EXAMPLES OF SMOOTH MANIFOLDS

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[1, No. 1.18]. *Given.* Let M be a topological manifold.

To prove. Two smooth atlases for M determine the same smooth structure if and only if their union is a smooth atlas.

Proof. ($\Rightarrow$) Let $\mathscr{A}$, $\mathscr{B}$ be two smooth atlases on a topological manifold. Say that $\mathscr{A} \cup \mathscr{B}$ is a smooth atlas. Then the unique maximal smooth atlas $\overline{\mathscr{A} \cup \mathscr{B}}$, which certainly contains $\mathscr{A}$ and $\mathscr{B}$, forces the conclusion:

$$\overline{\mathscr{A}} = \overline{\mathscr{A} \cup \mathscr{B}} = \overline{\mathscr{B}},$$

that is, both $\mathscr{A}$ and $\mathscr{B}$ determine the same smooth structure on M.

($\Leftarrow$) Say that $\mathscr{A}$ and $\mathscr{B}$ are smooth atlases that determine the same smooth structure on M. For contradiction, suppose that $\mathscr{A}$ and $\mathscr{B}$ are not compatible, so that their union contains at least $(U, \varphi) \in \mathscr{A}$ and $(V, \psi) \in \mathscr{B}$ for which $U \cap V \neq \varnothing$ and

$$\varphi(U \cap V) \xrightarrow{\varphi^{-1}} U \cap V \xrightarrow{\psi} \psi(U \cap V)$$

is *not* a diffeomorphism. Now

$$(U, \varphi) \in \mathscr{A} \subset \overline{\mathscr{A}}, \text{ and}$$
$$(V, \psi) \in \mathscr{B} \subset \overline{\mathscr{B}},$$

so either

- $\overline{\mathscr{A}} = \overline{\mathscr{B}}$ is not smooth, or
- $\overline{\mathscr{A}} \neq \overline{\mathscr{B}}$ are not the same smooth structures.

Both conclusions are contrary to our hypotheses, so it had better be that $\mathscr{A}$ and $\mathscr{B}$ are compatible! □

[1, Nos. 1–6]. *Given.* Let M be a nonempty topological manifold of dimension $n \geq 1$.

To prove. If M has a smooth structure, then it has uncountably many distinct ones.

Proof. We construct uncountably many distinct smooth structures on M.

step of construction	justification
1. Say $\mathscr{A}$ is a maximal smooth atlas on M.	Hypothesis.
2. Of the charts in $\mathscr{A}$, take a countable basis of regular coordinate balls and form the smooth atlas $\{(B_i, \varphi_i)\}$.	Every smooth manifold has a countable basis of regular coordinate balls. [1, p. 15]
3. Form an open, locally finite, refinement $\{V_j\}$ of the cover $\{B_i\}$.	$\{B_i\}$ is an open cover of paracompact M.
4. For each V_j, choose (B_i, φ_i) such that $V_j \subset B_i$ and define $\psi_j := \varphi_i\|_{V_j}$.	
5. Note that $\{(V_j, \psi_j)\}$ is a smooth atlas.	$\{V_j\}$ covers M. For each pair ψ_k, ψ_ℓ, in the atlas, the transition map $\varphi_k \circ \psi_\ell^{-1}$ is (the restriction of) a diffeomorphism.

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step of construction	justification
6. Consider a point in M, a neighborhood of this point, and take all $V_1, \ldots, V_n$ of $\{V_j\}$ which meet this neighborhood.	$\{V_j\}$ is locally finite, M is non-empty.
7. Remove an open set V_k from $V_1, \ldots, V_n$ if V_k is not covered by $\{V_1, \ldots, \hat{V}_k, \ldots, V_n\}$.	Choice.
8. Repeat step 7 until there's a open V from the initial n that's not covered by those open sets of the initial n remaining.	Finite recursion.
9. Choose a point $p \in V$ such that p is not in any other open set of those remaining after step 8.	
10. Note the atlas $\{(V_j, \psi_i)\}$ from step 5, with $\{(V_1, \psi_1), \ldots, (V_n, \psi_n)\}$ removed, with $\{$charts remaining after step 8$\}$ adjoined, is smooth.	
11. Moreover, the smooth atlas in step 10 has a particular chart (V, ψ) such that $p \in V$ alone.	$p \in V$, yet $p \notin V_1, \ldots, \hat{V}, \ldots, V_n$.
12. Consider the image of V and p in $\mathbf{R}^n$ under the map $\varphi \colon B \to \hat{B} \subset \mathbf{R}^n$, where B is the regular coordinate ball chosen in step 4 to contain V.	
13. Diffeomorph $\hat{B}$ to itself so that $\varphi(p) \in \hat{V}$ is mapped to the origin. Denote this diffeomorphism by g.	WLOG, say $\hat{B}$ is a unit ball centered at the origin. Then g is a tangent map $\hat{B} \to \mathbf{R}^n$, followed by a rigid translation in $\mathbf{R}^n$, finished with an inverse tangent map $\mathbf{R}^n \to \hat{B}$.
14. For all reals $s > 0$ such that $s \neq 1$, there's a function F_s that homeomorphs $\hat{B}$ to itself.	$F_s \colon x \mapsto \lvert x\rvert^{s-1} x$.
15. The restriction of $F_s \circ g$ to the punctured unit ball $\hat{B} \setminus \{0\}$, *is indeed* a diffeomorphism of $\hat{B} \setminus \{0\}$.	Partial derivatives of all orders exist for both F_s and its inverse $F_{1/s}$.
16. When $s \neq 1$, $F_s \circ g$ is either *not smooth* at the origin or *fails to have a smooth inverse.*	Either F_s or F_s^{-1} fails to be of class C^∞ on $\hat{B}$. There's no good linear approximation of the norm of a vector at the origin.
16. Define ψ_s as the restriction of the composite $F_s \circ g \circ \varphi$ to V.	

I claim (V, ψ_s) is *compatible* with any other chart (V', ψ') from the atlas constructed in step 10. Why? Because $V \cap V'$ does not contain p, whence the transition map $\psi' \circ \psi_s^{-1}$ where

$$\psi_s(V \cap V') \xleftarrow{\psi_s} V \cap V' \xrightarrow{\psi'} \psi'(V \cap V')$$

is a diffeomorphism. Yet, for distinct parameters $s, t \in \mathbf{R}^+ \setminus \{1\}$, the charts (V, ψ_s) and (V, ψ_t) are *not compatible* with each other. Why? Because $p \in V$. The transition map $\psi_t \circ \psi_s^{-1}$,

$$\hat{V} \xleftarrow{\psi_s} V \xrightarrow{\psi_t} \hat{V}$$

is not differentiable at $\psi_s(p)$. We conclude there's a unique smooth structure on the topological manifold M for each value of the parameter $s \in \mathbf{R}^+ \setminus \{1\}$. $\square$

[1, Nos. 1–7]. *Given.* Let N denote the *north pole* $(0, \ldots, 0, 1) \in S^n \subset \mathbf{R}^{n+1}$, and let S denote the *south pole.* Define the *stereographic projection* $\sigma \colon S^n \setminus \{N\} \to \mathbf{R}^n$

$$\sigma\left(x^1, \ldots, x^{n+1}\right) = \frac{(x^1, \ldots, x^n)}{1 - x^{n+1}}.$$

Let $\tilde{\sigma}(x) = -\sigma(-x)$ for $x \in S^n \setminus \{S\}$.

To prove.

a. For $x \in S^n \setminus \{N\}$, $\sigma(x) = u$, where $(u, 0)$ is the point where the line through N and x intersects the linear subspace where $x^{n+1} = 0$. (There's a similar intersection for $\tilde{\sigma}$. Find it.)

b. σ is bijective, and its inverse is

$$\sigma^{-1}(u^1, \ldots, u^n) = \frac{(2u^1, \ldots, 2u^n, |u|^2 - 1)}{|u|^2 + 1}.$$

c. We compute the transition map $\tilde{\sigma} \circ \sigma^{-1}$ and verify that the atlas consisting of the two charts $(S^n \setminus \{N\}, \sigma)$ and $(S \setminus \{S\}, \tilde{\sigma})$ defines a smooth structure on S^n.

d. This smooth structure is the same as the one defined in Example 1.31 (spheres).

[1, Nos. 1–8]. *Given.* By identifying $\mathbf{R}^2$ with $\mathbf{C}$, we can think of the unit circle S^1 as a subset of the complex plane. An *angle function* on a subset $U \subset S^1$ is a continuous map $\theta \colon U \to \mathbf{R}$ such that $e^{i\theta(z)} = z$ for all $z \in U$.

To prove.

- There exists an angle function θ on an open subset $U \subset S^1$ if and only if $U \neq S^1$.
- For any such angle function, (U, θ) is a smooth coordinate chart for S^1 with its standard smooth structure.

References

[1] J. M. Lee, *Introduction to Smooth Manifolds.* New York: Springer-Verlag, 2003.