MATHEMATICS 23a/E-23a, Fall 2018

Linear Algebra and Real Analysis I

Week 7 (Limits and continuity of functions)

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Reading from Ross

- Chapter 3, sections 17 and 18. (continuity)
- Chapter 3, sections 19 and 20 (uniform continuity and limits of functions)

Recorded Lectures

- Lecture 14 (Week 7, Class 1) (watch on October 23 or 24)
- Lecture 15 (Week 7, Class 2) (watch on October 25 or 26)

Proofs to present in section or to a classmate who has done them.

- 7.1 Suppose that $a < b$, f is continuous on $[a, b]$, and $f(a) < y < f(b)$. Prove that there exists at least one $x \in [a, b]$ such that $f(x) = y$.

  Use Ross's "no bad sequence" definition of continuity, not the epsilon-delta definition.

- 7.2 Using the Bolzano-Weierstrass theorem, prove that if function f is continuous on the closed interval $[a, b]$, then f is uniformly continuous on $[a, b]$.

Additional proofs(may appear on quiz, students will post pdfs or videos)

- 7.3 Prove that if f and g are real-valued functions that are continuous at $x_0 \in \mathbb{R}$, then $f + g$ is continuous at x_0. Do the proof twice: once using the "no bad sequence" definition of continuity and one using the epsilon-delta definition of continuity.

- 7.4 (Ross, page 146; uniform continuity and Cauchy sequences)

  Prove that if f is uniformly continuous on a set S and (s_n) is a Cauchy sequence in S, then $(f(s_n))$ is a Cauchy sequence. Invent an example where f is continuous but not uniformly continuous on S and $(f(s_n))$ is not a Cauchy sequence.

R Scripts

- Script 2.3A-Continuity.R
  Topic 1 - Two definitions of continuity
  Topic 2 – Uniform continuity

- Script 2.3B-IntermediateValue.R
  Topic 1 - Proving the intermediate value theorem
  Topic 2 - Corollaries of the IVT

1 Executive Summary

1.1 Two equivalent definitions of continuity

- Continuity in terms of sequences

  This definition is not standard: Ross uses it, but many authors use the equivalent epsilon-delta definition. Here is some terminology that students find useful when discussing the concept:

  - If $\lim x_n = x_0$ and $\lim f(x_n) = f(x_0)$, we call x_n a "good sequence."
  - If $\lim x_n = x_0$ but $\lim f(x_n) \neq f(x_0)$, we call x_n a "bad sequence."

  Then "function f is continuous at x_0" means "every sequence is a good sequence"; i.e. "there are no bad sequences."

- The more conventional definition:

  Let f be a real-valued function with domain $U \subset \mathbb{R}$. Then f is continuous at $x_0 \in U$ if and only if

  $\forall \epsilon > 0, \exists \delta > 0$ such that if $x \in U$ and $|x - x_0| < \delta, |f(x) - f(x_0)| < \epsilon$.

- Which definition to use?

  To prove that a function is continuous, it is often easier to use the second version of the definition. Start with a specified ϵ, and find a δ (not "the δ") that does the job. However, as Ross Example 1a on page 125 shows, the first definition, combined with the limit theorems that we have already proved, can let us prove that an arbitrary sequence is good.

  To prove that a function is discontinuous, the first definition is generally more useful. All you have to do is to construct one bad sequence.

1.2 Useful properties of continuous functions

- New continuous functions from old ones.
  - If f is continuous at x_0, then $|f|$ is continuous at x_0.
  - If f is continuous at x_0, then kf is continuous at x_0.
  - If f and g are continuous at x_0, then $f+g$ is continuous at x_0.
  - If f and g are continuous at x_0, then fg is continuous at x_0.
  - If f and g are continuous at x_0 and $g(x_0) \neq 0$, then $\frac{f}{g}$ is continuous at x_0.
  - If f is continuous at x_0 and g is continuous at $f(x_0)$, then the composite function $g \circ f$ is continuous at x_0.

  Once you know that the identity function and elementary functions like nth root, sine, cosine, exponential, and logarithm as continuous (Ross has not yet defined most of these functions!), you can state the casual rule

  "If you can write a formula for a function that does not involve division by zero, that function is continuous everywhere."

- Theorems about a continuous function on a closed interval $[a, b]$ (an example of a "compact set"), easy to prove by using the Bolzano-Weierstrass theorem.
  - f is a *bounded* function.
  - f achieves its maximum and minimum values on the interval (i.e. they are not just approached as limiting values).

- The Intermediate Value Theorem and some of its corollaries.
  It is impossible to do calculus without either proving these theorems or stating that they are obvious!

  Now f is assumed continuous on an interval I that is not necessarily closed (e.g. $\frac{1}{x}$ on (0, 1])
  - IVT: If $a < b$ and y lies between $f(a)$ and $f(b)$, there exists at least one x in (a, b) for which $f(x) = y$.
  - The image of an interval I is either a single point or an interval J.
  - If f is a strictly increasing function on I, there is a continuous strictly increasing inverse function $f^{-1} : J \to I$.
  - If f is a strictly decreasing function on I, there is a continuous strictly decreasing inverse function $f^{-1} : J \to I$.
  - If f is one-to-one on I, it is either strictly increasing or strictly decreasing.

1.3 Continuity versus uniform continuity

It's all a matter of the order of quantifiers. For continuity, y is agreed upon before the epsilon-delta game is played. For uniform continuity, a challenge is made using some $\epsilon > 0$, then a δ has to be chosen that meets the challenge independent of y.

For function f whose domain is a set S:

- Continuity: $\forall y \in S, \forall \epsilon > 0$,
  $\exists \delta > 0$ such that $\forall x \in S$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.
- Uniform continuity: $\forall \epsilon > 0$,
  $\exists \delta > 0$ such that $\forall x, y \in S$,$|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.
- On $[0, \infty]$ (not a bounded set), the squaring function is continuous but not uniformly continuous.
- On $(0, 1)$ (not closed) the function $f(x) = \frac{1}{x}$ is continuous but not uniformly continuous.
- On a closed, bounded interval $[a, b]$, continuity implies uniform continuity. The proof uses the Bolzano-Weierstrass theorem.
- By definition, if a function is continuous at $s \in S$ and (s_n) converges to s, then $(f(s_n))$ converges to $f(s)$. If (s_n) is merely Cauchy, we know that it converges, but not what it converges to. To guarantee that $(f(s_n))$ is also Cauchy, we must require f to be *uniformly* continuous.
- On an open interval (a, b) a function can be continuous without being uniformly continuous. However, if we can *extend* f to a function $\overline{f}$, defined so that $\overline{f}$ is continuous at a and b, then $\overline{f}$ is uniformly continuous on $[a, b]$ and f is uniformly continuous on (a, b). The most familiar example is $f(x) = \frac{\sin x}{x}$ on $(0, \infty)$, extended by defining $\overline{f}(0) = 1$.
- Alternative criterion for uniform continuity (sufficient but not necessary): f is differentiable on (a, b), with f' bounded on (a, b).

1.4 Limits of functions

1. Definitions of "limit"

   - Ross's definition of limit, consistent with the definition of continuity: S is a subset of $\mathbb{R}$, f is a function defined on S, and a and L are real numbers, ∞ or $-\infty$. Then $\lim_{x\to a^S} f(x) = L$ means

     for every sequence (x_n) in S with limit a, we have $\lim(f(x_n)) = L$.
   - The conventional epsilon-delta definition:

     f is a function defined on $S \subset \mathbb{R}$, a is a real number in the closure of S (not $\pm\infty$) and L is a real number (not $\pm\infty$). $\lim_{x\to a} f(x) = L$ means $\forall \epsilon > 0, \exists \delta > 0$ such that if $x \in S$ and $|x - a| < \delta$, then $|f(x) - L| < \epsilon$.

2. Useful theorems about limits, useful for proving differentiation rules.

   Note: a can be $\pm\infty$ but L has to be finite.

   Suppose that $L_1 = \lim_{x\to a^S} f_1(x)$ and $L_2 = \lim_{x\to a^S} f_2(x)$ exist and are finite.

   Then

   - $\lim_{x\to a^S}(f_1 + f_2)(x) = L_1 + L_2$.
   - $\lim_{x\to a^S}(f_1 f_2)(x) = L_1 L_2$.
   - $\lim_{x\to a^S}(\frac{f_1}{f_2})(x) = \frac{L_1}{L_2}$, provided $L_2 \neq 0$ and $f_2(x) \neq 0$ for $x \in S$.

3. Limit of the composition of functions

   Suppose that $L = \lim_{x\to a^S} f(x)$ exists and is finite.
   Then $\lim_{x\to a^S}(g \circ f)(x) = g(L)$ provided

   - g is defined on the set $\{f(x) : x \in S\}$.
   - g is defined at L
     (which may just be a limit point of the set $\{f(x) : x \in S\}$.)
   - g is continuous at L.

4. One-sided limits

   We can modify either definition to provide a definition for $L = \lim_{x\to a^+} f(x)$.

   - With Ross's definition, choose the set S to include only values that are greater than a.
   - With the conventional definition, consider only $x > a$: i.e.
     $a < x < a + \delta$ implies $|f(x) - L| < \epsilon$.

   It is easy to prove that

   $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$.

2 Lecture outline

1. Continuity defined in terms of sequences (Ross, page 124)

   For specified x_0 and function f, define the following terminology:

   - If $\lim x_n = x_0$ and $\lim f(x_n) = f(x_0)$, we call x_n a "good sequence."
   - If $\lim x_n = x_0$ but $\lim f(x_n) = f(x_0)$, we call x_n a "bad sequence."

   Then Ross's definition of continuity is "every sequence is a good sequence."

   Prove the following, which is the more conventional definition:

   Let f be a real-valued function with domain $U \subset \mathbb{R}$. Then f is continuous at $x_0 \in U$ if and only if

   $\forall \epsilon > 0, \exists \delta > 0$ such that if $x \in U$ and $|x - x_0| < \delta, |f(x) - f(x_0)| < \epsilon$.

2. Using the "bad sequence" criterion to show that a function is discontinuous.

   The "signum function" $\mathrm{sgn}(x)$ is defined as $\frac{x}{|x|}$ for $x \neq 0$, 0 for $x = 0$.

   Invent a "bad sequence," none of whose elements is zero, to prove that $\mathrm{sgn}(x)$ is discontinuous at 0, then show that for any positive x, no such bad sequence can be constructed.

   Restate this proof that $\mathrm{sgn}(x)$ is discontinous at $x = 0$, continuous for positive x, in terms of the epsilon-delta definition.

3. Proving that a function is continuous

   Using sequences, prove that the function $f(x) = x^2 - 2x + 1$ is continuous at any $x_0 \in \mathbb{R}$. It is important to realize that this argument is valid "for all sequences."

4. Proof 7.3 – two ways to prove a theorem about continuity (Ross, page 128)

   Prove that if f and g are real-valued functions that are continuous at $x_0 \in \mathbb{R}$, then $f + g$ is continuous at x_0.

   Approach 1 (the easy way). Use Ross's definition of continuity in terms of sequences, and invoke the appropriate therem about sequences.

   Approach 2(the hard way). Use the epsilon-delta definition of continuity.

5. An important theorem of calculus that is often just stated without proof (Ross, page 133)

   Let f be a continuous real-valued function on a closed interval $[a, b]$.
   Using the Bolzano-Weierstrass theorem, prove that f is bounded and that f achieves its maximum value: i.e. $\exists y_0 \in [a, b]$ such that $f(x) \leq f(y_0)$ for all $x \in [a, b]$.

6. Proof 7.1 – the intermediate value theorem (Ross, page 134)

   Suppose that $a < b$, f is continuous on $[a, b]$, and $f(a) < y < f(b)$. Prove that there exists at least one $x \in [a, b]$ such that $f(x) = y$.

   Use Ross's "no bad sequence" definition of continuity, not the epsilon-delta definition. Constructing the appropriate sequences requires some care.

7. Using the intermediate value theorem

   Prove that the function

$$C(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

   is equal to zero for one and only one value $x \in [1, 2]$.

   This result will be useful when we define π without trigonometry.

8. (Ross, pages 135 and 136; existence of inverse function. This is a composite of Corollary 18.3 and Theorem 18.4)

   Let f be a strictly increasing function on some interval I. Then $f(I)$ is an interval J and there exists a continuous strictly inverse function f^{-1} such that $f^{-1}(f(x)) = x$ for

   A diagram is a great help in visualizing this theorem!

9. Continuity versus uniform continuity (Ross, page 143)

   It's all a matter of the order of quantifiers. For continuity, y is agreed upon before the epsilon-delta game is played. For uniform continuity, a challenge is made using some $\epsilon > 0$, then a δ has to be chosen that meets the challenge independent of y.

   For function f whose domain is a set S:

   - Continuity: $\forall y \in S, \forall \epsilon > 0$,
     $\exists \delta > 0$ such that $\forall x \in S$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.
   - Uniform continuity: $\forall \epsilon > 0$,
     $\exists \delta > 0$ such that $\forall x, y \in S$,$|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

   Two standard counterexamples, using sets that are not both closed and bounded.

   - On $[0, \infty]$ (not a bounded set), the squaring function is continuous, Show that it is not uniformly continuous.

   - On $(0, 1)$ (not closed) the function $f(x) = \frac{1}{x}$ is continuous. Show that it is not uniformly continuous.

10. Proof 7.2 – when continuity implies uniform continuity (Ross, theorem 19.2)

    Using the Bolzano-Weierstrass theorem, prove that if function f is continuous on the closed interval $[a, b]$, then f is uniformly continuous on $[a, b]$.

11. Proof 7.4 – uniform continuity in terms of sequences (Ross, page 146)

    Prove that if f is uniformly continuous on a set S and (s_n) is a Cauchy sequence in S, then $(f(s_n))$ is a Cauchy sequence.

    Invent an example where f is continuous but not uniformly continuous on S and $(f(s_n))$ is not a Cauchy sequence.

12. Uniform continuity (or lack thereof)

Let $f(x) = x^2 + \frac{1}{x^2}$.

Determine whether f is or is not uniformly continuous on each of the following intervals:

(a) $[1, 2]$

(b) $(0, 1]$

(c) $[2, \infty)$

(d) $(1, 2)$

13. One way to show uniform continuity on an interval that is not closed

    On an open interval (a, b) a function can be continuous without being uniformly continuous. However, if we can *extend* f to a function $\overline{f}$, defined so that $\overline{f}$ is continuous at a and b, then $\overline{f}$ is uniformly continuous on $[a, b]$ and f is uniformly continuous on (a, b).

    Show that on the open interval $(0, \pi)$ the function

    $$f(x) = \frac{1 - \cos x}{x^2}$$

    is uniformly continuous by using the "extension" approach.

14. Ross's non-standard but excellent definition of limit (Ross, page 156)

    S is a subset of $\mathbb{R}$, f is a function defined on S, and a and L are real numbers, ∞ or $-\infty$.

    Then $\lim_{x \to a^S} f(x) = L$ means

    for every sequence (x_n) in S with limit a, we have $\lim(f(x_n)) = L$.

    Suppose that $L_1 = \lim_{x \to a^S} f_1(x)$ and $L_2 = \lim_{x \to a^S} f_2(x)$ exist and are finite.

    Prove that $\lim_{x \to a^S}(f_1 + f_2)(x) = L_1 + L_2$ and
    $\lim_{x \to a^S}(f_1 f_2)(x) = L_1 L_2$.

15. The conventional definition of limit (Ross, page 159;)

Let f be a function defined on $S \subset \mathbb{R}$, let a be in the closure of S, and let a be a real number.

Prove that $\lim_{x\to a} f(x) = L$ if and only if

$\forall\epsilon > 0, \exists\delta > 0$ such that

if $x \in S$ and $|x - a| < \delta$, then $|f(x) - L| < \epsilon$.

16. Evaluating limits by brute force

(a) Use the epsilon-delta definition of limit to prove that $\lim_{x\to 0} \sqrt{|x|} = 0$.

(b) Use the sequence definition of limit to show that $\lim_{x\to 0} \frac{x}{|x|}$ does not exist.

17. Useful rules for evaluation of limits

    Note:

    Suppose that $L_1 = \lim_{x \to a^S} f_1(x)$ and $L_2 = \lim_{x \to a^S} f_2(x)$ exist and are finite. a can be $\pm\infty$.

    - $\lim_{x \to a^S} (f_1 + f_2)(x) = L_1 + L_2$.
    - $\lim_{x \to a^S} (f_1 f_2)(x) = L_1 L_2$.
    - $\lim_{x \to a^S} (\frac{f_1}{f_2})(x) = \frac{L_1}{L_2}$, provided $L_2 \neq 0$ and $f_2(x) \neq 0$ for $x \in S$.

    Prove the second of these (limit of product = product of limits) by using the corresponding theorem about sequences.

18. Limit of the composition of functions

    Suppose that $L = \lim_{x \to a^S} f(x)$ exists and is finite.
    Then $\lim_{x \to a^S} (g \circ f)(x) = g(L)$ provided

    - g is defined on the set $\{f(x) : x \in S\}$.
    - g is defined at L
      (which may just be a limit point of the set $\{f(x) : x \in S\}$.)
    - g is continuous at L.

    Combine all of these rules and you can pretty much conclude that if you can write a formula for a function that does not involves division by zero, the function is continuous.

19. Limits that involve roots

    Use the sum and product rules for limits to evaluate

$$\lim_{x \to 1} \frac{x^{\frac{1}{3}} - 1}{x - 1}$$

20. Limits that involve trig functions

Using the definition of the sine and tangent functions from right-triangle trigonomtry and the principle that if region R_1 of the plane is a subregion of region R_2, then R_1 has a smaller area than R_2, prove that, for angle $\theta \geq 0$,

$$\sin\theta \leq \theta \leq \tan\theta.$$

Using the squeeze lemma for limits (proof left to the homework), prove that $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

By a clever rewrite to express everything in terms of $\frac{\sin x}{x}$, evaluate

$$\lim_{x\to 0} \frac{\sin 2x - 2\sin x}{x^3}$$

3 Seminar Topics

Your section instructor will either have emailed a list of topics to prepare or will have posted a signup list of appointments on the Calendar tab of Canvas. Either way, there will be one of the following topics that you should be prepared to present.

Practice your presentation so that it takes about 8 minutes. The text of the presentation will be projected onto a screen so that you need not recopy it. To save time, avoid writing long sentences on the chalkboard. You may use notes, but be discreet about it.

1. (Proof 7.3)
   Prove that if f and g are real-valued functions that are continuous at $x_0 \in \mathbb{R}$, then $f + g$ is continuous at x_0. Do the proof twice: once using the "no bad sequence" definition of continuity and one using the epsilon-delta definition of continuity.

2. (Ross, page 133)

   Let f be a continuous real-valued function on a closed interval $[a, b]$. Using the Bolzano-Weierstrass theorem, prove that f is bounded and that f achieves its maximum value: i.e. $\exists y_0 \in [a, b]$ such that $f(x) \leq f(y_0)$ for all $x \in [a, b]$.

3. (Proof 7.1)
   Suppose that $a < b$, f is continuous on $[a, b]$, and $f(a) < y < f(b)$. Prove that there exists at least one $x \in [a, b]$ such that $f(x) = y$.

   Use Ross's "no bad sequence" definition of continuity, not the epsilon-delta definition.

4. (Proof 7.2)
   Using the Bolzano-Weierstrass theorem, prove that if function f is continuous on the closed interval $[a, b]$, then f is uniformly continuous on $[a, b]$.

5. (Proof 7.4 – uniform continuity and Cauchy sequences)
   Prove that if f is uniformly continuous on a set S and (s_n) is a Cauchy sequence in S, then $(f(s_n))$ is a Cauchy sequence. Invent an example where f is continuous but not uniformly continuous on S and $(f(s_n))$ is not a Cauchy sequence.

6. (Extra topic) Here is Ross's definition of limit:
   S is a subset of $\mathbb{R}$, f is a function defined on S, and a and L are real numbers, ∞ or $-\infty$.

   Then $\lim_{x \to a^S} f(x) = L$ means

   for every sequence (x_n) in S with limit a, we have $\lim(f(x_n)) = L$.

   For the case where a is in the closure of S and L is finite, prove the conventional definition of limit:
   $\lim_{x \to a^S} f(x) = L$ if and only if

   $\forall \epsilon > 0, \exists \delta > 0$ such that

   if $x \in S$ and $|x - a| < \delta$, then $|f(x) - L| < \epsilon$.

4 Workshop Problems

1. Proofs about continuity

   (a) Prove that if f is continuous at $x_0 \in \mathbb{R}$, and g is continuous at $f(x_0)$, then the composite function $g \circ f$ is continuous at x_0.

   Do two different versions of the proof:

   - Use the "no bad sequence definition" (this is easy, but in case you get stuck, it is also Ross Theorem 7.5)
   - Use the epsilon-delta definition. Using ϵ in the codomain of g and δ in the codomain of f, you will need a third Greek letter in the domain of f. Ross often uses η (eta) in this role.

   (b)
   - The Heaviside function $H(x)$ is defined by $H(x) = 0$ for $x < 0, H(x) = 1$ for $x \geq 0$. Using the "no bad sequence" definition, prove that H is discontinuous at $x = 0$. (Hint: try a sequence of negative numbers.)
   - Using the epsilon-delta definition of continuity, prove that $f(x) = x^3$ is continuous for arbitrary x_0. (Hint: first deal with the special case $x_0 = 0$, then notice that for small enough δ, $|x| < 2|x_0|$.

2. Intermediate-value theorem

   (a) Using the intermediate-value theorem

   As a congressional intern, you are asked to propose a tax structure for families with incomes in the range 2 to 4 million dollars inclusive. Your boss, who feels that proposing a tax rate of exactly 50% for anyone would be political suicide, wants a function $T(x)$ with the following properties:

   - It is continuous.
   - Its domain is [2,4].
   - Its codomain is [1,2].
   - There is no x for which $2T(x) = x$.

   Prove that this set of requirements cannot be met by applying the intermediate-value theorem to the function $x - 2T(x)$, which is negative if the tax rate exceeds 50%.

   Then prove "from scratch" that this set of requirements cannot be met, essentially repeating the proof of the IVT. Hint: Consider the least upper bound s of the set of incomes $S \in [2, 4]$ for which the tax rate is greater than 50 %, and construct sequences (s_n) and (t_n) that converge to s from below and above.

   (b) Continuous functions on an interval that is not closed

   Let $S = [0, 1)$. Invent a sequence $x_n \in S$ that converges to a number $x_0 \notin S$. Hint: try $x_1 = \frac{1}{2}, x_2 = \frac{3}{4}$.

   Then, using this sequence, invent an unbounded continuous function on S and invent a bounded continuous function on S that has no maximum. Explain why you could not do either of these things if S were [0,1].

3. Calculation of limits

(a)
- For i, use the sequence definition of limit and the squeeze lemma.
- For ii, rewrite the expression so that it becomes easy to use the sum and product rules for limits.
- For iii, rewrite the expression so that it becomes easy to use the fact that $\lim_{x\to 0}\frac{\sin x}{x}=1$. along with the limit rules.

i. Prove that
$$\lim_{x\to 0} x\sin\frac{1}{x}=0.$$

ii. Evaluate
$$\lim_{h\to 0}\frac{(x+h)^{\frac{3}{2}}-x^{\frac{3}{2}}}{h}$$

iii. Evaluate
$$\lim_{x\to 0}\frac{\cos 2x-1}{x^2}$$

(b)
- For i, use the sequence definition of limit and invent sequences that would lead to two different values for L.
- For ii, rewrite the expression so that it becomes easy to use the sum and product rules for limits.
- For iii, rewrite the expression so that it becomes easy to use the fact that $\lim_{x\to 0}\frac{\sin x}{x}=1$. along with the limit rules.

i. Show that $\lim_{x\to 0}\sin\frac{1}{x}$ does not exist.

ii. Evaluate
$$\lim_{x\to\infty}(\sqrt{x+1}-\sqrt{x})$$

iii. Evaluate
$$\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}$$

5 Homework

Special offer – if you do the entire problem set, with one problem omitted, in LaTeX, you will receive full credit for the omitted problem. Alternatively, if you work all the problems in LaTeX, we will convert your lowest score to a perfect score.

1. Ross, exercises 19.2(b) and 19.2(c). Be sure that you prove *uniform* continuity, not just continuity!

2. Ross, exercise 19.4.

3. Ross, exercises 20.16 and 20.17. This squeeze lemma is a cornerstone of elementary calculus, and it is nice to be able to prove it!

4. Ross, exercise 20.18. Be sure to indicate where you are using various limit theorems.

5. Ross, exercise 17.4. It is crucial that the value of δ is allowed to depend on x.

6. Ross, exercises 17-13a and 17-14. These functions will be of interest when we come to the topic of integration in the spring term.

7. Ross, exercise 18-4. To show that something exists, describe a way to construct it.

8. Ross, exercise 18-10. You may use the intermediate-value theorem to prove the result.