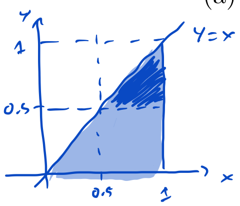
Name: KEY

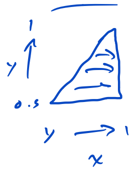
MATH 321: Review Part 1

1. Let (X, Y) be a bivariate continuous random vector with joint pdf $f(x, y) = 3x \quad \text{for } 0 \le y \le x \le 1.$

   (a) Find $P(X \ge 0.5, Y \ge 0.5)$.

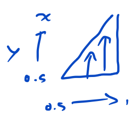


Way 1



$$\int_{0.5}^{1}\left[\int_{y}^{1} 3x\,dx\right]dy = \int_{0.5}^{1}\left[\frac{3}{2}x^2\right]_y^1 dy$$
$$= \int_{0.5}^{1} \frac{3}{2}(1-y^2)\,dy$$
$$= \frac{3}{2}y - \frac{1}{2}y^3\Big|_{0.5}^{1} = \boxed{0.3125 = 5/16}$$

Way 2



$$\int_{0.5}^{1}\left\{\int_{0.5}^{x} 3x\,dy\right\}dx = \int_{0.5}^{1}\left[3xy\right]_{0.5}^{x} dx$$
$$= \int_{0.5}^{1}\left(3x^2 - \frac{3}{2}x\right)dx$$
$$= x^3 - \frac{3}{4}x^2\Big|_{0.5}^{1} = \boxed{0.312}$$

   (b) Show if $X \perp\!\!\!\perp Y$ using the definition.

↳ True if $f(x)f(y) = f(x,y)$

→ $f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^x 3x\,dy = 3xy\Big|_0^x = 3x^2, \quad 0 \le x \le 1$

$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_y^1 3x\,dx = \frac{3}{2}x^2\Big|_y^1 = \frac{3}{2}(1-y^2), \quad 0 \le y \le 1$

→ $f(x)f(y) \overset{?}{=} f(x,y)$

$3x^2\left[\frac{3}{2}(1-y^2)\right] \ne 3x \implies \boxed{X \not\perp\!\!\!\perp Y}$

   (c) Find the conditional pdf $f(x \mid y)$. & $f(y \mid x)$

for $0 \le y \le 1$ → $f(x \mid y) = \dfrac{f(x,y)}{f(y)} = \dfrac{3x}{\frac{3}{2}(1-y^2)} = \boxed{\dfrac{2x}{(1-y^2)}, \quad y \le x \le 1}$

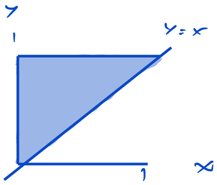
for $0 \le x \le 1$ → $f(y \mid x) = \dfrac{f(x,y)}{f(x)} = \dfrac{3x}{3x^2} = \boxed{\dfrac{1}{x}, \quad 0 \le y \le x}$

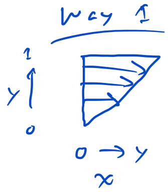
   (d) Using result from (c), find $E(X \mid Y = 0.5)$.

$$E(X \mid Y=0.5) = \int_{-\infty}^{\infty} x\, \underbrace{f(x \mid y=0.5)}_{\frac{2x}{(1-0.5^2)} = \frac{8}{3}x}\,dx = \int_{0.5}^{1} x\left[\frac{8}{3}x\right]dx = \int_{0.5}^{1} \frac{8}{3}x^2\,dx = \frac{8}{9}x^3\Big|_{0.5}^{1} = \frac{8}{9}\left(1 - \frac{1}{8}\right) = \boxed{7/9}$$

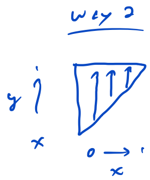
2. Let (X, Y) be a bivariate continuous random vector with joint pdf
$f(x, y) = 4xy \quad$ for $0 \leq x \leq 1, 0 \leq y \leq 1$.

(a) Find $P(Y \geq X)$.





$$\int_0^1 \left[\int_0^y 4xy\,dx\right]dy = \int_0^1 \left[2yx^2\right]_0^y dy$$
$$= \int_0^1 2y^3\,dy$$
$$= \frac{1}{2} y^4 \Big|_0^1 = \boxed{1/2}$$

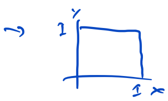


$$\int_0^1 \left[\int_x^1 4xy\,dy\right]dx = \int_0^1 \left[2xy^2\right]_x^1 dx$$
$$= \int_0^1 (2x - 2x^3)\,dx$$
$$= x^2 - \frac{1}{2}x^4 \Big|_0^1 = \boxed{1/2}$$

(b) Show if $X \perp\!\!\!\perp Y$ by inspection.

→ $f(x,y) = \underbrace{4x}_{g(x)}\underbrace{y}_{h(y)}$ ⇒ separable ✓

→ rectangular range ✓



$\Rightarrow \boxed{X \perp\!\!\!\perp Y}$

(c) Using result from (b), find $E(X^4 Y)$

→ $E(x^4 y) = E(x^4)\,E(y) = \frac{1}{3}\left(\frac{2}{3}\right) = \boxed{\frac{2}{9}}$

< b/c $X \perp\!\!\!\perp Y$ >

→ $f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 4xy\,dy = 2xy^2\Big|_0^1 = 2x$, $0 \le x \le 1$ ⇒ $E(x^4) = \int_{-\infty}^{\infty} x^4 f(x)\,dx = \int_0^1 x^4(2x)\,dx = \frac{1}{3}x^6\Big|_0^1 = 1/3$

→ $f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 4xy\,dx = 2yx^2\Big|_0^1 = 2y$, $0 \le y \le 1$ ⇒ $E(y) = \int_{-\infty}^{\infty} y f(y)\,dy = \int_0^1 y(2y)\,dy = \frac{2}{3}y^3\Big|_0^1 = 2/3$

3. Let (X, Y) be a bivariate discrete random vector with joint pmf table:

$y \backslash x$	0	1
0	2/9	3/9
1	2/9	1/9
2	1/9	0

(a) Find the following probabilities: $P(X \leq 1, Y = 0)$ ①, $P(X + 1 \leq Y)$ ②, and $P(Y^2 = X)$ ③.

① $P(\{(0,0), (1,0)\}) = \frac{2}{9} + \frac{3}{9} = \boxed{\frac{5}{9}}$

② $P(\{(0,1), (0,2), (1,2)\}) = \frac{2}{9} + \frac{1}{9} + 0 = \boxed{1/3}$

③ $P(\{(0,0), (1,1)\}) = \frac{2}{9} + \frac{1}{9} = \boxed{1/3}$

(b) Find the marginal pmfs of X and Y. Also find the conditional pmfs of $f(x \mid Y = 0)$ and $f(x \mid Y = 1)$.

$$\boxed{f(x) = \begin{cases} 5/9 & x=0 \\ 4/9 & x=1 \end{cases}}$$

$$\boxed{f(y) = \begin{cases} 5/9 & y=0 \\ 3/9 & y=1 \\ 1/9 & y=2 \end{cases}}$$

$$\boxed{f(x \mid y=0) = \frac{f(x,y)}{\underbrace{f(Y=0)}_{5/9}} = \begin{cases} \frac{2/9}{5/9} = \frac{2}{5} & x=0 \\ \frac{3/9}{5/9} = 3/5 & x=1 \end{cases}}$$

$$\boxed{f(x \mid y=1) = \frac{f(x,y)}{\underbrace{f(Y=1)}_{3/9}} = \begin{cases} \frac{2/9}{3/9} = \frac{2}{3} & x=0 \\ \frac{1/9}{3/9} = 1/3 & x=1 \end{cases}}$$

(c) Find the following: [redacted], $E(X \mid Y = 1)$, and [redacted]. $E(X+1 \mid Y=0)$

$$E(X \mid Y=1) = \sum_{x=0}^{1} x f(x \mid y=1) = 0\left(\tfrac{2}{3}\right) + 1\left(\tfrac{1}{3}\right) = \boxed{1/3}$$

$$E(X+1 \mid Y=0) = 1 + E(x \mid Y=1) = 1 + \left[0(2/5) + 1(3/5)\right] = \boxed{8/5}$$

(d) Find $\mathrm{Cov}(X, Y)$. $= E(XY) - E(X)E(Y) = 1/9 - 4/9\,(5/9) = \boxed{-0.1358}$

By hand

$\rightarrow E(XY) = \sum_{x=0}^{1} \sum_{y=0}^{2} xy f(x,y) = 1/9$

$\rightarrow$ * Find marginals OR use joint

$E(X) = \sum_{x=0}^{1} x f(x) = \sum_{x=0}^{1} \sum_{y=0}^{2} x f(x,y) = 4/9$

$E(Y) = \sum_{y=0}^{2} y f(y) = \sum_{y=0}^{2} \sum_{x=0}^{1} y f(x,y) = 5/9$

using calc

① Enter data

$L_1 = x$	$L_2 = y$	$L_3 = f(x,y)$
0	0	2/9
1	0	3/9
0	1	2/9
1	1	1/9
0	2	1/9
1	2	0

② First step — check $n=1$

1 var stats

$\rightarrow$ Find $E(X) \rightarrow L_1 = X,\ L_3 = f(X,Y) \rightarrow 4/9$

$E(Y) \rightarrow L_1 = Y,\ L_3 = f(X,Y) \rightarrow 5/9$

$\boxed{\text{using joint pmf} \rightarrow E(X) = \sum\sum x f(x,y)}$

Second step

one way $\rightarrow$ Find $E(XY)$ & do $E(XY) - E(X)E(Y) = \boxed{-0.1358}$

1 var stats

$E(XY) \rightarrow L_4 = XY = L_1 * L_2,\ L_3 = f(X,Y) \rightarrow 1/9$

another way $\rightarrow$ use $\mathrm{Cov} = E[(X-\mu_x)(Y-\mu_y)]$ def

↳ 1 var stats

$L_4 = (X - E(X))(Y - E(Y)),\ L_3 = f(X,Y) \rightarrow \boxed{-0.1358}$

$(L_1 - 4/9)(L_2 - 5/9)$

(e) Find $\mathrm{Corr}(X, Y)$.

$$\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sigma_x \sigma_y} \approx \frac{-0.1358}{0.4964\,(0.6849)} = \boxed{-0.399}$$

↳ From 1-var stats

(f) Find $V(X + Y)$.

$$V(X+Y) = V(X) + V(Y) + 2\,\mathrm{Cov}(X,Y)$$

$$= 0.4696^2 + 0.6849^2 + 2(-0.1358)$$

$$\boxed{\approx 0.444}$$