Applications of Galois Theory

Monodromy Groups and Fuchsian Differential Equations

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Abstract

Lazarus Immanuel Fuchs (1833-1902) was a German mathematician and a leading theorist of differential equations. In an attempt "to impose upon the inchoate world of differential equations the conceptual order of the emerging theory of complex functions" Fuchs created the ground work for what would become Poincaré's later theory of automorphic functions. From the 1870s to the turn of the century, two strands in the study of differential equations were brought together: applications of particular solutions (Legendre's equation, hypergeometric functions) and more abstract existence theorems. Following Michio Kuga's (1928-1990) pedagogical treatment of this "intersection," I present a theoretical approach to the solutions of certain Fuchsian differential equations utilizing Galois theory and monodromy groups. I consider the relation of Kuga's method [2, 3.2] to physically meaningful differential equations. Specifically, I discuss alternative solutions to problems arising in mathematical physics.

Mathematical Background

We work in the space $D = \mathbf{C} \setminus \{0, 1\}$, i.e., the plane with two points removed. While D is connected, it is not simply connected. Notice there are (at least) two closed loops in D which cannot be continuously deformed to a point—the loop around 0 and the loop around 1. This intuitive fact is captured by the algebraic structure of the fundamental group $\pi_1(D)$. Indeed, $\pi_1(D)$ is the free group with two generators [2, wk. 6]. Figure 1 demonstrates that $\pi_1(D)$ is non-abelian.

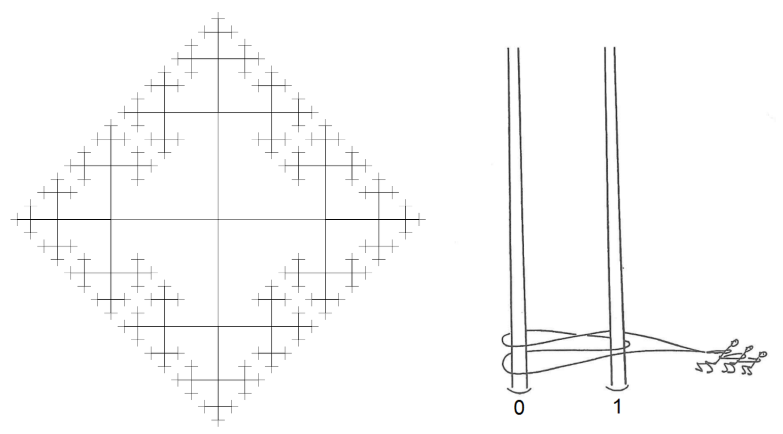


Figure 1: Cayley representation of $\pi_1(D)$, a non-commutative group

Because a loop around 0, around 1, back around 0 and then back around 1 is *not* null homotopic, we have the "branching" Cayley representation of $\pi_1(D)$.

Fortunately, there is a unique space, which naturally corresponds to D, where all closed loops null-homotopic. It is $\widetilde{D}$, the universal covering space of D.

Definition (universal covering space)**.** Given a space X, the *universal covering space* of X consists of a space $\widetilde{X}$ and a map $f : \widetilde{X} \to X$ such that $\widetilde{X}$ is simply connected and for each point $x \in X$ there is an open neighborhood U of x in X such that $f^{-1}(U)$ is a union of disjoint open sets each of which is mapped homeomorphically onto U by f.

Because $\widetilde{D}$ is simply connected we know [2, wk. 16]:

Theorem. *Let $\widetilde{D} \xrightarrow{z} D \subset \mathbf{C}$ be the universal covering of D with P and Q functions defined out of $\widetilde{D}$ (where there exist holomorphic functions P_0 and Q_0 on D such that $P_0 \circ z = P$ and $Q_0 \circ z = Q$). Let $\widetilde{p}_0$ be a point of $\widetilde{D}$ and α, β arbitrary points in $\mathbf{C}$. Then there is a unique holomorphic function $w(\widetilde{p})$ that satisfies both*

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2}(\widetilde{p}) + P(z)\frac{\mathrm{d}w}{\mathrm{d}z}(\widetilde{p}) + Q(z)w(\widetilde{p}) = 0 \qquad (\#)$$

and the initial conditions

$$w(\widetilde{p}_0) = \alpha, \frac{\mathrm{d}w(\widetilde{p}_0)}{\mathrm{d}z} = \beta.$$

Monodromy Representation

Let $V_\#$ denote the set of all solutions to $\#$, without considering the initial conditions. Then $V_\#$ is a two dimensional vector space over $\mathbf{C}$. Since D may not be simply connected, solutions in $V_\#$ may not be functions on D. In general, they are only functions on $\widetilde{D}$.

Denote the group of covering transformations $\Gamma(\widetilde{D} \xrightarrow{z} D)$ by Γ. The elements of this group are homeomorphisms from $\widetilde{D}$ unto itself [2, wk. 11]. Because $\widetilde{D}$ is the universal covering, the group Γ is isomorphic to $\pi_1(D)$ [2, wk. 13].

Let $K(\widetilde{D})$ be the field of meromorphic functions out of $\widetilde{D}$ onto $\mathbf{C}$. For each element $\gamma \in \Gamma$, we may define [2, wk. 15] the field endomorphism $\gamma^* : K(\widetilde{D}) \to K(\widetilde{D})$ by

$$\gamma^*(\psi) = \psi \circ \gamma \quad \text{for all} \quad \psi \in K(\widetilde{D}).$$

We must understand that γ^* acts *locally*. For $a \in \mathbf{C}$, let U_a be punctured open disk about a. Consider the class of curves γ which wind once around a in U_a. Then $\pi_1(U_a)$ is the cyclic group generated by γ. For the universal covering $\widetilde{U}_a \xrightarrow{z} U_a$, $\pi_1(U_a) \cong \Gamma(\widetilde{U}_a \xrightarrow{z} U_a)$. Identify γ as a covering transformation. It can be shown [2, wk. 18] that γ^* acts:

$$\gamma^*(\log(z-a)) = \log(z-a) + 2\pi i$$
$$\gamma^*((z-a)^\alpha) = e^{2\pi i\alpha}(z-a)^\alpha$$

Suppose that w is a solution to $\#$. Because w is meromorphic on $\widetilde{D}$, we know $w \in K(\widetilde{D})$. It can be shown [2, wk. 16] that $\gamma^* w$ is also a solution to $\#$. Whence it is known [2, wk. 16] that γ^* is a linear automorphism of $V_\#$.

Let $\mathcal{M}(\gamma)$ denote $(\gamma^{-1})^*$. The correspondence $\mathcal{M} : \gamma \mapsto \mathcal{M}(\gamma)$ from the group Γ to the group of linear automorphisms of $V_\#$ is a linear representation of Γ [2, wk. 16]. This representation is know as the *monodromy representation* of $\#$. When we fix a basis $[w_1, w_2]$ of $V_\#$, $\mathcal{M}(\gamma)$ can be expressed as a 2×2 matrix,

$$M(\gamma) = \begin{pmatrix} a(\gamma) & b(\gamma) \\ c(\gamma) & d(\gamma) \end{pmatrix}.$$

This matrix can be determined by

$$(\mathcal{M}(\gamma)w_1, \mathcal{M}(\gamma)w_2) = (w_1, w_2)\begin{pmatrix} a(\gamma) & b(\gamma) \\ c(\gamma) & d(\gamma) \end{pmatrix}.$$

The correspondence $M : \Gamma \ni \gamma \to M(\gamma) \in GL(2, \mathbf{C})$ is a matrix representation of Γ. We'll find $M(\gamma)$ around 0 and 1.

Hypergeometric Differential Equations

The hypergeometric differential equation (with $a, b, c \in \mathbf{R}$)

$$z(z-1)\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + [(a+b-1)z + c]\frac{\mathrm{d}w}{\mathrm{d}z} + abw = 0 \qquad (\bigstar)$$

is a *Fuchsian* differential equation (see either [1, 21.1] or [2, wk. 19]) as $\bigstar$ can be put in the form $\#$ by choosing

$$P(z) = \frac{c}{z} + \frac{a+b+1-c}{z-1},$$
$$Q(z) = \frac{-ab}{z} + \frac{ab}{z-1}.$$

The local exponents of $\bigstar$ (which we need) are found to be

0	1	∞
0	0	a
$1-c$	$c-a-b$	b

Table 1: Indicial Exponents [4, 1.4]

"A very effective way to study a new technique is to do some simple problems by hand in order to understand the process, and compare results with a computer solution" [1].

The First Example

Let $a = -1, b = 3/5, c = -2$; then $V_\bigstar$ is spanned by:

$$[w_1, w_2] = \left[z + \frac{10}{3}, \frac{15\left(z + \frac{10}{3}\right)(13z - 10)}{104(1-z)^{8/5}(3z+10)}\right]$$

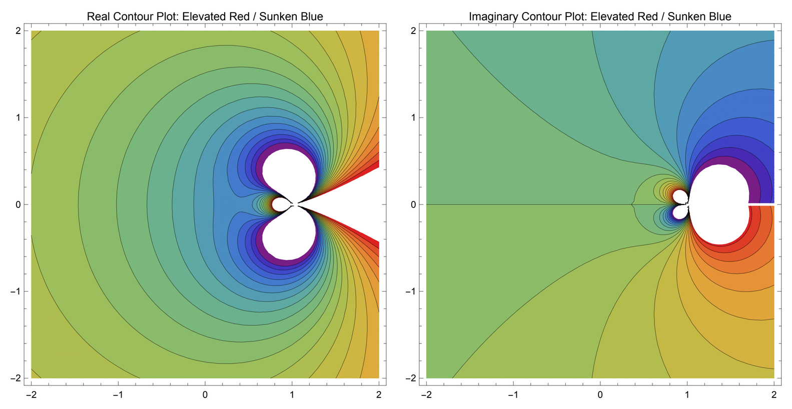
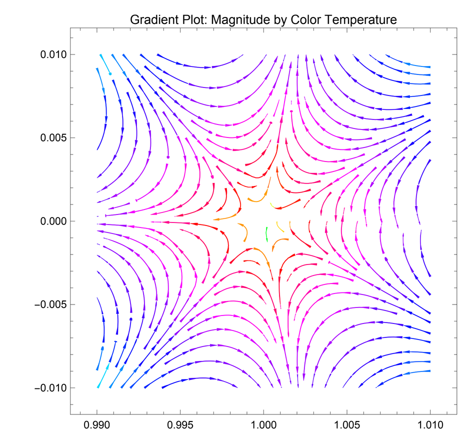


Figure 2: Re w_2 (Left) and Im w_2 (Right)



around zero	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
around one	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Figure 3: Streamlines of w_2 around 1 (see [1, p. 713] for definition of streamline); Generators of the monodromy group

The Second Example

Let $a = -3, b = 2, c = 4$; then $V_\bigstar$ is spanned by:

$$[w_1, w_2] = \left[\frac{(z-1)^5\left(z + \frac{1}{2}\right)}{z^3}, \frac{2(1-3z)\left(z + \frac{1}{2}\right)}{15z^3(2z+1)}\right]$$

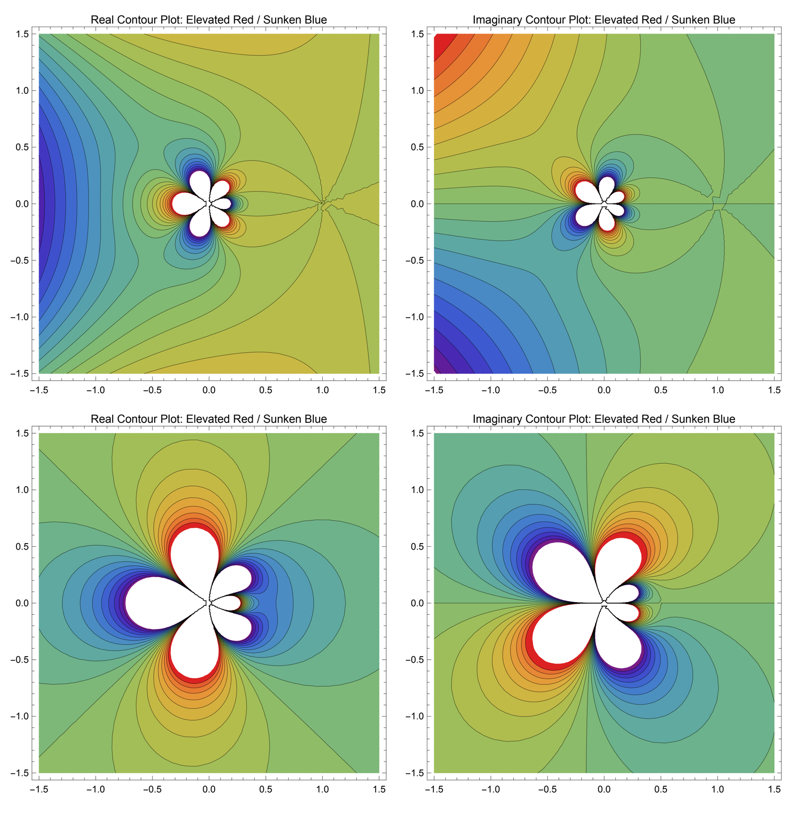
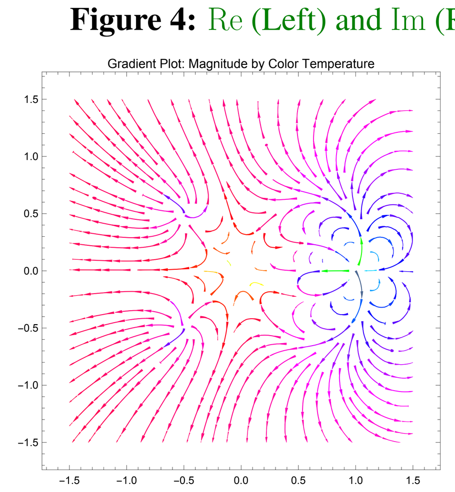


Figure 4: Re (Left) and Im (Right); w_1 (Top) and w_2 (Bottom)



around zero	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
around one	$\begin{pmatrix} 1 & 0 \\ 0 & e^{4\pi i/5} \end{pmatrix}$

Figure 5: Streamlines of w_1; Generators of the monodromy group

The Third Example

Let $a = -1/3, b = 6, c = 2/3$; then $V_\bigstar$ is spanned by:

$$[w_1, w_2] = \Bigg[z^{1/3}, \quad \frac{1}{3645}\Bigg(3640\sqrt[3]{z}\log\left(z^{2/3} + \sqrt[3]{z} + 1\right) - \frac{10905z}{z-1} + \frac{5445z}{(z-1)^2} - \frac{3105z}{(z-1)^3} + \frac{1701z}{(z-1)^4} - \frac{729z}{(z-1)^5} - 7280\sqrt[3]{z}\log\left(1 - \sqrt[3]{z}\right) - 7280\sqrt{3}\sqrt[3]{z}\tan^{-1}\left(\frac{2\sqrt[3]{z}+1}{\sqrt{3}}\right) - 10935\Bigg)\Bigg]$$

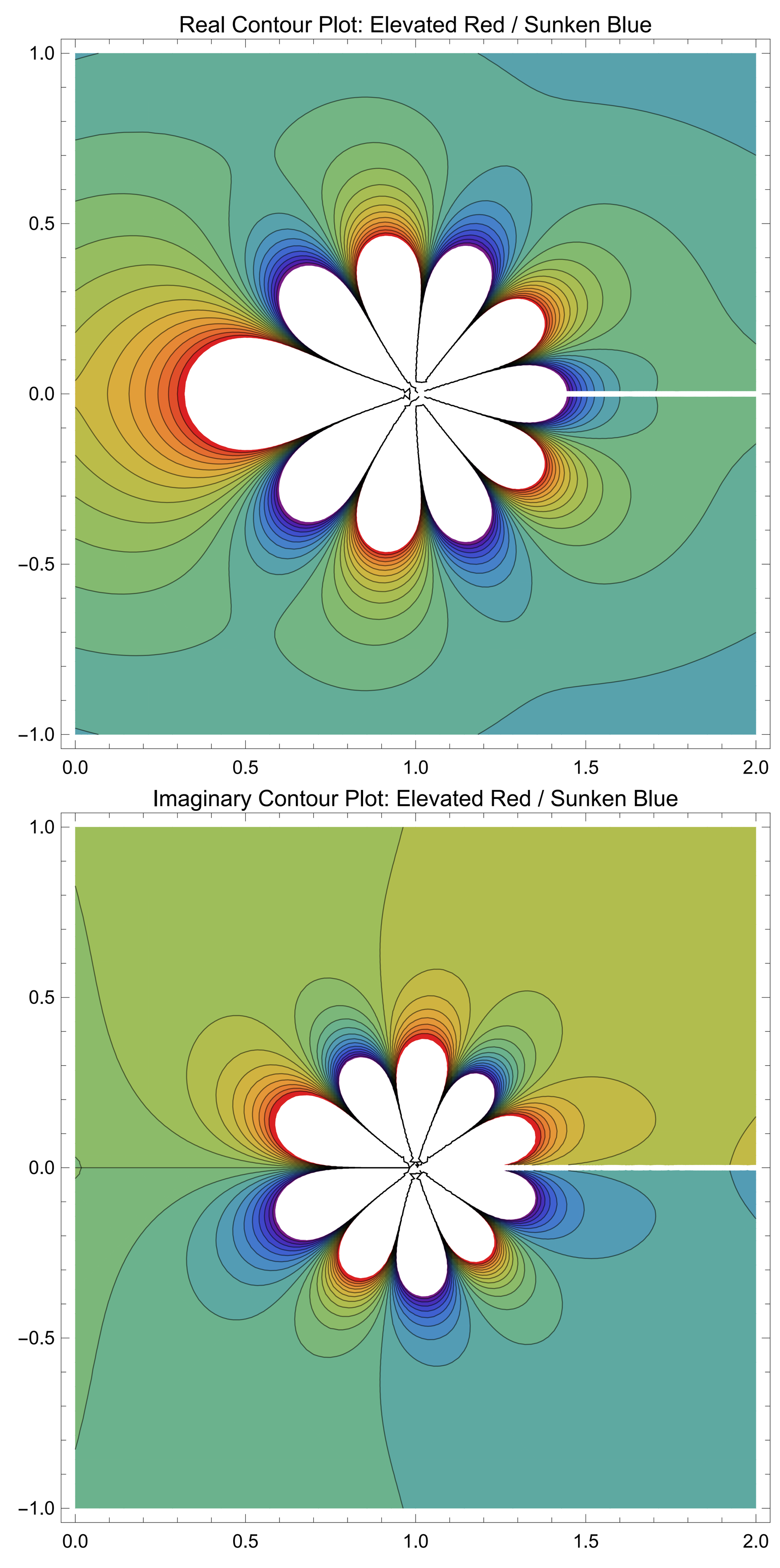


Figure 6: Re w_2 (Top) and Im w_2 (Bottom), around 1

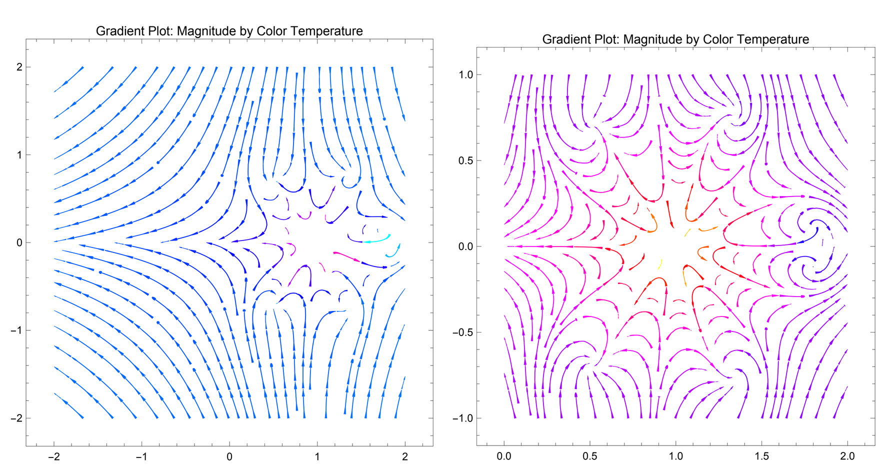


Figure 7: Streamlines of w_2, wide-view (Left), around 1 (Right)

around zero	$\begin{pmatrix} e^{2\pi i/3} & 1 \\ 0 & e^{2\pi i/3} \end{pmatrix}$
around one	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Figure 8: Generators of the monodromy group, Jordan form

Applications

Discussion of Fuchs's theorem in a real variable is given by Boas [1, ch. 12.21]. She remarks, "our main interest in series solutions is not to solve differential equations this way in general, but to study sets of functions. . . which are solutions of differential equations that occur in applications." In the reals, Fuchs's theorem identifies solutions $y(x)$ for

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + 2py = 0 \qquad \text{(the Hermite diffeq),}$$
$$x\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (1-x)\frac{\mathrm{d}y}{\mathrm{d}x} + py = 0 \qquad \text{(the Laguerre diffeq).}$$

Beyond the reals, a general knowledge of complex analysis is useful to a student of physics. The following theorem demonstrates [1, ch. 14.1].

Theorem. *If $f(z) = u + iv$ is analytic in a region, then u and v satisfy Laplace's equation in the region (that is, u and v are harmonic functions). Moreover, any function u (or v) satisfying Laplace's equation in a simply-connected region is the real or imaginary part of an analytic function $f(z)$.*

There are many systems in electrostatics, hydrodynamics and thermodynamics which obey Laplace's equation—in fact, outside of sources or sinks *all incompressible fluid flows do*. We have this result, almost by accident, through Clairaut and d'Alembert [3, ch. 12.5].

Pedagogy

The monodromy group is a mathematical object which begs for physical interpretation. Does it have any? Gauss's law for electric flux, $\oiint_S \mathbf{E} \cdot \mathrm{d}\mathbf{a} = q_{enc}/\varepsilon_0$, is an example of the relation between functional analysis and singular points (given point charges). Unfortunately, Gauss's law suffers from the description of the electric field as a *real* vector function $\mathbf{E}\colon \mathbf{R}^3 \to \mathbf{R}^3$. This form is prohibitive to students who wish to express the law as a complex function—especially because the conjugation required to produce the complex modulus is not an algebraic operation.

How might a mathematics-physics curriculum incorporate complex analysis and physical applications? Well, attention could be given to each of the two subjects' historical development. Early modern mathematicians such as the Bernoullis, d'Alembert, Lagrange and Laplace were often motivated by problems arising in mechanics [3, ch. 12-13]. Students may be able to stomach difficult mathematics in pursuit of some physically meaningful conclusion.

References

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Acknowledgments

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