2014 Algebra SEP $\sim$ Groups Day 2 Problems, by E. Dummit

1. (Aug-13.1): Let p be prime and V a 3-dimensional vector space over $\mathbb{Z}/p$.
   (a) Construct an explicit bijection between ordered bases of V and elements of $GL_3(\mathbb{Z}/p)$, and find the order of $GL_3(\mathbb{Z}/p)$.
   (b) Show that the kernel of the natural homomorphism from $GL_3(\mathbb{Z}/p^2)$ to $GL_3(\mathbb{Z}/p)$ is abelian of exponent p, and find the order of $GL_3(\mathbb{Z}/p^2)$.

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2. (Jan-97.3) Let $f(x) \in \mathbb{Q}[x]$ be a polynomial of degree 5 that is not solvable by radicals and let S be its splitting field over $\mathbb{Q}$.
   (a) Show that there exists at most one subfield E of S such that $[E : \mathbb{Q}] = 2$.
   (b) If $\alpha, \beta \in S$ are irrational and satisfy $\alpha^2 \in \mathbb{Q}$ and $\beta^2 \in \mathbb{Q}$, show that $\alpha\beta \in \mathbb{Q}$.

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3. (Jan-13.1): A finite group is said to have property C if, whenever $g \in G$ and n is relatively prime to the order of g, g and g^n are conjugate in G.
   (a) Give infinitely many nonisomorphic finite groups which have property C.
   (b) Give infinitely many nonisomorphic finite groups which do not have property C.
   (c) Show that if G has property C and $\rho : G \to GL_m(\mathbb{C})$ is a homomorphism, then the trace of $\rho(g)$ lies in $\mathbb{Q}$ for every $g \in G$.

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4. (Aug-92.5): Let G be the group of 2×2 integer matrices with determinant 1, and let G act by right multiplication on the set Ω of all 1-dimensional subspaces of $\mathbb{Q}^2$.
   (a) Find all elements of G which fix every element of Ω.
   (b) Prove that G acts transitively on Ω.

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5. (Aug-05.5): Let $G = GL_n(\mathbb{Z})$, fix a prime p, and let S be the subset of G of matrices of the form $I + pX$.
   (a) Prove that S is a subgroup of G.
   (b) Suppose $M \in S$ has prime order q. Show that $q = p$.
   (c) If $p > 2$, show that no element of S has order p. Conclude S has no nonidentity element of finite order.

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6. (Jan-91.5) Let G be a nontrivial group whose subgroups are totally ordered by inclusion: thus if H, K are subgroups, then either $H \subseteq K$ or $K \subseteq H$.
   (a) Show that G is abelian and that the orders of the elements of G are all powers of the same prime p.
   (b) If $G_n = \{g \in G : g^{p^n} = 1\}$, show that $|G_n| \leq p^n$.

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7. (Aug-11.5): Let A be an additive abelian group.
   (a) If A is free abelian, show that A contains no nonzero divisible element.
   (b) Now let $A = \prod \mathbb{Z}$ be the countably infinite direct product of copies of $\mathbb{Z}$, and B be the subgroup given by the direct sum (i.e., with all but finitely many coordinates equal to 0). Prove that A/B contains a nonzero divisible element and conclude that A/B is not free abelian.

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8. (Aug-09.5): Let A be a multiplicative abelian group.
   (a) If A is divisible, show that any homomorphic image $\bar{A}$ of A is divisible.
   (b) If A is a finite divisible group, prove that $A = 1$.
   (c) Suppose A is divisible and $A \subseteq B$. If $A \cap X > 1$ for all nonidentity subgroups X of B, show that $A = B$.
