HOMEWORK 9

COLTON GRAINGER (MATH 6230 INTRO TO DIFF GEO 1)

Assignment due 2019-04-03

[1] Exercise 8.18. *Given.* Let $F\colon \mathbf{R}^2 \to \mathbf{R}$ be the smooth immersion

$$F(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}.$$

Let $X \in \Gamma(T\mathbf{R})$ be the smooth section (vector field)

$$X_a = \left.\frac{\partial}{\partial t}\right|_a$$

and let $Y \in \gamma(T\mathbf{R}^2)$ be the smooth section (vector field)

$$Y_{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]} = u \left.\frac{\partial}{\partial y}\right|_{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]} - v \left.\frac{\partial}{\partial x}\right|_{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]}.$$

To prove. The vector fields X and Y are F-related, i.e., following diagram commutes.

$$\begin{tikzcd}
T\mathbf{R} \arrow[r, "dF"] & T\mathbf{R}^2 \\
\mathbf{R} \arrow[u, "X"] \arrow[r, "F"] & \mathbf{R}^2 \arrow[u, "Y"']
\end{tikzcd}$$

Proof.

1. Directly, we take the differential

$$dF_a = \begin{bmatrix} -\sin a \\ \cos a \end{bmatrix}$$

with respect to standard coordinates on $\mathbf{R}$ and $\mathbf{R}^2$. Then dF_a acts on $\left.\frac{\partial}{\partial t}\right|_a$ so that (w.r.t. coords)

$$\begin{aligned}
dF_a \left.\frac{\partial}{\partial t}\right|_a &= \begin{bmatrix} \left.\frac{\partial}{\partial x}\right|_{F(a)} & \left.\frac{\partial}{\partial y}\right|_{F(a)} \end{bmatrix} \begin{bmatrix} -\sin a \\ \cos a \end{bmatrix} \\
&= \begin{bmatrix} -\left.\frac{\partial}{\partial x}\right|_{F(a)} & \left.\frac{\partial}{\partial y}\right|_{F(a)} \end{bmatrix} \begin{bmatrix} F_2(a) \\ F_1(a) \end{bmatrix} \\
&= Y_{F(a)}.
\end{aligned}$$

2. Equivalently, for any f representing a *germ* of a $C^\infty(U)$ function out of N, take

$$\text{a point} \quad \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] \in U \quad \text{and } a \in \mathbf{R} \text{ s.th. } F(a) = \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right].$$

For clarity, f a smooth function $f\colon U \subset \mathbf{R}^2 \to \mathbf{R}$, dependent on the coordinates x and y. So

$$f \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{is a real number.}$$

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To show that X and Y are F-related, we need to show that the action of X and Y on fF and f, respectively, is intertwined. It suffices to check at the arbitrary point $a \in \mathbf{R}$, that is, to check

$$X_a(fF) = (Yf)_{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]} F.$$

First,

$$\begin{aligned}
X_a(fF) &= \left.\frac{\partial}{\partial t}\right|_a (f \circ F) \\
&= \begin{bmatrix} \left.\frac{\partial f}{\partial x}\right|_{F(a)} & \left.\frac{\partial f}{\partial y}\right|_{F(a)} \end{bmatrix} \begin{bmatrix} \left.\frac{\partial \cos t}{\partial t}\right|_a \\ \left.\frac{\partial \sin t}{\partial t}\right|_a \end{bmatrix} \\
&= \begin{bmatrix} \left.\frac{\partial f}{\partial x}\right|_{F(a)} & \left.\frac{\partial f}{\partial y}\right|_{F(a)} \end{bmatrix} \begin{bmatrix} -\sin a \\ \cos a \end{bmatrix}.
\end{aligned}$$

And likewise (TODO: this is wrong),

$$\begin{aligned}
(Y_{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]} f)F &= \left(u \left.\frac{\partial f}{\partial y}\right|_{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]} - v \left.\frac{\partial f}{\partial x}\right|_{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]} \right) \quad \text{such that} \quad \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos a \\ \sin a \end{bmatrix} \\
&= \cos a \left.\frac{\partial f}{\partial y}\right|_{F(a)} - \sin a \left.\frac{\partial f}{\partial x}\right|_{F(a)}.
\end{aligned}$$

□

References

[1] J. M. Lee, *Introduction to Smooth Manifolds*. New York: Springer-Verlag, 2003.