MODULE THEORY: BASIC RESULTS

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[1, No. 10.1.1]. *Given.* R is a unital ring and M is a left R-module.

To prove. $0m = 0$ and $(-1)m = -m$ for all $m \in M$.

Proof. Let $0_R \in R$ and $0_M \in M$ be the respective additive identities. Then for all $m \in M$, by distributivity, $1.m = (1 + 0_R).m = 1.m + 0_R.m$. We have

$$m + 0_R.m = m \quad \text{so by cancellation in } M \quad 0_R.m = 0_M.$$

Knowing $0_R.m = 0_M$ implies $0_M = (1 + (-1))_R.m = 1.m + (-1).m$, so that

$$-m = -(1.m) = (-1).m.$$

□

[1, No. 10.1.3]. *Given.* Say $rm = 0$ for some $r \in R$ and some $m \in M$ with $m \neq 0$.

To prove. There is no $s \in R$ such that $sr = 1$.

Proof by contradiction. Suppose there's $s \in R$ such that $sr = 1$. Then both

$$sr.m = m$$
$$r.m = 0$$

Try adding:

$$\begin{aligned} s.m + sr.m &= s.m + s.(r.m) && \text{by module axioms} \\ &= s.(m + r.m) && \text{by distributivity of scalar multiplication} \\ &= s.(m + 0) && \text{by hypothesis} \\ &= s.m, \end{aligned}$$

so $sr.m$ had better be 0_M. But it's not, for $sr = 1$ implies $sr.m = m$, which together with the last argument is absurd. □

[1, No. 10.1.4]. *Given.* Let M be the modules R^n and let $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n$ be left ideals of R.

To prove. Both of the following are submodules of M:

a. $P = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathfrak{a}_i\}$,
b. $N = \{(x_1, x_2, \ldots, x_n) : x_i \in R \text{ and } \sum_i x_i = 0\}$.

Proof.

a. P is nonempty, for $\prod 0_R$ is in each ideal, so in P as well. Now say x and y are in P. The ith components of x and y are in each $\mathfrak{a}_i$, so the sum $x + y$ has ith component in each $\mathfrak{a}_i$ by closure of ideals under addition. Whence $x + y \in P$. Lastly, take a ring element $\alpha \in R$. Then $\alpha x = \prod \alpha x_i$. By closure of ideals under left multiplication, $\alpha x_i \in \mathfrak{a}_i$. We conclude $\alpha x \in P$.

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b. N is nonempty, as it contains the additive identity of R^n. Let x, y be in N. Then $x+y$ has components summing

$$\sum_i (x_i + y_i) = \sum_i x_i + \sum_i y_i = 0.$$

So $x+y \in N$. Consider $\alpha \in R$. Then αx has the sum

$$\sum_i \alpha x_i = \alpha \left(\sum_i x_i \right) = \alpha.0 = 0,$$

by distributivity of multiplication over addition and previous argument [1, No. 10.1.3]. So $\alpha x \in N$. So N is a submodule of R^n. □

[1, No. 10.1.5]. *Given.* Consider a left ideal $\mathfrak{a}$ of R. Let

$$\mathfrak{a}M = \left\{ \sum_{\text{finite}} a_i m_i : a_i \in \mathfrak{a}, m_i \in M \right\}.$$

To prove. We have $\mathfrak{a}M$ as a submodule of M.

Proof. First, both $0_R \in \mathfrak{a}$ and $0_M \in M$ so that $0 \in \mathfrak{a}M$. Second, let $x, y \in \mathfrak{a}M$ where $x = \sum_i a_i m_i$ and $y = \sum_j b_j n_j$. This sum of finite sums is finite, so that $x+y$ is in $\mathfrak{a}M$. Lastly, for any $r \in R$, consider $rx = r\left(\sum_i a_i m_i\right) = \sum_i (ra_i) m_i \in \mathfrak{a}M$. □

[1, No. 10.1.6]. *Given.* Let M be a module over R and $\{N_i\}$ be a nonempty collection of submodules.

To prove. The intersection $\bigcap_i N_i$ is a submodule of M.

Proof. Observe $0_M \in \cap_i N_i \neq \varnothing$. Next, suppose $x, y \in \cap_i N_i$. Then $x+y$ is in each N_i, by closure of submodules under addition. If $r \in R$, then so too $rx \in N_i$ for all i, by closure of submodules under scalar multiplication. Thence $x+y \in \cap_i N_i$ and $rx \in \cap_i N_i$. □

[1, No. 10.1.8]. *Given.* An element m of the R-module M is called a *torsion element* if $rm = 0$ for some nonzero element $r \in R$. The set of torsion elements is denoted

$$\mathrm{Tor}\,(M) = \{m \in M : rm = 0 \text{ for some nonzero } r \in R\}.$$

To prove.

a. If R is an entire ring, then $\mathrm{Tor}\,(M)$ is a submodule of M (called the *torsion submodule*).

b. If R has zero divisors, then every nonzero R-module has nonzero torsion elements.

Proof.

a. $\mathrm{Tor}\,(M)$ contains 0. Now say $x, y \in \mathrm{Tor}\,(M)$. Then some $\alpha, \beta \in R$ kill x and y respectively:

$$\alpha x = 0, \qquad \beta y = 0.$$

Consider $x+y$. Because R is commutative and by module axioms,

$$\alpha\beta(x+y) = \beta(\alpha x) + \alpha(\beta y) = 0 + 0.$$

So the sum $x+y \in \mathrm{Tor}\,(M)$. Next say $r \in R$ is nonzero. Consider rx. We have

$$\alpha r x = r(\alpha x) = 0$$

and because both r and α are nonzero elements in an entire ring, it can't be that $r\alpha = 0$ and it must be that $rx = 0$. So $rx \in \mathrm{Tor}\,(M)$.

b. Let R have at least the pair of zero divisors α and β with $\alpha\beta = 0$. Say that M is a non-trivial R-module, and take nonzero m in M. Either $\alpha m = 0$ or not—in the former case $m \in \mathrm{Tor}\,(M)$, and in the later $\beta(\alpha m) = \beta\alpha m = 0m = 0$, so that $\alpha m \in \mathrm{Tor}\,(M)$. Both cases force a nontrivial element in $\mathrm{Tor}\,(M)$. □

[1, No. 10.1.9]. *Given.* If N is a submodule of M, the *annihilator of* N *in* R is defined to be

$$\{r \in R : rn = 0 \text{ for all } n \in N\}.$$

To prove. The annihilator of N in R is a 2-sided ideal of R.

Proof. Denote the annihilator of N in R by $\mathfrak{a}$. First note $0 \in \mathfrak{a}$ as $0n = 0$ for all $n \in N$. Next take $\alpha, \beta \in \mathfrak{a}$. Then $(\alpha + \beta)n = \alpha n + \beta n = 0 + 0 = 0$. Then $\alpha + \beta \in \mathfrak{a}$; we see $\mathfrak{a}$ is closed under finite sums. Moreover, each $\alpha \in \mathfrak{a}$ has an additive inverse $-\alpha \in R$. Since $0 = -\alpha n = (-\alpha)n$, $\mathfrak{a}$ is closed under additive inverses. So $\mathfrak{a}$ is an abelian subgroup of R. Now say that $r \in R$ and $a \in \mathfrak{a}$. Then

$$(ra)n = r(an) = r(0) = 0,$$

so $ra \in \mathfrak{a}$. Consider ar. We have

$$(ar)n = a(rn) = a(n_0) = 0, \qquad \text{where } rn = n_0 \in N,$$

so that $ar \in \mathfrak{a}$. We conclude $\mathfrak{a}$ is a two-sided ideal. □

[1, No. 10.1.15]. *Given.* Say M is a finite abelian group. M is naturally a $\mathbf{Z}$-module.

To prove. This action cannot be extended to make M into a $\mathbf{Q}$-module.

Proof by contradiction. Suppose $\mathbf{Q}$ extends the action of $\mathbf{Z}$ on M. Pick any nonzero $m \in M$. I claim for each $n \in \mathbf{N}$, $\frac{1}{n}.m \neq 0$. To wit, by the unital module axiom,

$$m = n.\left(\frac{1}{n}.m\right) \neq n.(0) = 0 \quad \text{by assumption than } m \text{ is nonzero.}$$

I claim for each distinct pair $a, b \in \mathbf{N}$, the image of m under the action of $\frac{1}{a}$ and $\frac{1}{b}$ is distinct. Else we could find

$$0 = \frac{1}{a}.m - \frac{1}{b}.m = \left(\frac{1}{a} - \frac{1}{b}\right).m = \frac{1}{ab}.m,$$

which is prevented by the previous claim. Since $\mathbf{N}$ is infinite, the image of $\frac{1}{n}.m$ in M as n ranges through $\mathbf{N}$ must be infinite, which is absurd! (By hypothesis, M is a finite group.) □

[1, No. 10.1.18]. *Given.* Say we have a linear transformation $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ with the associated matrix

$$A_T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \text{with respect to the standard basis.}$$

Consider $V = \mathbf{R}^2$ as an $\mathbf{R}[x]$-module where scalar multiplication is obviously defined by constant polynomials and x acts by the linear transformation $x.v = Av$.

To prove. If M is a submodule of V, then M is trivial or V.

Proof. Say M is a nontrivial submodule of V. Take some nonzero $m \in M$. By closure under the ring action, $x.m \in M$. I claim both m and $x.m$ are nonzero and orthogonal, in the sense that

$$x.m = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} m_2 \\ -m_1 \end{bmatrix} \text{ and } (x.m)^T m = \begin{bmatrix} m_1 & m_2 \end{bmatrix} \begin{bmatrix} m_2 \\ -m_1 \end{bmatrix} = 0.$$

So $x.m$ and m span $\mathbf{R}^2$, which gives $M \supset \mathbf{R}^2$. □

[1, No. 10.1.19]. *Given.* Say we have a linear transformation $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ with associated matrix

$$A_T = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{with respect to the standard basis.}$$

As before, consider $V = \mathbf{R}^2$ as an $\mathbf{R}[x]$-module where x acts by the linear transformation $x.v = Av$.

To prove. If M is a submodule of V, then M is trivial, all of V, the x-axis, or the y-axis.

Proof. Let M be a nontrivial proper submodule of V. In particular, M is an $\mathbf{R}$-linear proper subspace, and as an $\mathbf{R}$-vector space, M has dimension exactly 1. It follows that M is of the form $\{\alpha v : \alpha \in \mathbf{R}\}$ for some nonzero $v \in M$. Consider Av, which has exactly one nonzero component. Because v is an eigenvector of the matrix A, it must be that v has exactly one nonzero component. So M is either the x- or y-axis. $\square$

[1, No. 10.1.20]. *Given.* Let $F = \mathbf{R}$, let $V = \mathbf{R}^2$, and let T be the linear transformation from V to V that is rotation clockwise about the origin by π radians.

To prove. Every subspace of V is an $F[x]$-submodule for this T.

Proof. Say $\mathbf{R}^2$ is an $\mathbf{R}[x]$-module with the action of $x \in \mathbf{R}[x]$ given by

$$x.v = Av \quad \text{where} \quad A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{w.r.t. the standard basis.}$$

The property of being an $\mathbf{R}[x]$-submodule is stronger than that of being an $\mathbf{R}$-linear subspace, so the submodules of $\mathbf{R}^2$ are all subspaces. Conversely, if $V \subset \mathbf{R}^2$ is a subspace, then V is an abelian group. For any $v \in V$, the image of v under the action of x is $x.v = -v$. So V is stable under the linear transformation; whence V is an $\mathbf{R}[x]$-submodule. $\square$

[1, No. 10.1.21]. *Given.* Let $n \in \mathbf{Z}^+$, $n > 1$, and R be the ring $\mathscr{M}_n(F)$ of $n \times n$ matrices from the field F. Let $M \subset \mathscr{M}_n(F)$ be

$$M = \left\{ (a_i^j) : a_i^j = 0 \text{ if } j > 1 \right\},$$

that is, the set of matrices with arbitrary elements of F in the first column and zeros elsewhere.

To prove.

- M is a submodule of R when R is considered as a left module over itself.
- M is *not* a submodule of R when R is considered as a right module.

Proof. In either case (of $\mathscr{M}_n(F)$ being a left or right module over itself), the set M is an abelian subgroup of the ring $\mathscr{M}_n(F)$.

- Now consider $\mathscr{M}_n(F)$ as a left module. Let $[a_{ij}] \in \mathscr{M}_n(F)$ and $[m_{jk}] \in M$ (with respect to the standard unit basis for F^n). Then

$$[a_{ij}][m_{jk}] = \left[\sum_{j=1}^{n} a_{ij} m_{jk} \right] \in M,$$

  as $m_{jk} = 0$ whenever the index $k > 0$. So M is closed under scalar multiplication. So M is a left submodule.
- On the other hand, consider $\mathscr{M}_n(F)$ as a right module. Then it's not necessarily true that for $[a_{jk}] \in \mathscr{M}_n(F)$ and $[m_{ij}] \in M$ that

$$[m_{ij}][a_{jk}] = \left[\sum_{j=1}^{n} m_{ij} a_{jk} \right]$$

  will be in N, e.g., when $a_{12} \neq 0$. Because M is not closed under scalar multiplication, M is *not* (right) submodule. $\square$

References

[1] D. Dummit and R. Foote, *Abstract algebra.* Prentice Hall, 2004.