COMPLEX PROJECTIVE SPACE & MAPPINGS BETWEEN MANIFOLDS

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Assignment due 2019-02-06

[1] Problem 1-9. *Given.* We assume the statement of Lee's smooth manifold chart lemma [1, No. 1.35]. We will compare this lemma to Do Carmo's definition of a differentiable manifold [2, Ch. 0.2].

To prove. Complex projective space $\mathbf{C}P^n$ is a smooth $2n$-dimensional manifold with structure generated by $n+1$ projective charts.

Proof.

Date: 2019-02-06.

[1] **Problem 1-11.** *Given.* The closed unit ball $\bar{\mathbf{B}}^n$ as a set of points in $\mathbf{R}^n$.

To prove. $\bar{\mathbf{B}}^n$ is a smooth n-dimensional manifold with boundary, which we can endow with a smooth structure such that:

- the interior $\mathbf{B}^n$ is parameterized by coordinates in the interior of the half space $\mathbf{H}^n$,
- the boundary $S^{n-1} = \partial\bar{\mathbf{B}}^n$ is parameterized by coordinates at the boundary of $\mathbf{H}^n$, and
- every interior chart is a chart for the standard structure on $\mathbf{B}^n$ as an open submanifold of $\mathbf{R}^n$.

Proof.

[1] **Exercise 2.3.** *Given.* Let M be a smooth manifold (with or without boundary). Say $f\colon M \to \mathbf{R}^k$ is a smooth function.

To prove. The composition $f \circ \varphi^{-1}\colon \varphi(U) \to \mathbf{R}^k$ is smooth for *every* open chart (U, φ) for M.

Proof.

[1] Exercise 2.9. *Given.* Say $F\colon M \to N$ is a smooth map between smooth manifolds (with or without boundary).

To prove. The coordinate representation of F with respect to *every* pair of smooth charts for M and N is smooth.

Proof.

[1] **Problem 2-1.** *Given.* Consider the function $f\colon \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0 \end{cases}.$$

To prove. For each real number $x \in \mathbf{R}$, there are smooth coordinate charts (U, φ) and (V, ψ) containing x and $f(x)$ respectively such that

$$\psi \circ f \circ \varphi^{-1} \quad \text{is smooth from } \varphi(U \cap f^{-1}(V)) \text{ to } \psi(V).$$

However, there are charts for which $U \cap f^{-1}(V)$ is *not* open. (By definition then, f is *not* a smooth map between manifolds [1, No. 2.5].)

Proof.

References

[1] J. M. Lee, *Introduction to Smooth Manifolds*. New York: Springer-Verlag, 2003.

[2] M. P. do Carmo, *Differential geometry of curves and surfaces*. Upper Saddle River, N.J.: Prentice-Hall, 1976.