METRICS, ALGEBRAS, AND MEASURES

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1. Assignment due 2018-09-12

1.1. **Open balls in metric spaces.**

Let (X, ρ) be a metric space. We will show that balls in the metric space are open.

Take an arbitrary point in the space $x \in X$ and a radius $\varepsilon > 0$. I claim

$$B_\rho(\varepsilon, x) := \{a \in X : \rho(a, x) < \varepsilon\}$$

is open in X.

Indeed, take a point $a \in B_\rho(\varepsilon, x)$ (note that $\rho(a, x) < \varepsilon$). Now consider the radius $\delta = \varepsilon - \rho(x, a) > 0$. We need to show the open ball $B_\rho(\delta, a)$ containing the point a, is nested in within a radius ε of x, i.e., we need to show[1] $B_\rho(\delta, a) \subset B_\rho(\varepsilon, x)$. Well, if $b \in B_\rho(\delta, a)$, then $\rho(a, b) < \delta = \varepsilon - \rho(x, a)$, with the triangle inequality we have $\rho(b, x) \leq \rho(a, b) + \rho(x, a) < \varepsilon$, hence $b \in B_\rho(\varepsilon, x)$.

So for every point a in the open ball $B_\rho(\varepsilon, x)$ there's a nested open ball $B_\rho(\delta, a)$ such that $a \in B_\rho(\delta, a) \subset B_\rho(\varepsilon, x)$. By definition, $B_\rho(\varepsilon, x)$ is open.

1.2. **New metric spaces from old.**

Let (X, d) be a metric space. Define $\rho \colon X \times X \to [0, \infty)$ by

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

We'll show that ρ is a metric on X.

Key idea. We have some transformation $f : [0, \infty) \to [0, \infty)$ of the metric d given by $f(d) = \frac{d}{1+d}$. Now f is strictly monotone, so order preserving, and concave, so subadditive. We'll make repeated use of the property that

$$\text{if } c, k \in [0, \infty) \text{ and } c \leq k, \text{ we have } c(1 + k) = c + ck \leq k + ck = k(1 + c) \text{ whence } \frac{c}{1 + c} \leq \frac{k}{1 + k}.$$

We have equality iff $c = k$.

Now to show that ρ is a metric, take three arbitrary points $x, y, z \in X$.

- We have $y = x$ iff $d(x, y) = 0$ iff $\rho(x, y) = 0$, since for the metric transformation defined above $f(d(x, y)) = \rho(x, y)$ and $f(0) = 0$ and $f^{-1}(0) = 0$.
- Symmetry of ρ is clear from the symmetry of d; consider $\frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)}$.
- To verify the triangle inequality holds in the metric space (X, ρ), let $a = d(y, x)$, $b = d(z, y)$ and $c = d(z, x)$. We need to show
$$\frac{c}{1 + c} \leq \frac{a}{1 + a} + \frac{b}{1 + b}.$$
Note $a, b, c \in [0, \infty)$. Since the triangle inequality holds in the space (X, d), we have $c \leq a + b$. Consider $k = a + b$. It follows that $c \leq k$ and thus $\frac{c}{1+c} \leq \frac{k}{1+k}$. That is, $\frac{c}{1+c} \leq \frac{a+b}{1+a+b}$ hence, with $a, b \geq 0$,
$$\frac{c}{1 + c} \leq \frac{a + b}{1 + a + b} = \frac{a}{1 + a + b} + \frac{b}{1 + a + b} \leq \frac{a}{1 + a} + \frac{b}{1 + b},$$
as desired.

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[1]We denote "X is a subset of Y" by $X \subset Y$ and "Y is a proper subset of Y" by $X \subsetneq Y$.

We conclude that ρ is a metric on X because we've demonstrated

- ρ is nonnegative in general and zero only between identical points,
- ρ is symmetric between points, and
- ρ has the triangle inequality for distances between any three points.

(a) Now, for any point $x \in X$ and radius $r > 0$,

$$B_d(r, x) = B_\rho\left(\frac{r}{1+r}, x\right).$$

We have set equality for points within a radius r of x in (X, d) and points within radius $f(r) = \frac{r}{1+r}$ of x in (X, ρ) where f is the above defended metric transformation.

(b) Further, if G is an (X, d)-open set, then G is also (X, ρ)-open. That is, G is open in (X, d) iff there's some ball of radius (in the d metric) ε around each point nested in G iff there's some ball of radius $\frac{\varepsilon}{1+\varepsilon}$ (in the ρ metric) around the same point, also nested in G iff G is open in (X, ρ).

1.3. **An algebra of sets.** Let X be an uncountable set. Consider the union

$$\mathscr{A} = \{U \subset X : U \text{ is finite}\} \cup \{V \subset X : X \setminus V \text{ is finite}\}.$$

(a) $\mathscr{A}$ is an algebra of sets of X. Closure under complements is apparent. We have closure under finite unions noting that either

- a union of the E_i is infinite with a finite complement when one of the E_i is infinite, or
- a finite union of the E_i is finite when all of the E_i are finite.

We have closure under intersections by noting that either

- a finite intersection of the E_i has with a finite complement when all of the E_i are infinite, or
- a finite intersection of the E_i is finite when one of the E_i are finite.

(b) Indeed $\mathscr{A}$ is *not* a σ-algebra of subsets of X, for consider that the infinite intersection of sets with a finite complement may no longer have a finite complement. The smallest σ-algebra containing $\mathscr{A}$ is the algebra where countable unions and countable intersections are allowed, and this is given by relaxing the requirements in the set builder notation to "U is countable" or $X \setminus V$ is countable".

1.4. **Rings and σ-rings [1, No. 1.1].** A family of sets $\mathscr{R} \subset \mathscr{P}(X)$ is called a ring if it is closed under finite unions and differences; a ring that is closed under countable unions is called a σ-ring.

a. A ring $\mathscr{R}$ is closed under finite intersections, for each pair of sets E, F in $\mathscr{R}$, write

$$E \cap F = (E \cup F) \setminus ((E \setminus F) \cup (F \setminus E)).$$

Now a point x is in both E and F if and only if x is in either E or F but not in *only* F or *only* F. Respectively, a σ-ring $\mathscr{R}$ is closed under countable intersections, for each countable family of sets E_j write for the

$$\bigcap E_j = \bigcup_{\forall j} E_j \setminus \left[\bigcup_{\forall j} \left(E_j \setminus \cup_{k \neq j} E_k\right)\right]$$

where the unions on the left hand side are countable.

b. A ring (respectively σ-ring) $\mathscr{R}$ on X is an algebra (respectively σ-algebra) iff $X \in \mathscr{R}$. ($\Rightarrow$) Suppose that $\mathscr{R}$ is an (σ) algebra. Then $X \in \mathscr{R}$ and so too $A \setminus B = A \cap B^c \in \mathscr{R}$, whence $\mathscr{R}$ is closed under differences, and so $\mathscr{R}$ is a (σ) ring. ($\Leftarrow$) Suppose that $\mathscr{R}$ is an (σ) ring and $X \in \mathscr{R}$. Then $E^c = X \setminus E \in \mathscr{R}$. So $\mathscr{R}$ is closed under complements, and therefore an (σ) algebra.

c. Let $\mathscr{R}$ be a σ-ring on X, and take

$$\mathscr{A} = \{E \subset X : E \in \mathscr{R} \text{ or } E^c \in \mathscr{R}\}.$$

- We assume $\mathscr{R}$ is nonempty.

- We have that $\mathscr{A}$ is closed under countable unions as $\mathscr{R}$ is.
- We have that $E \in \mathscr{A}$ implies $E^c \in \mathscr{A}$; as when $E \in \mathscr{A}$ either E or E^c are in the ring $\mathscr{R}$, whence, by *definition* of $\mathscr{A}$, we conclude $E^c \in \mathscr{A}$.

d. Let $\mathscr{R}$ be a σ-ring. Take $\mathscr{A} = \{E \subset X : E \cap F \in \mathscr{R} \text{ for all } F \in \mathscr{R}\}$. Note $\mathscr{A}$ contains $\mathscr{R}$ (we think of $\mathscr{A}$ as extending $\mathscr{R}$) and is therefore closed under countable unions. Further note that $X \in \mathscr{A}$, whence by previous argument can write complements as set differences with the ambient space X, so $\mathscr{A}$ is closed under complements. We conclude $\mathscr{A}$ is a σ-algebra.

1.5. **The Borel σ-algebra on R [1, No. 1.2].** To demonstrate the Borel σ-algebra $\mathscr{B}_{\mathbf{R}}$ is generated by the following collections of intervals in $\mathbf{R}$, note that we need only obtain an expression (in terms of countable unions, countable intersections, and complements) for the open balls as a basis for the standard topology on the real line. From there, we rely on the fact that every open set in the standard topology is a countable union of open balls.

- That the open intervals generate $\mathscr{B}_{\mathbf{R}}$ follows from the argument above.
- A countable union of closed intervals of the form $[a + \frac{1}{n}, b - \frac{1}{n}]$ is an open ball.
- A countable union of half-open intervals of the form $[a + \frac{1}{n}, b)$ or $(a, b - \frac{1}{n}]$ is an open ball.
- The union of an open ray with the complement of a open ray is either empty or an open ball.
- The union of an closed ray with the complement of a closed ray either empty or a closed ball, an infinite union of which form an open ball.

1.6. **Infinite σ-algebras [1, No. 1.3].** Let $\mathscr{M}$ be an infinite σ-algebra. Then $\mathscr{M}$ contains an infinite sequence of disjoint nonempty sets, and $\operatorname{card}(\mathscr{M}) \geq \mathfrak{c}$.

Proof.

We'll show the existence of a set $\{E_i\}_{i=1}^{\infty}$ of nonempty disjoint events in the σ algebra. We proceed by induction on the number of such disjoint sets, indexed by some function $f \colon \mathbf{N} \to \mathscr{M}$.

As a base for induction, consider two distinct nonempty events E_1 and E_2 in $\mathscr{M}$, which exist since $\mathscr{M}$ is infinite.

- Case: $E_2 \setminus E_1$ is empty. Then $E_2 \subsetneq E_1$ (since $E_2 \neq E_1$).
  - So let $f(1) = E_1 \setminus E_2$ (nonempty because E_2 is a proper subset) and
  - $f(2) = E_2$ (nonempty by construction).
- Case: $E_2 \setminus E_1$ is nonempty.
  - Then let $f(1) = E_1$ (nonempty by construction) and
  - $f(2) = E_2 \setminus E_1$ (nonempty in this case).

For the inductive step, suppose that we have a set of n disjoint nonempty events $\{E_i\}_{i=1}^{n}$. Since $\mathscr{M}$ is infinite, $\mathscr{M}$ contains an E_{n+1} distinct from the finite set of all possible unions of the E_i for $1 \leq i \leq n$ (there are 2^n of such possible unions, given that $\mathscr{P}(\{\{E_i\}_{i=1}^{n})$ has a natural one-to-one correspondence with the set of such possible unions).

- Case: $E_{n+1} \setminus \left(\cup_{i=1}^{n} E_i\right)$ is empty. Then $E_{n+1} \subsetneq \cup_{i=1}^{n} E_i$ (since E_{n+1} was chosen distinct from unions of the E_i).
  - So let $f(i) = E_i \setminus E_{n+1}$ for all $1 \leq i \leq n$ (nonempty because E_{n+1} is distinct from the E_i, and contained in their union) and
  - $f(n+1) = E_{n+1}$ (nonempty by construction).
- Case: $E_{n+1} \setminus \left(\cup_{i=1}^{n} E_i\right)$ is nonempty.
  - Then let $f(i) = E_i$ for all $1 \leq i \leq n$ (nonempty by construction) and
  - $f(n+1) = E_{n+1} \setminus \left(\cup_{i=1}^{n} E_i\right)$ (nonempty in this case).

In either case, the set $\{f(i)\}_{i=1}^{n+1}$ is a nonempty collection of disjoint events in the σ-algebra, which concludes our induction on n.

Now suppose we've parameterized the sequence of disjoint events in the σ algebra as $\{E_j\}_{j=1}^{\infty}$. There's a natural injection from $2^{\mathbf{N}}$ into $\mathscr{M}$, namely a bijection between $2^{\mathbf{N}}$ and unions of families in $\mathscr{P}(\{E_j\}_{j=1}^{\infty})$ (consider the possible values of the indicator function which indicates whether some set E_j *is or is not* included in the union). Whence $\operatorname{card}(\mathscr{M}) \geq 2^{\aleph_0} = \mathfrak{c}$.

1.7. **Closure under countable increasing unions [1, No. 1.4].** An algebra $\mathscr{A}$ is an σ-algebra iff $\mathscr{A}$ is closed under countable increasing unions. That is,

$$\text{if } \{E_j\}_1^{\infty} \subset \mathscr{A} \text{ and } E_1 \subset E_2 \subset \cdots, \text{ then } \cup_1^{\infty} \in \mathscr{A}.$$

1.8. **Linear combinations of measures [1, No. 1.7].** If $\mu_1, \ldots, \mu_n$ are measures on $(X, \mathscr{M})$ and $a_1, \ldots, a_n \in [0, \infty)$, then $\sum_1^n a_j \mu_j$ is a measure on (X, μ).

1.9. **Summing sets by union and intersection [1, No. 1.9].** If $(X, \mathscr{M}, \mu)$ is a measure space and $E, F \in \mathscr{M}$, then

$$\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F).$$

1.10. **A measure relative to set intersection [1, No. 1.10].** If $(X, \mathscr{M}, \mu)$ is a measure space and $E \in \mathscr{M}$, define

$$\mu_E(A) = \mu(A \cap E) \text{ for all } A \in \mathscr{M}.$$

Then μ_E is a measure.

References

[1] G. B. Folland, *Real analysis*, Second. John Wiley & Sons, Inc., New York, 1999.