MATH 6230 HOMEWORK 12

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Problems from *Introduction to Smooth Manifolds*, Chapter 11.

30. Say that S is an immersed k-dimensional submanifold $\iota\colon S \to M$, and consider $f \in C^\infty(M)$. Then the differential of f restricted to S is the pullback of the differential of f under ι^*, i.e.,

$$\mathrm{d}\left(f\Big|_S \right) = \iota^*(\mathrm{d}f). \tag{30.1}$$

Proof. Let $p \in S$. Choose a slice chart $(U, (x^i))$ centered about $\iota(p)$ such that the first $k = \dim S$ coordinates $\iota^{-1}(x^i)$ parameterize $S \cap \iota^{-1}U$. Identifying $S \cap \iota^{-1}U \hookrightarrow \iota(S) \cap U$, both f and $f\big|_S$ are determined (in the neighborhood U of p) by coordinates by $(x^1, \dots, x^k, x^{k+1}, \dots, x^j)$.

Now we evaluate the cotangent vectors on the LHS and RHS of (30.1) at an arbitrary tangent vector $v_p \in T_pS \cong T_p(S \cap U)$, where $v_p = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x_i}\big|_p$. On the LHS, $f\big|_S$ is constant with respect to $x^{k+1}, \dots, x^n$, therefore

$$\begin{aligned}
\mathrm{d}\left(f\Big|_S \right)(v_p) &= \begin{bmatrix} \frac{\partial f}{\partial x^1} & \cdots & \frac{\partial f}{\partial x^k} \end{bmatrix}\Bigg|_p \begin{bmatrix} \mathrm{d}x^1 \\ \vdots \\ \mathrm{d}x^k \end{bmatrix}\Bigg|_p \begin{bmatrix} \frac{\partial}{\partial x^1} & \cdots & \frac{\partial}{\partial x^n} \end{bmatrix}\Bigg|_p \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial f}{\partial x^1} & \cdots & \frac{\partial f}{\partial x^k} \end{bmatrix}\Bigg|_p \begin{bmatrix} I_k \mid \mathbf{0} \end{bmatrix} \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial f}{\partial x^1} & \cdots & \frac{\partial f}{\partial x^k} \end{bmatrix}\Bigg|_p \begin{bmatrix} v^1 \\ \vdots \\ v^k \end{bmatrix}.
\end{aligned}$$

On the RHS, pulling back v_p under ι^*, we have

$$\iota^*(\mathrm{d}f_p)(v_p) = v_p f = \begin{bmatrix} \frac{\partial f}{\partial x^1} & \cdots & \frac{\partial f}{\partial x^k} \end{bmatrix}\Bigg|_p \begin{bmatrix} v^1 \\ \vdots \\ v^k \end{bmatrix}.$$

This shows that the covectors on the LHS and RHS of (30.1) agree at each point p in S. $\square$

6. Let M be a smooth n-dimensional manifold, let $W \subset M$ be an open subset of M, and consider k smooth functions $y^i\colon W \to \mathbb{R}$ for $i = 1, \dots, k$.

(a) At a point $p \in W$, if $k = n$ and

$$\mathrm{d}y^1\Big|_p, \dots, \mathrm{d}y^k\Big|_p \quad \text{are linearly independent in } T_p^*M,$$

then

$$\begin{bmatrix} y^1 \\ \vdots \\ y^k \end{bmatrix} \quad \text{is a local coordinate system in a neighborhood } U \subset W \text{ of } p.$$

Proof. Choose local coordinates (x^j) about p. Then each y^i has differential w.r.t. the (x^j) given by

$$\begin{bmatrix} \mathrm{d}y^1 \big|_p \\ \vdots \\ \mathrm{d}y^n \big|_p \end{bmatrix} = \begin{bmatrix} \frac{\partial y^1}{\partial x^1}\big|_p & \cdots & \frac{\partial y^1}{\partial x^n}\big|_p \\ \\ \frac{\partial y^n}{\partial x^1}\big|_p & \cdots & \frac{\partial y^n}{\partial x^n}\big|_p \end{bmatrix} \begin{bmatrix} \mathrm{d}x^1 \big|_p \\ \vdots \\ \mathrm{d}x^n \big|_p \end{bmatrix}.$$

Our hypothesis is that the set $\left\{ \begin{bmatrix} \frac{\partial y^i}{\partial x^1}\big|_p & \cdots & \frac{\partial y^i}{\partial x^n}\big|_p \end{bmatrix} \right\}_{i=1}^n$ (vectors of evaluated partial derivatives) is linearly independent over $\mathbb{R}^n$. Therefore the linear transformation $\begin{bmatrix} \frac{\partial y^i}{\partial x^j}\big|_p \end{bmatrix} = [\varphi_*]$ is invertible, where φ is the map

$$\varphi \colon W \to \mathbb{R}^n \quad \text{such that} \quad q \overset{\varphi}{\longmapsto} \begin{bmatrix} y^1(q) \\ \vdots \\ y^n(q) \end{bmatrix}. \tag{6.1}$$

By the inverse function theorem, there's an open neighborhood $U \subset W$ of p such that $\varphi^{-1} \circ \varphi = \mathrm{id} \colon U \to U$, where φ^{-1} is smooth. So $\varphi\big|_U$ is a diffeomorphism into $\mathbb{R}^n$. Therefore the (y^i) are local coordinates on U. ☐

(b) Similarly, if $p \in W$, $k < n$, and

$$\mathrm{d}y^1 \Bigg|_p, \ldots, \mathrm{d}y^k \Bigg|_p \quad \text{are linearly independent in } T_p^*M,$$

then

$$\begin{bmatrix} y^1 \\ \vdots \\ y^k \end{bmatrix} \quad \text{can be } \textit{extended} \text{ to a local coordinates in a neighborhood } U \subset W \text{ of } p.$$

Proof. Again, take local coordinates (x^j) about p. Each y^i has differential w.r.t. the (x^j) given by

$$\begin{bmatrix} \mathrm{d}y^1 \big|_p \\ \vdots \\ \mathrm{d}y^k \big|_p \end{bmatrix} = \begin{bmatrix} \frac{\partial y^1}{\partial x^1}\big|_p & \cdots & \frac{\partial y^1}{\partial x^n}\big|_p \\ \\ \frac{\partial y^k}{\partial x^1}\big|_p & \cdots & \frac{\partial y^k}{\partial x^n}\big|_p \end{bmatrix} \begin{bmatrix} \mathrm{d}x^1 \big|_p \\ \vdots \\ \mathrm{d}x^n \big|_p \end{bmatrix}. \tag{6.2}$$

Assuming $k < n$ and $\left\{ \begin{bmatrix} \frac{\partial y^i}{\partial x^1}\big|_p & \cdots & \frac{\partial y^i}{\partial x^n}\big|_p \end{bmatrix} \right\}_{i=1}^k$ is linearly independent, there are precisely $n-k$ differentials $\mathrm{d}x^{j_\ell}$ in the nullspace of $\begin{bmatrix} \frac{\partial y^i}{\partial x^j}\big|_p \end{bmatrix}$ (6.2). Define

$$\varphi \colon W \to \mathbb{R}^n \quad \text{such that} \quad q \overset{\varphi}{\longmapsto} \begin{bmatrix} y^1(q) \\ \vdots \\ y^k(q) \\ x^{j_1} \\ \vdots \\ x^{j_{n-k}} \end{bmatrix}.$$

In particular, $k < n$ and 6.2 imply that $[\varphi_*]$ has full rank at p. Applying the inverse function theorem as before, there's a neighborhood U such that $p \in U \subset W$ for which $\varphi\big|_U$ is a coordinate chart extending the (y^j). ☐

(c) Lastly, if $p \in W$, $k > n$, and $\mathrm{d}y^1\big|_p, \ldots, \mathrm{d}y^k\big|_p$ are linearly independent in T_p^*M as before, then there are n indices i_ℓ such that $\begin{bmatrix} y^{i_1} & \vdots & y^{i_n} \end{bmatrix}$ is a chart on a neighborhood U of p.

Proof. With local coordinates (x^j) about p, each y^i has differential w.r.t. the (x^j)

$$\begin{bmatrix} \mathrm{d}y^1 \big|_p \\ \vdots \\ \mathrm{d}y^k \big|_p \end{bmatrix} = \begin{bmatrix} \frac{\partial y^1}{\partial x^1}\big|_p & \cdots & \frac{\partial y^1}{\partial x^n\big|_p} \\ \\ \frac{\partial y^k}{\partial x^1}\big|_p & \cdots & \frac{\partial y^k}{\partial x^n\big|_p} \end{bmatrix} \begin{bmatrix} \mathrm{d}x^1 \big|_p \\ \vdots \\ \mathrm{d}x^n \big|_p \end{bmatrix}, \tag{6.3}$$

and our hypothesis is that the linear transformation $\left[\frac{\partial y^i}{\partial x^j}\Big|_p\right]$ has rank n. Choose n differentials as a basis for the image of $\left[\frac{\partial y^i}{\partial x^j}\Big|_p\right]$ (a subspace of T_p^*M), call these y^{i_ℓ}. Defined φ as the map

$$\varphi\colon W \to \mathbb{R}^n \quad \text{such that} \quad q \xmapsto{\varphi} \begin{bmatrix} y^{i_1}(q) \\ \vdots \\ y^{i_n}(q) \end{bmatrix}.$$

Then $k > n$ and equation 6.3 imply that $[\varphi_*]$ has full rank at p. Apply the inverse function theorem to obtain a neighborhood U such that $p \in U \subset W$ and $\varphi\big|_U$ is a chart. □

11. Say that M is a smooth n-dim manifold, $C \subset M$ is embedded $n-k$ dim manifold, and $f \in C^\infty(M)$ is a smooth function. Suppose the restriction $f\big|_C$ has a (relative) maximum point at $p \in C$. Then fore any defining function $\Phi\colon U \to \mathbb{R}^k$ for C on a neighborhood of p, there exist k real numbers $\ell_1, \ldots, \ell_k$ such that

$$\mathrm{d}f_p = \begin{bmatrix}\ell_1 & \cdots & \ell_k\end{bmatrix} \begin{bmatrix} \mathrm{d}\Phi^1 \\ \vdots \\ \mathrm{d}\Phi^k \end{bmatrix}\Bigg|_p.$$

Proof. If $C = M$, then $k = 0$, the defining function is trivial, and $\mathrm{d}f_p = 0$. This follows because for any local chart (x^i) centered at p, the path $f(\mathbf{u}t)$ has a local maximum point at $t = 0$ for any unit vector $\mathbf{u} \in S^{n-1} \subset \mathbb{R}^n$, and therefore $\frac{\partial f}{\partial x^i} = 0$ for all $i = 1, \ldots, n$.

Let $\Phi\colon U \to \mathbb{R}^k$ be a local defining function for $C \cap U$. Because Φ is a submersion, the push-forwards Φ_* *surjects* onto the tangent space $T_0\mathbb{R}^k$. Dually, the pullback Φ^* is a rank $k = \dim T_0\mathbb{R}^k$ *injection* into the cotangent space T_pM. Because $\Phi\big|_C$ is constant, the image of the pullback $T_0^*\mathbb{R}^k \hookrightarrow T_p^*M$ must be orthogonal to every cotangent vector in T_p^*C. So the cotangent space over M splits

$$T_p^*\mathbb{R}^k \oplus T_p^*C \xrightarrow{\cong} T_p^*M. \tag{11.1}$$

Our hypothesis is that p is a local maximum point of $f\big|_C$. Consider that, for any smooth path $\gamma(t)$ in C passing through p at time $t = 0$, the composite $f(\gamma(t))$ has a local extremum at $t = 0$. It follows that

$$\text{every tangent vector } \gamma'(0) \text{ in } T_pC \text{ annihilates } f \quad \Longrightarrow \quad \text{the components of } \mathrm{d}f_p \text{ in } T_p^*C \text{ are trivial.}$$

We have shown that the differential $\mathrm{d}f_p$ is orthogonal to T_p^*C, so, by 11.1, $\mathrm{d}f_p$ lies in the image of the pullback Φ^*. The Lagrange multipliers $\ell_1, \ldots, \ell_k$ are just the coordinates of $\mathrm{d}f_p$ in the covector space $T^*\mathbb{R}^k \hookrightarrow T_p^*M$. □