2014 Algebra SEP $\sim$ Linear Algebra Day 2 Problems, by E. Dummit

1. (Jan-99.4) Let V be finite-dimensional over F algebraically closed, and let $ST = TS$, where the characteristic polynomial of S has distinct roots.
   (a) Show that every eigenvector of S is an eigenvector for T.
   (b) If T is nilpotent, prove that $T = 0$.

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2. (Jan 12.4): Let V be a finite-dimensional $\mathbb{C}$-vector space.
   (a) If S, T are commuting linear operators on V, show that each eigenspace of S is mapped onto itself by T.
   (b) If $A_1, \cdots, A_k$ are operators which commute pairwise, show they have a common eigenvector in V.
   (c) If V has dimension n, show there exists a nested sequence of subspaces $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ where $\dim(V_j) = j$ and each V_j is mapped onto itself by each of the operators $A_1, \cdots, A_k$.

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3. (Aug-05.4): Let F be a field and A, B nonsingular 3×3 matrices over F. Suppose $B^{-1}AB = 2A$.
   (a) Find the characteristic of F.
   (b) If n is a positive or negative integer not divisible by 3, prove that A^n has trace 0.
   (c) Prove that the characteristic polynomial of A is $X^3 - a$ for some $a \in F$.

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4. (Aug-08.4/85.4a): Let S, T, M be $n \times n$ matrices over $\mathbb{C}$ with $SM = MT$.
   (a) If $f(x)$ is the minimal polynomial of T, show $f(S)M = 0$.
   (b) If $M \neq 0$, deduce that S and T have a common eigenvalue.
   (c) Now let $S = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$, $T = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$. Find a nonzero M with $SM = MT$ and show that any such M cannot be invertible.

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5. (Aug-06.4): Let V be a nonzero finite dimensional vector space over F and let $T : V \to V$ be a linear transformation. We say T is regular if its characteristic polynomial and minimal polynomial are equal.
   (a) If there exists a vector $v \in V$ such that V is spanned by $v, T(v), T^2(v), \cdots$, prove that T is regular.
   (b) Assume that T is regular and let W be a subspace with $T(W) \subseteq W$. Show that T_W, the restriction of T to W, and $T_{V/W}$, the induced action of T on V/W, are both regular.

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6. (Jan-95.4) Let A be an $n \times n$ matrix over an algebraically closed field K and let $K[A]$ denote the K-linear span of $I, A, A^2, \cdots$. Show that A is diagonalizable iff $K[A]$ contains no nonzero nilpotent element.

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7. (Aug-03.4): Let A be a real $n \times n$ matrix. We say A is a "difference of two squares" if there exist real $n \times n$ matrices B and C for which $BC = CB = 0$ and $A = B^2 - C^2$.
   (a) If A is diagonal, show it is a difference of two squares.
   (b) If A is symmetric, show it is a difference of two squares.
   (c) If A is a difference of two squares with B and C as above, if B has a nonzero real eigenvalue, prove that A has a positive real eigenvalue.

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8. (Aug-12.4): Let V be an n-dimensional K-vector space and $T : V \to V$.
   (a) Suppose there exists $v \in V$ such that V is spanned by $v, Tv, T^2v, \cdots$. Prove that the minimal polynomial of T equals the characteristic polynomial of T.
   (b) As a partial converse, suppose the characteristic polynomial of T has distinct roots in K. Prove that there exists $v \in V$ such that V is spanned by $v, Tv, T^2v, \cdots$.
