2014 Algebra SEP $\sim$ Rings and Modules Day 2 Problems, by E. Dummit

1. (Jan-06.2) Let R be the subring of $\mathbb{Z}[x]$ consisting of all polynomials with zero x- and x^2-coefficients.
   (a) Show that $\mathbb{Q}(x)$ is the field of fractions of R.
   (b) Find the integral closure of R in $\mathbb{Q}(x)$.
   (c) Does there exist a polynomial $g(x) \in R$ such that R is generated as a ring by 1 and $g(x)$?

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2. (Aug-09.2/Jan-08.2a) Let $R \subseteq S$ be commutative rings with the same 1, and assume that every element of S is integral over R.
   (a) If $r \in R$ has an inverse in S, prove this inverse is in R.
   (b) Suppose R is a field and $s \in S$ is regular (i.e., if $sx = 0$ for some $x \in S$, then $x = 0$). Show that s is invertible in S.
   (c) If P is a prime ideal of S, prove that P is maximal in S iff $R \cap P$ is maximal in R.

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3. (Jan-01.3): Let $f(x) \in \mathbb{Z}[x]$ be monic and such that $f(\alpha) = f(2\alpha) = 0$ for some $\alpha \in \mathbb{C}$.
   (a) Show that $f(0) \neq 1$.
   (b) If f is irreducible, prove $\alpha = 0$.

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4. (Aug-12.5) Let R be a not necessarily commutative ring with 1, such that $x^5 = x$ for every $x \in R$.
   (a) Show that $J(R) = 0$.
   (b) Now assume R is right-Artinian. Prove that R is a direct sum of division rings.
   (c) Let D be a division ring direct summand of R. If F is any subfield of D, show that $F = \mathbb{F}_2$, $\mathbb{F}_3$, or $\mathbb{F}_5$.
   (d) Deduce that D above is isomorphic to $\mathbb{F}_2$, $\mathbb{F}_3$, or $\mathbb{F}_5$, and conclude that R is commutative.

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5. (Aug-04.2) Let R be a ring with 1, M be a finitely-generated (right) R-module, and $N \subset M$ a proper submodule of M.
   (a) Prove that there exists a maximal submodule of M containing N.
   (b) Show that $N + MJ$ is a proper submodule of M, where $J = J(R)$ is the Jacobson radical of R.

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6. (Aug-06.2) Let R be a ring with 1 and N a nil ideal of R such that R/N has no zero divisors.
   (a) Show that the only idempotents of R are 0 and 1.
   (b) If R/N is a division ring, show that every zero divisor in R is nilpotent.

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7. (Jan-14.1): Let R be a commutative ring and I an ideal of R.

   (a) Show that the radical of I, $\text{rad}(I)$, is an ideal of R. (Recall that the radical is given by the set of all elements $x \in R$ such that there exists an integer n such that $x^n \in I$.)

   (b) Give an example of an ideal I in $\mathbb{Q}[x, y]$ such that I is non-principal but $\text{rad}(I)$ is principal.

   (c) Suppose we try to define $\text{rad}(0)$ in $R = M_{2\times 2}(\mathbb{R})$ to be the set of all elements $r \in R$ such that there exists an integer n with $r^n = 0$. Show that this set $\text{rad}(0)$ is not an ideal of R.

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8. (Aug-08.2) Let $S = \mathbb{Z} \oplus \mathbb{Z}$, and $R = \{(a, b) \in S : a \equiv b \bmod 6\}$.

   (a) Show that R is a finitely-generated $\mathbb{Z}$-module and conclude that R is a Noetherian ring.

   (b) Prove that the ideal $P = \{(a, 0) \in R : a \equiv 0 \bmod 6\}$ is prime in R.

   (c) If Q is a primary ideal of R with $P = \text{rad}(Q)$, show that $Q = P$.

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9. (Jan-12.2) Let R be a commutative ring with 1 and Q be a primary ideal of R. Suppose that $Q = \bigcap X_i$ is a finite intersection of the ideals X_i.

   (a) If each X_i is prime, prove that $Q = X_j$ for some j. [Hint: Show that Q is prime.]

   (b) If R is Noetherian and each X_i is primary, and the radicals of the X_i are distinct, prove again that $Q = X_j$ for some j.

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10. (Aug-02.2) Let R be a commutative ring with 1 in which every proper ideal is primary.

   (a) If P is a prime ideal and I is any ideal, show that either $I \subseteq P$ or $P = IP \subseteq I$.

   (b) If M is a maximal ideal of R, show that M is the set of nonunits of R.

   (c) Show that J is prime in R iff for all $r \in R$, $r^2 \in J$ implies $r \in J$.
