# EXAMPLES OF SMOOTH MANIFOLDS

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## Assignment due 2019-01-30

**[1, No. 1.18].** *Given.* Let $M$ be a topological manifold.

*To prove.* Two smooth atlases for $M$ determine the same smooth structure if and only if their union is a smooth atlas.

*Proof.* ($\Rightarrow$) Let $\mathscr{A}$, $\mathscr{B}$ be two smooth atlases on a topological manifold. Say that $\mathscr{A} \cup \mathscr{B}$ is a smooth atlas. Then the unique maximal smooth atlas $\overline{\mathscr{A} \cup \mathscr{B}}$, which certainly contains $\mathscr{A}$ and $\mathscr{B}$, forces the conclusion:

$$\overline{\mathscr{A}} = \overline{\mathscr{A} \cup \mathscr{B}} = \overline{\mathscr{B}},$$

that is, both $\mathscr{A}$ and $\mathscr{B}$ determine the same smooth structure on $M$.

($\Leftarrow$) Say that $\mathscr{A}$ and $\mathscr{B}$ are smooth atlases that determine the same smooth structure on $M$. For contradiction, suppose that $\mathscr{A}$ and $\mathscr{B}$ are not compatible, so that their union contains at least $(U, \varphi) \in \mathscr{A}$ and $(V, \psi) \in \mathscr{B}$ for which $U \cap V \neq \varnothing$ and

$$\varphi(U \cap V) \xrightarrow{\varphi^{-1}} U \cap V \xrightarrow{\psi} \psi(U \cap V)$$

is *not* a diffeomorphism. Now

$$\begin{aligned} (U, \varphi) &\in \mathscr{A} \subset \overline{\mathscr{A}}, \text{ and} \\ (V, \psi) &\in \mathscr{B} \subset \overline{\mathscr{B}}, \end{aligned}$$

so either

- $\overline{\mathscr{A}} = \overline{\mathscr{B}}$ is not smooth, or
- $\overline{\mathscr{A}} \neq \overline{\mathscr{B}}$ are not the same smooth structures.

Both conclusions are contrary to our hypotheses, so it had better be that $\mathscr{A}$ and $\mathscr{B}$ are compatible! $\square$

**[1, Nos. 1–6].** *Given.* Let $M$ be a nonempty topological manifold of dimension $n \geq 1$.

*To prove.* If $M$ has a smooth structure, then it has uncountably many distinct ones.

*Proof.* We construct uncountably many distinct smooth structures on $M$.

| step of construction | justification |
|---|---|
| 1. Say $\mathscr{A}$ is a maximal smooth atlas on $M$. | Hypothesis. |
| 2. Of the charts in $\mathscr{A}$, take a countable basis of regular coordinate balls and form the smooth atlas $\{(B_i, \varphi_i)\}$. | Every smooth manifold has a countable basis of regular coordinate balls. [1, p. 15] |
| 3. Form an open, locally finite, refinement $\{V_j\}$ of the cover $\{B_i\}$. | $\{B_i\}$ is an open cover of paracompact $M$. |
| 4. For each $V_j$, choose $(B_i, \varphi_i)$ such that $V_j \subset B_i$ and define $\psi_j := \varphi_i\|_{V_j}$. | |
| 5. Note that $\{(V_j, \psi_j)\}$ is a smooth atlas. | $\{V_j\}$ covers $M$. For each pair $\psi_k$, $\psi_\ell$, in the atlas, the transition map $\varphi_k \circ \psi_\ell^{-1}$ is (the restriction of) a diffeomorphism. |

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| step of construction | justification |
|---|---|
| 6. Consider a point in $M$, a neighborhood of this point, and take all $V_1, \dots, V_n$ of $\{V_j\}$ which meet this neighborhood. | $\{V_j\}$ is locally finite, $M$ is non-empty. |
| 7. Remove an open set $V_k$ from $V_1, \dots, V_n$ if $V_k$ is not covered by $\{V_1, \dots, \hat{V}_k, \dots, V_n\}$. | Choice. |
| 8. Repeat step 7 until there's a open $V$ from the initial $n$ that's not covered by those open sets of the initial $n$ remaining. | Finite recursion. |
| 9. Choose a point $p \in V$ such that $p$ is not in any other open set of those remaining after step 8. | |
| 10. Note the atlas $\{(V_j, \psi_i)\}$ from step 5, with $\{(V_1, \psi_1), \dots, (V_n, \psi_n)\}$ removed, with $\{$charts remaining after step 8$\}$ adjoined, is smooth. | |
| 11. Moreover, the smooth atlas in step 10 has a particular chart $(V, \psi)$ such that $p \in V$ alone. | $p \in V$, yet $p \notin V_1, \dots, \hat{V}, \dots, V_n$. |
| 12. Consider the image of $V$ and $p$ in $\mathbf{R}^n$ under the map $\varphi\colon B \to \hat{B} \subset \mathbf{R}^n$, where $B$ is the regular coordinate ball chosen in step 4 to contain $V$. | |
| 13. Diffeomorph $\hat{B}$ to itself so that $\varphi(p) \in \hat{V}$ is mapped to the origin. Denote this diffeomorphism by $g$. | WLOG, say $\hat{B}$ is a unit ball centered at the origin. Then $g$ is a tangent map $\hat{B} \to \mathbf{R}^n$, followed by a rigid translation in $\mathbf{R}^n$, finished with an inverse tangent map $\mathbf{R}^n \to \hat{B}$. |
| 14. For all reals $s > 0$ such that $s \neq 1$, there's a function $F_s$ that homeomorphs $\hat{B}$ to itself. | $F_s\colon x \mapsto \lvert x \rvert^{s-1} x$. |
| 15. The restriction of $F_s \circ g$ to the punctured unit ball $\hat{B} \setminus \{0\}$, *is indeed* a diffeomorphism of $\hat{B} \setminus \{0\}$. | Partial derivatives of all orders exist for both $F_s$ and its inverse $F_{1/s}$. |
| 16. When $s \neq 1$, $F_s \circ g$ is either *not smooth* at the origin or *fails to have a smooth inverse.* | Either $F_s$ or $F_s^{-1}$ fails to be of class $C^\infty$ on $\hat{B}$. There's no good linear approximation of the norm of a vector at the origin. |
| 16. Define $\psi_s$ as the restriction of the composite $F_s \circ g \circ \varphi$ to $V$. | |

I claim $(V, \psi_s)$ is *compatible* with any other chart $(V', \psi')$ from the atlas constructed in step 10. Why? Because $V \cap V'$ does not contain $p$, whence the transition map $\psi' \circ \psi_s^{-1}$ where

$$\psi_s(V \cap V') \xleftarrow{\psi_s} V \cap V' \xrightarrow{\psi'} \psi'(V \cap V')$$

is a diffeomorphism. Yet, for distinct parameters $s, t \in \mathbf{R}^+ \setminus \{1\}$, the charts $(V, \psi_s)$ and $(V, \psi_t)$ are *not compatible* with each other. Why? Because $p \in V$. The transition map $\psi_t \circ \psi_s^{-1}$,

$$\hat{V} \xleftarrow{\psi_s} V \xrightarrow{\psi_t} \hat{V}$$

is not differentiable at $\psi_s(p)$. We conclude there's a unique smooth structure on the topological manifold $M$ for each value of the parameter $s \in \mathbf{R}^+ \setminus \{1\}$. $\square$

**[1, Nos. 1–7].** *Given.* Let $N$ denote the *north pole* $(0,\dots,0,1) \in S^n \subset \mathbf{R}^{n+1}$, and let $S$ denote the *south pole*. Define the *stereographic projection* $\sigma\colon S^n \setminus \{N\} \to \mathbf{R}^n$

$$\sigma\left(x^1,\dots,x^{n+1}\right) = \frac{\left(x^1,\dots,x^n\right)}{1-x^{n+1}}.$$

Let $\tilde{\sigma}(x) = -\sigma(-x)$ for $x \in S^n \setminus \{S\}$.

*To prove.*

a. For $x \in S^n \setminus \{N\}$, $\sigma(x) = u$, where $(u,0)$ is the point where the line through $N$ and $x$ intersects the linear subspace where $x^{n+1} = 0$. (There's a similar intersection for $\tilde{\sigma}$. Find it.)

b. $\sigma$ is bijective, and its inverse is

$$\sigma^{-1}(u^1,\dots,u^n) = \frac{(2u^1,\dots,2u^n,|u|^2-1)}{|u|^2+1}.$$

c. We compute the transition map $\tilde{\sigma}\circ\sigma^{-1}$ and verify that the atlas consisting of the two charts $(S^n \setminus \{N\},\sigma)$ and $(S \setminus \{S\},\tilde{\sigma})$ defines a smooth structure on $S^n$.

d. This smooth structure is the same as the one defined in Example 1.31 (spheres).

**[1, Nos. 1–8].** *Given.* By identifying $\mathbf{R}^2$ with $\mathbf{C}$, we can think of the unit circle $S^1$ as a subset of the complex plane. An *angle function* on a subset $U \subset S^1$ is a continuous map $\theta \colon U \to \mathbf{R}$ such that $e^{i\theta(z)} = z$ for all $z \in U$.

*To prove.*

- There exists an angle function $\theta$ on an open subset $U \subset S^1$ if and only if $U \neq S^1$.
- For any such angle function, $(U, \theta)$ is a smooth coordinate chart for $S^1$ with its standard smooth structure.

## References

[1] J. M. Lee, *Introduction to Smooth Manifolds.* New York: Springer-Verlag, 2003.