1. (January 2009 Problem 4) Let V be a finite-dimensional vector space over the complex numbers $\mathbb{C}$ and let $T : V \to V$ be a linear operator on V.

   (a) If T is diagonalizable on V, and if W is a subspace of V with $T(W) \subseteq W$, prove that T is diagonalizable on W.

Jordan Canonical Form:
The Jordan canonical form, which requires passing to an algebraic extension of the base field containing all the eigenvalues, says that any matrix is conjugate to a block-diagonal matrix whose diagonal entries are $k \times k$ "Jordan blocks," which have λ on the diagonal, 1 in the entries directly above the diagonal, and 0 elsewhere. The Jordan form is unique up to rearrng blocks.

Eigenvalue view of Diagonalizable:
Let V be a n-dimensional vector space, and let $A : V \to V$. There may not always exist a full set of n linearly independent eigenvectors of A that form a complete basis for V, that is, the matrix A may not be diagonalizable. This happens when the algebraic multiplicity of at least one eigenvalue λ is greater than the geometric multiplicity, that is, the dimension of the nullspace of $(A-\lambda I)$.

Since T is diagonalizable on V, the Jordan form of T (on V) is diagonal. Thus the eigenvalues of T each correspond only to ordinary eigenvectors.

The non-ordinary eigenvectors correspond to the Jordan blocks with 1's on the super-diagonal. So a matrix is diagonalizable (over a field containing all eigenvalues) if and only if there are no non-ordinary generalized eigenvectors.

Generalized Eigenvector:
A generalized eigenvector of rank m of an $n \times n$ matrix A corresponding to the eigenvector λ is a vector x such that

$$(A-\lambda I)^m x = 0$$

but $(A-\lambda I)^{m-1} x \neq 0$. If $m = 1$, then x is an ordinary eigenvector.

Each eigenvalue of $T|_W$ is also an eigenvalue of T. If an eigenvalue λ corresponded to a non-ordinary eigenvector of $T|_W$, then λ would also correspond to a non-ordinary eigenvector of T in V, which is a contradiction. Thus the Jordan form of $T|_W$ is also diagonal, and $T|_W$ is diagonalizable.

   (b) If T has the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

   with respect to some basis of V, decide (with proof) whether T is diagonalizable on V.

We have $T: e_1 \mapsto 2e_3,\ e_2 \mapsto e_2,\ e_3 \mapsto 0$. Thus $T^2 e_1 = 0$, and so e_1 is a non-ordinary eigenvector corresponding to the eigenvalue $\lambda = 0$. Thus T cannot be diagonalizable.

Generalized Eigenspace
Every $n \times n$ matrix A has n linearly independent generalized eigenvectors associated with it. The set spanned by all generalized eigenvectors is the generalized eigenspace of A.

2. Let k be any field.

(a) Let $a_1, \ldots, a_d, b_1, \ldots, b_d$ be elements of k. Prove that there is a unique non-zero polynomial $f \in k[x]$ of degree at most $d-1$ over k such that $f(a_i) = b_i$.

Let V be the vector space of polynomials of degree at most d.
Then V has dimension $d+1$ over k $\longleftarrow$ A basis is $\{1, x, x^2, \ldots, x^d\}$
Define $T: V \to k^{d+1}$ by $T(f) = (f(a_0), \ldots, f(a_d))$

We show that T is injective, and therefore surjective since it is a map of vector spaces with the same dimension. We have

$$T(f) = 0 \Rightarrow f(a_i) = 0 \Rightarrow a_i \text{ is a root of } f \text{ for } i = 0, \ldots, d$$

Thus we have found $d+1$ distinct roots of f. Since k is a field, and since f has degree at most d, we conclude that $f = 0$.

Thus T is an isomorphism, and so we must have a unique $f \in k[x]$ of degree at most d with $f(a_i) = b_i$.

(b) Is the previous theorem still true if we allow polynomials with more than one variable? Specifically, prove or disprove the following claim: Let $(a_1, b_1), \ldots, (a_{\binom{d+2}{2}}, b_{\binom{d+2}{2}}), c_1, \ldots, c_{\binom{d+2}{2}} \in k$, then there exists a unique non-zero polynomial $f \in k[x, y]$ of degree at most d such that $f(a_i, b_i) = c_i$.

Let $d=1$, consider the 3 points $(1,0)$, $(0,1)$, $(1,1)$, and let $c_1 = c_2 = c_3 = 0$.
If f has degree at most 1, then $f(x,y) = a + bx + cy$ for some a, b, c.
Suppose $f(1,0) = f(0,1) = f(1,1) = 0$. Then we have

$$a + b = 0, \qquad a + c = 0, \qquad \text{and} \qquad a + b + c = 0.$$

This implies $a = b = c = 0$, i.e. $f = 0$.

3. (January 2013 Problem 5) Let W_n be the set of $n \times n$ complex matrices C such that the equation

$$AB - BA = C$$

has a solution in $n \times n$ matrices A, B.

(a) Show that W_n is closed under scalar multiplication and conjugation.

$\lambda C = (\lambda A)B - B(\lambda A)$, $PCP^{-1} = (PAP^{-1})(PBP^{-1}) - (PBP^{-1})(PAP^{-1})$

(b) Show that the identity matrix is not in W_n.

We have $tr(AB-BA) = tr(AB) - tr(BA) = 0$, and $tr(I) = n \neq 0$

(c) Give a complete description of W_2 (i.e. a criterion for determining whether a matrix C is in W_2.)

By part a, we know C is in W_2 if and only if its Jordan form is in W_2.
By part b, we know every element of W has trace 0. Thus we only need to consider matrices of the form $\begin{bmatrix} a & b \\ 0 & -a \end{bmatrix}$ where $b=1$ or $b=0$

If $a \neq 0$, then $b=0$ since the eigenvalues are distinct. Since W_2 is closed under scaling, we only need to consider the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Clearly the zero matrix is in W_2. We also have

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and $$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Thus all matrices with trace zero are in W_2, and so W_2 consists precisely of these matrices.

4. (January 2014 Problem 2) Let F be a field and n a positive integer. Let A be an n by n matrix over F, such that $A^n = 0$ but $A^{n-1} \neq 0$. Show that any n by n matrix B over F that commutes with A is contained in the F-linear span of $I, A, \ldots, A^{n-1}$.

Since A is nilpotent, the eigenvalues are all zero

λ is an eigenvalue of A $\Leftrightarrow$ λ^n is an eigenvalue of A^n

So A can be put in Jordan form over F, say $PAP^{-1} = J$.

Now, note that matrices A, B commute if and only if their conjugates PAP^{-1}, PBP^{-1} commute. Also, B is in the F-linear span of $I, A, \ldots, A^{n-1}$ is equivalent to its conjugate PBP^{-1} being in the F-linear span of $I, J, \ldots, J^{n-1}$. So, without loss of generality, we can assume that A is in Jordan form.

Since the eigenvalues of A are 0, we can write

$$A = \begin{bmatrix} 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} = a_1 E_{1,2} + a_2 E_{2,3} + \cdots + a_{n-1} E_{n-1,n} \quad \text{where } a_i \in \{0,1\}.$$

We first show that $a_i = 1$ for $i = 1, \ldots, n-1$

We have $A^2 = \left(a_1 E_{1,2} + a_2 E_{2,3} + \cdots + a_{n-1} E_{n-1,n}\right)\left(a_1 E_{1,2} + a_2 E_{2,3} + \cdots + a_{n-1} E_{n-1,n}\right)$

$$= a_1 a_2 E_{1,3} + a_2 a_3 E_{2,4} + \cdots a_{n-2} a_{n-1} E_{n-2,n}$$

$$= \begin{bmatrix} 0 & 0 & a_1 a_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & a_2 a_3 & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Terms move up and to the right.

$$E_{i,j} \cdot E_{k,\ell} = \begin{cases} 0 & \text{if } j \neq k \\ E_{i,\ell} & \text{if } j = k \end{cases}$$

Proceding inductively, we get

$$A^{n-1} = \begin{bmatrix} 0 & 0 & \cdots & (a_1 a_2 \cdots a_{n-1}) \\ 0 & 0 & 0 \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & 0 \cdots & 0 \end{bmatrix} = \left(\prod_{i=1}^{n-1} a_i\right) E_{1,n}$$

Since $A^{n-1} \neq 0$, we must have $a_i = 1$ for each i, and so $A = E_{1,2} + E_{2,3} + \cdots E_{n-1,n}$

Now for any matrix $B = \sum_{i,j} b_{ij} E_{i,j}$, we have

$$E_{k,\ell} B = \sum_j b_{\ell j} E_{k,j} \qquad \text{and} \qquad B E_{k,\ell} = \sum_i b_{ik} E_{i,\ell}$$

In other words, $E_{k,\ell}$ moves the ℓ^{th} row to the k^{th} row (and kills everything else) when acting on the left, and it moves the k^{th} column to the ℓ^{th} column (and kills everything else) when acting on the right.

This tells us that $A^m = \sum_{i=1}^{m-1} E_{i,i+m}$ acts on the left by "moving all rows up m" (and replacing empty rows with 0), and acts on the right by "moving all columns right m".

So a matrix B commutes with A if and only if these right and left actions are equal on B, i.e. B must be upper triangular, and each minor diagonal must have equal values in each entry. We can write

$$B = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \\ 0 & b_1 & \cdots & b_{n-1} \\ \vdots & & & \\ 0 & \cdots & & b_1 \end{bmatrix} = b_1 I + b_2 A + \cdots + b_n A^{n-1}.$$

5. (January 2015 Problem 4) Let F be a field, and let V be a finite-dimensional vector space over F that has dimension n ($1 \leq n < \infty$). Let q be a nonzero element of F such that $q^i \neq 1$ for all $1 \leq i \leq n$. Suppose given F-linear transformations $X : V \to V$ and $Y : V \to V$ such that $XY = qYX$. Show that XY is nilpotent.

Let λ be an eigenvalue of YX, with $YXv = \lambda v$. Then $XYv = qYXv = q\lambda v$, so $q\lambda$ is an eigenvalue of XY.

The characterstic polynomials of AB and BA are the same, and thus AB and BA have the same eigenvalues.

However, we know that XY and YX share the same eigenvalues. Thus we can say that $\lambda, q\lambda, q^2\lambda, \dots, q^n\lambda$ are all eigenvalues of XY. If $\lambda \neq 0$, then these values are distinct, which contradicts contradicts the fact that XY can only have n eigenvalues. Thus all eigenvalues of XY are zero, which implies XY is nilpotent.

A matrix is nilpotent if and only if each of its eigenvalues is 0.

Note: This is only true if you look at all eigenvalues in the algebraic closure of the field. It follows from the fact that the characteristic polynomial must be x^n to have all zero eigenvalues.

1. (August 2014 Problem 3) This problem concerns eigenvectors of linear transformations.

   (a) Let $V \neq 0$ be a finite-dimensional vector space over $\mathbb{C}$ and let $T : V \to V$ be a linear transformation. Prove that T has an eigenvector.

Since V is finite-dimensional, we can take T to be an $n \times n$ matrix over $\mathbb{C}$.

A vector v is an eigenvector of T if and only if it is in the kernel of the transformation $T - \lambda I$ for some λ. Thus we only need to show that a λ exists so that the kernel of $T - \lambda I$ is nontrivial, which is equivalent to showing $\det(T - \lambda I) = 0$.

If $\det(T - \lambda I) = 0$, then $T - \lambda I$ has no inverse, so in particular, the transformation cannot be injective, thus the kernel is nontrivial.

Now $\det(T - \lambda I_n)$ is a n^{th} degree polynomial of λ with coefficients in $\mathbb{C}$. Since $\mathbb{C}$ is algebraically closed, this must have solutions $\lambda \in \mathbb{C}$, and so λ is an eigenvalue for T (thus an eigenvalue exists).

Theorem: A field k is algebraically closed if and only if every square matrix with coefficients in k has an eigenvector.

   (b) Give an example of a finite-dimensional vector space $V \neq 0$ over $\mathbb{R}$ and a linear transformation $T : V \to V$ which does not have an eigenvector.

Any matrix T so that $\det(T - \lambda I_n)$ has no real roots λ will work. In particular, $T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has $\det(T - \lambda I_2) = \lambda^2 + 1$, and so T has no eigenvalues in $\mathbb{R}$ and therefore no eigenvectors.

   (c) Does a linear transformation of an *infinite-dimensional* vector space have to have an eigenvector? Either prove this is the case, or give an example of a linear transformation of an infinite-dimensional vector space which has no eigenvector.

If we don't assume V is over an algebraically closed field, then any example for part b would work. (take a transformation on a finite subspace which has no eigenvector, and then define the transformation to fix V outside of this subspace).

Note that our proof of part a does not work because $\det(T - \lambda I)$ may be an infinite power series, rather than a polynomial.

Note that even in infinite-dimensional vector spaces, every vector is a finite linear combination of basis elements.

So let's assume they meant to specify a vector space over $\mathbb{C}$. Define $V = \mathbb{C}[x]$, the infinite-dimensional vector space with basis $\{1, x, x^2, \ldots\}$.

We want to find a transformation T so that $T - \lambda I$ has trivial kernel for any constant λ. ($T - \lambda I$ is always injective)

Define $T : V \to V$, $T(f(x)) = xf(x)$.

Now $xf(x) = \lambda f(x)$ for $\lambda \in \mathbb{C}$ if and only if $f(x) = 0$

Proof: degrees of $xf(x)$ and $\lambda f(x)$ must be the same

Thus T has no eigenvectors.

   (d) Suppose that T and U are two linear transformations of a finite-dimensional vector space V over $\mathbb{C}$ which commute with each other. Prove that there is some $v \in V$ which is an eigenvector for both T and U.

Let v be an eigenvector for T with eigenvalue λ. Then $U(v)$ is also an eigenvector of T with eigenvalue λ, since

$$T(U(v)) = U(T(v)) = U(\lambda v) = \lambda U(v).$$

In particular, the λ-eigenspace of T, call it V_λ, is invariant under U. Thus the map $U_\lambda = U|_{V_\lambda}$ is a linear transformation $V_\lambda \to V_\lambda$, and so U_λ has an eigenvector w.

Since $U_\lambda(w) = U(w)$, w is an eigenvector of U, and since $w \in V_\lambda$, w is also an eigenvector of T.

3. (August 1999 Problem 4) In this problem, all matrices are viewed over the complex numbers.

(a) For which complex numbers x, if any, is the matrix

$$\begin{pmatrix} 1 & -2 \\ 8 & x \end{pmatrix}$$

not similar to a diagonal matrix? Explain.

An $n \times n$ matrix A is diagonalizable over the field F if it has n distinct eigenvalues in F, i.e. if its characteristic polynomial has n distinct roots in F (converse is false).

The characteristic polynomial is $\det(A-\lambda I) = (1-\lambda)(x-\lambda)+16 = \lambda^2-(1+x)\lambda+(x+16)$.
The discriminant is $(1+x)^2-4(x+16) = x^2-4x-63$, which has roots $x=9$, $x=-7$.
Thus we can say when $x \neq 9, -7$, the discriminant of the characteristic polynomial is nonzero, and thus A has distinct eigenvalues, which implies A is diagonalizable.

A matrix or a linear map is diagonalizable over the field F if and only if its minimal polynomial is a product of distinct linear factors over F.

So we only need to consider the cases $x=9$ and $x=-7$. If $x=9$, then the characteristic polynomial of A is $\lambda^2-10\lambda+25=(\lambda-5)^2$. So the minimal polynomial of A is either $(\lambda-5)$ or $(\lambda-5)^2$. We have $A-5I = \begin{bmatrix} -4 & -2 \\ 8 & 4 \end{bmatrix} \neq 0$, so the minimal polynomial must be $(\lambda-5)^2$, which means A cannot be diagonalizable.

Similarly, if $x=-7$, the characteristic polynomial of A is $\lambda^2+6\lambda+9=(\lambda+3)^2$. We have $A+3I=\begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \neq 0$, thus the minimal polynomial has repeated roots, and A cannot be diagonalizable.

(b) Let J be the $n \times n$ matrix all of whose entries are equal to 1. Find a diagonal matrix similar to J, or prove that one does not exist.

Diagonalizable Matrices:
A linear map $T: V \to V$ is diagonalizable if and only if the sum of the dimensions of its eigenspaces is equal to $\dim(V)$, which is the case if and only if there exists a basis of V consisting of eigenvectors of T. With respect to this basis, T can be represented by a diagonal matrix. (If T is a matrix with respect to any other basis, then T is similar to the diagonal matrix representation.

First note that since the colums of J are equal, they are linearly dependent, and so J cannot be invertible. This implies that J has 0 as an eigenvalue.

A square matrix is invertible if and only if 0 is not an eigenvalue.

Rank-Nullity tells us that $\dim \ker J + \dim \operatorname{im} J = n$. Since the image of J has dimension 1, and since the 0-eigenspace is just the kernel of J, we can conclude that the dimension of the 0-eigenspace is $n-1$.

Each basis vector is sent to the corresponding column vector of J, which are all equal (also $\operatorname{Rank} J = 1$)

Thus J is diagonalizable if and only if it has a nonzero eigenvalue, since in that case, the sum of the dimensions of the eigenspaces will be n. This is indeed the case, since n is also an eigenvalue of J

If the sums of each row of a matrix are equal, then the common sum is an eigenvalue.

<u>proof</u>: If m_i is the sum of the entries in the i^{th} row of A, then $Ae_i = m_i e_i$, so in the case where $m_i = m$ for each i, we get $A(e_1+\cdots+e_n) = me_1+\cdots+me_n = m(e_1+\cdots+e_n)$. This is also true for columns, since A^T has the same eigenvalues as A.

We conclude that the vector space $\mathbb{C}^n$ is spanned by the eigenspace of J, and so J is similar to the diagonal matrix

$$\begin{bmatrix} n & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

The matrix representation of the transformation J with a basis consisting of eigenvectors.

4. (January 2003 Problem 4) Let V be a vector space over the field K and let $(,) : V \times V \to K$ be a bilinear form on V.

(a) If V is finite dimensional and if W is a proper subspace of V, show that there exists a nonzero vector $v \in V$ with $(w, v) = 0$ for all $w \in W$.

the image of v in $W^\vee$ is the map $W \to K$, $w \mapsto (v,w)$

The bilinear form induces a linear map $V \to W^\vee$ given by $v \mapsto (v, \bullet)$
The problem is equivalent to showing that the kernel of this map is nontrivial,
$v \mapsto 0 \in W^\vee \Leftrightarrow (v,w) = 0 \ \forall\, w \in W$
i.e. that the induced map is <u>not</u> injective.
Since $\dim W^\vee = \dim W < \dim V$, there can be no injective map $V \to W^\vee$.

Here we are using that V is finite-dimensional

(b) Now let V have an infinite basis $\mathfrak{B}$ and let $(,)$ be the unique bilinear form such that, for all $a, b \in \mathfrak{B}$, we have $(a, b) = 0$ if $a \neq b$ and $(a, b) = 1$ if $a = b$. If W is the subspace of V spanned by all vectors of the form $a - b$ with $a, b \in \mathfrak{B}$, show that W is a proper subspace of V and that there is no non-zero vector $v \in V$ with $(w, v) = 0$ for all $w \in W$.

First we show that W must be proper. If not, then for any $a \in \mathcal{B}$, there are $\lambda_i \in K$ such that $a = \sum_{i \in M} \lambda_i (b_i - c_i)$ for a finite index set M.
Now define $e = \sum_{i \in M} (b_i + c_i)$. Then we have

$$(a,e) = \Big(\sum_{i\in M} \lambda_i (b_i - c_i), \sum_{i \in M} (b_i + c_i)\Big) = \sum_{i \in M} \big(\lambda_i (b_i, b_i) + \lambda_i (-c_i, c_i)\big) = \sum_{i \in M} \lambda_i - \lambda_i = 0$$

This implies that the basis element a does not appear in the collection $\{b_i, c_i \mid i \in M\}$, since

$$\sum_{i \in M} (a, b_i) + \sum_{i \in M} (a, c_i) = \Big(a, \sum_{i \in M} (b_i + c_i)\Big) = (a,e) = 0$$

If this is the case, then

$$(a,a) = \Big(a, \sum_{i \in M} \lambda_i (b_i - c_i)\Big) = \sum_{i \in M} \lambda_i (a, b_i) - \sum_{i \in M} \lambda_i (a, c_i) = 0$$

which is a contradiction. Thus we have shown that W is a proper subspace of V.

Now we show that there can be no non-zero vector $v \in V$ such that $(w,v) = 0$ for every $w \in W$.
Let $v \in V$ be nonzero. Then $v = \sum_{i \in N} \lambda_i a_i$ for a finite index set N, where $a_i \in \mathcal{B}$, $\lambda_i \in K - \{0\}$.
Since N is finite, there exists $b \in \mathcal{B}$ with $b \notin \{a_i \mid i \in M\}$. Let $w = a_1 - b$. Then $w \in W$, and

$$(w,v) = \Big(a_1 - b, \sum_{i \in M} \lambda_i a_i\Big) = \sum_{i \in M} \lambda_i (a_1 - b, a_i) = \lambda_1 (a_1 - b, a_1) = \lambda_1 \neq 0.$$

5. (August 2003 Problem 4) Let A be a real $n \times n$ matrix. We say that A is a *difference of two squares* if there exist real $n \times n$ matrices B and C with $BC = CB = 0$ and $A = B^2 - C^2$.

(a) If A is a diagonal matrix, show that it is a difference of two squares.

For a real number x, we can define $x^+ = \max\{x, 0\}$, and $x^- = \max\{-x, 0\}$.
Then for any x, we have $x = x^+ - x^-$, $x^+ \cdot x^- = 0$, and $x^+, x^- \geq 0$.
For a matrix $A = (a_{ij})$, we can define matrices $B = (b_i)$ and $C = (c_i)$ so that $b_{ij} = \sqrt{a_{ij}^+}$ and $c_{ij} = \sqrt{a_{ij}^-}$.

Now B and C are both diagonal, so we have $B^2 = (b_{ij}^2) = (a_{ij}^+)$, $C^2 = (c_{ij}^2) = (a_{ij}^-)$, and so $A = B^2 - C^2$.

Also, since $(b_{ii} \cdot c_{ii})^2 = a_{ii}^+ \cdot a_{ii}^- = 0$, we get $BC = CB = 0$.

(b) If A is a symmetric matrix that is not necessarily diagonal, again show that it is a difference of two squares.

We know by the real spectral theorem that A is diagonalizable, say $A = PDP^{-1}$ for a diagonal matrix D.

Real Spectral Theorem:
A real symmetric matrix is diagonalizable.

By part a, there are B, C with $BC = CB = 0$ and $D = B^2 - C^2$.
Then $(PBP^{-1})(PCP^{-1}) = P(BC)P^{-1} = 0$, and similarly $(PCP^{-1})(PBP^{-1}) = 0$.
Also $(PBP^{-1})^2 - (PCP^{-1})^2 = PB^2P^{-1} - PC^2P^{-1} = P(B^2 - C^2)P^{-1} = PDP^{-1} = A$, thus A is also a difference of two squares.

(c) Suppose A is a difference of two squares, with corresponding matrices B and C as above. If B has a non-zero real eigenvalue, prove that A has a positive real eigenvalue.

Let λ be a nonzero eigenvalue of B with eigenvector v.
Then $B^2 v = \lambda^2 v$. Also, $C^2 v = \frac{1}{\lambda} C^2(\lambda v) = \frac{1}{\lambda} C^2(Bv) = \frac{1}{\lambda} C(CB)v = 0$.
We conclude that $Av = (B^2 - C^2)v = \lambda^2 v - 0 = \lambda^2 v$, so v is an eigenvector of A with positive eigenvalue λ^2.

1. (August 2013 Problem 4) Let $T_1, \ldots, T_k$ be a collection of linear transformations which act irreducibly on a finite-dimensional $\mathbb{C}$-vector space V (i.e. such that there is no nontrivial proper subspace W such that $T_i W \subseteq W$ for all i). Suppose $S : V \to V$ is a linear transformation which commutes with each of $T_1, \ldots, T_k$. Show that S is a scalar operator.

Since $\mathbb{C}$ is algebraically closed, every matrix over $\mathbb{C}$ has an eigenvector. Let v be an eigenvector of S with eigenvalue $\lambda \in \mathbb{C}$, and let V_λ be the λ-eigenspace in V. Note that $V_\lambda \neq 0$ (it contains v). Now since $ST_i = T_i S$, we know that $T_i(V_\lambda) \subseteq V_\lambda$ for each $i = 1, \ldots, k$.

Matrices which commute with each other preserve each others' eigenspaces.

proof: Let $w \in V_\lambda$. We show that $T_i(w)$ is a λ-eigenvector of S for each i. We have $S(T_i(w)) = T_i(Sw) = T_i(\lambda w) = \lambda T_i(w)$

Thus we have shown that V_λ is invariant under each T_i, which implies $V_\lambda = V$. So S acts as the scalar operator $Sv = \lambda v$ for all $v \in V$.

2. (August 2013 Problem 5) Let V be a k-vector space and $V^\vee$ be its dual vector space. Consider the map $\psi : V^\vee \otimes_k V \to Hom_k(V, V) = End(V)$ given by $\sum_i \phi \otimes v_i \mapsto f$ where $f(v) = \sum_i \phi_i(v)v_i$.

(a) Characterize the image of this map.

We claim that the image of ψ is the subspace $E \subseteq End(V)$ of linear transformations with finite-dimensional images (i.e. $E = \{\varphi \in End(V) \mid \dim \varphi(V) < \infty\}$.)

Given bases $\{v_i\}$, $\{w_i\}$ for V, W, the tensors $\{v_i \otimes v_j\}$ form a basis for $V \otimes W$.

First we show that $\operatorname{im}\psi \subseteq E$. If $\{v_i\}$ is a basis of V, then an element of $V^\vee \otimes_k V$ can be written as a finite sum $\sum_j \varphi_j \otimes v_{i_j}$ for some $\varphi_j \in V^\vee$.
If $f = \psi(\sum \varphi_j \otimes v_{i_j})$, then we have $f(v) = \sum_j \varphi_j(v) v_{i_j} \in \operatorname{Span}\{v_{i_j}\}$, so $f \in E$.

Now we show $E \subseteq \operatorname{im}\psi$. Let $f \in E$. By Rank-Nullity, we can find a basis $\{w_i\} \cup \{v_j\}$ of V, where $\{w_i\}$ is a (possibly infinite) basis of $\ker f$, and $\{f(v_j)\}$ is a (finite) basis for $\operatorname{im}(f)$. Define $\varphi_j \in V^\vee$ to be the functional which is 1 at v_j, but 0 at all other basis elements.
Then we have

$$\psi\Big(\sum_j \varphi_j \otimes f(v_j)\Big)(w_i) = \sum_j \varphi_j(w_i) f(v_j) = 0 = f(w_i)$$
$$\psi\Big(\sum_j \varphi_j \otimes f(v_j)\Big)(v_i) = \sum_j \varphi_j(v_i) f(v_j) = f(v_i).$$

We conclude that $\psi\big(\sum_j \varphi_j \otimes f(v_j)\big) = f$, and so $E \subseteq \operatorname{im}\psi$.

(b) Fill in the blank, and prove your answer: "The above map is an isomorphism if and only if V is ____."

ψ is an isomorphism if and only if V is finite dimensional.
First, assume V is finite-dimensional. Then every endomorphism of V has finite-dimensional image, so ψ must be surjective. Moreover, since $\dim(V^\vee \otimes_k V) = \dim(V^\vee)\cdot\dim(V) = \dim(V^2) = \dim(End(V))$, we can conclude that $\psi : V^\vee \otimes_k V \to End(V)$ is in fact an isomorphism.
Conversely, suppose ψ is an isomorphism. Then every endomorphism on V has finite-dimensional image. In particular, the identity map on V has finite-dimensional image, so V must be finite-dimensional.

(c) Note that $End(V)$ is a ring, and the elements of $End(V)$ act on the left, making V a left $End(V)$-module. There is also a natural right action of $End(V)$ on $V^\vee$ given by $(\phi \cdot f)(v) = \phi(f(v))$, for $f \in End(V)$ and $\phi \in V^\vee$. With these assumptions, compute the k-vector space $V^\vee \otimes_{End(V)} V$ under the assumption that V is finite-dimensional.

Note that, in general, the tensor product $M \otimes_R N$ of R-modules M, N (right, left respectively) is simply an abelian group with added structure, but it is not necessarily an R-module if R is not abelian. However, it will be an S-module for any subring S contained in the center of R. This gives the structure of $V^\vee \otimes_{End(V)} V$ as a k-vector space, where scalar multiplication has the form $\lambda \sum_i \varphi_i \otimes v_i = \sum_i \varphi_i(\lambda I) \otimes v_i = \sum_i \varphi_i \otimes \lambda v_i$

Dual Set:
Given a vector space V with a basis $B = \{v_i\}$, its dual set is a set $B^\vee = \{\varphi_i\}$ of vectors in $V^\vee$, with the same index set, defined by

$$\varphi_i(v_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

The dual set is always linearly independent, it always exists, and it gives an injection $V \hookrightarrow V^\vee$, $v_i \to \varphi_i$

Let $v_1, \dots, v_n$ be a basis for V, and let $\varphi_1, \dots, \varphi_n$ be the dual basis, so $\varphi_i(v_i) = 1$, and $\varphi_i(v_j) = 0$ for $i \neq j$. We claim that the set $\{\varphi_i \otimes v_j\}$ spans $V^\vee \otimes_{End(V)} V$ as a k-vector space. For general tensor products, we know that every element can be written as a sum of pure tensors. If $\varphi \otimes v$ is a pure tensor, then $\varphi = \alpha_1\varphi_1 + \dots + \alpha_n \varphi_n$, $v = \beta_1 v_1 + \dots + \beta_n v_n$, and so we can write $\varphi \otimes v$ as a linear combination of pure tensors of the form $\varphi_i \otimes v_j$.

coefficients in k

Dual Basis
The dual set spans $V^\vee$ if and only if V is finite-dimensional, then we call it the dual basis.

Fix i, j with $i \neq j$. We show that $\varphi_i \otimes v_j = 0$, and $\varphi_i \otimes v_i = \varphi_j \otimes v_j$. Let $T \in End(V)$ be the transformation which sends $v_i \mapsto v_j$, and sends all other basis elements to 0. Define $S \in End(V)$ similarly, with $v_j \mapsto v_i$. Then we have

$$\varphi_i \otimes v_j = \varphi_i \otimes T v_i = \varphi_i T \otimes v_i = 0 \otimes v_i = 0$$

$$\varphi_i \otimes v_i = \varphi_i \otimes S v_j = \varphi_i S \otimes v_j = \varphi_j \otimes v_j$$

Thus we have shown that the element $\varphi_1 \otimes v_1$ spans $V^\vee \otimes V$ as a k-vector space. This implies the dimension is at most 1.

To be done, we need to show $\varphi_1 \otimes v_1 \neq 0$.

To finish the proof, we construct a surjective map of k-vector spaces $V^\vee \otimes_{End(V)} V \longrightarrow k$. Define $\psi: V^\vee \times V \longrightarrow k$ by $\psi(\varphi, v) = \varphi(v)$. This is clearly linear in both variables, and it is $End(V)$-bilinear, since, if $T \in End(V)$, we have

$$\psi(\varphi, Tv) = \varphi(Tv) = (\varphi T)v = \psi(\varphi T, v).$$

Thus by the universal property of tensor products, ψ induces a map $V^\vee \otimes_{End(V)} V \to k$. Since $\psi(\varphi_1 \otimes v_1) = \varphi_1(v_1) = 1$, we see that $\varphi_1 \otimes v_1 \neq 0$, and so $V^\vee \otimes V$ must have dimension 1 as a k-vector space, i.e. $V^\vee \otimes_{End(V)} V = k$.

The Universal Property only says this is a map of groups, but it is clearly also a map of k-vector spaces.

ψ is actually an isomorphism of vector spaces, which also implies the result, but we haven't shown that explictly.

3. (January 1991 Problem 4) Let F be an algebraically closed field of prime characteristic p and let V be an F-vector space of dimension precisely p. Suppose A and B are linear operators on V such that $AB - BA = B$. If B is nonsingular, prove that V has a basis $\{v_1, \ldots, v_p\}$ of eigenvectors of A such that $Bv_i = v_{i+1}$ for $1 \le i \le p-1$ and $Bv_p = \lambda v_1$ for some $\lambda \in F - \{0\}$.

Since F is algebraically closed, A has an eigenvector $v_1 \in V$, with eigenvalue $\lambda \in F$. We want to show that Bv_1 is also an eigenvector of A. We have

$$ABv_1 = BAv_1 + Bv_1 = B(\lambda v_1) + Bv_1 = (\lambda+1)Bv_1.$$

So Bv_1 is an eigenvector of A, with eigenvalue $\lambda+1 \neq \lambda$.

Here we use B nonsingular: these vectors are nonzero.

Proceeding inductively, we see that $v_n = B^{n-1}v_1$ is an eigenvector with eigenvalue $\lambda+n-1$. Also we clearly see that $Bv_n = v_{n+1}$ for each n. For $1 \le n \le p-1$, each eigenvalue is distinct, and so $v_1, \ldots, v_p$ are linearly independent, and therefore form a basis for V.

Thus each eigenspace has dimension 1, and so since v_{p+1}, v_1 both have eigenvalue λ, they must be scalar multiples of each other, say $v_{p+1} = \alpha v_1$. Since $v_{p+1} = Bv_p$, we are done.

5. (January 1994 Problem 4) Let X be a subspace of $M_n(\mathbb{C})$, the $\mathbb{C}$-vector space of all $n \times n$ complex matrices. Assume that every nonzero matrix in X is invertible. Prove that $\dim_{\mathbb{C}} X \leq 1$.

Let $A, B \in X$. Then $B^{-1}A \in M_n(\mathbb{C})$. $B^{-1}A$ has an eigenvector v with eigenvalue λ. So we have $B^{-1}Av = \lambda v$, so $Av = \lambda Bv$. This gives $(A - \lambda B)v = 0$, so v is an eigenvalue of $A - \lambda B$ with eigenvalue 0.

Thus $A - \lambda B$ cannot be invertible, since it has 0 as an eigenvalue. But $A - \lambda B \in X$, and so by assumption we must have $A - \lambda B = 0$.

It follows that $A = \lambda B$, and so every matrix is X is a scalar multiple of B, and therefore X has dimension at most 1

1. (January 1995 Problem 4) Let A be an $n \times n$ matrix over an algebraically closed field K, and let $K[A]$ denote the K-linear span of the matrices $I = A^0, A^1, A^2, \ldots$. Show that A is diagonalizable if and only if $K[A]$ has no non-zero nilpotents.

Since K is algebraically closed, we can put A in Jordan form, with $PAP^{-1} = J$. Note that any element of $K[A]$ has the form $f(A)$ for a polynomial $f(x) \in K[X]$. Note that $f(PAP^{-1}) = Pf(A)P^{-1}$, and so $f(PAP^{-1})$ is nilpotent if and only if $f(A)$ is nilpotent. Thus without loss of generality, we assume that A is in Jordan form.

Given an eigenvalue λ, its multiplicity in the minimal polynomial is the size of its largest Jordan block.

Let m be the minimal polynomial of A. Then $m(x) = \prod_i (x-\lambda_i)^{r_i}$, where λ_i are distinct eigenvalues of A, and r_i is the size of the largest Jordan block associated to λ_i.

For each λ_i, the geometric and algebraic multiplicities are equal if and only if each Jordan block associated to λ_i have size 1, i.e. if and only if $r_i = 1$.

Given an eigenvalue λ, its geometric multiplicity is the dimension of $\ker(A-\lambda I)$, and is the number of Jordan blocks corresponding to λ.

Thus we can conclude that A is diagonalizable if and only if $r_i = 1$ for each λ_i, i.e. if and only if the minimal polynomial has distinct factors.

A is diagonalizable if and only if, for every eigenvalue λ of A, its geometric and algebraic multiplicities coincide.

Define $f(x) = \prod_i (x-\lambda_i)$, i.e. $f(x)$ has each λ_i as a root with multiplicity 1. Certainly $f(x) \mid m(x)$. If $r = \max_i \{r_i\}$, then $m(x) \mid f(x)^r$.

If $r > 1$, then $f(A) \neq 0$, but $f(A)^r = 0$, so $f(A)$ is nilpotent.

Conversely, any nilpotent $g(A) \in K[A]$ with $g(A)^n = 0$ must have $m(x) \mid g(x)^n$, so $f(x) \mid g(x)$. If $r = 1$, then $f(x) = m(x)$, and so $g(A) = 0$.