

MATH 6210 HOMEWORK 11

COLTON GRAINGER

APRIL 16, 2019

Problems set from Chapter 9, Lee, *Introduction to Smooth Manifolds*.

1. 9-1. Suppose M is a smooth manifold, X is a vector field in $\mathfrak{X}M$, and γ is a maximum integral curve of X .
 - (a) *Either γ is constant, γ is injective, or γ is periodic and nonconstant.*
 - (b) *If γ is periodic and nonconstant, then there's a unique positive number T (called the period of γ) such that $\gamma(t) = \gamma(t')$ if and only if $t - t' \in T\mathbb{Z}$.*
 - (c) *The image of γ is an immersed submanifold of M , diffeomorphic to \mathbb{R} , S^1 , or $\text{pt} \approx \mathbb{R}^0$.*

2. 9-7. Say M is a connected smooth manifold. *The group of diffeomorphisms $\text{Diff}(M)$ acts transitively on M .*

Lemma. Let $B \subset \mathbb{R}^n$ be an open unit ball. If $p, q \in B$, there is a compactly supported smooth vector field $X \in \mathfrak{X}B$ whose flow θ satisfies $\theta_1(p) = q$.

- 3.** 9-10. *For each of the following vector fields, find smooth coordinates in a neighborhood of $\begin{bmatrix} 1 & 0 \end{bmatrix}^t$ such that the given vector field is a coordinate vector field.*

(a) $V = y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}.$

(b) $W = x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y}.$

(c) $X = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}.$

(d) $Y = x \frac{\partial}{\partial y} + y \frac{\partial}{\partial x}.$

4. 9-16. *Exhibit smooth vector fields V , \tilde{V} , and W on \mathbb{R}^2 such that $V = \tilde{V} = \partial/\partial x$ along the x -axis but $\mathcal{L}_V W \neq \mathcal{L}_{\tilde{V}} W$ at the origin.*

Note. This shows that it is necessary to know the vector field V to compute $\mathcal{L}_V W$ at a point $p \in M$; it is not sufficient to just know the vector V_p , or even to know the values of V along an integral curve of V . \blacktriangleleft

5. 9-18. Define vector fields $X, Y \in \mathfrak{X}\mathbb{R}^2$ by

$$X = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}, \quad Y = x \frac{\partial}{\partial y} + y \frac{\partial}{\partial x}.$$

(a) Compute the flows $x\text{fl}$ and $y\text{fl}$ of X and Y .

(b) Find open intervals xB and yB containing 0 such that

$x\text{fl}_t \circ y\text{fl}_s - y\text{fl}_s \circ x\text{fl}_t$ is defined for all $\begin{bmatrix} t & s \end{bmatrix}^t$ in $xB \times yB$,
but nonzero at some time $\begin{bmatrix} \bar{t} & \bar{s} \end{bmatrix}^t$.

6. 9-19. Consider $M = \mathbb{R}^3 \setminus \{z\text{-axis}\}$, the open submanifold of \mathbb{R}^3 formed by removing the z -axis. Define $V, W \in \mathfrak{X}M$ by

$$V = \frac{\partial}{\partial x} - \frac{y}{x^2 + y^2} \frac{\partial}{\partial y}, \quad W = \frac{\partial}{\partial y} + \frac{x}{x^2 + y^2} \frac{\partial}{\partial z},$$

and let vfl , wfl be the flows of V and W , respectively. *Then V and W commute, but (caveat!) there exist a point $p \in M$ and times $s, t \in \mathbb{R}$ such that*

$$(\text{vfl}_t \circ \text{wfl}_s)(p) - (\text{wfl}_s \circ \text{vfl}_t)(p) \quad \text{is defined and nonzero.}$$