2014 Algebra SEP $\sim$ Fields Day 2 Problems, by E. Dummit

1. (Jan-10.3) Let $F \subseteq E$ be finite fields where $|F| = q$ and $[E : F] = n$.

   (a) Show that every monic irreducible polynomial in $F[x]$ of degree dividing n is the minimal polynomial over F of some element of E.

   (b) Compute the product of all the monic irreducible polynomials in $F[x]$ of degree dividing n.

   (c) If $|F| = 2$, find the number of monic irreducible polynomials of degree 10 in $F[x]$.

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2. (Jan-09.3): Suppose $f(x) = x^m + 1$ is irreducible over $\mathbb{F}_p[x]$ where p is an odd prime.

   (a) Show that every root of f in a splitting field of f has multiplicative order $2m$.

   (b) Show that $2m$ divides $p^m - 1$ but does not divide $p^n - 1$ for any n with $0 < n < m$.

   (c) Show that $m \neq 4$.

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3. (Jan-13.2): Let k be a field. We say a polynomial $f(x) \in k[x]$ is "consecutive-root" if it has two roots x_0, x_1 (not necessarily in k) such that $x_1 - x_0 = 1$.

   (a) Show that there is no irreducible consecutive-root polynomial in $\mathbb{Q}[x]$.

   (b) Let p be a prime. Show that $x^p - x - 1$ is consecutive-root and irreducible in $\mathbb{F}_p[x]$.

   (c) Characterize all irreducible monic consecutive-root polynomials in $\mathbb{F}_p[x]$ of degree $\leq p$.

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4. (Jan-89.3) Prove that $x^9 - 2$ is an irreducible factor of $x^{27} - 1$ over $\mathbb{F}_7$.

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5. (Aug-08.3): Let $E \subseteq \mathbb{C}$ be the splitting field of $x^3 - 2$ over $\mathbb{Q}$.

   (a) Show that $[E : \mathbb{Q}] = 6$.

   (b) If $\alpha \in E$ and $\alpha^5 \in \mathbb{Q}$ show that $\alpha \in \mathbb{Q}$.

   (c) Show that there exists $\beta \in E$ with $\beta^2 \in \mathbb{Q}$ but $\beta \notin \mathbb{Q}$.

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6. (Aug-96.3): Let $f(x) = x^6 + 3 \in \mathbb{Q}[x]$, let α be a root of f over $\mathbb{C}$, and set $E = \mathbb{Q}[\alpha]$.

   (a) Show that E contains a primitive 6th root of unity.

   (b) Show that E is Galois over $\mathbb{Q}$.

   (c) Find the number of intermediate fields F with $\mathbb{Q} \subset F \subset E$ with $[F : \mathbb{Q}] = 3$.

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7. (Aug-09.3): Let F be a field and $f(x) \in F[x]$ irreducible with splitting field E. Choose $\alpha \in E$ with $f(\alpha) = 0$ and a positive integer n and let $g(x) \in F[x]$ irreducible polynomial with $g(\alpha^n) = 0$.

   (a) Show that $\deg(g)$ divides $\deg(f)$ and $\deg(f)/\deg(g) \leq n$.

   (b) If $\deg(f)/\deg(g) = n$ and the characteristic of F does not divide n, show E contains a primitive nth root of unity.
