2014 Algebra SEP $\sim$ Groups Day 1 Solutions, by E. Dummit

1. (Jan-12.1) Let $|G| = 4312 = 2^3 7^2 11$.

   (a) Show that G has a subgroup of order 77.

   (b) Show that G has a subgroup of order 7 whose normalizer in G has index dividing 8.

   (c) Conclude that G is not simple.

   Solution: By Sylow we have $n_7 \in \{1, 8, 22\}$ and $n_{11} \in \{1, 56\}$.

   a) If $n_{11} = 1$ then taking the product of a 7-Sylow subgroup with an 11-Sylow subgroup gives a subgroup of order $7^2 11$, and any group of this order is abelian (it must have unique 7-Sylow and 11-Sylow subgroups, each of which is abelian), so it has a subgroup of order 77. If $n_{11} = 56$, then the normalizer of an 11-Sylow has order 77.

   b) Let H be the subgroup of order 77 from part (a). Since groups of order 77 are abelian (indeed, cyclic), we see that its Sylow 7-subgroup H_7 is centralized (hence in particular, normalized) by H. Furthermore, since groups of order 49 are abelian, any Sylow 7-subgroup containing H_7 centralizes (hence normalizes) H_7. So we see that the normalizer $N_G(H_7)$ contains a subgroup of order 77 and 49, so $|N_G(H_7)|$ is divisible by $7^2 11$ – meaning its index divides $|G| / (7^2 11) = 8$.

   c) If G were simple then the permutation representation of G on the cosets of the subgroup of index 8 from part (b) would give an injective homomorphism $\phi : G \to S_8$. But this is impossible, since S_8 contains no element of order 11, and G does.

---

2. (Jan-89.5) Let G be a nonabelian finite simple group of order divisible by p. If G has no more than $2p$ Sylow p-subgroups, determine the number of elements of G whose order is a power of p, in terms of p.

   Solution: We have $n_p = 1 \bmod p$, it cannot be 1, and it is $\leq 2p$, so it must be $p+1$. Hence the normalizer H of a p-Sylow subgroup has index $p+1$, so we get an injective homomorphism $\phi : G \to S_{p+1}$ from the action of G by left-multiplication on the cosets of H. So we see that the p-Sylow subgroups of G must have order p, since $|S_{p+1}| = (p+1)!$ is divisible by p but not p^2. Hence any two distinct p-Sylow subgroups of G intersect in the identity, so since there are $p+1$ of them each having $p-1$ elements of order p, there are a total of $(p+1)(p-1) + 1 = p^2$ elements of order a power of p.

   Remark An example of such a group is A_5, which has 6 Sylow 5-subgroups. And indeed, it contains 24 elements of order 5 (the 5-cycles), along with the identity.

---

3. (Jan-14.3) Let G be a finite group.

   (a) If H is a proper subgroup of G, show that there is some element $x \in G$ which is not contained in any subgroup conjugate to H.

   (b) Use part (a) to show that if all maximal subgroups of G are conjugate, then G is cyclic.

   Solution:

   a) If H is trivial there is nothing to do (take any element in G that is not the identity). Otherwise, consider the conjugation action of elements of G, on H. All elements of H stabilize H, so by the orbit-stabilizer lemma, there are at most $[G : H]$ conjugates of H in G. All of them contain the identity, so there are at most $(|H| - 1) \cdot [G : H] + 1 = |G| - |H| + 1$ elements in the union of these conjugates. Since $|H| > 1$ we see that some elements of G must therefore be missing.

   b) Let H be a maximal subgroup of G. By part (a), there exists some $x \in G$ which is not contained in any conjugate of H – thus, x is not contained in any maximal subgroup. But then $\langle x \rangle = G$, so G is cyclic. (In fact, since G is therefore abelian, we see that G must have prime-power order, as otherwise it would have two subgroups of differing prime index.)

---

4. (Jan-01.1): Let X and Y be distinct subgroups of a finite group G. We say X and Y are a "weird pair" if $|X| = |Y|$ and no other subgroups of G have this same order.

   (a) If G has a weird pair of subgroups, show that some subgroup of G has a weird pair of normal subgroups.

   (b) If $G = A \times B$ is a direct product of solvable groups, show that $A \times 1$ and $1 \times B$ cannot be a weird pair.

   (c) Show that a solvable group cannot contain a weird pair of subgroups.

Solution:

a) Clearly any subgroup of G containing both X and Y has a weird pair (namely, X and Y), so we need only find one in which X and Y are normal, and if a subgroup has X and Y normal then any subgroup containing them both will also have X and Y normal: so, if the result is to hold, it must hold where H is equal to the subgroup generated by X and Y.
We claim that X and Y are indeed normal in H, if X and Y are a weird pair: clearly X normalizes itself, and then if $y \in Y$, we see that yXy^{-1} is either X or Y (they are the only subgroups of that size), but it cannot be Y because $yXy^{-1} = Y$ implies $y^{-1}Yy = X$, which is not possible since $y^{-1}Yy = Y$. Hence Y also normalizes X in H, so X is normal in H. Similarly, Y is normal in H.

b) If G is trivial the result obviously doesn't hold, so assume otherwise. If $A \times 1$ and $1 \times B$ are a weird pair, then clearly $|A| = |B|$; we want to find another subgroup of G of the same order. If $|A| = |B| = p$, take the diagonal subgroup generated by $(1, 1)$. If $|A| = p^d$ is a prime power with $d > 1$, then since p-groups have subgroups of any order, take P of order p^{d-1} in A and H of order p in B; then $|P \times H| = |A|$. Now assume $|A|$ is not a prime power. Hall's theorem on solvable groups says a group is solvable if and only if for each n dividing $|G|$ with $\gcd(n, |G|/n) = 1$, G has a subgroup of order n. Let p^d be the order of a p-Sylow subgroup of A: then the theorem implies that B has a subgroup H of order $n = |A|/p^d$. Then we clearly have $|P \times H| = |A|$, where P and H are both proper. (Note that this proof still works even if only one of A and B is solvable.)

b-alt) As above we need only deal with the case where $|A|$ is not a prime power. Since A is solvable, it has a subgroup A_1 for which A/A_1 is nontrivial and abelian. If p is any prime dividing $[A : A_1]$ then since A/A_1 has a subgroup of index p (abelian groups have subgroups of every order), we can lift to get a subgroup A' of index p in A. By Cauchy's theorem, B has an element b of order p; then $A' \times \langle b \rangle$ has order equal to $|A|$ and is equal to neither A nor B.

c) Suppose otherwise and let G be a solvable group with a weird pair of subgroups of minimal order. By part (a) since subgroups of solvable groups are solvable, it must be the case that G is generated by the normal subgroups X and Y, so that $G = XY$. Now $X \cap Y$ is normal in G and $|X : X \cap Y| = |Y : X \cap Y|$, so $G/(X \cap Y)$ is also solvable and has a weird pair of subgroups, so it must be the case that $X \cap Y = 1$. But now we have $G = X \times Y$, whereupon we get a contradiction from part (b).

Remark Given part (c), it is not easy to see that there are any groups at all with a "weird pair" of subgroups! By the arguments above, we see that a minimal such example must be of the form $A \times B$ where $|A| = |B|$ and, for each n dividing $|A|$, there exists no subgroup of B of index n. These conditions are already rather restrictive: for example, by Sylow, we see that neither A nor B can contain any subgroups of prime-power index.

---

5. (Jan-11.1) Let $G = H \times K$. Suppose there exists a group X with surjective homomorphisms $\theta : H \to X$ and $\phi : K \to X$, and define $U = \{hk \in G \,:\, h \in H,\, k \in K,\, \theta(h) = \phi(k)\}$.

   (a) Show that U is a subgroup of G with $UH = G = UK$, $U \cap H = \ker(\theta)$, and $U \cap K = \ker(\phi)$.

   (b) If V is a subgroup of G with $V \supseteq U$, show that $V \cap H$ and $V \cap K$ are normal in G.

   (c) If X is simple, show that U is a maximal subgroup of G containing neither H nor K.

Solution:

a First, U is a subgroup since $1 \in U$ and if (h_1, k_1) and $(h_2, k_2) \in U$ then $\theta(h_1 h_2^{-1}) = \theta(h_1)\theta(h_2^{-1}) = \phi(k_1)\phi(k_2^{-1}) = \phi(k_1 k_2^{-1})$. Then for any $g = (h, k) \in G$, if k_1 is any element of k with $\phi(k_1) = \theta(h)$ (which exists since ϕ is surjective) we can write $g = (h, k_1) \cdot (1, k_1^{-1}k) \in UK$. Hence $G = UK$, and by the same argument we see $G = UH$. Furthermore, if $g = (h, 1) \in U \cap H$, then we require $\theta(g) = \phi(1)$ so $g \in \ker(\theta)$, and conversely if $(h, 1) \in \ker(\theta)$ then clearly $\theta(h) = \phi(1)$ so $(h, 1) \in U$, and clearly also $(h, 1) \in H$ so $(h, 1) \in U \cap H$. Symmetrically, we see $U \cap K = \ker(\phi)$.

b) By (a) we know that $VH = G$ so we need only show that $V \cap H$ is normalized by V and H, which (since $G = H \times K$) is equivalent to showing merely that H normalizes $V \cap H$, since everything in the K-component can be ignored. Now let $(h_1, 1) \in V \cap H$ and $(h_2, 1) \in H$: then since V contains U and θ is surjective there exists $(1, k) \in K$ with $\theta(h_2) = \phi(k)$: then $(h_2^{-1}, k^{-1}) \cdot (h_1, 1) \cdot (h_2, k)$ is in V since $(h_2, k) \in U$, and multiplying out shows it is $(h_2^{-1} h_1 h_2, 1)$ so we conclude $(h_2^{-1} h_1 h_2, 1) \in V \cap H$.

c) First observe that U cannot contain all of H (or K), since by (a) $U \cap H = \ker(\theta)$ (or $U \cap K = \ker(\phi)$), so θ (or ϕ) would have to be the zero map hence X would have to be the trivial group; in particular, we see that U cannot be G. Now suppose $V \supset U$ be a proper subgroup of G properly containing U. Then since $UH = VH = G$ by (a), we see that H is not contained in V since V is not G. By part (b) we see that $V \cap H$ is normal in G hence normal in H, and so since $U \cap H = \ker(\theta)$ by part (a), and $V \cap H$ is properly contained in H, we see that $\theta(V \cap H)$ is a nontrivial proper normal subgroup of X, which is a contradiction. Hence we conclude that no such V can exist, so U is maximal.

---

6. (Jan-10.1)

   (a) Find the number of elements of order 7 in S_7 and the order of the centralizer in S_7 of one of these elements.

   (b) Find the order of the normalizer of a 7-Sylow subgroup in A_7.

   (c) Show that S_7 does not contain a simple subgroup of order $504 = 2^3 3^2 7$.

Solution:

a) An element has order 7 in S_7 iff it is a 7-cycle, which can be written uniquely in the form $(1\,b\,c\,d\,e\,f\,g)$. There are 6! ways to order the numbers 2-7, so there are 6! 7-cycles. If we consider the conjugation action of G on the set of 7-cycles, since the action of G is transitive (since permutations are conjugate iff they have the same cycle type), the orbit-stabilizer lemma says that the size of the centralizer of any 7-cycle is $|G|/6! = 7$: thus the centralizer is simply the powers of the 7-cycle.

b) The number of 7-Sylow subgroups is $6!/6 = 5!$ since each of them is cyclic hence contains 6 nonidentity elements. Then the normalizer has order $(7!/2)/5! = 21$.

Remark In fact it is not hard to work out precisely what the normalizer is: for $H = \langle(1\,2\,3\,4\,5\,6\,7)\rangle$, the normalizer is generated by $x = (1\,2\,3\,4\,5\,6\,7)$ and $y = (2\,3\,5)(4\,7\,6)$. (It is easy to see that $yxy^{-1} = (1\,3\,5\,7\,2\,4\,6) = x^2$.)

c) Suppose otherwise: this subgroup G must then be inside A_7 (else $A_7 \cap G$ would be index-2 hence normal in G). Now we have $n_7 \in \{8, 36\}$. If $n_7 = 8$ then the normalizer of a 7-Sylow subgroup of G would have order $3^2 7$, but this is impossible because the order of the normalizer of any 7-Sylow subgroup of A_7 is only 21 by part (b). If $n_7 = 36$ then the normalizer of a 7-Sylow subgroup of G would have order 14, but this is also impossible because, by part (b) again, no element of order 2 normalizes a 7-cycle.

---

7. (Jan-09.1) Let G be a group of order $p(p+1)$ where p is an odd prime, and assume that G does not have a normal p-Sylow subgroup.

   (a) Find the number of elements of order different from p in G.

   (b) Show that each nonidentity conjugacy class of elements of order different from p has size at least p, and conclude there is precisely one such conjugacy class.

   (c) Show that $p+1$ is a power of 2.

Solution:

a) We have $n_p = p+1$, and each p-Sylow subgroup contains $p-1$ elements of order p. Since they are cyclic, they don't share any nonidentity elements, so there are $(p+1)(p-1)$ elements of order p and hence $p+1$ elements of order not equal to p.

b) The normalizer of a p-Sylow subgroup has index $p+1$ hence order p, so every p-Sylow subgroup is self-normalizing hence self-centralizing. Thus, any element of order p does not commute with any element of order not 1 or p. Now consider any nonidentity conjugacy class and let g be an element of order p that acts on it by conjugation. If the size of the conjugacy class were $k < p$, then g^k would act as the identity on it. But this is impossible since g^k also has order p, so it cannot commute with any elements not of the form g^n. Finally, since there are only p elements to fit into conjugacy classes of size at least p, there is only 1 conjugacy class.

c) From part (b), the p elements not of order 1 or p are conjugate (by an element of order p), hence they all have the same order, which must therefore be some prime q. But now there are $k(q-1)$ nonidentity elements (k cyclic subgroups of order q), so $p = k(q-1)$, whence $q = 2$. Then by Cauchy's theorem, the only prime divisors of $p(p+1)$ are 2 and p, so we conclude $p+1$ must be a power of 2. Alternatively, one could use Cauchy's theorem to see that $p+1$ must be a power of q, and then observe that $q^k - 1$ is never prime unless $q = 2$.
