MATHEMATICS 23a/E-23a, Fall 2018
Linear Algebra and Real Analysis I
Week 5 (Number Systems and Sequences)

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R scripts by Paul Bamberg
Last modified: October 11, 2018 by Paul Bamberg (special offer on HW)

Reading

- Ross, Chapter 1, sections 1 through 5 (number systems)
- Ross, Chapter 2, sections 7 through 9 (sequences)
- Hubbard, section 0.2 (quantifiers and negation)
- Hubbard, section 0.6 (infinite sets)

Recorded Lectures

- Lecture 10 (Week 5, Class 1) (watch on October 9 or 10)
- Lecture 11 (Week 5, Class 2) (watch on October 11 or 12)

Proofs to present in section or to a classmate who has done them.

- 5.1 Define "countably infinite." Prove that the set of positive rational numbers is countably infinite, but that the set of real numbers in the interval [0,1], as represented by infinite decimals, is not countable.
- 5.2 Suppose that $s_n \neq 0$ for all n and that $s = \lim s_n > 0$.
Prove that $\exists N$ such that $\forall n > N, s_n > s/2$, and that $\frac{1}{s_n}$ converges to $\frac{1}{s}$.

Additional proofs(may appear on quiz, students will post pdfs or videos)

- 5.3 (Ross, p. 25; the Archimedean Property of $\mathbb{R}$)

  The completeness axiom for the real numbers states that every nonempty subset $S \subset \mathbb{R}$ that is bounded above has a least upper bound sup S. Use it to prove that for any two positive real numbers a and b, there exists a positive integer n such that $na > b$.

- 5.4 (Ross, page 52)

  Suppose that $\lim s_n = +\infty$ and $\lim t_n > 0$. Prove that $\lim s_n t_n = +\infty$.

R Scripts

- Script 2.1A-Countability.R
  Topic 1 - The set of ordered pairs of natural numbers is countable
  Topic 2 - The set of positive rational numbers is countable

- Script 2.1B-Uncountability.R
  Topic 1 - Cantor's proof of uncountability
  Topic 2 - A different-looking version of the same argument

- Script 2.1C-Denseness.R
  Topic 1 - Placing rational numbers between any two real numbers

- Script 2.1D-Sequences.R
  Topic 1 - Limit of an infinite sequence
  Topic 2 - Limit of sum = sum of limits
  Topic 3 - Convergence of sequence of inverses (proof 5.2)

1 Executive Summary

1.1 Natural Numbers and Rational Numbers

- The natural numbers $\mathbb{N}$ are $1, 2, 3, \cdots$. They have the following rather obvious properties. What is not obvious is that these five properties (the "Peano axioms") are sufficient to prove any other property of the natural numbers.
  - N1. 1 belongs to $\mathbb{N}$.
  - N2. If $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$.
  - N3. 1 is not the successor of any element of $\mathbb{N}$.
  - N4. If n and $m \in \mathbb{N}$ have the same successor, then $n = m$.
  - N5. A subset $S \in \mathbb{N}$ which contains 1, and which contains $n + 1$ whenever it contains n, must equal $\mathbb{N}$.
- Axiom N5 is related to "proof by induction," where you want to prove an infinite set of propositions $P_1, P_2, P_3, \cdots$.
  You do this by proving P_1 (the "base case") and then proving that P_n implies P_{n+1} (the "inductive step").
- The "least number principle" states that any nonempty subset of $\mathbb{N}$ has a least element. This statement, along with the assumption that any natural number except 1 has a predecessor, can be used to replace N5.
  Practical application: instead of doing a proof by induction, you can assert that $k > 1$ is the smallest integer for which P_k is false, then get a contradiction by showing that P_{k-1} is also false, thereby proving that the set for which P_k is false must be empty.
- The familiar rational numbers can be regarded as fractions in lowest terms: e.g. $\frac{m}{n}$ and $\frac{2m}{2n}$ represent the same rational number. The rational number $r = \frac{m}{n}$ satisfies the first-degree polynomial equation $nx - m = 0$. More generally, a number that satisfies a polynomial equation of any (finite) degree, like $x^2 - 2 = 0$ or $x^5 + x - 1 = 0$, is called an algebraic number.
- The rational numbers form a "countably infinite set," which means that there is a bijection between them and the natural numbers. Many proofs rely on the fact that the rational numbers, or a subset of them, can be enumerated as $q_1, q_2, \cdots$.

1.2 Rational Numbers and Real Numbers

- The rational numbers and the real numbers each form an *ordered field*, which means that there is a relation $\leq$ with properties
  O1. Given a and b, either $a \leq b$ or $b \leq a$.
  O2. If $a \leq b$ and $b \leq a$, then $a = b$.
  O3. If $a \leq b$ and $b \leq c$ then $a \leq c$.
  O4. If $a \leq b$, then $a + c \leq b + c$.
  O5. If $a \leq b$ and $0 \leq c$, then $ac \leq bc$.
  Many important properties of infinite sequences of real numbers can be proved on the basis of ordering.

- If we think of the rational numbers or the real numbers as lying on a number line, we can interpret the absolute value $|a - b|$ as the distance between point a and point b: $\text{dist}(a, b) = |a - b|$. In two dimensions the statement $\text{dist}(\mathbf{a}, \mathbf{b}) \leq \text{dist}(\mathbf{a}, \mathbf{c}) + \text{dist}(\mathbf{c}, \mathbf{b})$ means that the length of one side of a triangle cannot exceed the sum of the lengths of the other two sides. The name "triangle inequality" is also applied to the one-dimensional special case where $c = 0$; i.e. $|a + b| \leq |a| + |b|$.

- Many well-known rules of algebra are not included on the list of field axioms. Usually, as for $(-a)(-b) = ab$, this is because they are easily provable theorems. However, there are properties of the real numbers that cannot be proved from the field axioms alone because they rely on the axiom that the real numbers are *complete*. The Completeness Axiom states that
  Every nonempty subset S of $\mathbb{R}$ that is bounded above has a least upper bound.
  This least upper bound $\sup S$ is not necessarily a member of the set S.

- The Archimedean property of the real numbers states that
  for any two positive real numbers a and b, there exists a positive integer n such that $na > b$. Its proof requires the Completeness Axiom.

- The rational numbers are a "dense subset" of the real numbers. This means if $a, b \in \mathbb{R}$ and $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$.
  Again the proof relies of the completeness of the real numbers.

- It is not unreasonable to think of real numbers as infinite decimals (though there are complications). In this view, π (which is not even algebraic) is the least upper bound of the set
  $S = \{3, 3.1, 3.14, 3.141, 3.1415, 3.14159, \cdots\}$

- The real numbers form an uncountable set. This means that there is no bijection between them and the natural numbers: they cannot be enumerated as $r_1, r_2, \cdots$.

1.3 Quantifiers and Negation

- Quantifiers are not used by Ross, but they are conventional in mathematics and save space when you are writing proofs.

  $\exists$ is read "there exists." It is usually followed by "such that" or "s.t."

  Example: the proposition "$\exists x$ s.t. $x^2 = 4$" is true since either 2 or -2 has the desired property.

  $\forall$ is read "for all" or "for each" or "for every." It is used to specify that some proposition is true for every member of a possibly infinite set or sequence.

  Example: $\forall x \in \mathbb{R}, x^2 \geq 0$ is true, but $\forall x \in \mathbb{R}, x^2 > 0$ is false.

- Quantifiers and negation: useful in doing proofs by contradiction.

  - The negation of "$\exists x$ such that $P(x)$ is true" is "$\forall x, P(x)$ is false."
  - The negation of "$\forall x$, $P(x)$ is true" is "$\exists x$ such that $P(x)$ is false."

1.4 Sequences and their limits

- A sequence is really a function whose domain is a subset $n \geq m$ of the integers, usually starting with $m = 0$ or 1, and whose codomain (in this module) is $\mathbb{R}$. Later we will consider sequences of vectors in $\mathbb{R}^n$.

  A specific element is denoted s_n. The entire sequence can be denoted $(s_1, s_2, \cdots)$ or $(s_n)_{n \in \mathbb{N}}$ or even just (s_n).

  Although a sequence is infinite, the set of values in the sequence may be finite; e.g. for $s_n = \cos n\pi$ the set of values is just $\{-1, 1\}$.

- "Limit of a sequence" always refers to the limit as n becomes very large; so it is unambiguous to write it $\lim s_n$ instead of $\lim_{n\to\infty} s_n$.

  Sequence (s_n) is said to converge to the limit s if

  $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n > N, |s_n - s| < \epsilon$.

  To prove that a sequence (s_n) converges by using this definition, we have to know or guess the value of the limit s. The rest is algebra, frequently rather messy algebra.

- If the limit of a sequence exists, it is unique. The proof is a classic application of the triangle inequality.

- A "formal proof" should be as concise as possible while omitting nothing that is essential. Sometimes it obscures the chain of thought that led to the invention of the proof. Formal proofs are nice, and you should learn how to write them (Ross has six examples in section 8 and six more in section 9), but if your goal is to convince or instruct the reader, a longer version of the proof may be preferable.

1.5 Theorems about sequences and their limits

- Theorems about limits, all provable from the definition. These will be especially useful for us after we define continuity in terms of sequences.
  - If $\lim s_n = s$ then $\lim(ks_n) = ks$.
  - If $\lim s_n = s$ and $\lim t_n = t$, then $\lim(s_n + t_n) = s + t$.
  - Any convergent sequence is bounded: if $\lim s_n = s$, $\exists M$ such that $\forall n, |s_n| < M$.
  - If $\lim s_n = s$ and $\lim t_n = t$, then $\lim(s_n t_n) = st$.
  - If $\lim s_n = 0$ and (t_n) is bounded, then $\lim(s_n t_n) = 0$.
  - If $s_n \neq 0$ for all n and $s = \lim s_n \neq 0$, then $\inf |s_n| > 0$ and $\frac{1}{s_n}$ converges to $\frac{1}{s}$.
- Using the limit theorems above is usually a much more efficient way to find the limit of the sequence than doing a brute-force calculation of N in terms of ϵ. Ross has six diverse examples.
- The symbol $+\infty$ has a precise meaning when used to specify a limit. We say that "the sequence s_n diverges to $+\infty$" if

  $\forall M > 0, \exists N$ such that $\forall n > N, s_n > M$.

  Similarly, we say that "the sequence s_n diverges to $-\infty$" if

  $\forall M < 0, \exists N$ such that $\forall n > N, s_n < M$.
- Theorems about infinite limits:
  - If $\lim s_n = +\infty$ and $\lim t_n > 0$ (could be $+\infty$), then $\lim s_n t_n = +\infty$.
  - If (s_n) is a sequence of *positive* real numbers, then $\lim s_n = +\infty$ if and only if $\lim \frac{1}{s_n} = 0$.
  - If $\lim s_n = +\infty$, then $\lim s_n + t_n = +\infty$ if t_n has any of the following properties:
    * $\lim t_n > -\infty$
    * t_n is bounded (but does not necessarily converge).
    * $\inf t_n > -\infty$ (who cares whether t_n is bounded above?).

2 Lecture Outline

1. Peano axioms for the natural numbers ——$\mathbb{N} = 1, 2, 3, \cdots$

   - N1. 1 belongs to $\mathbb{N}$.
   - N2. If $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$.
   - N3. 1 is not the successor of any element of $\mathbb{N}$.
   - N4. If n and $m \in \mathbb{N}$ have the same successor, then $n = m$.
   - N5. A subset $S \in \mathbb{N}$ which contains 1, and which contains $n + 1$ whenever it contains n, must equal $\mathbb{N}$.

   Axiom N5 is related to "proof by induction," where you want to prove an infinite set of propositions $P_1, P_2, P_3, \cdots$.
   You do this by proving P_1 (the "base case") and then proving that P_n implies P_{n+1} (the "inductive step").

   A well known example: the formula $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n + 1)$

   For proposition P_1 simply set $n = 1$: it is true that $1 = \frac{1}{2}n(n + 1)$

   Write down proposition P_n, and use a little algebra to show that if P_n is in the sequence of true propositions, then so is P_{n+1}

A surprising replacement for axiom N5:

- Every subset of $\mathbb{N}$ has a smallest element.
- Any element of $\mathbb{N}$ except 1 has a predecessor.

Use these two statements (plus N1 through N4) to prove N5.

Practical application: instead of doing a proof by induction, you can denote by k the smallest integer for which P_k is false, then get a contradiction by showing that P_{k-1} is also false, thereby proving that the set for which P_k is false must be empty.

How this works in our example-

Suppose that $1+2+3+\cdots+n = \frac{1}{2}n(n+1)$ is not always true. Then there is a nonempty subset of natural numbers for which it is false. This subset includes a smallest number k.

Using our analysis from the previous page:

How do we know that k cannot be 1?

Given that k cannot be 1, how do we know that k cannot in fact be the smallest element for which P_k is false?

There is less to this approach than meets the eye. Instead of proving that P_k implies P_{k+1} for $k \geq 1$, we showed that NOT P_k implies NOT P_{k-1} for $k \geq 2$,

But these two statements are logically equivalent: quite generally, for propositions p and q, $p \implies q$ if and only if $\neg q \implies \neg p$. (principle of contraposition)

A practical rule of thumb:

- If it is easier to prove that $P_k \implies P_{k+1}$, use induction.
- If it is easier to prove that $\neg P_k \implies \neg P_{k-1}$, use the least-number principle.

2. Proof by induction and least number principle – an example

   Students of algebra are aware that for any positive integer n, $x^n - y^n$ is divisible by $x - y$.

   - Give a formal inductive proof of this theorem by induction ("formal" means no use of $\cdots$).

   - Give an alternative proof using the fact that any nonempty set of positive integers contains a smallest element.

3. (Ross, page 16; consequences of the ordered field axioms)

   Using the fact that a set of numbers F (could be $\mathbb{Q}$ or $\mathbb{R}$) satisfies the ordered field axioms
   O1. Given a and b, either $a \leq b$ or $b \leq a$.
   O2. If $a \leq b$ and $b \leq a$, then $a = b$.
   O3. If $a \leq b$ and $b \leq c$ then $a \leq c$.
   O4. If $a \leq b$, then $a + c \leq b + c$.
   O5. If $a \leq b$ and $0 \leq c$, then $ac \leq bc$.
   prove the following:

   - If $a \leq b$ then $-b \leq -a$.

   - $\forall a \in F$, $a^2 \geq 0$.

4. Rules for "less than," derived from the ordered field axioms
   L1. If $b \leq a$ is false, then $a < b$ (definition)
   L2. If $a \neq b$, then either $a < b$ or $b < a$ (contrapositive of O2)
   L3. If $a < b$ and $b < c$, then $a < c$.
   L4. If $a < b$, then $a + c < b + c$.
   L5. If $a < b$ and $0 < c$, then $ac < bc$.

   (a) Show that L2 is the contrapositive of O2.

   (b) Prove that the sum of two positive numbers is a positive number.

   (c) Prove that the product of two positive numbers is a positive number.

   (d) Prove that $\mathbb{Z}_5$ is not an ordered field.

   (e) Use a theorem based on the ordered field axioms to show that the field of complex numbers is not ordered.

5. Countable sets

A set A is finite, with cardinality n, if there is a bijective correspondence between it and the set $\{1, 2, \cdots, n\}$. Naming the elements $A_1, A_2, \cdots, A_n$ implies that the set is finite.

A set is countably infinite if there is a bijective correspondence between it and the set $\mathbb{N} = \{0, 1, 2, \cdots\}$. Naming the elements $A_0, A_1, A_2, \cdots$ implies that the set is countably infinite. (Including 0 as a subscript is a matter of taste, consistent with Hubbard.)

Sets that "look much bigger" than $\mathbb{N}$ may be countable. Examples:

- The Cartesian product $A^2 = \mathbb{N} \times \mathbb{N}$.

  List the elements "diagonally" in increasing order of the sum of the elements in the pair.

  $(A^2)_0 = (0, 0)$
  $(A^2)_1 = (0, 1)$
  $(A^2)_2 = (1, 0)$
  $(A^2)_3 = (0, 2)$
  $(A^2)_4 = (1, 1)$
  $(A^2)_5 = (2, 0)$
  etc.

- The Cartesian product A^k of k copies of $\mathbb{N}$. An element of A^k is an ordered list of k non-negative integers, like (2,3,0,9,5) if $k = 5$.

  Proof strategy: induction on k. You should have learned how to do this in high school.

  What is the base case?

  Do the "inductive step": assume that A^k is countably infinite (inductive hypothesis), and prove that A^{k+1} is countably infinite.

- A countably infinite union of disjoint countably infinite sets:

$$A = \bigcup_{i=0}^{\infty} B_i$$

.

When a set is countably infinite, its elements can be indexed by the positive integers. Name the jth element of the ith set $B_{(i,j)}$. Now there is a bijective correspondence between the elements of A and the countably infinite set $\mathbb{N} \times \mathbb{N}$.

$A_0 = B_{(0,0)}$
$A_1 = B_{(0,1)}$
$A_2 = B_{(1,0)}$
$A_3 = B_{(0,2)}$
$A_4 = B_{(1,1)}$
$A_5 = B_{(2,0)}$
etc.

- The positive rational numbers

Proof: List them "diagonally" by numerator and denominator, omitting any that are not in lowest terms.

$Q_0 = \frac{1}{1}$
$Q_1 = \frac{1}{2}$
$Q_2 = \frac{2}{1}$
$Q_3 = \frac{1}{3}$
$Q_4 = \frac{3}{1}$
etc.

- All the integers, positive, negative, and zero.

Proof: List them as follows:

$Z_0 = 0$
$Z_1 = 1$
$Z_2 =$ -1
$Z_3 = 2$
$Z_4 =$ -2
etc.

- All the rational numbers, positive, negative, and zero.

- The set $\mathbb{Q}^n$ of n-element lists of rational numbers.

A sample element of $\mathbb{Q}^3$ is $(\frac{1}{2}, -\frac{8}{3}, \frac{22}{7})$.

One-line proof: this is a finite Cartesian product of countable sets.

6. Uncountable sets:

- The real numbers in [0, 1]

  The “diagonal” trick for this proof was invented by Georg Cantor in the late nineteenth century.

  Assume that these real numbers are a countable set: i.e. assume that there is an indexed list of infinite decimals. For example,

  $A_1 = .141592...$

  $A_2 = .428571...$

  $A_3 = .122333...$

  $A_4 = .331314...$

  $A_5 = .271828...$

  $A_6 = .137036...$

  Circle all the digits on the diagonal.

  There are many ways to modify the sequence of digits on the diagonal to make a number x that cannot possibly appear on the list. All that is required is to change every digit. Two possibilities:

  – Make a number x where the kth digit of x is greater by 1 (modulo 10) than the kth digit of A_k. Here, $x = .233437...$
  – Make the kth digit of x be 5 if the kth digit of A_k is 2. Otherwise make it 2. In this case, $x = .255252$.

  For either choice of x, the real number x cannot appear in any position k on the list, because it differs from A_k in the kth digit. So we have a contradiction with the assumption that every real number in [0, 1] is on the list. These real numbers are an uncountable set.

- The set of infinite lists of elements of $\mathbb{Z}_3$

  Proof: the same – assume an indexed list, and circle the diagonal elements.

  $A_1 = (1, 2, 1, 0, 2, ...)$

  $A_2 = (0, 2, 1, 2, 1, ...)$

  $A_3 = (1, 1, 1, 2, 1, ...)$

  $A_4 = (2, 1, 0, 2, 1, ...)$

  $A_5 = (2, 1, 1, 1, 0, ...)$

  Construct $x = (2, 0, 2, 0, 1, ...)$

  by adding 1 to the kth item in list A_k. This x cannot be the same as any of the A_k, so the set of infinite lists is not countable.

7. Proof 5.1 – start to finish

   Define "countably infinite." Prove that the set of positive rational numbers is countably infinite, but that the set of real numbers in the interval [0,1], as represented by infinite decimals, is not countable.

8. (Ross, p. 25; the Archimedean Property of $\mathbb{R}$– proof 5.3)

   The completeness axiom for the real numbers states that every nonempty subset $S \in \mathbb{R}$ that is bounded above has a least upper bound $\sup S$. Use it to prove that for any two positive real numbers a and b, there exists a positive integer n such that $na > b$.

Corollary – the denseness of $\mathbb{Q}$.
In between any two real numbers a and b there is a rational number r.(Illustrated in script 2.1C; proved as a section problem)

9. The least upper bound principle works for $\mathbb{R}$ but not for $\mathbb{Q}$.

   Your students at Springfield North are competing with a rival team from Springfield South to draw up a business plan for a company with m scientists and n other employees. Entries with $m^2 > 2n^2$ get rejected. The entry with the highest possible ratio of scientists to other employees wins the contest. Will this competition necessarily have a winner?

10. Sequences and their limits

- A sequence is really a function whose domain is a subset $n \geq m$ of the integers, usually starting with $m = 0$ or 1, and whose codomain (in this module) is $\mathbb{R}$. Later we will consider sequences of vectors in $\mathbb{R}^n$.

  A specific element is denoted s_n. The entire sequence can be denoted $(s_1, s_2, \cdots)$ or $(s_n)_{n\in\mathbb{N}}$ or even just (s_n).

  Although a sequence is infinite, the set of values in the sequence may be finite; e.g. for $s_n = \cos n\pi$ the set of values is just $\{-1, 1\}$.

- "Limit of a sequence" always refers to the limit as n becomes very large; so it is unambiguous to write it $\lim s_n$ instead of $\lim_{n\to\infty} s_n$.

  Sequence (s_n) is said to converge to the limit s if

  $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n > N, |s_n - s| < \epsilon$.

  To prove that a sequence (s_n) converges by using this definition, we have to know or guess the value of the limit s. The rest is algebra, frequently rather messy algebra.

- If the limit of a sequence exists, it is unique. The proof is a classic application of the triangle inequality.

- A "formal proof" should be as concise as possible while omitting nothing that is essential. Sometimes it obscures the chain of thought that led to the invention of the proof. Formal proofs are nice, and you should learn how to write them (Ross has six examples in section 8 and six more in section 9), but if your goal is to convince or instruct the reader, a longer version of the proof may be preferable.

- Using the definition of the limit of a sequence, invent a formal proof that

$$\lim \frac{1}{\sqrt{n}} = 0.$$

11. Using quantifiers to describe sequences

    Let s_n denote the number of inches of snowfall in Cambridge in year n, e.g. $s_{2013} = 90$. Using the quantifiers $\exists$ (there exists) and $\forall$ (for all), convert the following English sentences into mathematical notation.

    (a) There will be infinitely many years in which the Cambridge snowfall exceeds 100 inches.

    (b) If you wait long enough, there will come a year after which Cambridge never again gets more than 20 inches of snow.

    (c) The snowfall in Cambridge will approach a limit of zero.

12. Tricks of the trade for real analysis proofs

    - To prove that real number $a = 0$, prove that $\forall \epsilon > 0$, $a < \epsilon$.
    - "Triangle inequality trick": write $|a - b| = |a - c + c - b| = |(a - c) + (c - b)| \leq |(a - c)| + |(c - b)|$
    - "Epsilon over two trick": to prove that $|a - b| < \epsilon$,, show that $|(a - c)|$ and $|(c - b)|$ are both less than $\frac{\epsilon}{2}$.

13. Uniqueness of limits (Ross, page 37)

    Prove that if $\lim s_n = s$ and $\lim s_n = t$, then $s = t$.

14. Sums and products of infinite sequences

    (Ross, page 46)

    Prove that if $\lim s_n = s$ and $\lim t_n = t$, then $\lim(s_n + t_n) = s + t$.

    (Ross, pages 45 and 47)

    Prove that any convergent sequence is bounded, then use this result to show that if $\lim s_n = s$ and $\lim t_n = t$, then $\lim(s_n t_n) = st$.

15. Some examples with decreasing sequences

    - Invent an example of a sequence (s_n) of positive numbers that is strictly decreasing ($s_{n+1} < s_n$ for all n) but whose limit is not zero.

    - Students of calculus readily accept the statement
      "if $|a| < 1$, then $\lim_{n\to\infty} a^n = 0$"
      on the basis that "when I add 1 to n, a^n gets smaller in magnitude."
      However, the preceding example shows that this observation is not good enough! Do a correct proof.

16. (Proof 5.2 – a sequence of inverses)

    Suppose that $s_n \neq 0$ for all n and that $s = \lim s_n > 0$.
    Prove that $\exists N$ such that $\forall n > N, s_n > s/2$ and that $\frac{1}{s_n}$ converges to $\frac{1}{s}$.

17. Theorems about limits, all provable from the definition. These will be especially useful for us after we define continuity in terms of sequences.

    - If $\lim s_n = s$ then $\lim(ks_n) = ks$.
    - If $\lim s_n = s$ and $\lim t_n = t$, then $\lim(s_n + t_n) = s + t$.
    - If $\lim s_n = s$, $\exists M$ such that $\forall n, |s_n| < M$.
    - If $\lim s_n = s$ and $\lim t_n = t$, then $\lim(s_n t_n) = st$.
    - If $\lim s_n = 0$ and (t_n) is bounded, then $\lim(s_n t_n) = 0$.
    - If $s_n \neq 0$ for all n and $s = \lim s_n \neq 0$, then $\inf |s_n| > 0$ and $\frac{1}{s_n}$ converges to $\frac{1}{s}$.
    - If $\lim s_n = 0$ and $\exists N$ such that $0 < t_n < s_n \forall n > N$ then $\lim t_n = 0$

18. (Ross, page 48) Using the binomial expansion, prove that $\lim(n^{\frac{1}{n}}) = 1$.

19. Show that if $0 < a < 1$, $\lim na^n = 0$.

20. Infinite limits

    "Infinity" is not a well-defined real number since, for example, we cannot assign a value to $\infty - \infty$ or to $0 \times \infty$. However, ∞ has a precise meaning when used as a limit.

    $\lim s_n = +\infty$ means that

    $\forall M \in \mathbb{R}$, $\exists N \in \mathbb{N}$ such that $\forall n > N, s_n > M$.

    Prove that $\lim \log_{10} n = \infty$.

    True or false? $\lim n \sin n = \infty$.

21. (Ross, page 52 – proof 5.4)

    Suppose that $\lim s_n = +\infty$ and $\lim t_n > 0$. Prove that $\lim s_n t_n = +\infty$.

22. Expressing limits in terms of “infinitely many” and “only finitely many.”

(a) “No matter how large a positive number M you choose, the sequence (s_n) has infinitely many elements that are greater than M.”
Does this statement imply that $\lim s_n = +\infty$?

(b) “No matter how large a positive number M you choose, the sequence (s_n) has only finitely many elements that are less than M.”
Does this statement imply that $\lim s_n = +\infty$?

(c) “No matter how small a positive number ϵ you choose, the sequence (s_n) has only finitely many elements that lie outside the interval $(a - \epsilon, a + \epsilon)$.”
Does this statement imply that $\lim s_n = a$?

(d) “No matter how small a positive number ϵ you choose, the sequence (s_n) has infinitely many elements that lie inside the interval $(a - \epsilon, a + \epsilon)$.”
Does this statement imply that $\lim s_n = a$?

23. Proving limits by brute force

    Prove by brute force that the sequence

$$\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \dots$$

    converges to

$$\frac{1}{2}.$$

24. Using limit theorems and trickery to prove limits

(a) Evaluate

$$\lim \frac{1}{n(\sqrt{n^2+1}-\sqrt{n^2-1})}.$$

$$\text{Note: } \frac{1}{100(\sqrt{10001}-\sqrt{9999})} = 0.99999999874999999...$$

(b) Evaluate

$$\lim((n+1)^{\frac{4}{3}} - n^{\frac{4}{3}}).$$

Note: $101^{\frac{4}{3}} - 100^{\frac{4}{3}} = 6.19907769....$; $\sqrt[3]{100} = 4.6415....$

3 Seminar Topics

Your section instructor will either have emailed a list of topics to prepare or will have posted a sign-up list of appointments on the Calendar tab of Canvas. Either way, there will be one of the following topics that you should be prepared to present.

Practice your presentation so that it takes about 8 minutes. The text of the presentation will be projected onto a screen so that you need not recopy it. To save time, avoid writing long sentences on the chalkboard. You may use notes, but be discreet about it.

1. (Alternate version of the Peano Axioms)
   Assume the following properties of the natural numbers $\mathbb{N}$:

   - N1. 1 belongs to $\mathbb{N}$.
   - N2. If $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$.
   - N3. 1 is not the successor of any element of $\mathbb{N}$.
   - N4. If n and $m \in \mathbb{N}$ have the same successor, then $n = m$.
   - Every nonempty subset $S \in \mathbb{N}$ has a least element.
   - A natural number n_0 that is not equal to 1 is the successor to some number in $\mathbb{N}$.

   Using these assumptions, prove
   N5. A subset $S \in \mathbb{N}$ which contains 1, and which contains $n + 1$ whenever it contains n, must equal $\mathbb{N}$.

   Explain the connection between N5 and the technique of proof by induction.

2. (Proof 5.1) Define "countably infinite." Prove that the set of positive rational numbers is countably infinite, but that the set of real numbers in the interval [0,1], as represented by infinite decimals, is not countable.

3. (Proof 5.3 – the Archimedean Property of $\mathbb{R}$)
   The completeness axiom for the real numbers states that every nonempty subset $S \subset \mathbb{R}$ that is bounded above has a least upper bound $\sup S$. Use it to prove that for any two positive real numbers a and b, there exists a positive integer n such that $na > b$.

4. (Proof 5.2) Suppose that $s_n \neq 0$ for all n and that $s = \lim s_n > 0$.
   Prove that $\exists N$ such that $\forall n > N, s_n > s/2$, and that $\frac{1}{s_n}$ converges to $\frac{1}{s}$.

5. (Proof 5.4)
   Suppose that $\lim s_n = +\infty$ and $\lim t_n > 0$. Prove that $\lim s_n t_n = +\infty$.

6. (Extra topic) Prove that if $\lim s_n = s$ and $\lim t_n = t$, then $\lim(s_n + t_n) = s + t$.

4 Workshop Problems

1. Proofs that use induction (in the second step)

   (a)
   - Starting from $xy \le |xy|$, which looks like Cauchy-Schwarz, prove the triangle inequality $|a + b| \le |a| + |b|$ for an ordered field
   - Starting from the triangle inequality, prove that for n numbers $a_1, a_2, \cdots, a_n$
     $|a_1 + a_2 + \cdots + a_n| \le |a_1| + |a_2| + \cdots + |a_n|$.

   (b)
   - Use the Archimedean property of the real numbers to prove if a and b are positive real numbers and $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$.
     If you need a hint, look at section 4.8 in Ross or run script 2.1C. The fact that a and b are positive makes the proof easier than the one in Ross.
   - By induction, prove that in any open interval (a, b) there are infinitely many rational numbers.

2. Properties of sequences (type your group's proof into the Canvas math editor)

   (a) The "squeeze lemma"
   Consider three sequences $(a_n), (b_n), (c_n)$ such that $a_n \leq s_n \leq b_n$ for all n and $\lim a_n = \lim b_n = s$. Prove that $\lim s_n = s$.

   (b) Prove that if sequence (t_n) is bounded and $\lim(s_n) = 0$, then $\lim(t_n s_n) = 0$.

3. Some slightly computational problems

   (a) Proving limits by brute force

   Let

   $$s_n = \frac{6n-4}{2n+8}.$$

   Determine $\lim s_n$ and prove your answer by brute force, directly from the definition of limit. (For a model, see Ross, Example 2 on page 39.)

   Then get the same answer more easily by using limit theorems.

   (b) Finding limits by using limit theorems

   Determine $\lim(\sqrt{n(n+2)} - n)$, stating what limit theorems you are using in each step.

   Hint: Use the same trick of "irrationalizing the denominator" as in Ross, section 8, example 5. However, that example requires using the definition of limit. You can invoke limit theorems, which makes things much easier.

5 Homework

Special offer – if you do the entire problem set, with one problem omitted, in LaTeX, you will receive full credit for the omitted problem. Alternatively, if you work all the problems in LaTeX, we will convert your lowest score to a perfect score.

1. Ross, exercise 1.1. Do the proof both by induction (with "base case" and "inductive step") and by the least number principle (show that the assumption that there is a nonempty set of positive integers for which the formula is not true leads to a contradiction)

2. Using quantifiers to describe infinite sequences

   A Greek hero enters the afterlife and is pleased to learn that the goddess Artemis is going to be to training him for eternity. He will be shooting an infinite sequence of arrows. The distance that the nth arrow travels is s_n. Use quantifiers $\exists$ and $\forall$ to convert the following to mathematical notation.

   (a) He will shoot only finitely many arrows more than 200 meters.

   (b) The negation of (a): he will shoot infinitely many arrows more than 200 meters. (You can do this mechanically by using the rules for negation of statements with quantifiers.)

   (c) No matter how small a positive number ϵ Artemis chooses, all the rest of his shots will travel more than $200 - \epsilon$ meters.

   (d) He will become so consistent that eventually any two of his subsequent shots will differ in distance by less than 1 meter. (This idea will resurface next week as the concept of "Cauchy sequence.")

3. Denseness of $\mathbb{Q}$

   This problem is closely related to group problem 1b.

   (a) Find a rational number x such that $\frac{355}{113} < x < \frac{22}{7}$.

   (b) Find a rational number x such that $\pi < x < \frac{355}{113}$.
   Hint: $\pi = 4 \arctan 1$, which any decent calculator can evaluate.

4. Ross, exercise 3.6.

5. Ross, exercise 4.8. If you like this problem, you might enjoy reading enrichment section 6 in Ross, which explains how to construct the real numbers using "Dedekind cuts."

6. Ross, Exercise 8.2(c) and 8.2(e). You might want to use the limit theorems from section 9 to determine the limit, but then do a "Formal Proof" in the style of the examples from section 8, working directly from the definition of limit.

7. Ross, Exercise 8.9. The star on the exercise means that it is “referred to in many places.”

8. Ross, Exercise 9.12. This “ratio test” may be familiar from a calculus course. There is a similar, better known test for infinite *series* that is slightly more difficult to prove.

9. Ross, Exercises 9.15 and 9.16(a). The first of these results is invoked frequently in calculus courses, especially in conjunction with Taylor series, but surprisingly few students can prove it. If you are working the problems in order, both should be easy.