POLYNOMIAL RINGS

COLTON GRAINGER (MATH 6130 ALGEBRA)

12. Assignment due 2018-12-12

12.1. **[1, No. 9.1.4].** *Given.* Let (x) and (x, y) be ideals in the ring of polynomials $\mathbf{Q}[x, y]$.

To prove.

- Both (x) and (x, y) are prime.
- (x, y) is maximal.
- (x) is not maximal.

12.2. **[1, No. 9.1.10]. Lemma.** Let $A = (a_i, a_2, ..., a_n)$ be a nonzero finitely generated ideal of R. There is an ideal B which is maximal with respect to the property that it does not contain A.

Given. Let R be the polynomial ring $\mathbf{Z}[x_1, x_2, x_3, \ldots]/(x_1x_2, x_3x_4, x_5x_6, \ldots)$.

To prove. R contains infinitely many minimal prime ideals.

12.3. **[1, No. 9.1.13].** *Given.* Let F be a field.

To prove. The rings $F[x, y]/(y^2 - x)$ and $F[x, y]/(y^2 - x^2)$ are not isomorphic.

12.4. **[1, No. 9.2.2]. Lemma.** Let $f(x) \in F[x]$ be a polynomial of degree $n \geqslant 1$ and let bars denote passage to the quotient $F[x]/(f(x))$. For each $\overline{g(x)}$ there's a unique polynomial $g_0(x)$ of degree strictly less than n such that $\overline{g(x)} = \overline{g_0(x)}$. In other words, the elements $\overline{1}, \overline{x}, \ldots, \overline{x^{n-1}}$ are a basis of the vector space $F[x]/(f(x))$ over F. In particular, the dimension of this space is n. (Hint: division algorithm.)

Given. Let F be a finite field of order q and let $f(x)$ be a polynomial in $F[x]$ of degree $n \geqslant 1$.

To prove. $F[x]/(f(x))$ has q^n elements.

12.5. **[1, No. 9.2.3].** *Given.* Let $f(x)$ be a polynomial in $F[x]$.

To prove. $F[x]/(f(x))$ is a field if and only if $f(x)$ is irreducible. (Hint: every nonzero prime ideal in a PID is maximal.)

12.6. **[1, No. 9.2.4].** *Given.* Let F be a finite field.

To prove. $F[x]$ contains infinitely many primes. (Note that over an infinite field the polynomials of degree 1 are an infinite set of primes in the ring of polynomials.)

12.7. **[1, No. 9.2.10].** *To find.* The greatest common divisor of $a(x) = x^3 + 4x^2 + x - 6$ and $b(x) = x^5 - 6x + 5$ in $\mathbf{Q}[x]$, expressed as a $\mathbf{Q}[x]$-linear combination of $a(x)$ and $b(x)$.

12.8. **[1, No. 9.3.3].** *Given.* Let F be a field.

To prove. The set R of polynomials in $F[x]$ whose coefficient of X is equal to 0 is a subring of $F[x]$ and R is not a UFD. (Hint: show that $x^6 = (x^2)^3 = (x^3)^2$ gives two distinct factorization of x^6 into irreducibles.)

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12.9. **[1, No. 9.4.7].** *Given.* The ring of polynomials $\mathbf{R}[x]$ and the ideal generated by x^2+1.

To prove. $\mathbf{R}[x]/(x^2+1)$ is a field that's isomorphic to the complex numbers.

12.10. **[1, No. 9.4.12].** *Given.* The ring of polynomials $\mathbf{Z}[x]$ and the polynomial $x^{n-1}+x^{n-2}+\cdots+x+1$.

To prove. $x^{n-1}+x^{n-2}+\cdots+x+1$ is irreducible in $\mathbf{Z}[x]$ if and only if n is a prime.

12.11. **[1, No. 9.4.16].** *Given.* Let F be a field and let $f(x)$ be a polynomial of degree n in $F[x]$. The polynomial $g(x) = x^n f(1/x)$ is called the reverse of $f(x)$.

To demonstrate.

(a) Describe the coefficients of g in terms of the coefficients of f.
(b) f is irreducible if and only if g is irreducible.

12.12. **A variant of Eisenstein's Criterion [1, No. 9.4.17].** *Given.* et P be a prime ideal in the Unique Factorization Domain R and let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial in $R[x]$, $n \geqslant 1$. Suppose $a_n \notin P$, $a_{n-1}, \ldots, a_0 \in P$ and $a_0 \notin P^2$.

To prove. $f(x)$ is irreducible in $F[x]$, where F is the quotient field of R.

References

[1] D. Dummit and R. Foote, *Abstract algebra*. Prentice Hall, 2004.