Homework #1 Solutions

(C.12) a) By the chain rule for total derivatives,

$$D(G\circ F)(x) = DG(F(x)) \circ DF(x)$$

In matrix form, we can write this as

$$\left[\frac{\partial (G\circ F)^i}{\partial x^j}(x)\right] = \left[\frac{\partial G^i}{\partial y^k}(F(x))\right]\left[\frac{\partial F^k}{\partial x^j}(x)\right]$$

Since $(G\circ F)^i = G^i \circ F$, the (i,j)th entry of this matrix equation says

$$\frac{\partial G^i\circ F}{\partial x^j}(x) = \sum_{k=1}^{m} \frac{\partial G^i}{\partial y^k}(F(x))\,\frac{\partial F^k}{\partial x^j}(x).$$

b) The chain rule says that if F and G are differentiable, then so is $G\circ F$. By induction, if F and G are C^k, so is $G\circ F$. Therefore if F and G are smooth, then $G\circ F$ is C^k for all $k \geq 0$, and hence smooth.

(C.39) Polar coordinates: The map $F : (0,\infty)\times(-\pi,\pi) \to \mathbb{R}^2$ defined by

$$F(r,\theta) = (r\cos\theta,\ r\sin\theta)$$

is C^∞, and a bijection from $(0,\infty)\times(-\pi,\pi)$ to $\mathbb{R}^2 - \{(x,0) \mid x \leq 0\}$. Its Jacobian is

$$J = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

which has determinant $r \neq 0$, so by Cor C.36 it is a diffeomorphism (Its inverse map is

$$F^{-1}(x,y) = \left(\sqrt{x^2+y^2},\ \frac{1}{i}\log\left(\frac{x+iy}{\sqrt{x^2+y^2}}\right)\right)$$

for an appropriately chosen branch of the log function on $\mathbb{C} - \{z \mid \operatorname{Re}(z) \leq 0,\ \operatorname{Im}(z) = 0\}$)

<u>Spherical coordinates</u>: The map

$$G(\rho, \varphi, \theta) = (\rho \sin\varphi \cos\theta,\ \rho \sin\varphi \sin\theta,\ \rho \cos\varphi)$$

is C^∞, and a bijection from

$$\{(\rho, \varphi, \theta) : \rho > 0,\quad 0 < \varphi < \pi,\quad -\pi < \theta < \pi\}$$

onto its image. Its Jacobian is

$$J = \begin{bmatrix} \sin\varphi \cos\theta & \rho\cos\varphi \cos\theta & -\rho \sin\varphi \sin\theta \\ \sin\varphi \sin\theta & \rho \cos\varphi \sin\theta & \rho \sin\varphi \cos\theta \\ \cos\varphi & -\rho \sin\varphi & 0 \end{bmatrix}$$

which has determinant $\rho^2 \sin\varphi \neq 0$ on the given domain. So by Cor. C.36 it is a diffeomorphism.

Additional Problem 1

We can write

$$g(\vec{v}, \vec{w}) = \begin{bmatrix} a^1 & \cdots & a^n \end{bmatrix} \begin{bmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & & \\ g_{n1} & \cdots & g_{nn} \end{bmatrix} \begin{bmatrix} b^1 \\ \vdots \\ b^n \end{bmatrix}$$

$$= \vec{a}^t A_g \vec{b}$$

Under the given change of basis, we have

$$\tilde{\vec{a}} = R^{-1} \vec{a}, \quad \tilde{\vec{b}} = R^{-1} \vec{b}$$

$$\Rightarrow \vec{a} = R \tilde{\vec{a}}, \quad \vec{b} = R \tilde{\vec{b}}.$$

The matrix $\tilde{A}_g$ should satisfy

$$g(\vec{v}, \vec{w}) = \tilde{\vec{a}}^t \tilde{A}_g \tilde{\vec{b}} = \vec{a}^t A_g \vec{b}$$

$$= \tilde{\vec{a}}^t R^t A_g R \tilde{\vec{b}}$$

for all vectors $\tilde{\vec{a}}, \tilde{\vec{b}}$. Therefore,

$$\tilde{A}_g = R^t A_g R.$$

Contrast with a linear transformation $T: V \to V$ which would transform as

$$\tilde{A}_T = R^{-1} A_T R.$$

So no, a metric is <u>not</u> the same kind of object as a linear transformation, because they transform differently under a change of basis for V.