With thanks to Kate Penner for her Fall 2015 review session.

Week 9

1. Axioms of topology
   - Empty set and closed set are open.
   - Finite/infinite union of open sets is open.
   - Finite intersection of open sets is open.
2. Website topology
3. Prove axioms of topology from definitions
4. Vocabulary related to sets
   - Open set
   - Closed set
   - Boundary
   - Closure
   - Interior
5. Redefining convergence topologically in $\mathbb{R}^n$
6. Hausdorff space (Proof 9.1: $\mathbb{R}$ is Hausdorff)
7. Prove convergence in $\mathbb{R}^n$
8. Proof 9.2: Infinite triangle inequality
9. Solve differential equations using diagonalization

Week 10

1. Limit of a sequence in $\mathbb{R}$
2. Limit of a function $f : \mathbb{R}^m \to \mathbb{R}^n$
3. *Skill:* Show a limit exists or does not exist
   - Exists: try using polar coordinates (only depends on θ)
   - DNE: depends on angle of approach (θ in polar coordinates), or two paths disagree
4. Continuity in $\mathbb{R}^n$ (Proof 10.1: f is continuous iff every sequence $\vec{x}_n \to \vec{x}_0$ is good)
5. Compact sets: closed and bounded
6. Bounded: wholly contained within a ball centered at the origin
7. Bolzano-Weierstrass: on a compact set, any sequence has a convergent subsequence
8. Proof 10.2: A continuous function on a compact set has and achieves its supremum

9. Nested Compact Set Theorem: If you have a decreasing sequence of compact sets $U_1 \supseteq U_2 \supseteq U_3 \cdots$, the infinite intersection is **not** empty
10. Heine-Borel: On a compact set, any open cover has a finite subcover
11. Directional derivatives
$$\nabla_{\vec{v}} f(\vec{a}) = \lim_{h \to 0} \frac{f(\vec{a} + h\vec{v}) - f(\vec{a})}{h}$$
12. Partial derivatives
13. Gradient vector: column of partial derivatives
14. Jacobian matrix ($f : \mathbb{R}^n \to \mathbb{R}$):
$$\begin{bmatrix} D_1 f & \cdots & D_n f \end{bmatrix}$$
15. : Linear approximation: $f(\vec{a} + h\vec{v}) = f(\vec{a}) + [Jf(\vec{a})](h\vec{v})$

Week 11

1. Use remainder $\to 0$ to prove a derivative (Proof 11.1: product rule)
2. Prove differentiability with remainder technique
3. Proof 11.2: Derivative = Jacobian
4. When Jacobian exists but function is not differentiable (Not differentiable: linearity of the derivative breaks down)
5. Matrix derivatives with the chain rule (e.g. inverse and squaring functions)
6. Newton's Method (use and relation to tangent line approximation)
7. Inverse function theorem: if f is strictly increasing/decreasing, there exists a local inverse g
$$g'(y_0) = \frac{1}{f'(g(y_0))} \quad \to \quad [Dg(\vec{y})] = [Df(g(\vec{y}))]^{-1}$$
8. Uses of the inverse function theorem

Week 12

1. Implicit function theorem and use
2. Manifolds: particularly well-behaved smooth curves/surfaces in an arbitrary number of dimensions
3. 3 ways to describe manifolds
    - Graph
    - Locus function (F = 0)

- Parametrization

4. Smooth: locally the graph of a C^1 function (or, for locus functions, $[DF]$ is onto)
5. Parametrizations should be one-to-one and onto
6. Tangent **space**: $\dot{x} = [Dg(\vec{z})]\dot{y}$
7. Tangent **plane**: $\vec{x} - \vec{a} = [Dg(\vec{z})](\vec{y} - \vec{b})$
8. ker $[DF(\vec{c})]$ or img $[D\gamma]$ gives a basis for the tangent space
9. Unconstrained critical points: Find by setting all partials equal to zero, and classify by using the Hessian matrix
10. Constricted optimization
    - $\vec{c}$ is a critical point of F restricted to M iff$Df = \lambda_1 DF_1 + \lambda_2 DF_2 + ... + \lambda_k DF_k$
    - Apply Lagrange Multipliers
    - f is the function we want to maximize
    - F is our constraint (written as a locus), which forms a manifold
    - Or parametrize the manifold using a parametrization γ and consider critical points of $f(\gamma)$.