Math/Phys/Engr 428, Math 529/Phys 528
Numerical Methods - Spring 2018

Homework 2

Due: **Monday, February 12, 2017**

Taylor Polynomials

1. Consider the function $f(x) = \cos(\pi x/2)$.

   (a) Expand $f(x)$ in a Taylor series about the point $x_0 = 0$.

   (b) Find an expression for the remainder.

   (c) Estimate the number of terms that would be required to guarantee accuracy for $f(x)$ within 10^{-5} for all x in the interval $[-1, 1]$.

   (d) Plot $f(x)$ and its 1st, 3rd, 5th and 7th degree Taylor polynomials over $[-2, 2]$. (Use the Matlab command *subplot* to generate a number of plots on the same page).

Root Finding Methods

2. Which of the following iterations $x_{n+1} = g(x_n)$ will converge to the indicated fixed point α (provided x_0 is sufficiently close to α)? If it does converge, give the order of convergence; for linear convergence, give the rate of linear convergence (i.e., the asymptotic constant). In the case that $g'(\alpha) = 0$, try expanding $g(x)$ in a Taylor polynomial about $x = \alpha$ to determine the order of convergence. (See Section 2.3 (pg. 90-91) for more details on convergence of fixed point iteration schemes.)

   (a) $x_{n+1} = -16 + 6x_n + \dfrac{12}{x_n}, \qquad \alpha = 2$

   (b) $x_{n+1} = \dfrac{2}{3}x_n + \dfrac{1}{x_n^2}, \qquad \alpha = 3^{1/3}$

   (c) $x_{n+1} = \dfrac{12}{1 + x_n}, \qquad \alpha = 3$

3. Let α be a fixed point of $g(x)$. Consider the fixed-point iteration $x_{n+1} = g(x_n)$ and suppose that $\max |g'(x)| = k < 1$. Prove the following error estimate

$$|\alpha - x_{n+1}| \leq \frac{k}{1-k}|x_{n+1} - x_n|.$$

   (hint: by MVT, $|\alpha - x_{n+1}| = |g'(\xi)||\alpha - x_n| \leq k|\alpha - x_n|$)

4. The function $f(x) = 27x^4 + 162x^3 - 180x^2 + 62x - 7$ has a zero at $x = 1/3$. Perform ten iterations of Newton's method on this function, starting from $p_0 = 0$. What is the apparent order of convergence of the sequence of approximations? What is the multiplicity of the zero at $x = 1/3$? Would the sequence generated by the bisection method converge faster?

5. Newton's method approximates the zero of $f(x) = x^3 + 2x^2 - 3x - 1$ on the interval $(-3, -2)$ to within 9.436×10^{-11} in 3 iterations and 6 function evaluations. How many iterations and how many function evaluations are needed by the secant method to approximate this zero to a similar accuracy? Take $p_0 = -2$ and $p_1 = -3$.

6. **(Gaussian Elimination)**

   Let A be the 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Use Gaussian elimination to obtain A^{-1} by solving the two systems $Ax_1 = e_1$ and $Ax_2 = e_2$, where e_1 and e_2 are the columns of the 2×2 identity matrix. Note that you can perform both at the same time by considering the augmented system $[A|I]$. Show that A^{-1} exists if and only if $\det(A) \neq 0$.

7. **(*LU* Decomposition)**

   Find the LU decomposition of A and use it to solve $Ax = b$.

$$A = \begin{pmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ -3 \\ 3 \\ -2 \end{pmatrix}.$$

8. **(Back and Forward Substitution: Matlab Program)**

   Write two programs, one that performs back substitution on an upper triangular matrix and another that performs forward substitution on a lower triangular matrix (you may assume that the diagonal entries are all 1). Both files should begin:

```
function [x] = forwardsub(L, b)
n=length(b);
(your code here)
```

   In the above, $L\mathbf{x} = \mathbf{b}$ and A is lower triangular. Test your code on the following systems:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \end{bmatrix} \mathbf{y} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

   Remember, in Matlab you can solve matrix equations as follows (assuming you have defined the matrix A and the rhs vector $\mathbf{b}$):

```
>> A\b
```

   Print and hand-in the text file containing your program.

9. **(Special Matrices)**

   Consider the problem $Ax = b$ where A is a tridiagonal matrix. What is the operation count for the forward elimination and the back substitution steps of Gaussian elimination in this case? Count add/sub and mult/div operations separately, then give the overall order of the total operations needed. (Use $O(n^p)$ notation).

Suggested / Additional problems for Math 529/Phys 528 students:

10. Let $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -m_{2,1} & 1 & 0 \\ -m_{3,1} & 0 & 1 \end{pmatrix}$, $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -m_{3,2} & 1 \end{pmatrix}$, $P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

    (a) Show that
    $$E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ m_{2,1} & 1 & 0 \\ m_{3,1} & 0 & 1 \end{pmatrix} .$$

    (b) Show that
    $$E_1^{-1}E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ m_{2,1} & 1 & 0 \\ m_{3,1} & m_{3,2} & 1 \end{pmatrix} .$$

    (c) Show that P_1^{-1}=P_1.

11. (**Accelerating convergence of Newton's method**) (Please read Section 2.6 on Accelerating Convergence, in particular, on Restoring Quadratic Convergence to Newton's method (pages 120–122))

    The function $f(x) = 27x^4 + 162x^3 - 180x^2 + 62x - 7$ has a zero of multiplicity 3 at $x = 1/3$. Apply both techniques for restoring quadratic convergence to Newton's method, discussed on pages 120–122, to this problem. Use $p_0 = 0$, and verify that both resulting frequencies converge quadratically.

12. **Elementary Matrices, from Trefethen–Bau 1997**

    Let B be a 4×4 matrix to which we apply the following operations.

    - Double column 1,
    - halve row 3,
    - add row 3 to row 1,
    - interchange columns 1 and 4,
    - subtract row 2 from each of the other rows,
    - replace column 4 by column 3,
    - delete column 1 (so that the column dimension is reduced by 1).

(a) Write the result as a product of eight matrices, including B.

(b) Write it again as a product ABC (same B) of three matrices.

Note: You may find useful using a handout *on elementary matrices* posted on the course web site:

`http://www.webpages.uidaho.edu/~barannyk/Teaching/elem_matr.pdf`