MATH 320: Test 1 Study Guide

Lecture 1 – Set Theory (1.1)

How to calculate probability

- Probability by counting equally likely outcomes:

  Probability of an event $= \dfrac{\textit{Number of outcomes in the event}}{\textit{Total number of possible outcomes}}$

- Empirical probability, relative frequency estimate of the probability of an event

  Probability of an event $= \dfrac{\textit{Number of times the event occurs in } n \textit{ trials}}{n}$

Set identities

- Commutative Law (reordering):

  $A \cup B = B \cup A \quad \& \quad A \cap B = B \cap A$

- Associative Law (changing location of parentheses):

  $A \cup (B \cup C) = (A \cup B) \cup C \quad \& \quad A \cap (B \cap C) = (A \cap B) \cap C$

- Distributive Law (distributing union or intersection):

  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \quad \& \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

- De Morgan's Law (distributing complement; flip everything):

  $\sim(A \cup B) = \sim A \cap \sim B \quad \& \quad \sim(A \cap B) = \sim A \cup \sim B$

Relationships among sets

- Mutually exclusive (disjoint) if $A \cap B = \emptyset$ (no overlap)
- Pairwise mutually exclusive if $A_i \cap A_j = \emptyset$ for all $i \neq j$ (no overlap of any pairs)
- Exhaustive if $\displaystyle\bigcup_{i=1}^{k} A_i = A_1 \cup \cdots \cup A_k = S$ (complete S)
- Form a partition if exhaustive and pairwise mutually exclusive

Lecture 2 – Counting (1.2)

Basic rules

- Complements counting rule: $n(\sim A) = n(S) - n(A)$
- General union counting rule: $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
- Special case union counting rule: If $A \cap B = \emptyset$, $n(A \cup B) = n(A) + n(B)$
- Union of three events counting rule:

  $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

Counting principles

- Multiplication principle for counting: If a job consists of k separate tasks, the ith of which can be done in n_i ways ($i = 1, \ldots, k$), then the entire job can be done in $n_1 \times n_2 \times \cdots \times n_k$ ways.
- Ordered with replacement:

  Given n distinguishable objects, there are n^r ways to choose with replacement an ordered sample of r objects.
- Ordered without replacement (all n):

  The number of permutations of n objects is $n! = n(n-1)(n-2)\cdots 2(1)$.
- Ordered without replacement ($r \leq n$):

  The number of permutations of n objects taken r at a time is $P\binom{n}{r} = \frac{n!}{(n-r)!}$
- Unordered without replacement ($r \leq n$):

  The number of combinations of n objects taken r at a time is $\binom{n}{r} = \frac{n!}{(n-r)!r!}$
- Useful identity (binomial coefficient): $\binom{n}{r} = \binom{n}{n-r}$
- Counting partitions (multinomial coefficient):

  The number of partitions of n objects into k distinct groups of sizes $n_1, n_2, \ldots, n_k$ (where $n_1 + \cdots + n_k = n \iff$ splitting up entire group) is given by: $\binom{n}{n_1,\, n_2,\, \ldots,\, n_k} = \frac{n!}{n_1! n_2! \cdots n_k!}$

Lecture 3 – Probability (1.1)

Probability definition based on counting equally likely outcomes

- $P(A) = \dfrac{n(A)}{n(S)}$

Probability when outcomes are not equally likely

- Sample point method:

  Let $S = \{O_1, \ldots, O_n\}$ be a finite set, where all O_i are individual outcomes each with probability $P(O_i) \geq 0$ and $\sum P(O_i) = 1$. For any $A \in S$,
- $P(A) = \displaystyle\sum_{O_i \in A} P(O_i)$

General definition of probability (axioms)

- If you define a way to assign a probability $P(A)$ to any event A, the following axioms must be true
  1. $P(A) \geq 0$
  2. $P(S) = 1$
  3. $P\big(\displaystyle\bigcup_{i=1}^{\infty} A_i\big) = \displaystyle\sum_{i=1}^{\infty} P(A_i)$

Probability theorems

- Complement probability: $P(\sim A) = 1 - P(A)$
- Probability of any event: $P(A) \leq 1$
- Probability of null set: $P(\emptyset) = 0$
- $P(A \cap \sim B) = P(A) - P(A \cap B)$
- General union probability: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- Special case union probability: If $A \cap B = \emptyset$, $P(A \cup B) = P(A) + P(B)$
- Subset probability: If $B \subset A$, then $P(B) \leq P(A)$
- Union of three events probability:

  $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

Lecture 4 – Conditional Probability (1.3)

Defining conditional probability

- Conditional probability by counting equally likely outcomes: $P(A \mid B) = \dfrac{n(A \cap B)}{n(B)}$
- General definition of conditional probability: $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$,

  provided $P(B) > 0$

Probability rules for conditional probability

- All probability theorems hold in conditional probability. Examples below:
- Conditional complement probability: $P(\sim A \mid B) = 1 - P(A \mid B)$
- Conditional general union probability:

  $P(A \cup B \mid C) = P(A \mid C) + P(B \mid C) - P(A \cap B \mid C)$

Multiplication rule for probability

- $P(A \cap B) = P(B)P(A \mid B)$, provided $P(B) > 0$
- $P(A \cap B) = P(A)P(B \mid A)$, provided $P(A) > 0$
- General multiplication rule for probability of k events:

  $P(A_1 \cap \cdots \cap A_k) = P(A_1)P(A_2 \mid A_1) \cdots P(A_k \mid A_1 \cap \cdots \cap A_{k-1})$

Lecture 5 – Independent Events (1.4)

Definition of independence

- Two events A and B, are independent if $P(A \cap B) = P(A)P(B)$

  If $P(A) > 0$ and $P(B) > 0$, then $A \perp\!\!\!\perp B \iff P(A \mid B) = P(A)$, or $P(B \mid A) = P(B)$

  Otherwise, events are said to be dependent. If one condition is true, all are true.

- Special cases of independence:
  - If $P(A) = 0$ or $P(B) = 0$, $A \perp\!\!\!\perp B$
  - If $A \cap B = \emptyset$, $A \perp\!\!\!\perp B$ only if $P(A) = 0$ or $P(B) = 0$
  - If $B \subset A$, $A \perp\!\!\!\perp B$ only if $P(B) = 0$, $P(A) = 0$ or $P(A) = 1$

- Independence of three events: Events A, B, and C are mutually independent if and only if they are pairwise independent (i.e. (A, B), (A, C) and (B, C) are independent pairs) and if $P(A \cap B \cap C) = P(A)P(B)P(C)$.

Applying independence

- Multiplication rule for independent events: If A and B are independent events, $P(A \cap B) = P(A)P(B)$
- Theorems: If A and B are independent events, then the following pairs of events are also independent: A and $\sim B$; $\sim A$ and B; $\sim A$ and $\sim B$

Lecture 6 – Bayes' Theorem (1.5)

Law of total probability

- Let B be an event. If $A_1, \ldots, A_n$ partition the sample space, then

  Law of total probability = P(Second stage event) = $\sum$ Branches of interest

  $$P(B) = P\Big[\bigcup_{i=1}^{n}(A_i \cap B)\Big] = \sum_{i=1}^{n} P(A_i)\, P(B \mid A_i)$$

Bayes' Theorem

- Let B be an event. If $A_1, \ldots, A_n$ partition the sample space, then

  Bayes' Theorem = P(First stage event | Second stage event) = $\frac{\text{Main branch of interest}}{\sum \text{All branches of interest}}$

  $$P(A_i \mid B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)\, P(B \mid A_i)}{\sum_{j=1}^{n} P(A_j)\, P(B \mid A_j)}$$