2014 Algebra SEP $\sim$ Linear Algebra Day 1 Solutions, by E. Dummit

1. (Aug-04.4): Let V be an n-dimensional vector space over K spanned by $v_0, \cdots, v_n$ where $v_0 + v_1 + \cdots + v_n = 0$. Let W be a second K-vector space and let $w_0, \cdots, w_n \in W$. Find necessary and sufficient conditions on $w_0, \cdots, w_n$ so that there exists a linear transformation $T : V \to W$ with $T(v_i) = w_i$ for $i = 0, \cdots, n$.

   Solution: Clearly, since $T(0) = 0$ we must have $w_0 + \cdots + w_n = T(v_0 + \cdots + v_n) = 0$. This condition is in fact sufficient: we have $v_0 = -(v_1 + \cdots + v_n)$, so $v_1, \cdots, v_n$ span V. But since V is n-dimensional, they are actually a basis for V. We then construct T by setting $T(v_i) = w_i$ for $i = 1, \cdots, n$ and then extending linearly to all of V. We then observe $w_0 = T(v_0) = T(-v_1 - \cdots - v_n) = -T(v_1 + \cdots + v_n) = -w_1 - \cdots - w_n$, as required.

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2. (Aug-09.4): Let V be a vector space over F and $\langle \cdot, \cdot \rangle : V \times V \to F$ be a bilinear form. For each $x \in V$ define $A_x = \{y \in V : \langle x, y \rangle = -\langle y, x \rangle\}$. Now suppose v is a fixed element of V with $\langle v, v \rangle \neq 0$.

   (a) For all $x \in V$ show that A_x is a subspace of V of codimension at most 1.

   (b) If $\text{char}(F) \neq 2$ prove that A_v is a subspace of V of codimension exactly 1.

   (c) If F is algebraically closed and $\text{char}(F) \neq 2$, show that either $\langle a, a \rangle = 0$ for every $a \in A_v$, or there exists $y \in V \backslash A_v$ with $\langle y, y \rangle = 0$.

   Solution:

   a) Fix x and let $z \in V$. If $\langle x, y \rangle + \langle y, x \rangle = 0$ for all $y \in V$ then $A_v = V$; otherwise choose z such that $\langle x, z \rangle + \langle z, x \rangle \neq 0$. Then for any $a \in F$ and $y \in V$ we have $\langle x, y + az \rangle + \langle y + az, x \rangle = a(\langle x, z \rangle + \langle z, x \rangle) + \langle y, z \rangle + \langle z, y \rangle$, so in particular if we choose $a = -\dfrac{\langle y, z \rangle + \langle z, y \rangle}{\langle x, z \rangle + \langle z, x \rangle}$ we see that $\langle x,\, y + az \rangle + \langle y + az,\, x \rangle = 0$. Therefore, we see that in the quotient space V/z, the image of A_v is all of V, so by the first isomorphism theorem that means A_x has codimension 1.

   a-alt) Let $T : V \to F$ be defined via $y \mapsto \langle x, y \rangle + \langle y, x \rangle$. Then T is a linear transformation and by definition, A_x is contained the kernel of T, so by the first isomorphism theorem, $T/A_x \cong \text{im}(T)$, which has dimension 0 or 1 since it is an F-subspace of F.

   b) Since the characteristic is not 2 we have $2\langle v, v \rangle \neq 0$ which eliminates the codimension-0 case above.

   c) Suppose that there exists $a \in A_v$ with $\langle a, a \rangle \neq 0$: we then have $\langle a, v \rangle + \langle v, a \rangle = 0$, so then for $t \in F$ we have $\langle a + tv,\, a + tv \rangle = \langle a, a \rangle + t^2 \langle v, v \rangle$. Since F is algebraically closed and neither $\langle v, v \rangle$ nor $\langle a, a \rangle$ is zero, we see that there is a nonzero γ with $\langle v, v \rangle \gamma^2 + \langle a, a \rangle = 0$: then $\langle a + \gamma v,\, a + \gamma v \rangle = 0$, and we see that $y = a + \gamma v$ is not in A_v by either the observation in part (b) or the explicit calculation $\langle y, v \rangle + \langle v, y \rangle = 2\gamma \langle v, v \rangle \neq 0$.

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3. (Jan 89.4): Let V be a finite-dimensional F-vector space.

   (a) If $T : V \to V$ is a linear transformation with $T^2 = T$, show that V is the direct sum $V = V_0 \oplus V_1$ where $V_0 = \{v : T(v) = 0\}$ and $V_1 = \{v : T(v) = v\}$.

   (b) If $|F| = q$ and $\dim_F V = 3$, determine in terms of q the number of linear transformations T with $T^2 = T$.

Solution:

a) Observe that $\varphi : V \to V_0 \oplus V_1$ defined via $v \mapsto (v - T(v), T(v))$ is a homomorphism, since $T(v - T(v)) = T(v) - T^2(v) = 0$ so $v - T(v) \in V_0$ and $T(v) \in V_1$, and it has an inverse homomorphism given by $\psi : V_0 \oplus V_1 \to V$ defined via $(x, y) \mapsto x + y$.

a-alt) Since $T^2 - T = 0$, we see that the minimal polynomial $m(x)$ must divide $x^2 - x$: thus, the only eigenvalues of T are 0 and 1. Furthermore, since the minimal polynomial does not have repeated roots, T is diagonalizable, and the diagonalization of T has only zeroes and ones on the diagonal. Hence V is the direct sum of the 0-eigenspace of T and the 1-eigenspace of T, as desired.

Remark A linear transformation with $T^2 = T$ is called a projection; part (a) shows that such a map is in fact simply projection onto some subspace (namely, the image of T).

b) By part (a), the map T is uniquely defined by the pair of subspaces (V_0, V_1). If $\dim(V_0) = 0$ then there is clearly only 1 choice. If $\dim(V_0) = 1$ then there are $\dfrac{(q^3-1)(q^3-q)(q^3-q^2)}{(q-1)(q^2-q)(q^2-1)} = q^4 + q^3 + q^2$ possible choices for the pair (V_0, V_1): we choose three basis elements for V sequentially (the first to generate V_0 and the others to generate V_1; this gives the numerator by the usual calculation of $|GL_3(\mathbb{F}_q)|$) – but there are $q - 1$ different bases that yield the same V_0 and $(q^2 - q)(q^2 - 1)$ different bases that yield the same V_1 (this gives the denominator). The calculations are the same when $\dim(V_0) = 2$ and $\dim(V_0) = 3$ – interchange V_0 and V_1 – so the answer is $2(q^4 + q^3 + q^2 + 1)$.

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4. (Jan 94.4): Let V be a vector subspace of $M_n(\mathbb{C})$. If every nonzero matrix in V is invertible, show $\dim_{\mathbb{C}} V \le 1$.

Solution: Suppose that A and B are linearly independent (invertible) matrices in V; then we want to find a linear combination $sA+tB$ which has determinant zero. Consider $f(s,t) = \det(sA+tB)$ as a function of s and t: it will be a homogeneous polynomial of degree n in s and t, and we see that $f(s,0) = s^n \det(A)$ and $g(0,t) = t^n \det(B)$; since these coefficients are both nonzero, we see that the polynomial $f(1,t)$ is therefore of positive degree, hence it has a zero λ over $\mathbb{C}$: then $A+\lambda B$ has determinant zero, contradiction.

Remark For a more difficult challenge, try this problem with $M_n(\mathbb{R})$ instead of $M_n(\mathbb{C})$. (If n is odd, then the dimension cannot be bigger than 1, but if n is even, the dimension can be larger.)

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5. (Aug-13.4) Let $T_1, \cdots, T_k$ be a collection of linear transformations which act irreducibly on a finite-dimensional $\mathbb{C}$-vector space V (i.e., such that there is no nontrivial proper subspace W such that $T_i W \subseteq W$ for all i). Suppose $S : V \to V$ is a linear transformation which commutes with each of $T_1, \cdots, T_k$. Show that S is a scalar operator.

Solution: Since we are over an algebraically closed field, S has an eigenvalue $\lambda \in \mathbb{C}$. Then if W is the λ-eigenspace of S and $w \in W$, we have $ST_i w = T_i S w = \lambda T_i w$, so $T_i w \in W$. Hence $T_i W \subseteq W$ for all the T_i, so since $W \neq 0$ we see $W = V$, meaning that S acts on V as multiplication by λ.

Remark This result is false if we do not assume that the eigenvalues of S are in the base field of V. A counterexample is given by $S = T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ over $\mathbb{Q}$: this matrix does not map any 1-dimensional subspace into itself, since its eigenvalues are not in $\mathbb{Q}$.

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6. (Aug 88.8): Let V be n-dimensional over F and $T : V \to V$. Let k be an integer with $1 \le k < n$ and suppose that $T(W) \subseteq W$ for all subspaces W with $\dim_F W = k$. Prove that T is multiplication by some scalar.

Solution: If $k > 1$ then take any $k-1$-dimensional subspace W' and then extend a basis of W' to a basis of W, including the vectors v_1 and v_2. Then the two k-dimensional subspaces $\langle W', v_1\rangle$ and $\langle W', v_2\rangle$ are both sent inside themselves by T, hence so is their intersection W'. We conclude that the property then holds for all $k-1$-dimensional subspaces too, so we may now assume $k = 1$. The property then means every vector is an eigenvector for T: then if v and w are any nonzero linearly-independent vectors with $Tv = \lambda v$, $Tw = \mu w$, $T(v+w) = \delta(v+w)$ we get $(\lambda - \delta)v + (\mu - \delta)w = 0$ so independence implies $\lambda = \mu = \delta$, so all vectors have eigenvalue λ.

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7. (Jan 96.4): Let V be a K-vector space and $S, T : V \to V$ such that S is one-to-one, $T(v) = 0$ for some $v \neq 0$, and $TS - ST = S$.

   (a) For every $n \ge 0$ show that $S^n(v)$ is an eigenvector for T and find its corresponding eigenvalue.

   (b) If $\mathrm{char}(K) = 0$ show $\dim_K V = \infty$.

   (c) If $\mathrm{char}(K) = p$ show that $\dim_K V$ can be finite, and give a concrete example when $p = 3$.

Solution:

a) The eigenvalue is n by induction on n. For the base case we have $TS = ST + S$ so $TSv = (ST+S)v = Sv$. For the inductive step we get $TS^n v = (ST+S)S^{n-1}v = STS^{n-1}v + S^n v = S(n-1)S^{n-1}v + S^n v = n\, S^n v$.

b) Eigenvectors with different eigenvalues are linearly independent, so $v, Sv, \cdots, S^k v$ are linearly independent for every k, which means V cannot be finite-dimensional.

c) By part (a) the vectors $v, Sv, \cdots, S^{p-1}v$ are linearly independent. If we try taking these to be a basis for V, then T is diagonal with diagonal entries $\{0, 1, 2, \cdots, p-1\}$, and S is a matrix with 1s in the first subdiagonal and something in the last column. If we let the last column of S be $[a_1, \cdots, a_p]$ then when we do the multiplication we will eventually see $ST - TS - S = B$ where the first $p-1$ columns of B are zeroes, and the last column of B is $[-pa_1, -(p-1)a_2, \cdots, -2a_{p-1}, -a_p]$: thus we can take $a_2 = a_3 = \cdots = a_p = 0$ and $a_1 = 1$. (Alternatively, we could observe that $S^p v$ has eigenvalue $p = 0$, and thus would have to be a multiple of v.)
