# INTRODUCTION TO MODULE THEORY: BASIC DEFINITIONS AND EXAMPLES

COLTON GRAINGER (MATH 6140 ALGEBRA 2)

## 1. Assignment due 2019-01-13

1.1. **[1, No. 10.1.1].** *Given.* $R$ is a unital ring and $M$ is a left $R$-module.

*To prove.* $0m = 0$ and $(-1)m = -m$ for all $m \in M$.

1.2. **[1, No. 10.1.3].** *Given.* Say $rm = 0$ for some $r \in R$ and some $m \in M$ with $m \neq 0$.

*To prove.* There is no $s \in R$ such that $sr = 1$.

1.3. **[1, No. 10.1.4].** *Given.* Let $M$ be the modules $R^n$ and let $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n$ be left ideals of $R$.

*To prove.* Both of the following are submodules of $M$:

a. $P = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathfrak{a}_i\}$,
b. $N = \{(x_1, x_2, \ldots, x_n) : x_i \in R$ and $\sum_i x_i = 0\}$.

1.4. **[1, No. 10.1.5].** *Given.* Consider a left ideal $\mathfrak{a}$ of $R$. Let

$$\mathfrak{a}M = \left\{ \sum_{\text{finite}} a_i m_i : a_i \in \mathfrak{a}, m_i \in M \right\}.$$

*To prove.* We have $\mathfrak{a}M$ as a submodule of $M$.

1.5. **[1, No. 10.1.6].** *Given.* Let $M$ be a module over $R$ and $\{N_i\}$ be a nonempty collection of submodules.

*To prove.* The intersection $\bigcap_i N_i$ is a submodule of $M$.

1.6. **[1, No. 10.1.8].** *Given.* An element $m$ of the $R$-module $M$ is called a *torsion element* if $rm = 0$ for some nonzero element $r \in R$. The set of torsion elements is denoted

$$\text{Tor}\,(M) = \{m \in M : rm = 0 \text{ for some nonzero } r \in R\}.$$

*To prove.*

a. If $R$ is an integral domain, then Tor $(M)$ is a submodule of $M$ (called the *torsion submodule*).

b. If $R$ has zero divisors, then every nonzero $R$-module has nonzero torsion elements.

1.7. **[1, No. 10.1.9].** *Given.* If $N$ is a submodule of $M$, the *annihilator of* $N$ *in* $R$ is defined to be

$$\{r \in R : rn = 0 \text{ for all } n \in N\}.$$

*To prove.* The annihilator of $N$ in $R$ is a $2$-sided ideal of $R$.

1.8. **[1, No. 10.1.15].** *Given.* Say $M$ is a finite abelian group. $M$ is naturally a $\mathbf{Z}$-module.

*To prove.* This action cannot be extended to make $M$ into a $\mathbf{Q}$-module.

---

*Date:* 2019-01-14.
*Compiled:* 2019-01-14.

1.9. **[1, No. 10.1.18].** *Given.* Let $F = \mathbf{R}$, let $V = \mathbf{R}^2$, and let $T$ be the linear transformation from $V$ to $V$ that is rotation clockwise about the origin by $\pi/2$ radians.

*To prove.* $V$ and $0$ are the only $F[x]$-submodules for this $T$.

1.10. **[1, No. 10.1.19].** *Given.* Let $F = \mathbf{R}$, let $V = \mathbf{R}^2$, and let $T$ be the linear transformation from $V$ to $V$ that is projection onto the $y$-axis.

*To prove.* $V$, $0$, the $x$-axis and the $y$-axis are the only $F[x]$-submodules for this $T$.

1.11. **[1, No. 10.1.20].** *Given.* Let $F = \mathbf{R}$, let $V = \mathbf{R}^2$, and let $T$ be the linear transformation from $V$ to $V$ that is rotation clockwise about the origin by $\pi$ radians.

*To prove.* Every subspace of $V$ is an $F[x]$-submodule for this $T$.

1.12. **[1, No. 10.1.21].** *Given.* Let $n \in \mathbf{Z}^+$, $n > 1$, and $R$ be the ring $\mathscr{M}_n(F)$ of $n \times n$ matrices from the field $F$. Let $M \subset \mathscr{M}_n(F)$ be

$$M = \left\{ (a_i^j) : a_i^j = 0 \text{ if } j > 1 \right\},$$

that is, the set of matrices with arbitrary elements of $F$ in the first column and zeros elsewhere.

*To prove.*

- $M$ is a submodule of $R$ when $R$ is considered as a left module over itself.
- $M$ is *not* a submodule of $R$ when $R$ is considered as a right module.

## References

[1] D. Dummit and R. Foote, *Abstract algebra*. Prentice Hall, 2004.