

Math/Phys/Engr 428, Math 529/Phys 528
Numerical Methods - Spring 2018

Homework 6

Due: Monday, April 30, 2018

1. (One-Step Methods).

Find the solution of the equation $y' = 2y$, $y(0) = 1$ by

- (a) Euler method;
- (b) Modified Euler (Runge-Kutta method of order 2);
- (c) Runge-Kutta method of order 4.
- (d) Adam-Bashforth method of order 2 (use exact solution to start the method).

Use step size $h = 0.1$, then repeat with $h = 0.05, 0.001$. In cases (a)–(c), perform the first 2 steps by hand (i.e. compute y_1 and y_2 by hand). Write a small program to find the solution at $x = 1$. Find the exact solution, compute the error and error/ h^p for the various step sizes, where $p = 1, 2$ and 4 in the respective cases. Put your results in a table and analyze them.

2. (Richardson Extrapolation Applied to Solving IVPs). Perform one step of Richardson's extrapolation to get an improved solution at $x = 1$ for the initial value problem in problem 1, using values obtained with $h = 0.1$ and $h = 0.05$ by the second order Runge-Kutta method (Improved Euler method). Compare with exact solution.

3. Solve $y' = 2y$, $y(0) = 1$ by the second order predictor-corrector method:

- (a) Predictor: $u_{n+1}^0 = u_n + hf(u_n)$
- (b) Corrector: $u_{n+1}^{k+1} = u_n + \frac{h}{2}(f(u_n) + f(u_{n+1}^k))$

Use $h = 0.2$ and (a) one correction iteration per step (b) two correction iterations per step. Compute your solution at $x = 1$.

4. (Systems of Equations: See Section 7.8).

Consider the IVP for a nonlinear pendulum, $y''(t) + \sin(y(t)) = 0$, $y(0) = 1$, $y'(0) = 0$. Convert the problem into a first order system and use 4th order Runge-Kutta with $h = 0.5$ to solve for $y(1)$ and $y'(1)$.

(Eigenvalues and Eigenvectors: see Sections 4.1, 4.2 and class notes)

5. For matrix

$$A = \begin{pmatrix} 1 & 4 & 5 \\ 4 & -3 & 0 \\ 5 & 0 & 7 \end{pmatrix}$$

- (a) use the Power method with the initial vector $\frac{1}{\sqrt{3}}(1, 1, 1)^T$ and a convergence tolerance of 5×10^{-5} to estimate the dominant eigenvalue and its associated eigenvector. How many iterations are needed for convergence of eigenvector? Compare this with the number of iterations for convergence of the eigenvalue. Had convergence of the eigenvalue been used for the stopping criterion, what would the error have been for the eigenvector estimate?
- (b) approximate the eigenvalue of the given matrix that is nearest to $q = 1$ using the Inverse Power method.
- (c) starting from the same initial vector, apply the Rayleigh Quotient Iteration method with the same stopping criterion. Which eigenvalue and eigenvector do you approximate? How many iterations would be needed compared with the previously used methods?

(Multistep Methods: see Section 7.5)

6. **(Two-step Adams-Moulton method)** Derive the two-step Adams-Moulton method by constructing a quadratic interpolating polynomial through (t_{n-1}, f_{n-1}) , (t_n, f_n) and (t_{n+1}, f_{n+1}) , where $f_k = f(y_k)$, and integrating it from t_n to t_{n+1} :

$$u_{n+1} = u_n + \frac{h}{12} [5f(t_{n+1}, u_{n+1}) + 8f(t_n, u_n) - f(t_{n-1}, u_{n-1})]$$

What is the truncation error term associated with this method?

7. **(Trapezoidal method)** Derive the trapezoidal method for solving IVPs (i.e. the Adams-Moulton one step method):

$$u_{n+1} = u_n + \frac{h}{2} [f(t_{n+1}, u_{n+1}) + f(t_n, u_n)]$$

What is the truncation error term associated with this method?

8. **(Least-Squares Method (see Section 5.8))**

Experimental data relating the oxide thickness, measured in Angstroms, of a thin film to the baking time of the film, measured in minutes, is given in the table below (see problem # 4 on page 426).

Baking time	20	30	40	60	70	90	100	120	150	180
Oxide thickness	3.5	7.4	7.1	15.6	11.1	14.9	23.5	27.1	22.1	32.9

- (a) Construct a scatter plot of this data. What functional form is most appropriate for fitting this data?
- (b) Fit the data to the function indicated in part (a). What physical significance do the model parameters have?
- (c) Predict the oxide thickness for a film which is baked for 45 minutes.

Suggested / Additional Problems for Math 529 / Phys 528 students:

9. (Shooting Method (see Sections 8.4 and 8.5))

Suppose we use the nonlinear shooting method to approximate the solution of the given boundary value problem (see problem # 1 on page 722). Write out the initial value problem that must be solved and the objective function for the corresponding rootfinding problem.

$$y'' + (y')^2 = 1, \quad y(0) = 1, \quad y(1) = 2$$

10. (Absolute Stability and Stiff Equations: Section 7.9)

Compare the approximate solutions of the initial value problem (see problem # 16, pg. 654)

$$y' = 5e^{5t}(y - t)^2 + 1, \quad y(0) = 1$$

obtained using

- (a) the trapezoid method;
- (b) the backward Euler method;
- (c) the second-order backward differentiation formula;
- (d) the classical fourth-order Runge-Kutta method.

Use the step size $h = 0.25$ for all methods and advance each solution out to $t = 1$. The exact solution of the initial value problem is $y(t) = t - e^{-5t}$.

- (a) Starting with only one subinterval, construct the four row Romberg integration table for $\int_3^{3.5} \frac{x}{\sqrt{x^2-4}} dx$ (see problem # 8, pg. 503).
- (b) What is the error estimate for the final approximation? How does this compare with the actual error?
- (c) How many subintervals would have been necessary to achieve the same accuracy using the composite trapezoid rule without extrapolation?

11. Three-step Adams-Bashforth method

Construct the quadratic interpolation polynomial through (t_{n-2}, f_{n-2}) , (t_{n-1}, f_{n-1}) and (t_n, f_n) , where $f_k = f(y_k)$. Integrate this polynomial from t_n to t_{n+1} to obtain the three-step Adams-Bashforth method.

$$u_{n+1} = u_n + \frac{h}{12} [23f(t_n, u_n) - 16f(t_{n-1}, u_{n-1}) + 5f(t_{n-2}, u_{n-2})]$$