MATHEMATICS 23a/E-23a, Fall 2018
Linear Algebra and Real Analysis I
Week 9 (Topology, sequences and series of vectors and matrices)

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Reading

- Hubbard, Section 1.5. The only topology that is treated is the "open-ball topology."

Alas, Hubbard does not mention either finite topology or differential equations. I have included a set of notes on these topics that I wrote for Math 121.

Recorded Lectures

- Lecture 18 (Week 9, Class 1) (watch on November 6 or 7)
- Lecture 19 (Week 9, Class 2) (watch on November 8 or 9)

Proofs to present in section or to a classmate who has done them.
Proofs:

- 9.1
  - Define "Hausdorff space," and prove that in a Hausdorff space the limit of a sequence is unique.
  - Prove that $\mathbb{R}^n$, with the topology defined by open balls, is a Hausdorff space.
- 9.2 Starting from the triangle inequality for two vectors, prove the triangle inequality for m vectors in $\mathbb{R}^n$, then prove the "infinite triangle inequality:"

$$|\sum_{i=1}^{\infty} \vec{\mathbf{a}}_{\mathbf{i}}| \leq \sum_{i=1}^{\infty} |\vec{\mathbf{a}}_{\mathbf{i}}|$$

  You may assume that the series $\sum_{i=1}^{\infty} \vec{\mathbf{a}}_{\mathbf{i}}$ is "absolutely summable" (the infinite series of lengths on the right is convergent) but you must prove that this series is "summable" (infinite sum of vectors on the left is convergent.) As on page 100 of the textbook, you may use theorems 0.5.8 (if $\sum_{n=1}^{\infty} |a_n|$ converges, then so does $\sum_{n=1}^{\infty} a_n$) and 1.5.13 (a sequence of vectors in $\mathbb{R}^n$ converges if and only if each component converges).

R Scripts

- Script 3.1A-FiniteTopology.R
  Topic 1 - The "standard" Web site graph, used in notes and examples
  Topic 2 - Drawing a random graph to create a different topology on the same set

- Script 3.1B-SequencesSeriesRn.R
  Topic 1 - A convergent sequence of points in $\mathbb{R}^2$
  Topic 2 - A convergent infinite series of vectors
  Topic 3 - A convergent geometric series of matrices

- Script 3.1C-DiffEquations.R
  Topic 1 - Two real eigenvalues
  Topic 2 - A repeated real eigenvalue
  Topic 3 - Complex conjugate eigenvalues

1 Executive Summary

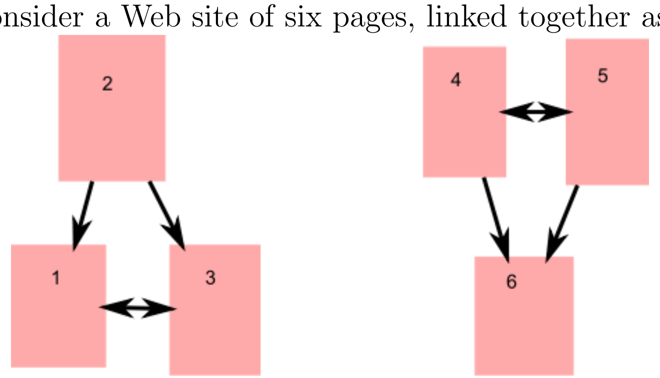
1.1 Axioms of Topology

In topology, we start with a set X and single out some of its subsets as "open sets." The only requirement on a topology is that the collection of open sets satisfies the following rules (axioms)

- The empty set and the set X are both open.
- The union of any finite or infinite collection number of open sets is open.
- The intersection of two open sets is open. It follows by induction that the intersection of n open sets is open, but the intersection of infinitely many open sets is not necessarily open.

1.2 A Web-site model for finite topology

A model for a set of axioms is a set of real-world objects that satisfy the axioms. Consider a Web site of six pages, linked together as follows:



In this model, an "open set" is defined by the property that no page in the set can be reached by a link from outside the set. We need to show that this definition is consistent with the axioms for open sets.
–The empty set is open. Since it contains no pages, it contains no page that can be reached by an outside link.
–The set X of all six pages is open, because there is no other page on the site from which an outside link could come.
–If sets A and B are open, no page in either can be reached by an outside link, and so their union is also open.
–If sets A and B are open, so is their intersection $A \cap B$. Proof by contraposition:

Suppose that $A \cap B$ is not open. Then it contains a page that can be reached by an outside link. If that link comes from A, then B is not open. If that link comes from B, then A is not open. If that link comes from outside both A and B, then both A and B are not open.

1.3 Topology in $\mathbb{R}$ and $\mathbb{R}^n$

The usual way to introduce a topology for the set $\mathbb{R}$ is to decree that any open interval is an open set and so is the empty set. Equivalently, we can decree that the set of points for which
$|x - x_0| < \epsilon$, with $\epsilon > 0$, is an open set. Notice that the infinite intersection of the open sets $(-1/n, 1/n)$ is the single point 0, a closed set!

The usual way to introduce a topology for the set $\mathbb{R}^n$ is to decree that any "open ball," the set of points for which $|\mathbf{x} - \mathbf{x_0}| < \epsilon$, with $\epsilon > 0$, is an open set.

1.4 More concepts of general topology

These definitions are intuitively reasonable for $\mathbb{R}$ and $\mathbb{R}^n$, but they also apply to the Web-site finite topology,

- Closed sets

  A closed set A is one whose complement $A^c = X - A$ is open. Careful: this is different from "one that is not open." There are lots of sets that are neither open nor closed, and there are sets that are both open and closed.

- A *neighborhood* of a point is any set that has as a subset an open set containing the point. A neighborhood does not have to be open.

- The *closure* of set $A \subset \mathbb{R}^n$, denoted $\overline{A}$, is "the smallest closed set that contains A," i.e. the intersection of all the closed sets that contain A

- The *interior* of a set $A \subset \mathbb{R}^n$, denoted $\mathring{A}$, is "the largest open set that is contained in A," i.e. the union of all the open subsets of A.

- The *boundary* of A, denoted ∂A, is the set of all points $\mathbf{x}$ with the property that any neighborhood of $\mathbf{x}$ includes points of A and also includes points of the complement A^c.
  The boundary of A is the difference between the closure of A and its interior.

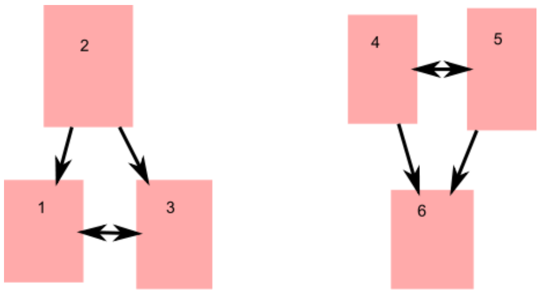
1.5 A topological definition of convergence

Sequence s_n converges to a limit s if for every open set A containing s, $\exists N$ such that $\forall n > N$, $a_n \in A$. In other words, the points of the sequence eventually get inside A and stay there.
Specialize to $\mathbb{R}$ and $\mathbb{R}^n$.

A sequence a_n of real numbers converges to a limit a if $\forall \epsilon > 0, \exists N$ such that $\forall n > N$, $|a - a_n| < \epsilon$. (open sets defined as open intervals)

A sequence $\mathbf{a_1}, \mathbf{a_2}, ...$ in $\mathbb{R}^n$ converges to the limit $\mathbf{a}$ if $\forall \epsilon > 0, \exists M$ such that if $m > M$, $|\mathbf{a_m} - \mathbf{a}| < \epsilon$. (open sets defined by open balls)
The sequence converges if and only if the sequences of coordinates all converge.

1.6 Something special about the open ball topology



For the Web diagram above, the sequence (6,5,4,6,5,4,5,4,5,4,...) converges both to 4 and to 5. Both {456} and {45} are open sets (no incoming links) but {4} {5}, {46}, and {56} are not.

This cannot happen in $\mathbb{R}^n$. If the sequence $\mathbf{a_1}, \mathbf{a_2}, \ldots$ in $\mathbb{R}^n$ converges to $\mathbf{a}$ and same sequence also converges to the limit $\mathbf{b}$, we can prove that $\mathbf{a} = \mathbf{b}$.

Why? The open ball topological space is *Hausdorff*. Given any two distinct points a and b, we can find open sets A and B with $a \in A$, $b \in B$, and $A \cap B = \emptyset$. In a Hausdorff space, the limit of a sequence is unique.

1.7 Infinite sequences and series of vectors and matrices

- We need something that can be made "less than ϵ." For vectors the familiar length is just fine. The "infinite triangle inequality" (proof 9.2) states that

$$|\sum_{i=1}^{\infty} \vec{\mathbf{a_i}}| \leq \sum_{i=1}^{\infty} |\vec{\mathbf{a_i}}|$$

- We define the "length of a matrix" by viewing the matrix as a vector.

  Since an $m \times n$ matrix A is an element of $\mathbb{R}^{mn}$, we can view it as a vector and define its length $|A|$ as the square root of the sum of the squares of all its entries. This definition has the following useful properties:

  - $|A\vec{\mathbf{b}}| \leq |A||\vec{\mathbf{b}}|$
  - $|AB| \leq |A||B|$

  Let A be a square matrix, and define its exponential by

$$\exp(At) = \sum_{r=0}^{\infty} \frac{(A)^r t^r}{r!}.$$

  Denoting the length of matrix A by $|A|$, we have

$$|\exp(At)| \leq \sum_{r=0}^{\infty} \frac{(|A|t)^r}{r!}.$$

  or $|\exp(At)| \leq \exp(|A|t) + \sqrt{n} - 1$, so the series is convergent for all t.

1.8 Calculating the exponential of a matrix

–If $D = \begin{bmatrix} b & 0 \\ 0 & c \end{bmatrix}$, then $Dt = \begin{bmatrix} bt & 0 \\ 0 & ct \end{bmatrix}$ and

$$\exp(Dt) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} bt & 0 \\ 0 & ct \end{bmatrix} + \frac{1}{2}\begin{bmatrix} (bt)^2 & 0 \\ 0 & (ct)^2 \end{bmatrix} + \cdots = \begin{bmatrix} e^{bt} & 0 \\ 0 & e^{ct} \end{bmatrix}$$

–If there is a basis of eigenvectors for A,

then $A = PDP^{-1}$, $A^r = PD^rP^{-1}$., and $\exp(At) = P\exp(Dt)P^{-1}$.

–Replace D by a conformal matrix $C = aI + bJ$ where $J^2 = -I$ and

$\exp(Ct) = \exp(aIt)\exp(bJt)$ can be expressed in terms of $\sin t$ and $\cos t$.

–If $A = bI + N$, and $N^2 = 0$, $\exp(At) = \exp bt \exp(Nt) = \exp bt(I + Nt)$.

1.9 Solving systems of linear differential equations

We put a dot over a quantity to denote its time derivative.

The solution to the differential equation $\dot{x} = kx$ is $x = \exp(kt)x_0$.

Suppose that there is more than one variable, for example

$\dot{x} = x + y$

$\dot{y} = -2x + 4y$.

If we set $\vec{\mathbf{v}} = \begin{bmatrix} x \\ y \end{bmatrix}$ then this pair of equations becomes

$\dot{\vec{\mathbf{v}}} = A\vec{\mathbf{v}}$, where $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$

The solution is the same as in the single-variable case: $\vec{\mathbf{v}} = \exp(At)\vec{\mathbf{v}}_0$

Proof:

$$\exp At = \sum_{r=0}^{\infty} \frac{A^r t^r}{r!}.$$

$$\frac{d}{dt}\exp At = \sum_{r=1}^{\infty} \frac{rA^r t^{r-1}}{r!}.$$

Set $s = r - 1$.

$$\frac{d}{dt}\exp At = \sum_{s=0}^{\infty} \frac{A^{s+1} t^s}{s!} = A\sum_{s=0}^{\infty} \frac{A^s t^s}{s!} = A\exp At.$$

So

$$\dot{\vec{\mathbf{v}}} = \frac{d}{dt}\exp At\vec{\mathbf{v}}_0 = A\exp At\vec{\mathbf{v}}_0 = A\vec{\mathbf{v}}.$$

2 Lecture outline

1. Constructing a finite topology
   Axioms for general topology
   – The empty set and the set X are both open.
   – The union of any finite or infinite collection number of open sets is open.
   – The intersection of two open sets is open.

   Suppose that we start with $X = \{123456\}$ and choose a "subbasis," consisting of $\{123\}$, $\{245\}$, and $\{456\}$.

   - Find all the other sets that must be open because of the intersection axiom and the empty-set axiom.

   - Find all the other sets that must be open because of the union axiom and the axiom that set X is open.

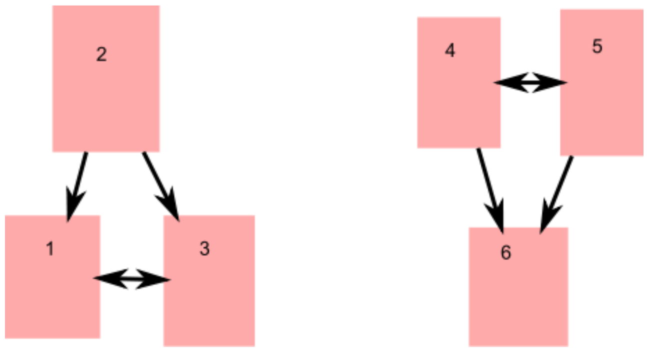
   - We now have the smallest collection of open sets that satisfies the axioms and includes the subbasis. A closed set is one whose complement is open. List all the closed sets.

   - What is the smallest legal collection of open sets in the general case?

   - What is the largest legal collection of open sets in the general case?

2. A Web-site model for finite topology

   A model for a set of axioms is a set of real-world objects that satisfy the axioms. Consider a Web site X of six pages, linked together as follows:

   In this model, an "open set" is defined by the property that no page in the set can be reached by a link from outside the set. We need to show that this definition is consistent with the axioms for open sets.
   Use an "11-legged alligator" (if you cannot construct a counterexample, it is true) argument for the following:

   The empty set is open.

   The set X of all six pages is open.

   Prove that if sets A and B are open, their union $A \cup B$ is also open.

   Do this one by contraposition:
   If sets A and B are open, so is their intersection $A \cap B$.

3. Topology in $\mathbb{R}$ and $\mathbb{R}^n$

   One way to introduce a topology for the set $\mathbb{R}$ is to decree that any open interval is an open set and so is the empty set.
   Prove that (a, b) is open if we decree that the set of points for which $|x - x_0| < \epsilon$, with $\epsilon > 0$, is an open set.

   Now the rule that only finite intersections of open sets have to be open becomes meaningful. Show that the infinite intersection of the open sets $(-1/n, 1/n)$ is not an open set!

   The usual way to introduce a topology for the set $\mathbb{R}^n$ is to decree that any "open ball," the set of points for which
   $|\mathbf{x} - \mathbf{x_0}| < \epsilon$, with $\epsilon > 0$, is an open set.

4. More concepts of general topology

   These definitions are intuitively reasonable for $\mathbb{R}$ and $\mathbb{R}^n$, but they also apply to the Web-site finite topology,

   - Closed sets

     A closed set A is one whose complement $A^c = X - A$ is open. Careful: this is different from "one that is not open." There are lots of sets that are neither open nor closed, and there are sets that are both open and closed.

   - A *neighborhood* of a point is any set that has as a subset an open set containing the point. A neighborhood does not have to be open.

   - The *closure* of set $A \subset \mathbb{R}^n$, denoted $\overline{A}$, is "the smallest closed set that contains A," i.e. the intersection of all the closed sets that contain A

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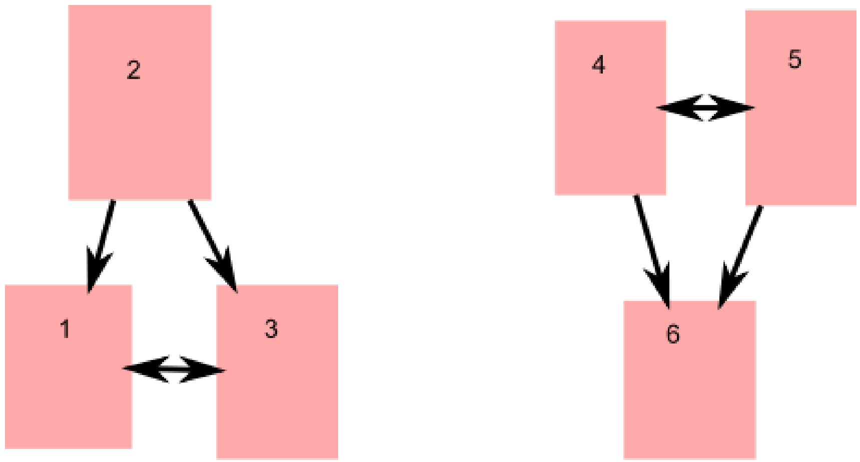
   - The *boundary* of A, denoted ∂A, is the set of all points $\mathbf{x}$ with the property that any neighborhood of $\mathbf{x}$ includes points of A and also includes points of the complement A^c.
     The boundary of A is the difference between the closure of A and its interior.

5. Applying these new definitions to the Web site topology



Open:$\{2\}, \{45\}, \{123\}, \{456\}, \{245\}, \{12345\}, \{2456\}$
Closed:$\{13456\}, \{1236\}, \{456\}, \{123\}, \{136\}, \{6\}, \{13\}$
Both: Empty set and $\{123456\}$

- Is $\{345\}$ a neighborhood of page 4?
- What is the closure of $\{23\}$?
- Of $\{26\}$?
- What is the interior of $\{23\}$?
- Of $\{23456\}$?
- What is the boundary of $\{23\}$?

6. The “open ball” definition of an open set satisfies the axioms of topology. A set $U \in \mathbb{R}^n$ is open if $\forall \mathbf{x} \in U, \exists r > 0$ such that the open ball $B_r(\mathbf{x}) \subset U$.

   - Prove that the empty set is open.

   - Prove that all of $\mathbb{R}^n$ is open.

   - Prove that the union of any collection of open sets is open.

   - Prove that the intersection of two open sets is open.

   - Prove that in $\mathbb{R}^2$, the boundary of the open disc $x^2 + y^2 < 1$ is the circle $x^2 + y^2 = 1$.

   - Find the infinite intersection of open balls of radius $\frac{1}{n}$ around the origin, for all positive integers. Is it open, closed, or neither?

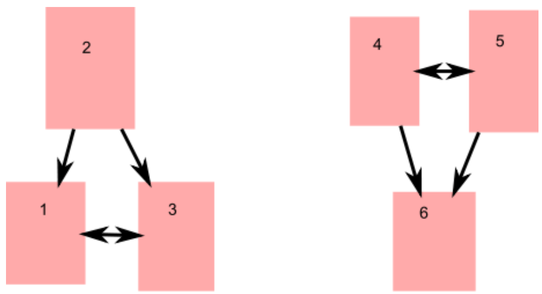
7. A topological definition of convergence

   Sequence s_n converges to a limit s if for every open set A containing s, $\exists N$ such that $\forall n > N$, $s_n \in A$. In other words, the points of the sequence eventually get inside A and stay there.
   Specialize to $\mathbb{R}$ and $\mathbb{R}^n$.

   A sequence s_n of real numbers converges to a limit s if $\forall \epsilon > 0, \exists N$ such that $\forall n > N$, $|s - s_n| < \epsilon$. (open sets defined as open intervals)

   A sequence $\mathbf{a_1}, \mathbf{a_2}, ...$ in $\mathbb{R}^n$ converges to the limit $\mathbf{a}$ if $\forall \epsilon > 0, \exists M$ such that if $m > M$, $|\mathbf{a_m} - \mathbf{a}| < \epsilon$. (open sets defined by open balls)
   We will prove that the sequence of points converges if and only if the sequences of coordinates all converge.

8. Something surprising about the open ball topology



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This cannot happen in $\mathbb{R}^n$. If the sequence $\mathbf{a_1}, \mathbf{a_2}, ...$ in $\mathbb{R}^n$ converges to $\mathbf{a}$ and same sequence also converges to the limit $\mathbf{b}$,we can prove that $\mathbf{a} = \mathbf{b}$.

Why? The open ball topological space is *Hausdorff*. Given any two distinct points a and b, we can find open sets A and B with $a \in A$, $b \in B$, and $A \cap B = \emptyset$. In a Hausdorff space, the limit of a sequence is unique.

9. Proof 9.1

- Define "Hausdorff space," and prove that in a Hausdorff space the limit of a sequence is unique.
- Prove that $\mathbb{R}^n$, with the topology defined by open balls, is a Hausdorff space.

10. A closed subset contains all its limit points

    We defined a closed subset to be the complement of an open set.

    Using this definition, prove that if every element of the convergent sequence $(\mathbf{x}_n)$ is in the closed subset $C \subset \mathbb{R}^n$, then the limit x_0 of the sequence is also in C.

11. Convergent sequences in $\mathbb{R}^n$:

A sequence $\mathbf{a_1}, \mathbf{a_2}, ...$ in $\mathbb{R}^n$ converges to the limit $\mathbf{a}$ if
$\forall \epsilon > 0, \exists M$ such that if $m > M$, $|\mathbf{a_m} - \mathbf{a}| < \epsilon$.

Prove that the sequence converges if and only if the sequences of coordinates all converge.

12. Proof 9.2

Starting from the triangle inequality for two vectors, prove the triangle inequality for m vectors in $\mathbb{R}^n$, then prove the "infinite triangle inequality:"

$$|\sum_{i=1}^{\infty} \vec{\mathbf{a_i}}| \leq \sum_{i=1}^{\infty} |\vec{\mathbf{a_i}}|$$

You may assume that the series $\sum_{i=1}^{\infty} \vec{\mathbf{a_i}}$ is "absolutely summable" (the infinite series of lengths on the right is convergent) but you must prove that this series is "summable" (infinite sum of vectors on the left is convergent.) As on page 100 of the textbook, you may use theorems 0.5.8 and 1.5.13.

13. Proof of inequalities involving matrix length

The length of a matrix is calculated by treating it as a vector: take the square root of the sum of the squares of all the entries.

Show that if matrix A consists of a single row, then $|A\vec{\mathbf{b}}| \leq |A||\vec{\mathbf{b}}|$ is just the Cauchy-Schwarz inequality.

Prove the following:

- $|A\vec{\mathbf{b}}| \leq |A||\vec{\mathbf{b}}|$ when A is an $m \times n$ matrix.
- $|AB| \leq |A||B|$
- $|I| = \sqrt{n}$ for the $n \times n$ identity matrix.

14. A geometric series of matrices

The geometric series formula for a square matrix A is

$$(I - A)^{-1} = I + A + A^2 + ...$$

Let $A = \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} -\frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \end{bmatrix}$.

(a) Evaluate $I + A^2 + A^4.....$

(b) Evaluate $A + A^3 + A^5.... = A(I + A^2 + A^4....)$.

(c) Evaluate $I + A + A^2 + ...$

(d) Evaluate $(I - A)^{-1}$ and compare.

15. The exponential of a matrix

Let A be a square matrix, and define

$$\exp(At) = \sum_{r=0}^{\infty} \frac{(A)^r t^r}{r!}.$$

To show that this series converges, we use the infinite-sum version of the triangle inequality (proof 9.2), which applies to matrices if we treat them as vectors by using matrix length.

$$|\exp(At)| \leq \sum_{r=0}^{\infty} |\frac{(A)^r t^r}{r!}|.$$

Denoting the length of matrix A by $|A|$. we have

$$|\exp(At)| \leq \sum_{r=0}^{\infty} \frac{(|A|t)^r}{r!}.$$

or

$$|\exp(At)| \leq \exp(|A|t) + \sqrt{n} - 1.$$

It is easy to calculate the exponential of a diagonal matrix directly from this definition. If, for example, $D = \begin{bmatrix} b & 0 \\ 0 & c \end{bmatrix}$, then $Dt = \begin{bmatrix} bt & 0 \\ 0 & ct \end{bmatrix}$ and

$$\exp(Dt) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} bt & 0 \\ 0 & ct \end{bmatrix} + \frac{1}{2} \begin{bmatrix} (bt)^2 & 0 \\ 0 & (ct)^2 \end{bmatrix} + \cdots = \begin{bmatrix} e^{bt} & 0 \\ 0 & e^{ct} \end{bmatrix}$$

Now suppose that there is a basis of eigenvectors for A.

Then $A = PDP^{-1}$, where D is diagonal and P is the change of basis matrix whose columns are the eigenvectors.

Prove by induction that $A^r = PD^rP^{-1}$.

Prove that $\exp(At) = P\exp(Dt)P^{-1}$.

16. Calculating an exponential

We have already found that $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$ has

eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with eigenvalue 2 and

eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ with eigenvalue 3.

Write A in the form $A = PDP^{-1}$.

Work out $\exp(At) = P \exp(Dt) P^{-1}$.

17. Solving systems of linear differential equations

We adopt the convention of putting a dot over a quantity to denote its time derivative.

The solution to the differential equation $\dot{x} = kx$ is $x = \exp(kt)x_0$, where x_0 can have any value.

Suppose that there is more than one variable, for example

$\dot{x} = x + y$

$\dot{y} = -2x + 4y$.

If we set $\vec{\mathbf{v}} = \begin{bmatrix} x \\ y \end{bmatrix}$ then this pair of equations becomes

$\dot{\vec{\mathbf{v}}} = A\vec{\mathbf{v}}$, where $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$

The solution is the same as in the single-variable case:

$\vec{\mathbf{v}} = \exp(At)\vec{\mathbf{v}}_0$

Proof:

$$\exp At = \sum_{r=0}^{\infty} \frac{A^r t^r}{r!}.$$

$$\frac{d}{dt} \exp At = \sum_{r=1}^{\infty} \frac{r A^r t^{r-1}}{r!}.$$

Set $s = r - 1$.

$$\frac{d}{dt} \exp At = \sum_{s=0}^{\infty} \frac{A^{s+1} t^s}{s!} = A \sum_{s=0}^{\infty} \frac{A^s t^s}{s!} = A \exp At.$$

So

$$\dot{\vec{\mathbf{v}}} = \frac{d}{dt} \exp At\vec{\mathbf{v}}_0 = A \exp At\vec{\mathbf{v}}_0 = A\vec{\mathbf{v}}.$$

18. Checking the solution

The equation is

$\dot{\vec{\mathbf{v}}} = A\vec{\mathbf{v}}$, where $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$

We earlier found that $A = PDP^{-1}$,

where $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

Therefore $\exp At = P \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix} P^{-1}$

As "initial conditions," take $\vec{\mathbf{v}}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

- Calculate $P^{-1}\vec{\mathbf{v}}_0$. This element of $\mathbb{R}^2$ expresses the initial conditions relative to the basis of eigenvectors.

- Calculate $\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix} P^{-1}\vec{\mathbf{v}}_0$. This element of $\mathbb{R}^2$ expresses the vector at time t, still relative to the basis of eigenvectors.

- Calculate $P \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix} P^{-1}\vec{\mathbf{v}}_0$. This element of $\mathbb{R}^2$ expresses the vector at time t, but now relative to the standard basis.

- Differentiate the answer with respect to t and check that
  $\dot{x} = x + y$
  $\dot{y} = -2x + 4y$.

19. Solving a differential equation when there is no eigenbasis.

The system of differential equations

$\dot{x} = 3x - y$

$\dot{y} = x + y$

can be written $\dot{\vec{\mathbf{v}}} = A\vec{\mathbf{v}}$, where $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$.

Our standard technique leads to $p(t) = t^2 - 4t + 4 = (t-2)^2$, so there is one only eigenvalue.

Let $N = A - 2I = \begin{bmatrix} 1 - 1 \\ 1 - 1 \end{bmatrix}$.

We have found that $p(A) = A^2 - 4A + 4I = (A - 2I)^2 = 0$, so $N^2 = 0$.

The addition formula for the exponential function,
$\exp(a + b) = \exp(a)\exp(b)$, which you proved on the homework by using Taylor series, is valid whenever $ab = ba$. It is therefore valid for commuting matrices. In particular, it is valid for the sum of a multiple of the identity matrix and any other matrix.

Since matrices $2I$ and N commute, $\exp(At) = \exp(2It)\exp(Nt)$

Show that $\exp At = e^{2t}(I + Nt)$,and confirm that $(\exp At)\vec{\mathbf{e}}_1$ is a solution to the differential equation.

20. Dealing with complex eigenvalues

Consider the 2×2 case where the eigenvalues of matrix A are complex. In this case we have learned how to express A in the form

$A = PCP^{-1}$, where C is the conformal matrix

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}$$

Then $Ct = atI + btJ$, where $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $J^2 = -I$.

Since the matrices atI and btJ commute, the addition formula for the exponential function holds, and

$\exp(Ct) = \exp(atI + btJ) = \exp(atI)\exp(btJ)$

Calculate $\exp(btJ)$ by substituting into the series for the exponential function and using $J^2 = -I$.

So $\exp(Ct) = \exp(atI)\exp(btJ) = \exp(atI)[(\cos bt)I + (\sin bt)J]$

and the solution to $\dot{\vec{\mathbf{v}}} = A\vec{\mathbf{v}}$

is $\vec{\mathbf{v}} = P\exp(Ct)P^{-1}\vec{\mathbf{v}}_{\mathbf{0}}$.

21. Solving the "harmonic oscillator" differential equation

Applying Newton's second law of motion to a mass of 1 attached to a spring with "spring constant" 4 leads to the differential equation

$$\ddot{x} = -4x.$$

Solve this equation by using matrices for the case where $x(0) = 1, v(0) = 0$. The trick is to consider a vector

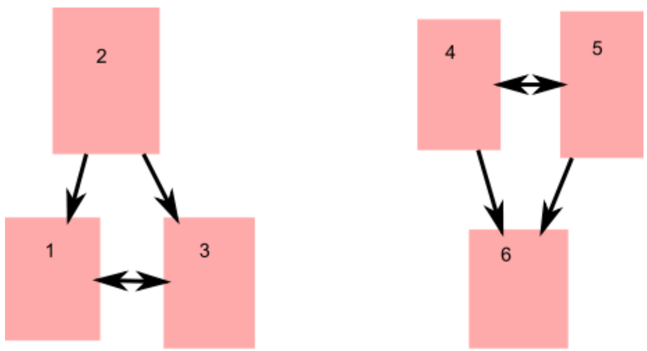
$$\vec{\mathbf{w}} = \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}, \text{ where } v = \dot{x}.$$

3 Seminar Topics

Your section instructor will either have emailed a list of topics to prepare or will have posted a signup list of appointments on the Calendar tab of Canvas. Either way, there will be one of the following topics that you should be prepared to present.

Practice your presentation so that it takes about 8 minutes. The text of the presentation will be projected onto a screen so that you need not recopy it. To save time, avoid writing long sentences on the chalkboard. You may use notes, but be discreet about it.

1. State the three axioms for general topology, and show that they are satisfied in the topology of subsets of this set X of six linked Web pages:



where an "open set" is a subset $S \subset X$ with the property that no page in S can be reached by a link from $X - S$.

Define "interior," "closed set," and "closure," and show how you can apply these definitions to the set $A = \{1, 2\}$ to determine $\mathring{A}$ and $\overline{A}$.

In the "open ball" topology for $\mathbb{R}$, any open interval is an open set and any closed interval is a closed set. Apply the definitions of interior and closure to find $\mathring{A}$ and $\overline{A}$ for the subset $A = (2, 3] \subset \mathbb{R}$.

2. (Proof 9.1)

   - Define "Hausdorff space," and prove that in a Hausdorff space the limit of a sequence is unique.
   - Prove that $\mathbb{R}^n$, with the topology defined by open balls, is a Hausdorff space.

3. A sequence $\mathbf{a_1}, \mathbf{a_2}, ...$ in $\mathbb{R}^n$ converges to the limit $\mathbf{a}$ if
$\forall \epsilon > 0, \exists M$ such that if $m > M$, $|\mathbf{a_m} - \mathbf{a}| < \epsilon$.

   Prove that the sequence converges if and only if, for all j, the sequence of jth coordinates $(a_m)_j$ converges to a_j.

4. (Proof 9.2) Starting from the triangle inequality for two vectors, prove the triangle inequality for m vectors in $\mathbb{R}^n$, then prove the "infinite triangle inequality:"

$$|\sum_{i=1}^{\infty} \vec{\mathbf{a_i}}| \leq \sum_{i=1}^{\infty} |\vec{\mathbf{a_i}}|$$

   You may assume that the series $\sum_{i=1}^{\infty} \vec{\mathbf{a_i}}$ is "absolutely summable" (the infinite series of lengths on the right is convergent) but you must prove that this series is "summable" (infinite sum of vectors on the left is convergent.) As on page 100 of the textbook, you may use theorems 0.5.8 (if $\sum_{n=1}^{\infty} |a_n|$ converges ,then so does $\sum_{n=1}^{\infty} a_n$) and topic 4 (a sequence of vectors in $\mathbb{R}^n$ converges if and only if each component converges)

5. Let A be an $n \times n$ matrix, and let $\vec{\mathbf{v}}(t)$ be a vector in $\mathbb{R}^n$ whose components are functions of time. Define $\exp At$ as an infinite series, and prove that

$$\frac{d}{dt} \exp At = A \exp At.$$

   Thereby show that the vector $\vec{\mathbf{v}}(t) = (\exp At)\vec{\mathbf{v}}_0$ is the solution to the differential equation $\dot{\vec{\mathbf{v}}} = A\vec{\mathbf{v}}$ that satisfies the initial condition $\vec{\mathbf{v}}(0) = \vec{\mathbf{v}}_0$.

6. (Extra topic)
Give a topological definition, one that uses only the concept of open set, for convergence of a sequence s_n. Show that, according to this definition, the sequence (6,5,4,6,5,4,5,4,5,4,...) in the Web site topology converges both to page 4 and to page 5. Then show, given that open intervals in $\mathbb{R}$ are open sets, that the sequence $(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \cdots)$ converges to 0 according to your definition.

4 Workshop Problems

1. Topology

   (a) Properties of closed sets
   Recall the axioms of topology, which refer only to open sets:

   - The empty set and the set X are both open.
   - The union of any collection of open sets is open.
   - The intersection of two open sets is open.

   A closed set C is defined as a set whose complement C^c is open.

   You may use the following well-known properties of set complements, sometimes called "De Morgan's Laws":
   $(A \cup B)^c = A^c \cap B^c, (A \cap B)^c = A^c \cup B^c$.

   i. Prove directly from the axioms of topology that the union of two closed sets is closed.
   ii. In the Web site topology, a closed set of pages is one that has no outgoing links to other pages on the site. Prove that in this model, the union of two closed sets is closed.
   iii. Prove that if A and B are closed subsets of $\mathbb{R}^2$ (with the topology specified by open balls), their union is also closed.

   (b) Subsets of $\mathbb{R}$

   i. Let $A = \{0\} \cup (1, 2]$. Determine A^c, $\mathring{A}$, $\overline{A}$, and ∂A.
   ii. What interval is equal to $\bigcup_{n=2}^{\infty}[-1 + \frac{1}{n}, 1 - \frac{1}{n}]$? Is it a problem that this union of closed sets is not a closed set?
   iii. Let $\mathbb{Q}_1$ denote the set of rational numbers in the interval $(-1, 1)$. Determine the closure, interior, and boundary of this set.

2. Convergence in $\mathbb{R}^n$

   (a) Suppose that the sequence $\mathbf{a_1}, \mathbf{a_2}, \ldots$ in $\mathbb{R}^n$ converges to $\mathbf{0}$, and the sequence of real numbers $k_1, k_2, \cdots$, although not necessarily convergent, is bounded: $\exists K > 0$ such that $\forall n \in \mathbb{N}, |k_n| < K$.

   Prove that the sequence $k_1\mathbf{a_1}, k_2\mathbf{a_2}, \ldots$ in $\mathbb{R}^n$ converges to $\mathbf{0}$.

   (b) Prove that if $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, then $\exp(Jt) = I\cos t + J\sin t$. Show that this is consistent with the Taylor series for e^{it}.

3. Differential equations

   (a) The original patriarchal differential equation problem

   Isaac has established large flocks of sheep for his sons Jacob and Esau. Anticipating sibling rivalry, he has arranged that the majority of the growth of each son's flock will come from lambs born to the other son. So, if $x(t)$ denotes the total weight of all of Jacob's sheep and $y(t)$ denotes the total weight of all of Esau's sheep, the time evolution of the weight of the flocks is given by the differential equations

   $\dot{x} = x + 2y$

   $\dot{y} = 2x + y$

   i. Calculate $\exp(At)$, where $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

   ii. Show that if the flocks are equal in size, they will remain that way. What has this got to do with the eigenvectors of A?

   iii. Suppose that when $t = 0$, the weight of Jacob's flock is S while the weight of Esau's flock is $2S$. Find formulas for the sizes as functions of time, and show that the flocks will become more nearly equal in weight as time passes.

   (b) Suppose that $\dot{\vec{\mathbf{v}}} = A\vec{\mathbf{v}}$, where $A = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$. Since $p(t) = (t-2)^2$, there is no basis of eigenvectors. By writing A as the sum of a multiple of the identity matrix and a nilpotent matrix, calculate $\exp(At)$.

5 Homework

1. Suppose that you want to construct a Web site of six pages numbered 1 through 6, where the open sets of pages, defined as in lecture, include $\{126\}$, $\{124\}$, and $\{56\}$.

   (a) Prove that in the Web site model of finite topology, the intersection of two open sets is open.

   (b) What other sets must be open in order for the family of open sets to satisfy the intersection axiom?

   (c) What other sets must be open in order for the family of open sets to satisfy the union axiom?

   (d) List the smallest family of open sets that includes the three given sets and satisfies all three axioms. (You have already found all but one of these sets!)

   (e) Draw a diagram showing how six Web pages can be linked together so that only the sets in this family are open. This is tricky. First deal with 5 and 6. Then deal with 1 and 2. Then incorporate 4 into the network, and finally 3. There are many correct answers since, for example, if page 1 links to page 2 and page 2 links to page 3, then adding a direct link from page 1 to page 3 does not change the topology.

2. More theorems about limits of sequences

   The sequence $\vec{\mathbf{a_1}}, \vec{\mathbf{a_2}}, ...$ in $\mathbb{R}^n$ converges to $\vec{\mathbf{a}}$.

   The sequence $\vec{\mathbf{b_1}}, \vec{\mathbf{b_2}}, ...$ in $\mathbb{R}^n$ converges to $\vec{\mathbf{b}}$.

   (a) Prove that the sequence of lengths $|\vec{\mathbf{b_1}}|, |\vec{\mathbf{b_2}}|, ...$ in $\mathbb{R}$ is bounded: $\exists K$ such that $\forall n, |\vec{\mathbf{b_n}}| < K$. Hint: write $\vec{\mathbf{b_m}} = \vec{\mathbf{b_m}} - \vec{\mathbf{b}} + \vec{\mathbf{b}}$, then use the triangle inequality.

   (b) Define the sequence of dot products: $c_n = \vec{\mathbf{a_n}} \cdot \vec{\mathbf{b_n}}$.
   Prove that $c_1, c_2, \cdots$ converges to $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$.
   Hint: Subtract and add $\vec{\mathbf{a}} \cdot \vec{\mathbf{b_n}}$, then use the triangle inequality and the Cauchy-Schwarz inequality.

3. Let $A = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$

   (a) By considering the length of A, show that

   $$\lim_{n\to\infty} A^n$$

   must be the zero matrix.

   (b) Find a formula for A^n when $n \geq 1$, and prove it by induction. Note that the formula is not valid for $n = 0$.

   (c) Verify the formula

   $$(I - A)^{-1} = I + A + A^2 + ...$$

   for this choice of A. As was the case in the example on page 20, you can evaluate the infinite sum on the right by summing a geometric series, but you should split off the first term and start the geometric series with the second term.

4. The differential equation $\ddot{x} = -3\dot{x} - 2x$ describes the motion of an "overdamped oscillator." The acceleration $\ddot{x}$ is the result of the sum of a force proportional to $\dot{x}$, supplied by a shock absorber, and a force proportional to x, supplied by a spring.

   (a) Introduce $v = \dot{x}$ as a new variable, and define the vector $\vec{\mathbf{w}} = \begin{bmatrix} x \\ v \end{bmatrix}$.

   Find a matrix A such that $\dot{\vec{\mathbf{w}}} = A\vec{\mathbf{w}}$.

   (b) Calculate the matrix $\exp(At)$.

   (c) Graph $x(t)$ for the following three sets of initial values that specify position and velocity when $t = 0$:

   Release from rest: $\vec{\mathbf{w}}_\mathbf{0} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

   Quick shove: $\vec{\mathbf{w}}_\mathbf{0} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

   Push toward the origin: $\vec{\mathbf{w}}_\mathbf{0} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

5. Suppose that A is a matrix of the form $S = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$. Prove that

$$\exp(St) = \exp(at) \begin{bmatrix} \cosh(bt) & \sinh(bt) \\ \sinh(bt) & \cosh(bt) \end{bmatrix}.$$

Then use this result to solve

$\dot{x} = x + 2y$

$\dot{y} = 2x + y$

without having to diagonalize the matrix S.

6. Let $B = \begin{bmatrix} -1 & 9 \\ -1 & 5 \end{bmatrix}$. Show that there is only one eigenvalue λ and find an eigenvector for it. Then show that $N = B - \lambda I$ is nilpotent.

   (a) By writing $B = \lambda I + N$, calculate B^2.

   (b) By writing $B = \lambda I + N$, solve the system of equations

   $\dot{x} = -x + 9y$

   $\dot{y} = -x + 5y$

   for arbitrary initial conditions $\vec{\mathbf{v}}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$.

7. In Week 4, we wrote $A = \begin{bmatrix} 7 & -10 \\ 2 & -1 \end{bmatrix}$ in the form $A = PCP^{-1}$, where $C = \begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix}$ is conformal and $P \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

   Follow up on this analysis to solve the differential equation $\dot{\vec{\mathbf{v}}} = A\vec{\mathbf{v}}$ for initial conditions $\vec{\mathbf{v}}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

8. Let A be a 2×2 matrix which has two distinct real eigenvalues λ_1 and λ_2, with associated eigenvectors $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$.

   (a) Show that the matrix $P_1 = \frac{A - \lambda_2 I}{\lambda_1 - \lambda_2}$ is a projection onto the subspace spanned by eigenvector $\vec{\mathbf{v}}_1$. Find its image and kernel, and show that $P_1^2 = P_1$.

   (b) Similarly, the matrix $P_2 = \frac{A - \lambda_1 I}{\lambda_2 - \lambda_1}$ is a projection onto the subspace spanned by eigenvector $\vec{\mathbf{v}}_2$. Show that $P_1 P_2 = P_2 P_1 = 0$, that $P_1 + P_2 = I$, and that $\lambda_1 P_1 + \lambda_2 P_2 = A$.

   (c) Show that $\exp(t\lambda_1 P_1 + t\lambda_2 P_2) = \exp(\lambda_1 t) P_1 + \exp(\lambda_2 t) P_2$, and use this result to solve the equations

   $\dot{x} = -4x + 5y$

   $\dot{y} = -2x + 3y$

   for arbitrary initial conditions $\vec{\mathbf{v}}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$.