Rules for Math 23a/E-23a Online Quizzes
Fall 2018

"Members of the Harvard College community commit themselves to producing academic work of integrity that is, work that adheres to the scholarly and intellectual standards of accurate attribution of sources, appropriate collection and use of data, and transparent acknowledgement of the contribution of others to their ideas, discoveries, interpretations, and conclusions. Cheating on exams or problem sets, plagiarizing or misrepresenting the ideas or language of someone else as ones own, falsifying data, or any other instance of academic dishonesty violates the standards of our community, as well as the standards of the wider world of learning and affairs."

1. No references are allowed except for the excerpts from the Executive Summaries that are on the last page.
2. After the quiz is downloaded and printed, computers and cell phones must be switched off and calculators should be put away.
3. No discussion with classmates or others is permitted during the entire period during which the quiz is available.
4. The proctor does not have to remain continuously in the room but should look in unexpectedly from time to time.
5. Although there is no explicit time limit, the quiz must be completed in a single sitting.
6. When the quiz is done, switch on the computer and scanner and upload the completed quiz, along with this signed form.

Date of quiz______________________________
Start time______________________________
End time______________________________
List any unusual circumstances in the administration of the quiz (e.g. the scanner was not located in the room where the quiz was taken).

I am aware of the Harvard College Honor Code, and I certify that I complied with the rules.

(Student signature)______________________________

I am aware of the Harvard College Honor Code, and I observed that the student complied with the rules.

(Proctor signature)______________________________

Proctor email______________________________

Proctor relationship to student ______________________________

Name: ______________________________
Section (if any): ___________________

MATHEMATICS 23a/E-23a, Fall 2018
Quiz #2
November 9-11, 2018

You must complete this quiz at a single sitting immediately after downloading and printing it. While you are taking the quiz, your computers and cell phone must be switched off.

The last page of the quiz contains useful information extracted from the Executive Summaries.

Calculators, which would be of no use, are not allowed.
You and your proctor must sign the statement on the front page of the exam.
You may omit one multiple-choice question in Part I and one question in Part II.
If you are doing proof logging, check here ____ and omit one proof in Part III.
If you opt out of proof logging, check here ____ and do all proofs in Part III.

There are three blank pages at the end of the exam. If your answer does not fit in the space provided on the page with the question, write "Continued on page XX" and finish the answer on the specified page. That way, your exam can be scanned without having to check for answers on the back of a page.

Problem	Points	Answer	Score
$I-1$	2		
$I-2$	2		
$I-3$	2		
$I-4$	2		
$I-5$	2		
$I-6$	2		
$II-1$	5	$--$	
$II-2$	5	$--$	
$II-3$	5	$--$	
$II-4$	5	$--$	
$II-5$	5	$--$	
$III-1$	5	$--$	
$III-2$	5	$--$	
$III-3$	5	$--$	
$III-4$	5	$--$	
$III-5$	5	$--$	
Total	50 or 55	$--$	

Part I. Answer five of the six multiple-choice questions. Transcribe your answers onto page 2, and mark an X in the score box on page 2 to indicate which question you have omitted.

If you answer all six questions, the last one will be ignored.

1. Let s_n be a sequence of real numbers on a bounded set S, where $\liminf s_n \neq \limsup s_n$. Which of the following is not necessarily true?
   (a) $\lim s_n$ does not exist.
   (b) s_n is not Cauchy.
   (c) $\liminf s_n < \limsup s_n$
   (d) There exists a convergent subsequence of s_n.
   (e) s_n has an infinite number of dominant terms.

2. Which of the following is not true about $s_n = \frac{1}{n}$?
   (a) The sequence converges to 0.
   (b) $\lim_{n\to\infty} \sum_{i=1}^{n} s_i = L$, for some finite L.
   (c) $\limsup s_n = 0$.
   (d) The series $\sum(-1)^n s_n$ converges.
   (e) The series $\sum s_n^2$ converges.

3. Which of the following must be true of a continuous function on (a, b)?
   (a) The function achieves its maximum on (a, b).
   (b) The function is bounded.
   (c) For all Cauchy sequences s_n on the set (a, b), $f(s_n)$ is also Cauchy.
   (d) If $f(a) = 2$, and $f(b) = 5$, then $f(c) = 3$, for some $c \in (a, b)$.
   (e) None of the above are true

4. Find $\lim_{x\to b} \frac{\sqrt{x}-\sqrt{b}}{x-b}$ for $b > 0$.
   (a) ∞
   (b) $\frac{1}{2\sqrt{b}}$
   (c) 0
   (d) $2\sqrt{b}$
   (e) b

5. Let f be a differentiable function, where all derivatives exist, such that $f(0) = 0$, $f'(0) = 0$, and $|f''(x)| \leq M, \forall x$. Which of the following is not necessarily true?

   (a) $|f(1)| \leq \frac{M}{2}$

   (b) 0 is neither a maximum nor a minimum.

   (c) $\forall \epsilon > 0, \exists \delta > 0$ s.t. if $x \in (-\delta, \delta), |f(x)| < \epsilon$

   (d) If $\lim s_n = 0$, then $\lim f(s_n) = 0$.

   (e) None of the above. They're all necessarily true.

6. Let $\sum a_n$ be a conditionally convergent alternating series. Which of the following is not necessarily true?

   (a) The series converges to some finite L.

   (b) The series sum is independent of order of terms.

   (c) $\sum |a_n|$ diverges.

   (d) $\lim a_n = 0$.

   (e) None of the above. They're all necessarily true.

Part II. Answer four of the five questions. Mark an X in the score box on page 1 to indicate which question you have omitted.

1. (Inspired by Week 5, workshop problems # 3)
   Consider the sequence:

$$s_n = \frac{n^p + 1}{n^p}$$

   where $p \in \mathbb{R}$.

   (a) Find $\lim s_n$ when $p = 1$ from the definition of the limit of a sequence.

   (b) Find $\lim s_n$ when $p > 0$ using whatever method you want. (Note that your answer should be independent of p, so the numerical answer to the limit will be the same as your answer to the first part)

   (c) For $p < 0$, prove that $\lim s_n = \infty$ from the definition of a limit of a sequence being infinity.

2. (Inspired by Week 6, workshop problems # 1)

   Consider a sequence $s_n \in \mathbb{R}$.

   (a) Prove that if s_n is bounded (both above and below), then $\limsup s_n$ exists and is finite. (Note by a similar argument $\liminf s_n$ exists and is finite.)

   (b) If s_n is unbounded below, then by definition, we say $\liminf s_n = -\infty$. However, $\limsup s_n$ is not determined from this information. Invent a sequence that is unbounded below, where $\limsup s_n$ is finite. Invent one where $\limsup s_n = -\infty$.

3. (Inspired by Week 7, workshop problems # 1)
Let's consider a function $h(x)$ constructed in a piecewise manner from two other functions $f(x)$ and $g(x)$.

$$h(x) = \begin{cases} f(x) & \text{for } a \leq x \leq b \\ g(x) & \text{for } b < x \leq c \end{cases}$$

Assume that $f(x)$ and $g(x)$ are defined on $[a, b]$ and $(b, c]$ respectively, so that $h(x)$ is defined on $[a, c]$.

(a) Invent (drawing a picture is okay) functions $f(x)$, $g(x)$ where $f(x)$ is continuous on $[a, b]$ and $g(x)$ is continuous on $[b, c]$ (note that $g(x)$ is also defined at b in this case), *but* $h(x)$ is not continuous on $[a, c]$.

(b) Let $f(x)$ be continuous at b. Prove that $h(x)$ is continuous at b, if and only if

$$\lim_{x \to b^-} f(x) = \lim_{x \to b^+} g(x)$$

(c) The condition that $f(x)$ be continuous at b is crucial! Show (using a picture is fine) a situation where

$$\lim_{x \to b^-} f(x) = \lim_{x \to b^+} g(x)$$

but $h(x)$ is not continuous at b, when we allow $f(x)$ to be discontinuous at b.

4. (Inspired by Week 8, workshop problems # 3)
Recall that the exponential function is the amazing function whose derivative is itself! Namely if $f(x) = e^x$, then $f'(x) = e^x$. Also, note that $f(0) = e^0 = 1$.

   (a) Compute the Taylor series of e^x expanded around $x = 0$. Prove using Taylor's theorem with remainder that this Taylor series converges to e^x.

   (b) Prove that the radius of convergence of the Taylor series is ∞.

   (c) Recall that the hyperbolic sine is given by

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + ... = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

   Show using Taylor series representations that

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

5. Consider a function $f(x)$ with domain $U = [a, b]$.

   (a) Prove that if $f(x)$ is differentiable at a point $x_0 \in U$, then $f(x)$ is continuous at x_0. You may also take for granted that $\lim_{x \to x_0} f(x)$ exists. (Hint: start from the limit which defines the derivative as existing and then multiply both sides by $\lim_{x \to x_0}(x - x_0)$.)

   (b) Show that the converse is not necessarily true! Invent (a picture is fine) a function $f(x)$ that is continuous at x_0, but not differentiable at x_0.

   (c) Using the Mean Value Theorem, show that if $f'(x)$ is bounded on (a, b), then $f(x)$ is uniformly continuous on $[a, b]$.

Part III. If you are doing proof logging, do four of the five proofs. Mark an X in the score box on page 1 to indicate which proof you have omitted.
If you are opting out of proof logging, do all five proofs.

1. (Proof 5.3)
   The completeness axiom for the real numbers states that every nonempty subset $S \subset \mathbb{R}$ that is bounded above has a least upper bound $\sup S$. Use it to prove that for any two positive real numbers a and b, there exists a positive integer n such that $na > b$.

2. (Proof 6.1)

- Prove that any bounded increasing sequence converges. (You may assume without additional proof the corresponding result, that any bounded decreasing sequence converges.)
- Prove that every sequence (s_n) has a monotonic subsequence.
- Prove the Bolzano-Weierstrass Theorem: every bounded sequence has a convergent subsequence.

3. (Proof 7.4)
   Prove that if f is uniformly continuous on a set S and (s_n) is a Cauchy sequence in S, then $(f(s_n))$ is a Cauchy sequence. Invent an example where f is continuous but not uniformly continuous on S and $(f(s_n))$ is not a Cauchy sequence.

4. (Proof 8.1)

- Prove Rolle's Theorem: if f is a continuous function on $[a, b]$ that is differentiable on (a, b) and satisfies $f(a) = f(b)$, then there exists at least one x in (a, b) such that $f'(x) = 0$.

- Using Rolle's Theorem, prove the Mean Value Theorem: if f is a continuous function on $[a, b]$ that is differentiable on (a, b), then there exists at least one x in (a, b) such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

5. (Proof 8.4, chosen at random from the remaining 12)
Let f be defined on (a, b) with $a < 0 < b$.
Suppose that the nth derivative $f^{(n)}$ exists on (a, b).

Define the remainder

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k.$$

Prove, by repeated use of Rolle's theorem, that for each $x \neq 0$ in (a, b), there is some y between 0 and x for which

$$R_n(x) = \frac{f^{(n)}(y)}{n!} x^n.$$