MATH 6210 HOMEWORK 11

COLTON GRAINGER
APRIL 16, 2019

Problems set from Chapter 9, Lee, *Introduction to Smooth Manifolds.*

1. 9-1. Suppose M is a smooth manifold, X is a vector field in $\mathfrak{X}M$, and γ is a maximum integral curve of X.

(a) *Either γ is constant, γ is injective, or γ is periodic and nonconstant.*

(b) *If γ is periodic and nonconstant, then there's a unique positive number T (called the* period *of γ) such that $\gamma(t) = \gamma(t')$ if and only if $t - t' \in T\mathbb{Z}$.*

(c) *The image of γ is an immersed submanifold of M, diffeomorphic to $\mathbb{R}$, S^1, or* $\mathrm{pt} \approx \mathbb{R}^0$.

2. 9-7. Say M is a connected smooth manifold. *The group of diffeomorphisms* $\mathrm{Diff}(M)$ *acts transitively on M.*

Lemma. Let $B \subset \mathbb{R}^n$ be an open unit ball. If $p, q \in B$, there is a compactly supported smooth vector field $X \in \mathfrak{X}B$ whose flow θ satisfies $\theta_1(p) = q$.

3. 9-10. *For each of the following vector fields, find smooth coordinates in a neighborhood of $\begin{bmatrix}1 & 0\end{bmatrix}^t$ such that the given vector field is a coordinate vector field.*

(a) $V = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$.

(b) $W = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}$.

(c) $X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$.

(d) $Y = x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}$.

4. 9-16. *Exhibit smooth vector fields V, $\widetilde{V}$, and W on $\mathbb{R}^2$ such that $V = \widetilde{V} = \partial/\partial x$ along the x-axis but $\mathcal{L}_V W \neq \mathcal{L}_{\widetilde{V}} W$ at the origin.*

Note. This shows that it is necessary to know the vector field V to compute $\mathcal{L}_V W$ at a point $p \in M$; it is not sufficient to just know the vector V_p, or even to know the values of V along an integral curve of V. ◂

5. 9-18. Define vector fields $X, Y \in \mathfrak{X}\mathbb{R}^2$ by

$$X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \quad Y = x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}.$$

(a) *Compute the flows ${}_X\mathfrak{fl}$ and ${}_Y\mathfrak{fl}$ of X and Y.*

(b) *Find open intervals ${}_XB$ and ${}_YB$ containing 0 such that*

$${}_X\mathfrak{fl}_t \circ {}_Y\mathfrak{fl}_s - {}_Y\mathfrak{fl}_s \circ {}_X\mathfrak{fl}_t \quad \textit{is defined for all } \begin{bmatrix}t & s\end{bmatrix}^t \textit{ in } {}_XB \times {}_YB,$$

but nonzero at some time $\begin{bmatrix}\bar{t} & \bar{s}\end{bmatrix}^t$.

6. 9-19. Consider $M = \mathbb{R}^3 \setminus \{z\text{-axis}\}$, the open submanifold of $\mathbb{R}^3$ formed by removing the z-axis. Define $V, W \in \mathfrak{X}M$ by

$$V = \frac{\partial}{\partial x} - \frac{y}{x^2+y^2}\frac{\partial}{\partial y}, \quad W = \frac{\partial}{\partial y} + \frac{x}{x^2+y^2}\frac{\partial}{\partial z},$$

and let ${}_V\mathfrak{fl}$, ${}_W\mathfrak{fl}$ be the flows of V and W, respectively. *Then V and W commute, but (caveat!) there exist a point $p \in M$ and times $s, t \in \mathbb{R}$ such that*

$$({}_V\mathfrak{fl}_t \circ {}_W\mathfrak{fl}_s)(p) - ({}_W\mathfrak{fl}_s \circ {}_V\mathfrak{fl}_t)(p) \quad \textit{is defined and nonzero.}$$