2014 Algebra SEP $\sim$ Fields Day 1 Problems, by E. Dummit

1. (Aug-13.2): Let K be the splitting field of $x^4 - 2$ over $\mathbb{Q}$.
   (a) Find $[K : \mathbb{Q}]$.
   (b) Give an example of an ideal I of $\mathbb{Q}[x, y]$ such that K is isomorphic to $\mathbb{Q}[x, y]/I$.
   (c) Find $\mathrm{Gal}(K/\mathbb{Q})$.

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2. (Aug-04.3):
   (a) Show that $x^4 - 2$ is irreducible over $\mathbb{Q}[i]$.
   (b) If $\sqrt[4]{2} + i$ is a root of a polynomial $f(x) \in \mathbb{Q}[x]$, show that $i\sqrt[4]{2} + i$ is also a root of $f(x)$.
   (c) Find the degree of the minimal polynomial of $\sqrt[4]{2} + i$ over $\mathbb{Q}$.

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3. (Aug-12.3):
   (a) Suppose $K, L \subseteq \mathbb{C}$ are Galois over $\mathbb{Q}$. Show that $E = KL$ is Galois over $\mathbb{Q}$.
   (b) If additionally $[K : \mathbb{Q}]$ and $[L : \mathbb{Q}]$ are coprime, show that $\mathrm{Gal}(E/\mathbb{Q}) \cong \mathrm{Gal}(K/\mathbb{Q}) \times \mathrm{Gal}(L/\mathbb{Q})$, and deduce $|E : \mathbb{Q}| = |K : \mathbb{Q}| \cdot |L : \mathbb{Q}|$.
   (c) Prove there is a subfield F of $\mathbb{C}$, Galois over $\mathbb{Q}$, with $[F : \mathbb{Q}] = 55$.

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4. (Jan-00.3): Let L/K be a finite-degree Galois extension with Galois group G, and $K \subseteq E \subseteq L$. E is said to be a "2-tower" over K if there exists a chain of fields $K = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_n = E$ with each extension of degree 2.
   (a) If G is abelian, show that E is a 2-tower over K iff $|E : K|$ is a power of 2.
   (b) Show by example that (a) is false if G is not abelian.

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5. (Aug-07.3): Let F be a field of characteristic 0 and E a finite Galois extension of F.
   (a) If $0 \neq \alpha \in E$ with $E = F[\alpha]$, show that $F[\alpha^2] \neq E$ iff there exists $\sigma \in \mathrm{Gal}(E/F)$ with $\alpha^\sigma = -\alpha$.
   (b) Prove there exists an element $\alpha \in E$ with $E = F[\alpha^2]$.

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6. (Jan-03.3): Let $F/\mathbb{Q}$ be a finite abelian Galois extension of $\mathbb{Q}$, embedded in $\mathbb{C}$. Let $\alpha \in F$ and $f(x) \in \mathbb{Q}[x]$ be its minimal monic polynomial. Assume that $|\alpha| = 1$.
   (a) Show F is closed under complex conjugation.
   (b) Show that $|\beta| = 1$ for every root β of f.
   (c) For $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, show that $|a_i| \leq 2^n$ for each i.
   (d) Show that F contains only finitely many algebraic integers having absolute value 1, and that each of these is a root of unity.

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7. (Aug-10.3): We say a polynomial $f \in \mathbb{Q}[x]$ is "special" if f is irreducible in $\mathbb{Q}[x]$, its degree is at least 2, and f splits over $\mathbb{Q}[\alpha]$ where α is some root of f in some extension of $\mathbb{Q}$.
   (a) If $f \in \mathbb{Q}[x]$ is irreducible with degree at least 2, with splitting field $L/\mathbb{Q}$ whose Galois group is abelian. Show that f is special.
   (b) If $L/\mathbb{Q}$ is finite and Galois (and not trivial), show that there exists a special polynomial f with a root in L.
   (c) Show that $x^n - 2$ is not special for any $n \geq 3$.
