MEASURE SPACES

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2. Assignment due 2018-09-26

2.1. **Finite coverings of a measure space.** Let $\{E_i\}_{i=1}^n$ be a finite sequence of measurable subsets of the measure space $(X, \mathscr{M}, \mu)$, where $\mu(X) = 1$. If each point of X belongs to at least three of these sets, then at least one of the E_i has measure $\geq \frac{3}{n}$.

2.2. **Equivalence relations on a finite space [1, No. 2.12].** Let $(X, \mathscr{M}, \mu)$ be a finite measure space.

(a) If $E, F \in \mathscr{M}$ and $\mu(E\Delta F) = 0$, then $\mu(E) = \mu(F)$.
(b) Say that $E \sim F$ iff $\mu(E\Delta F) = 0$: then $\sim$ is an equivalence relation on $\mathscr{M}$.
(c) If $E, F \in \mathscr{M}$, define $\rho(E, F) = \mu(E\Delta F)$. Then $\rho(E, G) \leq \rho(E, F) + \rho(F, G)$ for all $E, F, G \in \mathscr{M}$, and hence ρ defines a metric on the quotient space $\mathscr{M}/\sim$.

Proof. Let E, F be measurable sets, and suppose that $\mu(E\Delta F) = 0$. Since

$$E\Delta F = (E \setminus F) \sqcup (F \setminus E)$$

from countable additivity we have $0 = \mu(E\Delta F) = \mu(E \setminus F) + \mu(F \setminus E)$. Now μ maps into $[0, \infty]$, so both $\mu(E \setminus F) = \mu(F \setminus E) = 0$. Lastly, break into disjoint sets $E = (E \setminus F) \sqcup (E \cap F)$ and F likewise, so that, by countable additivity,

$$\mu(E) = \mu(E \setminus F) + \mu(E \cap F) \tag{1}$$
$$= \mu(E \cap F) + \mu(F \setminus F) \tag{2}$$
$$= \mu(F). \tag{3}$$

2.3. **Large sets in semifinite measures [1, No. 2.14].** If μ is a semifinite measure and $\mu(E) = \infty$, then for any $C > 0$ there exists $F \subset E$ with $C < \mu(F) < \infty$.

2.4. **Semifinite parts [1, No. 2.15].** If μ is a measure on $(X, \mathscr{M})$, define μ_0 on $\mathscr{M}$ for each $E \subset X$ by

$$\mu_0(E) = \sup\{\mu(F) : F \subset E \text{ and } \mu(F) \leq \infty\}.$$

(a) μ_0 is a semifinite measure.[1]
(b) If μ is semifinite, then $\mu_0 = \mu$.

2.5. **Countable additivity for intersections in outer measures [1, No. 2.17].** If μ^* is an outer measure and $\{A_j\}_1^\infty$ is a sequence of disjoint μ^*-measurable sets, then $\mu^*(E \cap (\cup_1^\infty A_j)) = \sum_1^\infty \mu^*(E \cap A_j)$ for any $E \subset X$.

2.6. **[1, No. 2.18].** Let $\mathscr{A} \subset \mathscr{P}(X)$ be an algebra, $\mathscr{A}_\sigma$ the collection of countable unions of sets in $\mathscr{A}$, and $\mathscr{A}_{\sigma\delta}$ the collection of countable intersections of sets in $\mathscr{A}_\sigma$. Let μ be a premeasure on $\mathscr{A}$ and μ^* the induced outer measure.

(a) For any $E \subset X$ and $\epsilon > 0$ there exists $A \in \mathscr{A}_\sigma$ with $E \subset A$ and $\mu^*(A) \leq \mu^*(E) + \epsilon$.
(b) If $\mu^*(E) < \infty$, then E is μ^* measurable iff there exists $B \in \mathscr{A}_{\sigma\delta}$ with $E \subset B$ and $\mu^*(B \setminus E) = 0$.
(c) If μ is σ-finite, the restriction $\mu^*(E) < \infty$ in (b) is superfluous.

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[1]It's called the **semifinite part** of μ.

2.7. **[1, No. 2.19].** Let μ^* be an outer measure on X induced from a premeasure, such that $\mu^*(X) < \infty$. If $E \subset X$, define the inner measure of E to be $\mu_*(E) = \mu^*(X) - \mu^*(E^c)$. Then E is μ^*-measurable iff $\mu^*(E) = \mu_*(E)$.

2.8. **[1, No. 2.22].** Let $(X, \mathscr{M}, \mu)$ be a measure space, μ^* the outer measure induced by μ according to

$$\mu^*(E) = \inf\left\{\sum_1^\infty \mu(A_j) : A_j \in \mathscr{A}, E \subset \bigcup_1^\infty A_j\right\},$$

and $\mathscr{M}^*$ the σ-algebra of μ^*-measurable sets, and $\tilde{\mu} = \mu^*|\mathscr{M}^*$.

(a) If μ is σ-finite, then $\tilde{\mu}$ is the completion of μ.

2.9. **[1, No. 2.23].** Let $\mathscr{A}$ be the collection of finite unions of sets of the form $(a, b] \cap \mathbf{Q}$ with $-\infty \leq a \leq b \leq \infty$.

(a) Knowing "if $\mathscr{E}$ is an elementary family, the collection $\mathscr{A}$ of finite disjoint unions of members of $\mathscr{E}$ is an algebra" we have that $\mathscr{A}$ is an algebra of $\mathbf{Q}$.

1. Revisions

Problems revised for completion.

1.1. **Closure under countable increasing unions [1, No. 1.4].** An algebra $\mathscr{A}$ is an σ-algebra iff $\mathscr{A}$ is closed under countable increasing unions. That is,

$$\text{if } \{E_j\}_1^\infty \subset \mathscr{A} \text{ and } E_1 \subset E_2 \subset \cdots, \text{ then } \cup_1^\infty \in \mathscr{A}.$$

1.2. **Linear combinations of measures [1, No. 1.7].** If $\mu_1, \ldots, \mu_n$ are measures on $(X, \mathscr{M})$ and $a_1, \ldots, a_n \in [0, \infty)$, then $\sum_1^n a_j \mu_j$ is a measure on (X, μ).

1.3. **Summing sets by union and intersection [1, No. 1.9].** If $(X, \mathscr{M}, \mu)$ is a measure space and $E, F \in \mathscr{M}$, then

$$\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F).$$

1.4. **A measure relative to set intersection [1, No. 1.10].** If $(X, \mathscr{M}, \mu)$ is a measure space and $E \in \mathscr{M}$, define

$$\mu_E(A) = \mu(A \cap E) \text{ for all } A \in \mathscr{M}.$$

Then μ_E is a measure.

References

[1] G. B. Folland, *Real analysis*, Second. John Wiley & Sons, Inc., New York, 1999.