MATH 6140: First midterm examination. Wednesday, 14 February 2007.

Put **your name** on each answer sheet. Answer **all four** questions.

Show your working in full.

Formula sheets, calculators, notes and books are not permitted.

1. Let $R = F[x]$ be a polynomial ring over a field. Show that the map

$$\phi : f(x) \mapsto f(x^2)$$

is not an R-module homomorphism.

2. Prove that, with the usual notational conventions, we have $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} = 0$. (You may use any standard properties of tensor products without comment.)

3. Prove that the short exact sequence of $\mathbb{Z}$-modules

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

does not split. (As usual, 2 denotes multiplication by 2, and π is the canonical projection.)

4. Using the result of problem 3, together with any standard theorems you require, prove the following:

(i) $\mathbb{Z}/2\mathbb{Z}$ is not a projective $\mathbb{Z}$-module;

(ii) $\mathbb{Z}$ is not an injective $\mathbb{Z}$-module;

(iii) $\mathbb{Z}/2\mathbb{Z}$ is not a free $\mathbb{Z}$-module.

Determine also whether $\mathbb{Z}$ is (iv) projective and (v) flat as a module over itself.