MATH 6130: Final examination. Wednesday, 18 December 2013.

Put **your name** on each answer sheet. Answer **all four** questions.

Justify all your answers. Formula sheets, calculators, notes and books are not permitted.

1. Let $n \geq 2$ and let $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ be a factorization of n into distinct prime powers.

(i) Define the group $(\mathbb{Z}/n\mathbb{Z})^\times$, and state a theorem describing the relationship between the group $(\mathbb{Z}/n\mathbb{Z})^\times$ and the groups $(\mathbb{Z}/p_i^{a_i}\mathbb{Z})^\times$.

(ii) Find the isomorphism type of the group $\mathrm{Aut}(\mathbb{Z}/15\mathbb{Z}) \cong (\mathbb{Z}/15\mathbb{Z})^\times$.

(iii) Let x be a generator of $\mathbb{Z}/15\mathbb{Z}$. Show (preferably without checking all 15 cases) that the only automorphisms of $\mathbb{Z}/15\mathbb{Z}$ of order 2 are ϕ_1, ϕ_2 and ϕ_3, where $\phi_1(x) = x^4$, $\phi_2(x) = x^{11}$ and $\phi_3(x) = x^{14}$.

2. Let G be a group of order 30, let Z_n denote the cyclic group of order n, and let D_{2n} denote the dihedral group of order $2n$.

(i) Show that either G has a normal subgroup of order 3, or G has a normal subgroup of order 5 (or both). Hence (or otherwise) show that in either case, G has a subgroup H of order 15. Show that H is normal in G, that H is cyclic, and that H is the unique subgroup of order 15.

(ii) Show that G is the semidirect product of H and a subgroup K of order 2. Show that the possible homomorphisms ϕ associated to this semidirect product are the trivial homomorphism ϕ_0 together with the three homomorphisms defined in Problem 1. Match these four homomorphisms to the groups Z_{30}, $S_3 \times Z_5$, $D_{10} \times Z_3$ and D_{30}. Prove that these four groups are pairwise nonisomorphic.

3. Let R be the ring $\mathbb{Z}[x]$. Prove that the subset I consisting of all elements of R having constant term zero is an ideal of R. Determine whether or not I is (a) principal, (b) prime and/or (c) maximal.

4. Let T be the subset of $\mathbb{Q}$ consisting of fractions (in their lowest terms) whose denominators are not integer multiples of 3. (For example, $9/7 \in T$ but $7/9 \notin T$.) You may assume that T is a subring of $\mathbb{Q}$ and an integral domain.

(i) Find a necessary and sufficient condition for an element of T to be a unit. Deduce that any element of T is an associate (i.e., unit multiple) of 3^k for a unique nonnegative integer k.

(ii) Let $q = a/b \in T$ be a fraction in lowest terms. Find a necessary and sufficient condition involving a and/or b ensuring that $3|q$ in T.

(iii) Prove that 3 is an irreducible element of T and that the only irreducible elements of T are the associates of 3.

(iv) Show that 3 is a prime element of T, and prove that the irreducible elements of T coincide with the prime elements.

(v) Prove that T is a unique factorization domain, and factorize the element $\dfrac{54}{5}$ into irreducibles in T.
