MATH 320: Probability

Lecture 2: Counting

Chapter 1: Probability (1.2)

Counting basics

Motivation

- There are ways to count the number of outcomes in certain types of random experiments. Thus, we need to develop some counting principles.

  This is useful in finding probabilities of events associated with these random experiments.

- Example: Suppose we have a shuffled deck and we deal seven cards. What is the probability that we draw no queens?

$$P(\text{no Queens}) = \frac{\#\text{ successes}}{\#\text{ possibilities}} \underset{<\text{ in Context }>}{=} \frac{\#\text{ hands with no queens}}{\#\text{ possible hands}}$$

Simple counting examples

1. Suppose our class 100 students. 78 students are mathematical science majors and 50 students are actuarial science majors. 41 students are double majors in mathematical science and actuarial science.

   (a) How many students are not mathematical science majors?

   $100 - 78 = 22$

   (b) How many students major in mathematical sciences or actuarial sciences?

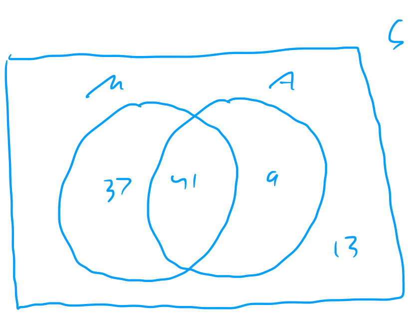
   $78 + 50 - \underbrace{41}_{\text{doubles}} = 87$

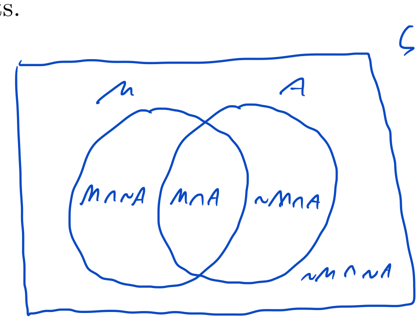
2. A single card is drawn from a well-shuffled deck. How many cards are hearts or clubs?

   $13 + 13 - 0 = 26$

Venn diagrams

- **Venn Diagrams** are helpful for visualizing all of the components of a counting problem and can easily extended to three events.





Basic rules

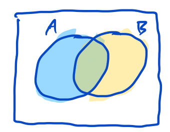
- **Notation**: The number of elements in the event (set) $A =$ $n(A)$

- **Complements counting rule**: For any finite sample space S and event A

  $\rightarrow n(\sim A) = n(S) - n(A)$

- **General union counting rule**:

  For any two events A and B in any finite sample space $\rightarrow n(A \cup B) = n(A) + n(B) - n(A \cap B)$



- **Special case union counting rule**: If A and B are mutually exclusive

  $n(A \cap B) = 0$

  $\rightarrow n(A \cup B) = n(A) + n(B)$

Counting outcomes of an experiment

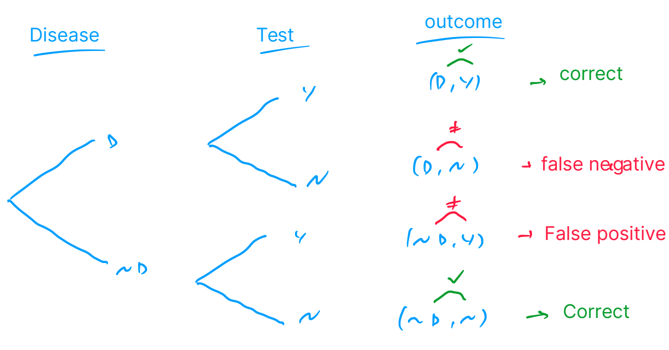
- **Tree diagrams** give a simple graphical display of all possible cases (pairs of outcomes) in problem/experiments if the number of outcomes is not unreasonably large.

  When drawing tree diagrams, think about the stages of the experiment.

- Example: Suppose we are testing for the presence of a disease. There are two things two consider, if the person has the disease (which is unknown) and the result of the test, positive or negative. Let's define:

  $D =$ the person tested has the disease
  $\sim D =$ the person tested does not have the disease
  $Y =$ the test is positive
  $N =$ the test is negative

  Find how many outcomes are possible and what each of them are.



$2 \times 2 = 4$ total outcomes

- When experiments get larger, we can use the following idea.
- **Multiplication principle for counting**:

  If a job consists of k separate tasks, the ith of which can be done in n_i ways ($i = 1, \ldots, k$), then the entire job can be done in $n_1 \times n_2 \times \cdots \times n_k$ ways.

Task 1	Task 2	$\cdots$	Task k	Total outcomes
n_1	n_2	$\cdots$	n_k	$n_1 \times n_2 \times \cdots \times n_k$

- Example: Sally has 6 pairs of socks, 4 shorts, 5 shirts, and 3 sunglasses. How many ways can she get dressed?

  $\underset{1}{\underline{6}} \times \underset{2}{\underline{4}} \times \underset{3}{\underline{5}} \times \underset{4}{\underline{3}} = 360$ total ways

  Task: 1 2 3 4

- It is very important to correctly define the sub experiments, then can just use the rule. Each "task / sub experiment" is like a level in our tree diagram.

  This is also an important principle because we can use it to develop some more counting techniques.

Permutations, combinations and partitions

Counting number of ways

- After defining tasks in an experiment, often we need to count the number of possible ways to perform each task. In doing so, there are four important criteria to consider:
  1. The number of distinguishable items $\rightarrow n$
  2. The number of objects we are going to select $\rightarrow r$
  3. Order matters or not?
  4. With replacement or without?

- Possible methods of counting

	Ordered	Unordered
With replacement	n^r	
Without replacement	$P\binom{n}{r}$	$\binom{n}{r}$

Ordered, with replacement

- Example: How many four-letter words can the letters A through Z produce?

$n = 26$
$r = 4$

$\underline{26} \times \underline{26} \times \underline{26} \times \underline{26} = 26^4$

- **Ordered, with replacement**: Given n distinguishable objects, there are $\underline{n^r}$ ways to choose with replacement an ordered sample of r objects.

- Strategy: When doing counting problems, think about (and actually draw) the "slots". This will help with what numbers to use AND to determine if order matters.

  This illustrates an application of the multiplication principle where each "slot" is a separate task.

ALWAYS
* across

Ordered, without replacement

- Example: How many ways can Bob, Mary and Jane sit in three seats?

  This question is really asking how many $\underline{\text{Permutations}}$ of these three are there?

$$\underset{1}{\underline{3}} \times \underset{2}{\underline{2}} \times \underset{3}{\underline{1}} = 6 = 3!$$
seat

- **Ordered without replacement (all n)**: The number of permutations of n distinct objects is $\underline{n! = n(n-1)(n-2)\cdots(2)(1)}$. $\rightarrow 0! = 1$

  "n - factorial"

- Example: What is the number of four-letter code words selecting from the 26 letters of the alphabet without replacement?

$$\underset{1}{\underline{26}} \times \underset{2}{\underline{25}} \times \underset{3}{\underline{24}} \times \underset{4}{\underline{23}} = P\binom{26}{4}$$
position

- **Ordered without replacement ($r \leq n$)**: The number of r-permutations of n distinct objects (aka **permutation** of n objects taken r at a time) is $\underline{P\binom{n}{r},\ {}_nP_r}$.

$$P\binom{n}{r} = n(n-1)\cdots(n-r+1)$$
$$= \frac{n!}{(n-r)!}$$

$$P\binom{n}{n} = \frac{n!}{\cancel{(n-n)!}} = n!$$
$$P\binom{n}{0} = \frac{n!}{(n-0)!} = 1$$

Example: $P\binom{10}{3} = \frac{10!}{7!} = \frac{10 \cdot 9 \cdot 8 \cdot 7!}{7!} = 10 \times 9 \times 8 = 720$

$\downarrow$
$10-3+1$

Unordered, without replacement

- Two scenarios: Among 8 students, (a) selecting 1st, 2nd and 3rd place winners (b) selecting 3 committee members among 8 students. What is the difference?

order matters in a + not in b

- **Unordered without replacement ($r \leq n$)**: The number of combinations of n objects taken r at a time is $\binom{n}{r}$, nCr.

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$ "n Choose r"

A **combination** is an unordered group (more formally, an r-element subset of the original n distinct objects), and $\binom{n}{r}$ counts the total number of different groups possible.

- Useful property: $\binom{n}{r} = \binom{n}{n-r}$

If a group of r iS made, then a group of $n-r$ is made and vice versa.

Relationship between combinations and permutations

- Both of these can be thought of as two sub experiments involving the other and demonstrates how **order** impacts the counting tool.

Permutation $\longrightarrow$ Combination

1. ordered selection of r from n
2. Remove order from groups

$\Rightarrow$ Divide out duplicates

Example: Committee of 3 from 7

A B C
B A C
⋮

6 times = 3!
ALL same group

$$\rightarrow \underbrace{P\binom{n}{r}}_{1} \times \underbrace{\frac{1}{r!}}_{2} = \frac{n!}{(n-r)!} * \frac{1}{r!} = \binom{n}{r}$$

$$\rightarrow \binom{n}{r} = \frac{P\binom{n}{r}}{r!}$$

Combination $\longrightarrow$ Permutation

1. unordered selection of r from n
2. order all r

$\Rightarrow$ multiply by possible arrangements

Example: Rank 3 from 7

A B C
A B D
⋮

each can happen 6 =3! ways

$$\Rightarrow \underbrace{\binom{n}{r}}_{1} * \underbrace{P\binom{r}{r}}_{2} = \frac{n!}{(n-r)!\,\cancel{r!}} * \cancel{r!} = P\binom{n}{r}$$

$$\Rightarrow P\binom{n}{r} = \binom{n}{r} r!$$

Examples

1. Determining ordered vs unordered.

   Find the number of ways to do each of the following.

   (a) Rank your favorite 4 desserts from the menu of 10 items.

   order ✓ $\Rightarrow P\binom{10}{4}$

   (b) Select which 3 side dishes to serve out of the 15 from your cookbook.

   order ✗ $\Rightarrow \binom{15}{3}$

   (c) Determine the jobs for 3 members out of 8 at the dinner party: set the table, serve the food, do the dishes.

   $\Rightarrow P\binom{8}{3}$

   meaning to slots
   $\Rightarrow$ order ✓   table   serve   dishes

   - In some problems involving ordering, the ordering is not obvious or implied, but rather implicit (like when making an assignment list).

     BEST way to think about it: If the "slots" have meaning, then order matters.

2. Combined problems

   A company has 20 male employees and 30 female employees. They are forming a committee that will have two male members and three female members. In how many ways can this committee be chosen?

   order ✗

   $\binom{20}{2} \times \binom{30}{3}$

   males   females

   Choose: 2 from 20   3 from 30

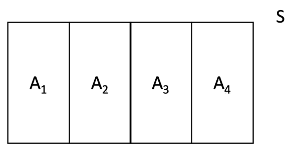
   - Many counting problems include combined the use of the multiplication principle, permutations and combinations.

     So just separate a scenario into tasks, count each task individually and then multiply each tasks total ways to get the total overall number of ways.

More than two groups

- **Partitioning** refers to the process of breaking a large group into separate smaller groups.

  Will learn how to count number of ways to divide all available objects:



  The combination problems previously discussed are simple examples of partitioning problems.

- Example: Flip 5 coins. How many observation sequences are there in which there are two heads and three tails?

  order X $\Rightarrow \binom{5}{2}$

  H H T T T

  T T T H H

  ⋮

- The basic idea of a combination divides n distinct objects into two groups: a group of chosen objects and a group of unchosen objects.

  This is why $$\binom{n}{r} = \binom{n}{n-r}$$ is called a **binomial coefficient**.

- This can be extended to more than two groups.

  Example: There are 10 students. How many ways can we make three groups with sizes [redacted] 3, 3 and [redacted] 4.

  $$\underbrace{\binom{10}{3}}_{\text{3 from 10}} \times \underbrace{\binom{7}{3}}_{\text{3 from 7 left}} = \frac{10!}{3!\cancel{7!}} \times \frac{\cancel{7!}}{3!\,4!} = \frac{10!}{3!\,3!\,4!}$$

  choose:

  group sizes

- **Counting partitions**: The number of partitions of n objects into k distinct groups of sizes $n_1, n_2, \ldots, n_k$ (where $n_1 + \cdots + n_k = n \Longleftrightarrow$ splitting up entire group) is given by:

  $$\binom{n}{n_1,\ n_2,\ \ldots,\ n_k} = \frac{n!}{n_1!\, n_2! \cdots n_k!}$$

  This is called the **multinomial coefficient**.

OR partition of Positions → $\underset{1}{M}\ \underset{2}{I}\ \underset{3}{S}\ \underset{4}{S}\ \underset{5}{I}\ \underset{6}{S}\ \underset{7}{S}\ \underset{8}{I}\ \underset{9}{P}\ \underset{10}{P}\ \underset{11}{I}$ → $S = \{1, 2, \dots, 11\}$

MISSISSIPPI $= \underset{M}{\{1\}}\ \underset{I}{\{2, 5, 8, 11\}}\ \underset{S}{\{3, 4, 6, 7\}}\ \underset{P}{\{9, 10\}}$

IPPISSISSIM $= \underset{M}{\{11\}}\ \underset{I}{\{1, 4, 7, 10\}}\ \underset{S}{\{5, 6, 8, 9\}}\ \underset{P}{\{2, 3\}}$

- Example: Find the number of ways to rearrange the letters in the word MISSISSIPPI.

$$\frac{\text{total } \#}{\text{dups}} = \frac{11!}{1!\,4!\,4!\,2!} = \binom{11}{1, 4, 4, 2}$$

↳ M → 1, I → 4, S → 4, P → 2; total 11

- Counting partitions can also be thought of as the number of ways to arrange n objects where n_1 objects are alike, n_2 objects are alike and so forth.

  To account for the repetitions when counting distinct permutations (arrangements), we need to divide out the duplicates.

☆ It is the groups that matter, not the order within the groups.

Summary: When to use which counting tool (formula)

- **Ordered**
  - **With replacement**: n^r ex) how many 6 digit passwords can the digits 0-9 make?
  - **Without replacement**: $P(n, r)$ ex) 7 possible vacation destinations, rank your top 3.
- **Unordered**
  - **Without replacement**: $\binom{n}{r}$ ex) select 5 out of 12 players to start the basketball game.

    More than two groups: $\binom{n}{n_1, \dots, n_k}$ ex) make smaller teams of size 2, 3, and 3 from 8 players.
  - With replacement: This exists, but we won't worry about it.