EXAMPLES OF SMOOTH MANIFOLDS

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[1, No. 1.18]. *Given.* Let M be a topological manifold.

To prove. Two smooth atlases for M determine the same smooth structure if and only if their union is a smooth atlas.

[1, Nos. 1–6]. *Given.* Let M be a nonempty topological manifold of dimension $n \geq 1$.

To prove. If M has a smooth structure, then it has uncountably many distinct ones. (Hint: for any $s > 0$, $F_s(x) = |x|^{s-1}x$ defines a homeomorphism from B^n to itself, which is a diffeomorphism if and only if $s = 1$.)

[1, Nos. 1–7]. *Given.* Let N denote the *north pole* $(0, \ldots, 0, 1) \in S^n \subset \mathbf{R}^{n+1}$, and let S denote the *south pole*. Define the *stereographic projection* $\sigma\colon S^n \setminus \{N\} \to \mathbf{R}^n$

$$\sigma\left(x^1, \ldots, x^{n+1}\right) = \frac{(x^1, \ldots, x^n)}{1 - x^{n+1}}.$$

Let $\tilde{\sigma}(x) = -\sigma(-x)$ for $x \in S^n \setminus \{S\}$.

To prove.

a. For $x \in S^n \setminus \{N\}$, $\sigma(x) = u$, where $(u, 0)$ is the point where the line through N and x intersects the linear subspace where $x^{n+1} = 0$. (There's a similar intersection for $\tilde{\sigma}$. Find it.)

b. σ is bijective, and its inverse is

$$\sigma^{-1}(u^1, \ldots, u^n) = \frac{(2u^1, \ldots, 2u^n, |u|^2 - 1)}{|u|^2 + 1}.$$

c. We compute the transition map $\tilde{\sigma} \circ \sigma^{-1}$ and verify that the atlas consisting of the two charts $(S^n \setminus \{N\}, \sigma)$ and $(S \setminus \{S\}, \tilde{\sigma})$ defines a smooth structure on S^n.

d. This smooth structure is the same as the one defined in Example 1.31 (spheres).

[1, Nos. 1–8]. *Given.* By identifying $\mathbf{R}^2$ with $\mathbf{C}$, we can think of the unit circle S^1 as a subset of the complex plane. An *angle function* on a subset $U \subset S^1$ is a continuous map $\theta\colon U \to \mathbf{R}$ such that $e^{i\theta(z)} = z$ for all $z \in U$.

To prove.

- There exists an angle function θ on an open subset $U \subset S^1$ if and only if $U \neq S^1$.
- For any such angle function, (U, θ) is a smooth coordinate chart for S^1 with its standard smooth structure.

References

[1] J. M. Lee, *Introduction to Smooth Manifolds.* New York: Springer-Verlag, 2003.

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