

# Math 6210 NOTES: DIFFERENTIAL GEOMETRY

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Lecture 1.

2019-04-17

Last time, we defined covectors, cotangent vectors, and a basis for the *cotangent space at a point*, given a basis of tangent vectors for the tangent space.

That is, by choosing a coordinate chart  $(U, (x^i))$  about a point  $p \in U$ , we have a basis for  $T_p M$  (and so a dual basis for  $T_p^* M$ ).

$$\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}, \quad \left\{ dx^1 \Big|_p, \dots, dx^n \Big|_p \right\}.$$

**Example 1.1** (Changing the cotangent basis). Let  $(\tilde{x}^j)$  be another local coordinate chart in a neighborhood of  $p$ . Then

$$\frac{\partial}{\partial x^i} \Big|_p = \sum_{j=1}^n \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial}{\partial \tilde{x}^j} \Big|_p.$$

Now let  $(d\tilde{x}^j)$  be the dual basis to  $(\frac{\partial}{\partial \tilde{x}^j} \Big|_p)$ . Consider a cotangent vector  $w \in T_p^* M$ . Written in terms of both the original and the new cotangent bases,  $w$  is expressed as

$$w = \sum_{i=1}^n a_i dx^i \Big|_p = \sum_{j=1}^n \tilde{a}_j d\tilde{x}^j \Big|_p.$$

To determine the relation between the original basis representation  $[a_i]$  and the new representation  $[\tilde{a}_j]$ , evaluate the covector  $w$  at each tangent vector  $\frac{\partial}{\partial x^i} \Big|_p$  in the original basis. Then

$$\begin{aligned}
 a_i &= w \left( \frac{\partial}{\partial x^i} \Big|_p \right) && \text{because } dx^i \frac{\partial}{\partial x^j} = \delta_j^i \\
 &= w \left( \sum_{j=1}^n \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial}{\partial \tilde{x}^j} \Big|_p \right) && \text{by the chain rule} \\
 &= \sum_{j=1}^n \frac{\partial \tilde{x}^j}{\partial x^i} w \left( \frac{\partial}{\partial \tilde{x}^j} \Big|_p \right) && \text{by linearity} \\
 &= \sum_{j=1}^n \tilde{a}_j \frac{\partial \tilde{x}^j}{\partial x^i} && \text{because } d\tilde{x}^r \frac{\partial}{\partial \tilde{x}^j} = \delta_j^r.
 \end{aligned}$$

◀

**Exercise 1.2.** Give a numerical representation of the change of basis map taking  $[a_i]$  to  $[\tilde{a}_j]$ . Compare this to the change of basis on  $T_p M$  with  $\tilde{v}^j$  written in terms of a basis  $v^i$ .

We prefer to express old covectors in terms of a new covector basis. Whereas with tangent vectors, the chain rule gives us a means (and hence a preference) to express the new vectors in terms of the old tangent vector basis.

**Example 1.3** (Basis covectors). Let  $w$  be the cotangent vector in example 1.1. Then

$$(1.4) \quad w = \sum_{j=1}^n \tilde{a}_j d\tilde{x}^j \Big|_p = \sum_{i=1}^n a_i dx^i \Big|_p \quad \text{and} \quad w = \sum_{i,j=1}^n \tilde{a}_j \frac{\partial \tilde{x}^j}{\partial x^i} dx^i \Big|_p.$$

As the scalars  $[\tilde{a}_j]$  range through  $\mathbb{R}^n$ , the relation in (1.4) determines the representation of the new basis covector in terms of the old basis covectors:

$$(1.5) \quad d\tilde{x}^j \Big|_p = \sum_{i=1}^n \frac{\partial \tilde{x}^j}{\partial x^i} dx^i \Big|_p.$$

◀

*Note.* We have to hurry on to define covector fields, in order to state Stokes' theorem by the end of the semester. So we're glossing over the theory of vector bundles. ◀

**Definition 1.6** (Cotangent bundle). Let  $M \in \text{Man}^n$ . The *cotangent bundle* over  $M$  is the disjoint union

$$T^*M = \coprod_{p \in M} T_p^*M,$$

with projection  $\pi: T^*M \rightarrow M$  sending cotangent vectors  $w_p \mapsto p$  onto  $M$ .

**Proposition 1.7.** The cotangent bundle is a smooth vector bundle over  $M$ .

*Proof.* By definitions, fiber over  $p \in M$  is the cotangent vector space at  $p$ . We claim  $\pi$  is a smooth projection with the fibers over  $M$  varying smoothly.

Now consider a local coordinate chart  $(x^i)$  on  $U \subset M$ . We will show that  $U$  is a trivializing neighborhood for  $\pi$ . The coordinate cotangent vectors determine  $n$  (local) sections back into the cotangent bundle (called *coordinate covector fields* on  $M$ ). These are

$$(1.8) \quad dx^i : U \rightarrow T^*U \quad \text{such that} \quad dx^i(p) = dx^i \Big|_p \in T_p^*M.$$

The coordinate chart  $(U, (x^i))$  then determines a chart for the open neighborhood  $\pi^{-1}(U) = T^*U$ . Each covector  $w \in \pi^{-1}(U)$  is in the fiber of some  $p \in U$ , and can be expressed in terms of the cotangent basis at  $p$

$$w_p = [\xi_1 \quad \cdots \quad \xi_n] \begin{bmatrix} dx^1 \Big|_p \\ \vdots \\ dx^n \Big|_p \end{bmatrix}.$$

We'll define our chart  $\pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{R}^n$  pointwise

$$\sum_{i=1}^n \xi_i dx^i \Big|_p \longmapsto (x^1, \dots, x^n, \xi_1, \dots, \xi_n).$$

Taking for granted that this correspondence is smooth, we have shown the atlas  $\{U_i, \varphi_i\}$  for  $M$  induces a local trivialization of the cotangent bundle.  $\square$

**Definition 1.9** (Covector fields). A global section of  $T^*M$  is a *global covector field* (a.k.a., a differential 1-form), and a local section of  $T^*M$  is a *local covector field*.

If  $\omega \in \Gamma(T^*M)$  is a covector field, and  $X \in \mathfrak{X}M$  is a vector field, then we can define a scalar function  $\omega(X): M \rightarrow \mathbb{R}$  such that  $\omega(X)(p) = \omega|_p(X_p)$ . In local coordinates,

$$\text{if } X = X^i \frac{\partial}{\partial x^i}, \text{ and } \omega = \xi_j dx^j, \text{ then } \omega(X) = \sum_i \xi_i(x) X^i(x).$$

**Definition 1.10** (Pullbacks of covector fields). Let  $M \xrightarrow{F} N$  be a map in  $\mathbf{Man}$ , with  $p$  a point of  $M$ . The differential  $dF_p: T_p M \rightarrow T_{F(p)} N$  has a dual linear map  $dF_p^*: T_{F(p)}^* N \rightarrow T_p^* M$  called the *pullback map* by  $F$  at  $p$ , or the *cotangent map* at  $p$ . It's characterized by the property that for all  $\mathbf{v} \in T_p M$ , and covectors  $\omega \in T_{F(p)}^* N$ ,

$$(dF_p)^*(\omega)(\mathbf{v}) = \omega(dF_p(\mathbf{v})).$$

**Definition 1.11** (Coframe fields). A *local coframe field* on  $U \subset M$  is an ordered  $n$ -tuple  $(\varepsilon^1, \dots, \varepsilon^n)$  of covector fields such that, for all  $p \in U$ , the evaluation of  $(\varepsilon^1, \dots, \varepsilon^n)$  at  $p$  forms a basis for  $T_p^* M$ .

Suppose  $(E^1, \dots, E^n)$  is a frame field on  $U \subset M$ , then the *dual coframe field*  $(\varepsilon^1, \dots, \varepsilon^n)$  is defined by

$$\varepsilon^i(E_j) = \delta_j^i.$$

We write  $\mathfrak{X}^*M$  for the *smooth covector fields* on  $M$ .

**Exercise 1.12.** Does each smooth manifold  $M$  admit a smooth covector field?<sup>1</sup>

We proceed to “discover” differential 1-forms.

**Definition 1.13** (Differential of a scalar function). Let  $M \in \mathbf{Man}$  and  $f \in C^\infty(M)$ . The *differential* of  $f$  is the covector field  $df$  on  $M$  defined by

$$df \Big|_p (\mathbf{v}) = \mathbf{v}(f) \quad \text{for all } \mathbf{v} \in T_p M.$$

We've specified how  $df$  acts at each point, for each tangent vector. Doesn't this *look* like a Kröner pairing?

**Example 1.14** (Coordinate representation of a differential). Consider local coordinates  $(x^i)$  on  $M$ . If we write  $\mathbf{v} = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i} \Big|_p$  and  $df \Big|_p = \sum_{j=1}^n a_j dx^j \Big|_p$ , then

$$\begin{aligned} \sum_{i=1}^n v^i a_i &= \left( a_j dx^j \Big|_p \right) \left( v^i \frac{\partial}{\partial x^i} \Big|_p \right) \\ &= df \Big|_p (\mathbf{v}) \\ &= \mathbf{v} \Big|_p (f) \\ &= \sum_{i=1}^n v^i \frac{\partial f}{\partial x^i} \Big|_p \end{aligned}$$

Varying the scalars  $[v^1 \ \dots \ v^n]^t$  for the tangent vector  $\mathbf{v}$ , we determine each scalar component

$$a_i = \frac{\partial f}{\partial x^i} \Big|_p$$

in the basis representation of  $df \Big|_p$ . Therefore the differential of  $f$  in terms of local coordinates is

$$df \Big|_p = \frac{\partial f}{\partial x^i} dx^i \Big|_p.$$

<sup>1</sup>Patrick asked if it's true that every  $M \in \mathbf{Man}$  admits a smooth nonvanishing covector field. I have no idea.

**Remark 1.15.** We have also shown  $dx^i = d(x^i)$  for the coordinate functions  $x^i$  of chart  $(U, (x^i))$ . That is,  $dx^i$  is the differential of  $x^i: U \rightarrow \mathbb{R}$ . ◀

Defining the differential of a scalar function as an *evaluation* extends the notion of the “differential” given for  $dx$  in high school calculus. The two concepts are the same for a function  $f: M \rightarrow \mathbb{R}$  if we choose to make the identification  $df: T_p M \rightarrow T_{f(p)} \mathbb{R} \cong \mathbb{R}$ .

**Proposition 1.16.** Let  $\gamma: J \rightarrow M$  be a smooth curve,  $f \in C^\times(M)$ . Then the differential of the function  $f \circ \gamma: J \rightarrow \mathbb{R}$  is given by (the familiar evaluation)

$$(f \circ \gamma)'(t) = df_{\gamma(t)}(\gamma'(t)).$$

**Exercise 1.17.** Soft question: why should a covector represent an infinitesimal path? Can this be connected with local 1-parameter group actions?

Lecture 2.

2019-04-19

“Covectors are just linear maps on vector spaces. They’re not necessarily Krönecker pairings.”

Last time, we defined pullbacks of smooth vector fields.

**Note.** For a map  $M \xrightarrow{F} N$ , regardless<sup>2</sup> of whether  $F$  is surjective, injective, etc., the pullback of  $\omega \in \mathfrak{X}^*(N)$  is always a smooth covector field on  $M$ . ◀

For  $\omega \in \mathfrak{X}^*(N)$ , we denote the pullback via  $F$  as  $F^*\omega$ . If we let  $F^*(\mathbf{v})$  denote  $dF_p(\mathbf{v})$ , then  $(F^*\omega)(\mathbf{v}) = \omega(F_*(\mathbf{v}))$ .

**Proposition 2.1.** Let  $M \xrightarrow{F} N$  be smooth,  $\omega \in \mathfrak{X}^*(N)$ , and  $u \in C^\times(N)$ . Then both:

- (1)  $F^*(u\omega) = (u \circ F)F^*\omega$ .
- (2)  $F^*(du) = d(u \circ F)$ .

**Example 2.2** (Helicoids). Define  $F: \mathbb{R}^2 \rightarrow \mathbb{R}^3$  by

$$F \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} v \cos u \\ v \sin u \\ u \end{bmatrix}.$$

Let  $\omega \in \mathfrak{X}^*(\mathbb{R}^3)$  be the covector field

$$\omega = x \, dx - y \, dy + z^2 \, dz$$

Then (TODO)

$$F^*(\omega) = (x \circ F) \, d(y \circ F) - (y \circ F) \, d(x \circ F) + (z \circ F)^2 \, d(z \circ F).$$

**Proposition 2.3** (Restricting the covector field to submanifolds). Say  $\iota: S \hookrightarrow M$  is the injection of an immersed submanifold. The pullback of  $\omega \in \mathfrak{X}^*(M)$  is  $\iota^*\omega \in \mathfrak{X}^*(S)$ , and, at a point  $p \in M$ , where  $\omega$  is in the cotangent space  $T_p^*M$ , its pullback is just the restriction

$$\iota^*\omega = \omega \Big|_{T_p S}.$$

Explicitly,<sup>3</sup> if  $\mathbf{v} \in T_p S$ , then

$$\iota^*\omega(\mathbf{v}) = \omega(\iota_*(\mathbf{v})) = \omega(\mathbf{v})$$

**Example 2.4.** Let  $M = \mathbb{R}^2$ , and consider the  $x$ -axis as  $S \hookrightarrow M$ . Say  $\omega = dy$ . For  $p \in S$ , any  $\mathbf{v} \in T_p S$  has the form  $\mathbf{v} = a \frac{\partial}{\partial x}$ . But then  $\omega(\mathbf{v}) = da \frac{\partial}{\partial x} = 0$ ! So  $\iota^*\omega = 0$ , although  $\omega$  is a nonzero covector field on  $M$ . ◀

**Remark 2.5** (Making ends meet). What are *exterior differential systems*? Translate the partial differential equations in terms of covector fields on smooth manifolds. The solutions to the differential equations correspond to the ideals on the manifold for which the pullback map is trivial (the kernel). ◀

**Example 2.6** (Line integrals). Let  $[a, b] \subset \mathbb{R}$  be a closed bounded interval of the real line, and let  $w$  be a smooth covector field of  $[a, b]$ . If  $t$  is a local coordinate, then  $dt$  gives a basis and we can write  $w = f(t)dt$  for some  $f \in C^\infty([a, b])$ . ◀

<sup>2</sup>Much nicer than pushforwards! Think about preimages as respecting set-theoretic operations.

<sup>3</sup>Much nicer than restricting vector fields!

**Proposition 2.7.** If  $w \in \mathfrak{X}^*([a, b])$  and  $\varphi: [c, d] \rightarrow [a, b]$  is an increasing diffeomorphism, then  $\int_{[c, d]} \varphi^* w = \int_{[a, b]} w$ .

We'll have to introduce sign conventions for orientation when taking surface integrals.<sup>4</sup>

**Definition 2.8.** Now let  $M$  be a smooth manifold. A *smooth curve segment* is a smooth map  $\gamma: [a, b] \rightarrow M$ . The *line integral* of  $w$  over  $\gamma$  is

$$\int_{\gamma} w := \int_{[a, b]} \gamma^* w.$$

Note that by 2.7 the line integral is well defined, regardless of parameterization  $\gamma$ .

**Exercise 2.9.** Let  $M$  be the plane punctured at the origin. Let  $w = \frac{xdy - ydx}{x^2 + y^2}$ . Then compute  $\int_{\gamma} w$  for the parameterization  $\gamma: [0, 2\pi] \rightarrow M$  given by

$$\gamma(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}.$$

**Theorem 2.10** (Fundamental theorem of line integrals). If  $w = df$ ,  $f \in C^\infty(M)$ , and  $\gamma: [a, b] \rightarrow M$  is a smooth curve segment, then

$$\int_{\gamma} w = \int_{\gamma} df = f(\gamma(b)) - f(\gamma(a)).$$

**Definition 2.11** (Exact covector fields). If  $w = df$  for some  $f \in C^\infty(M)$ ,  $w$  is called *exact*, and  $f$  is called a *potential function* for  $w$ . In this case,  $\int_{\gamma} w$  depends only on the endpoints of  $\gamma$ . This integral is called *path-independent*. (We'd also say that the covector field is conservative.)

Working in  $\mathbb{R}^n$ , we often call vector fields conservative—but in  $\mathbb{R}^n$  this amounts to a vector field being the gradient of a smooth function. Apparently the Euclidean metric gives a structure for  $\nabla$  and  $\text{div}$  to produce a correspondence between covector fields and vector fields.

**Theorem 2.12.** A smooth covector field  $w$  is conservative if and only if it is exact.

*Proof.* On the one hand, exact implies conservative. Conversely, choose  $p_0 \in M$  and define  $f(p) = \int_{\gamma} w$  where  $\gamma$  is a path from  $p_0$  to  $p$ . (There are some requirements for  $\gamma$ —it need be continuous, piecewise smooth, and with well-behaved one-sided limits.)  $\square$

**Proposition 2.13** (Necessary conditions for exactness). If  $w = df$  is exact and  $w = a_i(x) dx^i$  for  $a_i(x) = \frac{\partial f}{\partial x^i}$ , then for all indices  $i, j$  we have

$$\frac{\partial a_i}{\partial x_j} = \frac{\partial a_j}{\partial x_i} = \frac{\partial^2 f}{\partial x^i \partial x^j}$$

**Definition 2.14** (Closed covector fields). A covector field  $w = a_i(x) dx^i$  for  $a_i(x) = \frac{\partial f}{\partial x^i}$  is *closed* if for all indices  $i, j$ ,

$$\frac{\partial a_i}{\partial x_j} = \frac{\partial a_j}{\partial x_i}.$$

**Exercise 2.15.** Show that the non-exact covector field in 2.9 is closed.

Lecture 3.

2019-04-22

**Lemma 3.1** (Poincaré lemma for covector fields). If  $M \in \text{Man}$  is a simply connected manifold, then a covector field  $w$  is closed if and only if  $w$  is exact.

*Note* (deRham cohomology). So, if  $M$  is a simply connected manifold, then the first deRham cohomology group is trivial. (Where deRham cohomology groups are the quotient of the group of closed  $n$ -forms by exact  $n$ -forms.)  $\blacktriangleleft$

**Proposition 3.2** (Closed covector fields).

- (1) Let  $w$  be a closed covector field on  $M$ . Then every  $p \in M$  has a neighborhood  $U \subset M$  on which  $w|_U$  is exact.
- (2) Closedness of covector fields is independent of the choice of local coordinate charts.
- (3) In fact,  $w$  is closed iff for all  $X, Y \in \mathfrak{X}M$ , we have  $X(w(Y)) - Y(w(X)) = w([X, Y])$ .
- (4) Pullbacks preserve closedness and exactness.

<sup>4</sup>Hope it's as much fun as for defining sign conventions for homology on CW-complexes.

**Definition 3.3** (Tensor products). Let  $V, W \in \mathbf{Vec}$  be finite dimensional vector spaces over  $\mathbb{R}$ . Let  $a^*, b^* \in V^*, W^*$  be covectors. The *tensor product* of  $a^* \otimes b^*$  is the bilinear functional defined on vectors  $\mathbf{v} \times \mathbf{w} \in V \times W$  by

$$(a^* \otimes b^*)(\mathbf{v}, \mathbf{w}) = a^*(\mathbf{v})b^*(\mathbf{w}).$$

Then the space  $V^* \otimes W^*$  is the space of the tensors  $\{a^* \otimes b^* : a \in V, b \in W\}$ .

**Proposition 3.4** (Properties of the tensor product). *The dimension of the tensor product of dual spaces  $V^*$  and  $W^*$  is the product  $\dim V^* \cdot \dim W^*$ . There's a basis for  $V^* \otimes W^*$  given by the tensors of basis elements for  $V^*$  and  $W^*$  respectively. Note that  $V \otimes W := (V^* \otimes W^*)^*$  is the dual.*

We'll mostly work with  $n$ -isomorphic copies of a single  $V \in \mathbf{Vect}$ .

**Definition 3.5** (Wedge product). Let  $\alpha, \beta \in V^*$ . The *wedge product* of  $\alpha$  and  $\beta$  is defined as

$$\alpha \wedge \beta := \alpha \otimes \beta - \beta \otimes \alpha.$$

In particular, evaluated on two vectors  $(v, w) \in V^2$ , we have

$$\alpha \wedge \beta(v, w) = \det \begin{bmatrix} \alpha(v) & \alpha(w) \\ \beta(v) & \beta(w) \end{bmatrix}.$$

More generally, the wedge product of  $k$  covectors in  $V^*$  is defined pointwise for  $v^k$  in  $V$  by

$$\alpha^1 \wedge \cdots \wedge \alpha^k[v_1 \quad \cdots \quad v_k] = \det \left( \sum_{i,j} \alpha^i(v_j) \right).$$

The *space of alternating  $k$ -tensors* or  *$k$ -vectors* of  $V^*$  is the defined by

$$\bigwedge^k V^* := \text{span}\{\alpha^1 \wedge \cdots \wedge \alpha^i : \alpha^i \in V^*\}.$$

**Proposition 3.6** (Properties of the wedge product).

- (1) *It's bilinear,*
- (2) *it's associative,*
- (3) *it's anti-commutative.*
- (4) *For  $\omega \in \bigwedge^k V^*$  and  $\eta \in \bigwedge^\ell V^*$ , we have  $\omega \wedge \eta \in \bigwedge^{\ell+k} V^*$  and  $\omega \wedge \eta = (-1)^{k\ell} \eta \wedge \omega$ .*

**Definition 3.7** (Decomposable). A  $k$ -covector  $\eta$  is said to be *decomposable* if it can be expressed as a wedge of  $k$  covectors.

**Example 3.8** (An indecomposable 2-covector on  $\mathbb{R}^4$ ).  $\eta = dx_1 \wedge dx_2 + dx_3 + dx_4$  is not decomposable. (Hint: consider that any decomposable  $k$ -covector wedged with itself is the 0-function.)  $\blacktriangleleft$

**Definition 3.9** (Interior multiplication). There's a linear map from the space of alternating  $k$ -tensors to the space of alternating  $k-1$ -tensors called *interior multiplication* by  $\mathbf{v}$ .

$$\begin{aligned} \iota_{\mathbf{v}} : \bigwedge^k V^* &\rightarrow \bigwedge^{k-1} V^* \\ \omega &\mapsto \iota_{\mathbf{v}} \omega = \iota_{\mathbf{v}} \omega : (\mathbf{w}_1, \dots, \mathbf{w}_{k-1}) \mapsto \omega(\mathbf{v}, \mathbf{w}_1, \dots, \mathbf{w}_{k-1}). \end{aligned}$$

The interior multiplication of  $\omega$  and  $\mathbf{v}$  is referred to as “ $\mathbf{v}$ -left hooked with  $\omega$ .”

Ryan asked if there's a connection to, e.g., the front or back face maps. I asked if there was a relation to the cap product. **TODO**. . .

In smooth charts, a  $k$ -form  $\omega$  has a local expression

$$\omega = \sum_{\forall i_1, \dots, i_k} f_{i_1, \dots, i_k}(x) dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

Lecture 4.

2019-04-24

“We’re in the algebra of differential forms on a manifold,  $\Omega(M)$ .”

**Definition 4.1** (Multi-index notation). In local coordinates  $(x^i)$ , any  $k$ -form  $\omega$  can be expressed as

$$\omega = \sum_{(i_1, \dots, i_k) \in I_n^k} f_{(i_1, \dots, i_k)}(x) dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

where  $I_n^k$  is the product of tuples  $(1, \dots, n) \times (1, \dots, k)$ .

We can compute the action of a covector  $\omega$  on  $f_I(x)$  **TODO**. (I got lost in the indices.)

$$f_I(x) = \omega \left[ \frac{\partial}{\partial x^{i_1}} \quad \dots \quad \frac{\partial}{\partial x^{i_k}} \right]$$

We’ll start in  $\mathbb{R}^n$ . Let  $\omega$  be a  $k$ -form, of the form,

$$\omega = \sum_{|I|=k} f_I(x) dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

The exterior derivative of  $\omega$  is the  $k+1$ -form on  $\mathbb{R}^n$

$$d\omega = \sum_{|I|=k} df_I \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

In  $\mathbb{R}^2$ , we have  $dx dy = dx \wedge dy$ .

If  $\omega$  is a 1-form, where  $\omega = f_i(x) dx^i$ , then

$$\begin{aligned} d\omega &= df_i \wedge dx^i \\ &= () \end{aligned}$$

**Example 4.2** (Exterior derivative). ◀

Lecture 5.

2019-04-29

Today we’ll barrel through orientations (so to take integrals over forms).

Let  $M$  be a smooth manifold of dimension  $n$ . An orientation of  $M$  is determined by a smooth non-vanishing  $n$ -form  $\omega$  on  $M$  (called a *top-form*). Two non-vanishing  $n$ -forms  $\omega$  and  $\tilde{\omega}$  on  $M$  determine the same orientation if  $\omega = f(\tilde{\omega})$  for a strictly positive  $f \in C^\infty(M)$ .

**Example 5.1** (Standard orientation on Euclidean space). On  $\mathbb{R}^n$ , the standard orientation is determined by  $\omega = dx^1 \wedge \dots \wedge dx^n$ . The *only* other orientation is determined by, e.g.,  $-\omega = dx^1 \wedge \dots \wedge dx^n \wedge dx^{n-1}$ . ◀

How is one to orient a manifold with boundary? Well, given an orientation on the interior of the manifold, what would a decent convention be to orient the boundary?

Let  $M$  be a  $n$ -dim manifold with boundary. The *induced orientation* on  $\partial(M)$  is determined by the  $n-1$  form  $N \lrcorner \omega$ , where  $\omega$  is a representative determining the orientation on  $M$ , and  $N$  is any *outwards-pointing* nonzero vector field along  $\partial(M)$ . (Recall that our boundary convention has the *last* coordinate of  $\mathbb{H}^n$  to be  $\mathbb{R}_{\geq 0}$ .)

**Example 5.2** (Calc III orientations).

- (1) Let  $M$  be the unit disc in the plane  $\mathbb{R}^2$ . Then  $\partial(M) = S^1$ . Say  $\omega = dx \wedge dy$ . Choose  $N = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ , which is a nonzero vector field  $N \in \mathfrak{X}M$ . Taking the cap  $N \lrcorner (dx \wedge dy) = x dy - y dx$ , gives a representative for the (pretty standard) orientation on  $S^1$ .
- (2) If  $\omega$  is the standard representative for the orientation on  $\mathbb{R}^n$ , then on  $S^3 = \partial(B^4)$ , the induced orientation is given by

$$x dy \wedge dz + y dz \wedge dx + z dx \wedge dy.$$

Is it always possible to choose an outwards pointing normal vector? Nope. ◀

**Example 5.3** (Intuition for integration). We're integrating *forms* not functions. What's the differential? For notation's sake, " $dx dy = dx \wedge dy$ " refers to the *area of a parallelogram*, appropriately signed.

Then  $dx \wedge dy(\mathbf{v}, \mathbf{w}) = v^1 w^2 - v^2 w^1 = \det \begin{bmatrix} \mathbf{v} & \mathbf{w} \end{bmatrix}$ .

If we assume the standard orientation on  $\mathbb{R}^n$ , then computationally, we may take the iterated Riemann integrals in what ever order is useful. (TODO, why?)

$$\int_{\partial(M)} f dx \wedge dy := \int f dx dy = \int f dy dx.$$

◀

**Definition 5.4** (Domain of integration). In Euclidean space, a domain of integration  $D$  is a bounded subset whose boundary  $\partial(D)$  has measure zero.

Let  $D \subset \mathbb{R}^n$  be a domain of integration, and  $\omega$  a continuous  $n$ -form on  $D$ . We have in local coordinates

$$\omega = f dx^1 \wedge \cdots \wedge dx^n.$$

We define the *integral of  $\omega$  over  $D$*  to be

$$\int_D \omega := \int \cdots \int f dx^1 \cdots dx^n \quad (\text{Riemann integral}).$$

**Proposition 5.5** (Diffeomorphisms preserve orientation). Let  $D, E$  be open domains in Euclidean space. Let  $g: \bar{D} \rightarrow \bar{E}$  be a smooth map that restricts to  $D$  to be a diffeomorphism. Then  $g$  pulls back a  $g^*\psi$  for  $\psi \in \Omega^n(E)$  to the top-form on  $D$   $\det g_* g(\psi) \dots$  TODO.

$$\int_D g^* \omega = \pm \int_E \omega.$$

Suppose now  $M \in \text{Man}^n$  and  $\omega$  is a top-form. If  $\omega$  is compactly supported within the domain  $U$  of a chart, then the integral of  $\omega$  over  $M$  is

$$\int_M \omega := \varepsilon \int_{\varphi(U)} (\psi^{-1})^* \omega.$$

The sign  $\varepsilon$  is determined according to whether or not  $\varphi$  is orientation preserving:

$$\begin{cases} \det \varphi^* < 0 & \varepsilon = -1, \\ \det \varphi^* > 0 & \varepsilon = 1. \end{cases}$$

Since  $\varphi$  is a diffeomorphism, the determinant of  $\varphi^*$  is strictly non-zero.

**Example 5.6** (Low-dim manifolds). Say  $M$  is a countable collection of points. An orientation on  $M$  is determined by a function  $\{\pm 1\}^M$ .

Say  $M$  is a connected, compact, 1-dim manifold. Then  $\partial(M) = \emptyset$  or  $\{a, b\}$ , depending on whether or not  $M$  is  $\simeq S^1$ . Putting a chart on  $M$ , we have a diffeo  $M \simeq [a, b]$ . If  $\omega = dx$ , then at  $b$ , the outward pointing normal is  $\frac{\partial}{\partial x}$ , and the orientation at  $b$  induced by that of the interior of  $M$  is 1. On the other hand, the orientation at  $a$  is  $-1$ , since the outward pointing normal is  $-\frac{\partial}{\partial x}$ .

◀

Say that  $S \subset M^n$  is an oriented immersed  $k$ -dim submanifold. Then let  $\omega \in \Omega^k(M)$  with a restriction to  $S$  that's compactly supported. Then we define the integral of  $\omega$  over  $S$  to be

$$\int_S \omega := \int_S \iota^* \omega,$$

where  $\iota: S \hookrightarrow M$  is the (not necessarily injective) inclusion map.

**Example 5.7** (Path integrals). TODO, find  $\gamma$  a smooth path in the plane, then compute the integral of  $\omega = x dy - y dx$  with respect to  $\gamma$ .

◀

**Example 5.8** (Main motivating example). Say  $M$  is a compact, connected, oriented  $n$ -manifold with boundary and let  $S = \partial(M)$ . For  $\omega$  a top form on  $S$ , we have

$$\int_{\partial(M)} \omega := \int_{\partial(M)} \iota_{\partial(M)}^* \omega.$$

◀

**Proposition 5.9** (How integration interacts with other operations).

- (1) If  $\omega$  is a positively oriented top form on  $M$ , then  $\int_M \omega > 0$ .
- (2) If  $f: M \rightarrow N$  is a diffeomorphism, then  $\int_N \omega = \pm \int_M f^* \omega$ .



We'll really exploit that we only need a subset  $D \subset M$  whose boundary has measure zero. The two key examples here are for parameterizations of the circle in  $\mathbb{R}^2$  and the sphere in  $\mathbb{R}^3$ , which are:

$$\gamma: [0, 2\pi] \rightarrow \mathbb{R}^2$$

$$\gamma(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

$$F: [0, \pi] \times [0, 2\pi] \rightarrow \mathbb{R}^3$$

$$\begin{bmatrix} \varphi \\ \theta \end{bmatrix} \xrightarrow{F} \begin{bmatrix} \sin \varphi \cos \theta \\ \sin \varphi \sin \theta \\ \cos \varphi \end{bmatrix}$$

Lecture 6.

2019-05-01

Today we state (without proof) Stokes' theorem.

**Example 6.1** (Essay). Describe how the fundamental theorem of calculus, Green's theorem, and the fundamental theorem of line integrals are all consequences of Stokes' theorem. Do so as if you were explaining it to a lovable dog. ◀

Suppose that  $M \in \mathbf{Man}^n$  is a manifold with boundary  $\partial(M)$  and that  $M$  is *orientable*.

Let  $\omega \in \Omega^{n-1}(M)$  be a top differential form, such that  $\omega$  is compactly supported on  $M$ .

Then

$$\int_M d\omega = \int_{\partial(M)} \omega,$$

where  $\partial(M)$  has the induced orientation from  $M$ . Note that if  $\partial(M) = \emptyset$ , then  $\int_{\partial(M)} \omega = 0$ . If  $M$  is a 1-dimensional manifold, then  $\int_{\partial(M)} \omega$  is a finite linear combination (of TODO residues?).

*Proof.* Let  $M = \mathbb{H}^n$  as the basic case. Then say  $\omega \in \Omega^{n-1}(M)$  has compact support. There's a finite  $R > 0$  such that  $\text{supp} \omega \subset [-R, R]^{n-1} \times [0, R]$ .

In a coordinate basis (the standard one for  $\mathbb{H}^n$ ), we have  $\omega = \sum_{i=1}^n a_i(x) dx_{(i)}$  where  $dx_{(i)}$  is the wedge of all 1-forms in the basis for the cotangent space of  $\mathbb{H}^n$ , omitting  $dx^i$  and adding the sign  $(-1)^{i-1}$ .

Then

$$d\omega = \sum_{i=1}^n \frac{\partial a_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n.$$

Sketch. Compare  $\iota_{\partial(\mathbb{H}^n)}^* \omega = a_n(x^1, \dots, x^{n-1}, 0) dx_{(n)}$ . Finish for  $\mathbb{H}^n$ .

Similarly, if  $M = \mathbb{R}^n$ , then  $\text{supp} \omega$  is contained in some cube  $[-R, R]^n$ , and a similar computation shows that

$$\int_M d\omega = 0 = \int_{\partial(M)} \omega \quad \text{because } \partial(M) \text{ is empty.}$$

Now let  $M$  be a general manifold with boundary and orientable. Suppose  $\omega$  is an  $n-1$  form compactly supported in  $\text{supp}(U)$ . Then from above, we see that  $\int_M d\omega = \int_{\partial(M)} \omega$  pulling back  $\omega$  and  $d\omega$  with the diffeomorphism  $(U, \varphi)$ .

Finally, let  $\omega$  be a compactly supported  $n-1$  form on  $M$ . Covering the compact set  $\text{supp} \omega$  with finitely many positively oriented chart domains  $(U_i, \varphi_i)$  and let  $\{\psi_i\}$  be a partition of unity subordinate to this open cover. Integrating  $\omega$  over  $\partial(M)$  given  $\int_{\partial(M)} \omega = \sum_i \psi_i \int_{\partial(M)} \omega$ . □

**Corollary 6.2** (Exact and closed forms).

- (1) If  $\omega$  is an exact  $n$  form on  $M$ , a compact orientable manifold without boundary, then  $\int_M \omega = 0$ .
- (2) If  $\omega$  is a closed  $n-1$  form on  $M$ , a compact orientable manifold without boundary, then  $\int_{\partial(M)} \omega = -$ .
- (3) If  $M$  is a smooth manifold and  $S \subset M$  is an oriented compacted  $k$ -dim submanifold without boundary TODO