2014 Algebra SEP $\sim$ Groups Day 1 Problems, by E. Dummit

1. (Jan-12.1) Let $|G| = 4312 = 2^3 7^2 11$.
   (a) Show that G has a subgroup of order 77.
   (b) Show that G has a subgroup of order 7 whose normalizer in G has index dividing 8.
   (c) Conclude that G is not simple.

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2. (Jan-89.5) Let G be a nonabelian finite simple group of order divisible by p. If G has no more than $2p$ Sylow p-subgroups, determine the number of elements of G whose order is a power of p, in terms of p.

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3. (Jan-14.3) Let G be a finite group.
   (a) If H is a proper subgroup of G, show that there is some element $x \in G$ which is not contained in any subgroup conjugate to H.
   (b) Use part (a) to show that if all maximal subgroups of G are conjugate, then G is cyclic.

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4. (Jan-01.1): Let X and Y be distinct subgroups of a finite group G. We say X and Y are a "weird pair" if $|X| = |Y|$ and no other subgroups of G have this same order.
   (a) If G has a weird pair of subgroups, show that some subgroup of G has a weird pair of normal subgroups.
   (b) If $G = A \times B$ is a direct product of solvable groups, show that $A \times 1$ and $1 \times B$ cannot be a weird pair.
   (c) Show that a solvable group cannot contain a weird pair of subgroups.

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5. (Jan-11.1) Let $G = H \times K$. Suppose there exists a group X with surjective homomorphisms $\theta : H \to X$ and $\phi : K \to X$, and define $U = \{hk \in G : h \in H, k \in K, \theta(h) = \phi(k)\}$.
   (a) Show that U is a subgroup of G with $UH = G = UK$, $U \cap H = \ker(\theta)$, and $U \cap K = \ker(\phi)$.
   (b) If V is a subgroup of G with $V \supseteq U$, show that $V \cap H$ and $V \cap K$ are normal in G.
   (c) If X is simple, show that U is a maximal subgroup of G containing neither H nor K.

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6. (Jan-10.1)
   (a) Find the number of elements of order 7 in S_7 and the order of the centralizer in S_7 of one of these elements.
   (b) Find the order of the normalizer of a 7-Sylow subgroup in A_7.
   (c) Show that S_7 does not contain a simple subgroup of order $504 = 2^3 3^2 7$.

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7. (Jan-09.1) Let G be a group of order $p(p+1)$ where p is an odd prime, and assume that G does not have a normal p-Sylow subgroup.
   (a) Find the number of elements of order different from p in G.
   (b) Show that each nonidentity conjugacy class of elements of order different from p has size at least p, and conclude there is precisely one such conjugacy class.
   (c) Show that $p+1$ is a power of 2.
