\,(\mu = 0, \sigma^2 = 1)$.
- Can standardize any random variable.

Normal probabilities and percentiles

- Z-table: Gives $F_Z(z) = P(Z \leq z)$.
- $P(x_1 \leq X \leq x_2) = P\Big(\frac{x_1-\mu}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{x_2-\mu}{\sigma}\Big) = P(z_1 \leq Z \leq z_2)$,

  where $z_1 = \frac{x_1-\mu}{\sigma}$ and $z_2 = \dfrac{x_2 - \mu}{\sigma}$.
- $F_Z(z_p) = p \quad \to \quad z_p = \frac{x_p-\mu}{\sigma} \quad \to \quad x_p = \sigma z_p + \mu$.

Sums of independent, identically distribution random variables

- Central Limit Theorem (CLT): If $X_i \overset{iid}{\sim} f(x)$ with mean μ and variance σ^2 and $S = \displaystyle\sum_{i=1}^{n} X_i$

  $\to S \overset{approx}{\sim} \text{Normal}\,(n\mu, n\sigma^2)$ for large n.

Lecture 12 – Moment Generating Functions (2.3, 2.4, 2.6, 2.7, 3.1, 3.2, 3.3)

Moments

- Definition: n^{th} moment of $X = E(X^n), \qquad n = 1, 2, 3 \ldots$
- n^{th} moment of X about $b = E[(X-b)^n], \qquad n = 1, 2, 3 \ldots$
- Central moments $= E[(X-\mu)^n], \qquad n = 1, 2, 3 \ldots$

Moment generating functions (mgf)

- Definition:

$$M_X(t) = \underbrace{E(\mathrm{e}^{tx})}_{\text{In general}} \quad \to \quad \underbrace{\sum_x \mathrm{e}^{tx} f(x)}_{\text{Discrete}} \quad \underbrace{\int_{-\infty}^{\infty} \mathrm{e}^{tx}\, f(x)\, \mathrm{d}x}_{\text{Continuous}}$$

- How to find moments from mgf:

  $M_X'(0) = E(X), \quad M_X''(0) = E(X^2), \quad \ldots \quad , \quad M_X^{(n)}(0) = E(X^n)$

  * If X is discrete $\to M_X^{(n)}(t) = \sum x^n \mathrm{e}^{tx}\, f(x)$ and $M_X^{(n)}(0) = \sum x^n\, f(x) = E(X^n)$
- Mgf of $Y = aX + b$

  $M_Y(t) = M_{aX+b}(t) = \mathrm{e}^{tb} M_X(at)$
- Mgfs are unique.
- Another variance definition: Using mgfs: $M_X''(0) - \big[M_X'(0)\big]^2 = M_X''(t)\big|_{t=0} - \big[M_X'(t)\big|_{t=0}\big]^2$

Distributions

Discrete Distributions	
Discrete uniform (N_0, N_1)	
Pmf	$P(X = x \mid N_0, N_1) = \frac{1}{N_1 - N_0 + 1}; \qquad x = N_0, \dots, N_1; \qquad N_0 \leq N_1$
Mean and Variance	$E(X) = \frac{N_0 + N_1}{2}, \qquad V(X) = \frac{(N_1 - N_0 + 1)^2 - 1}{12}$
Mgf	$M_X(t) = \frac{1}{N_1 - N_0 + 1} \sum_{x=N_0}^{N_1} \mathrm{e}^{tx}$
Notes	
Bernoulli (p)	
Pmf	$P(X = x \mid p) = p^x (1-p)^{1-x}; \qquad x = 0, 1; \qquad 0 < \mathrm{p} < 1$
Mean and Variance	$E(X) = p, \qquad V(X) = p(1-p) = pq$
Mgf	$M_X(t) = (1-p) + p\mathrm{e}^t = q + p\mathrm{e}^t$
Notes	Special case of binomial with $n = 1$.
Binomial (n, p)	
Pmf	$P(X = x \mid n, p) = \binom{n}{x} p^x (1-p)^{n-x}; \qquad x = 0, 1, \dots, n; \qquad 0 < p < 1$
Mean and Variance	$E(X) = np, \qquad V(X) = np(1-p) = npq$
Mgf	$M_X(t) = (q + p\mathrm{e}^t)^n$
Notes	Sum of *iid* bernoulli RVs.
Geometric (p)	
Pmf	$P(X = x \mid p) = q^{x-1} p; \qquad x = 1, 2, \dots; \qquad 0 < p < 1$
Cdf	$F_X(x \mid p) = 1 - q^x$
Mean and Variance	$E(X) = \frac{1}{p}, \qquad V(X) = \frac{1-p}{p^2} = \frac{q}{p^2}$
Mgf	$M_X(t) = \frac{p\mathrm{e}^t}{1 - q\mathrm{e}^t}; \qquad t < -\ln(q)$
Notes	Special case of negative binomial with $r = 1$. * See other geometric probabilities. Alternate form $Y = X - 1$. This distribution is *memoryless*: $P(X > s \mid X > t) = P(X > s - t); \qquad s > t$.

Negative binomial (r, p)		
Pmf	$P(X = x \mid r, p) = P(X = x \mid r, p) = \binom{x-1}{r-1} p^r q^{x-r}; \qquad x = r, r+1, \ldots; \qquad 0 < p < 1$	
Mean and Variance	$E(X) = \frac{r}{p}, \qquad V(X) = \frac{r(1-p)}{p^2} = \frac{rq}{p^2}$	
Mgf	$M_X(t) = \left[\frac{pe^t}{1-qe^t}\right]^r; \qquad t < -\ln(q)$	
Notes	Sum of *iid* geometric RVs.	
Hypergeometric (N, M, K)		
Pmf	$P(X = x \mid r, p) = P(X = x \mid N, M, K) = \frac{\binom{M}{x}\binom{N-M}{K-x}}{\binom{N}{K}}; \qquad x = 0, 1, \ldots, \min(M, K)$	
Mean and Variance	$E(X) = K\left(\frac{M}{N}\right), \qquad V(X) = K\left(\frac{M}{N}\right)\left(\frac{N-M}{N}\right)\left(\frac{N-K}{N-1}\right)$	
Mgf		
Notes	If do not require $M \geq K$, $\mathcal{X} = \{\max(0, K+M-N), \ldots, \min(M, K)\}$, mean and variance converge to that of binomial $(n = K, p = M/K)$ when $N \to \infty$.	
Poisson (λ)		
Pmf	$P(X = x \mid \lambda) = \frac{e^{-\lambda}\lambda^x}{x!}, \qquad x = 0, 1, 2, \ldots; \qquad \lambda > 0$	
Mean and Variance	$E(X) = \lambda, \qquad V(X) = \lambda$	
Mgf	$M_X(t) = e^{\lambda(e^t - 1)}$	
Notes	If $X_i \overset{\perp\!\!\!\perp}{\sim}$ Poisson (λ_i), then $\sum X_i \sim \text{Poisson}(\lambda = \sum \lambda_i)$.	

Other geometric probabilities

- Let $X \sim \text{Geometric}(p)$.

$$
\begin{aligned}
P(X < \infty) &= 1 \\
P(X > x) &= q^x \\
P(X \geq x) &= q^{x-1} \\
P(a < X \leq b) &= q^a - q^b \\
P(a \leq X \leq b) &= q^{a-1} - q^b
\end{aligned}
$$

Continuous Distributions

Continuous uniform (a, b)

Pdf	$f(x \mid a, b) = \frac{1}{b-a}, \qquad a \leq x \leq b; \qquad a, b \in \mathbb{R}, \qquad a \leq b$
Cdf	$F(x) = \frac{x-a}{b-a} \qquad a \leq x \leq b$
Survival	$S(t) = \frac{b-t}{b-a} \qquad a \leq t \leq b \qquad$ if $T \sim$ Uniform (a, b)
Mean and Variance	$E(X) = \frac{a+b}{2}; \qquad V(X) = \frac{(b-a)^2}{12}$
Mgf	$M_X(t) = \frac{e^{tb} - e^{ta}}{t(b-a)} \qquad t \neq 0$
Notes	

Exponential (λ)

Pdf	$f(t \mid \lambda) = \lambda e^{-\lambda t}, \qquad t \geq 0; \qquad \lambda > 0$
Cdf	$F(t) = 1 - e^{-\lambda t} \qquad t \geq 0$
Survival	$S(t) = e^{-\lambda t} \qquad t \geq 0$
Mean and Variance	$E(X) = \frac{1}{\lambda}; \qquad V(X) = \frac{1}{\lambda^2}$
Mgf	$M_X(t) = \frac{\beta}{\beta - t} \qquad t < \beta; \qquad$ if $T \sim$ Exp (β)
Notes	Special case of gamma with $\alpha = 1, \beta$. This distribution is *memoryless*: $P(T > a + b \mid T > a) = P(T > b); \qquad a, b > 0$. Rate parameterization is given; alternate parameterization is with scale $\theta = 1/\lambda$.

Gamma (α, β)

Pdf	$f(x \mid \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \qquad x \geq 0; \qquad \alpha, \beta > 0$
Cdf	N/A
Mean and Variance	$E(X) = \frac{\alpha}{\beta} \qquad V(X) = \frac{\alpha}{\beta^2}$
Mgf	$M_X(t) = \left(\frac{\beta}{\beta - t}\right)^\alpha \qquad t < \beta$
Notes	$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \, dx$ Sum of *iid* exponential RVs. A special case is exponential $(\alpha = 1, \beta)$. Rate parameterization is given; alternate parameterization is with scale $\theta = 1/\beta$.

Normal (μ, σ^2)

Pdf	$f(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], \qquad -\infty < x < \infty; \qquad -\infty < \mu < \infty, \quad \sigma > 0$
Cdf	N/A
Mean and Variance	$E(X) = \mu, \qquad V(X) = \sigma^2$
Mgf	$M_X(t) = \exp\left[\mu t + \frac{\sigma^2 t^2}{2}\right]$
Notes	Special case: Standard normal $Z \sim$ Normal $(\mu = 0, \sigma^2 = 1)$.

Lognormal (μ, σ^2)	
Pdf	$f(y \mid \mu, \sigma^2) = \frac{1}{y\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(\ln(y)-\mu)^2}{2\sigma^2}\right]; \qquad y \geq 0; \qquad -\infty < \mu < \infty; \qquad \sigma > 0$
Mean and Variance	$E(Y) = e^{\mu+\frac{\sigma^2}{2}}, \qquad V(Y) = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$
Mgf	
Notes	If $Y \sim$ Lognormal $\Longrightarrow \ln(Y) \sim$ Normal (μ, σ^2); equivalently, if $X \sim$ Normal (μ, σ^2) and $Y = e^X \Longrightarrow Y \sim$ Lognormal. μ and σ^2 represent the mean and variance of the normal random variable X which appears in the exponent.
Beta (α, β)	
Pdf	$f(x \mid \alpha, \beta) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}; \qquad 0 \leq x \leq 1; \qquad \alpha, \beta > 0$
Mean and Variance	$E(X) = \frac{\alpha}{\alpha+\beta}, \qquad V(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
Mgf	
Notes	$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$