Rules for Math 23a/E-23a Online Quizzes
Fall 2018

"Members of the Harvard College community commit themselves to producing academic work of integrity that is, work that adheres to the scholarly and intellectual standards of accurate attribution of sources, appropriate collection and use of data, and transparent acknowledgement of the contribution of others to their ideas, discoveries, interpretations, and conclusions. Cheating on exams or problem sets, plagiarizing or misrepresenting the ideas or language of someone else as ones own, falsifying data, or any other instance of academic dishonesty violates the standards of our community, as well as the standards of the wider world of learning and affairs."

1. No references are allowed except for the excerpts from the Executive Summaries that are on the last page.
2. After the quiz is downloaded and printed, computers and cell phones must be switched off and calculators should be put away.
3. No discussion with classmates or others is permitted during the entire period during which the quiz is available.
4. The proctor does not have to remain continuously in the room but should look in unexpectedly from time to time.
5. Although there is no explicit time limit, the quiz must be completed in a single sitting.
6. When the quiz is done, switch on the computer and scanner and upload the completed quiz, along with this signed form.

Date of quiz________________________________
Start time________________________________
End time________________________________
List any unusual circumstances in the administration of the quiz (e.g. the scanner was not located in the room where the quiz was taken).

I am aware of the Harvard College Honor Code, and I certify that I complied with the rules.

(Student signature)________________________________

I am aware of the Harvard College Honor Code, and I observed that the student complied with the rules.

(Proctor signature)________________________________

Proctor email________________________________

Proctor relationship to student ________________________________

Name: ______________________________
Section (if any): ___________________

MATHEMATICS 23a/E-23a, Fall 2018
Quiz #2
November 9-11, 2018

You must complete this quiz at a single sitting immediately after downloading and printing it. While you are taking the quiz, your computers and cell phone must be switched off.

The last page of the quiz contains useful information extracted from the Executive Summaries.

Calculators, which would be of no use, are not allowed.

You and your proctor must sign the statement on the front page of the exam.
You may omit one multiple-choice question in Part I and one question in Part II.
If you are doing proof logging, check here ____ and omit one proof in Part III.
If you opt out of proof logging, check here ____ and do all proofs in Part III.

There are three blank pages at the end of the exam. If your answer does not fit in the space provided on the page with the question, write "Continued on page XX" and finish the answer on the specified page. That way, your exam can be scanned without having to check for answers on the back of a page.

Problem	Points	Answer	Score
$I-1$	2		
$I-2$	2		
$I-3$	2		
$I-4$	2		
$I-5$	2		
$I-6$	2		
$II-1$	5	$--$	
$II-2$	5	$--$	
$II-3$	5	$--$	
$II-4$	5	$--$	
$II-5$	5	$--$	
$III-1$	5	$--$	
$III-2$	5	$--$	
$III-3$	5	$--$	
$III-4$	5	$--$	
$III-5$	5	$--$	
Total	50 or 55	$--$	

Part I. Answer five of the six multiple-choice questions. Transcribe your answers onto page 2, and mark an X in the score box on page 2 to indicate which question you have omitted.

If you answer all six questions, the last one will be ignored.

1. Let s_n be a sequence of real numbers on a bounded set S, where $\liminf s_n \neq \limsup s_n$. Which of the following is not necessarily true?

   (a) $\lim s_n$ does not exist.

   (b) s_n is not Cauchy.

   (c) $\liminf s_n < \limsup s_n$

   (d) There exists a convergent subsequence of s_n.

   (e) s_n has an infinite number of dominant terms.

   Solution: $\boxed{E}$ a) is true, because in order for a limit to exist, the lim sup must equal the lim inf, and it turns out that that is equal to the limit. Because the sequence can't converge, choice b) is also out, since all Cauchy sequences of real numbers converge. For choice c), notice that in general $\liminf s_n \leq \limsup s_n$, and we have excluded the equal case in the prompt of the question. For choice d), the Bolzano Weierstrass Theorem still holds, since the sequence is defined on a bounded set. e) is not necessarily true. An explicit counterexample is $s_n = 1 - \frac{1}{n}$ for even n, and $s_n = 0$ for odd n. In this case, there are no dominant terms.

2. Which of the following is not true about $s_n = \frac{1}{n}$?

   (a) The sequence converges to 0.

   (b) $\lim_{n\to\infty} \sum_{i=1}^{n} s_i = L$, for some finite L.

   (c) $\limsup s_n = 0$.

   (d) The series $\sum (-1)^n s_n$ converges.

   (e) The series $\sum s_n^2$ converges.

   Solution: $\boxed{B}$ This series is known as the harmonic series, which diverges. Choice b) is simply stating the definition of a series sum as the limit of the partial sums. a) is true, since when n goes to infinity, $1/n$ goes to zero. c) is true, since if the limit exists, then $\limsup s_n = \liminf s_n = \lim s_n$. Choice d) is true, because then, we are considering an alternating series, which has a less stringent convergence condition, namely that $\lim |s_n| = 0$, which is satisfied. Choice e) is true, because $1/n^2$ is a convergent p series.

3. Which of the following must be true of a continuous function on (a, b)?
   (a) The function achieves its maximum on (a, b).
   (b) The function is bounded.
   (c) For all Cauchy sequences s_n on the set (a, b), $f(s_n)$ is also Cauchy.
   (d) If $f(a) = 2$, and $f(b) = 5$, then $f(c) = 3$, for some $c \in (a, b)$.
   (e) None of the above are true

   Solution: $\boxed{E}$ None of these are true! Choice a) is not true, by a counterexample. If $a = 0$ and $b = 1$, then $1/x$ is continuous on $(0, 1)$, but does not achieve its maximum. The key here is that the statement only guarantees continuity on the open interval. Part b) is false by the counterexample above. Part c) is also not true by the counterexample above. Let $\lim s_n = 0$, and then $f(s_n)$ will diverge, and not be Cauchy. d) is not true, because it doesn't need to be continuous at the end points. The function $f(x) = 5$ except at a, where $f(a) = 2$, is a counterexample. Notice how important continuity on a **closed** interval is for most of the theorems we've studied!

4. Find $\lim_{x \to b} \frac{\sqrt{x} - \sqrt{b}}{x - b}$ for $b > 0$.
   (a) ∞
   (b) $\frac{1}{2\sqrt{b}}$
   (c) 0
   (d) $2\sqrt{b}$
   (e) b

   Solution: $\boxed{B}$ If you multiply both top and bottom by $\sqrt{x} + \sqrt{b}$, the numerator becomes $x - b$, and cancels the denominator. Taking the limit results in choice b). Alternatively, this is the definition of the derivative of the $\sqrt{x}$ function evaluated at $x = b$.

5. Let f be a differentiable function, where all derivatives exist, such that $f(0) = 0$, $f'(0) = 0$, and $|f''(x)| \leq M, \forall x$. Which of the following is not necessarily true?
   (a) $|f(1)| \leq \frac{M}{2}$
   (b) 0 is neither a maximum nor a minimum.
   (c) $\forall \epsilon > 0, \exists \delta > 0$ s.t. if $x \in (-\delta, \delta), |f(x)| < \epsilon$
   (d) If $\lim s_n = 0$, then $\lim f(s_n) = 0$.
   (e) None of the above. They're all necessarily true.

   Solution: $\boxed{B}$ Choice a) is true, since that is just the statement of Taylor's Theorem of Remainder. It says that

$$f(x) = f(0) + f'(0)x + \frac{f''(c)x^2}{2!} \text{ where } c \in [0, x]$$

Notice that $f(0) = f'(0) = 0$, and that the derivative is bounded by M. Then, this states:

$$\begin{aligned} f(1) &\leq 0 + 0(1) + \frac{M(1)^2}{2!} \\ &\leq \frac{M}{2} \end{aligned}$$

The same holds on the negative side. Choice b) is false by counterexample. Let $f(x) = x^2$, then 0 is a minimum, and the second derivative is bounded by 2. Choice c) is true, since it is just the statement of continuity at $x = 0$, and being differentiable implies continuity. Choice d) is another statement of continuity, just the sequence definition of continuity.

6. Let $\sum a_n$ be a conditionally convergent alternating series. Which of the following is not necessarily true?

   (a) The series converges to some finite L.

   (b) The series sum is independent of order of terms.

   (c) $\sum |a_n|$ diverges.

   (d) $\lim a_n = 0$.

   (e) None of the above. They're all necessarily true.

   Solution: $\boxed{B}$ Part a) is true by definition! If it converges, it converges to something. This does not contradict part b), since the order of terms in a conditionally convergent series needs to be respected! The order of terms as written defines the series sum, despite other sums being possible if the terms are moved around. Part c) is false, because, if $|a_n|$ converges, then the series would be absolutely convergent, not conditionally convergent. Part d) is true, since that is exactly the statement of the alternating series test.

Part II. Answer four of the five questions. Mark an X in the score box on page 1 to indicate which question you have omitted.

1. (Inspired by Week 5, workshop problems # 3)
   Consider the sequence:

$$s_n = \frac{n^p + 1}{n^p}$$

   where $p \in \mathbb{R}$.

   (a) Find $\lim s_n$ when $p = 1$ from the definition of the limit of a sequence.

   (b) Find $\lim s_n$ when $p > 0$ using whatever method you want. (Note that your answer should be independent of p, so the numerical answer to the limit will be the same as your answer to the first part)

   (c) For $p < 0$, prove that $\lim s_n = \infty$ from the definition of a limit of a sequence being infinity.

   Solution:

   (a) When $p = 1$, our sequence is given by:

$$s_n = \frac{n+1}{n} = 1 + \frac{1}{n}$$

   This looks like it will converge to 1, but we need to prove that!
   Scratch work: We eventually want to find N such that all $n > N$ result in $|s_n - 1| < \epsilon$ for any given epsilon. Let's find the N such that $|s_N - 1| = \epsilon$ and then all terms after that will be closer to 1 than s_N. We want to solve:

$$|s_N - 1| = \epsilon \implies |\frac{1}{N}| = \epsilon \implies N = \frac{1}{\epsilon}$$

   Formal proof: For any $\epsilon > 0$, choose $N = \frac{1}{\epsilon}$. Then, $\forall n > N$, we have:

$$|s_n - 1| = |\frac{1}{n}| < \frac{1}{N} \implies |s_n - 1| < \epsilon$$

   Thus, we have shown that s_n converges to 1.

   (b) We want to evaluate $\lim s_n$ when $p > 0$. Since we found in part a) that the sequence converges to 1 for $p = 1$, then it should converge to 1 for all p (based on the statement of the problem). We will use limit rules this time around. Note:

$$s_n = \frac{n^p + 1}{n^p} = 1 + \frac{1}{n^p}$$

   Now, we use a limit rule. We take the limit as $n \to \infty$ and the limit of a sum is the sum of the limits (as long as both limits exist). The limit of 1 is just 1, while the limit of $1/n^p$ is zero. Thus,

$$\lim s_n = \lim \left(1 + \frac{1}{n^p}\right) = 1 + 0 = 1$$

(c) Now for $p < 0$, we want to show that $\lim s_n = \infty$. For negative p, we can write our sequence as:

$$s_n = \frac{n^{-|p|} + 1}{n^{-|p|}} = 1 + n^{|p|}$$

We need to show that the limit is infinity from the definition.

Scratch work: We need to exhibit a N such that for all $n > N$, $s_n > M$ for any given M. Let's find the N such that $s_N = M$, then all terms after that are greater than M. We want to find:

$$1 + N^{|p|} = M \implies N = (M-1)^{1/|p|}$$

Formal proof: For any M, choose $N = (M-1)^{1/|p|}$. Then, $\forall n > N$:

$$\begin{aligned} |s_n| = (1 + n^{|p|}) &> 1 + N^{|p|} \\ &> 1 + (M - 1) \\ &> M \end{aligned}$$

Thus, we have shown $\lim s_n = \infty$ for $p < 0$.

2. (Inspired by Week 6, workshop problems # 1)

   Consider a sequence $s_n \in \mathbb{R}$.

   (a) Prove that if s_n is bounded (both above and below), then $\limsup s_n$ exists and is finite. (Note by a similar argument $\liminf s_n$ exists and is finite.)

   (b) If s_n is unbounded below, then by definition, we say $\liminf s_n = -\infty$. However, $\limsup s_n$ is not determined from this information. Invent a sequence that is unbounded below, where $\limsup s_n$ is finite. Invent one where $\limsup s_n = -\infty$.

   Solution:

   (a) By definition,

$$\limsup s_n = \lim_{N\to\infty} \sup\{s_n : n > N\}$$

   Notice that $\sup\{s_n : n > N\}$ forms a decreasing sequence in N. Namely, as I start excluding more and more elements, the supremeum can only go down. Put another way, $\{s_n : n > N\} \subset \{s_n : n > N - m\}$ for any m, so the supremum as I go further in the sequence in N can only go down. Furthermore, $\sup\{s_n : n > N\}$ is bounded below by the same bound as s_n is bounded by. The supremum can't possibly be below what the lower bound of the elements is. Thus, $\sup\{s_n : n > N\}$ is a decreasing sequence and is bounded below, so it converges and $\lim_{N\to\infty} \sup\{s_n : n > N\} = \limsup s_n$ exists.

   (b) We first invent a sequence that is unbounded below but $\limsup s_n$ is finite. Consider:

$$s_n = \begin{cases} 0 & \text{if n is even} \\ -n & \text{if n is odd} \end{cases}$$

   This sequence is $(-1, 0, -3, 0, -5, 0, ...)$. This is clearly unbounded, but the $\sup\{s_n : n > N\} = 0$ for any N. Thus, $\limsup s_n = 0$ and is finite. It does not diverge. For a sequence where $\limsup s_n = -\infty$, consider $s_n = -n$. This sequence is unbounded below and $\sup\{s_n : n > N\} = -(N + 1)$ so $\limsup s_n = -\infty$.

3. (Inspired by Week 7, workshop problems # 1)
Let's consider a function $h(x)$ constructed in a piecewise manner from two other functions $f(x)$ and $g(x)$.

$$h(x) = \begin{cases} f(x) & \text{for } a \leq x \leq b \\ g(x) & \text{for } b < x \leq c \end{cases}$$

Assume that $f(x)$ and $g(x)$ are defined on $[a, b]$ and $(b, c]$ respectively, so that $h(x)$ is defined on $[a, c]$.

(a) Invent (drawing a picture is okay) functions $f(x)$, $g(x)$ where $f(x)$ is continuous on $[a, b]$ and $g(x)$ is continuous on $[b, c]$ (note that $g(x)$ is also defined at b in this case), *but* $h(x)$ is not continuous on $[a, c]$.

(b) Let $f(x)$ be continuous at b. Prove that $h(x)$ is continuous at b, if and only if

$$\lim_{x \to b^-} f(x) = \lim_{x \to b^+} g(x)$$

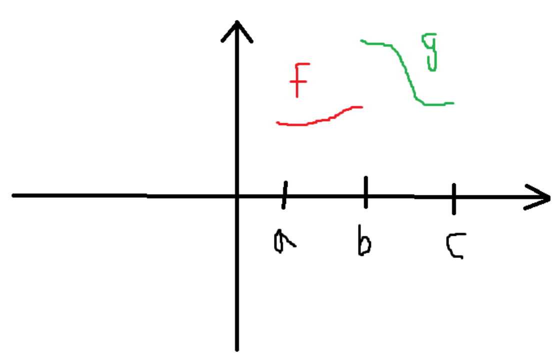
(c) The condition that $f(x)$ be continuous at b is crucial! Show (using a picture is fine) a situation where

$$\lim_{x \to b^-} f(x) = \lim_{x \to b^+} g(x)$$

but $h(x)$ is not continuous at b, when we allow $f(x)$ to be discontinuous at b.

Solution:

(a) First, we invent a function $f(x)$ and a function $g(x)$, which are continuous on $[a, b]$ and $[b, c]$ respectively, but the combined function $h(x)$ is not. One can imagine this occurs if $f(x)$ and $g(x)$ do not meet at b. The following is shown below:



(b) This is an if and only if proof, so we have to do this in both directions. Let's first show that if $h(x)$ is continuous, then:

$$\lim_{x\to b^-} f(x) = \lim_{x\to b^+} g(x)$$

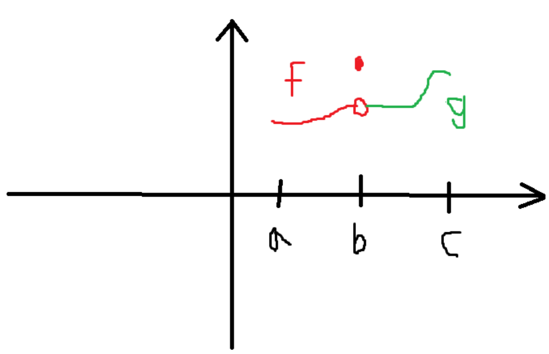
We let $h(b) = f(b) = L$. Since $f(x)$ is continuous at b, then $\lim_{x\to b^-} f(x) = L$. Thus, we only need to show that $\lim_{x\to b^+} g(x) = L$. If $h(x)$ is continuous at b, then $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $|x-b| < \delta$, then $|h(x) - L| < \epsilon$. To calculate $\lim_{x\to b^+} g(x)$, let's consider only $x > b$. In this region, $h(x) = g(x)$, and we have $x - b < \delta \implies |g(x) - L| < \epsilon$. This is exactly the definition for $\lim_{x\to b^+} g(x) = L$. Thus, we have shown

$$\lim_{x\to b^-} f(x) = \lim_{x\to b^+} g(x)$$

Now, we want to go the other way around and show that these limits being equal implies continuity of $h(x)$. We can do this in many ways, but I'll do this by contraposition. We instead show that if $h(x)$ is not continuous at b, then $\lim_{x\to b^-} f(x) \neq \lim_{x\to b^+} g(x)$. Let $h(b) = f(b) = L$. Once again, since $f(x)$ is continuous at b, then $\lim_{x\to b^-} f(x) = L$. If $h(x)$ is not continuous at b, then $\exists \epsilon > 0$, such that $\forall \delta > 0$, $|x - b| < \delta$, but $|h(x) - L| \geq \epsilon$. We restrict our attention to $x > b$, where $h(x) = g(x)$. Then, the statement of discontinuity says, $\exists \epsilon > 0$, such that $\forall \delta > 0$, $x - b < \delta$ but $|g(x) - L| \geq \epsilon$. This, by definition, is exactly the definition of $g(x)$ not converging to L. Thus, $\lim_{x\to b^+} g(x) \neq L$, so

$$\lim_{x\to b^-} f(x) \neq \lim_{x\to b^+} g(x)$$

(c) Continuity of $f(x)$ at b is crucial, and we used it several times in the previous proof! Here is an example, where if we relax that restriction, we can have a discontinuous $h(x)$ at b, despite having $\lim_{x\to b^-} f(x) = \lim_{x\to b^+} g(x)$.



4. (Inspired by Week 8, workshop problems # 3)
Recall that the exponential function is the amazing function whose derivative is itself! Namely if $f(x) = e^x$, then $f'(x) = e^x$. Also, note that $f(0) = e^0 = 1$.

   (a) Compute the Taylor series of e^x expanded around $x = 0$. Prove using Taylor's theorem with remainder that this Taylor series converges to e^x.

   (b) Prove that the radius of convergence of the Taylor series is ∞.

   (c) Recall that the hyperbolic sine is given by

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + ... = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

   Show using Taylor series representations that

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Solution:

(a) The Taylor series of e^x expanded around $x = 0$ is given by:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)x^k}{k!}$$

The first term is $e^0 = 1$. For subsequent terms, we need to calculate $f^{(k)}(0)$. All derivatives of $f(x)$ are e^x, since the derivative of e^x is itself. Thus, we have that $f^{(k)}(0) = 1$ for all k. Then, the Taylor series of e^x is:

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

To show that the Taylor series actually converges to e^x, recall Taylor's theorem with remainder:

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)x^k}{k!} + \frac{f^{(k)}(x_0)x^n}{n!}$$

where x_0 is in between 0 and x. This derivative at a given x is bounded by e^x, which is the derivative at $x_0 = x$. Thus, the remainder is at most:

$$\frac{e^x x^n}{n!}$$

To show the Taylor series converges, we need to show that the remainder as a sequence in n converges to 0. Thus, we want to evaluate:

$$\lim_{n\to\infty} \frac{e^x x^n}{n!}$$

We will use the ratio test **for sequences** (Ross Exercise 9.12, Problem Set 5 Question 8), and consider the ratio:

$$|\frac{x^{n+1}}{(n+1)!}\frac{n!}{x^n}| = |\frac{x}{n+1}|$$

The limit of this sequence as $n \to \infty$ is zero and thus we know that $e^x x^n/n!$ converges as a sequence in n to 0. Thus, the remainder goes to zero, and e^x is equal to its Taylor series.

(b) To find the radius of convergence, we use the ratio test for **series**. We evaluate:

$$\lim_{n\to\infty} |\frac{s_{n+1}}{s_n}| = \lim_{n\to\infty} |\frac{x^{n+1}}{(n+1)!}\frac{n!}{x^n}| = \lim_{n\to\infty} |\frac{x}{n+1}| = 0 < 1$$

for all x. Thus, this Taylor series which equals e^x converges for all x and has infinite radius of convergence.

(c) We calculate $(e^x - e^{-x})/2$ explicitly using Taylor series.

$$\begin{aligned}\frac{e^x - e^{-x}}{2} &= \frac{1}{2}\left(\sum_{k=0}^{\infty}\frac{x^k}{k!} - \sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\right)\\ &= \frac{1}{2}\left(\sum_{k=0}^{\infty}\frac{x^k}{k!} - \sum_{k=0}^{\infty}\frac{(-1)^k x^k}{k!}\right)\end{aligned}$$

Now, notice that all the even terms cancel, while we keep all the odd terms with a factor of 2. Thus, we have:

$$\begin{aligned}\frac{e^x - e^{-x}}{2} &= \frac{1}{2}\left(\sum_{j=0}^{\infty}\frac{2x^{2j+1}}{(2j+1)!}\right)\\ &= \sum_{j=0}^{\infty}\frac{x^{2j+1}}{(2j+1)!}\\ &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...\\ &= \sinh(x)\end{aligned}$$

which is exactly the series for the hyperbolic sine!

5. Consider a function $f(x)$ with domain $U = [a, b]$.

   (a) Prove that if $f(x)$ is differentiable at a point $x_0 \in U$, then $f(x)$ is continuous at x_0. You may also take for granted that $\lim_{x\to x_0} f(x)$ exists. (Hint: start from the limit which defines the derivative as existing and then multiply both sides by $\lim_{x\to x_0}(x - x_0)$.)

   (b) Show that the converse is not necessarily true! Invent (a picture is fine) a function $f(x)$ that is continuous at x_0, but not differentiable at x_0.

   (c) Using the Mean Value Theorem, show that if $f'(x)$ is bounded on (a, b), then $f(x)$ is uniformly continuous on $[a, b]$.

Solution:

(a) If $f(x)$ is differentiable at a point $x_0 \in U$, then we know that:

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

exists. Now, we multiply left and right side by $\lim_{x\to x_0}(x - x_0)$, and obtain:

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} \lim_{x\to x_0}(x - x_0) = f'(x_0) \lim_{x\to x_0}(x - x_0)$$

We know that $\lim_{x\to x_0}(x - x_0) = 0$. Furthermore, on the left hand side, we can use limit rules to combine the product of limits into a limit of the product:

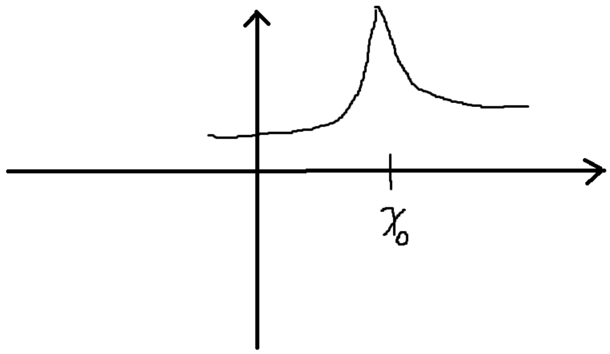
$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) = 0 \implies \lim_{x\to x_0}(f(x) - f(x_0)) = 0$$

Furthermore, we assume $\lim_{x\to x_0} f(x)$ exists, so we can use limit rules to separate the limit of a difference into the difference of the limits so

$$\lim_{x\to x_0} f(x) = f(x_0)$$

This is exactly what it means to be continuous, so we have shown $f(x)$ is continuous at x_0.

(b) A function can be continuous but not differentiable at a given point. The picture below is an example:



This "cusp" as it's called is continuous, but is not differentiable, because the slope when approaching from the left or the right is different and not the same.

(c) Now, we want to prove that a bounded derivative implies uniform continuity on an interval $[a, b]$.

Scratch work: To prove uniform continuity, we need to show that $\forall \epsilon > 0, \exists \delta > 0$ s.t. $\forall x, y \in [a, b], |y - x| < \delta \implies |f(y) - f(x)| < \epsilon$. Let's consider arbitrary $x, y \in [a, b]$, and without loss of generality we'll let $x < y$. Then, the Mean Value Theorem says

$$\frac{f(y) - f(x)}{y - x} = f'(c)$$

for some $c \in (x, y)$. Then, we have that:

$$|\frac{f(y) - f(x)}{y - x}| = |f'(c)|$$

Since the derivative is bounded, its absolute value has a least upper bound M, so that $|f'(c)| < M + 1$. Then, we have that:

$$|\frac{f(y) - f(x)}{y - x}| < M + 1 \implies |f(x) - f(y)| < |y - x|(M + 1)$$

Thus, if $|f(y) - f(x)|$ can only be $M + 1$ times bigger than $|y - x|$.

Formal proof: For any $\epsilon > 0$, choose $\delta = \frac{\epsilon}{M+1}$ where $M = \sup\{|f'(x)| : x \in [a, b]\}$ which exists since $f'(x)$ is bounded. Then, by the Mean Value Theorem for any $x, y \in [a, b]$:

$$|\frac{f(y) - f(x)}{y - x}| = |f'(c)| \leq M < M + 1 \implies |f(y) - f(x)| < |y - x|(M + 1)$$

for $c \in (x, y)$. Now, if $|y - x| < \delta$, then:

$$|f(y)-f(x)| < |y-x|(M+1) \implies |f(y)-f(x)| < \frac{\epsilon}{M+1}(M+1) \implies |f(y)-f(x)| < \epsilon$$

and thus, we have shown that $f(x)$ is uniformly continuous on $[a, b]$.

Part III. If you are doing proof logging, do four of the five proofs. Mark an X in the score box on page 1 to indicate which proof you have omitted.
If you are opting out of proof logging, do all five proofs.

1. (Proof 5.3)
   The completeness axiom for the real numbers states that every nonempty subset $S \subset \mathbb{R}$ that is bounded above has a least upper bound $\sup S$. Use it to prove that for any two positive real numbers a and b, there exists a positive integer n such that $na > b$.

2. (Proof 6.1)

- Prove that any bounded increasing sequence converges. (You may assume without additional proof the corresponding result, that any bounded decreasing sequence converges.)
- Prove that every sequence (s_n) has a monotonic subsequence.
- Prove the Bolzano-Weierstrass Theorem: every bounded sequence has a convergent subsequence.

3. (Proof 7.4)
   Prove that if f is uniformly continuous on a set S and (s_n) is a Cauchy sequence in S, then $(f(s_n))$ is a Cauchy sequence. Invent an example where f is continuous but not uniformly continuous on S and $(f(s_n))$ is not a Cauchy sequence.

4. (Proof 8.1)

- Prove Rolle's Theorem: if f is a continuous function on $[a, b]$ that is differentiable on (a, b) and satisfies $f(a) = f(b)$, then there exists at least one x in (a, b) such that $f'(x) = 0$.

- Using Rolle's Theorem, prove the Mean Value Theorem: if f is a continuous function on $[a, b]$ that is differentiable on (a, b), then there exists at least one x in (a, b) such that
$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

5. (Proof 8.4, chosen at random from the remaining 12)
   Let f be defined on (a, b) with $a < 0 < b$.
   Suppose that the nth derivative $f^{(n)}$ exists on (a, b).

   Define the remainder

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k.$$

   Prove, by repeated use of Rolle's theorem, that for each $x \neq 0$ in (a, b), there is some y between 0 and x for which

$$R_n(x) = \frac{f^{(n)}(y)}{n!} x^n.$$