# 4 HOUR PRACTICE FROM ED DUMMIT'S SEP

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## 1. Groups

1. (Jan-12.1) Let $|G| = 4312 = 2^3 7^2 11$.

   (a) Show that $G$ has a subgroup of order 77.

   (b) Show that $G$ has a subgroup of order 7 whose normalizer in $G$ has index dividing 8.

   (c) Conclude that $G$ is not simple.

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2. (Jan-89.5) Let $G$ be a nonabelian finite simple group of order divisible by $p$. If $G$ has no more than $2p$ Sylow $p$-subgroups, determine the number of elements of $G$ whose order is a power of $p$, in terms of $p$.

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3. (Jan-14.3) Let $G$ be a finite group.

   (a) If $H$ is a proper subgroup of $G$, show that there is some element $x \in G$ which is not contained in any subgroup conjugate to $H$.

   (b) Use part (a) to show that if all maximal subgroups of $G$ are conjugate, then $G$ is cyclic.

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## 2. Rings

1. (Aug-13.3) Let $I$ and $J$ be ideals in a commutative ring $R$.

   (a) Show that if $I + J = R$, then $I \cap J = IJ$.

   (b) Suppose that $I$ and $J$ are ideals in $\mathbb{C}[x]$, and suppose that $I + J = (x)$. Show that $(I \cap J)/IJ$ is 1-dimensional as a complex vector space, and moreover that it is isomorphic to $\mathbb{C}[x]/(x)$ as a $\mathbb{C}[x]$-module.

   (c) On the other hand, for general commutative rings $R$, once $R/(I + J)$ is not trivial, the difference between $I \cap J$ and $IJ$ can be large even if $R/(I \cap J)$ is small. Demonstrate this by showing that, if $R = \mathbb{C}[x, y]$, there exist ideals $I$ and $J$ in $R$ such that $I + J = (x, y)$ and the dimension of $(I \cap J)/IJ$ as a $\mathbb{C}$-vector space is at least 100.

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2. (Jan-04.2) Let $K$ be a field and $R$ be the subring of $K[x]$ of all polynomials with zero $x$-coefficient.

   (a) Show that $x^2$ and $x^3$ are irreducible but not prime in $R$.

   (b) Show that $R$ is Noetherian.

   (c) Show that the ideal of all polynomials of $R$ with zero constant term is not principal.

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## 3. Solutions

For groups, see Prob 1 to Prob 3 for the Groups Day 1 handout:

- https://github.com/coltongrainger/fy19alg1/raw/master/2014_algebra_sep_notes_groups_day_1_solutions.pdf

For rings, see Prob 1 to Prob 2 for the Rings Day 1 handout:

- https://github.com/coltongrainger/fy19alg1/raw/master/2014_algebra_sep_notes_rings_day_1_solutions.pdf