2014 Algebra SEP $\sim$ Grab Bag problems, by E. Dummit

1. (Jan-97.4) Let K be a field.

   (a) If $\text{char}(K) \neq 2$, show that $GL_n(K)$ has exactly n conjugacy classes of elements of order 2.

   (b) If $\text{char}(K) = 2$, show that $GL_n(K)$ has exactly $\lfloor n/2 \rfloor$ conjugacy classes of elements of order 2.

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2. (Aug-08.5): Let R be a subring of $M_n(\mathbb{C})$ and suppose R is finitely generated as a $\mathbb{Z}$-module. Let $M \in R$.

   (a) Show that M is contained in a commutative subring S of $M_n(\mathbb{C})$ that is finitely generated as a $\mathbb{Z}$-module.

   (b) Deduce that there is a monic polynomial $f(x) \in \mathbb{Z}[x]$ such that $f(M) = 0$.

   (c) Prove that $\text{tr}(M)$ is an algebraic integer.

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3. (Aug-94.5) Let F be a field and $S = M_n(F)$.

   (a) If $s \in S$ is nilpotent, show that $\text{tr}(S) = 0$.

   (b) If R is a ring (not necessarily commutative) and $\theta : R \to S$ is a surjective ring homomorphism, let I be an ideal of R such that every element of I is a sum of nilpotent elements of R. Show that $\theta(I) = 0$.

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4. (Aug-99.5) Let F be a field, $f(x)$ and $g(y)$ be nonconstant polynomials in $R = F[x, y]$, and $I = (f(x), g(y))$, the ideal generated by f and g.

   (a) Show that $I \neq R$.

   (b) If $f(x) = x - \alpha$ and $g(y) = y - \beta$ for $\alpha, \beta \in F$, show that I is a maximal ideal.

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5. (Jan-92.5) Let $\alpha_1, \cdots, \alpha_n$ be the roots of the polynomial $f(x) = 2x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x]$.

   (a) Show that $2\alpha_i$ is an algebraic integer for $1 \leq i \leq n$.

   (b) Show that $\mathbb{Z}[\alpha_1, \cdots, \alpha_n] \cap \mathbb{Q} \subseteq \mathbb{Z}[1/2]$.

   (c) If some a_j with $0 \leq j \leq n-1$ is odd, show that $1/2 \in \mathbb{Z}[\alpha_1, \cdots, \alpha_n] \cap \mathbb{Q}$, and deduce that the latter intersection is $\mathbb{Z}[1/2]$. What happens if all a_j are even?

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6. (Jan-12.5): Let K be a field where -1 is not a square, and let $G = GL_2(K)$.

   (a) If $g \in G$, show that g has order 4 iff $\det(g) = 1$ and $\text{tr}(g) = 0$.

   (b) Find explicitly an element $g \in G$ of order 4.

   (c) Suppose there exist elements $a, b \in K$ with $a^2 + b^2 = -1$. Show that G contains two elements g, h of order 4 such that gh also has order 4.

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7. (Jan-96.5) Let q be a prime power and $f(x) = \dfrac{x^5 - 1}{x - 1} = x^4 + x^3 + x^2 + x + 1 \in \mathbb{F}_q[x]$.

   (a) If f has a root in $\mathbb{F}_q$, show that f splits completely over $\mathbb{F}_q$ and show that this happens precisely when $q \equiv 0, 1 \bmod 5$.

   (b) If $f(x)$ has an irreducible monic factor $g(x)$ of degree 2, show that g has constant term 1.

   (c) Factor $f(x)$ into quadratic factors when $q = 29$.

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8. (Jan-01.5) Let V be a finite-dimensional F-vector space and $T : V \to V$. Assume that no nonzero proper subspace of V is mapped into itself by T.

   (a) If $S \in F[T]$ is nonzero, show that $\{v \in V : Sv = 0\}$ is the zero subspace.

   (b) Prove that $F[T]$ is a field.

   (c) Show that $|F[T] : F| = \dim_F V$.

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9. (Jan-11.2) Let R be a commutative ring with 1, $(a) = aR$, and P a prime ideal properly contained in (a).

   (a) Show that $P = aP$.

   (b) If P is finitely generated, prove there exists $b \in R$ with $(1 - ab)P = 0$.

   (c) If R is a domain, conclude that either $P = 0$ or $(a) = R$.

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10. (Jan-07.5) Let A be an additive abelian group and B a subgroup. We say B is essential in A (B ess A) if $B \cap X \neq 0$ for every nontrivial subgroup of A.

    (a) If B_1 ess A_1 and B_2 ess A_2 show that $(B_1 \oplus B_2)$ ess $(A_1 \oplus A_2)$.

    (b) If B ess A and B has no nonzero elements of finite order, show A has no nonzero elements of finite order.

    (c) If $\mathbb{Q}$ ess A for some abelian group A, show that $A = \mathbb{Q}$.

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11. (Jan-08.4) Let V be a finite-dimensional vector space over F of characteristic p, $T : V \to V$, and $W = \{v \in V : Tv = v\}$. Further suppose $T^p = I$ and $\dim_F W = 1$.

    (a) Show that $(T - I)^p = 0$ and that $\dim_F V \leq p$.

    (b) If $\dim_F V < p$ show that $(T - I)^{p-1} = 0$.

    (c) If there exists $v \in V$ with $v + Tv + T^2 v + \cdots + T^{p-1} v \neq 0$, show $\dim_F V = p$.

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12. (Aug-11.2) Let R be a commutative ring with 1 and Q a primary ideal of R. For any $a \in R \backslash Q$, define the ideal $I_a = \{r \in R : ar \in Q\}$.

    (a) Show that $\mathrm{rad}(I_a) = \mathrm{rad}(Q)$.

    (b) Show that I_a is a primary ideal of R.

    (c) If R is Noetherian, show that there exists an a such that that I_a is a prime ideal.

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13. (Aug-07.2) Let R be a commutative integral domain that is integrally closed in its field of fractions F.

    (a) Suppose K is a field containing F and $\alpha \in K$ is integral over R. Show that the minimal monic polynomial of α over F is in $R[x]$.

    (b) Let $f(x) \in R[x]$ be monic. Show that $f(x)$ is irreducible in $R[x]$ iff it is irreducible in $F[x]$.

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14. (Jan-04.5) Let R be a ring with 1 and $V = X \oplus Y$ for nonzero (right) R-modules X and Y.

    (a) Show that $0, X, Y, V$ are the only submodules of V iff X and Y are nonisomorphic simple R-modules.

    (b) If X and Y are nonisomorphic simple R-modules, show that $\mathrm{End}_R(V)$ is isomorphic to the direct sum of two division rings.
