MATH 320: Probability

Lecture 4: Conditional Probability

Chapter 1: Probability (1.3)

In some probability problems a condition is given which restricts your attention to a subset of the sample space.

Conditional probability by counting

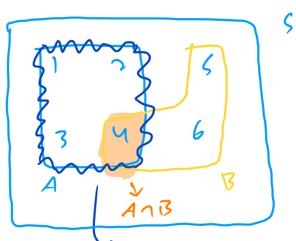
Motivating example

- Example: Roll a die. $A = \{1, 2, 3, 4\}$ and $B = \{4, 5, 6\}$.

  We have the following probabilities: $P(A) =$ 4/6 $\quad P(B) =$ 3/6

  If we know that A already occurred, what is the probability of B? $P(B|A) = 1/4$

- This is known as a conditional probability.



Updated sample spaces new sample space

- The sample space is the set of all possible outcomes.

  In other words, only the outcomes in the sample space can be observed.

- When an event occurs, we don't know specifically which outcome was observed.

- 'A occurred' means we know that the outcome is one of the elements in A. That is, there is no chance to get outcomes not in A.

  So these numbers are excluded from the sample space.

- Thus, an event which already occurred restricts the sample space and it becomes an updated sample space.

Examples with contingency tables

- Example: Here is attendance and college major of students:

	Statistics	Art	Chemistry	Total
Perfect	100	40	80	220
Good	20	50	70	140
Poor	30	15	30	75
Total	150	105	180	435

- Suppose we are told the selected student has good attendance, what is the probability the student is a chemistry major?

  extra info

  main event

  Said equivalently: What is the probability the student is a chemistry major *given that the student has good attendance*?

  denom $\neq n(S)$

  $$P(\text{Chem} \mid \text{Good}) = \frac{70}{140}$$

  $$= \frac{n(\text{Chem} \cap \text{Good})/n(S)}{n(\text{Good})/n(S)} = \frac{P(\text{Chem} \cap \text{Good})}{P(\text{Good})} = \frac{\text{Joint prob}}{\text{Marginal prob}}$$

  Another way we could interpret this result is: Of [redacted] good attenders, 50% [redacted] are chemistry majors.

- Thus the restricted sample space still applies to contingency tables, where now we look only within a single row or column because we have additional (GIVEN) information.

  And we see that these conditional probability problems could also be solved using probabilities.

- Types of probabilities:
  - **Marginal probabilities**: Refer to one event; use column / row totals to find these.
  - **Joint probabilities**: Refer to two (or more) events; use numbers in the middle of the table to find these.

	1	2	3	Total
A	Joint freq			Marginal freq
B				
Total	Marginal freq			Total Total

  - Examples) $P(\text{Stats}) = \frac{150}{435}$ $\qquad P(\text{Perfect} \cap \text{Stats}) = \frac{100}{435}$

  Denominator is the overall total

Defining conditional probability

- The previous examples showed two natural ways of finding conditional probability. The first was based on counting and the second on probabilities.

- **Conditional probability by counting** for [redacted] **equally likely outcomes**

$$P(A \mid B) = \frac{n(A \cap B)}{n(B)}, \quad \text{where } n(B) > 0$$

When outcomes are not equally likely, this rule does not apply. Then we need a general definition of conditional probability.

- Definition: The **conditional probability** of an event A given the occurrence of the event B is:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \quad \text{provided } P(B) > 0$$

$$\left\{ P(A \mid B) = \frac{\text{Joint prob } A \,\&\, B}{\text{marginal prob of } B} \right.$$

- The notation $P(A \mid B)$ is read "the conditional probability of A given B".
  - A is the main event of interest and B is called the conditioning event.

Examples

1. 100 cars will be painted in a production line. Of these cars 25 will be painted blue, 75 will be painted red, 12 will get a clear coat, and 9 of the blue cars will get a clear coat at random.

	C	~C	
B	9	16	25
R	3	72	75
	12	88	100

   (a) [redacted] What is the probability that a car will be painted blue?

$$P(B) = \frac{25}{100}$$

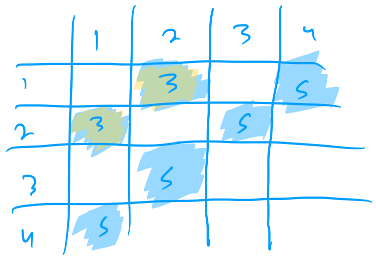
   (b) Given that a car is blue, what is the probability that it got a clear coat?

$$P(C \mid B) = \frac{9}{25}$$

   (c) What is the probability that the car is red, given that it did not get a clear coat?

$$P(R \mid \sim C) = \frac{72}{88}$$

2. A pair of fair four sided dice is rolled and the sum is determined. Find the probability that a sum of 3 is rolled given that a sum of 3 or 5 is rolled.



$P(3 \mid 3 \cup 5) = \frac{2}{6}$ → all 16 rolls are equally likely

$= \frac{2/16}{6/16}$ → probabilities

3. Probabilities for the number of auto insurance claims are given in the table below.

$\underbrace{1 \quad 2 \quad 3}_{= C}$

Number of claims	0	1	2	3
Probability	0.72	0.22	0.05	0.01

Find the probability that a policyholder files exactly 2 claims, *given that the policy holder has filed at least one claim.* → $C = \geq 1$ claim

→ $P(C) = \sum$ p robs for 1,2,3

$\downarrow = 0.28$

$$P(2 \mid C) = \frac{P(2 \cap C)}{P(C)}$$

$$= \frac{P(2)}{P(C)}$$

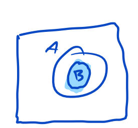
$$= \frac{0.05}{0.28}$$

$0 \; \boxed{1 \; 2 \; 3}$

$2 \subset C = \{1, 2, 3\} \Rightarrow 2 \cap C = 2$

This tells us $\approx$ <u>0.179</u> of policyholders who file a claim file exactly 2 claims.

- Note on conditional probability:
  - ALWAYS $P(B \mid A) = \frac{P(A \cap B)}{P(A)}$
  - ONLY SOMETIMES $P(B \mid A) = \frac{P(B)}{P(A)}$ → can simplify numerator ONLY IF $B \subset A$



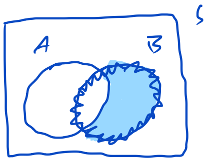
Applying conditional probability

Probability rules for conditional probability

- All of the rules (axioms) for a probability assignment from "Lecture 3 – Probability" apply to a conditional probability function $P(\cdot \mid B)$ as well.
- In addition, the theorems we proved also hold true for $P(\cdot \mid B)$.

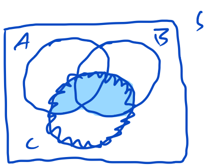
- Examples and (a few) new theorems:

  – We know $P(\sim A) = 1 - P(A)$, using the previous example we can find $P(\sim 2 \mid C)$.



$$P(\sim 2 \mid C) = P(C \mid C) - P(2 \mid C)$$
$$= 1 - 0.179 = 0.821$$

  In general: $P(\sim A \mid B) = 1 - P(A \mid B)$



  – We know $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Now applying this to the conditional probability of $A \cup B$ given C,

  $P(A \cup B \mid C) = P(A \mid C) + P(B \mid C) - P(A \cap B \mid C)$

Multiplication rule for probability

- The definition of conditional probability can be rewritten as a multiplication rule for probabilities, where we are trying to get an expression for the joint probability of A and B.

$$P(B) * P(A \mid B) = \frac{P(A \cap B)}{\cancel{P(B)}} * \cancel{P(B)}$$

- **Multiplication rule for probability**: Given events A and B,

  $P(A \cap B) = P(A)\,P(B \mid A)$, $P(A) > 0$

  $P(A \cap B) = P(B)\,P(A \mid B)$, $P(B) > 0$

- Interpretation: $\underbrace{P(A \cap B)}_{\text{Both events occurred}} = \underbrace{P(A) * P(B \mid A)}_{\text{A occurred , then B occurred later}}$

AND in probability $\Longrightarrow$ multiplication

- Example: At a country fair game there are 25 balloons on a board of which 10 balloons are yellow, 8 are red and 7 are green. A player throws darts at balloons to win a prize and randomly hits one of them. If a player throws two darts in a row what is the probability that both balloons hit are yellow?

Direct → $P(2Y) = \underbrace{\frac{10}{25}}_{P(Y1)} \times \underbrace{\frac{9}{24}}_{P(Y2|Y1)}$

Counting → $P(2Y) = \frac{\binom{10}{2}}{\binom{25}{2}} = \frac{\frac{10!}{8!\,2!}}{\frac{25!}{23!\,2!}} = \frac{\frac{10\cdot 9\cdot 8!}{8!}}{\frac{25\cdot 24\cdot 23!}{23!}} = \frac{10\cdot 9}{25\cdot 24}$ ✓

↳ nCr = w/o rep

The "direct way" of solving counting probability problems is just an application of the multiplication rule for probability.

- Often in a probability experiment, it can be easier to assign $P(A)$ and $P(B \mid A)$ rather than $P(A \cap B)$. Then we can easily compute $P(A \cap B)$ using these.

  Example: Two cards are drawn at random from a standard deck without replacement, as in the previous example. Find the probability that both are kings.

  $P(K1) * P(K2|K1) = \frac{4}{52} * \frac{3}{51} = P(K1 \cap K2)$

- This multiplication rule can be extended to any number of events as well.
  - More generally, given k events $A_1, \ldots, A_k$

    $$P(A_1 \cap \cdots \cap A_k) = P(A_1)\,P(A_2|A_1)\cdots P(A_k|A_1 \cap \cdots \cap A_{k-1})$$

  - Example: For $k = 3$ events, the multiplication rule is

    $$P(A_1 \cap A_2 \cap A_3) = P(A_1) \times P(A_2|A_1) \times P(A_3|A_1 \cap A_2)$$

- Example:

  (a) From an ordinary deck of playing cards, cards are to be drawn successively at random and without replacement. What is the probability that the third spade appears on the sixth draw?

  Counting $\underbrace{\frac{\binom{13}{2}\binom{39}{3}}{\binom{52}{5}}}_{\text{2 spaids from 5}} * \underbrace{\frac{11}{47}}_{\text{1 spaid}} \approx 0.0064$

  (b) Same question as (a), except now with replacement.

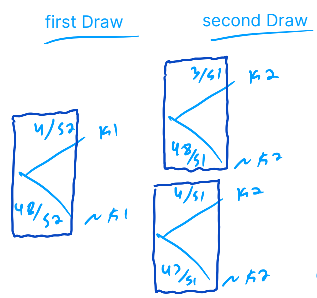
  direct

  $\overline{S}\ \ \overline{S}\ \ \overline{\sim S}\ \ \overline{\sim S}\ \ \overline{\sim S}\ \ \overline{\sim S}$

  $\Rightarrow \underbrace{\underbrace{\binom{5}{2}}_{\#\text{ seq}}\underbrace{\left(\frac{13}{52}\right)^2\left(\frac{39}{52}\right)^3}_{\text{Prob}}}_{\text{2 Spaids from 5}} \underbrace{\left(\frac{13}{52}\right)}_{\text{1spaid}}$

Using tree diagrams in probability problems

- Experiments involving multiple stages, such as drawing two cards without replacement can be summarized completely using trees!
- Examples:
  1. Lets continue the drawing two kings example from before:



Branch #	outcomes	Probability
1	$(K1, K2)$	$\frac{4}{52}\left(\frac{3}{51}\right) \approx 0.01$
2	$(K1, \sim K2)$	$\frac{4}{52}\left(\frac{48}{51}\right) \approx 0.07$
3	$(\sim K1, K2)$	$\frac{48}{52}\left(\frac{4}{51}\right) \approx 0.07$
4	$(\sim K1, \sim K2)$	$\frac{48}{52}\left(\frac{47}{51}\right) \approx 0.85$

Total $\Sigma = 1$

★ Probabilities must sum to 1 $\longleftrightarrow$ $\Sigma P(\cdot) = 1$

(a) What is the probability of exactly one king?

→ using tree → $P(\text{exactly 1 king}) = ② + ③$

→ Formally writing out → $P[(K1 \cap \sim K2) \cup (\sim K1 \cap K2)] = P(K1 \cap \sim K2) + P(\sim K1 \cap K2)$

$$= P(K1)P(\sim K2 | K1) + P(\sim K1)P(K2 | \sim K1)$$

$$= \frac{4}{52}\left(\frac{48}{51}\right) + \frac{48}{52}\left(\frac{4}{51}\right)$$

(b) What is the probability of two kings or no kings?

$$P(2K \text{ or } 0K) = ① + ④ = \frac{4}{52}\left(\frac{3}{51}\right) + \frac{48}{52}\left(\frac{47}{51}\right)$$

$P[(K1, K2) \cup (\sim K1, \sim K2)]$

(c) What is the probability of at least one king?

$$P(\text{At least 1 king}) = ① + ② + ③$$

$$= 1 - ④ = 1 - \frac{48}{52}\left(\frac{47}{51}\right)$$

can easily use complements rather than solving for many branches.

(d) Find the probability that the second card is a king ($K2$), given that the first card drawn was a king.

$$P(K2 | K1) = \frac{3}{51}$$

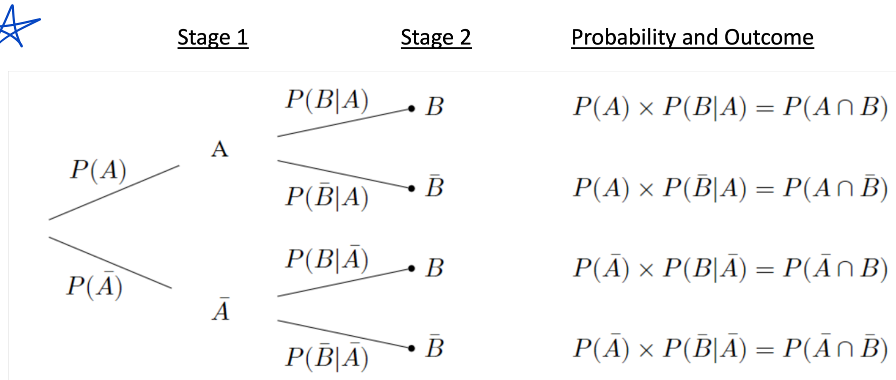
– This is just knowing parts of the tree. For problems that are just asking for "one level" of the tree or "one stage" of an experiment, it is often easier to find conditional probability by

(1) Taking into account the information by updating the scenario and then
(2) Solving like normal.

- Creating tree diagram and summing up the final probabilities of interest simplifies many harder problems!

  In most experiments, there will be a natural order of events, which will make setting up the tree intuitive.

- General tree



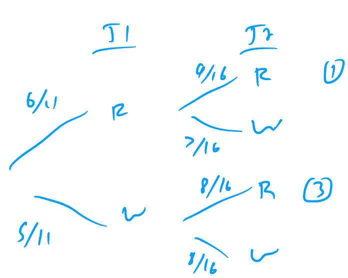
2. There are two jars, Jar 1 and Jar 2. Jar 1 has 6 red and 5 white chips and Jar 2 has 8 red and 7 white chips.

   (a) If you first pick a red chip from Jar 1 and transfer the chip to Jar 2, what is the probability of then picking a red chip from Jar 2?

   J1: 6R, 5W $\to -1$ J2: 8R, 7W $\to +1$

   $$P(R_2 \mid R_1) = \frac{9}{16}$$

   (b) If you first pick a chip (don't know which color) from Jar 1 and transfer the chip to Jar 2, what is the probability of then picking a red chip from Jar 2?



$$P(R_2) = ① + ③$$

$$= \frac{6}{11}\left(\frac{9}{16}\right) + \frac{5}{11}\left(\frac{8}{16}\right)$$

$$= 0.5341$$

3. An insurance company sells several types types of insurance policies including auto policies and homeowner policies.

   – Let A be those people with an auto policy only and $P(A) = 0.3$

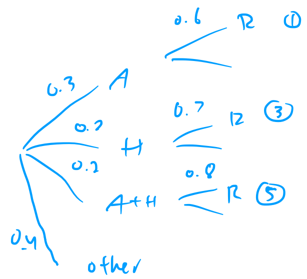
   – Let H be those people with an auto policy only and $P(H) = 0.2$

   – Let $A{+}H$ be those people with both an auto and homeowner's policy (but no other policies) and $P(A{+}H) = 0.2$

   Further let R be the event that the person will renew at least one of these policies. From past experience the following conditional probabilities are assigned:

   $P(R \mid A) = 0.6, P(R \mid H) = 0.7$ and $P(R \mid A{+}H) = 0.8$.

   Given that a person selected at random has an auto or homeowner policy, what is the conditional probability that a person will renew at least one of those policies?



$$P(R \mid A \cup H \cup A{+}H) = \frac{P(R \cap (A \cup H \cup A{+}H))}{P(A \cup H \cup A{+}H)}$$

$$= \frac{\text{(distribute, add, rewrite w/ mult rules)}}{\text{(adds)}}$$

$$= \frac{1 + 3 + 5}{0.3 + 0.2 + 0.2}$$

$$= \frac{0.3(0.6) + 0.2(0.7) + 0.2(0.8)}{0.3 + 0.2 + 0.2}$$

$$\approx 0.686$$