2014 Algebra SEP $\sim$ Fields Day 3 Problems, by E. Dummit

1. (Jan-08.3): Let ϵ be a primitive 16th root of unity and $\alpha = \epsilon\sqrt{2}$, with $E = \mathbb{Q}[\epsilon]$ and $f(x) = x^8 + 16$; observe that $f(\alpha) = 0$.

   (a) Show that $\sqrt{2} \in \mathbb{Q}[\epsilon^2]$.

   (b) Show that $f(x)$ splits in $E[x]$.

   (c) For $G = \text{Gal}(E/\mathbb{Q})$, show that no nonidentity element of G fixes α. Conclude that $f(x)$ is irreducible in $\mathbb{Q}[x]$.

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2. (Jan-11.3): Let E/F be an extension of char-0 fields with $E = F[\alpha]$ for some α with $\alpha^p \in F$ and some prime p. Let $E^* = E[\epsilon]$ where ϵ is a primitive pth root of unity.

   (a) Show that E^* is a Galois extension of F.

   (b) If E is Galois over F, show $E = F$ or $E = E^*$.

   (c) Show by example that it is possible to have $E = E^*$ without having $\epsilon \in F$.

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3. (Jan-14.5): Let L/K be a field extension of degree 4. We say K' is intermediate between K and L if K' properly contains K and is properly contained in L.

   (a) Show that L/K has at most 3 intermediate fields.

   (b) Give an explicit example to show that there can be 3 intermediate fields between L and K.

   (c) Give an explicit example to show that there can be 0 intermediate fields between L and K.

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4. (Aug-05.3): Let p, q be distinct primes and let $\mu = q^{1/p}$ and $\nu = p^{1/q}$.

   (a) Let F be a subfield of $\mathbb{R}$ not containing μ. If $\mu^n \in F$ for some $n > 0$, show that $p|n$.

   (b) With F as in (a), show that $|F[\mu] : F| = p$.

   (c) Show that $|\mathbb{Q}[\mu + \nu] : \mathbb{Q}| = pq$.

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5. (Aug-06.3): Let $\mathbb{Q} \subseteq K \subseteq E \subseteq \mathbb{C}$ be fields with $E = \mathbb{Q}[\alpha]$ with $\alpha^n \in \mathbb{Q}$, and K is generated by all roots of unity in E. Assume E is Galois over $\mathbb{Q}$.

   (a) Show that $\text{Gal}(E/K)$ is cyclic.

   (b) If the restriction τ of complex conjugation to E is in the center of $\text{Gal}(E/\mathbb{Q})$, prove that $|\alpha|^2 \in \mathbb{Q}$.

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6. (Jan-05.3): Let F be a field and $f(x) \in F[x]$ be irreducible. Suppose E/F is an extension containing a root α of $f(x)$ such that $f(\alpha^2) = 0$. Show that f splits over E.

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7. (Aug-11.3): Let $K \subseteq F \subseteq E$ be fields with $E = F[\alpha]$, $\alpha^n \in F$ for some n, and K containing a primitive nth root of unity. Let L be a field with $K \subseteq L \subseteq E$ with $L \cap F = K$.

   (a) If L is Galois over K, show that $L = K[\beta]$ for some β with $\beta^n \in K$.

   (b) Show by example that L need not be Galois over K.
