ASSIGNMENT 8

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Notation. Let A be an $n \times n$ matrix and let F be a field containing the eigenvalues of A. Then $A \in \mathscr{M}_n(F)$.

[1, No. 12.3.3]. *Given.* Let $A \in \mathscr{M}_n(F)$ have eigenvalues $\lambda_1, \dots, \lambda_n$.

To prove. For each integer $k \geq 1$, the eigenvalues of A^k are $\lambda_1^k, \dots, \lambda_n^k$.

Proof. Similar matrices have the same characteristic polynomial, and therefore the eigenvalues of A are the same as the eigenvalues of the Jordan canonical form $J = PAP^{-1}$. Now because J is a direct sum of its Jordan blocks, it suffices to prove:

> If an arbitrary Jordan block J_λ has eigenvalue λ with multiplicity ℓ, then J_λ^k has eigenvalue λ^k with multiplicity ℓ.

Consider such a Jordan block J_λ, with eigenvalue λ of multiplicity ℓ. It is an upper triangular $\ell \times \ell$ matrix. Recall that the $\ell \times \ell$ upper triangular matrices form a subring of $\mathscr{M}_\ell(F)$, and that the diagonal of a product of two upper triangular matrices is the entry-wise product of the two diagonals:

$$\begin{pmatrix} a_{11} & & \\ & \ddots & \\ & & a_{\ell\ell} \end{pmatrix} \begin{pmatrix} b_{11} & & \\ & \ddots & \\ & & b_{\ell\ell} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & & \\ & \ddots & \\ & & a_{\ell\ell}b_{\ell\ell} \end{pmatrix}.$$

Therefore, because J_λ^k is a finite product of upper triangular matrices, each entry of along the diagonal of J_λ^k is λ^k. Moreover, the matrix $xI - J_\lambda^k$ is upper triangular, and because the determinant of an upper triangular matrix is the product of its diagonal entries, J_λ^k has characteristic polynomial $\prod_{i=1}^{\ell}(x - \lambda^k)$. Therefore J_λ^k has eigenvalue λ^k with multiplicity ℓ. Applying this argument to each block of the Jordan canonical form $J = PAP^{-1}$, it is apparent that $J^k = PA^kP^{-1}$ has eigenvalues $\lambda_1^k, \dots, \lambda_n^k$. $\square$

[1, No. 12.3.17]. *Given.* Let A^t be the transpose of $A \in \mathscr{M}_n(F)$.

To prove. A is similar to A^t.

Proof. Say $J = PAP^{-1}$ is the JCF of A. Because P is invertible, the transpose P^t is also invertible. Therefore A^t is similar to $J^t = (P^{-1})^t A^t P^t$. Because similarity is an equivalence relation, A is similar to A^t if and only if J is similar to J^t.

As J and J^t are block diagonal matrices, they are similar if and only if their blocks are similar up to some permutation. Thus, to show J is similar to J^t, it suffices to demonstrate that an arbitrary Jordan block J_λ is similar to its transpose J_λ^t. In particular, appealing to Theorem 12.15 and the Chinese remainder theorem, J_λ and J_λ^t are similar if they have the same *single* elementary divisor.

Let $c(x)$, $m(x)$, $c_t(x)$, and $m_t(x)$ be the characteristic and minimal polynomials for J_λ and J_λ^t respectively. We need to show

$$c(x) = m(x) = m_t(x) = c_t(x).$$

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Because J_λ is a Jordan block, it has a single elementary divisor. Therefore

$$m(x) = c(x). \tag{1}$$

As the determinant of a matrix is invariant under transposition, the characteristic polynomial $c_t(x)$ of J_λ^t is

$$c_t(x) = \det\bigl(xI - J_\lambda^t\bigr) = \det\bigl((xI - J_\lambda)^t\bigr) = \det(xI - J_\lambda) = c(x). \tag{2}$$

Now suppose $m_t(x) = \sum_k a_k x^k$ is the minimal polynomial of J_λ^t. Then

$$\sum_k a_k (J_\lambda^t)^k = m_t(J_\lambda^t) = 0. \tag{3}$$

Because $(J_\lambda^t)^k = (J_\lambda^k)^t$, transposing each term in equation (3) implies

$$\sum_k a_k (J_\lambda)^k = m_t(J_\lambda) = 0. \tag{4}$$

Because $m(x)$ is minimal among monic polynomials that J_λ satisfies, equation (4) implies

$$m(x) \mid m_t(x).$$

By Cayley-Hamilton, $m_t(x) \mid c_t(x)$. So by (2),

$$m_t(x) \mid c(x),$$

and this with (1) implies

$$m_t(x) \mid m(x).$$

Therefore, by divisibility considerations, $m(x) = m_t(x)$. (And so by degree considerations, $m_t(x) = c_t(x)$.)

We have demonstrated that J_λ and J_λ have the same elementary divisor $m(x)$. Repeating this argument for each block of J and J^t, one may show J is similar to J^t. We conclude A is similar to A^t. □

[1, No. 12.3.18]. *Given.* Suppose T is a linear transformation with characteristic polynomial $c(x) = (x-2)^2(x-3)^2$.

To exhibit. A Jordan canonical form for each possible similarity class of T.

Exhibition. There are 3 integer partitions of 3 and 2 integers partitions of 2, corresponding to the possible elementary divisors of $c(x)$. The transformation T belongs to one of the following 6 similarity classes, represented by Jordan canonical form.

$$\begin{pmatrix} 2 &&&& \\ & 2 &&& \\ && 2 && \\ &&& 3 & \\ &&&& 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 &&& \\ & 2 &&& \\ && 2 && \\ &&& 3 & \\ &&&& 3 \end{pmatrix} \begin{pmatrix} 2 & 1 &&& \\ & 2 & 1 && \\ && 2 && \\ &&& 3 & \\ &&&& 3 \end{pmatrix},$$

$$\begin{pmatrix} 2 &&&& \\ & 2 &&& \\ && 2 && \\ &&& 3 & 1 \\ &&&& 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 &&& \\ & 2 &&& \\ && 2 && \\ &&& 3 & 1 \\ &&&& 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 &&& \\ & 2 & 1 && \\ && 2 && \\ &&& 3 & 1 \\ &&&& 3 \end{pmatrix}.$$

□

[1, No. 12.3.20]. *Given.* Let p be a prime and consider the following matrices in $\mathscr{M}_p(\mathbf{F}_p)$.

$$R = \begin{pmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ & 1 & \ddots & & \\ & & \ddots & 1 & \\ & & & 1 & 1 \end{pmatrix}$$

To prove. R and J are similar in $\mathscr{M}_p(\mathbf{F}_p)$.

Proof. Notice R is in RCF and J is in JCF, each with one block corresponding to a single invariant factor. Computing $\det(xI - R)$ and $\det(xI - J)$, we find

- R has invariant factor $x^p - 1$, and
- J has invariant factor $(x-1)^p$.

Because $\mathbf{F}_p$ is a field of characteristic p, the binomial $(x-1)^p$ in $\mathbf{F}_p[x]$ expands to

$$\sum_{k=1}^{p} \binom{p}{k} x^{p-k}(-1)^k \quad \text{with} \quad \binom{p}{k} = 0 \quad \text{for all} \quad 0 < k < p.$$

Then $x^p - 1 = (x-1)^p$ in $\mathbf{F}_p[x]$. So R and J have the same invariant factor. Therefore R and J are similar in $\mathscr{M}_p(\mathbf{F}_p)$. □

[1, No. 12.3.21]. *Given.* Let $A^2 = A$ be an idempotent $n \times n$ matrix over F.

To prove. A is diagonalizable and is similar to a matrix with only 1s and 0s along the diagonal.

Proof. The minimal polynomial $m(x)$ divides $x^2 - x = x(x-1)$, as our hypothesis is that $A^2 - A = 0$. So $m(x)$ is one of the following: x, $x-1$, or $x(x-1)$. If $m(x) = x$, then *all* of the elementary divisors of A are x. In this case, $A = 0$ is diagonalizable. In the case that $m(x) = x - 1$, then *all* of the elementary divisors of A are $x - 1$, and $A = I$ is diagonalizable. Lastly, in the case that $m(x) = x(x-1)$, the invariant factors are either

$$x \mid \ldots \mid x(x-1) \quad \text{or} \quad x - 1 \mid \ldots \mid x(x-1).$$

We deduce the elementary divisors are *all* either x or $x-1$. Thus A has a JCF consisting of n Jordan blocks, each block associated to either the eigenvalue 0 or the eigenvalue 1, with multiplicity 1. □

[1, No. 12.3.22]. *Given.* Let $A \in \mathscr{M}_n(F)$ and require $A^3 = A$.

To prove. If F is any field *not* of characteristic 2, then it is possible that A is diagonalizable.

Proof. The minimal polynomial $m(x)$ of A divides $x^3 - x = x(x-1)(x+1)$. Suppose the characteristic of F is not 2. Then $m(x)$ is the product of distinct linear factors. Because $m(x)$ has no repeated roots, by Corollary 25, A is diagonalizable.

Suppose on the other hand that F has characteristic 2. Then perhaps $m(x)$ has $(x+1)^2$ as a factor. If so, because $m(x)$ has repeated roots, A is *not* diagonalizable. □

References

[1] D. Dummit and R. Foote, *Abstract algebra*. Prentice Hall, 2004.