2014 Algebra SEP $\sim$ Standard Linear Algebra Facts (by Evan Dummit)

- Suggested reading: Dummit/Foote chapter 11 (vector spaces), and chapter 12 (Jordan and rational canonical forms).
- Basic facts about vector spaces:
  - Every vector space has a basis. A linear transformation is determined by its values on a basis.
  - A set of n vectors in an n-dimensional space is linearly independent iff their determinant is nonzero.
  - A pairing $\langle \cdot, \cdot \rangle : V \times V \to F$ is called a bilinear form if it is linear in each component: $\langle v_1 + av_2, w \rangle = \langle v_1, w \rangle + a \langle v_2, w \rangle$ and the reverse.
  - A bilinear form is symmetric if $\langle x, y \rangle = \langle y, x \rangle$ for all x, y.
  - A symmetric bilinear form is called an inner product if the base field is $\mathbb{R}$ and $\langle v, v \rangle \geq 0$ for all v, with equality iff $v = 0$. One can perform Gram-Schmidt in an inner product space to establish the existence of an orthonormal basis (i.e., one with $\langle \beta_i, \beta_j \rangle = \delta_{i,j}$.)
- Basic facts about linear transformations: Let $T : V \to V$ be a linear transformation and V be a finite-dimensional F-vector space:
  - An element $\lambda \in \bar{F}$ is an eigenvalue of T with eigenvector v (where $v \in V$ is nonzero) if $Tv = \lambda v$.
  - The vectors with a given eigenvalue form a subspace of V, the λ-eigenspace.
  - The characteristic polynomial of T is $p(x) = \det(xI - T) = \prod_i (x - \lambda_i)$.
  - The minimal polynomial of T is the monic polynomial $m(x) \in F[x]$ of smallest degree such that $m(T) = 0$. This polynomial is unique.
  - (Cayley-Hamilton) The minimal polynomial divides the characteristic polynomial.
  - The eigenvalues of T are the roots of the minimal polynomial and characteristic polynomial: the only possible difference is that the eigenvalues might appear with higher multiplicity in the characteristic polynomial. In particular, if the characteristic polynomial has no repeated roots, the minimal polynomial is equal to it.
  - To calculate the minimal polynomial, first find its roots (the eigenvalues of A) by calculating the characteristic polynomial. Then one may determine the correct multiplicities of each eigenvalue in the minimal polynomial by explicitly computing polynomials in A to see whether they yield the zero matrix.
    - Example: If $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, the characteristic polynomial is x^3, so the minimal polynomial is one of x, x^2, x^3. We can see that $A^2 \neq 0$ so the minimal polynomial cannot divide x^2, so it must be x^3.
  - The dimension of the λ-eigenspace is $\leq$ the power of $(x - \lambda)$ dividing the characteristic polynomial.
  - If V is an F-vector space, the dual space V^* is defined as $\text{Hom}_F(V, F)$: it is the set of linear transformations from V to the base field F.
    - If V is finite-dimensional, then V^* is (non-canonically) isomorphic to V: if $e_1, \cdots, e_n$ is any basis of V, the corresponding "dual basis" consists of the maps $\varphi_1, \cdots, \varphi_n$ where φ_i is 1 on e_i and 0 on the other basis elements. If V is infinite-dimensional, then $\dim_F V < \dim_F V^*$.
    - There is a natural injection from V to V^{**} (which is an isomorphism if V is finite-dimensional) called the evaluation map: it sends $v \in V$ to the map in V^{**} that sends $f \in V^*$ to $f(v)$.
- Basic facts about matrices:
  - If we choose a basis for a finite-dimensional V then we obtain a matrix A associated to $T : V \to V$.
  - If A and B are matrices such that there exists an invertible matrix T with $B = TAT^{-1}$ then A and B are conjugate (or similar). Conjugating a matrix is the same as changing the basis of the underlying vector space.

- Conjugate matrices have the same eigenvalues, eigenspace dimensions, characteristic polynomial, and minimal polynomial.
- A is diagonalizable if it is conjugate to a diagonal matrix.
- A matrix is diagonalizable if and only if its minimal polynomial has no repeated roots, if and only if V has a basis of eigenvectors of A. In such a case, A is diagonalized by the matrix whose columns are the eigenvectors of A.
- (Real spectral theorem) A real symmetric matrix is diagonalizable.
- Non-diagonalizable matrices can still be conjugated into one of several simple forms; see below.

• Basic facts about the Jordan form:

- The Jordan canonical form, which requires passing to an algebraic extension of the base field containing all of the eigenvalues, says that any matrix is conjugate to a block-diagonal matrix, whose diagonal entries are $k \times k$ "Jordan block" matrices, which has λs on the diagonal, 1s in the entries directly above the diagonal, and 0s elsewhere.
  - Examples: $A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$, and $D = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ are three matrices in Jordan form with all eigenvalues equal to 2; A has one Jordan block of size 4, B has a Jordan block of size 3 and size 1, C has two Jordan block of size 2, and D has a Jordan block of size 2 and two blocks of size 1. The characteristic polynomial for each matrix is $(x-2)^4$.
- Up to rearranging the order of the blocks, the Jordan form is unique. To compute it, one finds all the eigenvectors of A and then finds chains of generalized eigenvectors above each eigenvector (a generalized eigenvector is an element in the kernel of $(\lambda I - A)^k$).
- The existence of the Jordan form follows from the structure theorem for finitely-generated modules over PIDs: if A is a matrix with entries in F (where F is algebraically closed), then $F[A]$ has an $F[x]$-module structure where x acts as multiplication by A. Then invoking the structure theorem gives the existence of the Jordan form: one obtains an isomorphism $F[A] \cong \prod_k F[x]/(x-\lambda_k)^{j_k}$, and $F[x]/(x-\lambda_k)^{j_k}$ is isomorphic to the module $F[J]$ where J is the $j_k \times j_k$ Jordan block with eigenvalue λ_k.
- The total number of times λ appears as an eigenvalue in the Jordan form is equal to its multiplicity in the characteristic polynomial.
- The total number of Jordan λ-blocks is equal to the dimension of V_λ (there is one Jordan block for each linearly-independent eigenvector).
- The maximum size of the largest Jordan λ-block is equal to the multiplicity of λ as a root of the minimal polynomial.
  - Thus we see that the minimal polynomial for A is $(x-2)^4$, while the minimal polynomial for B is $(x-2)^3$ and the minimal polynomial for both C and D is $(x-2)^2$. In particular, observe that C and D have the same characteristic and minimal polynomials but are not similar.
- There are some limitations to the Jordan form: for example, if the eigenvalues are not elements of the base field F, then the Jordan form doesn't exist over F.

• Basic facts about the rational canonical form:

- One obtains the existence of the RCF from a different version of the structure theorem for modules over PIDs, in the same manner as the existence of the Jordan form.
- The RCF, which always exists over F (hence "rational"), says that every matrix is conjugate over F to a unique block-diagonal matrix whose blocks are "companion matrices" for monic polynomials $f_1|f_2|\cdots|f_m$ where f_m is the minimal polynomial, $f_1 \cdots f_m$ is the characteristic polynomial, and where the companion matrix to $x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is the matrix whose entries are all zeroes except for 1s on the first subdiagonal and the coefficients $[-a_0, -a_1, \cdots, -a_{n-1}]$ down the last column.

    * Example: $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{pmatrix}$ is the companion matrix for $x^4 - 4x^3 - 3x^2 - 2x - 1$.

  - The RCF always exists over F, and is unique. To compute it, one performs row-reduction on the matrix $xI - A$: the non-constant diagonal entries of the result are the invariant factors of F.
  - By putting conjugate matrices into their respective RCFs, one has an immediate proof that any two matrices with coefficients in F which are conjugate over any extension field are actually conjugate over F. In particular, if all of the eigenvalues of the matrix lie in the base field, then the Jordan form is actually conjugate to A over F (i.e., one does not need to pass to an algebraic closure of F).
  - The companion matrix to $x^n + a_{n-1}x^{n-1} + \cdots + a_0$ has characteristic and minimal polynomial equal to this polynomial. (In particular, it is a good thing to remember if you ever need to write down a rational matrix with a particular characteristic polynomial!)

- Useful tricks:
  - Most problems involving bilinear forms can be solved using element-wise calculations.
  - In general it is often useful to work with matrices rather than general linear transformations. (At the least, it is simpler to write things down that way.) Of course, the underlying vector space must be finite-dimensional.
  - If given a linear transformation on a finite-dimensional vector space, think about its eigenvalues, its minimal polynomial, and its characteristic polynomial: most problems involve at least some of them in some way, even if it's not obvious immediately how. Particularly, these are of use when working with traces and determinants.
  - If anything ever happens over an algebraically-closed field (in particular, if a problem happens over the complex numbers), there is a high chance the problem will be very simple if you use the Jordan form. Even if the field is not algebraically closed, you can very often get a good idea of what is happening if you think about Jordan forms – though if it's necessary to rework the argument to avoid the Jordan form, the rational canonical form may be of use.
  - If you are trying to figure out why something is true, write down some examples and see what happens. Using Jordan-block matrices is often good for this, as they capture every kind of behavior that can occur (up to changing basis).