MATHEMATICS 23a/E-23a, Fall 2018
Linear Algebra and Real Analysis I
Week 6 (Series, Convergence, Power Series)

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Last modified: August 13, 2018 by Paul Bamberg (special offer on HW)

Reading from Ross

- Chapter 2, sections 10 and 11(pp. 56-77) (monotone and Cauchy sequences, subsequences, introduction to lim sup and lim inf)
- Chapter 2, sections 14 and 15 (pp. 95-109) (series and convergence tests)
- Chapter 4, section 23 (pp.187-192) (convergence of power series)

Recorded Lectures

- Lecture 12 (Week 6, Class 1) (watch on October 16 or 17)
- Lecture 13 (Week 6, Class 2) (watch on October 18 or 19)

Proofs to present in section or to a classmate who has done them.

- 6.1 Bolzano-Weierstrass
  - Prove that any bounded increasing sequence converges. (You may assume without additional proof the corresponding result, that any bounded decreasing sequence converges.)
  - Prove that every sequence (s_n) has a monotonic subsequence.
  - Prove the Bolzano-Weierstrass Theorem: every bounded sequence has a convergent subsequence.
- 6.2 The Root Test
  Consider the infinite series $\sum a_n$ and $\limsup |a_n|^{1/n}$, referred to as α. Prove the following statements about $\sum a_n$:
  - The series converges absolutely if $\alpha < 1$.
  - The series diverges if $\alpha > 1$.
  - If $\alpha = 1$, then nothing can be deduced conclusively about the behavior of the series.

Additional proofs(may appear on quiz, students wiill post pdfs or videos

- 6.3 (Cauchy sequences) A Cauchy sequence is defined as a sequence where $\forall \epsilon > 0$, $\exists N$ s.t. $\forall m, n > N \implies |s_n - s_m| < \epsilon$

  - Prove that any Cauchy sequence is bounded.
  - Prove that any convergent sequence is Cauchy.
  - Prove that any Cauchy sequence of real numbers is convergent. You will need to use something that follows from the completeness of the real numbers. This could be the Bolzano-Weierstrass theorem, or it could the fact that, for a sequence of real numbers, if $\liminf s_n = \limsup s_n = s$, then $\lim s_n$ is defined and

$$\lim s_n = s$$

- 6.4 (Ross, p.188, Radius of Convergence)
  Consider the power series $\sum a_n x^n$. Refer to $\limsup |a_n|^{1/n}$ as β and $1/\beta$ as R. If $\beta = 0, R = +\infty$ and if $\beta = +\infty, R = 0$.)
  Prove the following:

  - If $|x| < R$, the power series converges.
  - If $|x| > R$, the power series diverges.

R Scripts

- Script 2.2A-MoreSequences.R
  Topic 1 – Cauchy Sequences
  Topic 2 – Lim sup and lim inf of a sequence

- Script 2.2B-Series.R
  Topic 1 – Series and partial sums
  Topic 2 – Passing and failing the root test
  Topic 3 – Why the harmonic series diverges

1 Executive Summary

1.1 Monotone sequences

A sequence (s_n) is **increasing** if $s_n \leq s_{n+1}\ \forall n$.
A sequence (s_n) is **strictly increasing** if $s_n < s_{n+1}\ \forall n$.
A sequence (s_n) is **decreasing** if $s_n \geq s_{n+1}\ \forall n$.
A sequence (s_n) is **strictly decreasing** if $s_n > s_{n+1}\ \forall n$.
A sequence that is either increasing or decreasing is called a **monotone** sequence.
All bounded monotone sequences converge.
For an unbounded increasing sequence, $\lim_{n\to\infty} s_n = +\infty$.
For an unbounded decreasing sequence, $\lim_{n\to\infty} s_n = -\infty$.

1.2 Supremum, infimum, maximum, minimum

The **supremum** of a subset S (which is a subset of some set T) is the least element of T that is greater than or equal to all of the elements that are in the subset S. The supremum of the subset S definitely lives in the set T. It may also be in S, but that is not a requirement.
The **supremum** of a sequence the least upper bound of its set of elements.
The **maximum** is the largest value attained within a set or sequence.
It is easy to find examples of sets or sequences for which no supremum exists, or for which a supremum exists but a maximum does not.

The **infimum** of a sequence is the "greatest lower bound," or the greatest element of T that is less than or equal to all of the elements that are in the subset S. It is not the same as a **minimum**, because the minimum must be achieved in S, while the infimum may be an element of only T.

1.3 Cauchy sequences

A sequence is a *Cauchy sequence* if

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall m, n > N,\ |s_n - s_m| < \epsilon$$

Both convergent and Cauchy sequences must be bounded.
A convergent sequence of real numbers or of rational numbers is Cauchy.
A Cauchy sequence *of real numbers* is convergent.
It is easy to invent a Cauchy sequence of rational numbers whose limit is an irrational number.
Off the record: quantum mechanics is done in a "Hilbert space," one of the requirements for whichis that every Cauchy sequence is convergent. Optimization problems in economics are frequently formualated in a "Banach space," which has the same requirement.

1.4 lim inf and lim sup

Given any bounded sequence, the "tail" of the sequence, which consists of the infinite number of elements beyond the Nth element, has a well-defined supremum and infimum.

Let us combine the notion of limit with the definitions of supremum and infimum. The "limit infimum" and "limit supremum" are written and defined as follows:

$$\liminf s_n = \lim_{N\to\infty} \inf\{s_n : n > N\}$$

$$\limsup s_n = \lim_{N\to\infty} \sup\{s_n : n > N\}$$

The limit supremum is defined in a parallel manner, only considering the supremum of the sequences instead of the infimum.

Now that we know the concepts of lim inf and lim sup, we find the following properties hold:

- If $\lim s_n$ is defined as a real number or $\pm\infty$, then

$$\liminf s_n = \lim s_n = \limsup s_n$$

- If $\liminf s_n = \limsup s_n$, then $\lim s_n$ is defined and

$$\lim s_n = \liminf s_n = \limsup s_n$$

- For a Cauchy sequence of real numbers, $\liminf s_n = \limsup s_n$, and so the sequence converges.

1.5 Subsequences and the Bolzano-Weierstrass theorem

A subsequence is a sequence obtained by selecting an infinite number of terms from the "parent" sequence in order.
If (s_n) converges to s, then any subsequence selected from it also converges to s.

Given any sequence, we can construct from it a monotonic subsequence, either an increasing whose limit is $\limsup s_n$, a decreasing sequence whose limit is $\liminf s_n$, or both. If the original sequence is bounded, such a monotonic sequence must converge, even if the original sequence does not.
This construction proves one of the most useful results in all of mathematics, the Bolzano-Weierstrass theorem:
Every bounded sequence has a convergent subsequence.

1.6 Infinite series, partial sums, and convergence

Given an infinite series Σa_n we define the **partial sum**

$$s_n = \sum_{k=m}^{n} a_k$$

The lower limit m is usually either 0 or 1.

The series $\sum_{k=m}^{\infty} a_k$ is said to **converge** when the limit of its partial sums as $n \to \infty$ equals some number S. If a series does not converge, it is said to **diverge**. The sum $\sum a_n$ has no meaning unless its sequence of partial sums either converges to a limit S or diverges to either $+\infty$ or $-\infty$.
A series with all positive terms will either converge or diverge to $+\infty$.
A series with all negative terms will either converge or diverge to $-\infty$.
For a series with both positive and negative terms, the sum $\sum a_n$ may have no meaning.
A series is called **absolutely convergent** if the series $\sum |a_n|$ converges.
Absolutely convergent series are also convergent.

1.7 Familiar examples

A **geometric series** is of the form

$$a + ar + ar^2 + ar^3 + \dots$$

If $|r| < 1$, then

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

A **p-series** is of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

for some positive real number p. It converges if $p > 1$, diverges if $p \leq 1$.

1.8 Cauchy criterion

. We say that a series satisfies the Cauchy criterion if the sequence of its partial sums is a Cauchy sequence. Writing this out with quantifiers, we have

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall m, n > N, \; |s_n - s_m| < \epsilon$$

Here is a restatement of the Cauchy criterion, which proves more useful for some proofs:

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall n \geq m > N, \; |\sum_{k=m}^{n} a_k| < \epsilon$$

A series converges if and only if it satisfies the Cauchy criterion.

1.9 Convergence tests

- **Limit of the terms.** If a series converges, the limit of its terms is 0.

- **Comparison Test.** Consider the series $\sum a_n$ of all positive terms.
  If $\sum a_n$ converges and $|b_n| \leq a_n$ for all n then $\sum b_n$ also converges.
  If $\sum a_n$ diverges to $+\infty$ and $|b_n| > a_n$ for all n, then $\sum b_n$ also diverges to $+\infty$

- **Ratio Test.** Consider the series $\sum a_n$ of nonzero terms.
  This series converges if $\limsup |\frac{a_{n+1}}{a_n}| < 1$
  This series diverges if $\liminf |\frac{a_{n+1}}{a_n}| > 1$
  If $\liminf |\frac{a_{n+1}}{a_n}| \leq 1 \leq \limsup |\frac{a_{n+1}}{a_n}|$, then we have no information and need to perform another test to determine convergence.

- **Root Test.** Consider the series $\sum a_n$, and evaluate $\limsup |a_n|^{1/n}$.
  If $\limsup |a_n|^{1/n} < 1$, the series $\sum a_n$ converges absolutely.
  If $\limsup |a_n|^{1/n} > 1$, the series $\sum a_n$ diverges.
  If $\limsup |a_n|^{1/n} = 1$, the test gives no information.

- **Integral Test.** Consider a series of nonnegative terms for which the other tests seem to be failing. In the event that we can find a function $f(x)$, such that $f(n) = a_n\ \forall n$, we may look at the behavior of this function's integral to tell us whether the series converges.
  If $\lim_{n\to\infty} \int_1^n f(x)dx = +\infty$, then the series will diverge.
  If $\lim_{n\to\infty} \int_1^n f(x)dx < +\infty$, then the series will converge.

- **Alternating Series Test.** If the absolute value of the each term in an alternating series is decreasing and has a limit of zero, then the series converges.

1.10 Convergence tests for power series

Power series are series of the form

$$\sum_{n=0}^{\infty} a_n x^n$$

where the sequence (a_n) is a sequence of real numbers. A power series defines a function of x whose domain is the set of values of x for which the series converges. That, of course, depends on the coefficients (a_n). There are three possibilities:
– Converges $\forall x \in \mathbb{R}$.
– Converges only for $x = 0$.
– Converges $\forall x$ in some interval, centered at 0. The interval may be open $(-R, R)$, closed $[-R, R]$, or a mix of the two like $[-R, R]$. The number R is called the *radius of convergence*. Frequently the series converges absolutely in the interior of the interval, but the convergence at an endpoint is only conditional.

2 Lecture Outline

1. Supremum, infimum, maximum, minimum
   The **supremum** of a subset S (which is a subset of some set T) is the least element of T that is greater than or equal to all of the elements that are in the subset S. The supremum of the subset S definitely lives in the set T. It may also be in S, but that is not a requirement.
   The **supremum** of a sequence the least upper bound of its set of elements.
   The **maximum** is the largest value attained within a set or sequence.

   Invent a sequence for which no supremum exists.

   Invent a sequence for which a supremum exists but a maximum does not.

   The **infimum** of a subset $T \subset S$ is the "greatest lower bound," or the greatest element of T that is less than or equal to all of the elements that are in the subset S. It is not the same as a **minimum**, because the minimum must be achieved in S, while the infimum may be an element only of T.

2. Monotone sequences

Terminology used in the latest edition of Ross (available through HOLLIS):

(a) A sequence (s_n) is **increasing** if $s_n \leq s_{n+1}$ $\forall n$.

(b) A sequence (s_n) is **strictly increasing** if $s_n < s_{n+1}$ $\forall n$.

(c) A sequence (s_n) is **decreasing** if $s_n \geq s_{n+1}$ $\forall n$.

(d) A sequence (s_n) is **strictly decreasing** if $s_n > s_{n+1}$ $\forall n$.

(e) A sequence that is either increasing or decreasing is called a **monotone** sequence.

If you own an earlier edition of Ross, beware:
He used "nondecreasing" for what is now called "increasing" .
He used "increasing" for what is now called "strictly increasing."

Useful and easy-to-prove results for increasing sequences of real numbers.
A bounded increasing sequence converges to its least upper bound.

For an unbounded increasing sequence, $\lim s_n = +\infty$.

3. Defining a sequence recursively (model for group problems, set 1)

   John's rich parents hope that a track record of annual gifts to Harvard will enhance his chance of admission. On the day of his birth they set up a trust fund with a balance $s_0 = 1$ million dollars. On each birthday they add another million dollars to the fund, and the trustee immediately donates 1/3 of the fund to Harvard in John's name. After the donation, the balance is therefore

   $$s_{n+1} = \frac{2}{3}(s_n + 1).$$

   - Find the annual fund balance up through s_2.
   - Use induction to show $s_n < 2$ for all n.
   - Show that (s_n) is an increasing sequence.
   - Show that $\lim s_n$ exists and find $\lim s_n$.

4. (Ross, p. 62, convergent & Cauchy sequences)
A Cauchy sequence is defined as a sequence where
$\forall \epsilon > 0$, $\exists N$ s.t. $\forall m, n > N \implies |s_n - s_m| < \epsilon$.

(a) Prove that any Cauchy sequence is bounded.

(b) Prove that any convergent sequence is Cauchy.

5. (Ross, pp. 60-62, limits of the supremum and infimum)
The limit of the supremum, written "lim sup" is defined as follows:

$$\limsup s_n = \lim_{N\to\infty} \sup\{s_n : n > N\}$$

The limit of the infimum, written "lim inf" is defined as follows:

$$\liminf s_n = \lim_{N\to\infty} \inf\{s_n : n > N\}$$

(We do not restrict s_n to be a bounded sequence, so if it is not bounded above, $\limsup s_n = +\infty$, and if it is not bounded below, $\liminf s_n = -\infty$)
With these definitions, $\limsup s_n$ and $\liminf s_n$ exist for every sequence (s_n). These are difficult concepts, but we will need them in order to make correct statements of convergence tests for infinite series.

An easy example: If $s_n = (0, \frac{1}{2}, -2, \frac{3}{4}, -4, \frac{7}{8}, -8, \cdots)$

What is $\limsup s_n$?

What is $\liminf s_n$?

A slightly harder example: If $s_n = (\frac{3}{2}, -\frac{3}{2}, \frac{4}{3}, -\frac{4}{3}, \frac{5}{4}, -\frac{5}{4}, \cdots)$

What is $\limsup s_n$?

What is $\liminf s_n$?

6. Existence of a limit in terms of lim sup and lim inf

   Let (s_n) be a sequence in $\mathbb{R}$. Prove that if $\liminf s_n = \limsup s_n = s$, then $\lim s_n$ is defined and
$$\lim s_n = s.$$

7. Examples of subsequences

   Consider the sequence
   $$s_n = \frac{n+2}{n+1}\sin(\frac{n\pi}{4}),$$
   give three examples of a subsequence, find the $\limsup$ and the $\liminf$, and determine whether it converges.

   To get started, write out the first few terms.

8. Determine the $\limsup$ and the $\liminf$ for this sequence.

9. Invent a subsequence for which the $\limsup$ is positive but less than 1 and the $\liminf$ is 0.

10. Invent a subsequence that converges to a negative number.

11. (Ross, p. 64, convergent and Cauchy sequences)

    Using the result of the preceding proof, which relies on the completeness axiom for the real numbers, prove that any Cauchy sequence of real numbers is convergent.

12. (Convergent subsequences, Bolzano Weierstrass)
Given a sequence $(s_n)_{n\in\mathbb{N}}$, a subsequence of this sequence is a sequence $(t_k)_{k\in\mathbb{N}}$, where for each k, there is a positive integer n_k such that

$$n_1 < n_2 < \ldots < n_k < n_{k+1} \ldots$$

and $t_k = s_{n_k}$. So (t_k) is just a sampling of some, or all, of the (s_n) terms, with order preserved.
A term s_n is called *dominant* if it is greater than any term that follows it.

(a) Use the concept of dominant term to prove that every sequence (s_n) has a monotonic subsequence.

(b) Prove the Bolzano-Weierstrass Theorem: every bounded sequence has a convergent subsequence.

13. Infinite series, partial sums, and convergence
Given an infinite series Σa_n we define the **partial sum**

$$s_n = \sum_{k=m}^{n} a_k$$

The lower limit m is usually either 0 or 1.

The series $\sum_{k=m}^{\infty} a_k$ is said to **converge** when the limit of its partial sums as $n \to \infty$ equals some number S. If a series does not converge, it is said to **diverge**. The sum $\sum a_n$ has no meaning unless its sequence of partial sums either converges to a limit S or diverges to either $+\infty$ or $-\infty$.
A series with all positive terms will either converge or it will diverge to $+\infty$.
A series with all negative terms will either converge or it will diverge to $-\infty$.
For a series with both positive and negative terms, the sum $\sum a_n$ may have no meaning.
A series is called **absolutely convergent** if the series $\sum |a_n|$ converges.
Absolutely convergent series are always convergent.

A series is called **conditionally convergent** if the series $\sum a_n$ converges but the series $\sum |a_n|$ diverges.

14. A cautionary tale about conditional convergence.

What is the fallacy in the following argument?

- $$\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots .$$
- $$\frac{1}{2}\log_e 2 = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots .$$
- $$\frac{3}{2}\log_e 2 = 1 - \frac{1}{4} - \frac{1}{4} + \frac{1}{3} - \frac{1}{8} - \frac{1}{8} + \frac{1}{5} - \frac{1}{12} - \frac{1}{12} + \cdots = \log_e 2.$$
- $$\frac{3}{2} = 1; 3 = 2; 1 = 0.$$

In fact, the general situation is even worse that this example suggests. In a conditionally convergent series

- The sum of the positive terms is $+\infty$.
- The sum of the negative terms is $-\infty$.

The Riemann series theorem states that it is possible to reorder the terms of a conditionally convergent series to achieve any of the following:

- The sequence of partial sums converges to any desired value a.
- The sequence of partial sums diverges to $+\infty$.
- The sequence of partial sums diverges to $-\infty$.

Suppose that the terms a_n are a conditionally convergent infinite series of revenue items (positive) and expense items (negative) for your startup company. How could you exploit the Riemann series theorem to create alternative business plans with any desired long-term outcome?
(Beware: CFOs who do this sometimes end up in prison!)

15. Non-negative terms: order does not matter

If all the a_i are non-negative and the series converges, the order of terms in the series is irrelevant. We can express the sum in a way that is independent of the order of the terms as

$$S' = \sup \sum_{i \in A \subset \mathbb{N}} a_i$$

where the supremum is over all finite subsets of $\mathbb{N}$.

We need to prove that S' is equal to S, the limit of the sequence $S_0, S_1, ...$

The sum S is the limit of a nondecreasing sequence, so it can be expressed as

$$S = \sup S_N$$

How does this establish that $S' \geq S$?

Any finite subset A is a subset of S_N for some N. How does this establish that $S' \leq S$?

But if $S' \geq S$ and $S' \leq S$, then $S' = S'$, and the proof is complete.

On the homework you can prove that if a series has both positive and negative terms but $\sum_i |a_i|$ converges, then the order in which the terms are summed is irrelevant. Such a series is absolutely convergent, not conditionally convergent.

16. (Ross, p. 96, Example 1, geometric series (refers also to p. 98))

Prove that

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} \text{ if } |r| < 1,$$

and that the series diverges if $|r| \geq 1$.

For the sake of novelty, do the first part of the proof by using the least-number principle insted of by induction.

Given a repeating decimal, you can write this number as a geometric series. Write the repeating decimal $0.363636363\cdots$ as a geometric series, and use the formula

$$\frac{a}{1-r}$$

to show that it is equal to $4/11$.

17. Cauchy criterion

We say that a series satisfies the Cauchy criterion if the sequence of its partial sums is a Cauchy sequence. Writing this out with quantifiers, we have

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall m, n > N, \ |s_n - s_m| < \epsilon$$

Here is a restatement of the Cauchy criterion, which proves more useful for some proofs:

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall n \geq m > N, \ |\sum_{k=m}^{n} a_k| < \epsilon$$

A series converges if and only if it satisfies the Cauchy criterion.

Use the Cauchy criterion to prove two simple but useful convergence tests.

- **Limit of the terms.** For a series to converge, the limit of its terms must be 0.

- **Comparison Test.** Consider the series $\sum a_n$ of all positive terms.
  If $\sum a_n$ converges and $|b_n| \leq a_n \ \forall n$ then $\sum b_n$ also converges.
  If $\sum a_n$ diverges to $+\infty$ and $b_n > a_n \ \forall n$, then $\sum b_n$ diverges to $+\infty$.

18. Clever proofs for p-series.

   (a) Prove that $\sum \frac{1}{n} = +\infty$ by showing that the sequence of partial sums is not a Cauchy sequence.

   (b) Evaluate

$$\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$$

   by exploiting the fact that this is a "telescoping series."

   (c) Prove that

$$\sum_{n=2}^{\infty} \frac{1}{n^2}$$

   is convergent.

19. (Ross, p.99, The Root Test)
Consider the infinite series $\sum a_n$ and $\limsup |a_n|^{1/n}$, referred to as α.
Prove the following statements about $\sum a_n$:
(you may assume the Comparison Test as proven)

- The series converges absolutely if $\alpha < 1$.
- The series diverges if $\alpha > 1$.
- If $\alpha = 1$, then nothing can be deduced conclusively about the behavior of the series.

20. (Ross, pp. 99-100, The Ratio Test)
Let $\sum a_n$ be an infinite series of nonzero terms.
Prove the following, assuming the Root Test as proven.
We will use without proof the following result from Ross (theorem 12.2):

$$\liminf |\frac{s_{n+1}}{s_n}| \leq \liminf |s_n|^{\frac{1}{n}} \leq \limsup |s_n|^{\frac{1}{n}} \leq \limsup |\frac{s_{n+1}}{s_n}|$$

- If $\limsup |a_{n+1}/a_n| < 1$, then the series converges absolutely.
- If $\liminf |a_{n+1}/a_n| > 1$, then the series diverges.
- If $\liminf |a_{n+1}/a_n| \leq 1 \leq \limsup |a_{n+1}/a_n|$, then the test gives no information.

21. A case where the root test outperforms the ratio test
(Ross, Example 8 on page 103)

$$\sum_{n=0}^{\infty} 2^{(-1)^n - n} = 2 + \frac{1}{4} + \frac{1}{2} + \frac{1}{16} + \frac{1}{8} + \frac{1}{64} + \cdots .$$

(a) Show that the ratio test fails totally.

(b) Show that the root test correctly concludes that the series is convergent.

(c) Find a simpler argument using the comparison test.

22. (Ross, p.188, Radius of Convergence)
Consider the power series $\sum a_n x^n$. Let us refer to $\limsup |a_n|^{1/n}$ as β and $1/\beta$ as R.
Limiting cases: if $\beta = 0, R = +\infty$ and if $\beta = +\infty, R = 0$.
Prove the following:

- If $|x| < R$, the power series converges.
- If $|x| > R$, the power series diverges.

(You may recognize R here as the radius of convergence.)

23. Tests that are useful when the root test and the ratio test fail, which is often the case at the endpoints of the interval of convergence of a power series.

Integral Test. Consider a series of nonnegative terms for which the other tests seem to be failing. In the event that we can find a function $f(x)$, such that $f(n) = a_n \ \forall n$, we may look at the behavior of this function's integral to tell us whether the series converges.
If $\lim_{n\to\infty} \int_1^n f(x)dx = +\infty$, then the series will diverge.
If $\lim_{n\to\infty} \int_1^n f(x)dx < +\infty$, then the series will converge.
Use this test to provide an alternate proof that $\sum \frac{1}{n}$ diverges but $\sum \frac{1}{n^2}$ converges.

Alternating Series Test. If the absolute value of the each term in an alternating series a_n is decreasing and has a limit of zero, then the series converges.
Ross proves this on page 108 by using the Cauchy criterion. Prove it instead by considering the sequence of partial sums, (s_n), and showing that $\liminf s_n = \limsup s_n$.

24. (Model for group problems, set 3 – to be done as time permits) Find the radius of convergence and the exact interval of convergence for the series

$$\sum_{n=0}^{\infty} \frac{n}{2^n} x^{3n} \text{ by using the Root Test.}$$

Then use appropriate tests for the following examples.

$$\sum (\frac{2^n}{n!}) x^n$$

$$\sum n! x^n.$$

3 Seminar Topics

Your section instructor will either have emailed a list of topics to prepare or will have posted a signup list of appointments on the Calendar tab of Canvas. Either way, there will be one of the following topics that you should be prepared to present.

Practice your presentation so that it takes about 8 minutes. The text of the presentation will be projected onto a screen so that you need not recopy it. To save time, avoid writing long sentences on the chalkboard. You may use notes, but be discreet about it.

1. (Proof 6.1 – Bolzano-Weierstrass)

   - Prove that any bounded increasing sequence converges. (You may assume without additional proof the corresponding result, that any bounded decreasing sequence converges.)
   - Prove that every sequence (s_n) has a monotonic subsequence.
   - Prove the Bolzano-Weierstrass Theorem: every bounded sequence has a convergent subsequence.

2. For sequence (s_n), define $\limsup s_n$, and illustrate your definition for these two sequences, neither of which converges:

   - $(s_n) = (2, 0, \frac{3}{2}, 0, \frac{4}{3}, 0, \frac{5}{4}, 0, \cdots)$
   - $(t_n) = (0, 0, \frac{1}{2}, 0, \frac{2}{3}, 0, \frac{3}{4}, 0, \cdots)$

   Then prove the following properties of lim and lim sup:

   - If $\lim s_n = s$, then $\forall \epsilon > 0$,
     there are **only finitely many** s_n for which $s_n \geq s + \epsilon$
     and there are **only finitely many** s_n for which $s_n \leq s - \epsilon$.
   - If $\limsup s_n = s$, then $\forall \epsilon > 0$,
     there are **only finitely many** s_n for which $s_n \geq s + \epsilon$
     and there are **infinitely many** s_n for which $s_n > s - \epsilon$.

3. (Proof 6.2 – The Root Test)
   Consider the infinite series $\sum a_n$ and $\limsup |a_n|^{1/n}$, referred to as α.
   Prove the following statements about $\sum a_n$:

   - The series converges absolutely if $\alpha < 1$.
   - The series diverges if $\alpha > 1$.
   - If $\alpha = 1$, then nothing can be deduced conclusively about the behavior of the series.

4. (Proof 6.3 – Cauchy sequences)
A Cauchy sequence is defined as a sequence where $\forall \epsilon > 0$, $\exists N$ s.t. $\forall m, n > N \implies |s_n - s_m| < \epsilon$

   - Prove that any Cauchy sequence is bounded.
   - Prove that any convergent sequence is Cauchy.
   - Prove that any Cauchy sequence of real numbers is convergent. You will need to use something that follows from the completeness of the real numbers. This could be the Bolzano-Weierstrass theorem, or it could the fact that, for a sequence of real numbers, if $\liminf s_n = \limsup s_n = s$, then $\lim s_n$ is defined and $\lim s_n = s$.

     A simpler strategy than the one that Ross uses is to prove the contrapositive: assume that (s_n) is not convergent, so that
     $\limsup s_n - \liminf s_n = 3\epsilon > 0$,
     and prove that (s_n) is not Cauchy.

5. (Proof 6.4 – Radius of Convergence)
Consider the power series $\sum a_n x^n$. Refer to $\limsup |a_n|^{1/n}$ as β and $1/\beta$ as R. If $\beta = 0, R = +\infty$ and if $\beta = +\infty, R = 0$.)
Prove the following:

   - If $|x| < R$, the power series converges.
   - If $|x| > R$, the power series diverges.

6. (Extra topic – see pages 21 and 22 in the lecture outline)
State the Cauchy criterion for convergence of series, and use it to prove the following results, which are traditionally proved by the integral test:

   - The series

$$\sum_{k=1}^{\infty} \frac{1}{k} \text{ diverges. (Hint: consider } \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k}.\text{)}$$

   - The series

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \text{ converges.}$$

$$\text{Hint: } \sum_{k=m}^{n} \frac{1}{k^2} < \sum_{k=m}^{n} \frac{1}{k(k-1)} = \sum_{k=m}^{n} \left(\frac{1}{k-1} - \frac{1}{k}\right).$$

4 Workshop Problems

1. Working with lim sup and lim inf

   (a) (Ross, 12.4)
   Show that $\limsup(s_n + t_n) \le \limsup s_n + \limsup t_n$ for bounded sequences (s_n) and (t_n), and invent an example where
   $\limsup(s_n + t_n) < \limsup s_n + \limsup t_n$.
   Here is the hint from page 82 of Ross : first show that

   $$\sup\{s_n + t_n : n > N\} \le \sup\{s_n : n > N\} + \sup\{t_n : n > N\}$$

   You may use Ross Exercise 9.9 c, which can be stated as
   If $(a_n) \to a, (b_n) \to b$, and $\exists N_0$ s.t. $\forall n > N_0, a_n \le b_n$, then $a \le b$.
   A useful trick might be to replace t_n on the left-hand side by
   $v_N = \sup\{t_n : n > N\}$, which does not depend on n.

   (b) The following famous series, known as Gregory's series but discovered by the priest-mathematicians of southwest India long before James Gregory (1638-1675) was born, converges to $\frac{\pi}{4}$.

   $$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots .$$

      i. For the sequence of partial sums (s_n), find an increasing subsequence and a decreasing subsequence.
      ii. Prove convergence by showing that $\limsup s_n = \liminf s_n$.
      Hint: Show that $\sup\{s_n : n > N\} - \inf\{s_n : n > N\}$ is equal to the magnitude of one term in the series.
      iii. Prove that the series is not absolutely convergent by showing that it fails the Cauchy test with $\epsilon = 1/4$.

2. Sequences, defined recursively

   If someone in your group is skillful with R, you can use it to calculate a lot of terms of the sequence. By modifying script 2.2C, you can easily plot the first 20 or so terms. It you come up with a good R script, please upload it to the solutions page.

   Once you have shown that $\lim s_n$ exists, you can assert that $\lim s_{n+1} = \lim s_n = s$.

   (a) (Ross, 10.9) Let $s_1 = 1$ and $s_{n+1} = (\frac{n}{n+1})s_n^2$ for $n > 1$.
      - Find s_2, s_3, s_4 if working by hand. If using R, use a for loop to go at least as far as s_{20}.
      - Show that $\lim s_n$ exists.
      - Prove that $\lim s_n = 0$.

   (b) (Ross, 10.12) Let $t_1 = 1$ and $t_{n+1} = [1 - \frac{1}{(n+1)^2}]t_n$ for $n > 1$.
      - Find t_2, t_3, t_4 if working by hand. If using R, use a for loop to go at least as far as t_{20}.
      - Show that $\lim t_n$ exists.
      - Use induction to show $t_n = \frac{n+1}{2n}$ for all n.
      - Find $\lim t_n$.

This last set of problems should be done using LaTeX or in the Canvas editor. They provide good practice with summations, fractions, and exponents.

3. Applying convergence tests to power series (Ross, 23.1 and 23.2)

   Find the radius of convergence R and the exact interval of convergence.

   In each case, you can apply the root test (works well with powers) or the ratio test (works well with factorials) to get an equation that can be solved for x to get the radius of convergence R. Since you have an x^n, the root test, which you may not have encountered in AP calculus, is especially useful. At the endpoints you may need to apply something like the alternating series test or the integral test.

   Remember that $\lim n^{1/n} = 1$.

   (a)
   $$\sum(\frac{3^n}{n \cdot 4^n})x^n \text{ and } \sum \sqrt{n}x^n.$$

   (b)
   $$\sum(\frac{(-1)^n}{n^2 4^n})x^n \text{ and } \sum \frac{3^n}{\sqrt{n}}x^n.$$

5 Homework

Special offer – if you do the entire problem set, with one problem omitted, in LaTeX, you will receive full credit for the omitted problem. Alternatively, if you work all the problems in LaTeX, we will convert your lowest score to a perfect score.

1. Ross, 10.2 (Prove all bounded decreasing sequences converge.)
2. Ross, 10.6,
3. Ross, 11.8.
4. Suppose that (s_n) is a Cauchy sequence and that the subsequence $(s_1, s_2, s_4, s_8, s_{16}, \cdots)$ converges to s. Prove that $\lim s_n = s$. Hint – use the standard bag of tricks: the triangle inequality, epsilon-over-2, etc.
5. If a series $\sum_i a_i$ has both positive and negative terms but $\sum_i |a_i|$ converges, then the series is said to be absolutely convergent. If S_+ denotes the least upper bound for subsets of positive terms and S_- denotes the greatest lower bound for subsets of negative terms, then the sum $S = \sum_i a_i$ can be written as $S = S_+ + S_-$. This formula makes it clear that the order in which the terms are summed is irrelevant. As defined, S_- is a negative number.

   (a) Start with equation 0.5.7, which is proved at the bottom of page 20 of Hubbard. Re-express this formula by writing the sums on the right in terms of S_+ and S_-, and thereby show that $S = S_+ + S_-$.

   (b) Hubbard's proof of Theorem 0.5.8 treats positive and negative terms in an unsymmetrical way. Create a new, more symmetrical version of the proof by using Hubbard's b_m and also defining

   $$c_m = \sum_{n=1}^{m}(|a_n| - a_n).$$

   In your proof, be careful to change the order only in finite sums, and make it clear how you are using Theorem 0.5.7. You can use the result of theorem 1.5.16, which you proved in the previous problem, in the case $n = 1$.
6. Ross, 14.3 (Determining whether a series converges. Apologies to those who have already done hundreds of these in a high-school course.)
7. Ross, 14.8.
8. Ross, 15.6.

9. Ross, 23.4. You might find it useful to have R generate some terms of the series.
10. Ross, 23.5.