MATH 6140: Final examination. Monday, 4 May 2009.

Put **your name** on each answer sheet. Answer **all** 3 questions.

Justify your answers in full. Formula sheets, calculators, notes and books are not permitted.

1.

(i) Express the polynomial $x^6 - 1 \in \mathbb{Q}[x]$ as a product of cyclotomic polynomials.

(ii) Let F be a finite field of characteristic different from 3. Show that the polynomial $x^2 + x + 1 \in F[x]$ has a root in F if and only if F contains a cube root of unity other than 1.

(iii) Let F be a finite field of characteristic different from 3. Show that the polynomial $x^2 - x + 1 \in F[x]$ has a root in F if and only if F contains a cube root of -1 other than -1 itself.

(iv) Determine whether the statements of (ii) and/or (iii) are true or not if F is a finite field of characteristic 3.

(v) Classify all finite fields $F = \mathbb{F}_{p^n}$ in which $x^6 - 1$ splits into linear factors over F. (Hint: consider the residue class of p^n modulo 3.)

(vi) Classify all finite fields $F = \mathbb{F}_{p^n}$ in which $x^6 - 1$ splits into *distinct* linear factors over F.

(vii) What is the splitting field of $x^6 - 1$ over $\mathbb{F}_{p^n}$?

2. The group $G = SL(2, 5)$ consists of all 2 by 2 matrices of determinant 1 over the field with 5 elements, under matrix multiplication. Find the rational and Jordan canonical forms of all elements of G, by extending the field if necessary. How many conjugacy classes does G have? Find all integers k for which G has an element of order k. (For up to 5% extra credit, calculate the order of G.)

3. Show that $f(x) = x^5 - 10x + 5$ is an irreducible polynomial over $\mathbb{Q}$. Determine the number of real roots of $f(x)$, regarded as a polynomial over $\mathbb{R}$. Prove that $f(x)$, regarded as a polynomial over $\mathbb{Q}$, is not solvable by radicals. (You may use the fact that if x is a transposition in S_5 and y is a 5-cycle, then $\langle x, y \rangle = S_5$.)
