INTRODUCTION TO MODULE THEORY: BASIC DEFINITIONS AND EXAMPLES

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[1, No. 10.1.1]. Given. R is a unital ring and M is a left R-module.
To prove. $0m = 0$ and $(-1)m = -m$ for all $m \in M$.

[1, No. 10.1.3]. Given. Say $rm = 0$ for some $r \in R$ and some $m \in M$ with $m \neq 0$.
To prove. There is no $s \in R$ such that $sr = 1$.

[1, No. 10.1.4]. Given. Let M be the modules R^n and let $\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_n$ be left ideals of R.
To prove. Both of the following are submodules of M:

a. $P = \{(x_1, x_2, \dots, x_n) : x_i \in \mathfrak{a}_i\}$,
b. $N = \{(x_1, x_2, \dots, x_n) : x_i \in R \text{ and } \sum_i x_i = 0\}$.

[1, No. 10.1.5]. Given. Consider a left ideal $\mathfrak{a}$ of R. Let

$$\mathfrak{a}M = \left\{ \sum_{\text{finite}} a_i m_i : a_i \in \mathfrak{a}, m_i \in M \right\}.$$

To prove. We have $\mathfrak{a}M$ as a submodule of M.

[1, No. 10.1.6]. Given. Let M be a module over R and $\{N_i\}$ be a nonempty collection of submodules.
To prove. The intersection $\bigcap_i N_i$ is a submodule of M.

[1, No. 10.1.8]. Given. An element m of the R-module M is called a *torsion element* if $rm = 0$ for some nonzero element $r \in R$. The set of torsion elements is denoted

$$\mathrm{Tor}\,(M) = \{m \in M : rm = 0 \text{ for some nonzero } r \in R\}.$$

To prove.

a. If R is an integral domain, then $\mathrm{Tor}\,(M)$ is a submodule of M (called the *torsion submodule*).
b. If R has zero divisors, then every nonzero R-module has nonzero torsion elements.

[1, No. 10.1.9]. Given. If N is a submodule of M, the *annihilator of N in R* is defined to be

$$\{r \in R : rn = 0 \text{ for all } n \in N\}.$$

To prove. The annihilator of N in R is a 2-sided ideal of R.

[1, No. 10.1.15]. Given. Say M is a finite abelian group. M is naturally a $\mathbf{Z}$-module.
To prove. This action cannot be extended to make M into a $\mathbf{Q}$-module.

[1, No. 10.1.18]. Given. Let $F = \mathbf{R}$, let $V = \mathbf{R}^2$, and let T be the linear transformation from V to V that is rotation clockwise about the origin by $\pi/2$ radians.
To prove. V and 0 are the only $F[x]$-submodules for this T.

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[1, No. 10.1.19]. Given. Let $F = \mathbf{R}$, let $V = \mathbf{R}^2$, and let T be the linear transformation from V to V that is projection onto the y-axis.

To prove. V, 0, the x-axis and the y-axis are the only $F[x]$-submodules for this T.

[1, No. 10.1.20]. Given. Let $F = \mathbf{R}$, let $V = \mathbf{R}^2$, and let T be the linear transformation from V to V that is rotation clockwise about the origin by π radians.

To prove. Every subspace of V is an $F[x]$-submodule for this T.

[1, No. 10.1.21]. Given. Let $n \in \mathbf{Z}^+$, $n > 1$, and R be the ring $\mathscr{M}_n(F)$ of $n \times n$ matrices from the field F. Let $M \subset \mathscr{M}_n(F)$ be

$$M = \left\{ (a_i^j) : a_i^j = 0 \text{ if } j > 1 \right\},$$

that is, the set of matrices with arbitrary elements of F in the first column and zeros elsewhere.

To prove.

- M is a submodule of R when R is considered as a left module over itself.
- M is *not* a submodule of R when R is considered as a right module.

References. [1] D. Dummit and R. Foote, *Abstract algebra.* Prentice Hall, 2004.