2014 Algebra SEP $\sim$ Rings and Modules Day 3 Solutions, by E. Dummit

1. (DF-p369)

   (a) If m and n are relatively prime positive integers, prove that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$.

   (b) More generally, show that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$ where $d = \gcd(m, n)$.

   Solution: First we observe that by the properties of simple tensors we have $p \otimes q = pq(1 \otimes 1)$ so since any element of the tensor product is a sum of simple tensors, we see that the tensor product is a cyclic module generated by $1 \otimes 1$.

   a) If m and n are relatively prime then there exist a and b with $am + bn = 1$, so we see that $1 \otimes 1 = (am + bn) \otimes 1 = bn \otimes 1 = b \otimes n = b \otimes 0 = 0$. Hence by the observation above, the tensor product is zero.

   b) We have a natural map from $(\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z}) \to \mathbb{Z}/d\mathbb{Z}$ defined via $(p, q) \mapsto pq \bmod d$; it is well-defined since d divides m and d, and we see that it is clearly $\mathbb{Z}$-bilinear. Hence by the universal property of the tensor product we obtain a map $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \mapsto \mathbb{Z}/d\mathbb{Z}$ sending $1 \otimes 1 \mapsto 1$, hence the tensor product has size at least d. On the other hand, by the argument in part (a) we see that $d(1 \otimes 1) = 0$ hence its order is at most d. Hence it is exactly d, and so it is isomorphic to $\mathbb{Z}/d\mathbb{Z}$.

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2. (DF-10.4.17+19) Let $I = (2, x)$ in the ring $R = \mathbb{Z}[x]$. Observe that $\mathbb{Z}/2\mathbb{Z} \cong R/I$ is naturally an R-module.

   (a) Show that the map $\varphi : I \times I \to \mathbb{Z}/2\mathbb{Z}$ defined via $\varphi(a_0 + a_1 x + \cdots, b_0 + b_1 x + \cdots) = \dfrac{a_0}{2} b_1 \bmod 2$ is R-bilinear.

   (b) Show that there is an R-module homomorphism from $I \otimes_R I \to \mathbb{Z}/2\mathbb{Z}$ mapping $p(x) \otimes q(x)$ to $\dfrac{p(0)}{2} q'(0)$.

   (c) Show that $2 \otimes x \neq x \otimes 2$ in $I \otimes_R I$.

   (d) Show that the submodule of $I \otimes I$ generated by $\alpha = 2 \otimes x - x \otimes 2$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

   Solution:

   a) Clearly φ is additive in both components, so we need only see that $\phi(r(x) \cdot a(x), b(x)) = \phi(a(x), r(x) \cdot b(x))$. If $r(x) = r_0 + r_1 x + \cdots$, then $\varphi(ar, b) = \dfrac{r_0 a_0}{2} b_1$ while $\varphi(a, rb) = \dfrac{a_0}{2} [a_0 b_1 + a_1 b_0]$, and these are equal modulo 2 because $\dfrac{a_0}{2} \in \mathbb{Z}$ and $a_1 \in 2\mathbb{Z}$.

   b) This follows immediately from the universal property of tensor products applied to the map $\varphi : I \times I \to \mathbb{Z}/2\mathbb{Z}$ from part (a).

   c) The images of $2 \otimes x$ and $x \otimes 2$ are different under the map of part (b), since $\psi(2 \otimes x) = 1$ while $\psi(x \otimes 2) = 0$.

   d) We see that $x [2 \otimes x - x \otimes 2] = 2x \otimes x - x^2 \otimes 2 = 2x \otimes x - 2x \otimes x = 0$, and similarly $2 [2 \otimes x - x \otimes 2] = 2 \otimes 2x - 2 \otimes 2x = 0$, so 2 and x hence all of I annihilates α. Hence the submodule generated by α is isomorphic to a submodule of $R/I \cong \mathbb{Z}/2\mathbb{Z}$, and since it is nonzero by (c) we see it is all of $\mathbb{Z}/2\mathbb{Z}$.

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3. (Jan-14.4) Let R be a commutative ring and M and N be R-modules. Recall that M is "torsion" if, for each $m \in M$ there exists a nonzero $r \in R$ such that $rm = 0$, and that M is "torsion-free" if $rm = 0$ implies $r = 0$ or $m = 0$.

   (a) If M and N are torsion modules, show that $M \otimes_R N$ is torsion.
   If M and N are torsion-free, however, $M \otimes_R N$ is not necessarily torsion-free: let $I = (x, y)$ in $R = \mathbb{C}[x, y]$. Show that $I \otimes_R I$ is not torsion-free, as follows:

   (b) Show that $x \otimes y - y \otimes x \in I \otimes_R I$ is a torsion element.

   (c) Show that $x \otimes y - y \otimes x \neq 0$.

   Solution: As it was stated on the qualifying exam, the problem does not say that R is an integral domain. If R is not a domain, then the set of elements $m \in M$ such that there exists a nonzero $r \in R$ is not necessarily a submodule of M. (For example, if $R = M = \mathbb{Z}/6\mathbb{Z}$ with the natural action, then the torsion elements of M are $\{0, 2, 3, 4\}$, which is not closed under addition.)

a) Let $x = \sum_i m_i \otimes n_i$ be a sum of simple tensors, such that $r_i m_i = 0$. Then $\prod r_i$ annihilates x, since it kills each term, and $\prod r_i \neq 0$ since R is a domain. (In fact, we only need M to be a torsion module for this to work.)

b) Observe that $xy \cdot (x \otimes y - y \otimes x) = xy \otimes xy - xy \otimes xy = 0$, where we absorb the x and y on the right and left, and left and right, respectively.

c) Let $\varphi : I \times I \to \mathbb{C}$ be defined to send $(p, q) \mapsto p_x(0,0) \cdot q_y(0,0)$, where the subscripts denote partial derivatives. This is clearly a bilinear map, and if r is any polynomial in R, then $\varphi(rp, q) = [r_x(0,0)p(0,0) + r(0,0)\, p_x(0,0)]\, q_y(0,0) = r_x(0,0) \cdot q_y(0,0) \cdot r(0,0)$, as $p(0,0) = 0$ since $p \in I$, and similarly we see that $\varphi(p, rq) = r_x(0,0) \cdot q_y(0,0) \cdot r(0,0)$ as well. So this map is R-bilinear, hence by the universal property of the tensor product there exists a homomorphism $\psi : I \otimes I \to \mathbb{C}$ that agrees with it on simple tensors. But then we see $\psi(x \otimes y - y \otimes x) = 1$, so $x \otimes y - y \otimes x \neq 0$.

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4. (Aug-13.5): Let V be a k-vector space and V^* be its dual vector space. Consider the map $\psi : V^* \otimes_k V \to \mathrm{Hom}_k(V, V) = \mathrm{End}(V)$ given by $\sum_i \varphi_i \otimes v_i \mapsto f$ such that $f(v) = \sum_i \varphi_i(v) v_i$.

(a) Characterize the image of this map.

(b) Fill in the blank, and prove your answer: "The above map is an isomorphism if and only if the vector space V is ______".

(c) Note that $\mathrm{End}(V)$ is a ring, and elements of $\mathrm{End}(V)$ act on the left, making V a left $\mathrm{End}(V)$-module. There is also a natural right action of $\mathrm{End}(V)$ on V^* given by $(\varphi \cdot f)(v) = \varphi(f(v))$, for $f \in \mathrm{End}(V)$ and $\varphi \in V^*$. With these assumptions, compute the k-vector space $V^* \otimes_{\mathrm{End}(V)} V$ under the assumption that V is finite-dimensional.

Solution:

a) An endomorphism T is in the image of ψ if and only if $\mathrm{im}(T)$ is finite dimensional.

- $\Rightarrow$: Fix any $x \in V^* \otimes_k V$ with $x = \sum_{i=1}^{j} \varphi_i \otimes v_i$. Then for any $v \in V$, $\psi(x)(v) \in \mathrm{span}(v_1, \cdots v_j)$, hence $\mathrm{im}(T) \subseteq \mathrm{span}(v_1, \cdots, v_j)$, and the latter is a finite-dimensional vector space.
- $\Leftarrow$: Suppose $\mathrm{im}(T)$ is finite-dimensional and spanned by $e_1, \cdots, e_k$, and extend this to a basis of V by adjoining some other vectors B. Now let $\varphi_i \in V^*$ be the element of the dual space that sends all basis elements to 0 except for e_i which it sends to 1. Write $T(e_d) = \sum_i a_i e_i$ for constants $a_{i,d}$, and then set $x = \sum a_{i,i} \varphi_i \otimes e_i$: then we see that for any d, $\psi(x)(e_d) = \sum_i a_{i,d} \varphi_i(e_d) e_i = a_{d,d} e_d$, and $\psi(x)(b) = 0$ for any $b \in B$. Thus $\psi(x)$ agrees with T on a basis for V , hence on all of V.

b) The answer is "finite dimensional". This follows immediately from the result of part (a): if V is finite-dimensional, then the map is surjective since every endomorphism of V has a finite-dimensional image, hence it is an isomorphism since $V^* \otimes_k V$ and $\mathrm{End}(V)$ both have dimension $\dim(V)^2$. If V is infinite-dimensional, there are obviously linear transformations which do not have a finite-dimensional image (e.g., the identity map), so ψ is not surjective.

c) We claim that the tensor product is isomorphic to k (at least, ignoring the trivial case $V = 0$). To see this, choose any nonzero vector e in V. If w is any other nonzero vector in V then there exists an invertible linear transformation T sending w to e (extend w, e to a basis and then swap w and e), so $\varphi \otimes w = \varphi T^{-1} \otimes Tw = \varphi T^{-1} \otimes e$, meaning that every element in the tensor product is of the form $\varphi \otimes e$ for some $\varphi \in V^*$. Furthermore, we can see that $\varphi_1 \otimes e = 0$ if $\varphi_1(e) = 0$, because if T' is the linear transformation sending all basis elements of V other than e to 0 and fixes e, then $\varphi_1 \otimes e = \varphi_1 \otimes Te = \varphi_1 T \otimes e = 0 \otimes e = 0$ (since $\varphi_1 T$ kills everything). Hence $\varphi \otimes e = \varphi(e)\, [T' \otimes e]$, meaning that $\dim_k(V^* \otimes_k V) \leq 1$. But the map $V^* \times V \to k$ sending $(\varphi, v) \mapsto \varphi(v)$ is clearly R-bilinear and surjective, so it induces a surjection from the tensor product, which is therefore an isomorphism.

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5. (DF-10.3.11+10.4.16+X-1) Let R be a commutative ring with 1 and M and N be nonzero simple R-modules. (Recall that a simple module has no nontrivial proper submodules.)

   (a) Show that every nonzero element of $\mathrm{Hom}_R(M, N)$ is an isomorphism.

   (b) If I and J are any ideals of R, prove that $(R/I) \otimes_R (R/J) \cong R/(I + J)$. [Hint: First show every element of the tensor product is of the form $\bar{1} \otimes \bar{r}$ for some $r \in R$.]

   (c) Prove that $M \otimes_R N \neq 0$ implies that $\mathrm{Hom}_R(M, N) \neq 0$.

Solution:

a) This result (and a variety of similar statements) is known as Schur's lemma. If $\varphi : M \to N$ is an R-module homomorphism, then $\ker(\varphi)$ is a submodule of M, hence is either 0 or M. If it is M then φ is zero; otherwise φ is injective. Similarly, $\mathrm{im}(\varphi)$ is a submodule of N hence is either 0 or N; if it is 0 then φ is zero; otherwise φ is surjective. Hence: if φ is nonzero, then it is an isomorphism.

b) First we claim that every element of the tensor product is a simple tensor of the form $(1 + I) \otimes (r' + J)$ for $r \in R$; to see this we simply observe that $(a + I) \otimes (b + J) = (1 + I) \otimes (ab + J)$ and then extend to all tensors by linearity. We also have an R-bilinear map $\psi : (R/I) \times (R/J) \to R/(I + J)$ defined via $(r + I, r' + J) \mapsto rr' + (I + J)$, so by the universal property of the tensor product we have a surjective map from $(R/I) \otimes (R/J) \to R/(I + J)$ sending $(r + I) \otimes (r' + J) \mapsto rr' + (I + J)$. By the observation above, we can restrict to the tensors of the form $(1 + I) \otimes (r' + J)$, and from this we see that the map is also injective (since two tensors of this form are equal iff their second components differ by an element of I, so r' is defined up to an element of $I + J$), hence it is an isomorphism.

c) Let $x \in M$ and $y \in N$ be nonzero. Then $Rx = M$ since Rx is a nonzero submodule of M; similarly, $Ry = N$. Now define $\psi_1 : R \to M$ via $r \mapsto rx$; by the above we see this is a surjective R-module map, and its kernel I is the annihilator $\mathrm{ann}(M)$ of M: hence $M \cong R/I$ as an R-module; similarly, $N \cong R/J$ where $J = \mathrm{ann}(N)$. Now we can apply part (b) to see that $M \otimes_R N \cong R/(I + J)$, where $I = \mathrm{ann}(M)$ and $J = \mathrm{ann}(N)$. Furthermore, since M and N are simple, we see that I and J are both maximal ideals, so either $I = J$ or $I + J = R$. Then by part (b), the tensor product is nonzero if and only if $I = J$, hence $M \cong N$ and so $\mathrm{Hom}_R(M, N) \neq 0$ (since the isomorphism provides a nonzero element).

6. (Jan-13.4) Recall that a right R-module P is projective if for every surjection $f : N \to P$ there is a map $g : P \to N$ such that $f \circ g : P \to P$ is the identity.

   (a) Prove that a free R-module is projective.

   (b) Prove that a module M is projective iff there exists N such that $M \oplus N$ is free.

   (c) Assume R is commutative. If an "R-projection" is an R-module homomorphism $A : R^n \to R^n$ such that $A^2 = A$, prove that a finitely-generated R-module M is projective iff it is isomorphic to the image of some projection.

Solution: These are basically diagram chases.

a) If F is free with basis $\{b_i\}$ and $f : N \to F$ is a surjection, choose any preimage n_i of b_i. Then we can define $g : F \to N$ by setting $g(b_i) = n_i$; this map exists by the mapping property of free modules (which says: a map out of a free module is determined uniquely by the images of the basis elements). Then we have $f(g(\sum r_i b_i)) = f(\sum r_i n_i) = \sum r_i b_i$, so $f \circ g$ is the identity on P.

b) This is a standard property of projective modules that follows from the splitting criterion of exact sequences: specifically, choose any set of generators for M, and let F be the free module generated by those generators; by definition we have a surjection from F to M. Let K be the kernel of the map $f : F \to M$, and consider the short exact sequence $0 \to K \xrightarrow{i} F \xrightarrow{f} M \to 0$ where $i : K \to F$ is the natural inclusion map; the sequence is exact since by hypothesis f is surjective. Now we claim that the existence of the map $g : M \to F$ forces $F \cong K \oplus M$ (this is known as the splitting criterion of exact sequences): to see this, consider the map $\phi : K \oplus M \to F$ defined via $(x, y) \mapsto i(x) + g(y)$; it is clearly an R-module homomorphism. If $(x, y) \in \ker \phi$, then $i(x) + g(y) = 0$; applying f yields $0 = f \circ i(x) + f \circ g(y) = 0 + y$, so $y = 0$. Then $i(x) = 0$ so since i is injective we see $x = 0$, whence $\ker \phi$ is trivial. Further, ϕ is surjective: for any $z \in F$, we have $\phi(i^{-1}(z - g(f(z)), f(z)) = z - g(f(z)) + g(f(z)) = z$; note that $z - g(f(z)) \in \mathrm{im}(i)$ since it is in the kernel of f, so $i^{-1}(z - g(f(z)))$ is well-defined. Hence ϕ is an isomorphism so we can take $N = K$.
For the other direction, if M is such that $M \oplus N = F$ is free, then by part (a) if $f : Q \to M$ is a surjection, then we have a surjection $f' : Q \oplus N \to F$; then since F is projective by part (a), we have a map $g : F \mapsto Q \oplus N$ such that $f' \circ g : F \to F$ is the identity. Then $f \circ g|_M$ is the identity on M, so we are done.

c) First suppose $A^2 = A$, and consider the map $\phi : \ker(A) \oplus \mathrm{im}(A) \to R^n$ via $(v, w) \mapsto v + w$: if $\phi(v, w) = 0$ then if $w = Ax$, then we see $0 = A(v + w) = Av + A^2 x = Ax = w$, so $w = 0$, and hence also $v = 0$. Further, for any $y \in R^n$, we have $\phi(y - Ay, Ay) = y$ and $y - Ay \in \ker(A)$, so ϕ is surjective, hence an isomorphism. We conclude by part (b) that $\mathrm{im}(A)$ is projective.
Conversely, suppose M is projective, then by part (b) we know that there exists N with $M \oplus N$ free, and by the construction given in (b) we see that M being finitely-generated implies that F is finitely-generated hence is R^n for some n: hence we have an isomorphism $\phi : R^n \mapsto M \oplus N$. We can then define $A : R^n \to R^n$ as the restriction of ϕ to the first component of the direct sum; this map has image M and is clearly an R-projection.

7. (Hf-p179) Let R be an integral domain. Recall that if A, B, and C are R-modules, and $f : A \to B$ and $g : B \to C$ are homomorphisms, the sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is said to be exact (at B) if $\operatorname{im}(f) = \ker(g)$, and a longer sequence is exact if it is exact at each 3-term subsequence.

   (a) If A, B, and C are finitely-generated and $0 \to A \to B \to C \to 0$ is exact, show that $\operatorname{rank}(B) = \operatorname{rank}(A) + \operatorname{rank}(C)$. (Recall that the rank of an R-module is the maximal size of a linearly-independent subset.)

   (b) If $0 \to A \to B \to C \to 0$ is exact, show that the sequence $0 \to T(A) \xrightarrow{f|_{T(A)}} T(B) \xrightarrow{g|_{T(B)}} T(C)$ is also exact, where $T(M)$ is the torsion submodule of M, defined to be the set of $m \in M$ for which there exists a nonzero $r \in R$ with $rm = 0$.

   (c) Show by an explicit example that if $0 \to A \to B \to C \to 0$ is exact, then $T(B) \xrightarrow{g|_{T(B)}} T(C) \to 0$ need not be exact.

Solution:

a) Let $a_1, \cdots, a_d$ be maximal and independent in A and $c_1, \cdots, c_k$ be maximal and independent in C. Pull back the c_i along g to obtain preimages $b_1, \cdots, b_k$ with $g(b_i) = c_i$. Now if $b \in B$, there exists nonzero $r \in R$ such that $rg(b) = \sum r_i c_i$, so $g(rb - \sum r_i b_i) = 0$, hence $rb - \sum r_i b_i$ is in the image of f. Hence there exists some nonzero $s \in R$ with $s(rb - \sum r_i b_i) = \sum s_j f(a_j)$ for some s_j, so $srb = \sum s r_i b_i + \sum s_j f(a_j)$, whence $\operatorname{rank}(B) \le \operatorname{rank}(A) + \operatorname{rank}(C)$. On the other hand, if $\sum r_i b_i + \sum s_j f(a_j) = 0$, then applying g yields $\sum r_i c_i = 0$ hence all $r_i = 0$ since the c_i are independent in C. Then since f is injective, we obtain $\sum s_j a_j = 0$ hence all $s_j = 0$ since the a_j are independent in A.

b) First, $0 \to T(A) \to T(B)$ is exact, because if $a \in T(A)$ with $ra = 0$, then $rf(a) = 0$, so f maps $T(A)$ into $T(B)$, and $f_{T(A)}$ is injective since f is. For the exactness of $T(A) \to T(B) \to T(C)$, clearly $g|_{T(B)} \circ f|_{T(A)} = 0$ since $g \circ f = 0$, so now suppose $b \in \ker(g|_{T(B)})$: then since $b \in \ker(g) = \operatorname{im}(f)$, there exists a with $f(a) = b$. Now since $b \in T(B)$ there exists a nonzero $r \in R$ with $rb = 0$, whence we see $0 = rf(a) = f(ra)$. But since f is injective we see $ra = 0$, hence $a \in T(A)$, so $b \in \operatorname{im}(f|_{T(A)})$. Thus, we conclude that $T(A) \to T(B) \to T(C)$ is exact.

c) We can take the "multiplication by 2" exact sequence $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$; the corresponding sequence of torsion submodules is $0 \to 0 \to 0 \to \mathbb{Z}/2\mathbb{Z} \to 0$, which is not exact at the last term.

Remark The nonexactness of this sequence was the original motivation of defining the Tor functor, to generate the "missing" terms at the end of the torsion-module exact sequence.

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8. (Jan-08.5) Let R be a ring with 1. An R-module V is "strongly n-generated" if every submodule of V is generated by at most n elements.

   (a) If V is strongly n-generated and $W \subseteq V$, show that W and V/W are strongly n-generated.

   (b) If $W \subseteq V$ is strongly n-generated and V/W is strongly m-generated, prove that V is strongly $(n+m)$-generated.

   (c) If V has composition length n, prove that V is strongly n-generated.

Solution:

a) The statement about W is tautological; for V/W, observe that the preimage of any submodule M of V/W is a submodule of V , and so the images of the generators of that submodule of V will generate M.

b) Let X be any submodule of V. By hypothesis, there exist m generators $x_1 + W, \cdots, x_m + W$ of X/W. Then for any $x \in X$, we can write $x + W = \sum r_i x_i + W$, so $x - \sum r_i x_i \in W$. Since W can be generated by n generators, adjoining them to the set $\{x_1, \cdots, x_m\}$ gives a set of generators for X. Hence X is generated by $n + m$ elements, as desired.

c) It is immediate that a simple module is strongly 1-generated. Applying part (b) to the composition series of V shows via a trivial induction that the kth term is strongly k-generated, so V is strongly n-generated, where n is the length of its composition series.
