1. Given a manifold described in various ways, how can you determine a basis for the tangent space?
   (a) A parametrization?
   (b) A graph-making function?
   (c) A locus function?
   (d) What is the difference between a tangent space and a tangent plane?

2. Tangent spaces and tangent planes
   (a) The equation $F\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y - 1 = 0$ specifies a one-dimensional manifold in $\mathbb{R}^2$. From this, find a basis for the tangent space to the manifold at the point $(1, 0)$.
   (b) This manifold could also be described by a parametrization
   $$\gamma(t) = \begin{pmatrix} t \\ 1 - t^2 \end{pmatrix}$$
   Using this, find a basis for the tangent space to the manifold at the point $(1, 0)$.
   (c) Using the locus function from part **(a)**, find an equation for the tangent **plane** at this point like Paul did in lecture.
   (d) Do the same using your answer from part **(b)**.

3. Constrained critical points and Lagrange multipliers
   (a) Consider a one-dimensional manifold in $\mathbb{R}^2$ described as the locus of a function $F\begin{pmatrix} x \\ y \end{pmatrix} = 0$. By considering gradients, explain why the constrained critical points of a function $f\begin{pmatrix} x \\ y \end{pmatrix}$ restricted to this manifold will occur at a point where $[Df] = \lambda[DF]$.
   (b) Using Lagrange multipliers, find the constrained critical points of the function $f\begin{pmatrix} x \\ y \end{pmatrix} = y^2 - x$ subject to the constraint $F\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^2 - 4 = 0$.
   (c) The manifold in (b) is a circle with radius 2, and it could be equivalently described by the following parametrization, where $-2\pi \leq \theta \leq 0$:
   $$\gamma(\theta) = \begin{pmatrix} 2\cos(\theta) \\ 2\sin(\theta) \end{pmatrix}$$
   Use this parametrization to solve again for the constrained critical points of $f$.
   As a fun fact to help you here, $4\sin(x)\cos(x) + \sin(x) = 0$ at $x = \arccos\left(\frac{-1}{4}\right)$
   (d) True/False: A constrained critical point of $f$ on a manifold $M$ is always an unconstrained critical point of $f$ as well.
   (e) True/False: If an unconstrained critical point of $f$ occurs at some point $c$ that happens to lie on on a manifold $M$, then $c$ will be a constrained critical point of $f$ restricted to $M$ as well.