1 Multiple Choice and True/False

If a true-false statement is false, devise an explicit counterexample.

1. In a website consisting of six pages numbered 1 through 6, $\{12\}$ and $\{256\}$ are defined to be open. Using our standard website topology, which of the following sets is not necessarily open?
   (a) $\{123456\}$
   (b) $\emptyset$
   (c) $\{2\}$
   (d) $\{156\}$
   (e) $\{1256\}$
2. *True or false:* The intersection of any number, finite or infinite, of open sets is open.
3. *True or false:* From any sequence $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}, ...$ s.t. $\forall n \; \mathbf{x_n} \in \mathbb{R}^{\mathbf{3}}$, we can extract a convergent subsequence.
4. M is a three-dimensional manifold in $\mathbb{R}^7$. Which of the following is a way to describe M? **Choose all that apply.**
   (a) a parametrization $\gamma : \mathbb{R}^4 \to \mathbb{R}^3$
   (b) a parametrization $\gamma : \mathbb{R}^3 \to \mathbb{R}^7$
   (c) a parametrization $\gamma : \mathbb{R}^7 \to \mathbb{R}^4$
   (d) the graph of a function $g : \mathbb{R}^3 \to \mathbb{R}^7$
   (e) the graph of a function $g : \mathbb{R}^3 \to \mathbb{R}^4$
   (f) the graph of a function $g : \mathbb{R}^4 \to \mathbb{R}^7$
   (g) the set of points at which a function $F : \mathbb{R}^7 \to \mathbb{R}^4$ is zero
   (h) the set of points at which a function $F : \mathbb{R}^7 \to \mathbb{R}^3$ is zero
   (i) the set of points at which a function $F : \mathbb{R}^4 \to \mathbb{R}^7$ is zero

2 Problems

1. To protect your beloved rose bushes, you decide to hire Heine-Borel Fence Construction Company to build a fence along the front of your yard, forming a one-dimensional fence from location $x = 0$, on the left edge of your lawn, to location $x = 1$, on the right edge of your lawn. However, your cantankerous neighbor forbids you from touching his lawn, so the fence is not allowed to actually reach $x = 1$. (The left side of your lawn buts up against your driveway, so you don't care if the fence actually reaches $x = 0$ or even goes slightly past.) Also, for whatever reason, Heine-Borel Fence Co. really likes building fences on open intervals.
   a. Heine-Borel Construction proposes the following construction worker scheme. Construction worker 0 will build a fence on the interval $(-0.01, 0.6)$. Then, for $k \geq 1$, worker k will build a fence on the interval $\left(1 - \frac{1}{2^k}, 1 - \frac{1}{2^{k+1}}\right)$. Does this proposed scheme result in an open cover of the interval $[0, 1)$? Why or why not?
   b. Heine-Borel Construction decides to offer you a second construction option. Construction worker 0 will build a fence on the interval $(-0.01, 0.6)$. For $k \geq 1$, guard k will cover the interval $\left(1 - \frac{1}{2^k}, 1 - \frac{1}{2^{k+2}}\right)$. Explain why this scheme would enable Heine-Borel Construction to charge you an infinite amount of money to build the fence, even if each construction worker receives a finite salary.

c. You would really rather not spend an infinite amount of money building this fence. Propose a way to modify *your instructions* to Heine-Borel Fence Co. so that you only have to hire a finite number of construction workers.

2. The following limit does not exist:

$$\lim_{\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}} \frac{x^2 - y^2}{x^2 + y^2}$$

   a. Using polar coordinates, show that this limit does not exist.
   b. Using the "no bad sequence" definition, show that this limit does not exist.

3. The differential equation $\ddot{x} + 4\dot{x} + 3x = 0$ probably represents something interesting in physics and economics. Find a matrix A such that

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = A \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

   and then solve to determine $x(t)$ for initial conditions $\begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

4. Preparing for finals, the Harvard libraries have asked you to help them develop a new policy for library hours during finals period. They have realized that, in order to give their librarians enough rest, they need to close Widener and Lamont for a combined 20 hours. They want to minimize student frustration in the process. If Lamont is closed for x hours, student frustration will increase by $5x^2$. If Widener is closed for y hours, student frustration will increase by $10y$.

   (a) Use a parameter t to parametrize the manifold $x + y = 20$. Then determine how to minimize the student frustration function $f = 5x^2 + 10y$ given this constraint.
   (b) Use Lagrange multipliers to solve the same problem of minimizing the student frustration function $f = 5x^2 + 10y$ subject to the constraint $x + y = 20$.

5. Barnum and Bailey are planning to expand their Troupe of Remarkably Trained Pigs (which actually existed, apparently). Barnum trains his x pigs more efficiently than Bailey trains his y pigs, so the two agree that the number of pigs that each trains should increase as follows:

$$\dot{x} = 4x + 2y$$

$$\dot{y} = 2x + y$$

   (a) Diagonalize the matrix $A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$ to express A as PDP^{-1}.
   (b) At time 0, Barnum has 2 pigs and Bailey has 4. Exponentiate A to find formulas for x and y in terms of t.

6. The locus function

$$F \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x^2 + y^2 - z^2 \\ 2x + y + 4z + 8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

   determines a one-dimensional manifold in $\mathbb{R}^3$. As it happens, the point $\vec{c} = \begin{pmatrix} 4 \\ 0 \\ -4 \end{pmatrix}$ lies on this manifold.

   (a) Near this point, an implicit function g can express x and y in terms of z. Determine $g'(-4)$.
   (b) Using $g'(-4)$, which you just found, approximate the x- and y-coordinates of a point lying on the manifold for which $z = -3.98$.

7. You want to guess the time t, measured in years from now, at which Barden College's endowment will reach a whopping \$90 megabucks. You happen to know that the endowment size is s given by $s(t) = t^4 - 9t^3 + 10$. You guess that Barden's endowment will reach \$90 megabucks in around 10 years from now. Use one iteration of Newton's Method on the equation $t^4 - 9t^3 + 10 = 90$ to refine your estimate.