Name:

MATH 321: Homework 1

Due _____: Turn in a hard copy, neat and stapled.

1. Entrance exam scores of students at a certain college, X, are normally distributed with $X \sim \mathrm{N}(\mu = 1300, \sigma^2 = 2500)$. Suppose an admissions officer selects a random sample of 30 students at this college and checks their entrance exam scores.

   (a) If $\bar{X}$ is the sample mean of the 30 exam scores, find the distribution of $\bar{X}$.

   (b) Find $P(1290 \leq \bar{X} \leq 1310)$.

   (c) Find $P(X \leq 1230)$.

   (d) Let Y be the number of random variables (exam scores) in the sample that have values of at most 1230. Find the probability that less than 5 of the random variables in the sample have scores of at most 1230, that is, $P(Y < 5)$.

   HINT: Think of $P(X \leq 1230)$ as a success probability, and the result of checking this is for each random variable is either a success or failure.

   (e) Let S^2 be the sample variance of the 30 exam scores. Find $P(S^2 > 2000)$.

   HINT: Think about how to get the random variable of interest to follow a distribution we know.

2. Let $X_1 \sim \chi^2(10)$ and $X_2 \sim \chi^2(13)$ and $X_1 \perp\!\!\!\perp X_2$.

   (a) Let $Y_1 = \dfrac{X_1/10}{X_2/13}$. Find the distribution of Y_1 and $P(Y_1 > 5)$.

   (b) Let $Y_2 = 1/Y_1$. Find the distribution of Y_2 and the IQR of Y_2.

3. Suppose we take independent random samples of sizes $n_1 = 6$ and $n_2 = 10$ from two normal populations with equal population variances ($\sigma_1^2 = \sigma_2^2$). Let S_1^2 and S_2^2 be the sample variances from populations 1 and 2, respectively.

   Find $P(S_1^2/S_2^2 > 2)$.

4. **R simulation**: We are going to simulate a sampling distribution for a statistic and estimate some probabilities using empirical methods.

   Guidelines: Complete each of the following steps for the following pairs of distribution and statistic: 1) Gamma distribution and median and 2) Your choice (be creative!). For example, Binomial / sample variance or χ^2 / minimum.

   (a) Plot the population distribution of interest that you will be sampling from.

   (b) Generate $i = 10,000$ random samples of size $n = 30$ from the population distribution.

   (c) Calculate the sample statistic $Y = T(X_1, \ldots, X_{30})$ for each of the random samples.

   (d) Plot a histogram of the simulated sampling distribution of Y.

   (e) Calculate $\hat{\mu}_Y$ and $\hat{\sigma}_Y$, the estimated mean and standard deviation of the sampling distribution of the sample statistic based on the simulated results, respectively.

   (f) Calculate the estimated probability the sample statistic is within two standard deviations of its mean:
   $P(\hat{\mu}_Y - 2\hat{\sigma}_Y < Y < \hat{\mu}_Y + 2\hat{\sigma}_Y)$.

   Restrictions: Do not use the normal distribution or the sample mean $\bar{X}$.

   Submission: This problem will be worth 15 of the 30 points. Please submit your completed version of the starter .qmd file on canvas and rendered .html file.

Select answers

1. (a)
   (b) $P(1290 \leq \bar{X} \leq 1310) \approx 0.7267$
   (c) $P(X \leq 1230) \approx 0.0808$
   (d) $P(Y < 5) \approx 0.9097$
   (e) $P(S^2 > 2000) \approx 0.7673$
2. (a) $P(Y_1 > 5) \approx 0.0042$
   (b) IQR of $Y_2 \approx 0.8639$
3. $P(S_1^2/S_2^2 > 2) \approx 0.1727$
4.