Math/Phys/Engr 428, Math 529/Phys 528
Numerical Methods - Spring 2018

Homework 1
Due: **Friday, January 26, 2018**

- Include a cover page and a problem sheet.
- Always clearly label all plots (title, x-label, y-label, and legend).
- Use the `subplot` command from MATLAB when comparing 2 or more plots to make comparisons easier and to save paper.
- Include all of your script.
- Place a comment at the top of each function or script that you submit which includes the name of the function or script, your name, the date and MATH 428, PHYS 428, ENGR 428 or MATH 529, PHYS 528.

PROBLEMS:

1. Do the following calculations by hand.

   (a) Convert to base 10: $(1011011.001)_2$

   (b) Convert to base 2: $(2018)_{10}$

2. The floating point representation of a real number is $x = \pm(0.d_1 d_2 \ldots d_n)_\beta \cdot \beta^e$, where $d_1 \neq 0$, $-M \leq e \leq M$. Suppose that $\beta = 2$, $n = 6$, $M = 5$.

   (a) Find the smallest (positive) and largest floating point numbers that can be represented. Give the answers in decimal form.

   (b) Find the floating point number in this system which is closest to π.

3. Near certain values of x each of the following functions cannot be accurately computed using the formula as given due to cancellation error. Identify the values of x which are involved (e.g. near $x = 0$ or large positive x) and propose a reformulation of the function (e.g., using Taylor series, rationalization, trigonometric identities, etc.) to remedy the problem. This is problem # 12 from the textbook. Please also see pages 48-49 for examples and more details.

   (a) $f(x) = 1 + \cos x$ **(b)** $f(x) = \mathrm{e}^{-x} + \sin x - 1$
   (c) $f(x) = \ln x - \ln(1/x)$ **(d)** $f(x) = \sqrt{x^2+1} - \sqrt{x^2+4}$
   (e) $f(x) = 1 - 2\sin^2 x$ **(f)** $f(x) = \ln(x + \sqrt{x^2+1})$

4. The following algorithm

   step 1: $x_0 := x$; $j := 0$

   step 2: *while* $x_j \neq 0$, *do*

   $a_j :=$ remainder of integer divide $x_j/2$

   $x_{j+1} :=$ quotient of integer divide $x_j/2$

   $j := j+1$

   end while

   can be used to convert a positive decimal integer x to its binary equivalent,

$$x = (a_n a_{n-1} \cdots a_1 a_0)_2.$$

   Implement the algorithm (write a computer program) and apply it to convert the following integers to their binary equivalents.

   (a) 56 (b) 1543

   (The Matlab library functions *rem*, *mod* and *floor* might be helpful when you use Matlab. Try **help rem**, **help mod** and **help floor** to see how to use them.)

Finite Precision Arithmetic

5. Use three-digit rounding arithmetic to compute the following sums (sum in the given order):

$$\textbf{(a)} \quad \sum_{k=1}^{6} \frac{1}{3^k} \qquad\qquad \textbf{(b)} \quad \sum_{k=1}^{6} \frac{1}{3^{7-k}}$$

   Hint: use floating point representation, e.g. $\frac{1}{3^6} = 0.001371742112483 \approx 0.137 \times 10^{-2}$ but not 0.001; answers in (a) and (b) should be slightly different.

Finite-Difference Approximation

6. Let $f(x)$ be a given function and recall the forward difference approximation of $f'(x)$:

$$D_+ f(x) = \frac{f(x+h) - f(x)}{h},$$

   where $h > 0$ is the step size.

(a) Take $f(x) = \sin x$, $x = \pi/4$, $h = 2^{-n}$ for $n = 1, 2, \ldots, 6$. Following example in class, plot the error versus h (use command `loglog` instead of `plot`) and make a table with the following information:

h	D_+f	$f'(\frac{\pi}{4}) - D_+f$	$(f'(\frac{\pi}{4}) - D_+f)/h$	$(f'(\frac{\pi}{4}) - D_+f)/h^2$	$(f'(\frac{\pi}{4}) - D_+f)/h^3$

You may modify the Matlab code shown in class. Present at least eight decimal digits (type "`format long`" in Matlab to get the full 15 digits).

(b) Repeat for central difference approximation,

$$D_0 f(x) = \frac{f(x+h) - f(x-h)}{2h},$$

which also approximates $f'(x)$. Which approximation is more accurate? Explain why.

7. The forward and backward finite-difference operators are defined by

$$D_+ f(x) = \frac{f(x+h) - f(x)}{h}, \qquad D_- f(x) = \frac{f(x) - f(x-h)}{h}.$$

(a) Show that $D_+ D_- f(x) = \dfrac{f(x+h) - 2f(x) + f(x-h)}{h^2}$.

(b) Use Taylor expansions and the result in part (a) to show that $D_+ D_- f(x) = f''(x) + O(h^2)$. Find the asymptotic error constant.

Rootfinding

8. Consider $f(x) = x^2 - 5$. Since $f(2) < 0$, $f(3) > 0$, it follows that $f(x)$ has a root p in the interval $(2, 3)$. Compute an approximation to p by the following methods. Take 10 steps in each case. Use Matlab and print the answers to 15 digits.

(a) bisection method, starting interval $[a, b] = [2, 3]$;

(b) fixed-point iteration with $g_1(x) = 5/x$ and $g_2(x) = x - f(x)/3$, starting value $x_0 = 2.5$;

(c) Newton's method, starting value $x_0 = 2.5$.

Present the results in a table with columns as below for each method. Do the results agree with the theory discussed in class?

column 1 : n (step)

column 2 : x_n (approximation)

column 3 : $f(x_n)$ (residual)

column 4 : $|p - x_n|$ (error)

9. Consider the function $g(x) = 2x(1 - x)$.

(a) Verify $x = 0$ and $x = 1/2$ are fixed points of $g(x)$.

(b) Why should we expect that fixed point iteration, starting even with a value very close to zero, will fail to converge toward $x = 0$?

(c) Why should we expect that fixed point iteration, starting with $p_0 \in (0, 1)$ will converge toward $x = 1/2$? What order of convergence should we expect?

(d) Perform seven iterations starting from an arbitrary $p_0 \in (0, 1)$ and numerical confirm the order of convergence.

(Please include your program for completeness.)

10. **Suggested / Additional problems for Math 529/Phys 528 students**:

    (a) Plot the function $f(x) = 1 - \cos x$ over the interval $-5 \times 10^{-8} \leq x \leq 5 \times 10^{-8}$. Generate points at 1001 uniformly spaced abscissas and perform all calculations in IEEE standard double precision (in Matlab, for example).

    (b) Reformulate f to avoid cancellations error and then repeat part (a).

11. Verify that $x = \sqrt{a}$ is a fixed point of the function

$$g(x) = \frac{1}{2}\left(x + \frac{a}{x}\right).$$

Use the techniques of Section 2.3 to determine the order of convergence and the asymptotic error constant of the sequence $p_n = g(p_{n-1})$ toward $x = \sqrt{a}$.