1 Major Concepts

1. You will learn next week that inversion of a matrix A can be accomplished by multiplication on the left by some sequence of "elementary matrices" $E_k * ... * E_1$. For now, all that matters is that the determinant of an elementary matrix is nonzero. Given this, prove that a matrix A is invertible iff (if and only if) its determinant is nonzero. (You may take it as given that the result of Proof 2.2 generalizes to any number of matrices[1], not just 2, and generalizes beyond 3×3 matrices.)

2. $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation. For each of the following sets of three vectors, can we determine the matrix that represents T by T's actions on each vector in the set? (The intended solution involves calculating a **determinant**, so that's why we've put it here. This question anticipates next week's material and will likely seem more interesting after you watch those lectures.)

   (a) $\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}-2\\-4\\3\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$

   (b) $\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}-2\\-4\\3\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}$

3. Classic Inequality Results and interesting applications

   (a) Demonstrate that: $|x + y| \geq ||x| - |y||$

   (b) Demonstrate the Triangle Inequality for in (n) dimensions: $|x_1 + ... + x_n| \leq |x_1| + |x_2| + ...|x_n|$

   (c) Prove that $(x\cos(\theta) + y\sin(\theta))^2 \leq x^2 + y^2$

   (d) Prove that $\sum_{i=1}^{n} |x_i - [i/2]| \geq \sum_{i=1}^{n} |x_{i+1} - x_i|$

4. Making sense of rotation and reflection matrices

   (a) Let's think about rotating a vector through an angle θ. This is a linear transformation (let's call it f), so we can represent it by a matrix F. One way to think about building F is to think about how f would act on vectors in our *standard basis.*

   (b) f sends a vector extending straight along the x-axis, $\begin{bmatrix}1\\0\end{bmatrix}$, to the vector $\begin{bmatrix}\cos\theta\\\sin\theta\end{bmatrix}$. (Check this by geometry/trig.) f also sends a vector extending straight up along the y-axis to $\begin{bmatrix}-\sin\theta\\\cos\theta\end{bmatrix}$. Overall, we can then obtain a rotation matrix

   $$F = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

   (c) Now let's think about the linear transformation g that accomplishes reflection along a line that makes an angle θ with the positive x-axis. g sends $\begin{bmatrix}1\\0\end{bmatrix}$ to $\begin{bmatrix}\cos 2\theta\\\sin 2\theta\end{bmatrix}$. What g does to $\begin{bmatrix}0\\1\end{bmatrix}$ is less straightforward to figure out using trig identities and stuff, but looking at the special cases $\theta = \frac{\pi}{4}$ and $\theta = \frac{\pi}{2}$, it seems reasonable that g takes $\begin{bmatrix}1\\0\end{bmatrix}$ to $\begin{bmatrix}\sin 2\theta\\-\cos 2\theta\end{bmatrix}$. So we can obtain a reflection matrix

   $$G = \begin{bmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{bmatrix}$$

[1] To prove this, we would use a technique called *mathematical induction*, which we'll really get into in a couple weeks.

5. Parallelepiped, vectors, and volume
A parallelepiped is spanned by the following vectors:

$$\vec{v_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$$

   (a) Calculate the volume of the parallelepiped.
   (b) Find the cosine of the angle between $\vec{v}_1 and\ \vec{v}_2$.
   (c) Find the sine of the angle between $\vec{v}_1\ and\ \vec{v}_2$ (don't use the result from previous part).
   (d) check that the cosine and sine you have found are correct, by checking that they satisfy the identity $\sin^2\theta + \cos^2\theta = 1$.
   (e) Starting from the cross and dot product definitions of sine and cosine, prove the identity that you used above.