Name: ______________________________

Section: ____________________________

MATHEMATICS 23a/E-23a, Fall 2018
Final Examination
Monday, December 17, 2018

You may omit one multiple-choice question in Part I, one question in Part III, and one proof in Part IV.

You may use a calculator, but only for arithmetic, perhaps in support of Newton's method.

No other aids or references are allowed.

Problem	Answer	Points	Score
$I-1$		2	
$I-2$		2	
$I-3$		2	
$I-4$		2	
$I-5$		2	
$I-6$		2	
II	$--$	6	
$III-1$	$--$	5	
$III-2$	$--$	5	
$III-3$	$--$	5	
$III-4$	$--$	5	
$III-5$	$--$	5	
$III-6$	$--$	5	
$III-7$	$--$	5	
$III-8$	$--$	5	
$IV-1$	$--$	5	
$IV-2$	$--$	5	
$IV-3$	$--$	5	
Marked omit(s)	$--$	1	
Total		61	

Part I. Answer five of the six multiple-choice questions. Transcribe your answers onto page 1, and mark an X in the score box on page 1 to indicate which question you have omitted.

If you answer all six questions, the last one will be ignored, and you will lose the extra-credit point for marking omitted questions!

1. Consider set $X = \{123456\}$ where a finite topology is being constructed. The sets $\{12\}$, $\{34\}$ and $\{15\}$ are open. What are all the open sets, if this is a valid topology?

   (a) $\{12\}, \{34\}, \{1234\}, \{15\}, \{1345\}, \{125\}, \{12345\}$

   (b) $\{12\}, \{34\}, \{1234\}, \{15\}, \{1345\}, \{125\}, \{12345\}, \phi, X$

   (c) $\{12\}, \{34\}, \{1234\}, \{15\}, \{1345\}, \{125\}, \{12345\}, \{1\}, \phi, X$

   (d) $\{12\}, \{34\}, \{1234\}, \{15\}, \{1345\}, \{125\}, \{12345\}, \{1\}, \{134\}, \phi, X$

   (e) $\{12\}, \{34\}, \{1234\}, \{15\}, \{1345\}, \{125\}, \{12345\}, \{1\}, \{134\}, \{2\}, \phi, X$

   Solution: $\boxed{D}$ Here, we have used the three axioms of topology governing open sets. The empty and entire set are open. The union of open sets are open, and the intersection of open sets are open. Answer choice D includes all the possible open sets.

2. Which one of these functions is not continuous at the origin? For all these functions $f\begin{pmatrix}0\\0\end{pmatrix} = 0$.

   (a) $x^2y/(x^2+y^2)$

   (b) $(x^2y+xy^2)/(x^2+y^2)$

   (c) $xy^2/(x+y)$

   (d) $x^3y/(x^2+y^2)^{\frac{3}{2}}$

   (e) $x/(x+y)$

   Solution: $\boxed{E}$ The best approach here is to consider an approach in polar coordinates to the origin. Let $y = r\sin\theta$ and $x = r\cos\theta$, and take the limit as $r \to 0$. If our answer is independent of θ, then our limit is not sequence dependent, and we only have to check that the limit agrees with the defined function value. Using this technique on answer choice E, we obtain:

$$\lim_{r\to 0}\frac{r\cos\theta}{r\cos\theta + r\sin\theta} = \frac{\cos\theta}{\cos\theta+\sin\theta}$$

   This is dependent on the θ of approach! For all other answer choices, we are left with a r in the numerator which will make the function go to zero as $r \to 0$.

3. A manifold M is described by a parameterization function $\gamma : \mathbb{R}^5 \to \mathbb{R}^{43}$. The locus function's domain and codomain are:

   (a) $F : \mathbb{R}^{43} \to \mathbb{R}^{38}$
   (b) $F : \mathbb{R}^{38} \to \mathbb{R}^5$
   (c) $F : \mathbb{R}^{43} \to \mathbb{R}^5$
   (d) $F : \mathbb{R}^5 \to \mathbb{R}^{38}$
   (e) $F : \mathbb{R}^{43} \to \mathbb{R}$

   Solution: $\boxed{A}$ Because this manifold is described by 5 parameters, it is a 5 dimensional manifold. It lives in $\mathbb{R}^{43}$, because that is the number of output dimensions of the parameterization function. Therefore, the locus function goes from the entire space, $\mathbb{R}^{43}$ and imposes 38 constraints $\mathbb{R}^{38}$ to get a 5 dimensional manifold, so the locus function goes from $\mathbb{R}^{43}$ to $\mathbb{R}^{38}$.

4. Which of these is true about a continuous function on an open set U?

   (a) There exists only one good sequence at all points on U
   (b) $\forall \mathbf{x_0} \in U, \forall \epsilon > 0, \exists \delta > 0$ s.t. $\forall \mathbf{x} \in U, |\mathbf{x} - \mathbf{x_0}| < \delta \implies |f(\mathbf{x}) - f(\mathbf{x_0})| < \epsilon$
   (c) $\forall \epsilon > 0, \exists \delta > 0$ s.t. $, \forall \mathbf{x}, \mathbf{x_0} \in U, |\mathbf{x} - \mathbf{x_0}| < \delta \implies |f(\mathbf{x}) - f(\mathbf{x_0})| < \epsilon$
   (d) The function is bounded on U
   (e) The function is differentiable on U

   Solution: $\boxed{B}$ This is precisely the $\epsilon - \delta$ definition of continuity on a set U. A is not true, because **all** sequences must be good at the points in U. C is not true, because that has the stronger constraint that a given δ has to work for all points. This is the statement of uniform continuity. D is not correct. The function need not be bounded, because it could diverge at the boundary and still be continuous. If it was required to be continuous on the closed interval, then it would be bounded. E is also incorrect, because continuity does not imply differentiability, however the converse is true.

5. Which of these is not true about a function f that has continuous partial derivatives on an open set U?

   (a) The function achieves its maximum on the set U

   (b) The function is continuous on the set U

   (c) The function is C^1 on the set U

   (d) The derivative is a linear function of the direction

   (e) The directional derivative in the direction $\vec{v}$, where $\vec{v}$ is a unit vector, at point $\mathbf{c}$ is given by $[Df\mathbf{c}]\vec{v}$

   Solution: $\boxed{A}$ This would be true, if the set was closed! B is true, because continuous partial derivatives imply differentiability which implies continuous. C is true, because continuous partial derivatives is the definition of C^1. D and E are both true as consequences of differentiability.

6. What are all the critical points of the function $f\begin{pmatrix} y \\ x \end{pmatrix} = xy^2 + y^3 - 4x$? Note the change of order of the variables in the argument of the function!

   (a) The critical points are $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ and are both saddle points.

   (b) The critical points are $\begin{pmatrix} -2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and are both saddle points.

   (c) The critical points are $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ and are both maximums.

   (d) The critical points are $\begin{pmatrix} -2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and are both maximums.

   (e) The critical points are $\begin{pmatrix} -2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and are both minimums.

   Solution: $\boxed{A}$ To find the critical points, we set the partial derivative equal to zero. We obtain $y^2 - 4 = 0$ and $2xy + 3y^2 = 0$. From this, we get $y = 2, x = -3$ and $y = -2, x = 3$. To classify these critical points, we find the Hessian Matrix, which is:

$$\begin{bmatrix} 0 & 2y \\ 2y & 2x + 6y \end{bmatrix}$$

   For both critical points, the determinant of the Hessian matrix is negative, implying eigenvalues of different sign, hence a saddle point.

Part II(6 points, 2 per false statement) Of the following statements, more than three are false. Choose any three of the false statements and explain why they are false. For full credit you must both comment on what is wrong with the statement and also cite an explicit counterexample. Just ignore the true statements.

Example:

Statement: "Any two unequal nonzero vectors in $\mathbb{R}^2$ span $\mathbb{R}^2$."

Answer: "False: the vectors could be linearly dependent, like $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}2\\2\end{bmatrix}$"

1. For a compact set X, if there exists an infinite set of open sets, $\{U_i\}$, such that:

$$X \in \bigcup_{i=1}^{\infty} U_i$$

then there exists some finite subset of this set of sets, $\{V_i\} \subset \{U_i\}$, such that:

$$X \in \bigcup_{i=1}^{m} V_i$$

2. If $\mathbf{c}$ is a constrained critical point of function $f : \mathbb{R}^n \to \mathbb{R}$ on a k dimensional manifold M in $\mathbb{R}^n$ described by a locus function $\mathbf{F}$, then $\ker[Df(\mathbf{c})] \subset \ker[D\mathbf{F}(\mathbf{c})]$.

False: An unconstrained critical point that happened to coincidentally lie on the manifold we were constrained to, would have $\dim(\ker[Df(\mathbf{c})]) > \dim(\ker[DF(\mathbf{c})])$, because an unconstrained critical point has $\mathbb{R}^n$ as its kernel.

An explicit counterexample can be maximizing the function $f\begin{pmatrix}x\\y\end{pmatrix} = xe^{-x}$ on the manifold $F\begin{pmatrix}x\\y\end{pmatrix} = x^2 + y^2 - 1$. Notice that at the critical point $\mathbf{c} = \begin{pmatrix}1\\1\end{pmatrix}$, $[DF(\mathbf{c})] = \begin{bmatrix}2 & 2\end{bmatrix}$, and $[Df(\mathbf{c})] = \begin{bmatrix}0 & 0\end{bmatrix}$. This way, while $[DF(\mathbf{c})]$ has a 1 dimensional kernel, $[Df(\mathbf{c})]$ has a 2 dimensional kernel!

3. Given a k dimensional manifold M in $\mathbb{R}^n$, specified by locus function $\mathbf{F}(\mathbf{z})$, where $[D\mathbf{F}(\mathbf{z})]$ is onto, any ordering of the variables in $\mathbf{z}$ will result in the existence of an implicit function that expresses the first (first in order of variables of $\mathbf{z}$) $n-k$ passive variables in terms of the next k active variables.

   False: There were many examples on the homework where certain passive variables could be described by some active variables, but not in another permutation. The requirement is that whatever the ordering of the variables, the matrix A must be invertible for it to be valid!

4. A continuous real valued function defined on a compact set has its maximum on that set.

5. A function f is differentiable at $\mathbf{a}$ if:

$$\lim_{\vec{h}\to 0} \frac{1}{|\vec{h}|}(f(\mathbf{a}+\vec{h}) - f(\mathbf{a}) - [Df(\mathbf{a})]\vec{h}) = 0$$

6. If a real valued differentiable function f at a point $\mathbf{c}$ has directional derivatives on a set of basis vectors equal to zero, then $\mathbf{c}$ is a critical point of f.

7. Consider the sequence of open sets $X_1, X_2, X_3, ...$ where $X_1 \supset X_2 \supset X_3 \supset ....$

$$\bigcap_{k=1}^{\infty} X_k \neq \phi$$

   False: The nested compact set theorem, requires these sets to be compact not open! An explicit counterexample is the sequence of sets, $(0,1), (0,\frac{1}{2}), (0,\frac{1}{3}), ...$ The infinite intersection is empty!

8. When trying to solve a system of equations, that are set equal to zero, using Newton's method, any initial guess is guaranteed to eventually superconverge to one of the roots of the system.

   False: A guess where the derivative is zero is an invalid guess, because the derivative matrix cannot be inverted, and Newton's method will not work. Graphically in the 1 variable case, this is equivalent to a horizontal line that'll never touch the x axis.

Part III. Answer seven of the eight questions. Mark an X in the score box on page 1 to indicate which question you have omitted.

1. (Inspired by Week 9, group problems 3)

   Applying Newton's second law of motion to a mass of 1 attached to a spring of spring constant 4, and a damping force that is equal to 4 times the velocity leads us to the following differential equation

$$\ddot{x} = -4\dot{x} - 4x$$

   Solve this differential equation for $x(t)$ given $x(0) = 4$, and $\dot{x}(0) = 0$. (Hint: To get started let $\vec{w} = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$, and write the differential equation in the form $\dot{\vec{w}} = A\vec{w}$)

   Solution: Letting $\vec{w}$ equal the suggestion, we can write:

$$\dot{\vec{w}} = A\vec{w}$$

   where,

$$A = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}$$

   We know that the solution to this given a set of initial conditions, $\vec{w}_0$, is $\vec{w} = e^{At}\vec{w}_0$. To exponentiate the matrix A, we need to find the eigenvalues and eigenvectors. Now, we use Axler's method to attempt to find the eigenvalues and eigenvectors of this matrix. We use an initial vector $\vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We form our Axler's matrix, and obtain:

$$\begin{bmatrix} 1 & 0 & -4 \\ 0 & -4 & 16 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -4 \end{bmatrix}$$

   Therefore, we obtain that:

$$\begin{aligned} A^2\vec{w} + 4A\vec{w} + 4\vec{w} &= 0 \\ (A + 2I)^2 &= 0 \end{aligned}$$

   We notice that $N = A + 2I$ is nilpotent, and no eigenbasis exists! However, we do know that $A = N - 2I$, and therefore:

$$
\begin{aligned}
e^{At} &= e^{(N-2I)t} \\
&= e^{Nt}e^{-2It}
\end{aligned}
$$

To exponentiate a nilpotent matrix, we use the Series Definition of the exponential, and we obtain:

$$e^{At} = (I + Nt + \frac{N^2t^2}{2!} + ...) \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$$

However, because the matrix N is nilpotent, then $N^2 = 0$, and all further powers are zero, so we simply get:

$$e^{At} = \begin{bmatrix} 2t+1 & t \\ -4t & -2t+1 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-2t} \end{bmatrix} = \begin{bmatrix} (1+2t)e^{-2t} & te^{-2t} \\ -4te^{-2t} & (-2t+1)e^{-2t} \end{bmatrix}$$

Therefore, we obtain:

$$\vec{w} = \begin{bmatrix} (1+2t)e^{-2t} & te^{-2t} \\ -4te^{-2t} & (-2t+1)e^{-2t} \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

Finally, we know that the first component of $\vec{w}$ is $x(t)$, and we get:

$$\boxed{x(t) = 4e^{-2t} + 8te^{-2t}}$$

2. (Inspired by Week 10, group problems 3)

(a) Let $f\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^2 - 1$. Evaluate the Jacobian Matrix at $f\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and use it to find the best affine approximation to $\begin{pmatrix} 1 \\ 0.1 \end{pmatrix}$.

(b) Let $g\begin{pmatrix} r \\ t \end{pmatrix} = \begin{pmatrix} r\cos t \\ r\sin t \end{pmatrix}$. Evaluate the Jacobian Matrix at $g\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and find r and h such that $g\begin{pmatrix} 1+r \\ h \end{pmatrix} = \begin{pmatrix} 1 \\ 0.1 \end{pmatrix}$.

Solution: a) First, we evaluate the Jacobian Matrix and obtain:

$$\boxed{[Df\begin{pmatrix} 1 \\ 0 \end{pmatrix}] = \begin{bmatrix} 2 & 0 \end{bmatrix}}$$

To find the best affine approximation, we obtain:

$$f\begin{pmatrix} 1 \\ 0.1 \end{pmatrix} \approx f\begin{pmatrix} 1 \\ 0 \end{pmatrix} + [Df\begin{pmatrix} 1 \\ 0 \end{pmatrix}]\begin{bmatrix} 0 \\ 0.1 \end{bmatrix}$$

$$\boxed{f\begin{pmatrix} 1 \\ 0.1 \end{pmatrix} \approx 0}$$

b) Now, we evaluate the Jacobian matrix and obtain:

$$\boxed{[Dg\begin{pmatrix} 1 \\ 0 \end{pmatrix}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

Now, we use the affine approximation to obtain:

$$\begin{bmatrix} 1 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} r \\ h \end{bmatrix}$$

From this, we obtain $\boxed{r = 0}$ and $\boxed{h = 0.1}$

3. (Inspired by Week 11, group problems 1)

Consider the functions $f\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x^2y\\y+x^2\\3x^2+2y\end{pmatrix}$ and $g\begin{pmatrix}v\\w\end{pmatrix} = \begin{pmatrix}\sqrt{vw}\\v+w\end{pmatrix}$.

(a) Calculate the derivative of $f \circ g\begin{pmatrix}2\\2\end{pmatrix}$ by using the chain rule.

(b) Calculate the derivative of $f \circ g\begin{pmatrix}2\\2\end{pmatrix}$ by explicitly calculating a formula for $f \circ g$, and then taking the derivative of that composition function.

Solution: a) First, we use the chain rule, which states:

$$\begin{aligned}
[D(f\circ g)\begin{pmatrix}v\\w\end{pmatrix}] &= [Df(g\begin{pmatrix}v\\w\end{pmatrix})][Dg\begin{pmatrix}v\\w\end{pmatrix}] \\
&= \begin{bmatrix}2\sqrt{vw}(v+w) & vw\\ 2\sqrt{vw} & 1\\ 6\sqrt{vw} & 2\end{bmatrix}\begin{bmatrix}\frac{\sqrt{w}}{2\sqrt{v}} & \frac{\sqrt{v}}{2\sqrt{w}}\\ 1 & 1\end{bmatrix}
\end{aligned}$$

Now, we substitute the point $\begin{pmatrix}2\\2\end{pmatrix}$, and we get:

$$[D(f\circ g)\begin{pmatrix}2\\2\end{pmatrix}] = \begin{bmatrix}16 & 4\\ 4 & 1\\ 12 & 2\end{bmatrix}\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ 1 & 1\end{bmatrix}$$

$$\boxed{[D(f\circ g)\begin{pmatrix}2\\2\end{pmatrix}] = \begin{bmatrix}12 & 12\\ 3 & 3\\ 8 & 8\end{bmatrix}}$$

b) Now, we explicitly calculate a formula for $f \circ g$. We obtain:

$$f\circ g\begin{pmatrix}v\\w\end{pmatrix} = \begin{pmatrix}v^2w+vw^2\\ v+w+vw\\ 3vw+2v+2w\end{pmatrix}$$

Now, we calculate the derivative, and obtain:

$$[D(f\circ g)\begin{pmatrix}v\\w\end{pmatrix}] = \begin{bmatrix}2vw+w^2 & v^2+2vw\\ 1+w & 1+v\\ 3w+2 & 3v+2\end{bmatrix}$$

Now, we plug in our point, and obtain:

$$\boxed{\left[D(f \circ g)\begin{pmatrix} 2 \\ 2 \end{pmatrix}\right] = \begin{bmatrix} 12 & 12 \\ 3 & 3 \\ 8 & 8 \end{bmatrix}}$$

4. (Inspired by Week 12, group problems 2)

For your Freshman Seminar Class, "What Would Life be Like on a Deserted Island?" your term project is to analyze the question that the seminar is named after as the culmination of a semester's worth of insightful discussion and watching "Lost". You have come to a perfect model of how a group of survivors of a plane crash ought to behave. This deserted island has polar bears that can be hunted, guns, for no good reason, and unchartered territory. There are 40 survivors, and they will do one of three tasks, hunt polar bears, x, use the guns to defend the **deserted** island (just cause there may be other peple on the island, oxymoronic? Don't think too hard about this one.) y, or to explore the rest of the island, z. You are constrained by the number of survivors, so $x+y+z=40$. Let people's happiness be given by

$$f\begin{pmatrix}x\\y\\z\end{pmatrix} = 3x + y^3 + \frac{3}{2}z^2$$

As you can see, the survivors really like guns, and seem to be more concerned with exploring the dangerous island than getting food.

(a) Use Lagrange Multipliers to find the constrained critical points.

(b) This plane can also be parameterized by $\gamma\begin{pmatrix}s\\t\end{pmatrix} = \begin{pmatrix}40-s-t\\s\\t\end{pmatrix}$. Use this parameterization function to find the constrained critical points, and classify this critical point.

Solution: Our only constraint arises from the number of people on the island, and is given by the locus function:

$$F\begin{pmatrix}x\\y\\z\end{pmatrix} = x+y+z-40$$

We first find $[DF\begin{pmatrix}x\\y\\z\end{pmatrix}]$, and $[Df\begin{pmatrix}x\\y\\z\end{pmatrix}]$. We obtain:

$$[DF\begin{pmatrix}x\\y\\z\end{pmatrix}] = \begin{bmatrix}1 & 1 & 1\end{bmatrix}$$

$$[Df\begin{pmatrix}x\\y\\z\end{pmatrix}] = \begin{bmatrix}3 & 3y^2 & 3z\end{bmatrix}$$

Now, using a Lagrange Multiplier, we obtain that:

$$\begin{bmatrix} 3 & 3y^2 & 3z \end{bmatrix} = \lambda \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

Therefore, $\lambda = 3$, and our resulting equations are:

$$\begin{aligned} 3y^2 &= 3 \\ 3z &= 3 \\ x + y + z &= 40 \end{aligned}$$

From these, we obtain, $\boxed{x = 38, y = 1, z = 1}$

b) To use the parameterization function to find the constrained critical points, we find the unconstrained critical points of $f \circ \gamma$. We obtain:

$$\begin{aligned} f \circ \gamma \begin{pmatrix} s \\ t \end{pmatrix} &= 3(40 - s - t) + s^3 + \frac{3}{2}t^2 \\ &= 120 - 3s - 3t + s^3 + \frac{3}{2}t^2 \\ [D(f \circ \gamma) \begin{pmatrix} s \\ t \end{pmatrix}] &= \begin{bmatrix} -3 + 3s^2 & -3 + 3t \end{bmatrix} \end{aligned}$$

Setting these partials equal to zero, we obtain $s = 1$ and $t = 1$, which correspond to $\boxed{x = 38, y = 1, z = 1}$, which agrees with our previous results. To classify this critical point, we find the Hessian Matrix. We obtain:

$$H = \begin{bmatrix} 6s & 0 \\ 0 & 3 \end{bmatrix}$$

The determinant of this matrix at $s = 1$, $t = 1$ is 18, which is positive, so this is a $\boxed{\textit{minimum}}$.

5. (Inspired by a proof or problem from lecture or homework)
Manifold M is described by the following parameterization function near the point $\mathbf{c} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

$$\gamma \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} uv + v^2 \\ \sqrt{uv} \\ \sqrt{v} \end{pmatrix}$$

(a) Use this parameterization function to find a basis for the tangent space at the point $\mathbf{c}$ which corresponds to the set of parameters $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(b) Find a locus function that also describes this manifold near $\mathbf{c}$.

(c) Confirm your answer to a) by also using the locus function to find a basis for the tangent space $T_{\mathbf{c}}M$.

Solution: a) We know that $T_{\mathbf{c}}M = \text{img}[D\gamma \begin{pmatrix} 1 \\ 1 \end{pmatrix}]$. Therefore, we need to find the image of the derivative matrix. First, we find the derivative matrix, and obtain:

$$[D\gamma \begin{pmatrix} u \\ v \end{pmatrix}] = \begin{bmatrix} v & u + 2v \\ \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} \\ 0 & \frac{1}{2\sqrt{v}} \end{bmatrix}$$

$$[D\gamma \begin{pmatrix} 1 \\ 1 \end{pmatrix}] = \begin{bmatrix} 1 & 3 \\ \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$$

Therefore, a basis for the tangent space, consists of $\boxed{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 1 \\ 1 \end{bmatrix}}$

b) A locus function is a function that equals zero. We notice that:

$$\boxed{F \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x - y^2 - z^4}$$

c) We know that $T_{\mathbf{c}}M = \ker[DF \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}]$. First, we find the derivative of the locus function. We obtain:

$$[DF\begin{pmatrix}x\\y\\z\end{pmatrix}] = \begin{bmatrix}1 & -2y & -4z^3\end{bmatrix}$$

$$[DF\begin{pmatrix}2\\1\\1\end{pmatrix}] = \begin{bmatrix}1 & -2 & -4\end{bmatrix}$$

To find the vectors in the kernel, we use the vectors $\begin{bmatrix}a\\1\\0\end{bmatrix}, \begin{bmatrix}b\\0\\1\end{bmatrix}$. Acting this on the matrix $[DF\begin{pmatrix}2\\1\\1\end{pmatrix}]$, we get $a = 2$, and $b = 4$. Therefore, we have a basis for the tangent space, $\boxed{\begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}4\\0\\1\end{bmatrix}}$

Remark: We ought to check that our results for a) and c) are consistent. Therefore, let's make sure that the normal vector to the plane they define is the same (so that it's the same plane!). For our vectors in a), we get:

$$\begin{bmatrix}2\\1\\0\end{bmatrix} \times \begin{bmatrix}6\\1\\1\end{bmatrix} = \begin{bmatrix}1\\-2\\-4\end{bmatrix}$$

For our vectors in b), we get:

$$\begin{bmatrix}2\\1\\0\end{bmatrix} \times \begin{bmatrix}4\\0\\1\end{bmatrix} = \begin{bmatrix}1\\-2\\-4\end{bmatrix}$$

Yes, this is consistent!

6. (Inspired by a proof or problem from lecture or homework)

   Patrick is considering how many hours, x, he should devote to sleeping and how many hours, y he should devote to studying for his Math 23 exam. He is under a couple of constraints. His mom demands that he exercises everyday, spends time for a good dinner with friends, and that he leaves time for talking to them. His parents demand that these three activities ought to total 8 hours of his day. Therefore, the total number of hours he can devote to sleeping and studying is 16 hours, namely, $x + y = 16$. However, Patrick is also in dire need of studying for math, and believes that his math time is worth more than sleeping. Therefore, he imposes the constraint that $x^2 + y = 36$. Namely, he should not sleep more than 6 hours a day.

   (a) Use one iteration of Newton's method with an initial guess of 5.5 hours of sleep and 10.5 hours of studying to solve for the number of hours Patrick will sleep.

   (b) Solve the system of equations exactly for how many hours Patrick will sleep. Note: This is not a good way to spend your days prior to the exam!

   Solution: a) The function that we are trying to find the zeros of is a vector valued function:

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y - 16 \\ x^2 + y - 36 \end{pmatrix}$$

To use the formula for Newton's method, we need to calculate the function value at our initial guess, and the derivative at our initial guess. We obtain:

$$f\begin{pmatrix} 5.5 \\ 10.5 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{19}{4} \end{pmatrix}$$
$$[Df\begin{pmatrix} 5.5 \\ 10.5 \end{pmatrix}] = \begin{bmatrix} 1 & 1 \\ 11 & 1 \end{bmatrix}$$
$$[Df\begin{pmatrix} 5.5 \\ 10.5 \end{pmatrix}]^{-1} = \begin{bmatrix} -0.1 & 0.1 \\ 1.1 & -0.1 \end{bmatrix}$$

Now, applying the formula for Newton's method, we obtain:

$$\mathbf{a_1} = \begin{bmatrix} 5.5 \\ 10.5 \end{bmatrix} - \begin{bmatrix} -0.1 & 0.1 \\ 1.1 & -0.1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{19}{4} \end{bmatrix}$$
$$\mathbf{a_1} = \begin{pmatrix} 5.025 \\ 10.975 \end{pmatrix}$$

Patrick will get $\boxed{5.025}$ hours of sleep.

b) Solving this exactly, we can solve for y in the first equation, and substitute into the second equation to obtain:

$$\begin{aligned} x^2 - x - 20 &= 0 \\ (x-5)(x+4) &= 0 \end{aligned}$$

Because the amount of sleep you get must be positive, we obtain $x = 5$, so Patrick needs $\boxed{5}$ hours of sleep, which is very close to our Newton's method estimate.

7. (Inspired by a proof or problem from lecture or homework)

   Prove that if a sequence $\mathbf{a_1}, \mathbf{a_2}...$ in $\mathbb{R}^n$ converges, then all of its coordinates are bounded.

   Solution: Week 9 Proof 2 states that if a sequence of vectors is convergent, then the corresponding sequences of coordinates are also convergent. Using this fact, we will prove that a given coordinate is bounded, and the proof can be repeated for all n coordinates. Assume the sequence converges to some limit point $\mathbf{b}$. This means that for a given coordinate i,

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall n > N, |a_{n,i} - b_i| < \epsilon$$

   Choosing some $\epsilon = \epsilon_0$, and finding the guaranteed corresponding $N = N_0$, we get that all the coordinates of vectors beyond N_0 are bounded by $(b_i - \epsilon_0, b_i + \epsilon_0)$. Therefore, we can find the maximum of this set as follows:

$$M = \max(a_{1,i}, a_{2,i}, ..., a_{N_0,i}, b_i + \epsilon_0)$$

   and the corresponding minimum:

$$m = \min(a_{1,i}, a_{2,i}, ..., a_{N_0,i}, b_i - \epsilon_0)$$

   Hence, coordinate i is bounded, and this can be repeated for all the coordinates.

8. (Inspired by a proof or problem from lecture or homework)

Better Lawn Sprinkler Co. has been hired to install sprinklers to cover your entire 1 dimensional lawn. These sprinklers can only sprinkle water on an open set, for unknown reasons. Your lawn is the interval, $[0, 1)$. You have lost the boundary of the right half of your lawn due to a dispute with your neighbor, and therefore he decided to build a fence right on the boundary of your lawn. The manager of Better Lawn Sprinkler Co. is going to propose several plans of sprinkler installation.

(a) The manager is feeling sneaky today. Invent an infinite open cover of this interval, where there does exist an finite subcover, only if you can find it, after reviewing your Math 23 notes.

(b) The manager is feeling maliciously evil. Invent an infinite open cover of this interval, where no finite subcover exists, and the manager has succeeded in scamming you of all your money.

(c) How can you modify your lawn, so that the manager will always fail to force you to buy an infinite number of sprinklers?

Solution: a) Consider the open cover consisting of the sets defined by:

$$S_0 = (-0.1, 0.1)$$
$$S_k = (\frac{0.4}{2^k}, \frac{2.4}{2^k}), \text{ where } k \in \mathbb{N}$$

This open cover has a finite subcover, because there will be overlap between S_0 and all the S_k beyond a certain k.

b) Consider the open cover consisting of the sets defined by:

$$S_k = (-\frac{1}{k}, 1 - \frac{1}{k})$$

Here, you need all the sets in order to cover every point infinitesimally close to the right side boundary.

c) Declare war on your neighbor, knock down your neighbor's wall and claim it as your territory! Once, you do this, your lawn will be the interval $[0, 1]$, so it will be compact (closed and bounded), and thus the Heine-Borel Theorem guarantees the existence of a finite subcover for any open cover.

Remark: There can be many different correct answers for a and b!

Part V. Do two of the three proofs. Mark an X in the score box on page 1 to indicate which proof you have omitted.

1. (Proof 9.2)

   Starting from the triangle inequality for two vectors, prove the triangle inequality for n vectors, then prove the "infinite triangle inequality" for $\mathbb{R}^n$

$$|\sum_{i=1}^{\infty} \vec{\mathbf{a_i}}| \leq \sum_{i=1}^{\infty} |\vec{\mathbf{a_i}}|$$

   under the assumption that the infinite series on the right is convergent, which in turn implies that the infinite series of vectors on the left is convergent.

2. (Proof 11.2)

   Using the mean value theorem, prove that if a function $f : \mathbb{R}^2 \to \mathbb{R}$ has partial derivatives $D_1 f$ and $D_2 f$ that are continuous at $\mathbf{a}$, it is differentiable at $\mathbf{a}$ and its derivative is the Jacobian matrix $\begin{bmatrix} D_1 f(\mathbf{a}) & D_2 f(\mathbf{a}) \end{bmatrix}$.

3. (Proof 12.3)

   Let M be a manifold known by a real-valued C^1 function $F(\mathbf{x}) = 0$, where F goes from an open subset U of $\mathbb{R}^n$ to $\mathbb{R}$ and $[\mathbf{D}F(\mathbf{x})]$ is nowhere zero. Let $f : U \to \mathbb{R}$ be a C^1 function.

   Prove that $\mathbf{c} \in M$ is a critical point of f restricted to M if and only if there exists a Lagrange multiplier λ such that $[\mathbf{D}f(\mathbf{c})] = \lambda[\mathbf{D}F(\mathbf{c})]$.