2014 Algebra SEP $\sim$ Rings and Modules Day 3 Problems, by E. Dummit

1. (DF-p369)

   (a) If m and n are relatively prime positive integers, prove that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$.

   (b) More generally, show that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$ where $d = \gcd(m, n)$.

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2. (DF-10.4.17+19) Let $I = (2, x)$ in the ring $R = \mathbb{Z}[x]$. Observe that $\mathbb{Z}/2\mathbb{Z} \cong R/I$ is naturally an R-module.

   (a) Show that the map $\varphi : I \times I \to \mathbb{Z}/2\mathbb{Z}$ defined via $\varphi(a_0 + a_1 x + \cdots, b_0 + b_1 x + \cdots) = \dfrac{a_0}{2} b_1 \bmod 2$ is R-bilinear.

   (b) Show that there is an R-module homomorphism from $I \otimes_R I \to \mathbb{Z}/2\mathbb{Z}$ mapping $p(x) \otimes q(x)$ to $\dfrac{p(0)}{2} q'(0)$.

   (c) Show that $2 \otimes x \neq x \otimes 2$ in $I \otimes_R I$.

   (d) Show that the submodule of $I \otimes I$ generated by $\alpha = 2 \otimes x - x \otimes 2$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

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3. (Jan-14.4) Let R be a commutative ring and M and N be R-modules. Recall that M is "torsion" if, for each $m \in M$ there exists a nonzero $r \in R$ such that $rm = 0$, and that M is "torsion-free" if $rm = 0$ implies $r = 0$ or $m = 0$.

   (a) If M and N are torsion modules, show that $M \otimes_R N$ is torsion.
   If M and N are torsion-free, however, $M \otimes_R N$ is not necessarily torsion-free: let $I = (x, y)$ in $R = \mathbb{C}[x, y]$. Show that $I \otimes_R I$ is not torsion-free, as follows:

   (b) Show that $x \otimes y - y \otimes x \in I \otimes_R I$ is a torsion element.

   (c) Show that $x \otimes y - y \otimes x \neq 0$.

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4. (Aug-13.5): Let V be a k-vector space and V^* be its dual vector space. Consider the map $\psi : V^* \otimes_k V \to \mathrm{Hom}_k(V, V) = \mathrm{End}(V)$ given by $\sum_i \varphi_i \otimes v_i \mapsto f$ such that $f(v) = \sum_i \varphi_i(v) v_i$.

   (a) Characterize the image of this map.

   (b) Fill in the blank, and prove your answer: "The above map is an isomorphism if and only if the vector space V is ______".

   (c) Note that $\mathrm{End}(V)$ is a ring, and elements of $\mathrm{End}(V)$ act on the left, making V a left $\mathrm{End}(V)$-module. There is also a natural right action of $\mathrm{End}(V)$ on V^* given by $(\varphi \cdot f)(v) = \varphi(f(v))$, for $f \in \mathrm{End}(V)$ and $\varphi \in V^*$. With these assumptions, compute the k-vector space $V^* \otimes_{\mathrm{End}(V)} V$ under the assumption that V is finite-dimensional.

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5. (DF-10.3.11+10.4.16+X-1) Let R be a commutative ring with 1 and M and N be nonzero simple R-modules. (Recall that a simple module has no nontrivial proper submodules.)

   (a) Show that every nonzero element of $\mathrm{Hom}_R(M, N)$ is an isomorphism.

   (b) If I and J are any ideals of R, prove that $(R/I) \otimes_R (R/J) \cong R/(I + J)$. [Hint: First show every element of the tensor product is of the form $\bar{1} \otimes \bar{r}$ for some $r \in R$.]

   (c) Prove that $M \otimes_R N \neq 0$ implies that $\mathrm{Hom}_R(M, N) \neq 0$.

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6. (Jan-13.4) Recall that a right R-module P is projective if for every surjection $f : N \to P$ there is a map $g : P \to N$ such that $f \circ g : P \to P$ is the identity.

   (a) Prove that a free R-module is projective.

   (b) Prove that a module M is projective iff there exists N such that $M \oplus N$ is free.

   (c) Assume R is commutative. If an "R-projection" is an R-module homomorphism $A : R^n \to R^n$ such that $A^2 = A$, prove that a finitely-generated R-module M is projective iff it is isomorphic to the image of some projection.

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7. (Hf-p179) Let R be an integral domain. Recall that if A, B, and C are R-modules, and $f : A \to B$ and $g : B \to C$ are homomorphisms, the sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is said to be exact (at B) if $\operatorname{im}(f) = \ker(g)$, and a longer sequence is exact if it is exact at each 3-term subsequence.

   (a) If A, B, and C are finitely-generated and $0 \to A \to B \to C \to 0$ is exact, show that $\operatorname{rank}(B) = \operatorname{rank}(A) + \operatorname{rank}(C)$. (Recall that the rank of an R-module is the maximal size of a linearly-independent subset.)

   (b) If $0 \to A \to B \to C \to 0$ is exact, show that the sequence $0 \to T(A) \xrightarrow{f|_{T(A)}} T(B) \xrightarrow{g|_{T(B)}} T(C)$ is also exact, where $T(M)$ is the torsion submodule of M, defined to be the set of $m \in M$ for which there exists a nonzero $r \in R$ with $rm = 0$.

   (c) Show by an explicit example that if $0 \to A \to B \to C \to 0$ is exact, then $T(B) \xrightarrow{g|_{T(B)}} T(C) \to 0$ need not be exact.

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8. (Jan-08.5) Let R be a ring with 1. An R-module V is "strongly n-generated" if every submodule of V is generated by at most n elements.

   (a) If V is strongly n-generated and $W \subseteq V$, show that W and V/W are strongly n-generated.

   (b) If $W \subseteq V$ is strongly n-generated and V/W is strongly m-generated, prove that V is strongly $(n + m)$-generated.

   (c) If V has composition length n, prove that V is strongly n-generated.
