2014 Algebra SEP $\sim$ Rings and Modules Day 1 Problems, by E. Dummit

1. (Aug-13.3) Let I and J be ideals in a commutative ring R.

   (a) Show that if $I + J = R$, then $I \cap J = IJ$.

   (b) Suppose that I and J are ideals in $\mathbb{C}[x]$, and suppose that $I + J = (x)$. Show that $(I \cap J)/IJ$ is 1-dimensional as a complex vector space, and moreover that it is isomorphic to $\mathbb{C}[x]/(x)$ as a $\mathbb{C}[x]$-module.

   (c) On the other hand, for general commutative rings R, once $R/(I+J)$ is not trivial, the difference between $I \cap J$ and IJ can be large even if $R/(I \cap J)$ is small. Demonstrate this by showing that, if $R = \mathbb{C}[x, y]$, there exist ideals I and J in R such that $I + J = (x, y)$ and the dimension of $(I \cap J)/IJ$ as a $\mathbb{C}$-vector space is at least 100.

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2. (Jan-04.2) Let K be a field and R be the subring of $K[x]$ of all polynomials with zero x-coefficient.

   (a) Show that x^2 and x^3 are irreducible but not prime in R.

   (b) Show that R is Noetherian.

   (c) Show that the ideal of all polynomials of R with zero constant term is not principal.

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3. (Jan-13.3) A ring R is "von Neumann regular" if for every $a \in R$ there exists an $x \in R$ with $a = axa$. (The element x is called a weak inverse of a.) In particular, observe that every division ring is von Neumann regular: take $x = 0$ for $a = 0$ and $x = a^{-1}$ otherwise.

   (a) Give an example of a commutative von Neumann regular ring which is not a field.

   (b) Let $R = M_2(\mathbb{C})$ and $a = e_{12}$, the nilpotent matrix which sends $(e_1, e_2) \mapsto (0, e_1)$. Find a weak inverse for a.

   (c) Show that if V is a vector space over a field k, the ring of endomorphisms $\mathrm{End}_k V$ is von Neumann regular.

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4. (Aug-05.2) Let R be a ring with 1, V a Noetherian right R-module, and $\theta : V \to V$ a homomorphism.

   (a) Show that $\ker(\theta^{n+1}) = \ker(\theta^n)$ for some $n \geq 1$.

   (b) If θ is onto, prove that it is one-to-one.

   (c) If V has a unique maximal submodule M, and it is true that if $X \subseteq Y$ are any submodules with $Y/X \cong V/M$ then $Y = V$, prove that θ is either 0 or an isomorphism.

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5. (Aug-03.2) Let R be a commutative integral domain (with 1).

   (a) If K is the field of fractions of R and $t \in R$ is such that $K = R[1/t]$, show that t is contained in every nonzero prime ideal of R.

   (b) Let $R = F[x_1, \cdots, x_n]$ for a field F. If $f(x_1, \cdots, x_n)$ is contained in every nonzero prime ideal of R, show that $f(a_1, \cdots, a_n) = 0$ for all $a_1, \cdots, a_n \in F$.

   (c) Suppose $f(x_1, \cdots, x_n)$ is a polynomial with coefficients in F, where F is an infinite field. If $f(a_1, \cdots, a_n) = 0$ for all $a_1, \cdots, a_n \in F$, show that f is the zero polynomial.

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6. (Jan-09.2) Let S be the subring of $\mathbb{C}[x]$ consisting of all polynomials with real constant term.

   (a) Show that the ideal of S consisting of all polynomials with 0 constant term is not principal.

   (b) Let I be a nonzero ideal of S and choose $f \in I\backslash\{0\}$ to have minimal possible degree n. If $g \in I$, show that there exists $s \in S$ with $g - sf$ either equal to 0 or to a polynomial of degree n.

   (c) Conclude that any ideal I of S is either principal or generated by two elements of the same degree.

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7. (Aug-01.2) Let R be a commutative ring with 1 and M a maximal ideal of R.

   (a) Show that R/M^2 has no idempotents other than 0 and 1.

   (b) If R is Noetherian, show that M/M^2 is a finitely-generated R/M-module.

   (c) If $R = K[x_1, \cdots, x_t]$ where K is a field, show that $\dim_K(R/M^2) < \infty$.

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8. (Aug-00.5) Let R be a ring with 1 and Z be its center. A derivation $D : R \to R$ is an additive map such that $D(ab) = aD(b) + D(a)b$.

   (a) If $r \in R$, show that the map $A_r : R \to R$ given by $A_r(A) = ar - ra$ for all $a \in R$, is a derivation.

   (b) If D is a derivation, show that $D(Z) \subseteq Z$.

   (c) If D is a derivation of R and $e \in Z$ is an idempotent, show that $D(e) = 0$.

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9. (Aug-12.2) Let F be a field, $R = F[x, y]$, and $I = (x)$.

   (a) Prove that I/I^2 is infinite-dimensional as an F-vector space.

   (b) Let $S \subset R$ be the subring $S = F + I$, so that I is also an ideal of S. Show that I is not finitely-generated as an ideal of S.

   (c) Let M be a maximal ideal of R and $\theta : R \to R/M$ be the projection map. Then $\theta(S)$ is a ring with $\theta(F) \subseteq \theta(S) \subseteq \theta(R)$. Discuss the nature of the extension $\theta(F) \subseteq \theta(R)$, prove that $\theta(S)$ is a field, and conclude that $M \cap S$ is a maximal ideal of S.
