MATH 320: Probability

Lecture 1: Set Theory

Chapter 1: Probability (1.1)

Where does data come from

Definitions

- An **experiment** is the process by which an observation/outcome is made, which cannot be predicted with certainty (outcomes are random).
- An **outcome** of an experiment is any possible observation of that experiment (often called sample points).

Examples: Write the set of all possible outcomes for the following experiments.

1. Sampling students and computing the average number of study hours each day:
   $S = [0, 24] \rightarrow$ Continuous, uncountable
2. Roll die, record number that appears:
   $S = \{1, 2, 3, 4, 5, 6\} \rightarrow$ Discrete , countable
3. Roll die, record first role that a one appears:
   $S = \{1, 2, 3, \ldots\} \rightarrow$ Countable , infinite

What is probability and how to calculate it

Intuitively, probability is the likelihood of something occurring.

One approach (simplest case)

- **Probability by counting equally likely outcomes**

  $$\text{Probability of an event} = \frac{\#\ \text{outcomes in event}}{\text{total \# possible outcomes}}$$

  – Example: Flip a fair coin, $P(\text{Heads}) = \frac{1 \text{ success}}{2 \text{ possibilities}}$

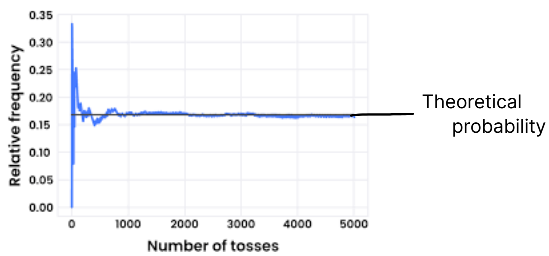
- Events are not always equally likely, so cannot determine probabilities by counting. But there is a simple way to estimate that probability.

Another approach

- **Empirical probability** (based on collected data).
- Relative frequency estimate of the probability of an event

  Probability of an event = $\dfrac{\#\ \text{times event occurs}}{\#\ \text{of trials}}$

- These are two ways of looking at probability.
  - If outcomes are equally likely, *relative frequency* $\approx$ *counting* for a very large number of trials (e.g. if we simulated rolling a fair die 10,000 times, $\frac{\#\,6s}{10{,}000} \approx \frac{1}{6}$).



Third approach

- **Subjective probability** is asking a well-informed person for his/her personal estimate of the probability of an event (relying on experience and personal recollections of relative frequencies in the past).
- The rest of this chapter will be building in more precise mathematical framework for probability.
- Counting will play a big role, but keep in mind that in practice many probability numbers actually used in calculations may come from relative frequencies or subjective estimates.

Set Theory

The mathematical basis of probability is sets. Set theory is useful here to provide a precise language for dealing with the outcomes in a probability experiment.

Definitions

- A **set** is a collection of objects (such as the numbers 1, 2, 3, 4, 5, 6).
  - The objects are called **elements** of the set. Outcomes of an experiment correspond to elements in a set.

– Writing sets: **Notation is important!**

We use capital letters to denote sets such as A, B, C.

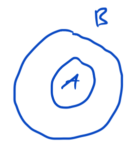
Can list the elements in braces if only a few, or use set-builder notation for large or infinite sets.

– Example: All positive numbers can be written as

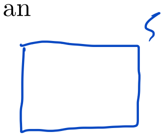
"A = the set of all x's such that (condition) $x > 0$"

$A = \{x : x > 0\}$

- **Subset** $A \subset B$ means that every element in A is also an element of B.



- The **sample space** S (aka outcome space) is the set of all outcomes of an experiment.



  – There are different types of sample spaces.

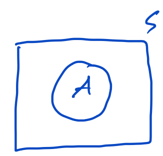
  – Countable or uncountable and if numeric: discrete or continuous.

- An **event** is a collection of possible outcomes of an experiment, that is, any subset of S.

  1. Examples: Roll die, record the number that appears.

     – Define notation: $A =$ roll even #

     – Show event: $A = \{2, 4, 6\} \subset S = \{1, 2, 3, 4, 5, 6\}$



     An event occurs when at least one element in the event has occurred.

  2. Is S an event? yes! $S \subseteq S$

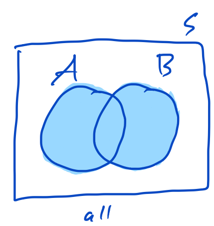
- The **null (empty) set** is the set containing no elements. $\emptyset = \{\ \}$

Basic operations (algebra of sets)

compound events

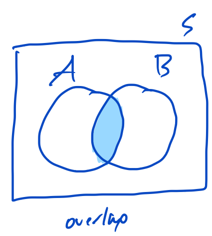
- **Union**: $A \cup B =$ "A or B"

  The set of elements that belong to either A or B or both



- **Intersection**: $A \cap B =$ "A and B"

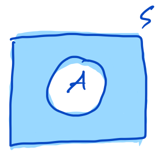
  The set of elements that belong to both A and B



- **Complement**: $\sim A$ (or A', A^c, $\bar{A}$) = "complement of A", "Not A"

  The set of elements (in the sample space) that are not in A

  So removing all elements of A from the original sample space S.



- Examples: Let $S = \{1, 2, 3, 4, 5, 6\}$ and $A_1 = \{1\}$, $A_2 = \{2, 3, 4\}$, $A_3 = \{4, 5, 6\}$.
  Find each of the following events:

$$A_1 \cup A_2 = \{1, 2, 3, 4\} \qquad A_2 \cup A_3 = \{2, 3, 4, 5, 6\}$$

$$A_1 \cap A_2 = \{\ \} = \emptyset \qquad A_2 \cap A_3 = \{4\}$$

$$\sim A_2 = \{1, 5, 6\} \qquad \sim(A_1 \cup A_2) = \sim\{2, 3, 4, 5, 6\} = \{1\}$$

Set identities

- These laws help simplify (rewrite) events that are stated verbally or in set notation when solving problems.

- **Commutative Law**: Reordering — Order doesn't matter

$$A \cup B = B \cup A \qquad a + b = b + a$$

$$A \cap B = B \cap A$$

- **Associative Law**:

  Changing location of parentheses — Order of operations with ( ) doesn't matter

$$(A \cup B) \cup C = A \cup (B \cup C) \qquad (a + b) + c = a + (b + c)$$

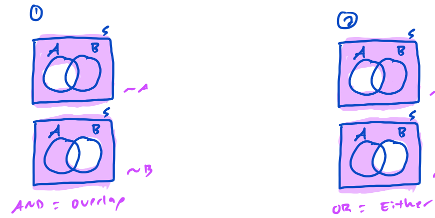
$$(A \cap B) \cap C = A \cap (B \cap C)$$

- **Distributive Law**: Distribute set operation — Distribute to items in ()

$$a(b + c) = ab + ac$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

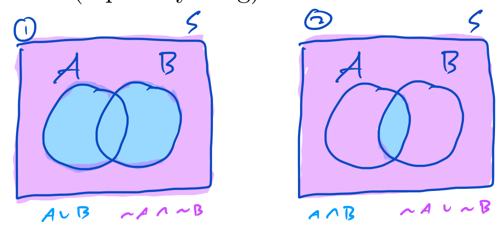
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$



- **De Morgan's Law**: Distributing complement (flip everything)

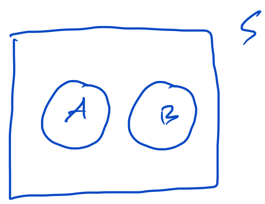
  ① $\sim(A \cup B) \quad = \quad \sim A \cap \sim B$

  ② $\sim(A \cap B) \quad = \quad \sim A \cup \sim B$



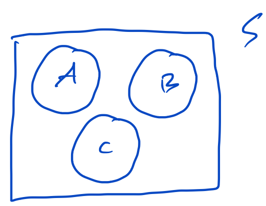
Relationships among sets

- Definitions
  - Two events are **mutually exclusive** (or **disjoint**) if No overlap $A \cap B = \emptyset$.



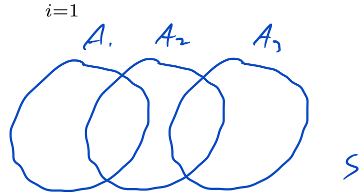
  - The events $A_1, A_2, \ldots$ are **pairwise mutually exclusive** if every pair is mutually exclusive.

    Formally, the condition is stated as: if $A_i \cap A_j = \emptyset$ for all $i \neq j$.



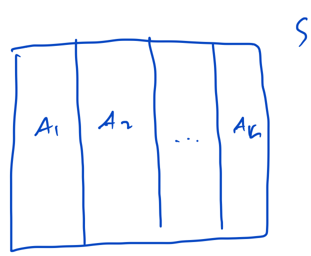
  - Events $A_1, \ldots, A_k$ are **exhaustive** if when combined, they form entire sample space.

    Formally, this is written as: $\bigcup_{i=1}^{k} A_i = A_1 \cup \cdots \cup A_k = S$



  - Events $A_1, \ldots, A_k$ form a **partition** of S if they are exhaustive and pairwise mutually exclusive.

    complete S   no overlap



- Examples: Given the sample space and events below, classify the relationship between events.

  $S = \{1, 2, 3, 4, 5, 6\}$ and $A_1 = \{1\}$, $A_2 = \{2, 3, 4\}$, $A_3 = \{4, 5, 6\}$, $A_4 = \{5, 6\}$.

  a. A_1 and A_2 are Mutually exclusive $\rightarrow$ No common elements

  b. A_2 and A_3 are not mutually exclusive → share element $\{4\}$

  c. A_1, A_2, A_3 are exhaustive → Completes

  but are Not a partition → not pairwise Mutually exclusive

  d. A_1, A_2, A_4 are a partition of S

  To check: 1. exhaustive $\rightarrow$ $A_1 \cup A_2 \cup A_3 = \{1\} \cup \{2,3,4\} \cup \{5,6\} \overset{\checkmark}{=} S$

  2. pairwise mutually exclusive $\rightarrow$ $A_1 \cap A_2 = A_1 \cap A_4 = A_2 \cap A_4 \overset{\checkmark}{=} \emptyset$