MATH 320: Probability

Lecture 5: Independent Events

Chapter 1: Probability (1.4)

The independence of events

Motivation

- For certain pairs of events, the occurrence of one of them may or may not change the probability of the occurrence of the other.
- Example: Roll a die. $A = \{1, 2, 3, 4\}$, $B = \{1, 3, 5\}$ and $C = \{4, 5, 6\}$.

  Compute the probabilities of $P(C \mid A)$ and $P(B \mid A)$, then compare them with $P(C)$ and $P(B)$, respectively.

  1. $P(C \mid A) = \frac{n(C \cap A)}{n(A)} = \frac{1}{4} \neq \frac{3}{6} = P(C)$
  2. $P(B \mid A) = \frac{2}{4} = \frac{3}{6} = P(B)$

- How can we interpret the results above?
  1. The knowledge of the occurrence of A has changed the probability of C.
  2. $P(B)$ is not affected by (does not depend on) the occurrence of A.

Definition of independence

- Two events A and B, are **independent** if $P(A \cap B) = P(A)\,P(B)$ ①

  If $P(A) > 0$ and $P(B) > 0$, then $A \perp\!\!\!\perp B$ $\iff$ $P(A \mid B) = P(A)$ ② & $P(B \mid A) = P(B)$ ③

  $\underbrace{A \perp\!\!\!\perp B}_{=\text{IND}}$

  $\hookrightarrow \frac{P(A \cap B)}{P(B)} = \frac{P(A)\cancel{P(B)}}{\cancel{P(B)}}$

  Otherwise, events are said to be dependent.

- To check for independence, we only need to check one of the three conditions. If one is true, then all are true.

Example: If a fair coin is tossed twice, then $S = \{HH, HT, TH, TT\}$. Let $H1$ be the event that the first toss is a head, and $H2$ be the event that the second toss is a head. Check if $H1$ and $H2$ are independent.

$\rightarrow H1 = \{HH, HT\},\ H2 = \{HH, TH\} \rightarrow P(H1) = P(H2) = \frac{2}{4}$

$\rightarrow P(H1 \mid H2) = \frac{1}{2} \longrightarrow P(H1 \mid H2) \stackrel{?}{=} P(H1)$

$\frac{1}{2} \stackrel{\checkmark}{=} \frac{2}{4} \Rightarrow H1 \perp\!\!\!\perp H2$

- Many experiments are best approached by assuming that successive trials are independent, just like successive tosses of a coin.

  There is another common problem in which independence and dependence are intuitively clear.

  with replacement $\Rightarrow \perp\!\!\!\perp$ & without replacement $\Rightarrow \not\perp\!\!\!\perp$ (not independent)

  Example: Drawing cards, probabilities change if the card drawn is not replaced.

Multiplication rule for independent events

- The general multiplication rule for any two events is

  $P(A \cap B) = P(A)\, P(B \mid A)$ (if $A \perp\!\!\!\perp B$, $P(B \mid A) = P(B)$)

  $= P(A)\, P(B)$

If $\perp\!\!\!\perp$, find Joint by multiplying Marginal probabilities.

- **Multiplication rule for independent events**

  If A and B are independent events, $P(A \cap B) = P(A) \times P(B)$

- This multiplication rule makes some problems very easy if independence is immediately recognized. However, it may be tricky to check for in practice. So in many problems when it is not intuitively obvious, it is simply given as an assumption.

  Example: Suppose the probability of hitting a target is 0.2 and ten shots are fired independently.

  (a) What is the probability the target is hit at least once?

  $P(\text{Hit} \geq 1) = 1 - P(\text{Hit} = 0)$

  $= 1 - \binom{10}{0}(0.2)^0(0.8)^{10}$

  $= 1 - 0.8^{10}$

  $\hookrightarrow P(\bar{H}_1)\, P(\bar{H}_2 \mid \bar{H}_1) \cdots$ Instead $P(\bar{H}_1)\, P(\bar{H}_2) \cdots$ b/c $\perp\!\!\!\perp$

  (b) What is the conditional probability the target is hit twice, given that it is hit at least once?

  $P(\text{Hit} \geq 2 \mid \text{Hit} \geq 1) = \dfrac{P(\text{Hit} \geq 2 \cap \text{Hit} \geq 1)}{P(\text{Hit} \geq 1)}$

  $= \dfrac{P(\text{Hit} \geq 2)}{P(\text{Hit} \geq 1)} \longrightarrow 1 - P(\text{Hit} = 0, 1)$

  $= 1 - \left[0.8^{10} + \binom{10}{1}(0.2)^1(0.8)^9\right]$

  $= \dfrac{1 - \left[0.8^{10} + \binom{10}{1}(0.2)^1(0.8)^9\right]}{1 - (0.8)^{10}}$

- Summary:
  - If A and B are independent, can easily compute $P(A \cap B)$ $= P(A) \times P(B)$.
  - If A and B are mutually exclusive, can easily compute $P(A \cup B)$ $= P(A) + P(B)$

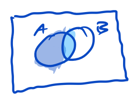
Theorems

- If A and B are independent events, then the following pairs of events are also independent:
  (a) A and $\sim B$
  (b) $\sim A$ and B
  (c) $\sim A$ and $\sim B$

  - Proof of (a)

    Setup: Want to show $P(A \cap \sim B) = P(A)\, P(\sim B)$

    One way: $A = A \cap \sim B \cup (A \cap B)$



$$P(A) = P[(A \cap \sim B) \cup (A \cap B)]$$
$$= P(A \cap \sim B) + P(A \cap B) \quad \text{(b/c disjoint)}$$

$$P(A \cap \sim B) = P(A) - P(A \cap B)$$
$$= P(A) - P(A)\,P(B) \quad \text{(} A \perp\!\!\!\perp B \text{ by assumption)}$$
$$= P(A)\,\{1 - P(B)\}$$
$$\stackrel{\checkmark}{=} P(A)\,P(\sim B) \implies A \perp\!\!\!\perp \sim B$$

Another way using conditional probabilities:

$$\rightarrow P(A \cap \sim B) = P(A)\,P(\sim B \mid A) \quad \text{(general multiplication rule)}$$
$$= [1 - P(B \mid A)] \quad \text{(conditional complement)}$$
$$= (1 - P(B)) \quad \text{(} A \perp\!\!\!\perp B \text{ by assumption)}$$
$$\stackrel{\checkmark}{=} P(A)\,P(\sim B) \implies A \perp\!\!\!\perp \sim B$$

  - Similar logic can be used to prove (b), and (a) and (b) imply (c).

- How independence relates to the other relationships for events.
  - These special cases of independence involve probabilities of zero. To check these, we have to use the most general definition of independence: $P(A \cap B) = P(A) \cdot P(B)$ because it can handle these cases.

    $P(B \mid A) = \dfrac{P(B \cap A)}{\underbrace{P(A)}_{=0}}$

    $\downarrow$ = undefined $\neq P(B)$

    This is why the definition of independence needed the additional restrictions when checking $P(A \mid B) = P(A)$ and $P(B \mid A) = P(B)$. $P(\cdot) > 0$

(a) If $P(A) = 0$ or $P(B) = 0$, the definition of independence always holds.

- Assume $P(A) = 0$

$\rightarrow$ $0 \le P(A \cap B) \le P(A) = 0$

axiom 1 $A \cap B \subset A$

$\rightarrow$ $P(A \cap B) \overset{?}{=} P(A)\, P(B)$

$0 \overset{\checkmark}{=} 0$ $\Rightarrow$ $A \perp\!\!\!\perp B$ if $P(A) = 0$

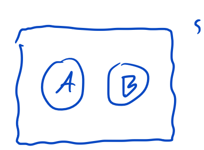
- similar logic for if $P(B) = 0$ $\Rightarrow$ $A \perp\!\!\!\perp B$

(b) If A and B are mutually exclusive, show if A and B are independent.

Need $P(A \cap B) = P(A) \cdot P(B)$ for independence:

If disjoint $\rightarrow$ $0 = P(A)\, P(B)$

ONLY TRUE if $P(A) = 0$ OR $P(B) = 0$

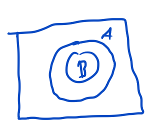


If events are mutually exclusive, there is a very dependent relationship (if one event occurs, the other cannot occur).

(c) If $B \subset A$, show if A and B are independent.

Need $P(A \cap B) = P(A) \cdot P(B)$ for independence.

If $B \subset A$ $\rightarrow$ $P(B) = P(A)\, P(B)$



ONLY TRUE if $P(A) = 0$, $P(B) = 0$, or $P(A) = 1$

$\downarrow$

$P(A) = 0 \Rightarrow P(B) = 0$ b/c subsets are smaller

Extending the independence definition

- Before extending the definition of independence to more than two events, let's see an example.
- Example: A jar contains four marbles numbered 1, 2, 3, and 4. One marble is to be drawn at random. Let the events A, B, and C be defined by $A = \{1, 2\}$, $B = \{1, 3\}$ and $C = \{1, 4\}$.

a) check if pairs are independent

$\rightarrow P(A)P(B) = P(A)P(C) = P(B)P(C) = \frac{1}{2}(\frac{1}{2}) = \frac{1}{4}$ } =

$\rightarrow P(A \cap B) = P(A \cap C) = P(B \cap C) = \frac{1}{4}$

$\Rightarrow A \perp\!\!\!\perp B, \quad A \perp\!\!\!\perp C, \quad B \perp\!\!\!\perp C$

b) check if all three are independent

$P(A \cap B \cap C) \stackrel{?}{=} P(A)P(B)P(C)$ } $\Rightarrow$ not all $\perp\!\!\!\perp$

$\frac{1}{4} \neq \frac{1}{2}(\frac{1}{2})(\frac{1}{2})$

This shows that A, B, and C are **pairwise independent**, but not **mutually independent**.

- Definition: Events A, B, and C are **mutually independent** if and only if they are pairwise independent (i.e. (A, B), (A, C) and (B, C) are independent pairs) and if $P(A \cap B \cap C) = P(A)P(B)P(C)$.

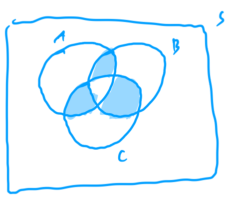
Examples

1. Suppose that A, B and C are mutually independent events and that $P(A) = 0.5$, $P(B) = 0.8$ and $P(C) = 0.9$. Find the following probabilities:

   (a) All three events occur.

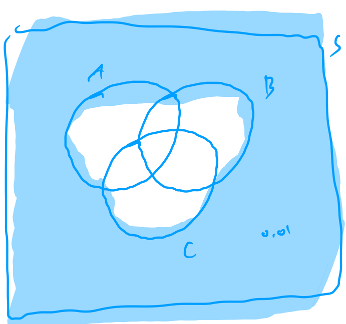
   $$P(A \cap B \cap C) = P(A)\,P(B)\,P(C) = 0.5(0.8)(0.9) = 0.36$$

   (b) Exactly two of the three events occur



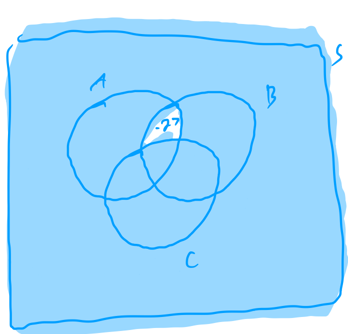
   $$P(A \cap B \cap \sim C) + P(A \cap \sim B \cap C) + P(\sim A \cap B \cap C)$$
   $$= 0.5(0.8)(0.1) + 0.5(0.2)(0.9) + 0.5(0.8)(0.9)$$
   $$= 0.04 + 0.09 + 0.36$$
   $$= 0.49$$

   (c) None of the events occur



   $$P(\sim A \cap \sim B \cap \sim C) = P(\sim A)\,P(\sim B)\,P(\sim C) = 0.5(0.2)(0.1) = 0.01$$

2. Let A, B and C be (mutually) independent events such that $P(A) = 0.5$, $P(B) = 0.6$ and $P(C) = 0.1$. Calculate $P(\sim A \cup \sim B \cup C)$.



   $$P(\sim A \cup \sim B \cup C) = 1 - P(A \cap B \cap \sim C) = 1 - 0.5(0.6)(0.9) = 1 - 0.27 = 0.73$$