TODO AND TO-REVISE

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0.1. **Minimal and maximal subgroups.** Suppose N is a nontrivial abelian subgroup of G, minimal with the property that it is normal in G. Let H be a proper subgroup of G such that $NH = G$. The intersection of N with H is trivial and H is a maximal subgroup of G.

0.2. **[1, No. 3.1.40].** Let G be a group, let N be a normal subgroup of G and let $\overline{G} = G/N$. The elements $\overline{x}$ and $\overline{y}$ commute in $\overline{G}$ if and only if $x^{-1}y^{-1}xy \in N$.

0.3. **[1, No. 3.4.7].** If G is a finite group and $H \lhd G$, there is a composition series of G one of whose terms in H.

0.4. **[1, No. 3.4.11].** If H is a nontrivial normal subgroup of the solvable group G, there is a nontrivial subgroup A of H with $A \lhd G$ and A abelian.

0.5. **[1, No. 3.5.10].** We find a composition series for A_4, and argue that A_4 is not solvable.

0.6. **[1, No. 4.1.9].** Assume G acts transitively on the finite set A and let H be a normal subgroup of G. Let $\mathscr{O}_1, \mathscr{O}_2, \dots, \mathscr{O}_r$ be the distinct orbits of H on A.

(a) G permutes the sets $\mathscr{O}_1, \mathscr{O}_2, \dots, \mathscr{O}_r$ in the sense that for each $g \in G$ and each $i \in \{1, \dots, r\}$ there is a j such that $g\mathscr{O}_i = \mathscr{O}_j$, where $g\mathscr{O} = \{g(a) : a \in \mathscr{O}\}$. Then G is transitive on $\{\mathscr{O}_1, \dots, \mathscr{O}_r\}$. Furthermore, all orbits of H on A have the same cardinality.

(b) If $a \in \mathscr{O}_1$, then $|\mathscr{O}_1| = |H : H \cap \mathrm{Stab}_G(a)|$. Furthermore, $r = |G : H\mathrm{Stab}_G(a)|$.

0.7. **[1, No. 4.4.20].** For any finite group P, let $d(P)$ be the minimum[1] number of generators of P. Let $m(P)$ be the maximum of the integers $d(A)$ as A runs[2] over all *abelian* subgroups of P. Define the *Thompson subgroup* of P as

$$J(P) = \langle A : A \text{ is an abelian subgroup of } P \text{ with } d(A) = m(P)\rangle.$$

(a) $J(P)$ is a characteristic subgroup of P.

(b) For each of the following groups P, we exhaustively list all abelian subgroups A of P that satisfy $d(A) = m(P)$: Q_8, D_8, D_{16}, QD_{16} (the quasidihedral group of order 16).

0.8. **[1, No. 6.2.25].** Let G be a simple group of order p^2qr where all p, q, r are prime. Then $|G| = 60$.

Proof sketch. By Feit-Thomposon, G must be of even order. Suppose that p is not 2. Then by "Erik's lemma", if G is a group of order $2k$ where k is odd, then G has a normal subgroup. Considering that p^2qr could be written as $2k$ with k odd if $p \neq 2$, we must have $p = 2$.

Without loss of generality, assume $q < r$. We can thus bound $n_r \in \{2q, 4q\}$. We want to show $n_r = 2q$. If we *could do so*, then we'd be able to consider $P \in \mathrm{Syl}_2(G)$. From here, we *could* argue that $p^2 \equiv 1 \pmod q$. Thence we'd find $q \mid (p-1)$ or $q \mid (p+1)$. Lastly, we'd observe $q = 2+1$. Moreover, if we could limit n_r to be $2q$, then we'd be forced by congruence, namely $rn + 1 = 2q$, to accept that $r = 5$. □

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[1]For example, $d(P) = 1$ if and only if P is a nontrivial cyclic group and $d(Q_8) = 2$.

[2]For example, $m(Q_8) = 1$ and $m(D_8) = 2$.

0.9. **[1, No. 5.5.23].** Let K and L be groups, let n be a positive integer, let $\rho\colon K \to S_n$ be a homomorphism and let H be the direct product of n copies of L. From [1, No. 5.1.8], we constructed an injective homomorphism ψ from S_n into $\operatorname{Aut}(H)$ by letting the elements of S_n permute the n factors of H. The compositions $\psi \circ \rho$ is a homomorphism from G into $\operatorname{Aut}(H)$. The *wreath product* of L by K is the semidirect product $H \rtimes_\psi K$ with respect to this homomorphism and is denoted by $L \wr K$. Note this wreath product depends on the choice of permutation representation ρ of K — if none is given explicitly, then φ is assumed to be the left regular representation of K.

(a) Assume K and L are finite groups and ρ is the left regular representation of K. We find $|L \wr K|$ in terms of $|K|$ and $|L|$.

(b) Let p be a prime, let $K = L = Z_p$. Suppose ρ is the left regular representation of K. Then $Z_p \wr Z_p$ is a non-abelian subgroup of order p^{p+1} and is isomorphic to a Sylow p-subgroup of S_{p^2}. [The p copies of Z_p whose direct products makes up H may be represented by p disjoint p-cycles; these are cyclically permuted by K.]

0.10. **[1, No. 6.1.20].** Let p be a prime, let P be a p-subgroup of the finite group G, let N be a normal subgroup of G whose order is relatively prime to p, and let $\bar{G} = G/N$.

(a) With Frattini's argument, $N_{\bar{G}}\left(\bar{P}\right) = \overline{N_G(P)}$.
(b) From above, $N_{\bar{G}}\left(\bar{P}\right) = \overline{N_G(P)}$.

0.11. **[1, No. 6.3.12].** Let S be a set and c a positive integer. Formulate the notion of a free nilpotent group on S of nilpotence class c and prove it has the appropriate universal property with respect to the nilpotent groups of class less than or equal to c.

Formulation. The free nilpotent group on S of nilpotence class c, denoted $N_c(S)$, ought to be given by the presentation $\langle S | \gamma_c(F(S))\rangle$ where $\gamma_c(F(S)) = [F(S), \gamma_{c-1}(S)]$. From the presentation, there's a surjection $\pi\colon F(S) \to N_c(S)$.

Universal property. Let G be a nilpotent group of class c. Let $\varphi\colon S \to G$ be a map of sets. Then there's a unique $\Psi\colon N_c(S) \to G$ such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "\pi \circ \iota"] \arrow[dr, "\varphi"'] & N_c(S) \arrow[d, dashed, "\exists! \Psi"] \\ & G \end{tikzcd}$$

Proof.[3] Observe $\Phi(\gamma_c(F(S))) \leqslant \gamma_c(G)$ as $\Phi([F(S), \gamma_{c-1}(F(S))]) = [\Phi(F(S)), \Phi(\gamma_{c-1}(F(S)))] \leqslant \gamma_c(G) = 1$. □

0.12. **[1, No. 6.3.14].** Prove that $G = \langle x, y : x^3 = y^3 = (xy)^3 = 1\rangle$ is an infinite group as follows. Let p be a prime congruent to $1 \mod 3$ and let G_p be the non-abelian group of order $3p$. Let $a, b \in G_p$ with $|a| = p$ and $|b| = 3$.

- Both ab and ab^2 have order 3.
- G_p is a homomorphic image of G.
- G is therefore an infinite group, as there are infinitely many primes $p \equiv 1 \mod 3$.

[1] D. S. Dummit and R. M. Foote, *Abstract algebra*, 3rd ed. Hardcover; Prentice Hall, 2004 [Online]. Available: http://www.worldcat.org/isbn/0471433349

[3] I consulted Erik, Hunter, Chris, and https://terrytao.wordpress.com/2009/12/21/the-free-nilpotent-group/ for this problem. The proof here is hardly sufficient, I'll admit—something to revise.