MATH 6140: First midterm examination. Wednesday, 11 February 2009.

Put **your name** on each answer sheet. Answer **all three** questions.

Justify all your answers in full.

Formula sheets, calculators, notes and books are not permitted.

Consider the short exact sequence of $\mathbb{Z}$-modules

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\iota} \mathbb{Q} \xrightarrow{\pi} \mathbb{Q}/\mathbb{Z} \longrightarrow 0\,,$$

where ι and π are the usual embedding and projection maps, respectively.

1.

(i) Show that $\mathbb{Q}/\mathbb{Z}$ is not finitely generated as a $\mathbb{Z}$-module.

(ii) Deduce that $\mathbb{Q}$ is not finitely generated as a $\mathbb{Z}$-module.

2.

(i) Prove that the given short exact sequence does not split.

(ii) What two pieces of information does the answer to 2(i) give us concerning whether the modules $\mathbb{Z}$ and $\mathbb{Q}/\mathbb{Z}$ are (a) projective $\mathbb{Z}$-modules and/or (b) injective $\mathbb{Z}$-modules?

3. Let A be a nontrivial finite abelian group equipped with the standard $\mathbb{Z}$-module structure.

(i) Explain why $A \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$ can be regarded as a quotient module of $A \otimes_{\mathbb{Z}} \mathbb{Q}$.

(ii) Prove that $A \otimes_{\mathbb{Z}} \mathbb{Z}$ is not (isomorphic to) a submodule of $A \otimes_{\mathbb{Z}} \mathbb{Q}$.

(iii) Which of the following terms apply to the functor $A \otimes_{\mathbb{Z}} -$? Indicate all that apply: (a) left exact; (b) right exact; (c) exact; (d) covariant; (e) contravariant.

(iv) Is A flat as a $\mathbb{Z}$-module?

(v) Is $A \otimes_{\mathbb{Z}} \mathbb{Q}$ flat as a $\mathbb{Z}$-module?