1 General Course Info

1.1 Office Hours

Office hours are sessions that your CAs and TFs hold to help you with anything regarding the course. They are spread all over the week, and the staff will announce the time and location of their office hours soon. **Different CAs have slightly different office hour styles, and you are more than welcome to attend any office hours, not just your own CA's.** Office hours are held in spaces like dining halls or classrooms, where students can sit, work on their homework/study other related material, collaborate with others, and ask CAs and TFs questions. Depending on the CA/TF, some may be a bit more centralized, with the CA trying to help with problems that most students might find a bit challenging.

1.2 Homework

Weekly problem sets are included at the end of the posted lecture notes on Canvas. They are usually about 8 problems long. They are due on Tuesday nights at 11:59 pm. You should submit either a LaTeX pdf file or a scanned pdf of your handwritten solutions.

At some point, there will probably be questions on a pset that you don't even know where to start with. This is ok and totally natural! The best things to do to make the most of the P-sets are: (Some of these may seem like obvious points, but they're just mentioned as reminders)

1. Start early! That way, you can give each question a good amount of thought on your own, and you can make the most of office hours too.

2. After you have worked on the problems individually, it is enjoyable and helpful to check answers with your friends. We want Math 23 to be a social experience in addition to an academic one. That said, there are certain points you need to be mindful about, just to make sure that you are complying with the academic integrity policy of the course, the Harvard College Honor Code, and the overall standards and values of our scholarly community. We ask you to engage with your classmates to learn from them, and learn how to express mathematical ideas clearly, and we also ask that you write up your answers individually. **Also, please list the name of the students you collaborated with at the top of each assignment in order to give them credit for their ideas.**

3. Come to office hours!

4. Make sure that you have studied the notes carefully! A lot of the times, questions are based off of examples covered in lecture (though they are not going to be identical). You can save yourself a lot of time and energy if you know the topics well and know where to look for each example.

1.3 Prooflog

Prooflog (174.138.58.67) is the course's web page for logging proof points that you earn by presenting proofs to other students in the course. There are a total of 26 "numbered" proofs (e.g. 1.1, 1.2, 2.1,...) listed in each week's lecture notes. Paul walks through each proof, and some of these proofs (chosen at random) will appear on each quiz and the final.

To help you learn these proofs, the course has a proof presentation system. **This year, the proof presentation scheme is optional, but we highly recommend doing it.** You present proofs to course staff or fellow students who have already presented those proofs to others. If you present the proof before the corresponding quiz/exam, you earn 0.95 points for presenting it. (If you present after the quiz, you earn 0.8 points.) For each person who presents a proof to you, you earn an additional 0.1 points. At the end of the semester (assuming you don't opt out), you will receive a grade for proof presentation calculated as [points you earned]/26. The total number of points you can earn is 30 (including points for listening to proofs), so you can earn extra credit by listening. Prooflog is the system that is used to log the points you accumulate.

If you opt out of proof logging, you will not receive a proof presentation grade. However, whereas if you participate in proof presentation you get to opt out of answering some of the proof questions on the quizzes and the final.

You can present proofs in office hours, proof parties (will be announced) or any casual gathering with the math23a people. Some of the seminar topics are also numbered proofs, and you can receive proof points for presenting them in seminar–just log the proof with your section leader or CA.

2 Major Concepts

1. Injectivity and surjectivity
   (a) Functions by definition act on everything in their domain.
   (b) Injective functions: each element in the domain gets mapped to a distinct element in the codomain. (But it's not necessarily the case that every element in the codomain gets "hit.")
   (c) Surjective functions: each element in the codomain gets mapped to by something. (But it's not necessarily the case that distinct elements map to distinct elements.)
   (d) Bijective functions: both injective and surjective
2. Domain and Codomain in Matrices
   (a) The number of columns is the dimension of the domain, the number of rows is the dimension of the codomain.
   (b) I always confuse which is which, so I'd like to think about them in terms of matrix multiplication and composition of functions:
   Example:

   We have a function f that is represented by an m x n matrix $[A]$, and function g, represented by an n x q matrix $[B]$. Since they have the dimension n in common, we can multiply the two and get AB which is an m x q matrix.
   At the same time, we know that matrix multiplication represents composition of functions. So, AB represents fog which means, we apply g first, and then apply f to the result:
   $$something1 \xrightarrow{\text{g}} something2 \xrightarrow{\text{f}} something3$$
   Now let's figure out what each of the "somethings" are! We can immediately figure out that something2 should be F^n, since the common dimension between $[A]$ and $[B]$ is n.

It follows that something1 is F^q, and something3 is F^m. we can see now, that dimensions of the domains for each function corresponds to the number of columns of the matrix that represents it. Similarly, the dimension of the codomain is equal to the number of rows of the representative matrices.

3. Injectivity, Surjecitvity, and matrix dimensions

   (a) There is no way that an m x n matrix with $m > n$ can be surjective, becuase codomain has some "extra dimensions" that the domain can't reach to, we can't "hit" all of them.

   (b) There is no way that a p x q matrix with $p < q$ can be injective. Since the codomain has smaller dimension, some elements of the domain hit the same thing in the codomain.

4. Matrices, and Matrices as Functions (Source: taken directly from executive summaries on Canvas)

   An $m \times n$ **matrix** over a field F has m rows and n columns.
   Matrices represent linear functions, also known as **linear transformations**:
   A function $\mathbf{g} : F^n \to F^m$ is called linear if

   $$g(a\vec{\mathbf{v}} + b\vec{\mathbf{w}}) = ag(\vec{\mathbf{v}}) + bg(\vec{\mathbf{w}}).$$

   For a linear function $\mathbf{g}$, if we know the value of $\mathbf{g}(\vec{\mathbf{e}}_i)$ for each standard basis vector $\vec{\mathbf{e}}_\mathbf{i}$, the value of $\mathbf{g}(\vec{\mathbf{v}})$ for any vector v follows by linearity:

   $$\mathbf{g}(v_1\vec{\mathbf{e}}_1 + v_2\vec{\mathbf{e}}_2 + \cdots + v_n\vec{\mathbf{e}}_n) = v_1\mathbf{g}(\vec{\mathbf{e}}_1) + v_2\mathbf{g}(\vec{\mathbf{e}}_2) + \cdots + v_n\mathbf{g}(\vec{\mathbf{e}}_n)$$

   The matrix G that represents the linear function $\mathbf{g}$ is formed by using $\mathbf{g}(\vec{\mathbf{e}}_k)$ as the kth column. Then, if $g_{i,j}$ denotes the entry in the ith row and jth column of matrix G, the function value $\vec{\mathbf{w}} = \mathbf{g}(\vec{\mathbf{v}})$ can be computed by the rule

   $$w_i = \sum_{j=1}^{n} g_{i,j} v_j$$

5. Categories
   Every category should have the following characteristics:

   (a) Have finite objects.
   (b) Have arrows from one object to another
   (c) Closed under composition of arrows
   (d) An identity arrow exists
   (e) Composition of arrows is associative

6. Finite fields

   (a) Is $\mathbb{Z}_8$ a (finite) field?

(b) Calculate $[3]_5 * [4]_5$.

(c) Calculate $\frac{[3]_5}{[4]_5}$.

7. A field axiom proof
We define $\frac{a}{b}$ by the equation $b * \frac{a}{b} = a$. Given this (and starting with it), use the field axioms to show that $\frac{a}{b} = a * b^{-1}$. Justify each step in your proof by listing the corresponding axiom.