Computational Investigation of Fixed Point Methods
Math 428: Numerical Methods HW 2

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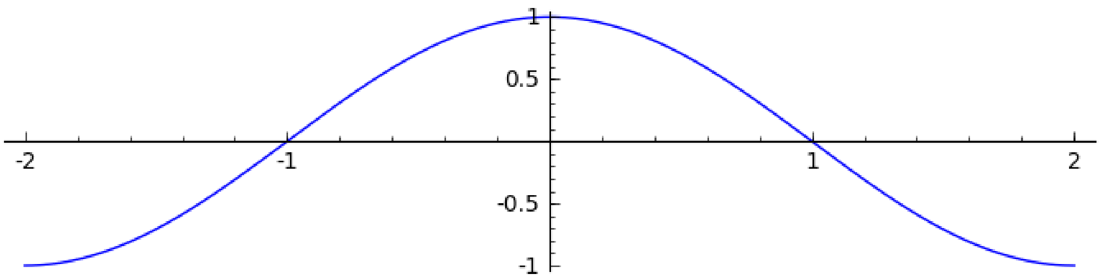
February 21, 2018

Taylor Polynomials

prob 1

Consider the function $f(x) = \cos(\pi x/2)$.

```
In [1]: f = cos(pi*x/2)
        p = f.plot((x, -2, 2))
        p.show(figsize = [7, 2])
```



We expand $f(x)$ in a Taylor series about the point $x_0 = 0$.
Since (about $x = 0$)

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

our Taylor expansion is just

```
In [2]: f.series(x, 8)

Out[2]: 1 + (-1/8*pi^2)*x^2 + (1/384*pi^4)*x^4 + (-1/46080*pi^6)*x^6 + Order(x^8)

In [3]: # here's the 6th degree Taylor polynomial
        f.taylor(x, 0, 6)

Out[3]: -1/46080*pi^6*x^6 + 1/384*pi^4*x^4 - 1/8*pi^2*x^2 + 1
```

We find an expression for the remainder $R_n = f - P_n$ between the function f and its nth degree Taylor P_n polynomial.

By Taylor's theorem, there exists an $\xi(x)$ between x and the origin such that

$$R_n(x) = \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(\xi).$$

Since $f(x) = \cos \frac{\pi}{2} x$, we can express derivatives as

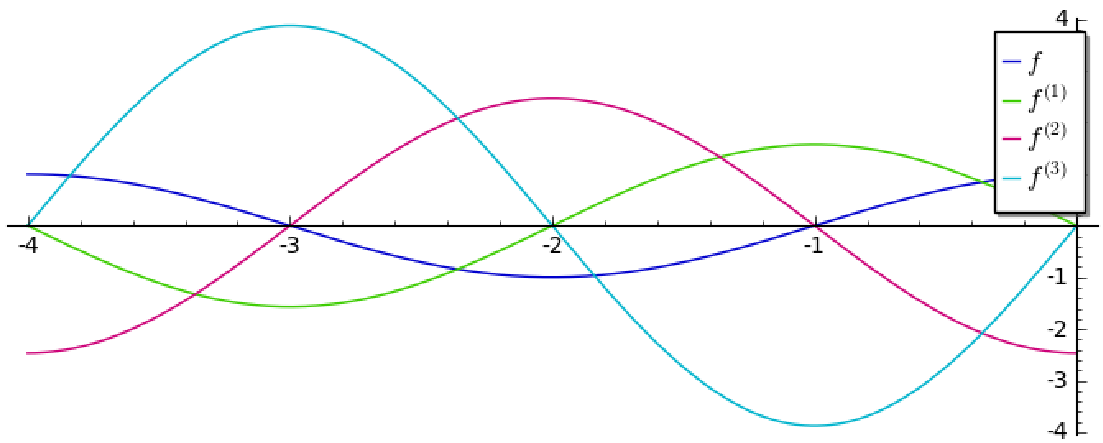
$$f^{(k)}(x) = (\pi/2)^k \cos\left(\frac{\pi}{2}(x+k)\right)$$

```
In [4]: # some evidence for the claim
        for k in range(4):
            error(x) = f.diff(x, k) - (pi/2)^k*cos((pi/2)*(x + k))
            print bool(error == 0)
```

```
True
True
True
True
```

```
In [5]: # and graphically, we see differentiation is just translation and scaling
        plot([f.diff(x, k) for k in range(4)], (x, -4, 0),
             legend_label = ['$f$','$f^{(1)}$', '$f^{(2)}$', '$f^{(3)}$'],
             figsize = [7,3])
```

Out[5]:



Whence

$$R_n(x) = \frac{x^{n+1}}{(n+1)!}(\pi/2)^{(n+1)} \cos\left(\frac{\pi}{2}(\xi + n + 1)\right) \quad \text{for some } \xi \text{ between } x \text{ and the origin.}$$

We estimate the number of terms required to guarantee accuracy for $f(x)$ within 10^{-5} for all x in the interval $[-1,1]$.

The error $|P_n(x) - f|$ is just $|R_n(x)|$.

Since $\cos\colon \mathbf{R} \to [-1,1]$, for whatever ξ, we have the inequality

$$|R_n(x)| \leq \left|\frac{x^{n+1}}{(n+1)!}(\pi/2)^{(n+1)}\right|.$$

Further, we notice

$$\max_{x\in[-1,1]} \left|\frac{x^{n+1}}{(n+1)!}(\pi/2)^{(n+1)}\right| = \frac{(\pi/2)^{(n+1)}}{(n+1)!}.$$

which is our desired error bound.

To find n such that $|R_n(x)| \leq \varepsilon$ with $\varepsilon = 10^{-5}$, we iterate.

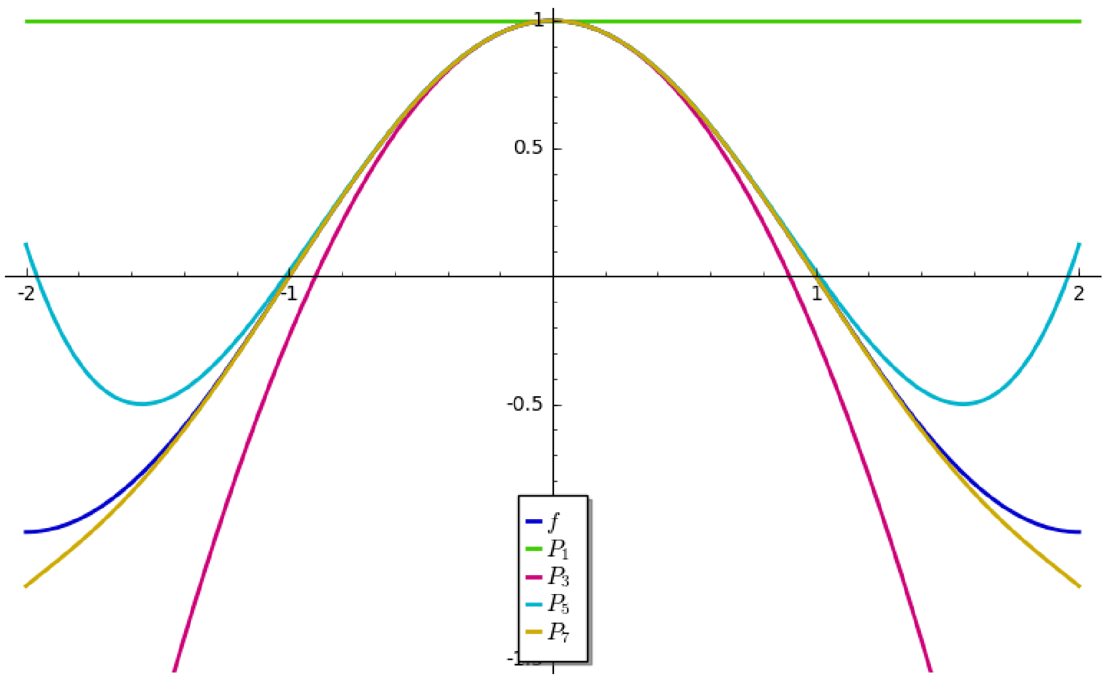
```
In [6]: from math import factorial
```

```
n = 0; eps = 10**(-5)
error_bound = lambda n : ((pi/2)**(n+1))/factorial(n+1)

while error_bound(n) > eps:
    n += 1
n
```

Out[6]: 10

We plot $f(x)$ and its 1st, 3rd, 5th and 7th degree Taylor polynomials over $[-2, 2]$.

```
In [7]: curves = [f] + [f.taylor(x,0,i) for i in [1,3,5,7]]
        p = plot(curves, (x,-2,2),
                 legend_label = ['$f$','$P_1$', '$P_3$', '$P_5$', '$P_7$'],
                 thickness = 2.0)
        p.show(ymin = -1.5, figsize = [8,5])
```



prob 2

We define a fixed point iteration function, which returns a list with 2 entries - a list of the first so many terms in the fixed point iteration sequence - the last term before the function halts fixed point iteration

```
In [8]: def fixedpt(g, approx, eps, Nmax, SAVELIST = True):
            i=0; previous = float("inf"); seq = [approx]
            while abs(approx - previous) > eps and i < Nmax:
                previous = approx
```

```
            approx = g(x=approx)
            i += 1
            if SAVELIST:
                seq = seq + [approx]
        return [seq, N(approx)]
```

We'll consider a few fixed point iteration schemes. Each of the following functions g have a fixed point α. But which converge to α (provided $|x_0 - \alpha|$ is sufficiently small)? (For discussion on *writing* such an iterator function, see fixed point iteration: finding g(x) on Stack Exchange.)

If it does converge, we'll give the order of convergence; for linear convergence, give the rate of linear convergence (i.e., the asymptotic constant). In the case that $g'(\alpha) = 0$, we'll expand $g(x)$ in a Taylor polynomial about $x = \alpha$ to determine the order of convergence.

prob 2 (a)

Consider the fixed point iteration (to find $\alpha = 2$).

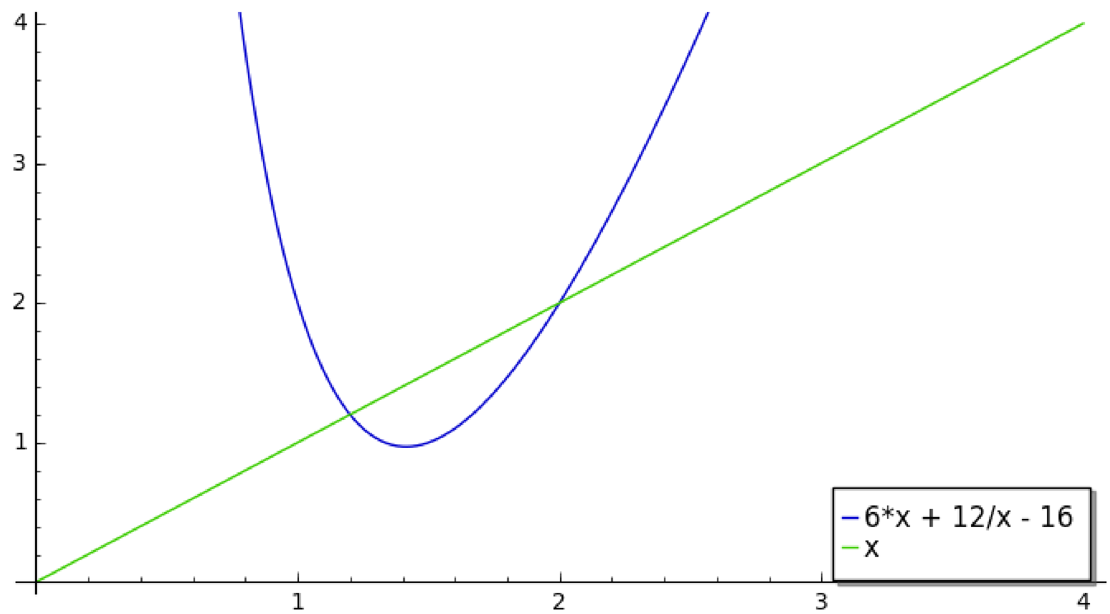
$$x_{n+1} = -16 + 6x_n + \frac{12}{x_n}$$

```
In [9]: g = -16 + 6*x + 12/x
        approx=2.1; eps=10^(-15); Nmax=10
        fixedpt(g, approx, eps, Nmax)
```

```
Out[9]: [[2.10000000000000,
          2.31428571428571,
          3.07089947089947,
          6.33304672201985,
          23.8931029233178,
          127.860854525974,
          751.258979179941,
          4491.56984826643,
          26933.4217612703,
          161584.531013165,
          969491.186153253],
         969491.186153253]
```

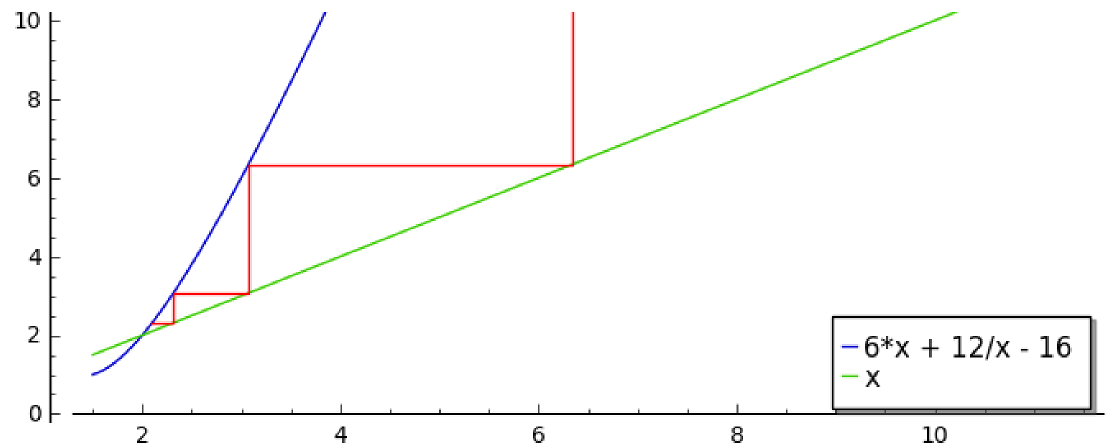
Starting with $x_0 = 2.1$, this iteration scheme diverges, as evinced by the above sequence of values and the following graphs.

```
In [10]: A = plot([g, x], (x, 0, 4), ymin=0, ymax=4, legend_label = 'automatic')
         show(A, figsize = [7,4])
```



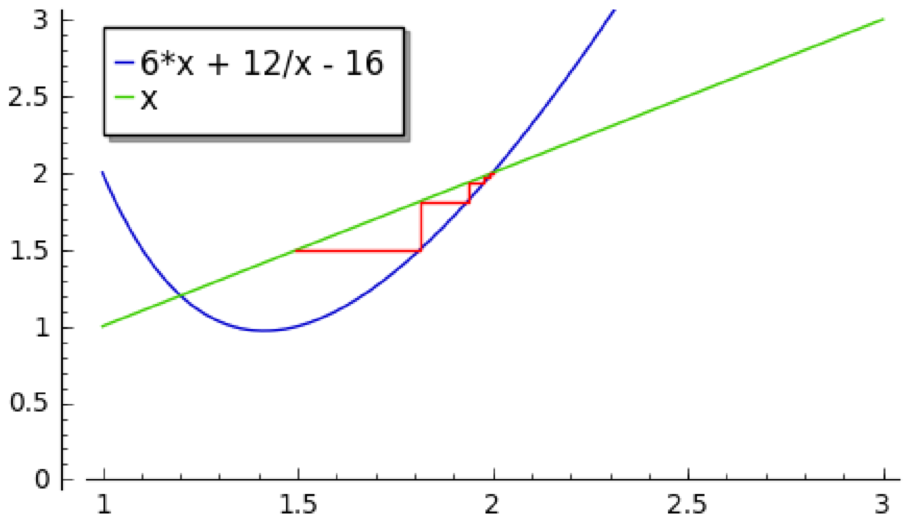
We'll generate cobweb plots to get some intuition as to why these iterations diverge.

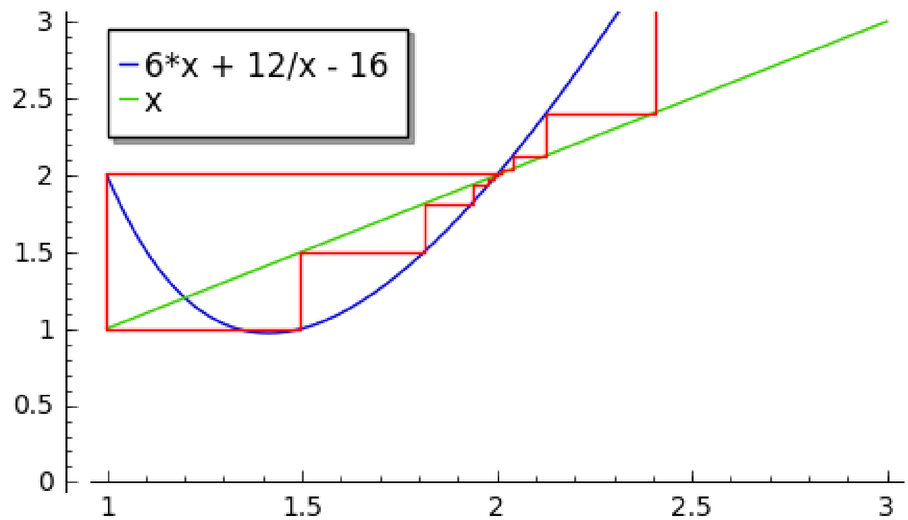
```
In [11]: approx=2.1; eps=10^(-15); Nmax=10

         seq = fixedpt(g, approx, eps, Nmax)[0]
         points = []
         for i in range(Nmax):
             points.append((seq[i], seq[i+1]))
             points.append((seq[i+1], seq[i+1]))

         h = 10
         A = plot([g, x], xmin=1.5, xmax=1.5+h, ymin = 0, ymax = h, legend_label = 'automatic')
         B = list_plot(points, xmin=1.5, xmax=1.5+h, ymin = 0, ymax = h, plotjoined = True, color = 'rec
         show(A+B, figsize = [7,3])
```



We can also try the iteration scheme from just below α; this sequence also diverges.

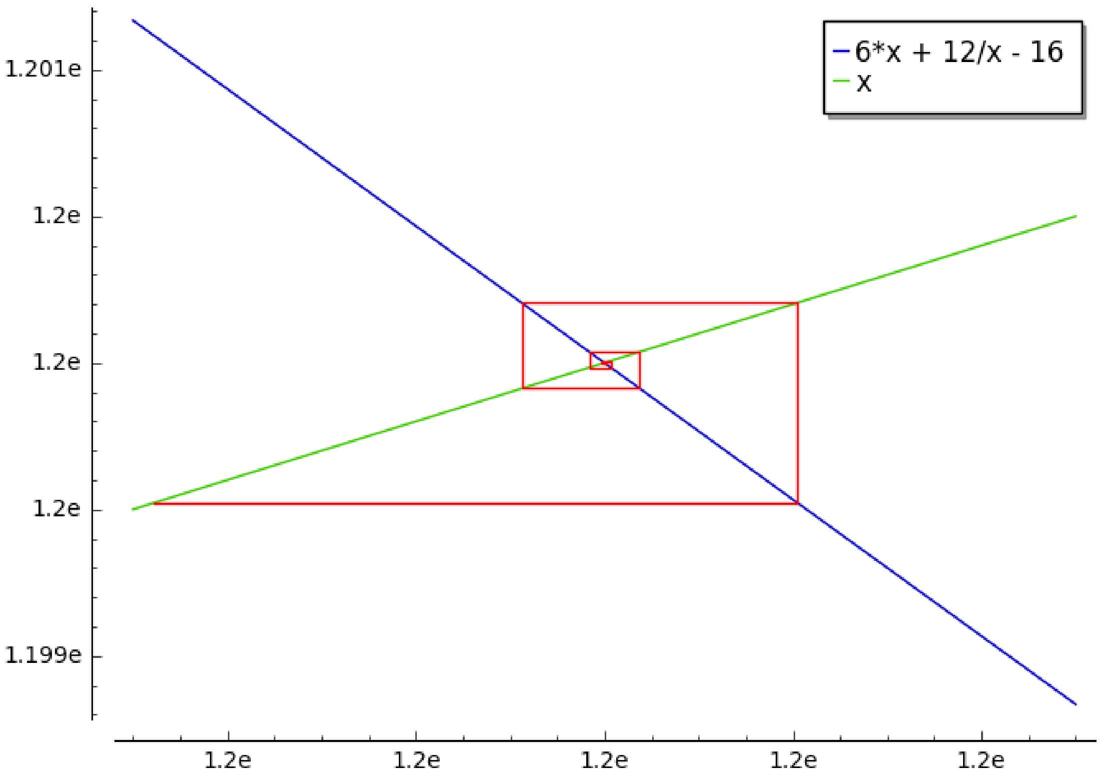
```
In [12]: approx=1.99999; eps=10^(-15); Nmax=10

         seq = fixedpt(g, approx, eps, Nmax)[0]
         points = []
         for i in range(Nmax):
             points.append((seq[i], seq[i+1]))
             points.append((seq[i+1], seq[i+1]))

         h = 1
         A = plot([g, x], xmin=2-h, xmax=2+h, ymin = 0, ymax = 2+h, legend_label = 'automatic')
         B = list_plot(points, xmin=2-h, xmax=2+h, ymin = 0, ymax = 2+h, plotjoined = True, color = 'rec
         show(A+B, figsize = [5,3])

         approx=1.99999; eps=10^(-15); Nmax=25

         seq = fixedpt(g, approx, eps, Nmax)[0]
         points = []
         for i in range(Nmax):
             points.append((seq[i], seq[i+1]))
             points.append((seq[i+1], seq[i+1]))

         h = 1
         A = plot([g, x], xmin=2-h, xmax=2+h, ymin = 0, ymax = 2+h, legend_label = 'automatic')
         B = list_plot(points, xmin=2-h, xmax=2+h, ymin = 0, ymax = 2+h, plotjoined = True, color = 'rec
         show(A+B, figsize = [5,3])
```



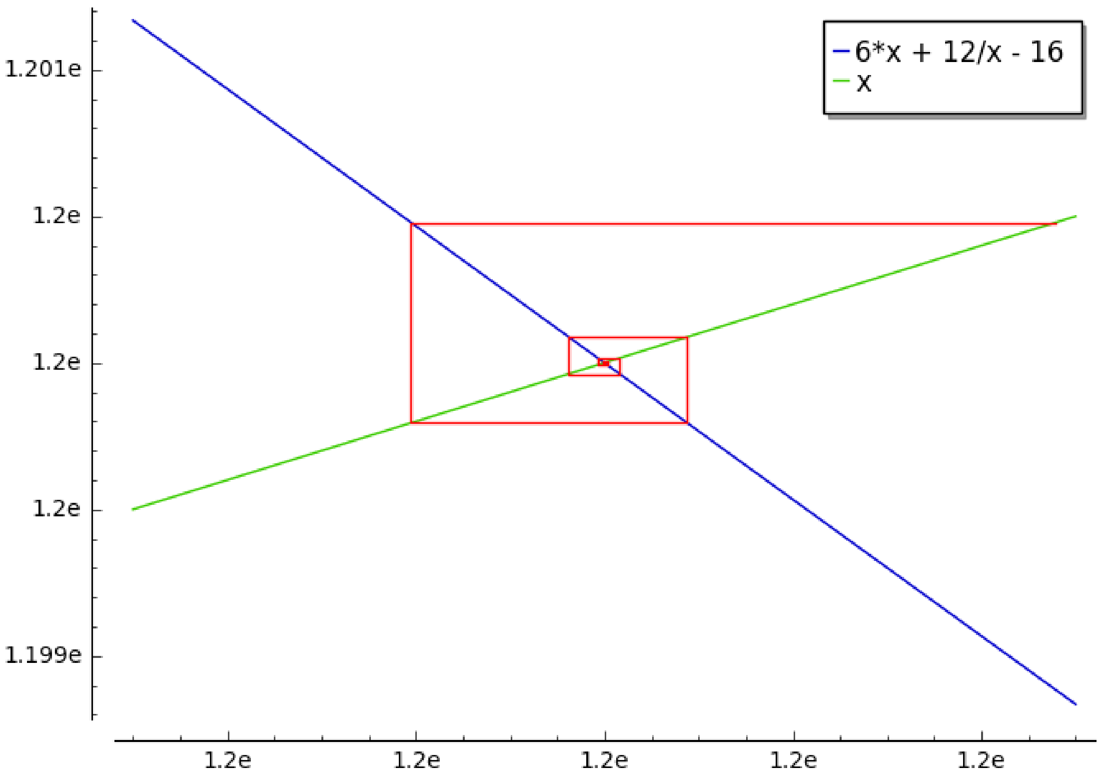


Additionally, we consider the other fixed point 1.2 (from below and from above); it seems the graph of g is *too steep* for convergence.

```
In [13]: #below
         approx=1.1999999; eps=10^(-15); Nmax=10
         seq = fixedpt(g, approx, eps, Nmax)[0]
         points = []
         for i in range(Nmax):
             points.append((seq[i], seq[i+1]))
             points.append((seq[i+1], seq[i+1]))

         h = 0.5*10^(-3)
         A = plot([g, x], (x, 1.2-h, 1.2+h), legend_label = 'automatic')
         B = list_plot(points, plotjoined = True, color = 'red')
         show(A+B, figsize = [7,5])
```




```
In [14]: # from above
         approx=1.2000001; eps=10^(-15); Nmax=10
         seq = fixedpt(g, approx, eps, Nmax)[0]
         points = []
         for i in range(Nmax):
             points.append((seq[i], seq[i+1]))
             points.append((seq[i+1], seq[i+1]))

         h = 0.5*10^(-3)
         A = plot([g, x], (x, 1.2-h, 1.2+h), legend_label = 'automatic')
         B = list_plot(points, plotjoined = True, color = 'red')
         show(A+B, figsize = [7,5])
```



Looking at g' at both 1.2 and 2, we see

```
In [15]: print "g'(1.2) = " + str(g.diff(x)(x=1.2))
         print "g'(2.0) = " + str(g.diff(x)(x=2))
```

```
g'(1.2) = -2.33333333333333
g'(2.0) = 3
```

prob 2 (b)

Now consider the iteration scheme (to find the fixed point $\alpha = 3^{1/3}$)

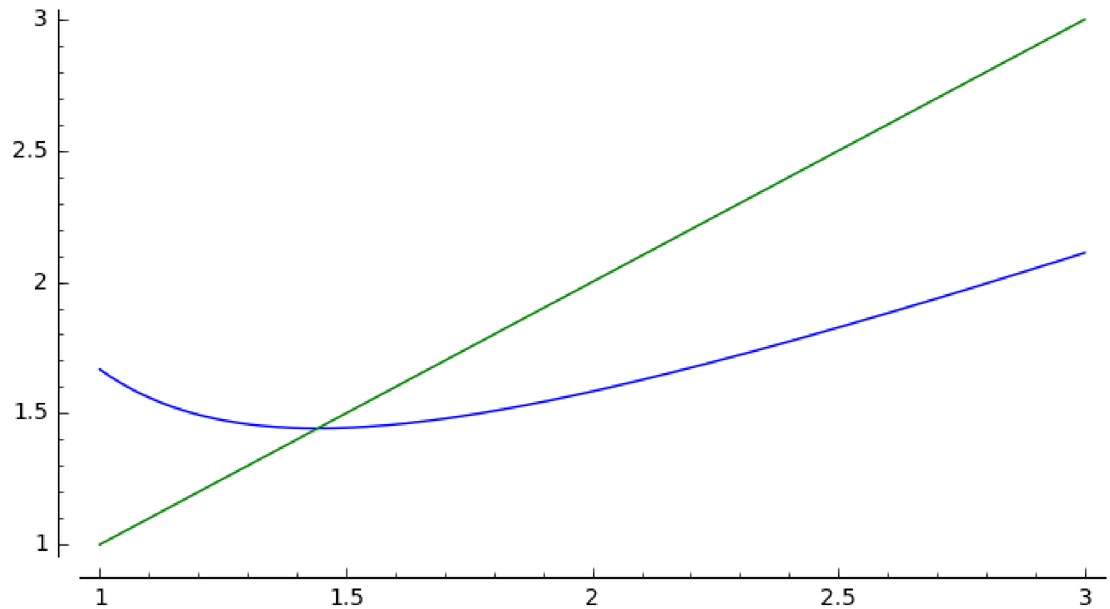
$$x_{n+1} = \frac{2}{3}x_n + \frac{1}{x_n^2}$$

To determine if this iteration scheme produces a convergent sequence, we'll state a simple existence theorem.

Let $g : [a, b] \to [a, b]$ be a continuous function with g differentiable on (a, b) and $|g'(x)| < 1$ for all x in (a, b). Then the sequence $\{p_n\}$ generated by $p_n = g(p_{n-1})$ converges to the fixed point p for any initial point $p_0 \in (a, b)$.

Looking at the following graph, it seems likely that the fixed point iterator will produce a convergent sequence. We'll construct a cobweb diagram before rigorously demonstrating the sequence converges.

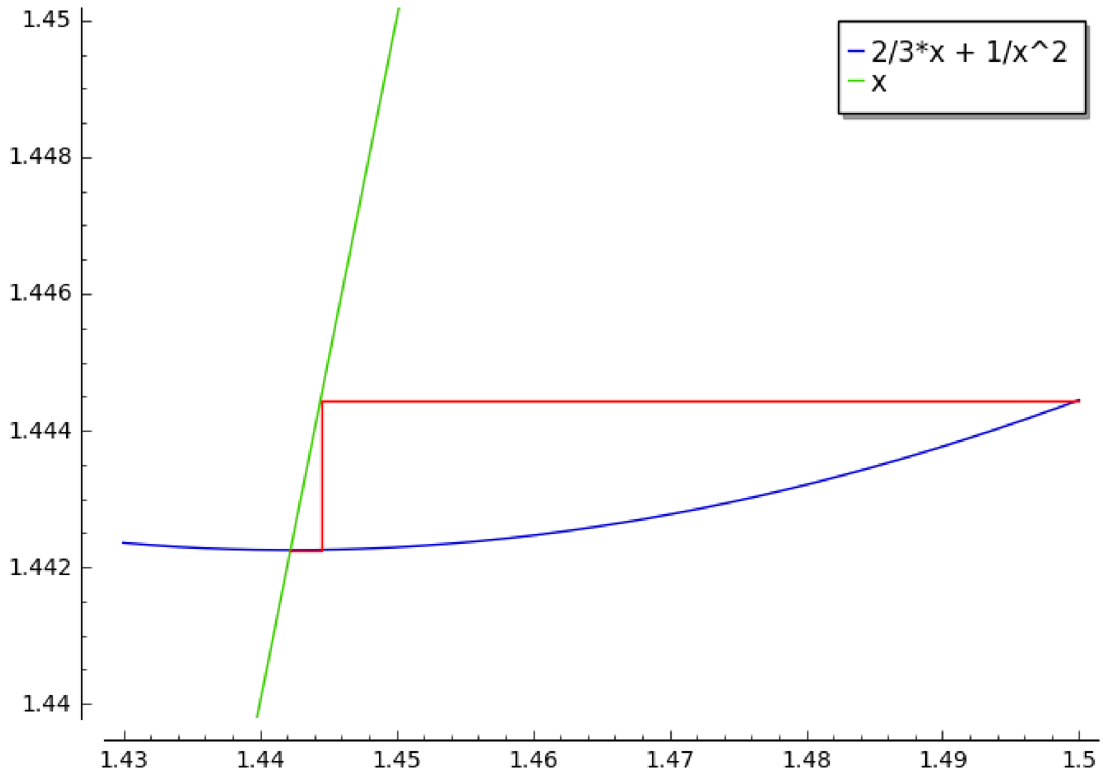
In [16]:
```
g = (2/3)*x + 1/(x^2)
xmin = 1; xmax = 3
a = plot(g, (x, xmin, xmax))
b = plot(x, (x, xmin, xmax), color = 'green')
show(a+b, figsize = [7,4])
```



In [17]:
```
# from above
approx=1.5; eps=10^(-15); Nmax=10
fixedpt(g, approx, eps, Nmax)
```

Out[17]:
```
[[1.50000000000000,
  1.44444444444444,
  1.44225290379137,
  1.44224957031511,
  1.44224957030741,
  1.44224957030741],
 1.44224957030741]
```

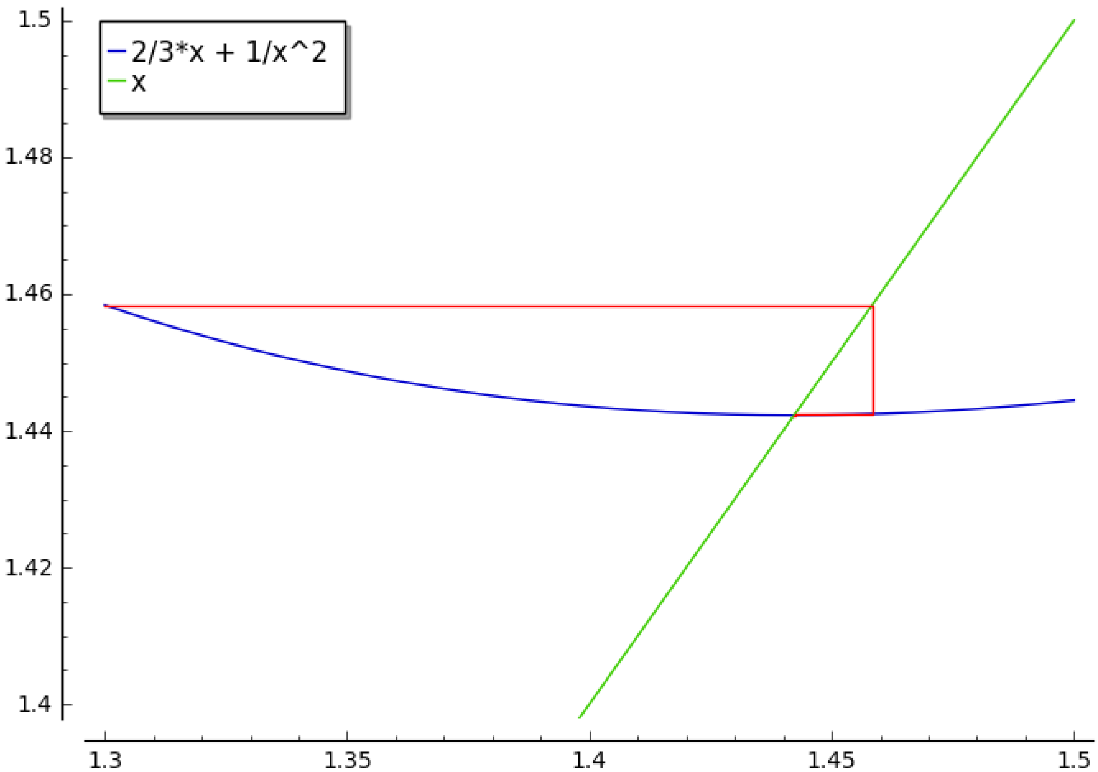
In [18]:
```
# cobweb from above
seq = fixedpt(g, approx, eps, Nmax)[0]
points = []
for i in range(len(seq)-1):
    points.append((seq[i], seq[i+1]))
    points.append((seq[i+1], seq[i+1]))

h = 0.07
A = plot([g, x], (x, approx-h, approx), ymax = 1.45, ymin =1.44, legend_label = 'automatic')
B = list_plot(points, plotjoined = True, color = 'red')
show(A+B, figsize = [7,5])
```



In [19]:
```
# from below
approx=1.3; eps=10^(-15); Nmax=10
fixedpt(g, approx, eps, Nmax)
```

Out[19]:
```
[[1.30000000000000,
  1.45838264299803,
  1.44242738117535,
  1.44224959222560,
  1.44224957030741,
  1.44224957030741],
 1.44224957030741]
```

In [20]:
```
# cobweb from below
seq = fixedpt(g, approx, eps, Nmax)[0]
points = []
for i in range(len(seq)-1):
    points.append((seq[i], seq[i+1]))
    points.append((seq[i+1], seq[i+1]))

A = plot([g, x], xmin=1.3, xmax=1.5, ymin=1.4, ymax=1.5, legend_label = 'automatic')
B = list_plot(points, plotjoined = True, color = 'red')
show(A+B, figsize = [7,5])
```



In this case $g'(\alpha) = 0$, so we'll try expanding $g(x)$ in a Taylor polynomial about $x = \alpha$ to determine the order of convergence.

```
In [21]: show(g.taylor(x, 3^(1/3), 4))
```

```
5/9*(x - 3^(1/3))^4 - 4/9*3^(1/3)*(x - 3^(1/3))^3 + 1/3*3^(2/3)*(x - 3^(1/3))^2 + 3^(1/3)
```

As the smallest non-zero power of g's Taylor series representation is 2, we ought to suspect the order of convergence is 2.

To numerically confirm this intuition, consider

$$r \approx \frac{\ln E_n}{\ln E_{n-1}},$$

where $E_n = |x_n - \alpha| \approx |x_n - x_{n+1}|$.

```
In [22]: # we find r approximated at different steps n
         rate_approx = []
         for n in [1..len(seq)-2]:
             rate_approx.append(log(abs(seq[n]-seq[n+1]))/log(abs(seq[n-1]-seq[n])))

         rate_approx
```

```
Out[22]: [2.24554929318992, 2.08675279479323, 2.04240018648047, 2.00445730406457]
```

Aside: is the asymptotic error constant given by $C = g''(3^{1/3}) \approx 1.386$? No.

```
In [23]: N(g.diff(x,2)(x=3^(1/3)))

Out[23]: 1.38672254870127
```

Why not? Consider a set of starting points for the fixed point iteration scheme

$$\texttt{starts} = \{1.1, 1.2, 1.3, 1.4, 1.5, 1.6\}$$

and look at the sequences of values generated.

```
In [24]: # halting parameters
         eps=10^(-20); Nmax=10

         # starting points
         starts = [1.1, 1.2, 1.3, 1.4, 1.5, 1.6]

         # sequences generated from starting points
         seqs = []
         for start in starts:
             seqs.append(fixedpt(g, start, eps, Nmax)[0])
```

Aside: is the order of this method $r = 2$?
By definition, a sequence x_n converges to α of *order* r with asymptotic error constant C iff

$$\lim_{n\to\infty} \frac{|x_n - \alpha|}{|x_{n-1} - \alpha|^r} = C.$$

We'll estimate C by evaluating the ratio $|x_{n+1} - x_n| : |x_n - x_{n-1}|^r$ for all n in $\{0, 1, 2, 3, 4\}$.

```
In [25]: r = 2

         for i in range(len(seqs)):
             print("starting at " + str(N(starts[i], digits = 2)))
             for n in range(5):
                 print("E_"+str(n+1)+":(E_"+str(n)+")^"+str(r)+" ~ "+str(abs(seqs[i][n+1]-seqs[i][n])/ab
             print("")

starting at 1.1
E_1:(E_0)^2 ~ 3.92521896530955
E_2:(E_1)^2 ~ 0.515127403734279
E_3:(E_2)^2 ~ 0.723722620291041
E_4:(E_3)^2 ~ 0.696087035559723
E_5:(E_4)^2 ~ 0.693377765497636

starting at 1.2
E_1:(E_0)^2 ~ 5.01738392104314
E_2:(E_1)^2 ~ 0.581252332057324
E_3:(E_2)^2 ~ 0.708606891994268
E_4:(E_3)^2 ~ 0.693936927079967
E_5:(E_4)^2 ~ 0.693356049491765

starting at 1.3
```

```
E_1:(E_0)^2 ~ 7.82718609595824
E_2:(E_1)^2 ~ 0.636046348028419
E_3:(E_2)^2 ~ 0.698388201565812
E_4:(E_3)^2 ~ 0.693418242131110
E_5:(E_4)^2 ~ 0.924401492637500

starting at 1.4
E_1:(E_0)^2 ~ 24.3903507867735
E_2:(E_1)^2 ~ 0.678813844638179
E_3:(E_2)^2 ~ 0.693773107480024
E_4:(E_3)^2 ~ 0.693423633803438
E_5:(E_4)^2 ~ 0.000000000000000

starting at 1.5
E_1:(E_0)^2 ~ 16.6577774285817
E_2:(E_1)^2 ~ 0.710059171597676
E_3:(E_2)^2 ~ 0.694062057080989
E_4:(E_3)^2 ~ 0.693365858877836
E_5:(E_4)^2 ~ 0.000000000000000

starting at 1.6
E_1:(E_0)^2 ~ 5.73466736902171
E_2:(E_1)^2 ~ 0.731003100856789
E_3:(E_2)^2 ~ 0.698057272096944
E_4:(E_3)^2 ~ 0.693410849037607
E_5:(E_4)^2 ~ 0.806030801736205
```

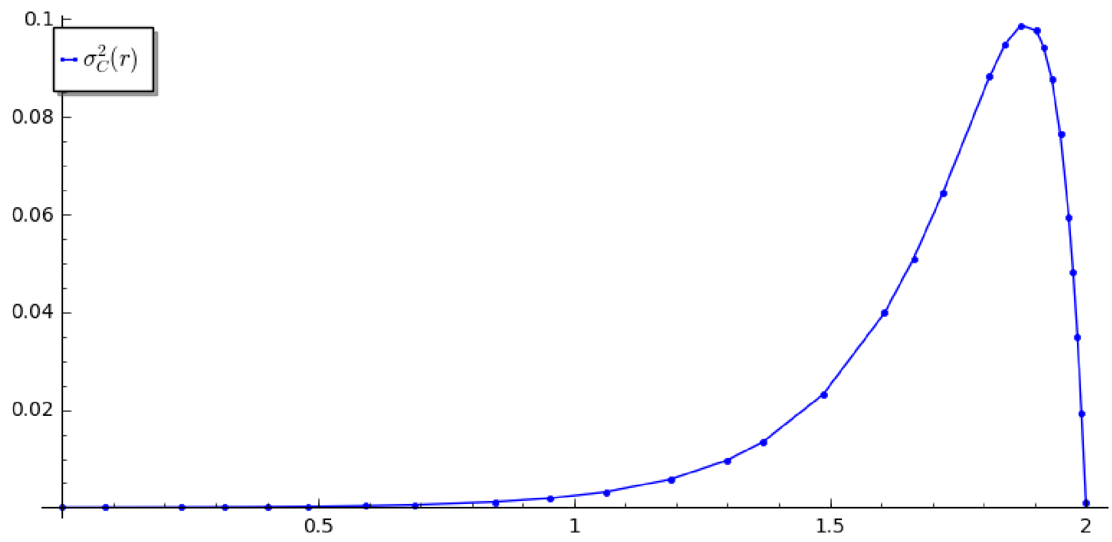
Do the above ratios converge? Well, the algorithm quits after a certain degree of precision, so we'll only consider the good middle values.

TODO: Firm up definition of asymptotic error constant, by varying r and while looking for convergent values C.

Plotting r against σ_C^2 (the standard deviation of the `E_4:(E_3)^r`'s generated above), we find:

```
In [26]: def std_C(r):
             return std([abs(seqs[i][3+1]-seqs[i][3])/abs(seqs[i][3]-seqs[i][3-1])^r for i in range(len(

         var('r')
         plot(std_C(r), (r,0,2), plot_points=20,
              marker='.', legend_label= '$\\sigma^2_C(r)$',
              figsize = [8,4])
```

Out[26]:



Well, while varying r, we find that the approximate C have *no deviation* about $r = 2$. As we need our iterative scheme to converge upon an asymptotic error constant, I continue to suspect $r = 2$. (Update: this can/will be theoretically shown.)

TODO: find r and C rigorously.

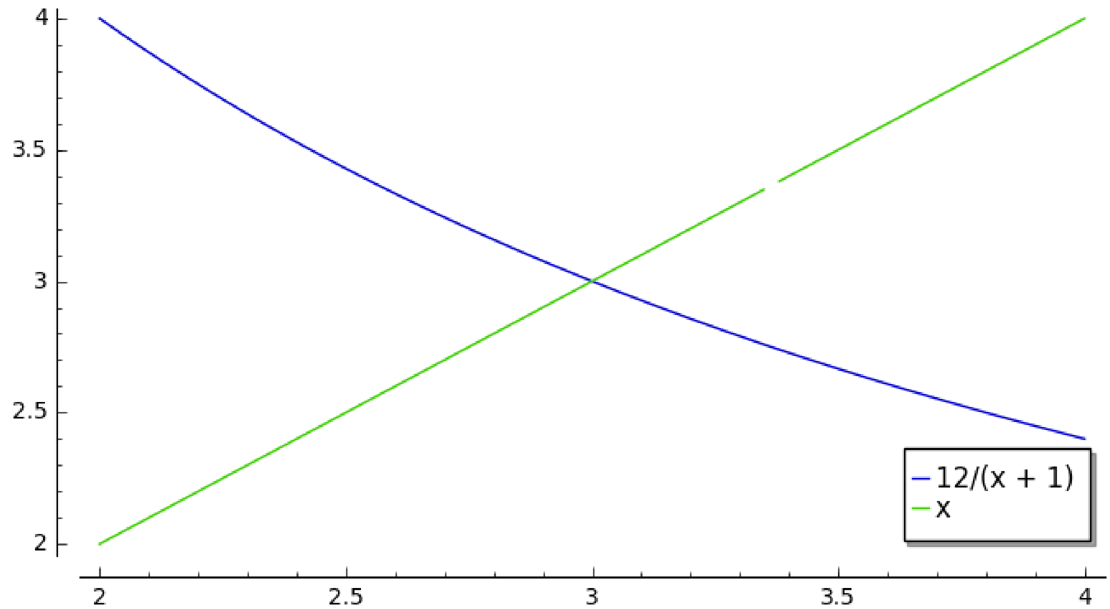
prob 2 (c)

Lastly, consider (to find the fixed point $\alpha = 3$)

$$x_{n+1} = \frac{12}{1 + x_n},$$

In [27]:
```
g = 12/(1+x)
xmin = 2; xmax = 4
plot([g,x], (x, xmin, xmax), legend_label = 'automatic', figsize=[7,4])
```

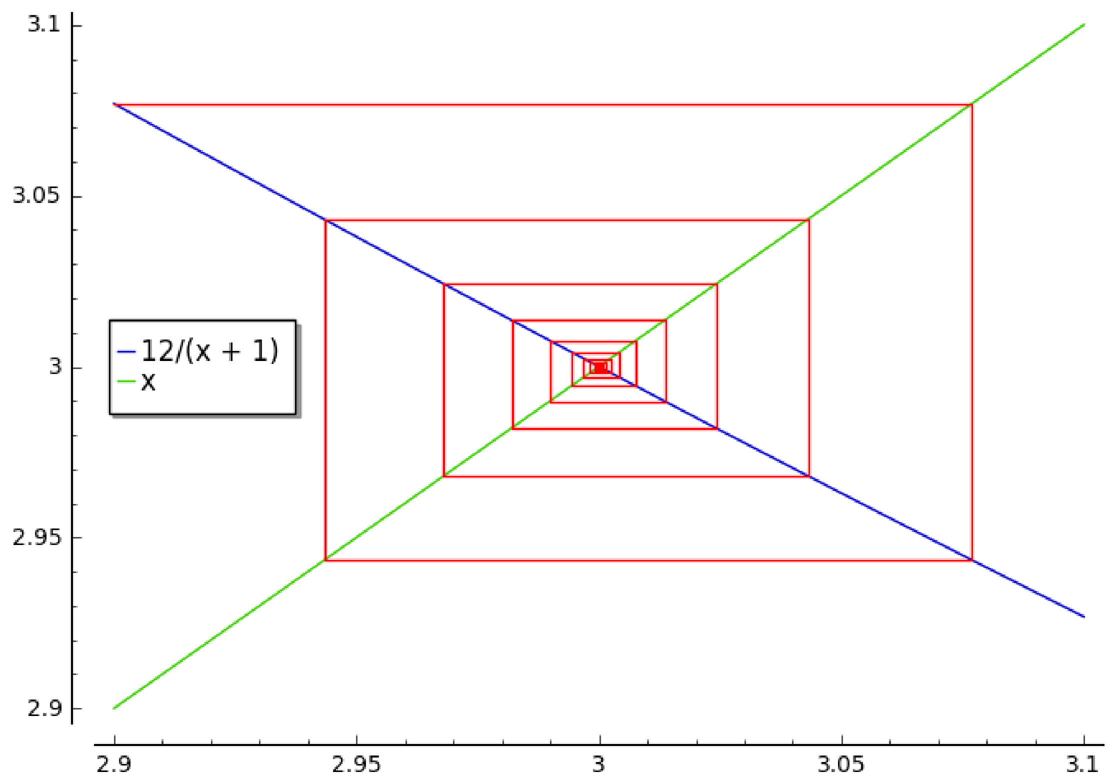
Out[27]:




```
In [28]: # notice the derivative at x = 3 is small enough, and this is likely the asymptotic error cons
         abs(g.diff(x,1)(x=3))
```

```
Out[28]: 3/4
```

```
In [29]: # cobweb diagram from starting point 2.9
         approx=2.9; eps=10^(-15); Nmax=20
         seq = fixedpt(g, approx, eps, Nmax)[0]
         points = []
         for i in range(len(seq)-1):
             points.append((seq[i], seq[i+1]))
             points.append((seq[i+1], seq[i+1]))

         # draw the figure
         h = 0.1
         A = plot([g, x], xmin=3-h, xmax=3+h, ymin=3-h, ymax=3+h, legend_label = 'automatic')
         B = list_plot(points, plotjoined = True, color = 'red')
         show(A+B, figsize = [7,5])
```




```
In [30]: # halting parameters
         eps=10^(-20); Nmax=20

         # starting points
         starts = [2.9, 2.95, 2.99, 3.01, 3.05, 3.1]

         # sequences generated from starting points
         seqs = []
         for start in starts:
             seqs.append(fixedpt(g, start, eps, Nmax)[0])
```

Again, I'm not sure how to determine the order of convergence r. Theoretically, I'm led to believe that $r = 1$ and the asymptotic error constant C is bounded by

$$C \leq \max_{x \in [2.8, 3.2]} |g'(x)|$$

We can verify this hunch by comparing the numerically determined asymptotic constants to the maximum slope on the interval $[2.8, 3.2]$.

```
In [31]: # numerically determined C from differnt starting points
         r = 1
         [abs(seqs[i][10+1]-seqs[i][10])/abs(seqs[i][10]-seqs[i][10-1])^r for i in range(len(starts))]

Out[31]: [0.749642720979530,
          0.749822700352781,
```

```
0.749964751571779,
 0.750035143620742,
 0.750174679326877,
 0.750346795995921]
```

```
In [32]: # max slope on [2.8,3.2]
         max([abs(g.diff(x,1)(x=start)) for start in starts])

Out[32]: 0.788954635108481
```

prob 3

Let α be a fixed point of $g(x)$. Consider the fixed-point iteration $x_{n+1} = g(x_n)$ and suppose that $\max_x |g'(x)| = k < 1$.

I claim $|\alpha - x_{n+1}| \leq \frac{k}{1-k} |x_{n+1} - x_n|$.

Proof. We desire $(1-k)|\alpha - x_{n+1}| \leq k |x_{n+1} - x_n|$.

Suppose $g\colon [\alpha, x_n] \to [\alpha, x_n]$ is a continuous function, differentiable on (α, x_n). Then the mean value theorem implies there's a ξ in (α, x_n) such that $\frac{g(\alpha)-g(x_n)}{\alpha - x_n} = g'(\xi)$.

Moving the denominator, taking absolute values, and noting $k = \max_x |g'(x)|$, we find $|g(\alpha) - g(x_n)| \leq k |\alpha - x_n|$.

We assume $\alpha = g(\alpha)$, and by definition $x_{n+1} = g(x_n)$. It follows that

$$|\alpha - x_{n+1}| \leq k |\alpha - x_n|.$$

Now consider that $\alpha - x_n = \alpha - x_{n+1} + x_{n+1} - x_n$.

Applying the triangle inequality, we find $|\alpha - x_n| \leq |\alpha - x_{n+1}| + |x_{n+1} - x_n|$.

With a bound for $|\alpha - x_n|$, we obtain

$$|\alpha - x_{n+1}| \leq k |\alpha - x_{n+1}| + k |x_{n+1} - x_n|.$$

from which the desired inequality results: $(1-k)|\alpha - x_{n+1}| \leq k |\alpha - x_{n+1}|$. $\square$