MATH 120 PRACTICE FINAL

Start each of the nine problems on a new page. Give complete proofs unless otherwise indicated. Each question is worth 6 points. They are not in order of difficulty. Please ask if you are unsure what can be assumed and what requires proof. No notes or calculators may be used. All rings are assumed to be commutative with 1.

1. For this question only, just the give the answer in a box, and don't show any work. No partial credit will be given.
(a) What is the largest order of an element of S_5?
(b) Find integers a and b such that $89x + 55y = 1$. (Remark: A large sunflower will have 89 spirals of seeds in one direction, and 55 spirals in the other. This is not a coincidence!)
(c) Give a Jordan-Holder decomposition of S_{100}. (Rough translation: factor it into simple groups.)

2. Let F be a field, and $n > 1$ an integer. Inside the group $GL_n(F)$ of $n \times n$ invertible matrices, are the upper triangular matrices (those whose entries "below the main diagonal" are 0) a subgroup? Is it normal?

3. Find the center of the dihedral group D_{20}.

4. (a) The classification of finite abelian groups implies that the group $(\mathbb{Z}/40)^\times$ (the group of units in the ring $\mathbb{Z}/40$) is a product of cyclic groups. Describe it as a product of explicit cyclic groups, stating whatever theorems you invoke.
(b) Find the *smallest* positive integer m such that whenever a is relatively prime to 40, then $a^m \equiv 1 \pmod{40}$.

5. Show that A_4 is not a simple group. (Do not just say that this was stated in the text!) Hint: show that the following subset of S_4 is a subgroup: $\{e, (12)(34), (13)(24), (14)(23)\}$ (this is known as the Klein 4-group).

6. Suppose G is a finite abelian group such that there are at most n elements of order dividing n, for all positive integers n. Show that G is cyclic.

7. Suppose you have a *transitive* action of a *finite* group G on a set A.
(a) How many elements of G fix a given element $a \in A$ (in terms of $|G|$ and $|A|$)?
(b) How many elements of A are fixed by a randomly chosen element of G? More precisely: what is

$$\frac{1}{|G|} \sum_{g \in G} \#\{\text{elements of } A \text{ fixed by } g\}?$$

(c) Show that there is an element of G fixing *no* elements of A.

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8. Suppose F is a field with a finite number n of elements. Show that n is a prime power. (Hint: if p and q are distinct prime factors of n, show that there are elements of order p and q in the group $(F, +)$.)

9. (Recall that $\mathbb{F}_2$ is the field with two elements.)
(a) In the ring $\mathbb{F}_2[x]$, show by induction on n that $(1+x)^{(2^n)} = 1 + x^{(2^n)}$. Use this to show that for any positive integer n, and $0 < k < 2^n$, $\binom{2^n}{k}$ is even.
(b) How many elements of the 2010th row of Pascal's triangle are odd? Translation: how many of

$$\binom{2010}{0}, \binom{2010}{1}, \ldots, \binom{2010}{2010}$$

are odd? Hint: $2010 = 1024 + 512 + 256 + 128 + 64 + 16 + 8 + 2$.