MATH 320: Probability

Lecture 13: Functions of Random Variables

Chapter 5: Distributions of Functions of Random Variables (5.1)

Deductibles and caps: Expected value of a function of a random variable

Expected value of a loss or claim

- These examples are in an insurance applications, but are just expected value of a function of a random variable problems.
- Insurance loss.
  - Example: (a) The amount of a single loss X for an insurance policy is exponential, with density function

    $$f(x) = 0.002\mathrm{e}^{-0.002x}, \quad x \geq 0 \qquad \Longrightarrow \qquad X \sim \text{Exp}\,(\lambda = 0.0002)$$

    So the (base) expected value of a single loss is: $E(X) = \frac{1}{\lambda} = 500$
- Insurance with a deductible.
  - Continuing example: (b) Suppose now the insurance policy has a deductible of \$100 for each loss. Find the expected value of a single claim.

    ** Now loss amount   claim amount
  - *STRATEGY*: We need to write a new function $g(X)$ that represents the new claim amount taking into account the deductible.

    $g(X)$ will be a piecewise function. So think about the values $g(X)$ takes in cases based on the range of X.

    NOTE: We are thinking about the values of the claim from the insurance company's perspective.

- Insurance with a deductible and a cap.
  - Continuing example: (c) Now suppose the insurance policy has a deductible of \$100 per claim AND a restriction that the largest amount paid on any claim will be \$700.
  - *STRATEGY*: Use the same strategy as before for the first case, then just need to take into account the cap.

- Another example: The amount of a single loss X for an insurance policy has the density function $f(x)$ for $x \geq 0$ with deductible of \$150 and cap of \$900.

  (a) Find a function $g(X)$ for the amount paid (claim amount) for a loss x.
  (b) Write the integral to solve for the expected claim amount.

- In general, if loss x with deductible d and cap c, we have

The distribution $Y = g(X)$

Transformations so far

- We have already seen simple methods for finding $E[g(X)]$ and $V[g(X)]$ for any type of variable.
- Example: The monthly maintenance cost for a machine $X \sim$ Exponential $(\lambda = 0.01)$. Next year costs will be increased 5% due to inflation. Thus next year's monthly cost is $Y = g(X) = 1.05X$.

  Find $E(Y)$.

- Note we did not need to to know the distribution of Y for this calculation.

  However, the mean and variance alone are not sufficient to enable us to calculate probabilities for $Y = g(X)$; we need the actual distribution function $f(y)$.

- Discrete example: Same X with a new (discrete) model and inflation costs $Y = g(X) = 1.05X$:

  (a) Find the distribution of $Y = g(X)$.

  (b) Find $P(Y < 100)$.

x	$f(x)$	$y = 1.05x$	$f(y)$
0	0.28		
50	0.43		
100	0.20		
150	0.09		

- For the original continuous model, it is not as simple to find the new distribution.

Continuous transformations example

- Continuing example: Using the original $X \sim$ Exponential $(\lambda = 0.01)$ model...
- Find $P(Y \leq 100)$.

  GOAL: Get the probability statement to be with with respect to X.

  STRATEGY: "Indirectly" find the probability for Y based on the known cdf of X and using some simple algebra. Note that this is the same strategy we used to find lognormal probabilities based on the normal cdf.

- Find the cdf $F_Y(y)$.

  STRATEGY: Use the same reasoning as above, just for a general y:
  $P(Y \leq 100) = F_Y(100) \longrightarrow P(Y \leq y) = F_Y(y)$ for any value $y \geq 0$.

- Note that the range of X is the interval $[0, \infty)$. The range for $Y = 1.05X$ is the same interval. This will not always be the case for transformations $g(X)$.

  STRATEGY: How to check range $\rightarrow$ Apply $g(x)$ to all pieces, ALWAYS need to check both sides.

Inverses

- Finding the distribution of $Y = g(X)$ like we did above is much simpler when the transformation function $g(X)$ has an inverse.
- Recall that the function $g(X)$ defines a mapping from the original ________________ to a ________________. That is,

  ** We do not know stuff (pdf, cdf, etc.); so we have to use the inverse function to go backwards. $\mathcal{Y}$ is completely determined by $\mathcal{X}$.

- When do inverse functions exist?

  If the function $g(x)$ is strictly **monotone** $\Longrightarrow$ one-to-one $\Longleftrightarrow$ inverse exists.

  $u > v \Rightarrow g(u) > g(v)$

  $u > v \Rightarrow g(u) < g(v)$

- Summary and results:

  For a function $g(x)$ that strictly increasing or strictly decreasing on the range of X, we can find an inverse function $h(y)$ defined on the range of Y. Thus we have:

** *STRATEGY* when problem solving:

1. Draw a figure of the transformation.

   If transformation is strictly increasing or strictly decreasing over $\mathcal{X}$, then use the methods described next.

2. Check range of Y (i.e. ALSO transform range of X to range of Y).

Using $F_X(x)$ to find $F_Y(y)$ for $Y = g(X)$

- We will only generalize the methods for when $g(X)$ has an inverse. If this is true, then there are two cases.
- **Case 1: $g(x)$ is strictly increasing on the range of X**
  - Let $h(y)$ be the inverse function of $g(x)$. The function $h(y)$ will also be strictly increasing. In this case, we can find $F_Y(y)$ as follows:

  - Example: Let $X \sim \text{Exponential}\,(\lambda = 3)$. Find the cdf of $Y = \sqrt{X}$.

    There are two ways that we can solve this.

    <u>Long way</u> <u>Short way</u>

- **Case 2: $g(x)$ is strictly decreasing on the range of X**
  - Let $h(y)$ be the inverse function of $g(x)$. The function $h(y)$ will also be strictly decreasing. In this case, we can find $F_Y(y)$ as follows:

  - Example: Let $X \sim \text{Exponential}\,(\lambda = 3)$. Find the cdf of $Y = 1 - X$.

    Again, we can do the long ("derivation") way or short way (skip to end result).

    <u>Long way</u> <u>Short way</u>

- **If $g(x)$ does NOT have an inverse**
  - Example: Let $X \sim \text{Uniform}\,(a = -2, b = 2)$. Find the cdf of $Y = X^2$.

  - It can be even more complicate if there isn't a "balanced" range of Y.

    Example: Let $X \sim \text{Uniform}\,(a = -2, b = 3)$. Find the cdf of $Y = X^2$.

  - Both of these cases will be left for grad school :)

Finding the density function $f_Y(y)$ for $Y = g(X)$

- Finding $F_Y(y)$ gives us all the information that is needed to calculate probabilities for Y, as shown below:

  $P(Y \leq y) =$ $P(Y \geq y) =$ $P(a \leq Y \leq b) =$

  Thus there is no real need to find the density function $f_Y(y)$. If the density function is required, it can be found by differentiating the cdf:

  $$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y)$$

- If X is continuous, it is usually easier to find the cdf of Y and then the pdf of Y (rather than skipping straight to the pdf). But we will learn both methods, which we shall name:

  1. Cdf method
  2. Pdf method

     (aka change of variable technique)

- Again when working in situations when $g(x)$ has an inverse, there are two cases:
- **Case 1: $g(x)$ is strictly increasing on the range of X**
  - Setup: $h(y)$ is the inverse of $g(x)$ and $h(y)$ is strictly increasing.
  - Previous results: $F_Y(y) = F_X(h(y))$
  - We can find the pdf $f_Y(y)$ as follows:

- **Case 2: $g(x)$ is strictly decreasing on the range of X**
  - Setup: $h(y)$ is the inverse of $g(x)$ and $h(y)$ is strictly decreasing.
  - Previous results: $F_Y(y) = 1 - F_X(h(y))$
  - We can find the pdf $f_Y(y)$ as follows:

  - Since $h(y)$ is decreasing, its derivative is negative. Thus the final expression above is actually positive.

- Theorem: Let X have cdf $F_X(x)$ with range $\mathcal{X}$, $Y = g(X)$ with and range $\mathcal{Y}$ and inverse $h(y)$.
  - If $g(x)$ is strictly increasing on $\mathcal{X} \longrightarrow F_Y(y) = F_X(h(y))$ for $y \in \mathcal{Y}$.
  - If $g(x)$ is strictly decreasing on $\mathcal{X} \longrightarrow F_Y(y) = 1 - F_X(h(y))$ for $y \in \mathcal{Y}$.
  - If $g(x)$ is strictly increasing or strictly decreasing on $\mathcal{X}$, then

$$f_Y(y) = f_X(h(y)) \cdot |h'(y)| \qquad \text{for } y \in \mathcal{Y}.$$

- Return to previous examples: Let $X \sim \text{Exponential}\,(\lambda = 3)$.

(a) Find the pdf of $Y = \sqrt{X}$.

<u>Cdf method</u> <u>Pdf method</u>

(b) Find the pdf of $Y = 1 - X$.

<u>Cdf method</u> <u>Pdf method</u>

More examples

1. Let X be the outcome when you roll a fair four sided die. If you get $Y = |X - 2|$ dollars based on your roll, find $f_Y(y)$.

2. Let $X \sim \text{Poisson}\,(\lambda = 4)$. If $Y = X^2$, find the pmf of Y.