MATH 6140: Second midterm examination. Wednesday, 18 March 2009.

Put **your name** on each answer sheet. Answer **all three** questions.

Justify all your answers in full.

Formula sheets, calculators, notes and books are not permitted.

1. Let $R = \mathbb{Z}$ and consider $M = \mathbb{Q}/\mathbb{Z}$ as an R-module in the usual way. Show that the tensor algebra $\mathcal{T}(\mathbb{Q}/\mathbb{Z})$ is isomorphic as a $\mathbb{Z}$-module to $\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z}$. Under these identifications, show that the subset of $\mathcal{T}(\mathbb{Q}/\mathbb{Z})$ given by

$$I = \{(5z, q + \mathbb{Z}) : z \in \mathbb{Z},\ q \in \mathbb{Q}\}$$

is a graded ideal of $\mathcal{T}(\mathbb{Q}/\mathbb{Z})$. Describe the quotient ring $\mathcal{T}(\mathbb{Q}/\mathbb{Z})/I$.

2. Let $n > 1$ and let $B \in M_n(\mathbb{Q})$ be the matrix all of whose entries are equal to 1. You may assume without proof that the vectors

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

form a basis of $\mathbb{Q}^n$ consisting of eigenvectors for B. Find the Jordan canonical form for B, and use it to deduce the rational canonical form for B and the minimal polynomial of B.

3. Let $F = \mathbb{Q}$ and $K = \mathbb{Q}(\sqrt{2}, i)$ be subfields of $\mathbb{C}$. Find the degree of K over F, and write down a basis for K as an F-vector space. Prove that the polynomial $x^2 - 2$ is irreducible over the field $\mathbb{Q}(i)$.