Multiple Choice

1. Here, we have to choose all that are countably infinite.

a. → NO, since the reals are uncountably infinite.

b → NO, again the reals are uncountably infinite.

c → YES! $\mathbb{Q}$ is countably infinite so $\mathbb{Q}^3$ (if that's the right way to write it) is countable as well.

d. → NO. We can show this using the same method we used to show $\mathbb{R}$ is uncountable.

$s_1 : (1, 2, -7, 17, 3, \dots)$

$s_2 : (-99, 7, 1, 1, 1, \dots)$

$s_3 : (2, -2, 2, -2, 2, \dots)$

Assume we've listed all of them like this. Then we construct an element that can't be on the list by taking the i^{th} element of s_i and subtracting 1 from it.

$s : (0, 6, 1, \dots)$

which by construction cannot be in our list. So the set must be uncountably infinite.

e. → YES! The primes are a subset of $\mathbb{N}$ which in turn is a subset of $\mathbb{Q}$. $\mathbb{Q}$ is countable, so $\mathbb{N}$ and by extension the set of primes must be countable.

2. Let's just solve this and then pick the correct answer at the end. I'll use the ratio test to find the radius of convergence.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x+2)^{n+1}\cdot 1/5^n}{n(x+2)^n \cdot 1/5^{n-1}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x+2)}{5n}\right| = \left|\frac{x+2}{5}\right|$$

This converges if $\left|\frac{x+2}{5}\right| < 1$ ie $-1 < \frac{x+2}{5} < 1$ ie our radius of convergence is 5 and we know the series converges for $x \in (-7, 3)$. Now we just need to check the behavior at the endpoints.

$x = -7$: $\sum_{n=1}^{\infty} \frac{n(-7+2)^n}{5^{n-1}} = \sum_{n=1}^{\infty} (-5)n$ diverges since $|5n| \not\to 0$. (blows up to ∞).

$x = 3$: $\sum_{n=1}^{\infty} \frac{n(3+2)^n}{5^{n-1}} = \sum_{n=1}^{\infty} 5n$ also diverges.

So (c) is the correct answer.

3\. a → converges! $\sin(n)$ is bounded, and $\frac{1}{n} \to 0$, so $\frac{\sin n}{n} \to 0$.

b → converges! First, we see that it's bounded, as $\forall n \in \mathbb{N}$ $2^n > 0$ and $n! > 0$. So $\lim_{n\to\infty} \frac{2^n}{n!} \geq 0$. Second, let's prove it's decreasing. WTS $\frac{2^{n+1}}{(n+1)!} \leq \frac{2^n}{n!}$. This holds since $\frac{2^{n+1}}{(n+1)!} = \frac{2^n}{n!}\left(\frac{2}{n+1}\right)$, and if $n \geq 1$, then $\frac{2}{n+1} \leq 1$. (The inequality is strict if $n > 1$.) So $\frac{2^{n+1}}{(n+1)!} = \frac{2^n}{n!}\left(\frac{2}{n+1}\right) \leq \frac{2^n}{n!}(1) = \frac{2^n}{n!}$. So the sequence is decreasing. And we know that decreasing & bounded below ⇒ convergent.

c → converges! Let's do some algebra to show it converges:

$$\lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n}) = \lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n})\left(\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right)$$
$$= \lim_{n\to\infty}\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$$

d → converges! This is a SEQUENCE (not a series) of terms whose limit is 0. (Note the difference between (d) and (e)!)

e → does NOT converge. This is the sequence of partial sums of the harmonic SERIES, which diverges, so the sequence of partial sums must diverge as well.

Problems

1a. WTS $\lim_{n\to\infty} n^a = \infty$.

ie WTS $\forall M>0, \exists N$ s.t. $\forall n>N$ $n^a > M$.

ie $n > M^{1/a}$

Given $M>0$ choose $N = M^{1/a}$. We know that $f(x) = x^a$ is an increasing function on $[0,\infty)$ as long as $a>0$. So $\forall n>N$, $n^a > N^a = (M^{1/a})^a = M$.

b. WTS $\lim_{n\to\infty} \frac{1}{n^a} = 0$.

ie WTS $\forall \varepsilon>0\ \exists N$ s.t. $\forall n>N$ $\left|\frac{1}{n^a}\right| < \varepsilon$.

$\frac{1}{\varepsilon} < n^a$

$\left(\frac{1}{\varepsilon}\right)^{1/a} < n$ so choose $N = \left(\frac{1}{\varepsilon}\right)^{1/a}$

So given $\varepsilon>0$ choose $N = \left(\frac{1}{\varepsilon}\right)^{1/a}$. Then $\forall n>N$ $\left|\frac{1}{n^a}\right| < \frac{1}{N^a} = \frac{1}{((1/\varepsilon)^{1/a})^a} = \varepsilon$.

(Ross thm. 9.10 also tells us that $\lim s_n = \infty \Leftrightarrow \lim \frac{1}{s_n} = 0$.)

For both (a) & (b), I didn't really worry about N being a natural number. If you want to deal with this, just use the ceiling function $\lceil x \rceil$ to round N up to the nearest whole number, or just say "choose $N \in \mathbb{N}$ s.t. $N > M^{1/a}$ or $N > \left(\frac{1}{\varepsilon}\right)^{1/a}$.

2. We assume $\exists$ a least number j for which $\sum_{i=1}^{j} i \neq \frac{j(j+1)}{2}$

Then we know $\sum_{i=1}^{j-1} i \neq \frac{j(j+1)}{2} - j$ (j is the last term in this sum)

This means $\sum_{i=1}^{j-1} i \neq \frac{j^2+j-2j}{2} = \frac{j^2-j}{2} = \frac{j(j-1)}{2}$

So the proposition also does not hold for $j-1<j$. So since we also know that the proposition holds for $n=1$ (ie $\frac{1(1+1)}{2} = 1$), we know we have a contradiction, and the property must hold for all numbers.

3. First, let's express the idea of a supremum using quantifiers.

If M is a supremum, it is the least upper bound:

$$\underbrace{\forall s\in S,\ s\le M}_{\text{upper bound}} \quad \text{and} \quad \underbrace{\forall N<M,\ \exists s_0\in S \text{ s.t. } N<s_0\le M}_{\text{least upper bound}}$$

To show that the supremum is unique, let's assume that S has two suprema, M_1 & M_2, and show they must be the same.

① $\forall s \in S,\ s \le M_1$ and $\forall N < M_1$, $\exists s_0 \in S$ s.t. $N < s_0 \le M_1$

② $\forall s \in S,\ s \le M_2$ and $\forall N < M_2$, $\exists s_0 \in S$ s.t. $N < s_0 \le M_1$

Assume $M_2 < M_1$. Then by ① $\exists s_0 \in S$ s.t. $M_2 < s_0 \le M_1$ so M_2 is not a supremum after all – contradiction! By ②, the same type of contradiction arises if we assume $M_1 < M_2$. So the only possibility is $M_1 = M_2$, which means the supremum of S is unique.

4. We'll do this by contradiction. Assume that $\exists x \in (a,b)$ s.t. $f(x) = \alpha \neq 0$. We WTS f is discontinuous at x, ie $\exists \varepsilon > 0$ s.t. $\forall \delta > 0$ $\exists y$ s.t. $|x-y| < \delta$ but $|f(x) - f(y)| < \varepsilon$.
Let's choose $\varepsilon = |\alpha|/2$. We haven't used our fact about $f(r) = 0\ \forall r \in \mathbb{Q}$, so let's think about how to use that. By the density of the rationals, we can construct an r arbitrarily close to x (say, by truncating the decimal expansion of x). Thus $\forall \delta > 0$ $\exists r \in \mathbb{Q}$ s.t. $x < r < x + \delta$. (We know that this construction is possible.) Then certainly $|x - r| < \delta$, since r is less than δ away from x. But $f(x) = \alpha$ and $f(r) = 0$, so $|f(x) - f(r)| = |\alpha| > |\alpha|/2 = \varepsilon$. So we have a contradiction, and our assumption must have been wrong. So $f(x) = 0\ \forall x \in (a,b)$.

5. We'll do this by contradiction. Assume there is an infinite element $x \in \mathbb{R}$, ie $\forall n \in \mathbb{N}$ $n < x$. But we also know that the reals are complete. Let's apply the Archimedean property to x and any $y \in \mathbb{N}$ and $y < x$. (By necessity, this means $y \in \mathbb{R}$ and $y > 0$ [I'm assuming $\mathbb{N}$ starts with 1, though you could have $\mathbb{N}$ start with 0 and just add in the condition $y > 0$]. So we have met the criteria to apply the Archimedean property.) Thus by the Archimedean property we know $\exists m \in \mathbb{N}$ s.t. $ym > x$. $y \in \mathbb{N}$ and $m \in \mathbb{N}$, so certainly $ym \in \mathbb{N}$ ($\mathbb{N}$ is closed under multiplication). So there exists an element $z = ym \in \mathbb{N}$ s.t. $z > x$. But earlier we said $n < x\ \forall n \in \mathbb{N}$, so we have a contradiction!

6a. First, we consider $f'(x)$ for $x \in (0,1)$, where $f(x) = x$

$$f'(x) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x - x_0}{x - x_0} = \lim_{x \to x_0} 1 = 1$$

Now we consider $f'(x)$ for $x \in (1,2)$, where $f(x) = x - 2$

$$f'(x) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x - 2 - x_0 + 2}{x - x_0} = \lim_{x \to x_0} \frac{x - x_0}{x - x_0} = 1$$

And at $x = 1$: we know that differentiable $\Rightarrow$ continuous. Equivalently, this means discontinuous $\Rightarrow$ not differentiable. So since f is obviously discontinuous at $x = 1$ (this is easily proved since

$$\lim_{x \to 1^-} f(x) = 1 \neq -1 = \lim_{x \to 1^+} f(x)$$

and if the limit (in $\mathbb{R}$) exists, it must be unique.), it must not be differentiable either.

b. This is not a counterexample to Rolle's Theorem! Rolle's Theorem would require that f be continuous on the closed interval $[0,2]$, which is certainly not the case.

c. $g(x)$ is clearly continuous on $(0,2)$. But Rolle's Theorem requires continuity on $[0,2]$, while $g(x)$ is discontinuous at $x = 0$. So Rolle's Theorem again does not apply! (We need to have a CLOSED interval for Rolle's to apply.)

d. In case you haven't figured out, the theme is that no, there aren't contradictions - rather, the MVT isn't applicable here. This is because although differentiability implies continuity, the converse does not hold. So continuity is a necessary but insufficient property for differentiability. Thus, though $h(x)$ is continuous everywhere, it is not differentiable at $x = 1$ and so the MVT does not apply.

since $\sin^2 y + \cos^2 y = 1$,
$\cos^2 y - 1 = -\sin^2 y$,
so $\cos^2 h - \sin^2 h - 1$
$= -2\sin^2 h$

7. $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(2(x+h)) - \sin(2x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(2x)\cos(2h) + \cos(2x)\sin(2h) - \sin(2x)}{h}$$

$$= \lim_{h \to 0}\left[\cos 2x \left(\frac{\sin(2h)}{h}\right)\right] + \lim_{h \to 0}\left[\sin 2x \left(\frac{\cos(2h) - 1}{h}\right)\right]$$

$$= \cos 2x \left(\lim_{h \to 0} \frac{2\sin h \cos h}{h}\right) + \sin 2x \left(\lim_{h \to 0} \frac{\cos^2 h - \sin^2 h - 1}{h}\right)$$

$$= 2\cos 2x \left(\lim_{h \to 0} \frac{\sin h}{h}\right)\left(\lim_{h \to 0} \cos h\right) - 2\sin 2x \left(\lim_{h \to 0} \frac{\sin h}{h}\right)\left(\lim_{h \to 0} \sin h\right)$$

$$= 2\cos 2x (1)(1) - 2\sin 2x (1)(0)$$

$$= 2\cos 2x \checkmark$$

8a The difference is the order of the quantifiers. For uniform continuity, we choose δ before x_0, so the δ must be "one-size-fits-all."

A proof of $\boxed{\text{continuity}}$ is of the form "Given any $\varepsilon > 0$ and x_0, choose $\delta = \delta(\varepsilon, x_0)$. Then $\forall x$, if $|x - x_0| < \delta$, $|f(x) - f(x_0)| < \varepsilon$."

A proof of $\boxed{\text{uniform continuity}}$ is of the form "Given $\varepsilon > 0$, choose $\delta = \delta(\varepsilon)$. Then $\forall x, y$, if $|x - y| < \delta$, $|f(x) - f(y)| < \varepsilon$."

b. First, we'll do scratch work. We want to have $|x^2 - y^2| < \varepsilon$. Factor:

$$|x^2 - y^2| = |x - y||x + y|.$$

Since $x, y \in (0,1)$, $x + y < 2$. So $|x^2 - y^2| = |x - y||x + y| < 2|x - y|$

The RHS of that is $< 2\delta$ by definition of δ. So if we set δ so that $2\delta < \varepsilon$, we'll have $|x^2 - y^2| < \varepsilon$. (This means $\delta < \varepsilon/2$).

FORMAL PROOF: Given $\varepsilon > 0$, choose $\delta < \varepsilon/2$. Then if $|x - y| < \delta$,

$$|x^2 - y^2| = |x - y||x + y| < 2\delta < \varepsilon. \checkmark$$

c. The above proof would suffice, provided that you just say that $x + y \leq 2$. (choosing $\delta \leq \varepsilon/2$ makes it so this isn't important.)

A perhaps easier way to do this, noting that $[0,1]$ is a closed interval, is to say that since $f(x)$ is continuous on a closed interval, it is uniformly continuous on that interval. (Proving that x^2 is continuous would look very similar to the above proof, but we don't need to worry about it for this question.)