MATHEMATICS 23a/E-23a, Fall 2018
Linear Algebra and Real Analysis I
Week 2 (Dot and Cross Products, Euclidean Geometry of $\mathbb{R}^n$)

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Reading

- Hubbard, section 1.4

Recorded Lectures

- Lecture 4 (Week 2, Class 1) (watch on September 18 or 19)
- Lecture 5 (Week 2, Class 2) (watch on September 20 or 21)

Proofs to present in section or to a classmate who has done them.

- 2.1 Given vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ in Euclidean $\mathbb{R}^n$, prove that $|\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}| \leq |\vec{\mathbf{v}}||\vec{\mathbf{w}}|$ (Cauchy-Schwarz) and that $|\vec{\mathbf{v}} + \vec{\mathbf{w}}| \leq |\vec{\mathbf{v}}| + |\vec{\mathbf{w}}|$ (triangle inequality). Use the distributive law for the scalar product and the fact that no vector has negative length.

  (The standard version of this proof is in the textbook. An alternative is in sections 1.3 and 1.4 of the Executive Summary.)

- 2.2 For a 3×3 matrix A, define $\det(A)$ in terms of the cross and dot products of the columns of the matrix. Then, using the definition of matrix multiplication and the linearity of the dot and cross products, prove that $\det(AB) = \det(A)\det(B)$.

R Scripts Scripts labeled A, B, ... are closely tied to the Executive Summary. Scripts labeled X, Y, ... are interesting examples. There is a narrated version on the Web site. Scripts labeled L are library scripts that you may wish to include in your own scripts.

- Script 1.2A-LengthDotAngle.R
  Topic 1 - Length, Dot Product, Angles
  Topic 2 - Components of a vector
  Topic 3 - Angles in Pythagorean triangles
  Topic 4 - Vector calculation using components

- Script 1.2B-RotateReflect.R
  Topic 1 - Rotation matrices
  Topic 2 - Reflection matrices

- Script 1.2C-ComplexConformal.R
  Topic 1 - Complex numbers in R
  Topic 2 - Representing complex numbers by 2x2 matrices

- Script 1.2D-CrossProduct.R
  Topic 1 - Algebraic properties of the cross product
  Topic 2 - Geometric properties of the cross product
  Topic 3 - Using cross products to invert a 3x3 matrix

- Script 1.2E-DeterminantProduct.R
  Topic 1 - Product of 2x2 matrices
  Topic 2 - Product of 3x3 matrices

- Script 1.2L-VectorLibrary.R
  Topic 1 - Some useful angles and basis vectors
  Topic 2 - Functions for working with angles in degrees

- Script 1.2X-Triangle.R
  Topic 1 - Generating and displaying a randomly generated triangle
  Topic 2 - Checking some formulas of trigonometry

- Script 1.2Y-Angles3D.R
  Topic 1 - Angles between vectors in $\mathbb{R}^3$
  Topic 2 - Angles and distances in a cube
  Topic 3 - Calculating the airline mileage between cities

1 Executive Summary

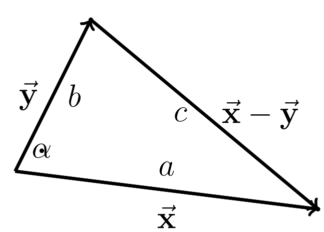
1.1 The dot product

The dot product of two vectors in $\mathbb{R}^n$ is $\vec{\mathbf{x}} \cdot \vec{\mathbf{y}} = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n = \sum_{i=1}^{n} x_i y_i$

- It requires two vectors and returns a scalar.
- It is commutative and it is distributive with respect to addition.
- In $\mathbb{R}^2$ or $\mathbb{R}^3$, the dot product of a vector with itself (a concept of algebra) is equal to the square of its length (a concept of geometry):
$$\vec{\mathbf{x}} \cdot \vec{\mathbf{x}} = |\vec{\mathbf{x}}|^2$$
- Taking the dot product with any standard basis vector $\vec{\mathbf{e_i}}$ extracts the corresponding component:
$$\vec{\mathbf{x}} \cdot \vec{\mathbf{e_i}} = x_i$$
- Taking the dot product with any unit vector $\vec{\mathbf{a}}$ (not necessarily a basis vector) extracts the component of $\vec{\mathbf{x}}$ along $\vec{\mathbf{a}}$:
$$\vec{\mathbf{x}} \cdot \vec{\mathbf{a}} = x_a$$
This means that the difference $\vec{\mathbf{x}} - x_a\vec{\mathbf{a}}$ is orthogonal to $\vec{\mathbf{a}}$.

1.2 Dot products and angles

We have the law of cosines, usually written $c^2 = a^2 + b^2 - 2ab\cos\alpha$.



Consider the triangle whose sides lie along the vectors $\vec{\mathbf{x}}$(length a), $\vec{\mathbf{y}}$ (length b), and $\vec{\mathbf{x}} - \vec{\mathbf{y}}$ (length c). Let α denote the angle between the vectors $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$.

By the distributive law,

$$(\vec{\mathbf{x}} - \vec{\mathbf{y}}) \cdot (\vec{\mathbf{x}} - \vec{\mathbf{y}}) = \vec{\mathbf{x}} \cdot \vec{\mathbf{x}} + \vec{\mathbf{y}} \cdot \vec{\mathbf{y}} - 2\vec{\mathbf{x}} \cdot \vec{\mathbf{y}} \implies c^2 = a^2 + b^2 - 2\vec{\mathbf{x}} \cdot \vec{\mathbf{y}}$$

Comparing with the law of cosines, we find that angles and dot products are related by:

$$\vec{\mathbf{x}} \cdot \vec{\mathbf{y}} = ab\cos\alpha = |\vec{\mathbf{x}}||\vec{\mathbf{y}}|\cos\alpha$$

1.3 Cauchy-Schwarz inequality

The dot product provides a way to extend the definition of length and angle for vectors to $\mathbb{R}^n$, but now we can no longer invoke Euclidean plane geometry to guarantee that $|\cos\alpha| \leq 1$.

We need to show that for any vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ in $\mathbb{R}^n$

$$|\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}| \leq |\vec{\mathbf{v}}||\vec{\mathbf{w}}|$$

This is generally known as the "Cauchy-Schwarz inequality."

For a short proof of the Cauchy-Schwarz inequality, make $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ into unit vectors and form their sum and difference.

$$(\frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|} \pm \frac{\vec{\mathbf{w}}}{|\vec{\mathbf{w}}|}) \cdot (\frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|} \pm \frac{\vec{\mathbf{w}}}{|\vec{\mathbf{w}}|}) \geq 0$$

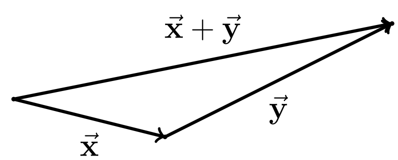
$$1 + 1 \pm 2\frac{\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}}{|\vec{\mathbf{v}}||\vec{\mathbf{w}}|} \geq 0, \text{ and by algebra } |\frac{\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}}{|\vec{\mathbf{v}}||\vec{\mathbf{w}}|}| \leq 1$$

We now have a useful definition of angle for vectors in $\mathbb{R}^n$ in general:

$$\alpha = \arccos\frac{\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}}{|\vec{\mathbf{v}}||\vec{\mathbf{w}}|}$$

1.4 The triangle inequality

If $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$, placed head-to-tail, determine two sides of a triangle, the third side coincides with the vector $\vec{\mathbf{x}} + \vec{\mathbf{y}}$.



We need to show that its length cannot exceed the sum of the lengths of the other two sides:

$$|\vec{\mathbf{x}} + \vec{\mathbf{y}}| \leq |\vec{\mathbf{x}}| + |\vec{\mathbf{y}}|$$

The proof uses the distributive law for the dot product.

$$|\vec{\mathbf{x}} + \vec{\mathbf{y}}|^2 = (\vec{\mathbf{x}} + \vec{\mathbf{y}}) \cdot (\vec{\mathbf{x}} + \vec{\mathbf{y}}) = (\vec{\mathbf{x}} + \vec{\mathbf{y}}) \cdot \vec{\mathbf{x}} + (\vec{\mathbf{x}} + \vec{\mathbf{y}}) \cdot \vec{\mathbf{y}}$$

Applying Cauchy-Schwarz to each term on the right-hand side, we have:

$$|\vec{\mathbf{x}} + \vec{\mathbf{y}}|^2 \leq |\vec{\mathbf{x}} + \vec{\mathbf{y}}||\vec{\mathbf{x}}| + |\vec{\mathbf{x}} + \vec{\mathbf{y}}||\vec{\mathbf{y}}|$$

In the special case where $|\vec{\mathbf{x}} + \vec{\mathbf{y}}| = 0$ the inequality is clearly true. Otherwise we can divide by the common factor of $|\vec{\mathbf{x}} + \vec{\mathbf{y}}|$ to complete the proof.

1.5 Isometries of $\mathbb{R}^2$

A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is completely specified by its effect on the basis vectors $\vec{\mathbf{e}}_1$ and $\vec{\mathbf{e}}_2$. These vectors are the two columns of the matrix that represents T. If you know what a transformation is supposed to do to each basis vector, you can simply use this information to fill out the necessary columns of its matrix representation.

Of special interest are **isometries**: transformations that preserve the distance between any pair of points, and hence the length of any vector.

Since dot products can be expressed in terms of lengths, it follows that any isometry also preserves dot products.

So the transformation T is an isometry if and only if for any pair of vectors:

$$T\vec{\mathbf{a}} \cdot T\vec{\mathbf{b}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$$

For the matrix associated with an isometry, both columns must be unit vectors and their dot product is zero.

Two isometries:

- A rotation, $R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, with $\det R = +1$.
- A reflection, $F(\theta) = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$, with $\det F = -1$.

Matrix R represents a counterclockwise rotation through angle θ about the origin. Matrix F represents reflection in a line through the origin that makes an angle θ with the first standard basis vector.

There are many other isometries of Euclidean geometry; translations, or rotations about points other than the origin. However, these do not hold the origin fixed, and so they are not linear transformations and cannot represented by 2×2 matrices.

Since the composition of isometries is an isometry, the product of any number of matrices of this type is another rotation or reflection. Remember that composition is a series of transformations acting on a vector in a specific order that must be preserved during multiplication.

1.6 Matrices and algebra: complex numbers

The same field axioms we reviewed on the first day apply here to the complex numbers, notated $\mathbb{C}$.

The real and imaginary parts of a complex number can be used as the two components of a vector in $\mathbb{R}^2$. The rule for addition of complex numbers is the same as the rule for addition of vectors in $\mathbb{R}^2$ (in that they are to be kept separate from each other), and the modulus of a complex number is the same as the length of the vector that represents it. So the triangle inequality applies for complex numbers: $|z_1 + z_2| \leq |z_1| + |z_2|$.

This property extends to vector spaces over complex numbers.

1.7 What about complex multiplication?

The geometrical interpretation of multiplication by a complex number $z = a + ib = re^{i\theta}$ is multiplication of the modulus by r combined with addition of θ to the angle with the x-axis.

This is precisely the geometrical effect of the linear transformation represented by the matrix

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix}$$

Such a matrix is the product of the constant matrix $\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$ and the rotation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

It is called a **conformal matrix** and it preserves angles even though it does not preserve lengths.

1.8 Complex numbers as a field of matrices

In general, matrices do not form a field because multiplication is not commutative. There are two notable exceptions: $n \times n$ matrices that are multiples of the identity matrix and 2×2 conformal matrices. Since multiples of the identity matrix and rotations all commute, the product of two conformal matrices $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and $\begin{bmatrix} c & -d \\ d & c \end{bmatrix}$ is the same in either order.

1.9 The cross product

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

Properties

1. $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = -\vec{\mathbf{b}} \times \vec{\mathbf{a}}$.
2. $\vec{\mathbf{a}} \times \vec{\mathbf{a}} = 0$.
3. For fixed $\vec{\mathbf{a}}$, $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is a linear function of $\vec{\mathbf{b}}$, and vice versa.
4. For the standard basis vectors, $\vec{\mathbf{e_i}} \times \vec{\mathbf{e_j}} = \vec{\mathbf{e_k}}$ if i, j and k are in cyclic increasing order (123, 231, or 312). Otherwise $\vec{\mathbf{e_i}} \times \vec{\mathbf{e_j}} = -\vec{\mathbf{e_k}}$.
5. $\vec{\mathbf{a}} \times \vec{\mathbf{b}} \cdot \vec{\mathbf{c}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{c}}$. This quantity is also the determinant of the matrix whose columns are $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, and $\vec{\mathbf{c}}$.
6. $(\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \times \vec{\mathbf{c}} = (\vec{\mathbf{a}} \cdot \vec{\mathbf{c}})\vec{\mathbf{b}} - (\vec{\mathbf{b}} \cdot \vec{\mathbf{c}})\vec{\mathbf{a}}$
7. $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is orthogonal to the plane spanned by $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.
8. $|\vec{\mathbf{a}} \times \vec{\mathbf{b}}|^2 = |\vec{\mathbf{a}}|^2|\vec{\mathbf{b}}|^2 - (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})^2$
9. The length of $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is $|\vec{\mathbf{a}}||\vec{\mathbf{b}}|\sin\alpha$.
10. The length of $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is equal to the area of the parallelogram spanned by $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

1.10 Cross product and determinants

If a 3×3 matrix A has columns $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, and $\vec{\mathbf{a}}_3$, then its determinant $\det(A) = \vec{\mathbf{a}}_1 \times \vec{\mathbf{a}}_2 \cdot \vec{\mathbf{a}}_3$.

1. $\det(A)$ changes sign if you interchange any two columns. (easiest to prove for columns 1 and 2, but true for any pair)
2. $\det(A)$ is a linear function of each column (easiest to prove for column 3, but true for any column)
3. For the identity matrix I, $\det(I) = 1$.

The magnitude of $\vec{\mathbf{a}} \times \vec{\mathbf{b}} \cdot \vec{\mathbf{c}}$ is equal to the volume of the parallelepiped spanned by $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$.
If $C = AB$, then $\det(C) = \det(A)\det(B)$

2 Lecture Outline

1. The dot product:

   This is defined for vectors in $\mathbb{R}^n$ as

   $$\vec{\mathbf{x}} \cdot \vec{\mathbf{y}} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

   It has the following properties. The proof of the first four (omitted) is brute-force computation.

   - Commutative law:
   $$\vec{\mathbf{x}} \cdot \vec{\mathbf{y}} = \vec{\mathbf{y}} \cdot \vec{\mathbf{x}}$$
   - Distributive law:
   $$\vec{\mathbf{x}} \cdot (\vec{\mathbf{y_1}} + \vec{\mathbf{y_2}}) = \vec{\mathbf{x}} \cdot \vec{\mathbf{y_1}} + \vec{\mathbf{x}} \cdot \vec{\mathbf{y_2}}$$
   - For Euclidean geometry, in $\mathbb{R}^2$ or $\mathbb{R}^3$, the dot product of a vector with itself (defined by algebra) is equal to the square of its length (a physically meaningful quantity).
   - Taking the dot product with any standard basis vector $\vec{\mathbf{e_i}}$ extracts the corresponding component:
   $$\vec{\mathbf{x}} \cdot \vec{\mathbf{e_i}} = x_i$$
   - Taking the dot product with any unit vector $\vec{\mathbf{a}}$ (not necessarily a standard basis vector) extracts the component of $\vec{\mathbf{x}}$ along $\vec{\mathbf{a}}$:
   $$\vec{\mathbf{x}} \cdot \vec{\mathbf{a}} = x_a$$
   This means that the difference $\vec{\mathbf{x}} - x_a\vec{\mathbf{a}}$ is orthogonal to $\vec{\mathbf{a}}$.

     Proof: Orthogonality of two vectors means that their dot product is zero. So to show orthogonality, evaluate
   $$(\vec{\mathbf{x}} - (\vec{\mathbf{x}} \cdot \vec{\mathbf{a}})\vec{\mathbf{a}}) \cdot \vec{\mathbf{a}}.$$

2. Dot products and angles

   From elementary trigonometry we have the law of cosines, usually written $c^2 = a^2 + b^2 - 2ab\cos\alpha$.

   In this formula, c denotes the length of the side opposite angle α. Just in case you forgot the proof, let's review it.

   Angles and dot products are related by the formula

   $\vec{\mathbf{x}} \cdot \vec{\mathbf{y}} = |\vec{\mathbf{x}}||\vec{\mathbf{y}}| \cos\alpha$

   Proof (Hubbard, page 69):

   Consider the triangle whose sides lie along the vectors $\vec{\mathbf{x}}, \vec{\mathbf{y}}$, and $\vec{\mathbf{x}} - \vec{\mathbf{y}}$, and let α denote the angle between the vectors $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$.

$$c^2 = (\vec{\mathbf{x}} - \vec{\mathbf{y}}) \cdot (\vec{\mathbf{x}} - \vec{\mathbf{y}}).$$

   Expand the dot product using the distributive law, and you can identify one of the terms as $2ab\cos\alpha$.

3. Cauchy-Schwarz inequality

The dot product provides a way to extend the definition of length and angle to vectors in $\mathbb{R}^n$, but now we can no longer invoke Euclidean plane geometry to guarantee that $|\cos\alpha| \le 1$.

We need to show that for any vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ in $\mathbb{R}^n$,

$$|\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}| \le |\vec{\mathbf{v}}||\vec{\mathbf{w}}|$$

This is generally known as the "Cauchy-Schwarz inequality." Hubbard points out that it was first published by Bunyakovsky. This fact illustrates Stigler's Law of Eponymy:
"No law, theorem, or discovery is named after its originator."
The law applies to itself, since long before Stigler formulated it, A. N. Whitehead noted that,

"Everything of importance has been said before, by someone who did not discover it."

The best-known proof of the Cauchy-Schwarz inequality incorporates two useful strategies.

- No vector has negative length.
- Discriminant of quadratic equation.

Define a quadratic function of the real variable t by

$$f(t) = |t\vec{\mathbf{v}} - \vec{\mathbf{w}}|^2 = (t\vec{\mathbf{v}} - \vec{\mathbf{w}}) \cdot (t\vec{\mathbf{v}} - \vec{\mathbf{w}})$$

Since $f(t)$ is the square of a length of a vector, it cannot be negative, so the quadratic equation $f(t) = 0$ does not have two real roots.

But by the quadratic formula, if the equation $at^2 + bt + c = 0$ does not have two real roots, its discriminant $b^2 - 4ac$ is not positive.

Complete the proof by writing out $b^2 - 4ac \le 0$ for quadratic function $f(t)$.

So we have a useful definition of angle for vectors in $\mathbb{R}^n$ in general:

$$\alpha = \arccos \frac{\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}}{|\vec{\mathbf{v}}||\vec{\mathbf{w}}|}$$

The function $\arccos(x)$ can be computed on your electronic calculator by summing an infinite series. It is guaranteed to return a value between 0 and π.

Example: In $\mathbb{R}^4$, what is the angle between vectors $\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}$?

4. The triangle inequality (second part of proof 2.1)

   If $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$, placed head-to-tail, determine two sides of a triangle, the third side coincides with the vector $\vec{\mathbf{x}}+\vec{\mathbf{y}}$. We need to show that its length cannot exceed the sum of the lengths of the other two sides:

   $|\vec{\mathbf{x}} + \vec{\mathbf{y}}| \le |\vec{\mathbf{x}}| + |\vec{\mathbf{y}}|$

   The proof uses the distributive law for the dot product and the Cauchy-Schwarz inequality.

   Express $|\vec{\mathbf{x}} + \vec{\mathbf{y}}|^2$ as a dot product:

   Apply the distributive law:

   Use Cauchy-Schwarz to get an inequality for lengths:

   Take the square root of both sides:

5. Proof 2.1 – start to finish, done in a slightly different way

   Given vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ in Euclidean $\mathbb{R}^n$, prove that $|\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}| \leq |\vec{\mathbf{v}}||\vec{\mathbf{w}}|$ (Cauchy-Schwarz) and that $|\vec{\mathbf{v}} + \vec{\mathbf{w}}| \leq |\vec{\mathbf{v}}| + |\vec{\mathbf{w}}|$ (triangle inequality). Use the distributive law for the scalar product and the fact that no vector has negative length.

6. Isometries of $\mathbb{R}^2$.

   A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is completely specified by its effect on the basis vectors $\vec{\mathbf{e}}_1$ and $\vec{\mathbf{e}}_2$. These vectors are the two columns of the matrix that represents T.

   Of special interest are "isometries:" transformations that preserve the distance between any pair of points, and hence the length of any vector.

   Since

   $$4\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = |\vec{\mathbf{a}} + \vec{\mathbf{b}}|^2 - |\vec{\mathbf{a}} - \vec{\mathbf{b}}|^2,$$

   dot products can be expressed in terms of lengths, and any isometry also preserves dot products.

   Prove this useful identity.

   So T is an isometry if and only if

   $T\vec{\mathbf{a}} \cdot T\vec{\mathbf{b}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$ for any pair of vectors.

   This means that the first column of T must be a unit vector, which can be written without any loss of generality as

   $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$.

   The second column must also be a unit vector, and its dot product with the first column must be zero. So there are only two possibilities:

   - A rotation,

     $$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

     which has $\det R = 1$.
   - A reflection,

     $$F(\theta) = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix},$$

     which has $\det F = -1$.

     This represents reflection in a line through the origin that makes an angle θ with the first basis vector.

   Since the composition of isometries is an isometry, the product of any number of matrices of this type is another rotation or reflection.

7. Using matrices to represent rotations and reflections

   (a) Use matrix multiplication to show that if a counterclockwise rotation though angle β is followed by a counterclockwise rotation though angle α, the net effect is a counterclockwise rotation though angle $\alpha + \beta$. (The proof requires some trig identities that you can rederive, if you ever forget them, by doing this calculation.)

   (b) Confirm, both by geometry and by matrix multiplication, that if you reflect a point P first in the line $y = 0$, then in the line $y = x$, the net effect is to rotate the point counterclockwise through $90°$.

8. Complex numbers as vectors and as matrices

 The field axioms that you learned last week apply also to the complex numbers, notated $\mathbb{C}$.

 The real and imaginary parts of a complex number can be used as the two components of a vector in $\mathbb{R}^2$. The rule for addition of complex numbers is the same as the rule for addition of vectors in $\mathbb{R}^2$, and the modulus of a complex number is the same as the length of the vector that represents it. So the triangle inequality applies for complex numbers: $|z_1+z_2| \leq |z_1|+|z_2|$.

 This property extends to vector spaces over complex numbers.

 The geometrical interpretation of multiplication by a complex number $z = a + ib = re^{i\theta}$ is multiplication of the modulus by r combined with addition of θ to the angle with the x-axis.
 This is precisely the geometrical effect of the linear transformation represented by the matrix

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix}$$

 Such a matrix is the product of the constant matrix $\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$ and the rotation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.
 It is called a **conformal matrix** and it preserves angles even though it does not preserve lengths.

 Example: Compute the product of the complex numbers $2+i$ and $3+i$ by useing matrix multiplication.

9. Complex numbers as a field of matrices

 In general, matrices do not form a field because multiplication is not commutative. There are two notable exceptions: $n \times n$ matrices that are multiples of the identity matrix and 2×2 conformal matrices. Since multiples of the identity matrix and rotations all commute, the product of two conformal matrices $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and $\begin{bmatrix} c & -d \\ d & c \end{bmatrix}$ is the same in either order.

10. Cross products:

At this point it is inappropriate to try to define the determinant of an $n \times n$ matrix. For $n = 3$, however, anything that can be done with determinants can also be done with cross products, which are peculiar to $\mathbb{R}^3$. So we will start with cross products:

Definition:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

Since this is a computational definition, the way to prove the following properties is by brute-force computation.

(a) $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = -\vec{\mathbf{b}} \times \vec{\mathbf{a}}$.

(b) $\vec{\mathbf{a}} \times \vec{\mathbf{a}} = \vec{\mathbf{0}}$.

(c) For fixed $\vec{\mathbf{b}}$, $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is a linear function of $\vec{\mathbf{b}}$, and vice versa.

(d) For the standard basis vectors, $\vec{\mathbf{e_i}} \times \vec{\mathbf{e_j}} = \vec{\mathbf{e_k}}$ if i, j and k are in cyclic increasing order (123, 231, or 312). Otherwise $\vec{\mathbf{e_i}} \times \vec{\mathbf{e_j}} = -\vec{\mathbf{e_k}}$.
You may find it easiest to calculate cross products in general as
$(a_1\vec{\mathbf{e_1}} + a_2\vec{\mathbf{e_2}} + a_3\vec{\mathbf{e_3}}) \times (b_1\vec{\mathbf{e_1}} + b_2\vec{\mathbf{e_2}} + b_3\vec{\mathbf{e_3}})$,
using the formula for the cross products of basis vectors. Try this approach for
$\vec{\mathbf{a}} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \vec{\mathbf{b}} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$.

(e) $\vec{\mathbf{a}} \times \vec{\mathbf{b}} \cdot \vec{\mathbf{c}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{c}}$. No parentheses are necessary, because the operations only make sense if the cross product is done first. This quantity is also the determinant of the matrix whose columns are $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, and $\vec{\mathbf{c}}$.

(f) $(\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \times \vec{\mathbf{c}} = (\vec{\mathbf{a}} \cdot \vec{\mathbf{c}})\vec{\mathbf{b}} - (\vec{\mathbf{b}} \cdot \vec{\mathbf{c}})\vec{\mathbf{a}}$
Physicists, memorize this formula ! The vector in the middle gets the plus sign.

11. Geometric properties of the cross product:

We can now prove these without messy calculations involving components. Justify each step, using properties of the dot product and properties (a) through (f) from the preceding page.

- $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is orthogonal to the plane spanned by $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.
  Proof: Let $\vec{\mathbf{v}} = s\vec{\mathbf{a}} + t\vec{\mathbf{b}}$ be a vector in this plane. Then

$$\vec{\mathbf{v}} \cdot \vec{\mathbf{a}} \times \vec{\mathbf{b}} = s\vec{\mathbf{a}} \cdot \vec{\mathbf{a}} \times \vec{\mathbf{b}} + t\vec{\mathbf{b}} \cdot \vec{\mathbf{a}} \times \vec{\mathbf{b}}$$

$$\vec{\mathbf{v}} \cdot \vec{\mathbf{a}} \times \vec{\mathbf{b}} = s\vec{\mathbf{a}} \cdot \vec{\mathbf{a}} \times \vec{\mathbf{b}} - t\vec{\mathbf{b}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{a}}$$

$$\vec{\mathbf{v}} \cdot \vec{\mathbf{a}} \times \vec{\mathbf{b}} = s\vec{\mathbf{a}} \times \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} - t\vec{\mathbf{b}} \times \vec{\mathbf{b}} \cdot \vec{\mathbf{a}}$$

$$\vec{\mathbf{v}} \cdot \vec{\mathbf{a}} \times \vec{\mathbf{b}} = 0 - 0 = 0.$$

- $|\vec{\mathbf{a}} \times \vec{\mathbf{b}}|^2 = |\vec{\mathbf{a}}|^2|\vec{\mathbf{b}}|^2 - (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})^2$
  Proof:

$$|\vec{\mathbf{a}} \times \vec{\mathbf{b}}|^2 = (\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \cdot (\vec{\mathbf{a}} \times \vec{\mathbf{b}})$$

$$|\vec{\mathbf{a}} \times \vec{\mathbf{b}}|^2 = ((\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \times \vec{\mathbf{a}}) \cdot \vec{\mathbf{b}}$$

$$|\vec{\mathbf{a}} \times \vec{\mathbf{b}}|^2 = ((\vec{\mathbf{a}} \cdot \vec{\mathbf{a}})\vec{\mathbf{b}} - (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})\vec{\mathbf{a}}) \cdot \vec{\mathbf{b}}$$

$$|\vec{\mathbf{a}} \times \vec{\mathbf{b}}|^2 = (\vec{\mathbf{a}} \cdot \vec{\mathbf{a}})(\vec{\mathbf{b}} \cdot \vec{\mathbf{b}}) - (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})$$

$$|\vec{\mathbf{a}} \times \vec{\mathbf{b}}|^2 = |\vec{\mathbf{a}}|^2|\vec{\mathbf{b}}|^2 - (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})^2$$

- The length of $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is $|\vec{\mathbf{a}}||\vec{\mathbf{b}}| \sin \alpha$.
  Proof:

$$|\vec{\mathbf{a}} \times \vec{\mathbf{b}}|^2 = |\vec{\mathbf{a}}|^2|\vec{\mathbf{b}}|^2(1 - \cos^2 \alpha) = |\vec{\mathbf{a}}|^2|\vec{\mathbf{b}}|^2(\sin^2 \alpha)$$

- The length of $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is equal to the area of the parallelogram spanned by $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.
  Proof: $|\vec{\mathbf{a}}|$ is the base of the parallelogram and $|\vec{\mathbf{b}}| \sin \alpha$ is its height. Draw a diagram to illustrate this property.

12. Cross products and determinants.

You should be familiar with 2×2 and 3×3 determinants from high-school algebra. The general definition of the determinant, to be introduced in the spring term, underlies the general technique for calculating volumes in $\mathbb{R}^n$ and will be used to define differential forms.

If a 2×2 matrix A has columns $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, then its determinant $\det(A) = a_1 b_2 - a_2 b_1$.

Equivalently,

$$\begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \end{bmatrix}$$

You can think of the determinant as a function of the entire matrix A or as a function of its two columns.

Matrix A maps the unit square, spanned by the two standard basis vectors, into a parallelogram whose area is $|\det(A)|$.

Let's prove this for the case where all the entries of A are positive and $\det(A) > 0$. The area of the parallelogram formed by the columns of A is twice the area of the triangle that has these columns as two of its sides. The area of this triangle can be calculated in terms of elementary formulas for areas of rectangles and right triangles.

13. Determinants in $\mathbb{R}^3$

Here is our definition:

If a 3×3 matrix A has columns $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, and $\vec{\mathbf{a}}_3$, then its determinant $\det(A) = \vec{\mathbf{a}}_1 \times \vec{\mathbf{a}}_2 \cdot \vec{\mathbf{a}}_3$.

Apply this definition to the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \end{bmatrix}.$$

Check the following properties of the definition.

(a) $\det(A)$ changes sign if you interchange any two columns. (easiest to prove for columns 1 and 2, but true for any pair)

(b) $\det(A)$ is a linear function of each column (easiest to prove for column 3, but true for any column)

(c) For the identity matrix I, $\det(I) = 1$.

14. Determinants, triple products, and geometry

The magnitude of $\vec{\mathbf{a}} \times \vec{\mathbf{b}} \cdot \vec{\mathbf{c}}$ is equal to the volume of the parallelepiped spanned by $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$.

Proof: $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is the area of the base of the parallelepiped, and $|\vec{\mathbf{c}}| \cos \alpha$, where α is the angle between $\vec{\mathbf{c}}$ and the direction orthogonal to the base, is its height.

Matrix A maps the unit cube, spanned by the three basis vectors, into a parallelepiped whose volume is $|\det(A)|$. You can think of $|\det(A)|$ as a "volume stretching factor." This interpretation will underly much of the theory for change of variables in multiple integrals, a major topic in the spring term.

If three vectors in $\mathbb{R}^3$ all lie in the same plane, the cross product of any two of them, which is orthogonal to that plane, is orthogonal to the third vector, so $\vec{\mathbf{v}}_1 \times \vec{\mathbf{v}}_2 \cdot \vec{\mathbf{v}}_3 = 0$.

Apply this test to $\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{\mathbf{v}}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \vec{\mathbf{v}}_3 = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$.

If four points in $\mathbb{R}^3$ all lie in the same plane, the vectors that join any one of the points to each of the other three points all lie in that plane. Apply this test to $\mathbf{p} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{q} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \mathbf{s} = \begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix}$.

15. Calculating angles and areas

Let $\vec{\mathbf{v}}_1 = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}, \vec{\mathbf{v}}_2 = \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$.

Both these vectors happen to be perpendicular to the vector $\vec{\mathbf{v}}_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

(a) Determine the angle between $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$.

(b) Determine the volume of the parallelepiped spanned by $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2,$ and $\vec{\mathbf{v}}_3$, and thereby determine the area of the parallelogram spanned by $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$.

16. Determinants and matrix multiplication

If $C = AB$, then $\det(C) = \det(A)\det(B)$

This useful result is easily proved by brute force for 2×2 matrices, and a brute-force proof in Mathematica would be valid for 3×3 matrices. Here is a proof that relies on properties of the cross product.

Recall that each column of a matrix is the image of a standard basis vector.

Consider the first column of the matrix $C = AB$, and exploit the fact that A is linear.

$$\vec{\mathbf{c}}_1 = A\vec{\mathbf{b}}_1 = A(\sum_{i=1}^{3} b_{i,1}\vec{\mathbf{e}}_i) = \sum_{i=1}^{3} b_{i,1}A(\vec{\mathbf{e}}_i)) = \sum_{i=1}^{3} b_{i,1}\vec{\mathbf{a}}_i.$$

The same is true of the second and third columns.

Now consider $\det C = \vec{\mathbf{c_1}} \times \vec{\mathbf{c_2}} \cdot \vec{\mathbf{c_3}}$.

$$\det C = (\sum_{i=1}^{3} b_{i,1}\vec{\mathbf{a_i}}) \times (\sum_{j=1}^{3} b_{j,2}\vec{\mathbf{a_j}}) \cdot (\sum_{k=1}^{3} b_{k,3}\vec{\mathbf{a_k}})$$

Now use the distributive law for dot and cross products.

$$\det C = \sum_{i=1}^{3} b_{i,1} \sum_{j=1}^{3} b_{j,2} \sum_{k=1}^{3} b_{k,3}(\vec{\mathbf{a_i}} \times \vec{\mathbf{a_j}} \cdot \vec{\mathbf{a_k}})$$

There are 27 terms in this sum, but all but six of them involve two subscripts that are equal, and these are zero because a triple product with two equal vectors is zero.

The six that are not zero all involve $\vec{\mathbf{a}}_1 \times \vec{\mathbf{a}}_2 \cdot \vec{\mathbf{a}}_3$, three with a plus sign and three with a minus sign. So

$\det C = f(B)(\vec{\mathbf{a}}_1 \times \vec{\mathbf{a}}_2 \cdot \vec{\mathbf{a}}_3) = f(B)\det(A)$, where $f(B)$ is some messy function of products of all the entries of B.

This formula is valid for any A. In particular, it is valid when A is the identity matrix, $C = B$, and $\det(A) = 1$.

So $\det B = f(B)\det(I) = f(B)$

and the messy function is the determinant!

17. Proof 2.2 – start to finish

    For a 3×3 matrix A, define $\det(A)$ in terms of the cross and dot products of the columns of the matrix. Then, using the definition of matrix multiplication and the linearity of the dot and cross products, prove that $\det(AB) = \det(A)\det(B)$.

18. Isometries of $\mathbb{R}^2$.

A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is completely specified by its effect on the basis vectors $\vec{\mathbf{e}}_1$ and $\vec{\mathbf{e}}_2$. These vectors are the two columns of the matrix that represents T.

Of special interest are "isometries:" transformations that preserve the distance between any pair of points, and hence the length of any vector.

Since

$$4\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = |\vec{\mathbf{a}} + \vec{\mathbf{b}}|^2 - |\vec{\mathbf{a}} - \vec{\mathbf{b}}|^2,$$

dot products can be expressed in terms of lengths, and any isometry also preserves dot products.

Prove this useful identity.

So T is an isometry if and only if

$T\vec{\mathbf{a}} \cdot T\vec{\mathbf{b}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$ for any pair of vectors.

This means that the first column of T must be a unit vector, which can be written without any loss of generality as

$\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$.

The second column must also be a unit vector, and its dot product with the first column must be zero. So there are only two possibilities:

- A rotation,

  $$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

  which has $\det R = 1$.
- A reflection,

  $$F(\theta) = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix},$$

  which has $\det F = -1$.

  This represents reflection in a line through the origin that makes an angle θ with the first basis vector.

Since the composition of isometries is an isometry, the product of any number of matrices of this type is another rotation or reflection.

19. Transposes and dot products

    Start by proving in general that $(AB)^T = B^T A^T$. This is a statement about matrices, and you have to prove it by brute force.

    The dot product of vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ can also be written in terms of matrix multiplication as

$$\vec{\mathbf{v}} \cdot \vec{\mathbf{w}} = \vec{\mathbf{v}}^T \vec{\mathbf{w}}$$

    where we think of $\vec{\mathbf{v}}^T$ as a $1 \times m$ matrix and think of $\vec{\mathbf{w}}$ as an $m \times 1$ matrix. The product is a 1×1 matrix, so it equals its own transpose.

    Prove that $\vec{\mathbf{v}} \cdot A\vec{\mathbf{w}} = A^T\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}$. This theorem lets you move a matrix from one factor in a dot product to the other, as long as you replace it by its transpose.

20. Orthogonal matrices

If a matrix R represents an isometry, then each column is a unit vector and the columns are orthogonal. Since the columns of R are the rows of R^T we can express this property as

$R^T R = I$

Perhaps a nicer way to express this condition for a matrix to represent an isometry is $R^T = R^{-1}$. Check that this is true for the 2×2 matrices that represent rotations and reflections.

For a rotation matrix

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

For a reflection matrix

$$F(\theta) = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}.$$

21. Isometries and cross products

Many vectors of physical importance (torque, angular momentum, magnetic field) are defined as cross products, so it is useful to know what happens to a cross product when an isometry is applied to each vector in the product.

Consider the matrix whose columns are $R\vec{\mathbf{u}}, R\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$.

Multiply this matrix by R^T to get a matrix whose columns are

$R^T R\vec{\mathbf{u}}, R^T R\vec{\mathbf{v}}$, and $R^T\vec{\mathbf{w}}$. In the process you multiply the determinant by $\det(R^T) = \det(R)$.

Now, since $R^T R = I$ for an isometry, $\vec{\mathbf{u}} \times \vec{\mathbf{v}} \cdot R^T\vec{\mathbf{w}} = \det(R) R\vec{\mathbf{u}} \times R\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}$

Equivalently, $R(\vec{\mathbf{u}} \times \vec{\mathbf{v}}) \cdot \vec{\mathbf{w}} = \det(R) R\vec{\mathbf{u}} \times R\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}$.

Since this is true for any $\vec{\mathbf{w}}$, in particular for any basis vector, it follows that

$R(\vec{\mathbf{u}} \times \vec{\mathbf{v}}) = \det(R) R\vec{\mathbf{u}} \times R\vec{\mathbf{v}}$

If R is a rotation, then $\det(R) = 1$ and $R(\vec{\mathbf{u}} \times \vec{\mathbf{v}}) = R\vec{\mathbf{u}} \times R\vec{\mathbf{v}}$

If R is a reflection, then $\det(R) = -1$ and $R(\vec{\mathbf{u}} \times \vec{\mathbf{v}}) = -R\vec{\mathbf{u}} \times R\vec{\mathbf{v}}$

This is reasonable. Suppose you are watching a physicist in a mirror as she calculates the cross product of two vectors. You see her apparently using a left-hand rule and think that she has got the sign of the cross-product wrong.

22. Using cross products to invert a 3×3 matrix

Thinking about transposes also leads to a formula for the inverse of a 3×3 matrix in terms of cross products. Suppose that matrix A has columns $\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2$, and $\vec{\mathbf{a}}_3$. Form the vector $\vec{\mathbf{s}}_1 = \vec{\mathbf{a}}_2 \times \vec{\mathbf{a}}_3$.

This is orthogonal to $\vec{\mathbf{a}}_2$ and $\vec{\mathbf{a}}_3$, and its dot product with $\vec{\mathbf{a}}_1$ is $\det(A)$.

Similarly, the vector $\vec{\mathbf{s}}_2 = \vec{\mathbf{a}}_3 \times \vec{\mathbf{a}}_1$

is orthogonal to $\vec{\mathbf{a}}_3$ and $\vec{\mathbf{a}}_1$, and its dot product with $\vec{\mathbf{a}}_2$ is $\det(A)$.

Finally, the vector $\vec{\mathbf{s}}_3 = \vec{\mathbf{a}}_1 \times \vec{\mathbf{a}}_2$

is orthogonal to $\vec{\mathbf{a}}_1$ and $\vec{\mathbf{a}}_2$, and its dot product with $\vec{\mathbf{a}}_3$ is $\det(A)$.

So if you form these vectors into a matrix S and take its transpose,

$S^T A = \det(A) I$.

If $\det A = 0$, A has no inverse. Otherwise

$$A^{-1} = \frac{S^T}{\det(A)}.$$

You may have learned this rule in high-school algebra in terms of 2×2 determinants.

Summarize the proof that this recipe is correct.

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3 Seminar Topics

Your section instructor will either have emailed a list of topics to prepare or will have posted a signup list of appointments on the Calendar tab of Canvas. Either way, there will be one of the following topics that you should be prepared to present.

Practice your presentation so that it takes about 8 minutes. The text of the presentation will be projected onto a screen so that you need not recopy it. To save time, avoid writing long sentences on the chalkboard. You may use notes, but be discreet about it.

1. (Proof 2.1) Given vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ in Euclidean $\mathbb{R}^n$, prove that $|\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}| \leq |\vec{\mathbf{v}}||\vec{\mathbf{w}}|$ (Cauchy-Schwarz) and that $|\vec{\mathbf{v}} + \vec{\mathbf{w}}| \leq |\vec{\mathbf{v}}| + |\vec{\mathbf{w}}|$ (triangle inequality). Use the distributive law for the scalar product and the fact that no vector has negative length.

   (The standard version of this proof is in the textbook. An alternative is in sections 1.3 and 1.4 of the Executive Summary.)

2. Let $F(\alpha)$ be the 2×2 matrix that represents reflection in an upward-sloping line through the origin that makes an angle α with the positive horizontal axis.
   Let $R(\theta)$ be the 2×2 matrix that represents rotation counterclockwise about the origin through an angle θ.

   Prove by matrix multiplication that $F(\beta)F(\alpha) = R(2(\beta - \alpha))$. You will need to use the trigonometric identities

   $$\sin(x + y) = \sin x \cos y + \cos x \sin y; \ \cos(x + y) = \cos x \cos y - \sin x \sin y,$$

   which every scientist should memorize!

   Optional: As you apply $F(\alpha)$ and $F(\beta)$ successively to a vector $\vec{\mathbf{v}}$, the tip of the vector moves along a circular arc. If you draw a diagram with $\beta > \alpha > 0$, it is not hard also to prove your result geometrically.

3. Using the formulas

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} \cdot \vec{\mathbf{c}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \times \vec{\mathbf{c}}$$

and

$$(\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \times \vec{\mathbf{c}} = (\vec{\mathbf{a}} \cdot \vec{\mathbf{c}})\vec{\mathbf{b}} - (\vec{\mathbf{b}} \cdot \vec{\mathbf{c}})\vec{\mathbf{a}},$$

which every physicist should memorize,
prove that $|\vec{\mathbf{a}} \times \vec{\mathbf{b}}|$ is equal to the area of a parallelogram in $\mathbb{R}^3$ spanned by vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, with angle α between them, namely $|\vec{\mathbf{a}}||\vec{\mathbf{b}}||\sin\alpha|$.

4. (Proof 2.2) For a 3×3 matrix A, define $\det(A)$ in terms of the cross and dot products of the columns of the matrix. Then, using the definition of matrix multiplication and the linearity of the dot and cross products, prove that $\det(AB) = \det(A)\det(B)$.

5. Given the rule that the transpose of a matrix product is the product of the transposes in reverse order, $(AB)^T = B^T A^T$ (if there is lots of time left, you may want to prove this rule), prove that

$$\vec{\mathbf{v}} \cdot A\vec{\mathbf{w}} = A^T\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}.$$

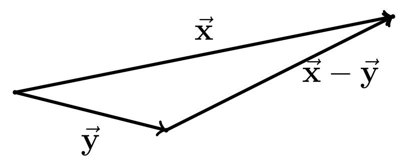
6. (Extra topic) A field isomorphism:
Let f be the function that converts the complex number $c = a + bi$ into the conformal matrix $C = f(c) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and the complex number $z = x + yi$ into the conformal matrix $Z = f(z) = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}$.

   - Prove that $f(c + z) = f(c) + f(z)$.
   - Prove that $f(cz) = f(c)f(z)$, where the multiplication on the left is standard multiplication of complex numbers and the multiplication on the right is matrix multiplication.

4 Workshop Problems

1. Proofs that use dot products

   (a) A triangle is formed by using vectors $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$, both anchored at one vertex. The vectors are labeled so that the longer one is called $\vec{\mathbf{x}}$: i.e. $|\vec{\mathbf{x}}| > |\vec{\mathbf{y}}|$. The vector $\vec{\mathbf{x}}-\vec{\mathbf{y}}$ then lies along the third side of the triangle. Prove that
   $|\vec{\mathbf{x}} - \vec{\mathbf{y}}| \geq |\vec{\mathbf{x}}| - |\vec{\mathbf{y}}|$.



   (b) A parallelogram has sides with lengths a and b. Its diagonals have lengths c and d, Prove the "parallelogram law," which states that

$$c^2 + d^2 = 2(a^2 + b^2).$$

2. Applying the dot product to parallelograms

   (a) A parallelogram is spanned by two vectors that meet at a 60 degree angle, one of which is twice as long as the other. Find the ratio of the lengths of the diagonals and the cosine of the acute angle between the diagonals. Confirm that the parallelogram law holds in this case.

   (b) Consider a parallelogram spanned by vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$. Using the dot product, prove that it is a rhombus if and only if the diagonals are perpendicular and that it is a rectangle if and only if the diagonals are equal in length.

3. Proofs and applications that involve cross products

   (a) Prove that the cross product is not associative but that it satisfies the "Jacobi identity"

$$(\vec{\mathbf{a}} \times \vec{\mathbf{b}}) \times \vec{\mathbf{c}} + (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) \times \vec{\mathbf{a}} + (\vec{\mathbf{c}} \times \vec{\mathbf{a}}) \times \vec{\mathbf{b}} = 0.$$

   (b) Consider a parallelepiped whose base is a parallelogram spanned by two unit vectors, anchored at the origin, with a 60 degree angle between them. The third side leaving the origin, also a unit vector, makes a 60 degree angle with each of the other two sides, so that each face is made of of a pair of equilateral triangles. Using dot and cross products, show that the angle α between the third side and a line that bisects the angle between the other two sides satisfies $\cos\alpha = 1/\sqrt{3}$ and that the volume of this parallelepiped is $\frac{1}{\sqrt{2}}$.

4. Problems that involve writing or editing R scripts
   Script 1.2A will be helpful for both of these. The library script 2L has functions for dealing with angles in degrees.

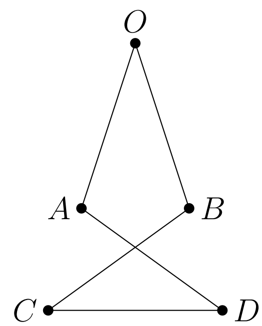
   (a) Construct a triangle where vector AB has length 5 and is directed east, while vector AC has length 10 and is directed 53 degrees north of east. On side BC, construct point D that is 1/3 of the way from B to C. Using dot products, confirm that the vector AD bisects the angle at A. Draw a diagram that illustrates this result.
   This is a special case of Euclid's Elements, Book VI, Proposition 3.

   (b) You are playing golf, and the hole is located 350 yards from the tee in a direction 18 degrees south of east. You hit a tee shot that travels 220 yards 14 degrees south of east, followed by an iron shot that travels 150 yards 23 degrees south of east. How far from the hole is your golf ball now located? Draw a diagram that illustrates this result.

5 Homework, due at 11:59 pm on Sept. 25

In working on these problems, you may collaborate with classmates and consult books and general online references. If, however, you encounter a posted solution to one of the problems, do not look at it, and email Paul, who will try to get it removed.

1. One way to construct a regular pentagon



Take five ball-point pens or other objects of equal length(call it 1) and arrange them symmetrically, as shown in the diagram above, so that O, A, C and O, B, D are collinear and $|OC| = |OD|$. Let $AO = \vec{\mathbf{v}}, |BO| = |\vec{\mathbf{v}}|$, $CD = \vec{\mathbf{w}}$, $CA = x\vec{\mathbf{v}}, |DB| = x|\vec{\mathbf{v}}|$.

(a) Express vectors AD and OB in terms of x, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$. By using the fact that these vectors have the same length 1 as $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$, get two equations relating x and $\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}$. (Use the distributive law for the dot product).

(b) Eliminate x to find a quadratic equation satisfied by $\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}$. Show that the angle α between $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ satisfies the equation $\sin 3\alpha = -\sin 2\alpha$ and that therefore $\alpha = \frac{2\pi}{5}$. (In case you have forgotten, $\sin 3\alpha = \sin\alpha(4\cos^2\alpha - 1)$).

(c) Explain how, given five identical ball-point pens, you can construct a regular pentagon. (Amazingly, the obvious generalization with seven pens lets you construct a regular heptagon. Crockett Johnson claims to have discovered this fact while dining with friends in a restaurant in Italy in 1975, using a menu, a wine list, and seven toothpicks)

2. One vertex of a quadrilateral in $\mathbb{R}^3$ is located at point $\mathbf{p}$. The other three vertices, going around in order, are located at $\mathbf{q} = \mathbf{p} + \vec{\mathbf{a}}$, $\mathbf{r} = \mathbf{p} + \vec{\mathbf{b}}$, and $\mathbf{s} = \mathbf{p} + \vec{\mathbf{c}}$.

   (a) Invent an expression involving cross products that is equal to zero if and only if the four vertices of the quadrilateral lie in a plane.

   (b) Prove that the midpoints of the four sides $\mathbf{pq}, \mathbf{qr}, \mathbf{rs}$, and $\mathbf{sp}$ are the vertices of a parallelogram.

3. Isometries and dot products

   The transpose of a (column) vector $\vec{\mathbf{v}}$ is a "row vector" $\vec{\mathbf{v}}^T$, which is also a $1 \times n$ matrix.

   Suppose that $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ are vectors in $\mathbb{R}^n$ and A is an $n \times n$ matrix.

   (a) Prove that $\vec{\mathbf{v}} \cdot A\vec{\mathbf{w}} = \vec{\mathbf{v}}^T A\vec{\mathbf{w}}$. (You can think of the right-hand side as the product of three matrices.)

   (b) Prove that $\vec{\mathbf{v}} \cdot A\vec{\mathbf{w}} = A^T\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}$. You can do this by brute force using summation notation, or you can do it by using part (a) and the rule for the transpose of a matrix product (Therem 1.2.17 in Hubbard).

   (c) Now suppose that $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ are vectors in $\mathbb{R}^3$ and R is an 3×3 isometry matrix. Prove that $R\vec{\mathbf{v}} \cdot R\vec{\mathbf{w}} = \vec{\mathbf{v}} \cdot \vec{\mathbf{w}}$. If you believe that physical laws should remain valid when you rotate your epcerimental apparatus, this result shows that dot products are appropriate to use in expressing physical laws.

4. Using vectors to prove theorems of trigonometry.

   (a) For vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$,
   $|\vec{\mathbf{a}} \times \vec{\mathbf{b}}| = |\vec{\mathbf{a}}||\vec{\mathbf{b}}| \sin\alpha$, where α is the angle between the vectors.
   By applying this formula to a triangle whose sides are $\vec{\mathbf{v}}, \vec{\mathbf{w}}$, and $\vec{\mathbf{v}} - \vec{\mathbf{w}}$, prove the Law of Sines.

   (b) Consider a parallelogram spanned by vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$.
   Its diagonal is $\vec{\mathbf{v}} + \vec{\mathbf{w}}$.

   Let α denote the angle between $\vec{\mathbf{v}}$ and the diagonal ; let β denote the angle between $\vec{\mathbf{w}}$ and the diagonal. By expressing sines and cosines in terms of cross products, dot products, and lengths of vectors, prove the addition formula

   $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

5. Let $R(\theta)$ denote the 2×2 matrix that represents a counterclockwise rotation about the origin through angle θ. Let $F(\alpha)$ denote the 2×2 matrix that represents a reflection in the line through the origin that makes angle α with the x axis. Using matrix multiplication and the trigonometric identities

   $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
   $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, prove the following:

   (a) $F(\beta)F(\alpha) = R(2(\beta-\alpha))$.

   (b) $F(\gamma)F(\beta)F(\alpha) = F(\gamma+\alpha-\beta)$. (If you are doing R, you might want to work problem 7 first.)

   (c) The product of any even number of reflections in lines through the origin is a rotation about the origin and the product of any odd number of reflections in lines through the origin is a reflection in a line through the origin. (Hint: use induction. First establish the base cases $n = 1$ and $n = 2$. Then do the "inductive step:" show that if the result is true for the product of n reflections, it is true for $n + 2$ reflections.)

6. Matrices that represent complex numbers

   (a) Confirm that $i^2 = -1$ using conformal matrices.

   (b) Represent $4 + 2i$ as a matrix. Square it and interpret its result as a complex number. Confirm your answer by checking what you get when expanding algebraically.

   (c) Show that using matrices to represent complex numbers still preserves addition as we would expect.

   That is, write two complex numbers as matrices. Then add the matrices, and interpret the sum as a complex number. Confirm your answer is correct algebraically.

The last two problems require R scripts. Feel free to copy and edit existing scripts and to use the library script 2L, which has functions for dealing with angles in degrees.

7. Vectors in two dimensions

   (a) You are playing golf and have made a good tee shot. Now the hole is located only 30 yards from your ball, in a direction 32 degrees north of east. You hit a chip shot that travels 25 yards 22 degrees north of east, followed by a putt that travels 8 yards 60 degrees north of east. How far from the hole is your golf ball now located? For full credit, include a diagram showing the relevant vectors.

   (b) The three-reflections theorem, whose proof was problem 5b, states that if you reflect successively in lines that make angle α, β, and γ with the $x-$axis, the effect is simply to reflect in a line that makes angle $\alpha + \gamma - \beta$ with the x-axis. Confirm this, using R, for the case where $\alpha = 40°$, $\beta = 30°$, and $\gamma = 80°$. Make a plot in R to show where the point $P = (1, 0)$ ends up after each of the three successive reflections.

8. Vectors in three dimensions (see script 2Y, topic 3)

   The least expensive way to fly from Boston (latitude 42.36° N, longitude 71.06° W) to Naples (latitude 40.84° N, longitude 14.26° E) is to buy a ticket on Aer Lingus and change planes in Dublin (latitude 53.35° N, longitude 6.26° W). Since Dublin is more than 10 degrees further north than either Boston or Naples, it is possible that the stop in Dublin might lengthen the journey substantially.

   (a) Construct unit vectors in $\mathbb{R}^3$ that represent the positions of the three cities.

   (b) By computing angles between these vectors, compare the length in kilometers of a nonstop flight with the length of a trip that stops in Dublin. Remember that, by the original definition of the meter, the distance from the North Pole to the Equator along the meridian through Paris is 10,000 kilometers. (You may treat the Earth as a sphere of unit radius.)

   (c) Any city that is on the great-circle route from Boston to Naples has a vector that lies in the same plane as the vectors for Boston and Naples. Invent a test for such a vector (you may use either cross products or determinants), and apply it to Dublin.