

Deductibles and caps: Expected value of a function of a random variable

Expected value of a loss or claim

- These examples are in an insurance applications, but are just expected value of a function of a random variable problems.
- Insurance loss.
  - Example: (a) The amount of a single loss  $X$  for an insurance policy is exponential, with density function

$$f(x) = 0.002e^{-0.002x}, \quad x \geq 0 \quad \implies \quad X \sim \text{Exp}(\lambda = 0.0002)$$

So the (base) expected value of a single loss is:  $E(X) = \frac{1}{\lambda} = 500$

- Insurance with a deductible.
  - Continuing example: (b) Suppose now the insurance policy has a deductible of \$100 for each loss. Find the expected value of a single claim.

\*\* Now loss amount      claim amount

- *STRATEGY*: We need to write a new function  $g(X)$  that represents the new claim amount taking into account the deductible.

$g(X)$  will be a piecewise function. So think about the values  $g(X)$  takes in cases based on the range of  $X$ .

*NOTE*: We are thinking about the values of the claim from the insurance company's perspective.

- Insurance with a deductible and a cap.
  - Continuing example: (c) Now suppose the insurance policy has a deductible of \$100 per claim AND a restriction that the largest amount paid on any claim will be \$700.
  - *STRATEGY*: Use the same strategy as before for the first case, then just need to take into account the cap.
  
- Another example: The amount of a single loss  $X$  for an insurance policy has the density function  $f(x)$  for  $x \geq 0$  with deductible of \$150 and cap of \$900.
  - (a) Find a function  $g(X)$  for the amount paid (claim amount) for a loss  $x$ .
  - (b) Write the integral to solve for the expected claim amount.
  
- In general, if loss  $x$  with deductible  $d$  and cap  $c$ , we have

### The distribution $Y = g(X)$

Transformations so far

- We have already seen simple methods for finding  $E[g(X)]$  and  $V[g(X)]$  for any type of variable.
- Example: The monthly maintenance cost for a machine  $X \sim \text{Exponential}(\lambda = 0.01)$ . Next year costs will be increased 5% due to inflation. Thus next year's monthly cost is  $Y = g(X) = 1.05X$ .

Find  $E(Y)$ .

- Note we did not need to know the distribution of  $Y$  for this calculation.  
However, the mean and variance alone are not sufficient to enable us to calculate probabilities for  $Y = g(X)$ ; we need the actual distribution function  $f(x)$ .
- Discrete example: Same  $X$  with a new (discrete) model and inflation costs  $Y = g(X) = 1.05X$ :
  - (a) Find the distribution of  $Y = g(X)$ .
  - (b) Find  $P(Y < 100)$ .

$x$	$f(x)$	$y = 1.05x$	$f(y)$
0	0.28		
50	0.43		
100	0.20		
150	0.09		

- For the original continuous model, it is not as simple to find the new distribution.

Continuous transformations example

- Continuing example: Using the original  $X \sim \text{Exponential}(\lambda = 0.01)$  model...
- Find  $P(Y \leq 100)$ .

*GOAL:* Get the probability statement to be with respect to  $X$ .

*STRATEGY:* “Indirectly” find the probability for  $Y$  based on the known cdf of  $X$  and using some simple algebra. Note that this is the same strategy we used to find lognormal probabilities based on the normal cdf.

- Find the cdf  $F_Y(y)$ .

*STRATEGY:* Use the same reasoning as above, just for a general  $y$ :

$$P(Y \leq 100) = F_Y(100) \longrightarrow P(Y \leq y) = F_Y(y) \text{ for any value } y \geq 0.$$

- Note that the range of  $X$  is the interval  $[0, \infty)$ . The range for  $Y = 1.05X$  is the same interval. This will not always be the case for transformations  $g(X)$ .

*STRATEGY:* How to check range  $\rightarrow$  Apply  $g(x)$  to all pieces, ALWAYS need to check both sides.

### Inverses

- Finding the distribution of  $Y = g(X)$  like we did above is much simpler when the transformation function  $g(X)$  has an inverse.
- Recall that the function  $g(X)$  defines a mapping from the original \_\_\_\_\_ to a \_\_\_\_\_. That is,

\*\* We do not know stuff (pdf, cdf, etc.); so we have to use the inverse function to go backwards.  $\mathcal{Y}$  is completely determined by  $\mathcal{X}$ .

- When do inverse functions exist?

If the function  $g(x)$  is strictly **monotone**  $\implies$  one-to-one  $\iff$  inverse exists.

$$u > v \Rightarrow g(u) > g(v)$$

$$u > v \Rightarrow g(u) < g(v)$$

- Summary and results:

For a function  $g(x)$  that strictly increasing or strictly decreasing on the range of  $X$ , we can find an inverse function  $h(y)$  defined on the range of  $Y$ . Thus we have:

\*\* *STRATEGY* when problem solving:

1. Draw a figure of the transformation.

If transformation is strictly increasing or strictly decreasing over  $\mathcal{X}$ , then use the methods described next.

2. Check range of  $Y$  (i.e. ALSO transform range of  $X$  to range of  $Y$ ).

Using  $F_X(x)$  to find  $F_Y(y)$  for  $Y = g(X)$

- We will only generalize the methods for when  $g(X)$  has an inverse. If this is true, then there are two cases.
- **Case 1:  $g(x)$  is strictly increasing on the range of  $X$** 
  - Let  $h(y)$  be the inverse function of  $g(x)$ . The function  $h(y)$  will also be strictly increasing. In this case, we can find  $F_Y(y)$  as follows:

- Example: Let  $X \sim \text{Exponential}(\lambda = 3)$ . Find the cdf of  $Y = \sqrt{X}$ .

There are two ways that we can solve this.

Long way

Short way

- **Case 2:  $g(x)$  is strictly decreasing on the range of  $X$** 
  - Let  $h(y)$  be the inverse function of  $g(x)$ . The function  $h(y)$  will also be strictly decreasing. In this case, we can find  $F_Y(y)$  as follows:

- Example: Let  $X \sim \text{Exponential}(\lambda = 3)$ . Find the cdf of  $Y = 1 - X$ .

Again, we can do the long (“derivation”) way or short way (skip to end result).

Long way

Short way

- If  $g(x)$  does **NOT** have an inverse

- Example: Let  $X \sim \text{Uniform}(a = -2, b = 2)$ . Find the cdf of  $Y = X^2$ .

- It can be even more complicate if there isn't a "balanced" range of  $Y$ .

- Example: Let  $X \sim \text{Uniform}(a = -2, b = 3)$ . Find the cdf of  $Y = X^2$ .

- Both of these cases will be left for grad school :)

Finding the density function  $f_Y(y)$  for  $Y = g(X)$

- Finding  $F_Y(y)$  gives us all the information that is needed to calculate probabilities for  $Y$ , as shown below:

$$P(Y \leq y) = \qquad \qquad P(Y \geq y) = \qquad \qquad P(a \leq Y \leq b) =$$

Thus there is no real need to find the density function  $f_Y(y)$ . If the density function is required, it can be found by differentiating the cdf:

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

- If  $X$  is continuous, it is usually easier to find the cdf of  $Y$  and then the pdf of  $Y$  (rather than skipping straight to the pdf). But we will learn both methods, which we shall name:

1. Cdf method

2. Pdf method

(aka change of variable technique)

- Again when working in situations when  $g(x)$  has an inverse, there are two cases:

- **Case 1:  $g(x)$  is strictly increasing on the range of  $X$**

- Setup:  $h(y)$  is the inverse of  $g(x)$  and  $h(y)$  is strictly increasing.
- Previous results:  $F_Y(y) = F_X(h(y))$
- We can find the pdf  $f_Y(y)$  as follows:

- **Case 2:  $g(x)$  is strictly decreasing on the range of  $X$**

- Setup:  $h(y)$  is the inverse of  $g(x)$  and  $h(y)$  is strictly decreasing.
- Previous results:  $F_Y(y) = 1 - F_X(h(y))$
- We can find the pdf  $f_Y(y)$  as follows:

- Since  $h(y)$  is decreasing, its derivative is negative. Thus the final expression above is actually positive.



- Theorem: Let  $X$  have cdf  $F_X(x)$  with range  $\mathcal{X}$ ,  $Y = g(X)$  with range  $\mathcal{Y}$  and inverse  $h(y)$ .

- If  $g(x)$  is strictly increasing on  $\mathcal{X} \rightarrow F_Y(y) = F_X(h(y))$  for  $y \in \mathcal{Y}$ .
- If  $g(x)$  is strictly decreasing on  $\mathcal{X} \rightarrow F_Y(y) = 1 - F_X(h(y))$  for  $y \in \mathcal{Y}$ .
- If  $g(x)$  is strictly increasing or strictly decreasing on  $\mathcal{X}$ , then

$$f_Y(y) = f_X(h(y)) \cdot |h'(y)| \quad \text{for } y \in \mathcal{Y}.$$

- Return to previous examples: Let  $X \sim \text{Exponential}(\lambda = 3)$ .

- (a) Find the pdf of  $Y = \sqrt{X}$ .

Cdf method

Pdf method

- (b) Find the pdf of  $Y = 1 - X$ .

Cdf method

Pdf method

More examples

1. Let  $X$  be the outcome when you roll a fair four sided die. If you get  $Y = |X - 2|$  dollars based on your roll, find  $f_Y(y)$ .
2. Let  $X \sim \text{Poisson}(\lambda = 4)$ . If  $Y = X^2$ , find the pmf of  $Y$ .