MATH 120 PRACTICE FINAL EXAM

There are 10 problems, on two pages. Each problem is worth 6 points. The problems are of widely varying difficulty, and the exam is intended to be challenging (some of the problems very much so), so do not be psyched out by this. The problems are not necessarily arranged in order of difficulty.

Please write neatly. Begin each problem on a new page, and write on one side of each page.

Good luck!

1. (a) For which primes p does $x^3 \equiv 1 \pmod{p}$ have a solution other than $x \equiv 1 \pmod{p}$?
(b) Give a composition series for S_4 (see problem 8 below for a reminder of the definition).
(c) Show that there does not exist a nonabelian group of order 15. Hint: show that every group of order 15 is a semidirect product.

2. Suppose n is a 17-digit positive integer whose last digit is 3. Show that the last digit of n^4 is 1. Show that the last two digits of n^{40} are 01. Show that the last three digits of n^{400} are 001.

3. For how many pairs of integers (x, y) is $x^2 + y^2 = 17^2 \times 73 \times 101$?

4. (a) Show that $F = (\mathbb{Z}/11)[x]/(x^2+1)$ is a field. How many elements does it have?
(b) $F^\times$ is a finite abelian group. Which one?

5. Suppose G is a group (possibly infinite), such that the index of the center $Z(G)$ in G is n (finite). Show that every conjugacy class of G contains at most n elements.

6. Show that there is no nonabelian simple group of order n, where $40 \leq n < 50$.

7. (a) Suppose N is a normal subgroup of G, and $S \subset G$ is a conjugacy class of G contained in N, and $x \in S$. Show that S is the union of $k = |G : NC_G(x)|$ conjugacy classes of equal size in N. Possible hint: the Second Isomorphism Theorem $AB/B \cong A/A \cap B$, with appropriate hypotheses.
(b) Show that a conjugacy class in S_n consisting of even prmutations is a single conjugacy class in A_n, or the union of two conjugacy classes in A_n.

8. Suppose G is a group, and

$$1 = M_0 \leq M_1 \leq \cdots \leq M_j = G$$

and

$$1 = N_0 \leq N_1 \leq \cdots \leq N_k = G$$

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are both *composition series* for G (M_i is normal in M_{i+1} and M_{i+1}/M_i is simple, and similarly for N_j).

(a) Show that $M_i \cap N_j$ is normal in $M_{i+1} \cap N_j$. Show that $M_{i+1} \cap N_j = M_i \cap N_j$ or $M_{i+1} \cap N_j / M_i \cap N_j$ is isomorphic to the simple group M_{i+1}/M_i.

(b) Use part (a) to prove the part of the Jordan-Holder Theorem not proved in class: that the quotients of the M_i's can be matched with the quotients of N_j's, but perhaps in a different order.

9. Recall that $\mathbb{F}_2 = \mathbb{Z}/2$ is a finite field with 2 elements. Let $G = GL(3, \mathbb{F}_2)$, the invertible 3×3 matrices with entries in the finite field of 2 elements. Recall from the midterm that $|G| = 168 = 2^3 \times 3 \times 7$: the first column vector $\vec{v}_1$ must be nonzero, so there are 7 choices; the second column vector $\vec{v}_2$ can be any nonmultiple of $\vec{v}_1$, giving 6 choices; and the third column vector $\vec{v}_3$ can be any vector not in the span of $\vec{v_1}$ and $\vec{v_2}$ (a two-dimensional subspace), yielding 4 choices.

In this problem, we show that G is a simple group of order 168 (the only nonabelian simple group of order < 360 other than A_5). Suppose N is a normal subgroup of G distinct from G and $\{e\}$. We will find a contradiction.

(a) G acts on the vectors vectors in $\mathbb{F}_2^3$, by matrix multiplication. Show the index of the stabilizer group H of a given nonzero vector is 7.

(b) Show that intersection of all the conjugates of H is $\{e\}$.

(c) Show that N is not contained in H. (HInt: otherwise, show that it is contained in all conjugates of H.)

(d) Show that $HN = G$.

(e) Show that $|N|$ is divisible by 7.

(f) Show that every 7-Sylow subgroup of G is contained in N.

(g) Show that there is more than one 7-Sylow of G. Possible hint: show that the matrices

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

both have order 7 and do not commute.

(h) Show that every 7-Sylow of G is also a 7-Sylow of N. Hence give the number of 7-Sylows of N, and show that $|N| = 56$.

(i) Show that N has no elements of order 3.

(j) Consider the map $\phi : S_3 \hookrightarrow G$ sending a permutation in S_3 to the corresponding "permutation matrix" (with one 1 in each row, and one 1 in each column). Show that the composition $S_3 \to G/N$ maps S_3 to the identity. Show that $\phi(S_3) \in N$. But $\phi((123))$ has order 3, yielding a contradiction.