2014 Algebra SEP $\sim$ Linear Algebra Day 1 Problems, by E. Dummit

1. (Aug-04.4): Let V be an n-dimensional vector space over K spanned by $v_0, \cdots, v_n$ where $v_0+v_1+\cdots+v_n = 0$. Let W be a second K-vector space and let $w_0, \cdots, w_n \in W$. Find necessary and sufficient conditions on $w_0, \cdots, w_n$ so that there exists a linear transformation $T : V \to W$ with $T(v_i) = w_i$ for $i = 0, \cdots, n$.

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2. (Aug-09.4): Let V be a vector space over F and $\langle \cdot, \cdot \rangle : V \times V \to F$ be a bilinear form. For each $x \in V$ define $A_x = \{y \in V : \langle x, y\rangle = -\langle y, x\rangle\}$. Now suppose v is a fixed element of V with $\langle v, v\rangle \neq 0$.

   (a) For all $x \in V$ show that A_x is a subspace of V of codimension at most 1.

   (b) If $\mathrm{char}(F) \neq 2$ prove that A_v is a subspace of V of codimension exactly 1.

   (c) If F is algebraically closed and $\mathrm{char}(F) \neq 2$, show that either $\langle a, a\rangle = 0$ for every $a \in A_v$, or there exists $y \in V \backslash A_v$ with $\langle y, y\rangle = 0$.

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3. (Jan-89.4): Let V be a finite-dimensional F-vector space.

   (a) If $T : V \to V$ is a linear transformation with $T^2 = T$, show that V is the direct sum $V = V_0 \oplus V_1$ where $V_0 = \{v : T(v) = 0\}$ and $V_1 = \{v : T(v) = v\}$.

   (b) If $|F| = q$ and $\dim_F V = 3$, determine in terms of q the number of linear transformations T with $T^2 = T$.

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4. (Jan-94.4): Let V be a vector subspace of $M_n(\mathbb{C})$. If every nonzero matrix in V is invertible, show $\dim_{\mathbb{C}} V \leq 1$.

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5. (Aug-13.4) Let $T_1, \cdots, T_k$ be a collection of linear transformations which act irreducibly on a finite-dimensional $\mathbb{C}$-vector space V (i.e., such that there is no nontrivial proper subspace W such that $T_i W \subseteq W$ for all i). Suppose $S : V \to V$ is a linear transformation which commutes with each of $T_1, \cdots, T_k$. Show that S is a scalar operator.

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6. (Aug-88.8): Let V be n-dimensional over F and $T : V \to V$. Let k be an integer with $1 \leq k < n$ and suppose that $T(W) \subseteq W$ for all subspaces W with $\dim_F W = k$. Prove that T is multiplication by some scalar.

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7. (Jan-96.4): Let V be a K-vector space and $S, T : V \to V$ such that S is one-to-one, $T(v) = 0$ for some $v \neq 0$, and $TS - ST = S$.

   (a) For every $n \geq 0$ show that $S^n(v)$ is an eigenvector for T and find its corresponding eigenvalue.

   (b) If $\mathrm{char}(K) = 0$ show $\dim_K V = \infty$.

   (c) If $\mathrm{char}(K) = p$ show that $\dim_K V$ can be finite, and give a concrete example when $p = 3$.
