DUAL VECTOR SPACES

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[1, No. 11.3.2] part (a). *Given.* Let V be the collection of polynomials with coefficients in $\mathbf{Q}$ in the variable x of degree at most 5 with $\mathscr{B} = \{1, x, x^2, \ldots, x^5\}$ as a basis. Consider the dual space $V* = \mathrm{Hom}_{\mathbf{Q}}(V, \mathbf{Q})$. Let the dual basis $\mathscr{B}^*$ for be defined on elements of $\mathscr{B}$ (letting the indices run over $0 \le i, j \le 5$) by

$$f_j(x^i) = \delta_{ij}.$$

To prove. The map $E\colon V \to \mathbf{Q}$ defined $E(p) = p(3)$ is a linear functional in $\mathrm{Hom}_{\mathbf{Q}}(V, \mathbf{Q})$ expressed in terms of the dual basis $\mathscr{B}^* = \{f_j\}_{j=0}^5$ as $\sum_{j=0}^5 3^j f_j$.

Proof. Let $a \in \mathbf{Q}$. Observe that the function $\mathrm{ev}_a|_V\colon V \to \mathbf{Q}$ defined by $\mathrm{ev}_a|_V(p) = p(a)$ is a restriction of the ring homomorphism $\mathrm{ev}_a\colon \mathbf{Q}[x] \to \mathbf{Q}$. In particular, taking $\lambda \in \mathbf{Q}$ as the constant polynomial and $p, q \in V$, we see

$$\begin{aligned} \mathrm{ev}_a(\lambda p) &= \lambda\, \mathrm{ev}_a(p) \\ \mathrm{ev}_a(p+q) &= \mathrm{ev}_a(p) + \mathrm{ev}_a(q). \end{aligned}$$

It follows that the restriction $\mathrm{ev}_a|_V$ is a $\mathbf{Q}$-linear functional in $\mathrm{Hom}_{\mathbf{Q}}(V, \mathbf{Q})$. Specifically, the evaluation at $3 \in \mathbf{Q}$,

$$E\colon V \to \mathbf{Q} \quad \text{defined} \quad E(p(x)) = p(3)$$

is a $\mathbf{Q}$-linear functional in $\mathrm{Hom}_{\mathbf{Q}}(V, \mathbf{Q})$. That is, E is a linear functional in the dual space V^*.

Now to express E as a linear combination of linear functionals in $\mathscr{B}^*$. Note for each $p \in V$ that $f_j(p)$ is the rational coefficient of x^j. Let $\mathbf{Q}$ be given the standard basis $\mathscr{E} = \{1\}$. For an arbitrary $a \in \mathbf{Q}$, the matrix representation of $\mathrm{ev}_a|_V \in \mathrm{Hom}_{\mathbf{Q}}(V, \mathbf{Q})$ with respect to $\mathscr{B}$ and $\mathscr{E}$ is

$$M_{\mathscr{B}}^{\mathscr{E}}(\mathrm{ev}_a|_V) = \begin{bmatrix} a^0 & a^1 & a^2 & a^3 & a^4 & a^5 \end{bmatrix}. \tag{1}$$

The matrix representation of E follows from (1) as

$$M_{\mathscr{B}}^{\mathscr{E}}(E) = \begin{bmatrix} 3^0 & 3^1 & 3^2 & 3^3 & 3^4 & 3^5 \end{bmatrix}.$$

To write E in terms of the dual basis:

$$E = \begin{bmatrix} 3^0 & 3^1 & 3^2 & 3^3 & 3^4 & 3^5 \end{bmatrix} \begin{bmatrix} f_0 \\ \vdots \\ f_5 \end{bmatrix} \quad \text{(a linear combination of functionals } f_j \text{ from the dual basis } \mathscr{B}^*\text{)}$$

We verify that E and the linear combination $M_{\mathscr{B}}^{\mathscr{E}}[f_j]$ agree on basis elements of $\mathscr{B}$ for $j = 0, \ldots, 5$:

$$\begin{aligned} \begin{bmatrix} 3^0 & 3^1 & 3^2 & 3^3 & 3^4 & 3^5 \end{bmatrix} \begin{bmatrix} f_0 \\ \vdots \\ f_5 \end{bmatrix} (x_j) &= \begin{bmatrix} 3^0 & 3^1 & 3^2 & 3^3 & 3^4 & 3^5 \end{bmatrix} \begin{bmatrix} \delta_{0j} \\ \vdots \\ \delta_{5j} \end{bmatrix} \\ &= 3^j. \end{aligned}$$

We have demonstrated that E and $M_{\mathscr{B}}^{\mathscr{E}}[f_j]$ agree on all x^j in the basis $\mathscr{B}$ for V. Extending linearly, $E \equiv M_{\mathscr{B}}^{\mathscr{E}}[f_j]$. $\square$

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[1, No. 11.3.2] part (b). *Given.* The same setup as [1, No. 11.3.2] part (a), but considering the function $\varphi\colon V \to \mathbf{Q}$ defined by $\varphi(p) = \int_0^1 p(t)\,dt$ instead of E.

To prove. The map $\varphi\colon V \to \mathbf{Q}$ is a linear functional in $\operatorname{Hom}_{\mathbf{Q}}(V, \mathbf{Q})$ expressed in terms of the dual basis $\mathscr{B}^* = \{f_j\}_{j=0}^5$ as $\sum_{j=0}^5 \frac{f_j}{j+1}$.

Proof. We first argue the following is true:

> Let $[a, b]$ be a closed interval in $\mathbf{R}$ with rational endpoints, and let $\mathbf{Q}[x]$ be considered as a rational vector space of polynomials. For each $g \in \mathbf{Q}[x]$, the function $\varphi_g\colon \mathbf{Q}[x] \to \mathbf{Q}$ defined by $\varphi_g(f) = \int_a^b g(t)f(t)\,dt$ is a linear functional in $\operatorname{Hom}_{\mathbf{Q}}(\mathbf{Q}[x], \mathbf{Q})$.

Why? Say φ_g is defined as integrating $f \in \mathbf{Q}[x]$ against g. Let $f_1, f_1 \in \mathbf{Q}[x]$ and $\lambda \in \mathbf{Q}$. Because $\mathbf{Q}[x]$ is also a ring under polynomial multiplication, both λf_i and gf_i (for $i = 1, 2$) are rational polynomials. The primitive F of a rational polynomial f is again a rational polynomial. It follows definite integral of a ration polynomial f is just the difference $F(b) - F(a)$ of the evaluations of F at b and a (in that order). So much to say that $\varphi_g\colon \mathbf{Q}[x] \to \mathbf{Q}$ is well defined. Now for $\mathbf{Q}$-linearity. Observe

$$\begin{aligned}
\varphi_g(f_1 + f_2) &= \int_a^b (f_1 + f_2)(t)g(t)\,dt \\
&= \int_a^b f_1(t)g(t)\,dt + \int_a^b f_2(t)g(t)\,dt \\
&= \varphi_g(f_1) + \varphi_g(f_2), \text{ and} \\
\varphi_g(\lambda f_1) &= \int_a^b (\lambda f_1)(t)g(t)\,dt \\
&= \lambda \int_a^b f_1(t)g(t)\,dt \\
&= \lambda\varphi_g(f_1).
\end{aligned}$$

We have shown φ_g is a linear functional in $\operatorname{Hom}_{\mathbf{Q}}(\mathbf{Q}[x], \mathbf{Q})$.

Onto our specific case. Let $a = 0$ and $b = 1$. Notice that φ as given is the restriction of $\varphi_{\mathbf{1}} \in \operatorname{Hom}_{\mathbf{Q}}(\mathbf{Q}[x], \mathbf{Q})$ (defined $\varphi_{\mathbf{1}}(f) = \int_0^1 f(t)\,dt$) to the subspace V of $\mathbf{Q}[x]$. Because the restriction of a morphism to a subobject is a morphism out of the subobject, φ is a linear functional in $\operatorname{Hom}_{\mathbf{Q}}(V, \mathbf{Q})$.

Lastly, for each x^i in the basis $\mathscr{B}$ of V, we have

$$\varphi(x^i) = \frac{t^{i+1}}{i+1}\bigg|_{t=0}^{t=1} = \frac{1}{i+1}.$$

The matrix representation (using $\mathscr{E} = \{1\}$ for $\mathbf{Q}$) is

$$M_{\mathscr{B}}^{\mathscr{E}}(\varphi) = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \end{bmatrix}.$$

The expression of φ is terms of the dual basis $\mathscr{B}^*$ is

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} f_0 \\ \vdots \\ f_5 \end{bmatrix} = \sum_{j=0}^5 \frac{f_j}{j+1}.$$

□

[1, No. 11.3.2] part (c). *Given.* The same setup as [1, No. 11.3.2] part (b), but considering a new definition of $\varphi\colon V \to \mathbf{Q}$ such that

$$\varphi(p) = \int_0^1 t^2 p(t)\,dt. \tag{2}$$

To prove. The map $\varphi\colon V \to \mathbf{Q}$ is a linear functional in $\operatorname{Hom}_{\mathbf{Q}}(V, \mathbf{Q})$ expressed in terms of the dual basis $\mathscr{B}^* = \{f_j\}_{j=0}^5$ as $\sum_{j=0}^5 \frac{f_j}{j+3}$.

Proof. In part (b) we demonstrated that for each $g \in \mathbf{Q}[x]$, the function $\varphi_g\colon \mathbf{Q}[x] \to \mathbf{Q}$ defined by $\varphi_g(f) = \int_a^b g(t)f(t)\,dt$ is a linear functional in $\operatorname{Hom}_{\mathbf{Q}}(\mathbf{Q}[x], \mathbf{Q})$. Let $g(t) = t^2$. We see that φ in (2) is the restriction of φ_g to the subspace V. Hence the restriction $\varphi_g|_V = \varphi$ is a linear functional in $\operatorname{Hom}_{\mathbf{Q}}(V, \mathbf{Q})$.

For each x^i in the basis $\mathscr{B}$ of V, we have

$$\varphi(x^i) = \frac{t^{i+3}}{i+3}\bigg|_{t=0}^{t=1} = \frac{1}{i+3}.$$

The matrix representation with respect to $\mathscr{B}$ and $\mathscr{E}$ is

$$M_{\mathscr{B}}^{\mathscr{E}}(\varphi) = \begin{bmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \end{bmatrix}.$$

So the expression of φ is terms of the dual basis $\mathscr{B}^*$ is

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \end{bmatrix} \begin{bmatrix} f_0 \\ \vdots \\ f_5 \end{bmatrix} = \sum_{j=0}^{5} \frac{f_j}{j+3}.$$

□

[1, No. 11.3.2] part (d). *Given.* The same setup as [1, No. 11.3.2] part (c), but considering $\varphi\colon V \to \mathbf{Q}$ such that

$$\varphi(p) = p'(5). \quad \text{(polynomial derivation)}$$

To prove. The map $\varphi\colon V \to \mathbf{Q}$ is a linear functional in $\operatorname{Hom}_{\mathbf{Q}}(V, \mathbf{Q})$ expressed in terms of the dual basis $\mathscr{B}^* = \{f_j\}_{j=0}^5$ as $\sum_{j=0}^5 j5^{j-1} f_j$.

Proof. In part (a) we demonstrated that for any $a \in \mathbf{Q}$ and $\mathbf{Q}$-subspace W of $\mathbf{Q}[x]$, the map $\operatorname{ev}_a|_W$ was a linear functional in $\operatorname{Hom}_{\mathbf{Q}}(W, \mathbf{Q})$. Now let W be the subspace of $\mathbf{Q}[x]$ of polynomials with degree at most 4. Let $D\colon V \to W$ be defined $D(p(x)) = p'(x)$ for all $p \in V$. Observe that D is linear. For $\lambda \in \mathbf{Q}$, $p, q \in V$,

$$\lambda D(p(x)) = \lambda p'(x) = D(\lambda p(x)), \quad \text{(scalar multiplication)}$$
$$D(p(x) + q(x)) = (p+q)'(x) = p'(x) + q'(x) = D(p(x)) + D(q(x)), \quad \text{(addition)}.$$

So D is in $\operatorname{Hom}_{\mathbf{Q}}(V, W)$.

By the argument in part (a), $\operatorname{ev}_5|_W$ is a linear functional in $\operatorname{Hom}_{\mathbf{Q}}(W, \mathbf{Q})$. Whence the composition $\operatorname{ev}_5|_W \circ D$ is in $\operatorname{Hom}_{\mathbf{Q}}(V, \mathbf{Q})$. But in fact $\operatorname{ev}_5|_W \circ D \equiv \varphi$. We conclude that φ is a linear functional in $\operatorname{Hom}_{\mathbf{Q}}(V, \mathbf{Q})$.

For each x^i in the basis $\mathscr{B}$ of V, we have

$$\varphi(x^i) = i5^{i-1}.$$

The matrix representation with respect to $\mathscr{B}$ and $\mathscr{E}$ is

$$M_{\mathscr{B}}^{\mathscr{E}}(\varphi) = \begin{bmatrix} 0 & 1 & 10 & 75 & 500 & 3125 \end{bmatrix}$$

So the expression of φ is terms of the dual basis $\mathscr{B}^*$ is

$$\begin{bmatrix} 0 & 1 & 10 & 75 & 500 & 3125 \end{bmatrix} \begin{bmatrix} f_0 \\ \vdots \\ f_5 \end{bmatrix} = \sum_{j=0}^{5} j5^{j-1} f_j.$$

□

[1, No. 11.3.3] part (a). *Given.* For any subset S of V^* for some finite dimensional space V, we define the *annihilator of S in V* as

$$\operatorname{Ann}(S) := \{v \in V : f(v) = 0 \text{ for all } f \in S\}.$$

Let V be such a finite dimensional vector space and S such a subset.

To prove. $\operatorname{Ann}(S)$ is a *subspace* of V.

Proof. Notice $0 \in \operatorname{Ann}(S)$, because ranging through all $f \in V^*$, it's always the case that $f(0) = 0$. Now suppose $u, v \in \operatorname{Ann}(S)$ and $\lambda \in F$. Then, for all $f \in S$,

$$\begin{aligned} f(u+v) &= f(u) + f(v) = 0 + 0 = 0 \\ \lambda f(u) &= \lambda 0 = 0. \end{aligned}$$

So $\operatorname{Ann}(S)$ is a subspace of V. □

[1, No. 11.3.3] part (c). *Given.* Let V be a finite dimensional vector space. Let $W_1, W_2 \subset V^*$ be subspaces of the dual space.

To prove. $W_1 = W_2$ if and only if $\operatorname{Ann}(W_1) = \operatorname{Ann}(W_2)$.

Proof. ($\Rightarrow$) Suppose $W_1 = W_2$. Then

$$\begin{aligned} \operatorname{Ann}(W_1) &= \{v \in V : f(v) = 0 \quad \text{for all} \quad f \in W_1\} \\ &= \{v \in V : f(v) = 0 \quad \text{for all} \quad f \in W_2\} \\ &= \operatorname{Ann}(W_2). \end{aligned}$$

($\Leftarrow$) Suppose $\operatorname{Ann}(W_1) = \operatorname{Ann}(W_2)$. From the forwards argument, it's apparent

(3) $$\operatorname{Ann}(\operatorname{Ann}(W_1)) = \operatorname{Ann}(\operatorname{Ann}(W_2)).$$

We desire that $W_1 = W_2$, so we will argue:

> For each subspace $U \subset V^*$, the annihilator of the annihilator $\operatorname{Ann}(\operatorname{Ann}(U))$ is naturally isomorphic (via the natural embedding of W into V^*, where $W^* = V$) to U itself.

Supposing the statement above has been proven, (3) implies that W_1 and W_2 in $V^* = W^{**}$ are natural images of the same subspace $\operatorname{Ann}(\operatorname{Ann}(W_1)) = \operatorname{Ann}(\operatorname{Ann}(W_2))$ in W. Naturality then implies $W_1 = W_2$.

Onto the argument. Say V^* be a finite dimensional vector space with $U \subset V^*$ a subspace of the dual. We might as well find a vector space W such that $W^* = V$ (certainly V^* is a candidate, but for notation's sake we'll write W). Here's a schematic (which is *not to imply* there are morphisms along the arrows):

(4)
$$\begin{tikzcd}
 & V^* \arrow[dl, dash, "\mathrm{id}"'] & & V \arrow[ll] \arrow[dl, dash, "\mathrm{id}"'] & \\
W^{**} & & W^* \arrow[ll] & & W \arrow[ll]
\end{tikzcd}$$

$$\begin{tikzcd}
 & U \arrow[rr] \arrow[dl, dash, "\mathrm{id}"'] & & \operatorname{Ann}(U) \arrow[dl, dash, "\mathrm{id}"'] & \\
U \arrow[rr] & & \operatorname{Ann}(U) \arrow[rr] & & \operatorname{Ann}(\operatorname{Ann}(U))
\end{tikzcd}$$

Choose $\tilde{U} \subset W$ as the preimage of U under the natural injection $W \to W^{**}$ given by $u \mapsto (\mathrm{ev}_u : W^* \to W)$. Because V is finite dimensional, $V^* = W^{**}$ is too. Whence $\tilde{U}$ naturally surjects onto U, which implies each linear functional in U is of the form $\mathrm{ev}_u : W^* \to F$ for a unique $u \in \tilde{U}$.

We can compute $\operatorname{Ann}(U)$ either thinking of $U \subset W^{**}$ or as $U \subset V^*$. That is,

$$\begin{aligned}\operatorname{Ann}(U) &= \{v \in V : f(v) = 0 \quad \text{for all} \quad f \in U \subset V^*\} \\ &= \{f \in W^* : \mathrm{ev}_u(f) = 0 \quad \text{for all} \quad \mathrm{ev}_u \in U \subset W^{**}\}.\end{aligned}$$

Proceeding in the $U \subset W^{**}$ style of thought,

$$\begin{aligned}\operatorname{Ann}(\operatorname{Ann}(U)) &= \operatorname{Ann}(\{f \in W^* : \mathrm{ev}_u(f) = 0 \quad \text{for all} \quad \mathrm{ev}_u \in U \subset W^{**}\}) \\ &= \{w \in W : f(w) = 0 \quad \text{for all} \quad f \in W^* \quad \text{s.th.} \quad \mathrm{ev}_u(f) = 0 \quad \text{for all} \quad \mathrm{ev}_u \in U \subset W^{**}\} \\ &= \{w \in W : f(w) = 0 \quad \text{for all} \quad f \in W^* \quad \text{s.th.} \quad f(u) = 0 \quad \text{for all} \quad u \in \tilde{U} \subset W\} \\ &= \tilde{U}.\end{aligned}$$

Because $\operatorname{Ann}(\operatorname{Ann}(U)) = \tilde{U}$ of each subspace $U \subset V^*$ is naturally isomorphic to U by way of including W into W^{**}, we conclude:

> Given subspaces W_1 and W_2 in V^* with identical annihilators $\operatorname{Ann}(W_1) = \operatorname{Ann}(W_2)$, it follows that $\operatorname{Ann}(\operatorname{Ann}(W_1)) = \operatorname{Ann}(\operatorname{Ann}(W_2))$, so the natural preimages W_1 and W_2 are in fact the same. □

[1, No. 11.3.3] part (d). *Given.* Let $S \subset V^*$ for V^* a finite dimensional vector space.

To prove. $\operatorname{Ann}(\operatorname{span}(S)) = \operatorname{Ann}(S)$.

Proof. We show both inclusions. Let $v \in \operatorname{Ann}(\operatorname{span}(S))$. Then for all $f \in \operatorname{span}(S)$, $f(v) = 0$. So clearly for all $f \in S$, $f(v) = 0$. Thus $v \in \operatorname{Ann}(\operatorname{span}(S))$.

On the other hand, let $v \in \operatorname{Ann}(S)$. For all $f \in S$, $f(v) = 0$. Because S is a generating set for $\operatorname{span}(S)$, there's a surjection

$$\bigoplus_{f \in S} F \to \operatorname{span}(S) \quad \text{such that} \quad \sum_{f \in S} \lambda_f \mapsto \sum_{f \in S} \lambda_f f. \tag{5}$$

Take an arbitrary functional in the image (e.g., the left hand side) of (5). Evaluating at v, it follows that $\sum_{f \in S} \lambda_f f(v) = 0$. So v annihilates all functionals in $\operatorname{span}(S)$. Thence $v \in \operatorname{Ann}(\operatorname{span}(S))$. □

[1, No. 11.3.3] part (e). *Given.* Assume V is finite dimensional with basis $v_1, \ldots, v_n$.

To prove. If $S = \{v_1^*, \ldots, v_k^*\}$ for some $k \leq n$, then $\operatorname{Ann}(S)$ is the subspace spanned by $\{v_{k+1}, \ldots, v_n\}$.

Proof. We show both inclusions for $\operatorname{Ann}(S) = \operatorname{span}(v_{k+1}, \ldots, v_n)$. Let

$$u = \lambda_{k+1} v_{k+1} + \cdots + \lambda_n v_n$$

be in $\operatorname{span}(v_{k+1}, \ldots, v_n)$. Consider the image of u under each dual basis functional $v_1^*, \ldots, v_k^*$. Because the indices $1, \ldots, k$ in the dual basis are disjoint from $k+1, \ldots, n$ of the (possibly) nonzero components of u, it's visible that u is a vector in $\operatorname{Ann}(S)$.

For the other inclusion, say $v \in \operatorname{Ann}(S)$. Then

$$v = \lambda_1 v_1 + \ldots + \lambda_k v_k + \lambda_{k+1} v_{k+1} + \cdots + \lambda_n v_n.$$

Evaluating each dual basis functional $v_1^*, \ldots, v_k^*$ at v, the condition $v_i^*(v) = 0$ implies $\lambda_i = 0$ for $i = 1, \ldots, k$. Whence $v \in \operatorname{span}(v_{k+1}, \ldots, v_n)$. □

[1, No. 11.3.3] part (f). *Given.* Assume V is a finite dimensional vector space.

To prove. If W^* is any subspace of V^*, then $\dim \operatorname{Ann}(W^*) = \dim V = \dim W^*$.

Proof. In part (e), we argued when $k \leq \dim V$, if $S = \{v_1^*, \ldots, v_k^*\}$ is a subset of the dual basis, then $\operatorname{Ann}(S)$ is spanned by $\dim V - k$ linearly independent vectors. In part (d), we proved that $\operatorname{Ann}(S) = \operatorname{Ann}(\operatorname{span}(S))$. Given $W^* \subset V^*$, both parts (d) and (e) imply: for a basis $\mathscr{B}$ of W^*, $\dim \operatorname{Ann}(S) = \dim V - |\mathscr{B}|$, whence $\dim \operatorname{Ann}(W^*) = \dim V - \dim W^*$. □

[1, No. 11.3.4]. *Given.* Let V be an infinite dimensional with basis $\mathscr{A}$.

To prove. $\mathscr{A}^* = \{v^* : v \in \mathscr{A}\}$ does *not* span V^*.

Proof. Suppose for contradiction that $\operatorname{span}(\mathscr{A}) = V^*$. Then for each $f \in V^*$, we would have a unique linear combination

$$f = \sum_{a \in \mathscr{A}} a^* f(a) \quad \text{such that all but finitely many of the } f(a) = 0.$$

Let $u \in F$ be a nonzero element of the base field. Then let $f \in \operatorname{Hom}_F(V, F)$ be defined on generators $a \in \mathscr{A}$ by $f(a) = u$. Extending linearly, f is a well defined homomorphism from V to F. Yet f cannot be expressed as a *finitary* linear combination $\sum_{a \in \mathscr{A}} a^* f(a)$, which is absurd. We conclude that $\mathscr{A}^*$ does *not* span V^*. □

References

[1] D. Dummit and R. Foote, *Abstract algebra*. Prentice Hall, 2004.