1. (Jan-99.4) Let V be finite-dimensional over F algebraically closed, and let $ST = TS$, where the characteristic polynomial of S has distinct roots.

   (a) Show that every eigenvector of S is an eigenvector for T.

   (b) If T is nilpotent, prove that $T = 0$.

   Solution:

   a) Since the characteristic polynomial of S has distinct roots, all of the eigenspaces are 1-dimensional. Now suppose $Sv = \lambda v$: then $STv = TSv = \lambda(Tv)$, so Tv is an eigenvalue of S also with eigenvalue λ, so it is a multiple of v, say $Tv = \mu v$. So v is an eigenvector of T.

   b) Since T has n linearly independent eigenvectors by (a), we see T is diagonalizable, hence its diagonalization must be the zero matrix (since that is the only nilpotent diagonal matrix). Hence T is also the zero matrix.

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2. (Jan 12.4): Let V be a finite-dimensional $\mathbb{C}$-vector space.

   (a) If S, T are commuting linear operators on V, show that each eigenspace of S is mapped onto itself by T.

   (b) If $A_1, \cdots, A_k$ are operators which commute pairwise, show they have a common eigenvector in V.

   (c) If V has dimension n, show there exists a nested sequence of subspaces $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ where $\dim(V_j) = j$ and each V_j is mapped onto itself by each of the operators $A_1, \cdots, A_k$.

   Solution:

   a) If $Sv = \lambda v$ then $S(Tv) = TSv = \lambda(Tv)$ so Tv is also in the λ-eigenspace of S.

   b) Induction on k: it is vacuously true for 1 operator. For the inductive step, let λ be any eigenvalue of A_k and let W be the λ-eigenspace of A_k (which is nonzero). By part (a), each of the operators $A_1, \cdots, A_{k-1}$ is a well-defined linear transformation on W, and they all commute with each other. So by the inductive hypothesis they have a common eigenvector w, which is also an eigenvector for A_k by construction.

   c) Inductive construction: Let $V_1 = \langle w \rangle$ where w is the eigenvector from part (b). Now suppose we have constructed V_{j-1} and consider the quotient space V/V_{j-1}. By hypothesis $A_1, \cdots, A_k$ are commuting linear operators on V/V_{j-1} so by part (b) again, they have a common eigenvector $\bar{v} = v + V_{j-1}$. Then we can take $V_j = V_{j-1} \oplus \langle v \rangle$. It is then immediate that $A_i : V_j \to V_j$ and that V_j is j-dimensional (since $\bar{v}$ is nonzero in V/V_{j-1}).

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3. (Aug-05.4): Let F be a field and A, B nonsingular 3×3 matrices over F. Suppose $B^{-1}AB = 2A$.

   (a) Find the characteristic of F.

   (b) If n is a positive or negative integer not divisible by 3, prove that A^n has trace 0.

   (c) Prove that the characteristic polynomial of A is $X^3 - a$ for some $a \in F$.

Solution:

a) We have $\det(A) = \det(2A) = 8\det(A)$ so since A is nonsingular we see that $8 = 1$ in F, so the characteristic is 7.

b) We have $B^{-1}A^nB = 2^nA^n$ and trace is unaffected by conjugation, so $(2^n - 1) \cdot \mathrm{tr}(A^n) = 0$. For $n \neq 0$ mod 3, $2^n - 1 \neq 0$ mod 7, so it is invertible in F; dividing by it gives $\mathrm{tr}(A^n) = 0$.

c) By part (b), we see that $\mathrm{tr}(A) = \mathrm{tr}(A^2) = 0$. If α, β, γ are the eigenvalues of A, then $\mathrm{tr}(A) = \alpha + \beta + \gamma$ and $\mathrm{tr}(A^2) = \alpha^2 + \beta^2 + \gamma^2$, so we can write $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{1}{2}\mathrm{tr}(A)^2 - \frac{1}{2}\mathrm{tr}(A^2) = 0$. The characteristic polynomial is then $(x-\alpha)(x-\beta)(x-\gamma) = x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\alpha\gamma+\beta\gamma)x - \alpha\beta\gamma = x^3 - \det(A)$.

alt) If α, β, γ are the eigenvalues of A, then since $2A$ is conjugate to A, $2\alpha, 2\beta, 2\gamma$ are also the eigenvalues of A, meaning that they are α, β, γ, possibly permuted. Since $\alpha \neq 0$ we see $\alpha \neq 2\alpha$ (and the same for β, γ), so it is easy to see that the only possibility is that the permutation is a 3-cycle. Thus, up to swapping β and γ, we have $\beta = 2\alpha$, $\gamma = 2\beta$, and $\alpha = 2\gamma$, meaning that the eigenvalues are $\alpha, 2\alpha, 4\alpha$, where $8\alpha = \alpha$. For (a), since $\alpha \neq 0$ we see the characteristic is 7. For (b), $\mathrm{tr}(A^n) = \alpha^n(1^n + 2^n + 4^n)$, and $1^n + 2^n + 4^n$ is 0 mod 7 for any n not divisible by 3. For (c), the characteristic polynomial is $(x-\alpha)(x-2\alpha)(x-4\alpha) = x^3 - (7\alpha)x^2 + (14\alpha)x - 8\alpha^3 = x^3 - 8\alpha^3$. (And the constant term is in F since it is just -1 times the determinant of A.)

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4. (Aug-08.4/85.4a): Let S, T, M be $n \times n$ matrices over $\mathbb{C}$ with $SM = MT$.

   (a) If $f(x)$ is the minimal polynomial of T, show $f(S)M = 0$.

   (b) If $M \neq 0$, deduce that S and T have a common eigenvalue.

   (c) Now let $S = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$, $T = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$. Find a nonzero M with $SM = MT$ and show that any such M cannot be invertible.

Solution:

a) We have $S^nM = MT^n$, so $f(S)M = Mf(T) = M \cdot 0 = 0$.

b) If $M \neq 0$ let $Mv \neq 0$: then $Mv \in \ker f(S)$, so $\det(f(S)) = 0$. If we write $f(x) = \prod_i (x - \lambda_i)$ where the λ_i are the eigenvalues of T, then $\det(f(S)) = \prod_i \det(S - \lambda_i I)$, hence $\det(S - \lambda_i I)$ is zero for some λ_i – but this means λ_i is an eigenvalue of both S and T.

c) Routine computation shows we can take $M = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. M cannot be invertible since if it were, S and T would be conjugate, but they're not since their eigenvalues are clearly different.

Remark In fact, the converse of (b) is also true: If S and T have a common eigenvalue, then such a nonzero matrix M does necessarily exist. To see this, observe that we can conjugate S and T independently (conjugate all three matrices, and then rescale M): so change variables to replace S with its Jordan form and T with the transpose of its Jordan form. We can then take M to be the diagonal matrix with a 1 in the first entry and 0s elsewhere.

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5. (Aug-06.4): Let V be a nonzero finite dimensional vector space over F and let $T : V \to V$ be a linear transformation. We say T is regular if its characteristic polynomial and minimal polynomial are equal.

   (a) If there exists a vector $v \in V$ such that V is spanned by $v, T(v), T^2(v), \cdots$, prove that T is regular.

   (b) Assume that T is regular and let W be a subspace with $T(W) \subseteq W$. Show that T_W, the restriction of T to W, and $T_{V/W}$, the induced action of T on V/W, are both regular.

Solution:

a) Let V be n-dimensional. We need to show that T does not satisfy a nonzero polynomial of degree less than n, so suppose it did, say $f(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$. Then in particular, $f(T)(v)$ is 0, and we can explicitly write $f(T)(v) = \left[T^{n-1} + a_{n-2}T^{n-2} + \cdots + a_1 T + a_0\right] v = a_{n-1}T^{n-1}(v) + a_{n-2}T^{n-2}(v) + \cdots + a_1 T(v) + a_0 v$. But $v, Tv, \ldots, T^{n-1}v$ are linearly-independent, as otherwise their span (which is the same as the span of $v, Tv, \ldots$ since any power above $T^{n-1}v$ is already dependent with the lower powers) would not be all of V. Therefore, $a_{n-1} = \cdots = a_0 = 0$, and f is the zero polynomial.

b) Choose a basis to make T block-upper-triangular, say $T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ where A corresponds to T_W and C corresponds to $T_{V/W}$. Then the characteristic polynomial $p_V(x)$ of T is $\det(xI - T) = \begin{vmatrix} xI - A & -B \\ 0 & xI - C \end{vmatrix} = \det(xI - A) \cdot \det(xI - C)$, which is the product of the characteristic polynomials $p_W(x)$ of T_W and $p_{V/W}(x)$ of $T_{V/W}$. So we have $m_W(x) \cdot m_{V/W}(x) = p_W(x) \cdot p_{V/W}(x)$, $m_W(x) | p_W(x)$, and $m_{V/W}(x) | p_{V/W}(x)$, hence equality must hold and both T_W and $T_{V/W}$ are regular.

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6. (Jan-95.4) Let A be an $n \times n$ matrix over an algebraically closed field K and let $K[A]$ denote the K-linear span of $I, A, A^2, \cdots$. Show that A is diagonalizable iff $K[A]$ contains no nonzero nilpotent element.

Solution: A is diagonalizable iff the minimal polynomial $m(x)$ of A has distinct roots. We see by definition of $m(x)$ that $K[A] \cong K[x]/m(x)$, and since K is algebraically closed we can factor to get $m(x) = \prod (x - \lambda_i)^{k_i}$. We claim that any nilpotent element in $K[A]$ must be a multiple of $q(x) = \prod (x - \lambda_i)$; to see this merely observe that if $\prod (x - \beta)$ is nilpotent then the minimal polynomial must divide some power of it, hence each root of $m(x)$ divides it hence $q(x)$ divides it. Conversely, $q(x) = \prod (x - \lambda_i)$ is indeed nilpotent, and it will be zero in $K[x]/m(x)$ if and only if all eigenvalue multiplicities are equal to 1.

Note In fact K does not even need to be algebraically closed as long as it is characteristic zero, for $q(x)$ above will actually have coefficients in K: if $f(x)$ is the characteristic polynomial of A, then the expression $q = f / \gcd(f, f')$ shows that q is a quotient of polynomials with coefficients in K.

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7. (Aug-03.4): Let A be a real $n \times n$ matrix. We say A is a "difference of two squares" if there exist real $n \times n$ matrices B and C for which $BC = CB = 0$ and $A = B^2 - C^2$.

   (a) If A is diagonal, show it is a difference of two squares.

   (b) If A is symmetric, show it is a difference of two squares.

   (c) If A is a difference of two squares with B and C as above, if B has a nonzero real eigenvalue, prove that A has a positive real eigenvalue.

Solution: Observe that we can conjugate each of A, B, C by any invertible matrix P and preserve the "difference of two squares" property. We will use this fact freely.

a) Reorder the basis to put $A = \begin{pmatrix} D & 0 \\ 0 & -E \end{pmatrix}$ where D and E are diagonal matrices with nonnegative entries. Then we can take $B = \begin{pmatrix} D^{1/2} & 0 \\ 0 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 0 \\ 0 & E^{1/2} \end{pmatrix}$.

b) Real symmetric matrices are diagonalizable, so by the observation we can reduce to part (a) to see symmetric matrices are also a difference of two squares.

c) Let $\lambda \neq 0$ be the eigenvalue of B with eigenvector v. Then $0v = CBv = \lambda(Cv)$, so $Cv = 0$. Then $Av = B^2 v - C^2 v = \lambda^2 v$ so A has an eigenvalue $\lambda^2 > 0$.

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8. (Aug-12.4): Let V be an n-dimensional K-vector space and $T : V \to V$.

   (a) Suppose there exists $v \in V$ such that V is spanned by $v, Tv, T^2v, \cdots$. Prove that the minimal polynomial of T equals the characteristic polynomial of T.

   (b) As a partial converse, suppose the characteristic polynomial of T has distinct roots in K. Prove that there exists $v \in V$ such that V is spanned by $v, Tv, T^2v, \cdots$.

Solution:

a) If $m(x)$ is the minimal polynomial of T, then for any vector w we know that $m(T)w = 0$, so in particular $m(T)v = 0$. But since $v, Tv, \cdots, T^{n-1}v$ are linearly independent, we see that the degree of $m(x)$ must be $\geq n$. But $m(x)$ divides the characteristic polynomial, which has degree n, so they must be equal since they are both monic.

b) Let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of T with corresponding basis of eigenvectors $v_1, \cdots, v_n$; by the given assumptions we know that the v_i are linearly independent. We claim that $v = v_1 + \cdots + v_n$ has the desired property: to see this, suppose that $p(T)v = 0$: then we see that $0 = p(T)v = \sum p(\lambda_i)v_i$, so since the v_i are linearly independent we see that $p(\lambda_i) = 0$ for each λ_i – now since the eigenvalues of T are distinct, we see that p must be divisible by the characteristic polynomial of T hence have degree $\geq n$. Thus, if p is any polynomial of degree $< n$ we see that $p(T)v \neq 0$, so $v, Tv, \cdots, T^{n-1}v$ are linearly independent, hence must span V.

b-alt) Alternatively, since the characteristic polynomial of T has distinct roots in K, this means T is diagonal with respect to an appropriate K-basis of V: this follows by observing that the Jordan form of (any) matrix corresponding to T is diagonal, and then using the fact that if two matrices with K-coefficients are conjugate over $\bar{K}$ then they are conjugate over K – this follows from properties of the rational canonical form. Then given such a diagonal matrix, it is easy to verify that $v = [1, 1, 1, \cdots, 1]$ has the desired property.