Math 23a

Quiz 2 Review

This handout is meant to serve as a guide to what you need to know in your studying, and is not meant to be all comprehensive and a sole replacement for your studies.

1 Week 5

Week 5 was all about setting the foundation for the real analysis module. We explored two main topics this week: the properties of real numbers and sequences. These properties of real numbers are important in our proofs as they allow us to declare the existence of some quantities that may be critical for our proofs (e.g. Denseness to declare the existence of a rational between any two reals). Sequences will form the foundation of all of our study in real analysis, and lead to nifty definitions of typically hard to understand concepts relating to functions. For this week, know/ know how to:

- **Properties of Numbers**
  - **Do inductive and least number proofs**
  - The difference between countable infinity and uncountable infinity
  - The statement and use of the Archimedan property of the real numbers (You can fill up a bathtub with a teaspoon).
  - **Denseness of the Rationals in the Reals**
  - **Least Upper Bound/Greatest Lower Bound property of the reals**
- **Sequences**
  - **Use the formal definition of limit of a sequence to prove convergence of a sequence**
  - **Use Quantifiers to express the concepts of infinitely many, finitely many, limit, etc.**
  - **Add and subtract the same thing, then use the triangle inequality to break up an unknown quantity into two known quantities, followed by the $\epsilon/2$ trick**
  - **Use the limit theorems**
  - **The formal definition of a divergent limit (one that goes to ∞)**

Notice almost all of these are bold... They are super important!

2 Week 6

Week 6 finished the tour of sequences by discussing the important concepts of supremum, $\limsup, \liminf$. We also talked about a useful alternate definition for convergence in the reals, namely being Cauchy, and then proved an important theorem for asserting the existence of a convergent sequence, the Bolzano Weierstrass Theorem. Then we discussed summing the elements of a sequence, formally known as a series, and defined it as a limit of a sequence (of partial sums). Then, we discussed several convergence tests for the convergence of a series. The stuff about sequences this week are all really important, and ubiquitous in real analysis. Series are important to know when the sum of an infinite number of terms can be finite. They also provide the theoretical backing for Taylor Series in week 8. For this week, know/ know how to:

- **Sequences (cont.)**
  - **The definition of supremum, infimum, maximum, minimum, and how to construct a set who's supremum is outside the set**
  - The definition of monotone
  - **The definition of Cauchy sequence, and using the formal definition to prove a sequence is Cauchy**
  - **The definition of** $\limsup, \liminf$**, and when they are equal** (when the overall limit exists, then they are equal)
  - **The Bolzano Weierstrass Theorem and how to use it** (use it when you need to construct a convergent subsequence out of points that are all over the map)
- **Series**
  - **Definition of partial sum, and how the limit of it is defined to be the series sum**
  - The difference between absolute and conditional convergence
  - Famous Series: geometric series, p-series, harmonic series
  - **Convergence Tests: Root Test, Ratio Test, Alternating Series Test**
  - **Find the radius of convergence**

3 Week 7

Week 7 introduced key concepts of **functions**: continuity, uniform continuity, and limits. One key theme throughout the week is that many of these concepts can be defined in terms of sequences, as well as in a formal δ, ϵ way. Knowing when to use which definition is key, and it is important to mix the two. This material will form the basis for talking about differentiability as well as functions in general. For this week, know/know how to:

- **Continuity**
  - **Concept of a good/bad sequence**
  - **The Two Equivalent Definitions: in terms of ϵ, δ, and sequences, and how to use them to prove continuity, discontinuity**
  - Sums of continuous functions are continuous, etc.
  - Theorems involving continuous functions: A function that is bounded on a closed interval achieves its maximum on the interval, and the **Intermediate Value Theorem** (if you start below the x axis, and end up above the x axis, you have to pass through the x axis, and generalizations of this concept)
  - **Using the Intermediate Value Theorem to show the existence of a quantity**
- **Uniform Continuity**
  - **The Two Equivalent Definitions: in terms of ϵ, δ, and Cauchy sequences**
  - An easy way to prove it: **Continuity on a closed interval**
  - How to cook up examples of non uniformly continuous functions
- **Limits**
  - **Two Equivalent Definitions: in terms of ϵ, δ, and sequences**
  - **Limit Rules:** Limits of sums is the sum of limits, etc.

4 Week 8

Week 8 was the beginning of calculus, as the derivative was introduced, and we learned the properties of differentiable functions. We also introduced the Taylor series, which is very important for computing quantities, as well as for making approximations (we will come to understand this in the future). This will prepare us for the next module of multivariable calculus. For this week, know/know how to:

- **The Derivative and Differentiability**
  - **Definition as a limit, and how to use it**
  - Rules of Differentiation: **Sums, Products, Quotients, Powers**, and the **Chain Rule**
  - Properties of Differentiable Functions: **Rolle's Theorem**, and it's generalization the **Mean Value Theorem**
  - Using Rolle's Theorem and the Mean Value Theorem to prove the existence of a quantity.
  - The derivative of a maximum or minimum must be zero.
  - **Derivative of an Inverse function**
  - **Using L'Hospital's Rule to evaluate indeterminate limits**
- **Taylor Series**
  - **Computing a Taylor Series**
  - **Proving Convergence of a Taylor Series using Taylor's Theorem with Remainder**
  - The idea that a function can be defined by its Taylor Series: hyperbolic trig functions