# Math 6210 NOTES: DIFFERENTIAL GEOMETRY

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These notes were taken in University of Colorado's Math 6210 (Differential Geometry) class in Spring 2019, taught by Prof. Jeanne Clelland. I live-TEXed them with `vim`, so there may be typos and failures of understanding. Any mistakes are my own. Please send questions, comments, complaints, and corrections to colton.grainger@colorado.edu. Thanks to `adebray` for the LATEX template, which I have forked from https://github.com/adebray/latex_style_files.

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Lecture 1.

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Last time, we defined covectors, cotangent vectors, and a basis for the *cotangent space at a point*, given a basis of tangent vectors for the tangent space.

That is, by choosing a coordinate chart $(U, (x^i))$ about a point $p \in U$, we have a basis for $T_pM$

$$\left.\frac{\partial}{\partial x^1}\right|_p, \dots, \left.\frac{\partial}{\partial x^n}\right|_p,$$

and therefore a dual basis for $T_p^*M$

$$\left.\mathrm{d}x^1\right|_p, \dots, \left.\mathrm{d}x^n\right|_p.$$

**Example 1.1** (Changing the cotangent basis)**.** Let $(\tilde{x}^j)$ be another local coordinate chart in a neighborhood of $p$. Then

$$\left.\frac{\partial}{\partial x^i}\right|_p = \sum_{j=1}^n \frac{\partial \tilde{x}^j}{\partial x^i} \left.\frac{\partial}{\partial \tilde{x}^j}\right|_p.$$

Now let $\left(\mathrm{d}\tilde{x}^j\right)$ be the dual basis to $\left(\frac{\partial}{\partial \tilde{x}^j}\big|_p\right)$. Consider a cotangent vector $w \in T_p^*M$. Written in terms of both the original and the new cotangent bases, $w$ is expressed as

$$w = \sum_{i=1}^n a_i \left.\mathrm{d}x^i\right|_p = \sum_{j=1}^n \tilde{a}_j \left.\mathrm{d}\tilde{x}^j\right|_p.$$

To determine the relation between the original basis representation $[a_i]$ and the new representation $[\tilde{a}_j]$, evaluate the covector $w$ at each tangent vector $\frac{\partial}{\partial x^i}\big|_p$ in the original basis. Then

$$\begin{aligned}
a_i &= w\left(\frac{\partial}{\partial x^i}\bigg|_p\right) && \text{because } \mathrm{d}x^i \frac{\partial}{\partial x^j} = \delta^i_j \\
&= w\left(\sum_{j=1}^n \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\partial}{\partial \tilde{x}^j}\bigg|_p\right) && \text{by the chain rule} \\
&= \sum_{j=1}^n \frac{\partial \tilde{x}^j}{\partial x^i} w\left(\frac{\partial}{\partial \tilde{x}^j}\bigg|_p\right) && \text{by linearity} \\
&= \sum_{j=1}^n \tilde{a}_j \frac{\partial \tilde{x}^j}{\partial x^i} && \text{because } \mathrm{d}\tilde{x}^r \frac{\partial}{\partial \tilde{x}^j} = \delta^i_j.
\end{aligned}$$

⮘

**Exercise 1.2.** Give a numerical representation of the change of basis map taking $[a_i]$ to $[\tilde{a}j]$. Compare this to the change of basis on $T_pM$ with $\tilde{v}^j$ written in terms of a basis $v^i$.

We prefer to express old covectors in terms of a new covector basis. Whereas with tangent vectors, the chain rule gives us a means (and hence a preference) to express the new vectors in terms of the old tangent vector basis.

**Example 1.3** (Basis covectors)**.** Let $w$ be the cotangent vector in example 1.1. Then

$$w = \sum_{j=1}^n \tilde{a}_j \,\mathrm{d}\tilde{x}^j \bigg|_p = \sum_{i=1}^n a_i \,\mathrm{d}x^i \bigg|_p \quad \text{and} \quad w = \sum_{i,j=1}^n \tilde{a}_j \frac{\partial \tilde{x}^j}{\partial x^i}\,\mathrm{d}x^i \bigg|_p . \tag{1.4}$$

As the scalars $[\tilde{a}_j]$ range through $\mathbb{R}^n$, the relation in (1.4) determines the representation of the new basis covector in terms of the old basis covectors:

$$\mathrm{d}\tilde{x}^j \bigg|_p = \sum_{i=1}^n \frac{\partial \tilde{x}^j}{\partial x^i}\,\mathrm{d}x^i \bigg|_p . \tag{1.5}$$

⮘

*Note.* We have to hurry on to define covector fields, in order to state Stokes' theorem by the end of the semester. So we're glossing over the theory of vector bundles. ⮘

**Definition 1.6** (Cotangent bundle)**.** Let $M \in \mathsf{Man}^n$. The *cotangent bundle* over $M$ is the disjoint union

$$T^*M = \coprod_{p\in M} T^*_pM,$$

with projection $\pi\colon T^*M \to M$ sending cotangent vectors $w_p \mapsto p$ onto $M$.

**Proposition 1.7.** *The cotangent bundle is a smooth vector bundle over $M$.*

*Proof.* By definitions, fiber over $p \in M$ is the cotangent vector space at $p$. We claim $\pi$ is a smooth projection with the fibers over $M$ varying smoothly.

Now consider a local coordinate chart $(x^i)$ on $U \subset M$. We will show that $U$ is a trivializing neighborhood for $\pi$. The coordinate cotangent vectors determine $n$ (local) sections back into the cotangent bundle (called *coordinate covector fields* on $M$). These are

$$\mathrm{d}x^i : U \to T^*U \quad \text{such that} \quad \mathrm{d}x^i\,(p) = \mathrm{d}x^i \bigg|_p \in T^*_pM. \tag{1.8}$$

The coordinate chart $(U, (x^i))$ then determines a chart for the open neighborhood $\pi^{-1}(U) = T^*U$. Each covector $w \in \pi^{-1}(U)$ is in the fiber of some $p \in U$, and can be expressed in terms of the cotangent basis at $p$

$$w_p = \begin{bmatrix} \xi_1 & \cdots & \xi_n \end{bmatrix} \begin{bmatrix} \mathrm{d}x^1\big|_p \\ \vdots \\ \mathrm{d}x^n\big|_p \end{bmatrix}.$$

We'll define our chart $\pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{R}^n$ pointwise

$$\sum_{i=1}^n \xi_i \, \mathrm{d}x^i \bigg|_p \longmapsto (x^1, \ldots, x^n, \xi_1, \ldots, \xi_n).$$

Taking for granted that this correspondence is smooth, we have shown the atlas $\{U_i, \varphi_i\}$ for $M$ induces a local trivialization of the cotangent bundle. □

**Definition 1.9** (Covector fields)**.** A global section of $T^*M$ is a *global covector field* (a.k.a., a differential 1-form), and a local section of $T^*M$ is a *local covector field.*

If $\omega \in \Gamma(T^*M)$ is a covector field, and $X \in \mathfrak{X}M$ is a vector field, then we can define a scalar function $\omega(X)\colon M \to \mathbb{R}$ such that $\omega(X)(p) = \omega\big|_p(X_p)$. In local coordinates,

$$\text{if} \quad X = X^i \frac{\partial}{\partial x^i}, \quad \text{and} \quad \omega = \xi_j \, \mathrm{d}x^j, \quad \text{then} \quad \omega(X) = \sum_i \xi_i(x) X^i(x).$$

**Definition 1.10** (Pullbacks of covector fields)**.** Let $M \xrightarrow{F} N$ be a map in $\mathsf{Man}$, with $p$ a point of $M$. The differential $\mathrm{d}F_p : T_pM \to T_{F(p)}N$ has a dual linear map $\mathrm{d}F_p^* : T^*_{F(p)} \to T^*_pM$ called the *pullback map* by $F$ at $p$, or the *cotangent map* at $p$. It's characterized by the property that for all $\mathbf{v} \in T_pM$, and covectors $\omega \in T^*_{F(p)}N$,

$$(\mathrm{d}F_p)^*(\omega)(\mathbf{v}) = \omega(\mathrm{d}F_p\,(\mathbf{v})).$$

**Definition 1.11** (Coframe fields)**.** A *local coframe field* on $U \subset M$ is an ordered $n$-tuple $(\varepsilon^1, \ldots, \varepsilon^n)$ of covector fields such that, for all $p \in U$, the evaluation of $(\varepsilon^1, \ldots, \varepsilon^n)$ at $p$ forms a basis for $T^*_pM$.

Suppose $(E^1, \ldots, E^n)$ is a frame field on $U \subset M$, then the *dual coframe field* $(\varepsilon^1, \ldots, \varepsilon^n)$ is defined by

$$\varepsilon^i(E_j) = \delta^i_j.$$

We write $\mathfrak{X}^*M$ for the *smooth covector fields* on $M$.

**Exercise 1.12.** Does each smooth manifold $M$ admit a smooth covector field?[1]

We proceed to "discover" differential 1-forms.

**Definition 1.13** (Differential of a scalar function)**.** Let $M \in \mathsf{Man}$ and $f \in C^\infty(M)$. The *differential of $f$* is the covector field $\mathrm{d}f$ on $M$ defined by

$$\mathrm{d}f\bigg|_p(\mathbf{v}) = \mathbf{v}(f) \quad \text{for all } \mathbf{v} \in T_pM.$$

We've specified how $\mathrm{d}f$ acts at each point, for each tangent vector. Doesn't this *look* like a Krönecker pairing?

[1]Patrick asked if it's true that every $M \in \mathsf{Man}$ admits a smooth nonvanishing covector field. I have no idea.

**Example 1.14** (Coordinate representation of a differential)**.** Consider local coordinates $(x^i)$ on $M$. If we write $\mathbf{v} = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}\Big|_p$ and $\mathrm{d}f\Big|_p = \sum_{j=1}^n a_j\,\mathrm{d}x^j\Big|_p$, then

$$\begin{aligned}\sum_{i=1}^n v^i a_i &= \left(a_j\,\mathrm{d}x^j\,\bigg|_p\right)\left(v^i\frac{\partial}{\partial x^i}\bigg|_p\right)\\ &= \mathrm{d}f\bigg|_p(\mathbf{v})\\ &= \mathbf{v}\bigg|_p(f)\\ &= \sum_{i=1}^n v^i\frac{\partial f}{\partial x^i}\bigg|_p\end{aligned}$$

Varying the scalars $\begin{bmatrix} v^1 & \cdots & v^n\end{bmatrix}^t$ for the tangent vector $\mathbf{v}$, we determine each scalar component

$$a_i = \frac{\partial f}{\partial x^i}\bigg|_p$$

in the basis representation of $\mathrm{d}f\Big|_p$. Therefore the differential of $f$ in terms of local coordinates is

$$\mathrm{d}f\bigg|_p = \frac{\partial f}{\partial x^i}\,\mathrm{d}x^i\,\bigg|_p.$$

◀

We have also shown $\mathrm{d}x^i = \mathrm{d}(x^i)$ for the coordinate functions $x^i$ of chart $(U,(x^i))$. That is, $\mathrm{d}x^i$ is the differential of $x^i\colon U \to \mathbb{R}$.

Defining the differential of a scalar function as an *evaluation* extends the notion of the "differential" given for $\mathrm{d}x$ in high school calculus. The two concepts are the same for a function $f\colon M \to \mathbb{R}$ if we choose to make the identification $\mathrm{d}f\colon T_pM \to T_{f(p)}\mathbb{R} \cong \mathbb{R}$.

**Proposition 1.15.** *Let $\gamma\colon J \to M$ be a smooth curve, $f \in C^(M)$. Then the differential of the function $f\circ\gamma\colon J \to \mathbb{R}$ is given by (the familiar evaluation)*

$$(f\circ\gamma)'(t) = \mathrm{d}f_{\gamma(t)}\,(\gamma'(t)).$$

**Exercise 1.16.** Soft question: why should a covector represent an infinitesimal path? Can this be connected with local 1-parameter group actions?