2014 Algebra SEP $\sim$ Linear Algebra Day 3 Problems, by E. Dummit

1. (Jan-00.4): Let $A \in M_n(\mathbb{C})$ and assume that A has rank 1.

   (a) What are the possible Jordan canonical forms for A?

   (b) For each of the forms in (a), find the characteristic and minimal polynomial of A.

---

2. (Jan-13.5): Let W_n be the set of $n \times n$ complex matrices C such that the equation $AB - BA = C$ has a solution in $n \times n$ matrices A and B.

   (a) Show that W_n is closed under scalar multiplication and conjugation.

   (b) Show that the identity matrix is not in W_n.

   (c) Give a complete description of W_2.

---

3. (Jan-11.4): Let V be a finite-dimensional $\mathbb{C}$-vector space and $T : V \to V$.

   (a) Suppose W is a subspace with $T(W) \subseteq W$. Show that the characteristic polynomial $f_S(x)$ of $S = T|_W$ divides the characteristic polynomial $f_T(x)$ of T on V.

   (b) Let λ be a root of $f_T(x)$ of multiplicity m and $V_\lambda = \{v \in V : T(v) = \lambda v\}$. Show that $1 \le \dim_{\mathbb{C}} V_\lambda \le m$.

   (c) Find (V, T, λ) such that λ has multiplicity 5 as a root of $f_T(x)$ but $\dim_{\mathbb{C}} V_\lambda = 1$.

---

4. (Jan-14.2): Let F be a field and n a positive integer. Let $A \in M_{n\times n}(F)$ such that $A^n = 0$ but $A^{n-1} \ne 0$. Show that any $B \in M_{n\times n}(F)$ that commutes with A is contained in the F-linear span of $I, A, A^2, \dots, A^{n-1}$.

---

5. (Aug-82.7): Let $A \in M_n(\mathbb{C})$. Show that the following are equivalent:

   (a) The ranks of A and A^2 are equal.

   (b) The multiplicity of 0 as a root of the minimal polynomial of A is at most 1.

   (c) There is an $n \times n$ matrix X such that $AXA = A$, $XAX = X$, $AX = XA$.

---

6. (Aug-06.5): Let $F = \mathbb{F}_q$ and $M_2(F)$ be the ring of 2×2 matrices over F.

   (a) If $A \in M_2(F)$ has equal eigenvalues in the algebraic closure of F, show that the eigenvalues of A belong to F.

   (b) Determine the number of nonzero nilpotent matrices in $M_2(F)$ as a function of q.

---

7. (Jan-10.4): Let V be finite-dimensional over F and $T : V \to V$, with characteristic polynomial $f(x) \in F[x]$.

   (a) Show that $f(x)$ is irreducible in $F[x]$ iff there are no proper nonzero subspaces W of V with $T(W) \subseteq W$.

   (b) If $f(x)$ is irreducible and $\text{char}(F) = 0$, show that T is diagonalizable over $\bar{F}$.

---

8. (Jan-05.4): Let F be an algebraically-closed field and $M_n(F)$ be the ring of $n \times n$ matrices over F. Describe those matrices $X \in M_n(F)$ such that all matrices that commute with X are diagonalizable.
