Math 6210 NOTES: DIFFERENTIAL GEOMETRY

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These notes were taken in University of Colorado's Math 6210 (Differential Geometry) class in Spring 2019, taught by Prof. Jeanne Clelland. I live-TEXed them with `vim`, so there may be typos and failures of understanding. Any mistakes are my own. Please send questions, comments, complaints, and corrections to colton.grainger@colorado.edu. Thanks to `adebray` for the LATEX template, which I have forked from https://github.com/adebray/latex_style_files.

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Lecture 1.

2019-04-17

Last time, we defined covectors, cotangent vectors, and a basis for the *cotangent space at a point*, given a basis of tangent vectors for the tangent space.

That is, by choosing a coordinate chart $(U, (x^i))$ about a point $p \in U$, we have a basis for T_pM (and so a dual basis for T_p^*M).

$$\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \dots, \frac{\partial}{\partial x^n}\bigg|_p \right\}, \quad \left\{ \mathrm{d}x^1 \bigg|_p, \dots, \mathrm{d}x^n \bigg|_p \right\}.$$

Example 1.1 (Changing the cotangent basis)**.** Let $(\tilde{x}^j)$ be another local coordinate chart in a neighborhood of p. Then

$$\frac{\partial}{\partial x^i}\bigg|_p = \sum_{j=1}^n \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial}{\partial \tilde{x}^j}\bigg|_p.$$

Now let $\left(\mathrm{d}\tilde{x}^j\right)$ be the dual basis to $\left(\frac{\partial}{\partial \tilde{x}^j}\big|_p\right)$. Consider a cotangent vector $w \in T_p^*M$. Written in terms of both the original and the new cotangent bases, w is expressed as

$$w = \sum_{i=1}^n a_i \, \mathrm{d}x^i \bigg|_p = \sum_{j=1}^n \tilde{a}_j \, \mathrm{d}\tilde{x}^j \bigg|_p.$$

To determine the relation between the original basis representation $\left[a_i\right]$ and the new representation $\left[\tilde{a}_j\right]$, evaluate the covector w at each tangent vector $\frac{\partial}{\partial x^i}\big|_p$ in the original basis. Then

$$\begin{aligned}
a_i &= w\left(\frac{\partial}{\partial x^i}\bigg|_p\right) && \text{because } \mathrm{d}x^i \frac{\partial}{\partial x^j} = \delta^i_j \\
&= w\left(\sum_{j=1}^n \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\partial}{\partial \tilde{x}^j}\bigg|_p\right) && \text{by the chain rule} \\
&= \sum_{j=1}^n \frac{\partial \tilde{x}^j}{\partial x^i} w\left(\frac{\partial}{\partial \tilde{x}^j}\bigg|_p\right) && \text{by linearity} \\
&= \sum_{j=1}^n \tilde{a}_j \frac{\partial \tilde{x}^j}{\partial x^i} && \text{because } \mathrm{d}\tilde{x}^r \frac{\partial}{\partial \tilde{x}^j} = \delta^i_j.
\end{aligned}$$

◂

Exercise 1.2. Give a numerical representation of the change of basis map taking $\left[a_i\right]$ to $\left[\tilde{a}j\right]$. Compare this to the change of basis on T_pM with $\tilde{v}^j$ written in terms of a basis v^i.

We prefer to express old covectors in terms of a new covector basis. Whereas with tangent vectors, the chain rule gives us a means (and hence a preference) to express the new vectors in terms of the old tangent vector basis.

Example 1.3 (Basis covectors)**.** Let w be the cotangent vector in example 1.1. Then

$$w = \sum_{j=1}^n \tilde{a}_j \,\mathrm{d}\tilde{x}^j \bigg|_p = \sum_{i=1}^n a_i \,\mathrm{d}x^i \bigg|_p \quad \text{and} \quad w = \sum_{i,j=1}^n \tilde{a}_j \frac{\partial \tilde{x}^j}{\partial x^i}\,\mathrm{d}x^i \bigg|_p. \tag{1.4}$$

As the scalars $\left[\tilde{a}_j\right]$ range through $\mathbb{R}^n$, the relation in (1.4) determines the representation of the new basis covector in terms of the old basis covectors:

$$\mathrm{d}\tilde{x}^j \bigg|_p = \sum_{i=1}^n \frac{\partial \tilde{x}^j}{\partial x^i}\,\mathrm{d}x^i \bigg|_p. \tag{1.5}$$

◂

Note. We have to hurry on to define covector fields, in order to state Stokes' theorem by the end of the semester. So we're glossing over the theory of vector bundles. ◂

Definition 1.6 (Cotangent bundle)**.** Let $M \in \mathsf{Man}^n$. The *cotangent bundle* over M is the disjoint union

$$T^*M = \coprod_{p \in M} T^*_pM,$$

with projection $\pi\colon T^*M \to M$ sending cotangent vectors $w_p \mapsto p$ onto M.

Proposition 1.7. *The cotangent bundle is a smooth vector bundle over M.*

Proof. By definitions, fiber over $p \in M$ is the cotangent vector space at p. We claim π is a smooth projection with the fibers over M varying smoothly.

Now consider a local coordinate chart (x^i) on $U \subset M$. We will show that U is a trivializing neighborhood for π. The coordinate cotangent vectors determine n (local) sections back into the cotangent bundle (called *coordinate covector fields* on M). These are

$$\mathrm{d}x^i : U \to T^*U \quad \text{such that} \quad \mathrm{d}x^i\,(p) = \mathrm{d}x^i \bigg|_p \in T^*_pM. \tag{1.8}$$

The coordinate chart $(U, (x^i))$ then determines a chart for the open neighborhood $\pi^{-1}(U) = T^*U$. Each covector $w \in \pi^{-1}(U)$ is in the fiber of some $p \in U$, and can be expressed in terms of the cotangent basis at p

$$w_p = \begin{bmatrix}\xi_1 & \cdots & \xi_n\end{bmatrix}\begin{bmatrix}\mathrm{d}x^1\big|_p \\ \vdots \\ \mathrm{d}x^n\big|_p\end{bmatrix}.$$

We'll define our chart $\pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{R}^n$ pointwise

$$\sum_{i=1}^n \xi_i \, \mathrm{d}x^i \bigg|_p \longmapsto (x^1, \ldots, x^n, \xi_1, \ldots, \xi_n).$$

Taking for granted that this correspondence is smooth, we have shown the atlas $\{U_i, \varphi_i\}$ for M induces a local trivialization of the cotangent bundle. □

Definition 1.9 (Covector fields)**.** A global section of T^*M is a *global covector field* (a.k.a., a differential 1-form), and a local section of T^*M is a *local covector field.*

If $\omega \in \Gamma(T^*M)$ is a covector field, and $X \in \mathfrak{X}M$ is a vector field, then we can define a scalar function $\omega(X)\colon M \to \mathbb{R}$ such that $\omega(X)(p) = \omega\big|_p(X_p)$. In local coordinates,

$$\text{if} \quad X = X^i \frac{\partial}{\partial x^i}, \quad \text{and} \quad \omega = \xi_j \, \mathrm{d}x^j \,, \quad \text{then} \quad \omega(X) = \sum_i \xi_i(x) X^i(x).$$

Definition 1.10 (Pullbacks of covector fields)**.** Let $M \xrightarrow{F} N$ be a map in Man, with p a point of M. The differential $\mathrm{d}F_p : T_pM \to T_{F(p)}N$ has a dual linear map $\mathrm{d}F_p^* : T^*_{F(p)} \to T^*_pM$ called the *pullback map* by F at p, or the *cotangent map* at p. It's characterized by the property that for all $\mathbf{v} \in T_pM$, and covectors $\omega \in T^*_{F(p)}N$,

$$(\mathrm{d}F_p)^*(\omega)(\mathbf{v}) = \omega(\mathrm{d}F_p\,(\mathbf{v})).$$

Definition 1.11 (Coframe fields)**.** A *local coframe field* on $U \subset M$ is an ordered n-tuple $(\varepsilon^1, \ldots, \varepsilon^n)$ of covector fields such that, for all $p \in U$, the evaluation of $(\varepsilon^1, \ldots, \varepsilon^n)$ at p forms a basis for T^*_pM.

Suppose $(E^1, \ldots, E^n)$ is a frame field on $U \subset M$, then the *dual coframe field* $(\varepsilon^1, \ldots, \varepsilon^n)$ is defined by

$$\varepsilon^i(E_j) = \delta^i_j.$$

We write $\mathfrak{X}^*M$ for the *smooth covector fields* on M.

Exercise 1.12. Does each smooth manifold M admit a smooth covector field?[1]

We proceed to "discover" differential 1-forms.

Definition 1.13 (Differential of a scalar function)**.** Let $M \in \mathsf{Man}$ and $f \in C^\infty(M)$. The *differential of* f is the covector field $\mathrm{d}f$ on M defined by

$$\mathrm{d}f\bigg|_p(\mathbf{v}) = \mathbf{v}(f) \quad \text{for all } \mathbf{v} \in T_pM.$$

We've specified how $\mathrm{d}f$ acts at each point, for each tangent vector. Doesn't this *look* like a Krönecker pairing?

Example 1.14 (Coordinate representation of a differential)**.** Consider local coordinates (x^i) on M. If we write $\mathbf{v} = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}\big|_p$ and $\mathrm{d}f\big|_p = \sum_{j=1}^n a_j \, \mathrm{d}x^j\,\big|_p$, then

$$\begin{aligned}
\sum_{i=1}^n v^i a_i &= \left(a_j \, \mathrm{d}x^j \,\bigg|_p\right)\left(v^i \frac{\partial}{\partial x^i}\bigg|_p\right) \\
&= \mathrm{d}f\bigg|_p(\mathbf{v}) \\
&= \mathbf{v}\bigg|_p(f) \\
&= \sum_{i=1}^n v^i \frac{\partial f}{\partial x^i}\bigg|_p
\end{aligned}$$

Varying the scalars $\begin{bmatrix} v^1 & \cdots & v^n \end{bmatrix}^t$ for the tangent vector $\mathbf{v}$, we determine each scalar component

$$a_i = \frac{\partial f}{\partial x^i}\bigg|_p$$

in the basis representation of $\mathrm{d}f\big|_p$. Therefore the differential of f in terms of local coordinates is

$$\mathrm{d}f\bigg|_p = \frac{\partial f}{\partial x^i} \, \mathrm{d}x^i \,\bigg|_p.$$

◀

[1] Patrick asked if it's true that every $M \in \mathsf{Man}$ admits a smooth nonvanishing covector field. I have no idea.

Remark 1.15. We have also shown $\mathrm{d}x^i = \mathrm{d}(x^i)$ for the coordinate functions x^i of chart $(U, (x^i))$. That is, $\mathrm{d}x^i$ is the differential of $x^i \colon U \to \mathbb{R}$. ⋖

Defining the differential of a scalar function as an *evaluation* extends the notion of the "differential" given for $\mathrm{d}x$ in high school calculus. The two concepts are the same for a function $f \colon M \to \mathbb{R}$ if we choose to make the identification $\mathrm{d}f \colon T_pM \to T_{f(p)}\mathbb{R} \cong \mathbb{R}$.

Proposition 1.16. *Let $\gamma \colon J \to M$ be a smooth curve, $f \in C^\times(M)$. Then the differential of the function $f \circ \gamma \colon J \to \mathbb{R}$ is given by (the familiar evaluation)*

$$(f \circ \gamma)'(t) = \mathrm{d}f_{\gamma(t)}\left(\gamma'(t)\right).$$

Exercise 1.17. Soft question: why should a covector represent an infinitesimal path? Can this be connected with local 1-parameter group actions?

Lecture 2.

2019-04-19

"Covectors are just linear maps on vector spaces. They're not necessarily Krönecker pairings."

Last time, we defined pullbacks of smooth vector fields.

Note. For a map $M \xrightarrow{F} N$, regardless[2] of whether F is surjective, injective, etc., the pullback of $\omega \in \mathfrak{X}^*(N)$ is always a smooth covector field on M. ⋖

For $\omega \in \mathfrak{X}^*(N)$, we denote the pullback via F as $F^*\omega$. If we let $F^*(\mathbf{v})$ denote $\mathrm{d}F_p(\mathbf{v})$, then $(F^*\omega)(\mathbf{v}) = \omega(F_*(\mathbf{v}))$.

Proposition 2.1. *Let $M \xrightarrow{F} N$ be smooth, $\omega \in \mathfrak{X}^*(N)$, and $u \in C^\times(N)$. Then both:*

(1) $F^*(u\omega) = (u \circ F)F^*\omega$.
(2) $F^*(\mathrm{d}u) = \mathrm{d}(u \circ F)$.

Example 2.2 (Helicoids)**.** Define $F \colon \mathbb{R}^2 \to \mathbb{R}^3$ by

$$F\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} v\cos u \\ v \sin u \\ u \end{bmatrix}.$$

Let $\omega \in \mathfrak{X}^*(\mathbb{R}^3)$ be the covector field

$$\omega = x\,\mathrm{d}x - y\,\mathrm{d}y + z^2\,\mathrm{d}z$$

Then (TODO)

$$F^*(\omega) = (x \circ F)\,\mathrm{d}(y \circ F) - (y \circ F)\,\mathrm{d}(x \circ F) + (z \circ F)^2\,\mathrm{d}(z \circ F)\,.$$

⋖

Proposition 2.3 (Restricting the covector field to submanifolds)**.** *Say $\iota \colon S \hookrightarrow M$ is the injection of an immersed submanifold. The pullback of $\omega \in \mathfrak{X}^*(M)$ is $\iota^*\omega \in \mathfrak{X}^*(S)$, and, at a point $p \in M$, where ω is in the cotangent space T_p^*M, its pullback is just the restriction*

$$\iota^*\omega = \omega\bigg|_{T_pS}.$$

Explicitly,[3] *if $\mathbf{v} \in T_pS$, then*

$$\iota^*\omega(\mathbf{v}) = \omega(\iota_*(\mathbf{v})) = \omega(\mathbf{v})$$

Example 2.4. Let $M = \mathbb{R}^2$, and consider the x-axis as $S \hookrightarrow M$. Say $\omega = \mathrm{d}y$. For $p \in S$, any $\mathbf{v} \in T_pS$ has the form $\mathbf{v} = a\frac{\partial}{\partial x}$. But then $\omega(\mathbf{v}) = \mathrm{d}a\frac{\partial}{\partial x} = 0$! So $\iota^*\omega = 0$, although ω is a nonzero covector field on M. ⋖

Remark 2.5 (Making ends meet). What are *exterior differential systems*? Translate the partial differential equations in terms of covector fields on smooth manifolds. The solutions to the differential equations correspond to the ideals on the manifold for which the pullback map is trivial (the kernel). ⋖

Example 2.6 (Line integrals)**.** Let $[a, b] \subset \mathbb{R}$ be a closed bounded interval of the real line, and let w be a smooth covector field of $[a, b]$. If t is a local coordinate, then $\mathrm{d}t$ gives a basis and we can write $w = f(t)\,\mathrm{d}t$ for some $f \in C^\infty([a, b])$. ⋖

[2] Much nicer than pushforwards! Think about preimages as respecting set-theoretic operations.
[3] Much nicer than restricting vector fields!

Proposition 2.7. *If $w \in \mathfrak{X}^*([a,b])$ and $\varphi\colon [c,d] \to [a,b]$ is an increasing diffeomorphism, then $\int_{[c,d]} \varphi^* w = \int_{[a,b]} w$.*

We'll have to introduce sign conventions for orientation when taking surface integrals.[4]

Definition 2.8. Now let M be a smooth manifold. A *smooth curve segment* is a smooth map $\gamma\colon [a,b] \to M$. The *line integral* of w over γ is
$$\int_\gamma w := \int_{[a,b]} \gamma^* w.$$
Note that by 2.7 the line integral is well defined, regardless of parameterization γ.

Exercise 2.9. Let M be the plane punctured at the origin. Let $w = \frac{x\mathrm{d}y - y\mathrm{d}x}{x^2+y^2}$. Then compute $\int_\gamma w$ for the parameterization $\gamma\colon [0, 2\pi] \to M$ given by
$$\gamma(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}.$$

Theorem 2.10 (Fundamental theorem of line integrals)**.** *If $w = \mathrm{d}f$, $f \in C^\infty(M)$, and $\gamma\colon [a,b] \to M$ is a smooth curve segment, then*
$$\int_\gamma w = \int_\gamma \mathrm{d}f = f(\gamma(b)) - f(\gamma(a)).$$

Definition 2.11 (Exact covector fields)**.** If $w = \mathrm{d}f$ for some $f \in C^\infty(M)$, w is called *exact*, and f is called a *potential function* for w. In this case, $\int_\gamma w$ depends only on the endpoints of γ. This integral is called *path-independent.* (We'd also say that the covector field is conservative.)

Working in $\mathbb{R}^n$, we often call vector fields conservative—but in $\mathbb{R}^n$ this amounts to a vector field being the gradient of a smooth function. Apparently the Euclidean metric gives a structure for $\boldsymbol{\nabla}$ and div to produce a correspondence between covector fields and vector fields.

Theorem 2.12. *A smooth covector field w is conservative if and only if it is exact.*

Proof. On the one hand, exact implies conservative. Conversely, choose $p_0 \in M$ and define $f(p) = \int_\gamma w$ where γ is a path from p_0 to p. (There are some requirements for γ—it need be continuous, piecewise smooth, and with well-behaved one-sided limits.) ☐

Proposition 2.13 (Necessary conditions for exactness)**.** *If $w = \mathrm{d}f$ is exact and $w = a_i(x)\,\mathrm{d}x^i$ for $a_i(x) = \frac{\partial}{\partial x^i}$, then for all indices i, j we have*
$$\frac{\partial a_i}{\partial x_j} = \frac{\partial a_j}{\partial x_i} = \frac{\partial^2 f}{\partial x^i \partial x^j}$$

Definition 2.14 (Closed covector fields)**.** A covector field $w = a_i(x)\,\mathrm{d}x^i$ for $a_i(x) = \frac{\partial}{\partial x^i}$ is *closed* if for all indices i, j,
$$\frac{\partial a_i}{\partial x_j} = \frac{\partial a_j}{\partial x_i}.$$

Exercise 2.15. Show that the non-exact covector field in 2.9 is closed.

Lecture 3.

2019-04-22

Lemma 3.1 (Poincaré lemma for covector fields)**.** *If $M \in \mathsf{Man}$ is a simply connected manifold, then a covector field w is closed if and only if w is exact.*

Note (deRham cohomology)*.* So, if M is a simply connected manifold, then the first deRham cohomology group is trivial. (Where deRham cohomology groups are the quotient of the group of closed n-forms by exact n-forms.) ◂

Proposition 3.2 (Closed covector fields)**.**

(1) *Let w be a closed covector field on M. Then every $p \in M$ has a neighborhood $U \subset M$ on which $w\big|_p$ is exact.*
(2) *Closedness of covector fields is independent of the choice of local coordinate charts.*
(3) *In fact, w is closed iff for all $X, Y \in \mathfrak{X}M$, we have $X(w(Y)) - Y(w(X)) = w([X,Y])$.*
(4) *Pullbacks preserve closedness and exactness.*

[4] Hope it's as much fun as for defining sign conventions for homology on CW-complexes.

Definition 3.3 (Tensor products)**.** Let $V, W \in$ be finite dimensional vector spaces over $\mathbb{R}$. Let $a^*, b^* \in V^*, W^*$ be covectors. The *tensor product* of $a^* \otimes b^*$ is the bilinear functional defined on vectors $\mathbf{v} \times \mathbf{w} \in V \times W$ by

$$(a^* \otimes b^*)(\mathbf{v}, \mathbf{w}) = a^*(\mathbf{v})b^*(\mathbf{w}).$$

Then the space $V^* \otimes W^*$ is the space of the tensors $\{a^* \otimes b^* : a \in V, b \in W\}$.

Proposition 3.4 (Properties of the tensor product)**.** *The dimension of the tensor product of dual spaces V^* and W^* is the product $\dim V^* \cdot \dim W^*$. There's a basis for $V^* \otimes W^*$ given by the tensors of basis elements for V^* and W^* respectively. Note that $V \otimes W := (V^* \otimes W^*)^*$ is the dual.*

We'll mostly work with n-isomorphic copies of a single $V \in \mathsf{Vect}$.

Definition 3.5 (Wedge product)**.** Let $\alpha, \beta \in V^*$. The *wedge product* of α and β is defined as

$$\alpha \wedge \beta := \alpha \otimes \beta - \beta \otimes \alpha.$$

In particular, evaluated on two vectors $(v, w) \in V^2$, we have

$$\alpha \wedge \beta(v, w) = \det \begin{bmatrix} \alpha(v) & \alpha(w) \\ \beta(v) & \beta(w) \end{bmatrix}.$$

More generally, the wedge product of k covectors in V^* is defined pointwise for v^k in V by

$$\alpha^1 \wedge \cdots \wedge \alpha^k \begin{bmatrix} v_1 & \cdots & v_k \end{bmatrix} = \det \left(\sum_{i,j}^{n} \alpha^i(v_i) \right).$$

The *space of alternating k-tensors* or k-vectors of V^* is the defined by

$$\bigwedge^k V^* := \operatorname{span}\left\{\alpha^1 \wedge \cdots \wedge \alpha : \alpha^i \in V^*\right\}.$$

Proposition 3.6 (Properties of the wedge product)**.**

(1) *It's bilinear,*
(2) *it's associative,*
(3) *it's anti-commutative.*
(4) *For $\omega \in \bigwedge^k V^*$ and $\eta \in \bigwedge^\ell V^*$, we have $\omega \wedge \eta \in \bigwedge^{\ell+k} V^*$ and $\omega \wedge \eta = (-1)^{k\ell} \eta \wedge \omega$.*

Definition 3.7 (Decomposable)**.** A k-covector η is said to be *decomposable* if it can be expressed as a wedge of k covectors.

Example 3.8 (An indecomposable 2-covector on $\mathbb{R}^4$)**.** $\eta = \mathrm{d}x_1 \wedge \mathrm{d}x_2 + \mathrm{d}x_3 + \mathrm{d}x_4$ is not decomposable. (Hint: consider that any decomposable k-covector wedged with itself is the 0-function.) ⋖

Definition 3.9 (Interior multiplication)**.** There's a linear map from the space of alternating k-tensors to the space of alternating $k-1$-tensors called *interior multiplication by* $\mathbf{v}$.

$$\iota_{\mathbf{v}} \colon \bigwedge^k V^* \to \bigwedge^{k-1} V^*$$
$$\omega \overset{\mapsto}{\iota_v} \iota_v \omega = \iota_v \omega : (\mathbf{w}_1, \ldots, \mathbf{w}_{k-1}) \longmapsto \omega(\mathbf{v}, \mathbf{w}_1, \ldots, \mathbf{w}_{k-1}).$$

The interior multiplication of ω and $\mathbf{v}$ is referred to as "$\mathbf{v}$-left hooked with ω."

Ryan asked if there's a connection to, e.g., the front or back face maps. I asked if there was a relation to the cap product. TODO. . . .

In smooth charts, a k-form ω has a local expression

$$\omega = \sum_{\forall i_1, \ldots, i_k} f_{i_1, \ldots, i_k}(x)\, \mathrm{d}x^{i_1} i \wedge \ldots \wedge \mathrm{d}x^{i_k}$$

.

Lecture 4.

2019-04-24

"We're in the algebra of differential forms on a manifold, $\Omega(M)$."

Definition 4.1 (Multi-index notation)**.** In local coordinates (x^i), any k-form $\omega(\ )$ can be expressed as

$$\omega(=) \sum_{(i_1,\dots,i_k)\in I_n^k} f_{(i_1,\dots,i_k)}(x)\,\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k}\,.$$

where I_n^k is the product of tuples $(1,\dots,n)\times(1,\dots,k)$.

We can compute the action of a covector $\omega(\ )$ on $f_I(x)$ TODO. (I got lost in the indices.)

$$f_I(x) = \omega(())[\frac{\partial}{\partial x^{i_1}}\cdots\frac{\partial}{\partial x^{i_k}}]$$

$\mathrm{f}_I(x)$

We'll start in $\mathbb{R}^n$. Let $\omega(\ )$ be a k-form, of the form,

$$\omega(=) \sum_{|I|=k} f_I(x)\,\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k}\,.$$

The exterior derivative of $\omega(\ )$ is the $k+1$-form on $\mathbb{R}^n$

$$\mathrm{d}$$

$\sum_{|I|=k} \mathrm{d}f_I\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k}\,.$

In $\mathbb{R}^2$, we have $\mathrm{d}x\,\mathrm{d}y = \mathrm{d}x\wedge\mathrm{d}y$.

If $\omega(\ )$ is a 1-form, where $\omega(=)f_i(x)\,\mathrm{d}x^i$, then

$$\mathrm{d}$$

$= \ \mathrm{d}f_i\wedge\mathrm{d}x^i$

$= ()$(4.2)

Example 4.3 (Exterior derivative)**.** ◄