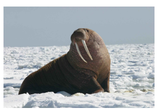
4 HOUR PRACTICE FROM MICKY'S SEP NOTES: GROUP THEORY

Attempt all 6 problems. From Micky Steinberg's notes on the 2014 Wisconsin Algebra SEP.

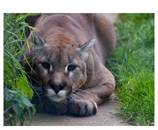
Solutions from Micky, *in color*: `https://www.math.wisc.edu/~micky/grouptheory.pdf`

2. Let G be a finite group, and write $|G| = p^a m$, where p is prime and m is relatively prime to p. Prove that for every $0 \leq b \leq a$, G has a subgroup of order p^b. Further prove that if P_b is a subgroup of order p^b, then there is some subgroup P_{b+1} of order p^{b+1} such that $P_b \lhd P_{b+1}$.



3. (August 2014 Problem 2) Let G be a finite group, and let A be a subgroup of $Aut(G)$.
   (a) Suppose G is the cyclic group $\mathbb{Z}/6\mathbb{Z}$ and A is the full automorphism group. What are the orbits of the action of A on G?
   (b) Let G be a non-trivial finite group. Show that two elements in the orbit of A on G must have the same order.
   (c) Show that for any non-trivial finite group G there are always at least two orbits of A on G. Prove that there are exactly two orbits for some A if and only if G is an elementary abelian p-group for some prime p.

4. (January 2015 Problem 1) This problem concerns expressing groups as unions of proper subgroups.
   (a) Show that no group is the union of two proper subgroups.
   (b) Show that $\mathbb{Z}$ is not the union of any number of proper subgroups.
   (c) For which n is $\mathbb{Z}^n$ the union of finitely many proper subgroups? What is the minimal number of such subgroups as a function of n?



5. (August 1996 Problem 1) We say that a group G has property (*) if every normal abelian subgroup of G is contained in its center.
   (a) Suppose that N and M are normal subgroups of a group G and that G/N and G/M have property (*). Prove that $G/(N \cap M)$ has property (*).
   (b) Let $N \lhd G$ and assume that G/N has property (*). If N has no non-trivial abelian normal subgroups, prove that G has property (*).
   (c) Show that a finite p-group with property (*) must be abelian.

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1. (January 2013 Problem 1) A finite group G is said to have property C if, whenever $g \in G$ and n is an integer relatively prime to the order of G, g and g^n are conjugate in G.

   (a) Give infinitely many non-isomorphic finite groups which have property C.

(b) Give infinitely many non-isomorphic finite groups which do not have property C.

3. (January 1991 Problem 5) Let G be a possibly infinite, non-trivial group whose subgroups are linearly ordered by inclusion. In other words, if H and K are subgroups of G, then either $H \subseteq K$ or $K \subseteq H$.

   (a) Prove that G is an abelian group, and that the orders of the elements of G are all powers of the same prime p.