Regression examples of matrices

Response vector

$$\rightarrow \ Y_{n\times 1} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}, \quad Y'_{1\times n} = \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_n \end{bmatrix}$$

design matrix

$$\rightarrow \ X_{n\times 2} = \begin{bmatrix} 1 & X_1 \\ 1 & X_2 \\ \vdots & \vdots \\ 1 & X_n \end{bmatrix}, \quad X'_{2\times n} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ X_1 & X_2 & \cdots & X_n \end{bmatrix}$$

mean responses

$$\rightarrow \ E(Y)_{n\times 1} = \begin{bmatrix} E(Y_1) \\ E(Y_2) \\ \vdots \\ E(Y_n) \end{bmatrix}$$

Error terms

$$\rightarrow \ \varepsilon_{n\times 1} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

Regression model

$$\rightarrow \ \underset{n\times 1}{Y} = \underset{n\times 1}{E(Y)} + \underset{n\times 1}{\varepsilon}$$

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} E(Y_1) \\ E(Y_2) \\ \vdots \\ E(Y_n) \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix} = \begin{bmatrix} E(Y_1) + \varepsilon_1 \\ E(Y_2) + \varepsilon_2 \\ \vdots \\ E(Y_n) + \varepsilon_n \end{bmatrix}$$

"squared" terms

$$\rightarrow \ Y'Y = \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_n \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} Y_1^2 + Y_2^2 + \cdots + Y_n^2 \end{bmatrix} = \begin{bmatrix} \sum Y_i^2 \end{bmatrix}_{1\times 1}$$

$$\rightarrow \ X'X = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ X_1 & X_2 & \cdots & X_n \end{bmatrix} \begin{bmatrix} 1 & X_1 \\ 1 & X_2 \\ \vdots & \vdots \\ 1 & X_n \end{bmatrix} = \begin{bmatrix} n & \sum X_i \\ \sum X_i & \sum X_i^2 \end{bmatrix}_{2\times 2}$$

$$\rightarrow \ X'Y = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ X_1 & X_2 & \cdots & X_n \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} \sum Y_i \\ \sum X_i Y_i \end{bmatrix}_{2\times 1}$$

Inverse

$$\rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\rightarrow \ (X'X)^{-1} = \begin{bmatrix} n & \sum X_i \\ \sum X_i & \sum X_i^2 \end{bmatrix}^{-1} = \frac{1}{n\sum X_i^2 - (\sum X_i)^2} \begin{bmatrix} \sum X_i^2 & -\sum X_i \\ -\sum X_i & n \end{bmatrix}$$

$$= n\left[\sum X_i^2 - \frac{(\sum X_i)^2}{n}\right]$$

$$= n\left[\sum X_i^2 - n\bar{X}^2\right] \longrightarrow \bar{X} = \frac{\sum X_i}{n} \Rightarrow \sum X_i = n\bar{X}$$

working backwards

$$\sum (X_i - \bar{X})^2$$
$$= \sum (X_i^2 - 2X_i\bar{X} + \bar{X}^2)$$
$$= \sum X_i^2 - 2\bar{X}\sum X_i + n\bar{X}^2$$
$$= \sum X_i^2 - 2n\bar{X}^2 + n\bar{X}^2$$
$$= \sum X_i^2 - n\bar{X}^2$$

$$= \frac{1}{n[\sum (X_i^2 - \bar{X})^2]} \begin{bmatrix} \sum X_i^2 & -\sum X_i \\ -\sum X_i & n \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\sum X_i^2}{n[\sum (X_i^2 - \bar{X})^2]} & \dfrac{-\sum X_i}{n[\sum (X_i^2 - \bar{X})^2]} \\ \dfrac{-\sum X_i}{n[\sum (X_i^2 - \bar{X})^2]} & \dfrac{1}{\sum (X_i^2 - \bar{X})^2} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1}{n} + \dfrac{\bar{X}^2}{\sum (X_i^2 - \bar{X})^2} & \dfrac{-\bar{X}}{\sum (X_i - \bar{X})^2} \\ \dfrac{-\bar{X}}{\sum (X_i - \bar{X})^2} & \dfrac{1}{\sum (X_i^2 - \bar{X})^2} \end{bmatrix}$$

< simplify after $\sum X_i = n\bar{X}$ +

$$\sum (X_i - \bar{X})^2 = \sum X_i^2 - n\bar{X}^2$$
$$\Rightarrow \sum X_i^2 = \sum (X_i - \bar{X})^2 + n\bar{X}^2 >$$

variance covariance matrix

$$\rightarrow \ \sigma^2\{Y\} = \begin{bmatrix} \sigma^2\{Y_1\} & \sigma\{Y_1, Y_2\} & \cdots & \sigma\{Y_1, Y_n\} \\ \sigma\{Y_2, Y_1\} & \sigma^2\{Y_2\} & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \sigma\{Y_n, Y_1\} & \vdots & \cdots & \sigma^2\{Y_n\} \end{bmatrix}_{n\times n}$$

Symmetric

$$= E\left[(Y - E(Y))(Y - E(Y))'\right]$$

$$= E\left[\begin{bmatrix} Y_1 - E(Y_1) \\ \vdots \\ Y_n - E(Y_n) \end{bmatrix} \begin{bmatrix} (Y_1 - E(Y_1)) & \cdots & (Y_n - E(Y_n)) \end{bmatrix}\right]$$

Random error variance covariance

$$\rightarrow \ \sigma^2\{\varepsilon\} = \begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ \vdots & \sigma^2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \sigma^2 \end{bmatrix}_{n\times n}$$

$$= \sigma^2 I_{n\times n}$$