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COMPLEX PROJECTIVE SPACE & MAPPINGS BETWEEN MANIFOLDS

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[1] Problem 1-9. *Given.* We assume the statement of Lee's smooth manifold chart lemma [1, No. 1.35]. We will compare this lemma to Do Carmo's definition of a differentiable manifold [2, Ch. 0.2].

To prove. Complex projective space $\mathbf{C}P^n$ is a smooth $2n$-dimensional manifold with structure generated by $n+1$ projective charts.

Proof.

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Consider the complex vect sp $\mathbb{C}^{n+1}$. Form the equiv relation on $\mathbb{C}^{n+1}$: $(z^1,\dots,z^{n+1}) \sim (\lambda z^1,\dots,\lambda z^{n+1})$ for any $\lambda \in \mathbb{C}^\times$. Let $\pi: \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}\setminus\{0\}/\sim \; := \mathbb{CP}^n$ be the nat'l projection; endow $\mathbb{CP}^n$ with the quotient topology. Observe when $z^i \neq 0$, we have a standard rep of an equiv class in $\mathbb{CP}^n$

$$[z^1 : \cdots : z^{n+1}] = \left[\frac{z^1}{z^i} : \cdots : \frac{z^{i-1}}{z^i} : 1 : \frac{z^{i+1}}{z^i} : \cdots : \frac{z^{n+1}}{z^i}\right].$$

Consider open subsets $V_1,\dots,V_{n+1}$ of $\mathbb{CP}^n$ defined $V_i := \{[z^1:\cdots:z^{n+1}] : z^i \neq 0\}$ for $i=1,\dots,n+1$ (open in $\mathbb{CP}^n$ because the hyperplane where $z^i=0$ is closed in $\mathbb{C}^{n+1}$ and $\mathbb{CP}^n$ has the quotient topology). Set up local coords on the V_i by letting $y^1 = \frac{z^1}{z^i}, \dots, y^{i-1} = \frac{z^{i-1}}{z^i}, y^i = \frac{z^{i+1}}{z^i}$ and $y^n = \frac{z^{n+1}}{z^i}$, then bijectively mapping V_i to $\mathbb{C}^n$ with the rule $[y^1 : \cdots : y^{i-1} : 1 : y^{i} : \cdots : y^n] \leftrightarrow (y^1,\dots,y^n)$ in $\mathbb{C}^n$.

Now for Lee's smooth mfld chart lemma.

(i) Define charts $\varphi_i : V_i \to \mathbb{C}^n$ by the rule. φ_i is clearly surj. and injectivity follows by the unique rep in V_i where $z_i = 1$.

(ii) Consider any two charts $(V_i,\varphi_i),(V_j,\varphi_j)$ and wlog say $i<j$. Then the image $\varphi_i(V_i\cap V_j) = \{(y^1,\dots,y^n)\in\mathbb{C}^n : y^j \neq 0\}$ is open in $\mathbb{C}^n$ as the hyperplane $y^j=0$ is cld.

(iii) Again say $i<j$ for arb charts φ_i,φ_j. Then $\varphi_i(V_i\cap V_j)$ of $\mathbb{C}^n$ is mapped along

$$(y^1,\dots,y^n) \xmapsto{\varphi_i^{-1}} [y^1:\cdots:y^{i-1}:1:y^i:\cdots:y^n]$$
$$= \left[\frac{y^1}{y^j}:\cdots:\frac{y^{j-1}}{y^j}:1:\frac{y^{j+1}}{y^j}:\cdots\frac{y^{i-1}}{y^j}:\frac{1}{y^j}:\frac{y^i}{y^j}\cdots\right]$$
$$\xmapsto{\varphi_j} \left(\frac{y^1}{y^j},\dots,\frac{y^{j-1}}{y^j},\hat{1},\frac{y^{j+1}}{y^j},\dots,\frac{y^n}{y^j}\right),$$

which is a dilation and permutation, so in particular a gen. linear transf of $\mathbb{C}^n$ and therefore smooth (real smooth).

(iv) $n+1$ of the V_i cover $M = \mathbb{CP}^n$

(v) Given any two distinct $[p],[q]\in\mathbb{CP}^n$, consider the intersection of the equiv classes in $\mathbb{C}^{n+1}$ with the surface of S^{2n+1}, a metric space. There are disjoint open sets (in S^{2n+1} w/ the subspace topology) which project to disjoint open saturated sets in $\mathbb{C}^{n+1}$ containing $[q]$ and $[p]$. So $\mathbb{CP}^n$ is Hsdf. □

By the smooth chart lemma, we conclude $\mathbb{CP}^n$ is a smooth $\mathbb{R}^{2n}$ mfld.

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[1] **Problem 1-11.** *Given.* The closed unit ball $\bar{\mathbf{B}}^n$ as a set of points in $\mathbf{R}^n$.

To prove. $\bar{\mathbf{B}}^n$ is a smooth n-dimensional manifold with boundary, which we can endow with a smooth structure such that:

- the interior $\mathbf{B}^n$ is parameterized by coordinates in the interior of the half space $\mathbf{H}^n$,
- the boundary $S^{n-1} = \partial\bar{\mathbf{B}}^n$ is parameterized by coordinates at the boundary of $\mathbf{H}^n$, and
- every interior chart is a chart for the standard structure on $\mathbf{B}^n$ as an open submanifold of $\mathbf{R}^n$.

Proof.

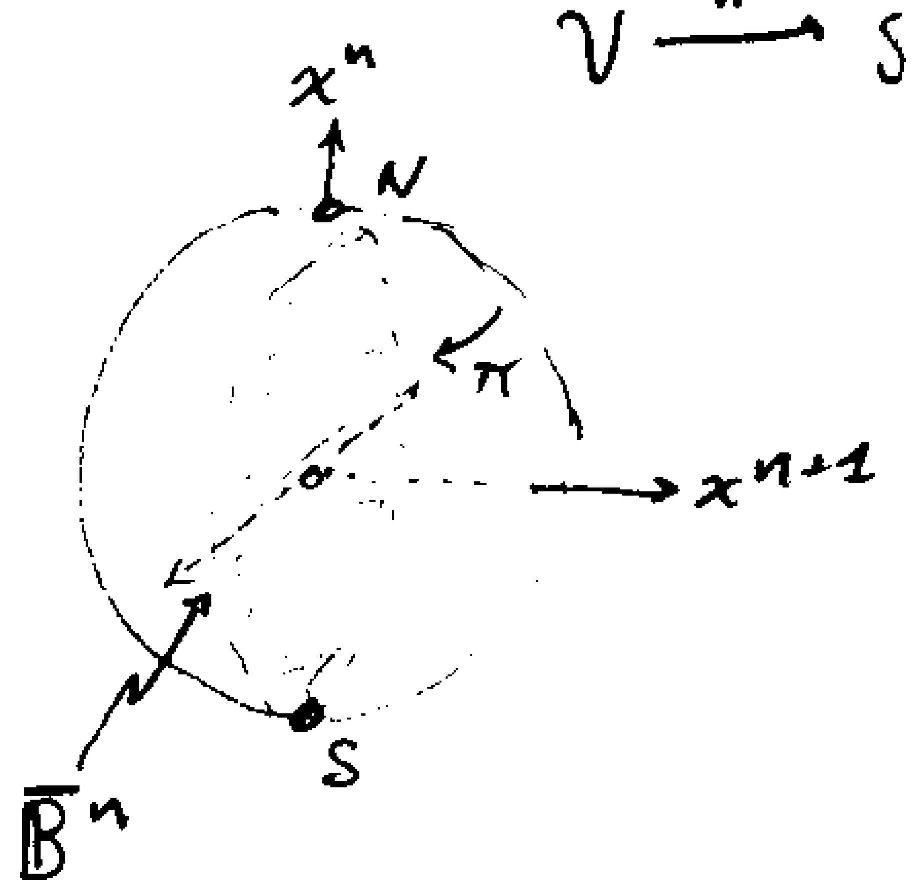
Say $\bar{\mathbb{B}}^n = \mathbb{B}^n \sqcup S^{n-1}$ and $\mathbb{H}^n = \{\mathbb{R}^n : x^n \geq 0\}$

Consider the two sets $\mathcal{U} = \bar{\mathbb{B}}^n \setminus \{N\}$ and $V = \bar{\mathbb{B}}^n \setminus \{S\}$, where $N = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $S = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$

Note that $\mathcal{U}$ and V form an open cover of $\bar{\mathbb{B}}^n$. Define charts as follows

$$\mathcal{U} \xrightarrow{\pi^{-1}} S^n \xrightarrow{\sigma} \mathbb{R}^n$$
$$V \xrightarrow{\pi^{-1}} S^n \xrightarrow{\tilde{\sigma}} \mathbb{R}^n$$

where π is the restriction of the projection $\mathbb{H}^{n+1} \longrightarrow \mathbb{R}^n$, $\begin{bmatrix} x' \\ x^{n+1} \end{bmatrix} \longmapsto \begin{bmatrix} x' \\ \hat{x}^{n+1} \end{bmatrix}$ to the closed half sphere $\{x \in S^n : x^{n+1} \geq 0\}$.



and σ, $\tilde{\sigma}$ resp. are non-standard stereographic projections from S^n to $\mathbb{R}^n$ along N and S as defined

- π is the restriction of a linear transf to a bijection, i.e., $\pi^{-1} \begin{bmatrix} x' \\ x^n \end{bmatrix} \longmapsto \begin{bmatrix} x' \\ x^n \\ \sqrt{1-|x|^2} \end{bmatrix}$.

  Both π and π^{-1} are smooth on the open half sphere $\{x \in S^n : x^{n+1} > 0\}$ and $\mathbb{B}^n$ respectively.
- The projections $\sigma, \tilde{\sigma}$ are diffeos by [Lee03, prob 1-7], when restricted to the closed half sphere $\{x \in S^n \; x^{n+1} \geq 0\}$.
- Call the composed, restricted charts $(\mathcal{U}, \varphi)$ and (V, ψ), mapping bijectively from $\mathcal{U}, V$ to $\mathbb{H}^n$.

- the transition maps are diffeos b/c φ and ψ can be extended to diffeos on an open set containing $\varphi(\mathcal{U} \cap V)$ and $\psi(\mathcal{U} \cap V)$ respectively.
- Because $\{\mathcal{U}, V\}$ covers $\bar{\mathbb{B}}^n$, these charts $(\mathcal{U}, \varphi)$ and (V, ψ) constitute a smooth atlas for $\bar{\mathbb{B}}^n$ as a mfld w/ boundary
- when we equip $\mathbb{B}^n$ w/ the std smooth structure as an open submfld of $\mathbb{R}^n$, the restriction of φ, ψ to $\mathbb{B}^n$ is a diffeo (note that π is not a diffeo on $\bar{\mathbb{B}}^n$) in the usual Euclidean sense, So that $\varphi \circ id^{-1} = \varphi|_{\mathbb{B}^n}$ is a diffeo. So the structures are compatible.
- Note $\pi^{-1}(x) = \begin{bmatrix} x' \\ x^n \\ \sqrt{1-|x|^2} \end{bmatrix} = \begin{bmatrix} x^1 \\ x^n \\ 0 \end{bmatrix}$ for $x \in \partial\mathbb{B}^n$

To list the local coords of boundary pts

$$x = N = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \psi(x) = \tilde{\sigma}(\pi^{-1}(x)) = 0$$

$$x = S = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \quad \varphi(x) = \sigma(\pi^{-1}(x)) = 0$$

$$x \in S^{n-1} \setminus \{N, S\}, \quad \varphi(x) = \sigma\left(\begin{bmatrix} x' \\ x^n \\ 0 \end{bmatrix}\right) = \frac{1}{1-x^n}\begin{bmatrix} x' \\ x^{n-1} \\ 0 \end{bmatrix}$$

Whereas on the interior, $|x| < 1$, so

$$\begin{bmatrix} x^1 \\ x^n \end{bmatrix} \xmapsto{\pi^{-1}} \begin{bmatrix} x' \\ x^n \\ \sqrt{1-|x|^2} \end{bmatrix} \xmapsto{\sigma} \frac{1}{1-x^n}\begin{bmatrix} x^1 \\ \vdots \\ x^{n-1} \\ \sqrt{1-|x|^2} \end{bmatrix} \in \mathrm{Int}(\mathbb{H}^n)$$

$\square$

[1] Exercise 2.3. *Given.* Let M be a smooth manifold (with or without boundary). Say $f\colon M \to \mathbf{R}^k$ is a smooth function.

To prove. The composition $f \circ \varphi^{-1}\colon \varphi(U) \to \mathbf{R}^k$ is smooth for *every* open chart (U, φ) for M.

Proof.

Consider any chart $(\mathcal{U}, \varphi)$ for M. Let $p \in \mathcal{U}$. Let (V_p, φ_p) contain p, s.th. $f \circ \varphi_p^{-1}$ is smooth. There's a diffeo from $\varphi(\mathcal{U} \cap V_p)$ to $\varphi_p(\mathcal{U} \cap V_p)$ given by $\varphi_p \circ \varphi$. Precomposing with the smooth map $\varphi_p \circ \varphi^{-1}$, we find a map from $\varphi(\mathcal{U} \cap V_p)$ to $\mathbb{R}^k$, $(f \circ \varphi_p^{-1}) \circ (\varphi_p \circ \varphi)$
$= f \circ \mathrm{id}_{\mathcal{U} \cap V_p} \circ \varphi^{-1} = f \circ \varphi^{-1}$, smooth.

Letting p run through $\mathcal{U}$, we may collect charts (V_p, φ_p) so that $f \circ \varphi^{-1}$ is smooth for all open nbhds of $\varphi(\mathcal{U})$ i.e., $f \circ \varphi^{-1}$ is smooth on $\varphi(\mathcal{U})$. $\square$

[1] Exercise 2.9. *Given.* Say $F\colon M \to N$ is a smooth map between smooth manifolds (with or without boundary).

To prove. The coordinate representation of F with respect to *every* pair of smooth charts for M and N is smooth.

Proof.

Let $(\tilde{\mathcal{U}}, \tilde{\varphi})$ and $(\tilde{V}, \tilde{\Psi})$ be any smooth charts for M, N resp. Let $p \in \tilde{\mathcal{U}}$. Find $(\mathcal{U}, \varphi), (V, \Psi)$ for M, N s.th. $p \in \mathcal{U}$ and $F(p) \in V$ s.th. $\mathcal{U} \cap F^{-1}(V)$ is open in M and $\Psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(\mathcal{U} \cap F^{-1}(V))$ to $\Psi(V)$. By precomposing with a transition map and restricting,

$\Psi \circ F \circ \varphi^{-1} \circ (\varphi \circ \tilde{\varphi}^{-1}) = \Psi \circ F \circ \tilde{\varphi}^{-1}$ is a smooth map from $\tilde{\varphi}(\mathcal{U} \cap F^{-1}(V) \cap \tilde{\mathcal{U}})$ to $\Psi(V)$.

Post composition yields $\tilde{\Psi} \circ \Psi^{-1} \circ \Psi \circ F \circ \tilde{\varphi}^{-1}$ $= \tilde{\Psi} \circ F \circ \tilde{\varphi}^{-1}$ as a smooth map from $\tilde{\varphi}(\mathcal{U} \cap \tilde{\mathcal{U}} \cap F^{-1}(V \cap \tilde{V}))$ to $\Psi(V)$, because the transition map $\tilde{\Psi} \circ \Psi^{-1}$ is a diffeo.

Letting p vary through $p \in \tilde{\mathcal{U}} \cap F^{-1}(\tilde{V})$ (either the coordinate rep is the empty function and vacuously smooth or) we find that $\tilde{\Psi} \circ F \circ \tilde{\varphi}^{-1} : \tilde{\varphi}(\tilde{\mathcal{U}} \cap F^{-1}(\tilde{V}))$ to $\tilde{\Psi}(\tilde{V})$ is smooth for each open nbhd of $\tilde{\varphi}(\tilde{\mathcal{U}} \cap F^{-1}(\tilde{V}))$, i.e., smooth. □

[1] **Problem 2-1.** *Given.* Consider the function $f\colon \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0 \end{cases}.$$

To prove. For each real number $x \in \mathbf{R}$, there are smooth coordinate charts (U, φ) and (V, ψ) containing x and $f(x)$ respectively such that

$$\psi \circ f \circ \varphi^{-1} \quad \text{is smooth from } \varphi(U \cap f^{-1}(V)) \text{ to } \psi(V).$$

However, there are charts for which $U \cap f^{-1}(V)$ is *not* open. (By definition then, f is *not* a smooth map between manifolds [1, No. 2.5].)

Proof.

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If $x \neq 0$, then take $\varepsilon' = \frac{|x|}{2}$ and the charts $(B_{\varepsilon'}(x), id), (B_{\varepsilon'}(f(x)), id)$ for $\mathbb{R}$ and $\mathbb{R}$. Then the coordinate representation $\hat{f}$ is smooth on open sets, so smooth in the Lee03 sense.

Else if $x = 0$, let $(B_\varepsilon(0), id)$ for $\mathbb{R}$ and $(B_\varepsilon(1), id)$ for $\mathbb{R}$, $0 < \varepsilon < 1$.

Then $f^{-1}(B_\varepsilon(1)) \cap B_\varepsilon(0) = [0, \varepsilon)$ on which f is const, so the coordinate rep is "smooth" (quasi) but cannot be extenceded to an smooth function on any open set containing $[0, \varepsilon)$. ← Actually, it's pretty easy to extend a constant function to a neighborhood. ☺ But the extension wouldn't be the same as f for $x < 0$, so it wouldn't help

Yet f is not continuous! ✓ $\square$

REFERENCES

[1] J. M. Lee, *Introduction to Smooth Manifolds.* New York: Springer-Verlag, 2003.

[2] M. P. do Carmo, *Differential geometry of curves and surfaces.* Upper Saddle River, N.J.: Prentice-Hall, 1976.