Math 23a

Quiz 2 Review

This handout is meant to serve as a guide to what you need to know in your studying, and is not meant to be all comprehensive and a sole replacement for your studies.

1 Week 5

Week 5 was all about setting the foundation for the real analysis module. We explored two main topics this week: the properties of real numbers and sequences. These properties of real numbers are important in our proofs as they allow us to declare the existence of some quantities that may be critical for our proofs (e.g. Denseness to declare the existence of a rational between any two reals). Sequences will form the foundation of all of our study in real analysis, and lead to nifty definitions of typically hard to understand concepts relating to functions. For this week, know/ know how to:

- **Properties of Numbers**
  - **Do inductive and least number proofs**
  - The difference between countable infinity and uncountable infinity
  - The statement and use of the Archimedan property of the real numbers (You can fill up a bathtub with a teaspoon).
  - **Denseness of the Rationals in the Reals**
  - **Least Upper Bound/Greatest Lower Bound property of the reals**
- **Sequences**
  - **Use the formal definition of limit of a sequence to prove convergence of a sequence**
  - **Use Quantifiers to express the concepts of infinitely many, finitely many, limit, etc.**
  - **Add and subtract the same thing, then use the triangle inequality to break up an unknown quantity into two known quantities, followed by the $\epsilon/2$ trick**
  - **Use the limit theorems**
  - **The formal definition of a divergent limit (one that goes to ∞)**

Notice almost all of these are bold... They are super important!

2 Week 6

Week 6 finished the tour of sequences by discussing the important concepts of supremum, $\limsup, \liminf$. We also talked about a useful alternate definition for convergence in the reals, namely being Cauchy, and then proved an important theorem for asserting the existence of a convergent sequence, the Bolzano Weierstrass Theorem. Then we discussed summing the elements of a sequence, formally known as a series, and defined it as a limit of a sequence (of partial sums). Then, we discussed several convergence tests for the convergence of a series. The stuff about sequences this week are all really important, and ubiquitous in real analysis. Series are important to know when the sum of an infinite number of terms can be finite. They also provide the theoretical backing for Taylor Series in week 8. For this week, know/ know how to:

- **Sequences (cont.)**
  - **The definition of supremum, infimum, maximum, minimum, and how to construct a set who's supremum is outside the set**
  - The definition of monotone
  - **The definition of Cauchy sequence, and using the formal definition to prove a sequence is Cauchy**
  - **The definition of** $\limsup, \liminf$**, and when they are equal** (when the overall limit exists, then they are equal)
  - **The Bolzano Weierstrass Theorem and how to use it** (use it when you need to construct a convergent subsequence out of points that are all over the map)
- **Series**
  - **Definition of partial sum, and how the limit of it is defined to be the series sum**
  - The difference between absolute and conditional convergence
  - Famous Series: geometric series, p-series, harmonic series
  - **Convergence Tests: Root Test, Ratio Test, Alternating Series Test**
  - **Find the radius of convergence**

3 Week 7

Week 7 introduced key concepts of **functions**: continuity, uniform continuity, and limits. One key theme throughout the week is that many of these concepts can be defined in terms of sequences, as well as in a formal δ, ϵ way. Knowing when to use which definition is key, and it is important to mix the two. This material will form the basis for talking about differentiability as well as functions in general. For this week, know/know how to:

- **Continuity**
  - **Concept of a good/bad sequence**
  - **The Two Equivalent Definitions: in terms of ϵ, δ, and sequences, and how to use them to prove continuity, discontinuity**
  - Sums of continuous functions are continuous, etc.
  - Theorems involving continuous functions: A function that is bounded on a closed interval achieves its maximum on the interval, and the **Intermediate Value Theorem** (if you start below the x axis, and end up above the x axis, you have to pass through the x axis, and generalizations of this concept)
  - **Using the Intermediate Value Theorem to show the existence of a quantity**
- **Uniform Continuity**
  - **The Two Equivalent Definitions: in terms of ϵ, δ, and Cauchy sequences**
  - An easy way to prove it: **Continuity on a closed interval**
  - How to cook up examples of non uniformly continuous functions
- **Limits**
  - **Two Equivalent Definitions: in terms of ϵ, δ, and sequences**
  - **Limit Rules:** Limits of sums is the sum of limits, etc.

4 Week 8

Week 8 was the beginning of calculus, as the derivative was introduced, and we learned the properties of differentiable functions. We also introduced the Taylor series, which is very important for computing quantities, as well as for making approximations (we will come to understand this in the future). This will prepare us for the next module of multivariable calculus. For this week, know/know how to:

- **The Derivative and Differentiability**
  - **Definition as a limit, and how to use it**
  - Rules of Differentiation: **Sums, Products, Quotients, Powers**, and the **Chain Rule**
  - Properties of Differentiable Functions: **Rolle's Theorem**, and it's generalization the **Mean Value Theorem**
  - Using Rolle's Theorem and the Mean Value Theorem to prove the existence of a quantity.
  - The derivative of a maximum or minimum must be zero.
  - **Derivative of an Inverse function**
  - **Using L'Hospital's Rule to evaluate indeterminate limits**
- **Taylor Series**
  - **Computing a Taylor Series**
  - **Proving Convergence of a Taylor Series using Taylor's Theorem with Remainder**
  - The idea that a function can be defined by its Taylor Series: hyperbolic trig functions

Rules for Math 23a/E-23a Online Quizzes
Fall 2018

"Members of the Harvard College community commit themselves to producing academic work of integrity that is, work that adheres to the scholarly and intellectual standards of accurate attribution of sources, appropriate collection and use of data, and transparent acknowledgement of the contribution of others to their ideas, discoveries, interpretations, and conclusions. Cheating on exams or problem sets, plagiarizing or misrepresenting the ideas or language of someone else as ones own, falsifying data, or any other instance of academic dishonesty violates the standards of our community, as well as the standards of the wider world of learning and affairs."

1. No references are allowed except for the excerpts from the Executive Summaries that are on the last page.
2. After the quiz is downloaded and printed, computers and cell phones must be switched off and calculators should be put away.
3. No discussion with classmates or others is permitted during the entire period during which the quiz is available.
4. The proctor does not have to remain continuously in the room but should look in unexpectedly from time to time.
5. Although there is no explicit time limit, the quiz must be completed in a single sitting.
6. When the quiz is done, switch on the computer and scanner and upload the completed quiz, along with this signed form.

Date of quiz________________________________
Start time________________________________
End time________________________________
List any unusual circumstances in the administration of the quiz (e.g. the scanner was not located in the room where the quiz was taken).

I am aware of the Harvard College Honor Code, and I certify that I complied with the rules.

(Student signature)________________________________

I am aware of the Harvard College Honor Code, and I observed that the student complied with the rules.

(Proctor signature)________________________________

Proctor email________________________________

Proctor relationship to student ________________________________

Name: ______________________________
Section (if any): ___________________

MATHEMATICS 23a/E-23a, Fall 2018
Quiz #2
November 9-11, 2018

You must complete this quiz at a single sitting immediately after downloading and printing it. While you are taking the quiz, your computers and cell phone must be switched off.

The last page of the quiz contains useful information extracted from the Executive Summaries.

Calculators, which would be of no use, are not allowed.

You and your proctor must sign the statement on the front page of the exam.
You may omit one multiple-choice question in Part I and one question in Part II.
If you are doing proof logging, check here ____ and omit one proof in Part III.
If you opt out of proof logging, check here ____ and do all proofs in Part III.

There are three blank pages at the end of the exam. If your answer does not fit in the space provided on the page with the question, write "Continued on page XX" and finish the answer on the specified page. That way, your exam can be scanned without having to check for answers on the back of a page.

Problem	Points	Answer	Score
$I-1$	2		
$I-2$	2		
$I-3$	2		
$I-4$	2		
$I-5$	2		
$I-6$	2		
$II-1$	5	$--$	
$II-2$	5	$--$	
$II-3$	5	$--$	
$II-4$	5	$--$	
$II-5$	5	$--$	
$III-1$	5	$--$	
$III-2$	5	$--$	
$III-3$	5	$--$	
$III-4$	5	$--$	
$III-5$	5	$--$	
Total	50 or 55	$--$	

Part I. Answer five of the six multiple-choice questions. Transcribe your answers onto page 2, and mark an X in the score box on page 2 to indicate which question you have omitted.

If you answer all six questions, the last one will be ignored.

1. Let s_n be a sequence of real numbers on a bounded set S, where $\liminf s_n \neq \limsup s_n$. Which of the following is not necessarily true?

   (a) $\lim s_n$ does not exist.

   (b) s_n is not Cauchy.

   (c) $\liminf s_n < \limsup s_n$

   (d) There exists a convergent subsequence of s_n.

   (e) s_n has an infinite number of dominant terms.

   Solution: $\boxed{E}$ a) is true, because in order for a limit to exist, the lim sup must equal the lim inf, and it turns out that that is equal to the limit. Because the sequence can't converge, choice b) is also out, since all Cauchy sequences of real numbers converge. For choice c), notice that in general $\liminf s_n \leq \limsup s_n$, and we have excluded the equal case in the prompt of the question. For choice d), the Bolzano Weierstrass Theorem still holds, since the sequence is defined on a bounded set. e) is not necessarily true. An explicit counterexample is $s_n = 1 - \frac{1}{n}$ for even n, and $s_n = 0$ for odd n. In this case, there are no dominant terms.

2. Which of the following is not true about $s_n = \frac{1}{n}$?

   (a) The sequence converges to 0.

   (b) $\lim_{n\to\infty} \sum_{i=1}^{n} s_i = L$, for some finite L.

   (c) $\limsup s_n = 0$.

   (d) The series $\sum(-1)^n s_n$ converges.

   (e) The series $\sum s_n^2$ converges.

   Solution: $\boxed{B}$ This series is known as the harmonic series, which diverges. Choice b) is simply stating the definition of a series sum as the limit of the partial sums. a) is true, since when n goes to infinity, $1/n$ goes to zero. c) is true, since if the limit exists, then $\limsup s_n = \liminf s_n = \lim s_n$. Choice d) is true, because then, we are considering an alternating series, which has a less stringent convergence condition, namely that $\lim |s_n| = 0$, which is satisfied. Choice e) is true, because $1/n^2$ is a convergent p series.

3. Which of the following must be true of a continuous function on (a, b)?
   (a) The function achieves its maximum on (a, b).
   (b) The function is bounded.
   (c) For all Cauchy sequences s_n on the set (a, b), $f(s_n)$ is also Cauchy.
   (d) If $f(a) = 2$, and $f(b) = 5$, then $f(c) = 3$, for some $c \in (a, b)$.
   (e) None of the above are true

   Solution: $\boxed{E}$ None of these are true! Choice a) is not true, by a counterexample. If $a = 0$ and $b = 1$, then $1/x$ is continuous on $(0, 1)$, but does not achieve its maximum. The key here is that the statement only guarantees continuity on the open interval. Part b) is false by the counterexample above. Part c) is also not true by the counterexample above. Let $\lim s_n = 0$, and then $f(s_n)$ will diverge, and not be Cauchy. d) is not true, because it doesn't need to be continuous at the end points. The function $f(x) = 5$ except at a, where $f(a) = 2$, is a counterexample. Notice how important continuity on a **closed** interval is for most of the theorems we've studied!

4. Find $\lim_{x\to b} \frac{\sqrt{x}-\sqrt{b}}{x-b}$ for $b > 0$.
   (a) ∞
   (b) $\frac{1}{2\sqrt{b}}$
   (c) 0
   (d) $2\sqrt{b}$
   (e) b

   Solution: $\boxed{B}$ If you multiply both top and bottom by $\sqrt{x}+\sqrt{b}$, the numerator becomes $x-b$, and cancels the denominator. Taking the limit results in choice b). Alternatively, this is the definition of the derivative of the $\sqrt{x}$ function evaluated at $x = b$.

5. Let f be a differentiable function, where all derivatives exist, such that $f(0) = 0$, $f'(0) = 0$, and $|f''(x)| \le M, \forall x$. Which of the following is not necessarily true?
   (a) $|f(1)| \le \frac{M}{2}$
   (b) 0 is neither a maximum nor a minimum.
   (c) $\forall \epsilon > 0, \exists \delta > 0$ s.t. if $x \in (-\delta, \delta), |f(x)| < \epsilon$
   (d) If $\lim s_n = 0$, then $\lim f(s_n) = 0$.
   (e) None of the above. They're all necessarily true.

   Solution: $\boxed{B}$ Choice a) is true, since that is just the statement of Taylor's Theorem of Remainder. It says that

$$f(x) = f(0) + f'(0)x + \frac{f''(c)x^2}{2!} \text{ where } c \in [0, x]$$

Notice that $f(0) = f'(0) = 0$, and that the derivative is bounded by M. Then, this states:

$$\begin{aligned} f(1) &\leq 0 + 0(1) + \frac{M(1)^2}{2!} \\ &\leq \frac{M}{2} \end{aligned}$$

The same holds on the negative side. Choice b) is false by counterexample. Let $f(x) = x^2$, then 0 is a minimum, and the second derivative is bounded by 2. Choice c) is true, since it is just the statement of continuity at $x = 0$, and being differentiable implies continuity. Choice d) is another statement of continuity, just the sequence definition of continuity.

6. Let $\sum a_n$ be a conditionally convergent alternating series. Which of the following is not necessarily true?

   (a) The series converges to some finite L.

   (b) The series sum is independent of order of terms.

   (c) $\sum |a_n|$ diverges.

   (d) $\lim a_n = 0$.

   (e) None of the above. They're all necessarily true.

Solution: $\boxed{B}$ Part a) is true by definition! If it converges, it converges to something. This does not contradict part b), since the order of terms in a conditionally convergent series needs to be respected! The order of terms as written defines the series sum, despite other sums being possible if the terms are moved around. Part c) is false, because, if $|a_n|$ converges, then the series would be absolutely convergent, not conditionally convergent. Part d) is true, since that is exactly the statement of the alternating series test.

Part II. Answer four of the five questions. Mark an X in the score box on page 1 to indicate which question you have omitted.

1. (Inspired by Week 5, workshop problems # 3)
   Consider the sequence:

$$s_n = \frac{n^p + 1}{n^p}$$

   where $p \in \mathbb{R}$.

   (a) Find $\lim s_n$ when $p = 1$ from the definition of the limit of a sequence.

   (b) Find $\lim s_n$ when $p > 0$ using whatever method you want. (Note that your answer should be independent of p, so the numerical answer to the limit will be the same as your answer to the first part)

   (c) For $p < 0$, prove that $\lim s_n = \infty$ from the definition of a limit of a sequence being infinity.

   Solution:

   (a) When $p = 1$, our sequence is given by:

$$s_n = \frac{n+1}{n} = 1 + \frac{1}{n}$$

   This looks like it will converge to 1, but we need to prove that!
   Scratch work: We eventually want to find N such that all $n > N$ result in $|s_n - 1| < \epsilon$ for any given epsilon. Let's find the N such that $|s_N - 1| = \epsilon$ and then all terms after that will be closer to 1 than s_N. We want to solve:

$$|s_N - 1| = \epsilon \implies |\frac{1}{N}| = \epsilon \implies N = \frac{1}{\epsilon}$$

   Formal proof: For any $\epsilon > 0$, choose $N = \frac{1}{\epsilon}$. Then, $\forall n > N$, we have:

$$|s_n - 1| = |\frac{1}{n}| < \frac{1}{N} \implies |s_n - 1| < \epsilon$$

   Thus, we have shown that s_n converges to 1.

   (b) We want to evaluate $\lim s_n$ when $p > 0$. Since we found in part a) that the sequence converges to 1 for $p = 1$, then it should converge to 1 for all p (based on the statement of the problem). We will use limit rules this time around. Note:

$$s_n = \frac{n^p + 1}{n^p} = 1 + \frac{1}{n^p}$$

   Now, we use a limit rule. We take the limit as $n \to \infty$ and the limit of a sum is the sum of the limits (as long as both limits exist). The limit of 1 is just 1, while the limit of $1/n^p$ is zero. Thus,

$$\lim s_n = \lim \left(1 + \frac{1}{n^p}\right) = 1 + 0 = 1$$

(c) Now for $p < 0$, we want to show that $\lim s_n = \infty$. For negative p, we can write our sequence as:

$$s_n = \frac{n^{-|p|} + 1}{n^{-|p|}} = 1 + n^{|p|}$$

We need to show that the limit is infinity from the definition.

Scratch work: We need to exhibit a N such that for all $n > N$, $s_n > M$ for any given M. Let's find the N such that $s_N = M$, then all terms after that are greater than M. We want to find:

$$1 + N^{|p|} = M \implies N = (M-1)^{1/|p|}$$

Formal proof: For any M, choose $N = (M-1)^{1/|p|}$. Then, $\forall n > N$:

$$\begin{aligned} |s_n| = (1 + n^{|p|}) &> 1 + N^{|p|} \\ &> 1 + (M-1) \\ &> M \end{aligned}$$

Thus, we have shown $\lim s_n = \infty$ for $p < 0$.

2. (Inspired by Week 6, workshop problems # 1)

   Consider a sequence $s_n \in \mathbb{R}$.

   (a) Prove that if s_n is bounded (both above and below), then $\limsup s_n$ exists and is finite. (Note by a similar argument $\liminf s_n$ exists and is finite.)

   (b) If s_n is unbounded below, then by definition, we say $\liminf s_n = -\infty$. However, $\limsup s_n$ is not determined from this information. Invent a sequence that is unbounded below, where $\limsup s_n$ is finite. Invent one where $\limsup s_n = -\infty$.

   Solution:

   (a) By definition,

$$\limsup s_n = \lim_{N\to\infty} \sup\{s_n : n > N\}$$

   Notice that $\sup\{s_n : n > N\}$ forms a decreasing sequence in N. Namely, as I start excluding more and more elements, the supremeum can only go down. Put another way, $\{s_n : n > N\} \subset \{s_n : n > N - m\}$ for any m, so the supremum as I go further in the sequence in N can only go down. Furthermore, $\sup\{s_n : n > N\}$ is bounded below by the same bound as s_n is bounded by. The supremum can't possibly be below what the lower bound of the elements is. Thus, $\sup\{s_n : n > N\}$ is a decreasing sequence and is bounded below, so it converges and $\lim_{N\to\infty} \sup\{s_n : n > N\} = \limsup s_n$ exists.

   (b) We first invent a sequence that is unbounded below but $\limsup s_n$ is finite. Consider:

$$s_n = \begin{cases} 0 & \text{if n is even} \\ -n & \text{if n is odd} \end{cases}$$

   This sequence is $(-1, 0, -3, 0, -5, 0, ...)$. This is clearly unbounded, but the $\sup\{s_n : n > N\} = 0$ for any N. Thus, $\limsup s_n = 0$ and is finite. It does not diverge. For a sequence where $\limsup s_n = -\infty$, consider $s_n = -n$. This sequence is unbounded below and $\sup\{s_n : n > N\} = -(N + 1)$ so $\limsup s_n = -\infty$.

3. (Inspired by Week 7, workshop problems # 1)
Let's consider a function $h(x)$ constructed in a piecewise manner from two other functions $f(x)$ and $g(x)$.

$$h(x) = \begin{cases} f(x) & \text{for } a \leq x \leq b \\ g(x) & \text{for } b < x \leq c \end{cases}$$

Assume that $f(x)$ and $g(x)$ are defined on $[a, b]$ and $(b, c]$ respectively, so that $h(x)$ is defined on $[a, c]$.

(a) Invent (drawing a picture is okay) functions $f(x)$, $g(x)$ where $f(x)$ is continuous on $[a, b]$ and $g(x)$ is continuous on $[b, c]$ (note that $g(x)$ is also defined at b in this case), *but* $h(x)$ is not continuous on $[a, c]$.

(b) Let $f(x)$ be continuous at b. Prove that $h(x)$ is continuous at b, if and only if

$$\lim_{x \to b^-} f(x) = \lim_{x \to b^+} g(x)$$

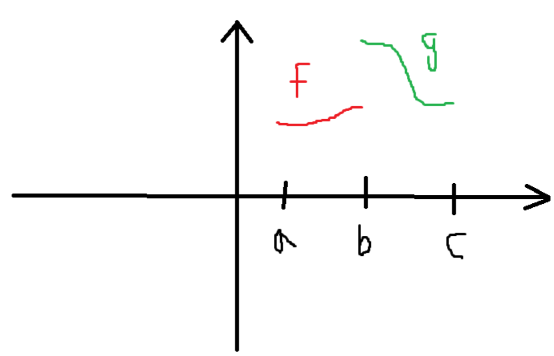
(c) The condition that $f(x)$ be continuous at b is crucial! Show (using a picture is fine) a situation where

$$\lim_{x \to b^-} f(x) = \lim_{x \to b^+} g(x)$$

but $h(x)$ is not continuous at b, when we allow $f(x)$ to be discontinuous at b.

Solution:

(a) First, we invent a function $f(x)$ and a function $g(x)$, which are continuous on $[a, b]$ and $[b, c]$ respectively, but the combined function $h(x)$ is not. One can imagine this occurs if $f(x)$ and $g(x)$ do not meet at b. The following is shown below:



(b) This is an if and only if proof, so we have to do this in both directions. Let's first show that if $h(x)$ is continuous, then:

$$\lim_{x\to b^-} f(x) = \lim_{x\to b^+} g(x)$$

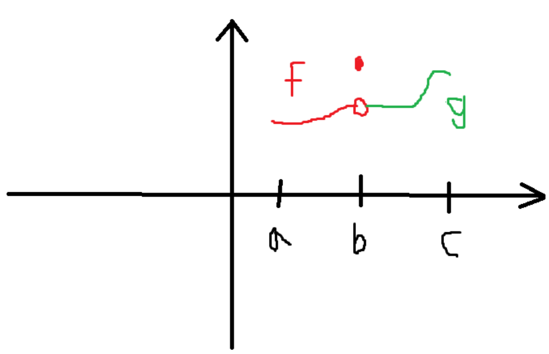
We let $h(b) = f(b) = L$. Since $f(x)$ is continuous at b, then $\lim_{x\to b^-} f(x) = L$. Thus, we only need to show that $\lim_{x\to b^+} g(x) = L$. If $h(x)$ is continuous at b, then $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $|x - b| < \delta$, then $|h(x) - L| < \epsilon$. To calculate $\lim_{x\to b^+} g(x)$, let's consider only $x > b$. In this region, $h(x) = g(x)$, and we have $x - b < \delta \implies |g(x) - L| < \epsilon$. This is exactly the definition for $\lim_{x\to b^+} g(x) = L$. Thus, we have shown

$$\lim_{x\to b^-} f(x) = \lim_{x\to b^+} g(x)$$

Now, we want to go the other way around and show that these limits being equal implies continuity of $h(x)$. We can do this in many ways, but I'll do this by contraposition. We instead show that if $h(x)$ is not continuous at b, then $\lim_{x\to b^-} f(x) \neq \lim_{x\to b^+} g(x)$. Let $h(b) = f(b) = L$. Once again, since $f(x)$ is continuous at b, then $\lim_{x\to b^-} f(x) = L$. If $h(x)$ is not continuous at b, then $\exists \epsilon > 0$, such that $\forall \delta > 0$, $|x - b| < \delta$, but $|h(x) - L| \geq \epsilon$. We restrict our attention to $x > b$, where $h(x) = g(x)$. Then, the statement of discontinuity says, $\exists \epsilon > 0$, such that $\forall \delta > 0$, $x - b < \delta$ but $|g(x) - L| \geq \epsilon$. This, by definition, is exactly the definition of $g(x)$ not converging to L. Thus, $\lim_{x\to b^+} g(x) \neq L$, so

$$\lim_{x\to b^-} f(x) \neq \lim_{x\to b^+} g(x)$$

(c) Continuity of $f(x)$ at b is crucial, and we used it several times in the previous proof! Here is an example, where if we relax that restriction, we can have a discontinuous $h(x)$ at b, despite having $\lim_{x\to b^-} f(x) = \lim_{x\to b^+} g(x)$.



4. (Inspired by Week 8, workshop problems # 3)
Recall that the exponential function is the amazing function whose derivative is itself! Namely if $f(x) = e^x$, then $f'(x) = e^x$. Also, note that $f(0) = e^0 = 1$.

   (a) Compute the Taylor series of e^x expanded around $x = 0$. Prove using Taylor's theorem with remainder that this Taylor series converges to e^x.

   (b) Prove that the radius of convergence of the Taylor series is ∞.

   (c) Recall that the hyperbolic sine is given by

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + ... = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

   Show using Taylor series representations that

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Solution:

(a) The Taylor series of e^x expanded around $x = 0$ is given by:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)x^k}{k!}$$

The first term is $e^0 = 1$. For subsequent terms, we need to calculate $f^{(k)}(0)$. All derivatives of $f(x)$ are e^x, since the derivative of e^x is itself. Thus, we have that $f^{(k)}(0) = 1$ for all k. Then, the Taylor series of e^x is:

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

To show that the Taylor series actually converges to e^x, recall Taylor's theorem with remainder:

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)x^k}{k!} + \frac{f^{(k)}(x_0)x^n}{n!}$$

where x_0 is in between 0 and x. This derivative at a given x is bounded by e^x, which is the derivative at $x_0 = x$. Thus, the remainder is at most:

$$\frac{e^x x^n}{n!}$$

To show the Taylor series converges, we need to show that the remainder as a sequence in n converges to 0. Thus, we want to evaluate:

$$\lim_{n \to \infty} \frac{e^x x^n}{n!}$$

We will use the ratio test **for sequences** (Ross Exercise 9.12, Problem Set 5 Question 8), and consider the ratio:

$$|\frac{x^{n+1}}{(n+1)!}\frac{n!}{x^n}| = |\frac{x}{n+1}|$$

The limit of this sequence as $n \to \infty$ is zero and thus we know that $e^x x^n/n!$ converges as a sequence in n to 0. Thus, the remainder goes to zero, and e^x is equal to its Taylor series.

(b) To find the radius of convergence, we use the ratio test for **series**. We evaluate:

$$\lim_{n\to\infty}|\frac{s_{n+1}}{s_n}| = \lim_{n\to\infty}|\frac{x^{n+1}}{(n+1)!}\frac{n!}{x^n}| = \lim_{n\to\infty}|\frac{x}{n+1}| = 0 < 1$$

for all x. Thus, this Taylor series which equals e^x converges for all x and has infinite radius of convergence.

(c) We calculate $(e^x - e^{-x})/2$ explicitly using Taylor series.

$$\begin{aligned}\frac{e^x - e^{-x}}{2} &= \frac{1}{2}\left(\sum_{k=0}^{\infty}\frac{x^k}{k!} - \sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\right)\\ &= \frac{1}{2}\left(\sum_{k=0}^{\infty}\frac{x^k}{k!} - \sum_{k=0}^{\infty}\frac{(-1)^k x^k}{k!}\right)\end{aligned}$$

Now, notice that all the even terms cancel, while we keep all the odd terms with a factor of 2. Thus, we have:

$$\begin{aligned}\frac{e^x - e^{-x}}{2} &= \frac{1}{2}\left(\sum_{j=0}^{\infty}\frac{2x^{2j+1}}{(2j+1)!}\right)\\ &= \sum_{j=0}^{\infty}\frac{x^{2j+1}}{(2j+1)!}\\ &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...\\ &= \sinh(x)\end{aligned}$$

which is exactly the series for the hyperbolic sine!

5. Consider a function $f(x)$ with domain $U = [a, b]$.

   (a) Prove that if $f(x)$ is differentiable at a point $x_0 \in U$, then $f(x)$ is continuous at x_0. You may also take for granted that $\lim_{x\to x_0} f(x)$ exists. (Hint: start from the limit which defines the derivative as existing and then multiply both sides by $\lim_{x\to x_0}(x - x_0)$.)

   (b) Show that the converse is not necessarily true! Invent (a picture is fine) a function $f(x)$ that is continuous at x_0, but not differentiable at x_0.

   (c) Using the Mean Value Theorem, show that if $f'(x)$ is bounded on (a, b), then $f(x)$ is uniformly continuous on $[a, b]$.

Solution:

(a) If $f(x)$ is differentiable at a point $x_0 \in U$, then we know that:

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

exists. Now, we multiply left and right side by $\lim_{x\to x_0}(x - x_0)$, and obtain:

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} \lim_{x\to x_0} (x - x_0) = f'(x_0) \lim_{x\to x_0} (x - x_0)$$

We know that $\lim_{x\to x_0}(x - x_0) = 0$. Furthermore, on the left hand side, we can use limit rules to combine the product of limits into a limit of the product:

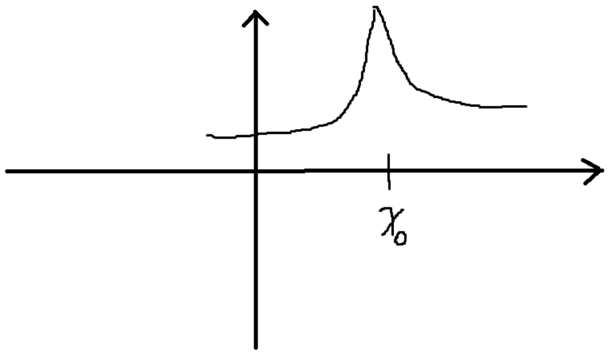
$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) = 0 \implies \lim_{x\to x_0} (f(x) - f(x_0)) = 0$$

Furthermore, we assume $\lim_{x\to x_0} f(x)$ exists, so we can use limit rules to separate the limit of a difference into the difference of the limits so

$$\lim_{x\to x_0} f(x) = f(x_0)$$

This is exactly what it means to be continuous, so we have shown $f(x)$ is continuous at x_0.

(b) A function can be continuous but not differentiable at a given point. The picture below is an example:



This "cusp" as it's called is continuous, but is not differentiable, because the slope when approaching from the left or the right is different and not the same.

(c) Now, we want to prove that a bounded derivative implies uniform continuity on an interval $[a, b]$.

Scratch work: To prove uniform continuity, we need to show that $\forall \epsilon > 0, \exists \delta > 0$ s.t. $\forall x, y \in [a, b], |y - x| < \delta \implies |f(y) - f(x)| < \epsilon$. Let's consider arbitrary $x, y \in [a, b]$, and without loss of generality we'll let $x < y$. Then, the Mean Value Theorem says

$$\frac{f(y) - f(x)}{y - x} = f'(c)$$

for some $c \in (x, y)$. Then, we have that:

$$|\frac{f(y) - f(x)}{y - x}| = |f'(c)|$$

Since the derivative is bounded, its absolute value has a least upper bound M, so that $|f'(c)| < M + 1$. Then, we have that:

$$|\frac{f(y) - f(x)}{y - x}| < M + 1 \implies |f(x) - f(y)| < |y - x|(M + 1)$$

Thus, if $|f(y) - f(x)|$ can only be $M + 1$ times bigger than $|y - x|$.

Formal proof: For any $\epsilon > 0$, choose $\delta = \frac{\epsilon}{M+1}$ where $M = \sup\{|f'(x)| : x \in [a, b]\}$ which exists since $f'(x)$ is bounded. Then, by the Mean Value Theorem for any $x, y \in [a, b]$:

$$|\frac{f(y) - f(x)}{y - x}| = |f'(c)| \leq M < M + 1 \implies |f(y) - f(x)| < |y - x|(M + 1)$$

for $c \in (x, y)$. Now, if $|y - x| < \delta$, then:

$$|f(y)-f(x)| < |y-x|(M+1) \implies |f(y)-f(x)| < \frac{\epsilon}{M+1}(M+1) \implies |f(y)-f(x)| < \epsilon$$

and thus, we have shown that $f(x)$ is uniformly continuous on $[a, b]$.

Part III. If you are doing proof logging, do four of the five proofs. Mark an X in the score box on page 1 to indicate which proof you have omitted.
If you are opting out of proof logging, do all five proofs.

1. (Proof 5.3)
   The completeness axiom for the real numbers states that every nonempty subset $S \subset \mathbb{R}$ that is bounded above has a least upper bound $\sup S$. Use it to prove that for any two positive real numbers a and b, there exists a positive integer n such that $na > b$.

2. (Proof 6.1)

- Prove that any bounded increasing sequence converges. (You may assume without additional proof the corresponding result, that any bounded decreasing sequence converges.)
- Prove that every sequence (s_n) has a monotonic subsequence.
- Prove the Bolzano-Weierstrass Theorem: every bounded sequence has a convergent subsequence.

3. (Proof 7.4)
   Prove that if f is uniformly continuous on a set S and (s_n) is a Cauchy sequence in S, then $(f(s_n))$ is a Cauchy sequence. Invent an example where f is continuous but not uniformly continuous on S and $(f(s_n))$ is not a Cauchy sequence.

4. (Proof 8.1)

- Prove Rolle's Theorem: if f is a continuous function on $[a, b]$ that is differentiable on (a, b) and satisfies $f(a) = f(b)$, then there exists at least one x in (a, b) such that $f'(x) = 0$.

- Using Rolle's Theorem, prove the Mean Value Theorem: if f is a continuous function on $[a, b]$ that is differentiable on (a, b), then there exists at least one x in (a, b) such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

5. (Proof 8.4, chosen at random from the remaining 12)
   Let f be defined on (a, b) with $a < 0 < b$.
   Suppose that the nth derivative $f^{(n)}$ exists on (a, b).

   Define the remainder

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k.$$

   Prove, by repeated use of Rolle's theorem, that for each $x \neq 0$ in (a, b), there is some y between 0 and x for which

$$R_n(x) = \frac{f^{(n)}(y)}{n!} x^n.$$

1 Multiple Choice

There **will not** be any "choose all that apply" questions on the exam, to the best of our knowledge; they are just included on this review for practice and discussion.

1. Which of the following sets are countably infinite? (**Choose all that apply.**)
   (a) All points in $\mathbb{R}^3$.
   (b) All points contained within the unit ball in $\mathbb{R}^3$ (i.e. all points (x, y, z) s.t. $x^2 + y^2 + z^2 \leq 1$).
   (c) All points in $\mathbb{R}^3$ whose coordinates are rational numbers.
   (d) All infinite sequences whose elements are all integers.
   (e) All prime numbers.

2. For what values of x does the power series

$$\sum_{n=1}^{\infty} \frac{n(x+2)^n}{5^{n-1}}$$

   converge?
   (a) Only for $x = -2$.
   (b) For all $x \in \mathbb{R}$.
   (c) For all $x \in (-7, 3)$.
   (d) For all $x \in [-7, 3]$.
   (e) For all $x \in [-7, 3)$.

3. Which of the following **sequences** converge as $n \to \infty$? (**Choose all that apply.**)
   (a) $\frac{\sin(n)}{n}$
   (b) $\frac{2^n}{n!}$
   (c) $\sqrt{n+1} - \sqrt{n}$
   (d) $\frac{1}{n}$
   (e) $\sum_{i=1}^{n} \frac{1}{i}$

2 Problems

1. First, prove that $\lim_{n\to\infty} n^a = +\infty$ for $a > 0$. Then, prove that $\lim_{n\to\infty} \frac{1}{n^a} = 0$ for $a > 0$.

2. Use a least number proof to show that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

   (For practice, you could also do an inductive proof to show the same result.)

3. Prove that the supremum of a set S is unique.

4. (Ross 17.12) Let f be a continuous real-valued function with domain (a, b). Show that if $f(r) = 0$ for every rational number $r \in (a, b)$, then $f(x) = 0\ \forall x \in (a, b)$.

5. Using the Archimedean Property, prove that there are no infinite elements (larger than all the natural numbers–you could also say larger than all real numbers and get the same conclusion) in the real numbers. (This shows that infinity is not a real number!)

6. Consider the function $f(x)$ defined on the interval $[0, 2]$ and given by

$$f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ x - 2 & 1 < x \leq 2 \end{cases}$$

   (a) Show (using the limit definition of the derivative) that for every point in the interval $(0, 1) \cup (1, 2)$, $\frac{d}{dx} f(x) = 1$. Also explain why $f'(x)$ is undefined at $x = 1$.

   (b) Certainly, $f(0) = 0$ and $f(2) = 0$, but we just saw that there is no $x \in (0, 2)$ satisfying $f'(x) = 0$. Is this a contradiction to Rolle's Theorem?

   (c) In order to try to solve the Rolle's Theorem issue in part (b), we could try building a new function $g(x)$ given by

$$g(x) = \begin{cases} 0 & x \leq 0 \\ x - 2 & x > 0 \end{cases}$$

   Again we see that $g(0) = 0$ and $g(2) = 0$. And this time, the function is continuous on $(0, 2)$. But again, there's no $x \in (0, 2)$ such that $g'(x) = 0$. This time, is there a contradiction with Rolle's Theorem?

   (d) Let's consider one more function $h(x)$ defined by

$$h(x) = \begin{cases} x & x \leq 1 \\ 1 & x > 1 \end{cases}$$

   $f(2) = 1$ and $f(0) = 0$, so the average slope between $x = 0$ and $x = 2$ is $\frac{f(2)-f(0)}{2-0} = \frac{1}{2}$. Clearly, we haven't met the conditions for Rolle's Theorem, since $f(0)$ doesn't equal $f(2)$. But there is no $x \in (0, 2)$ (or anywhere in $\mathbb{R}$, for that matter) for which $f'(x) = \frac{1}{2}$, even though $h(x)$ is continuous everywhere! Is this a contradiction to the Mean Value Theorem?

7. Use the $h \to 0$ limit definition of the derivative and some fun facts about $\frac{\sin h}{h}$ to prove that $\frac{d}{dx} \sin(2x) = 2\cos(2x)$.

8. Continuity and uniform continuity

   (a) What is the core difference between continuity and uniform continuity? (How would a formal proof of uniform continuity differ from a formal proof of continuity?)

   (b) Using the definition of uniform continuity, prove that $f(x) = x^2$ is uniformly continuous on the interval $(0, 1)$.

   (c) Prove that $f(x) = x^2$ is uniformly continuous on the interval $[0, 1]$. You may assume that x^2 is continuous on this interval.

Math 23a Practice Multiple Choice Questions

1. Which of these functions is **not** uniformly continuous on $(0,1)$?
   (a) x^2
   (b) $1/x^2$
   (c) $f(x) = 1$ for $x \in (0,1)$, $f(0) = f(1) = 0$
   (d) $\sin(x)$
   (e) $\frac{\sin(x)}{x}$

2. Let s_n be a sequence of real numbers on a bounded set S, where $\liminf s_n \neq \limsup s_n$. Which of the following is not necessarily true?
   (a) $\lim s_n$ does not exist.
   (b) s_n is not Cauchy.
   (c) $\liminf s_n < \limsup s_n$
   (d) There exists a convergent subsequence.
   (e) s_n has an infinite number of dominant terms.

3. Which of the following is not true about $s_n = \frac{1}{n}$?
   (a) The sequence converges to 0.
   (b) $\lim_{n\to\infty} \displaystyle\sum_{i=1}^{n} s_i = L$, for some finite L.
   (c) $\limsup s_n = 0$.
   (d) The series $\sum(-1)^n s_n$ converges.
   (e) The series $\sum s_n^2$ converges.

4. Let $\sum a_n$ be a conditionally convergent series. Which of the following is not necessarily true?
   (a) The series converges to some finite L.
   (b) The series sum is independent of order of terms.
   (c) $\displaystyle\sum |a_n|$ diverges.
   (d) $\lim(-1)^n a_n = 0$.
   (e) None of the above. They're all necessarily true.

5. Which of the following series converges? **THERE ARE TWO ANSWERS**

   (a) $\sum \frac{x^n}{n!}$, $\forall x$
   (b) $\sum \frac{1}{n+\sin(n)}$
   (c) $\sum(-1)^n n$
   (d) $\sum \sin(n)$
   (e) $\sum \frac{2^n}{\sqrt{n!}}$

6. Which of the following must be true of a continuous function on (a, b)?

   (a) The function achieves its maximum on (a, b).
   (b) The function is bounded.
   (c) For all Cauchy Sequences s_n on the set (a, b), $f(s_n)$ is also Cauchy.
   (d) If $f(a) = 2$, and $f(b) = 5$, then $f(c) = 3$, for some $c \in (a, b)$.
   (e) None of the above.

7. Which of the following is not necessarily true about a uniformly continuous function, f, on $[a, b]$? **THERE ARE TWO ANSWERS**

   (a) The function is bounded.
   (b) The function achieves its maximum on the set (a, b).
   (c) If $f(a) = 4$ and $f(b) = 6$, then $f'(c) = 2$ for some $c \in (a, b)$.
   (d) The derivative f' is bounded.
   (e) If $f'(a) = 3$, and $f'(b) = 4$, then $f'(c) = 3.5$ for some $c \in (a, b)$.

8. Find $\lim_{x\to b} \frac{\sqrt{x}-\sqrt{b}}{x-b}$ for $b > 0$.

   (a) ∞
   (b) $\frac{1}{2\sqrt{b}}$
   (c) 0
   (d) $2\sqrt{b}$
   (e) b

9. Let f be a differentiable function, where all derivatives exist, such that $f(0) = 0$, $f'(0) = 0$, and $|f''(x)| \leq M, \forall x$. Which of the following is not necessarily true?

   (a) $f(1) \leq \frac{M}{2}$
   (b) 0 is neither a maximum nor a minimum.
   (c) $\forall \epsilon > 0, \exists \delta > 0$ s.t. if $x \in (-\delta, \delta), |f(x)| < \epsilon$
   (d) If $\lim s_n = 0$, then $\lim f(s_n) = 0$.
   (e) None of the above.

Math 23a Practice Multiple Choice Questions

1. Which of these functions is **not** uniformly continuous on $(0, 1)$?

   (a) x^2

   (b) $1/x^2$

   (c) $f(x) = 1$ for $x \in (0, 1), f(0) = f(1) = 0$

   (d) $\sin(x)$

   (e) $\frac{\sin(x)}{x}$

   Solution: $\boxed{B}$. x^2 is uniformly continuous, because you can extend it so that $f(0) = 0$ and $f(1) = 1$, and then it's continuous on a closed interval, so it is uniformly continuous. The same extension could work for answer choice c, if we let $f(0) = f(1) = 1$. It does not matter that its discontinuous at the end points. Both parts d and e can also be extended to continuous functions on the closed interval. In particular, for part e) it is important to note that $\lim_{x \to 0} \sin(x)/x = 1$. The main issue with answer choice B is that it is unbounded on the open interval! Therefore, it cannot be uniformly continuous.

2. Let s_n be a sequence of real numbers on a bounded set S, where $\liminf s_n \neq \limsup s_n$. Which of the following is not necessarily true?

   (a) $\lim s_n$ does not exist.

   (b) s_n is not Cauchy.

   (c) $\liminf s_n < \limsup s_n$

   (d) There exists a convergent subsequence.

   (e) s_n has an infinite number of dominant terms.

   Solution: $\boxed{E}$ a) is true, because in order for a limit to exist, the lim sup must equal the lim inf, and it turns out that that is equal to the limit. Because the sequence can't converge, choice b) is also out, since all Cauchy sequences of real numbers converge. For choice c), notice that in general $\liminf s_n \leq \limsup s_n$, and we have excluded the equal case in the prompt of the question. For choice d), the Bolzano Weierstrass Theorem still holds, since the sequence is defined on a bounded set. e) is not necessarily true. An explicit counterexample is $s_n = 1 - \frac{1}{n}$ for even n, and $s_n = 0$ for odd n.

3. Which of the following is not true about $s_n = \frac{1}{n}$?

   (a) The sequence converges to 0.

   (b) $\lim_{n\to\infty} \sum_{i=1}^{n} s_i = L$, for some finite L.

   (c) $\limsup s_n = 0$.

   (d) The series $\sum(-1)^n s_n$ converges.

   (e) The series $\sum s_n^2$ converges.

   Solution: $\boxed{B}$ This series is known as the harmonic series, which diverges. Choice b) is simply stating the definition of a series sum as the limit of the partial sums. a) is true, since when n goes to infinity, $1/n$ goes to zero. c) is true, since if the limit exists, then $\limsup s_n = \liminf s_n = \lim s_n$. Choice d) is true, because then, we are considering an alternating series, which has a less stringent convergence condition, namely that $\lim |s_n| = 0$, which is satisfied. Choice e) is true, because $1/n^2$ is a convergent p series.

4. Let $\sum a_n$ be a conditionally convergent series. Which of the following is not necessarily true?

   (a) The series converges to some finite L.

   (b) The series sum is independent of order of terms.

   (c) $\sum |a_n|$ diverges.

   (d) $\lim(-1)^n a_n = 0$.

   (e) None of the above. They're all necessarily true.

   Solution: $\boxed{B}$ Part a) is true by definition! If it converges, it converges to something. This does not contradict part b), since the order of terms in a conditionally convergent series needs to be respected! The order of terms as written defines the series sum, despite other sums being possible if the terms are moved around. Part c) is false, because, if $|a_n|$ converges, then the series would be absolutely convergent, not conditionally convergent. Part d) is true, since that is exactly the statement of the alternating series test.

5. Which of the following series converges?

   (a) $\sum \frac{x^n}{n!}$, $\forall x$
   (b) $\sum \frac{1}{n+\sin(n)}$
   (c) $\sum (-1)^n n$
   (d) $\sum \sin(n)$
   (e) $\sum \frac{2^n}{\sqrt{n!}}$

   Solution: $\boxed{A, E}$ Oops, I put two convergent series! Choice a) converges. This can be done with the ratio test, or with the recognition, that a) is the Taylor series for e^x, which is valid everywhere. Choice b) diverges, because $\frac{1}{n+\sin(n)} \geq \frac{1}{n+1}$, which diverges, since that is just the harmonic series with a relabeling. Choice c) does not converge to any particular limit, as the terms go as: $-1, 2, -3, \ldots.$ For choice d), note that $\lim \sin(n) \neq 0$, so it can't converge. Choice e) converges by the ratio test.

6. Which of the following must be true of a continuous function on (a, b)?

   (a) The function achieves its maximum on (a, b).
   (b) The function is bounded.
   (c) For all Cauchy Sequences s_n on the set (a, b), $f(s_n)$ is also Cauchy.
   (d) If $f(a) = 2$, and $f(b) = 5$, then $f(c) = 3$, for some $c \in (a, b)$.
   (e) None of the above.

   Solution: $\boxed{E}$ None of these are true! Choice a) is not true, by a counterexample. If $a = 0$ and $b = 1$, then $1/x$ is continuous on $(0, 1)$, but does not achieve its maximum. The key here is that the statement only guarantees continuity on the open interval. Part b) is false by the counterexample above. Part c) is also true by the counterexample above. Let $\lim s_n = 0$, and then $f(s_n)$ will diverge, and not be Cauchy. d) is not true, because it doesn't need to be continuous at the end points. The function $f(x) = 5$ except at a, where $f(a) = 2$, is a counterexample.

7. Which of the following is not necessarily true about a uniformly continuous function, f, on $[a, b]$?

   (a) The function is bounded.

   (b) The function achieves its maximum on the set (a, b).

   (c) If $f(a) = 4$ and $f(b) = 6$, then $f'(c) = 2$ for some $c \in (a, b)$.

   (d) The derivative f' is bounded.

   (e) If $f'(a) = 3$, and $f'(b) = 4$, then $f'(c) = 3.5$ for some $c \in (a, b)$.

   Solution: $\boxed{C, E}$ Oops, two of these statements are false. Part a) is true, because any continuous function on a closed interval is bounded. Part b) is also true of any continuous function on a closed interval. Part c) is not necessarily true, since the continuity doesn't necessarily imply differentiability. Part d) is true about uniformly continuous functions. The unboundedness of the derivative is what prevents finding the one size fits all δ for all x. Part e) is not necessarily true, because, we do not know that the derivative is continuous!

8. Find $\lim_{x \to b} \frac{\sqrt{x}-\sqrt{b}}{x-b}$ for $b > 0$.

   (a) ∞

   (b) $\frac{1}{2\sqrt{b}}$

   (c) 0

   (d) $2\sqrt{b}$

   (e) b

   Solution: $\boxed{B}$ If you multiply both top and bottom by $\sqrt{x} + \sqrt{b}$, the numerator becomes $x - b$, and cancels the denominator. Taking the limit results in choice b).

9. Let f be a differentiable function, where all derivatives exist, such that $f(0) = 0$, $f'(0) = 0$, and $|f''(x)| \leq M, \forall x$. Which of the following is not necessarily true?

(a) $f(1) \leq \frac{M}{2}$

(b) 0 is neither a maximum nor a minimum.

(c) $\forall \epsilon > 0, \exists \delta > 0$ s.t. if $x \in (-\delta, \delta), |f(x)| < \epsilon$

(d) If $\lim s_n = 0$, then $\lim f(s_n) = 0$.

(e) None of the above.

Solution: $\boxed{B}$ Choice a) is true, since that is just the statement of Taylor's Theorem of Remainder. It says that

$$f(x) = f(0) + f'(0)x + \frac{f''(c)x^2}{2!} \text{ where } c \in [0, x]$$

Notice that $f(0) = f'(0) = 0$, and that the derivative is bounded by M. Then, this states:

$$\begin{aligned} f(1) &\leq 0 + 0(1) + \frac{M(1)^2}{2!} \\ &\leq \frac{M}{2} \end{aligned}$$

Choice b) is false by counterexample. Let $f(x) = x^2$, then 0 is a minimum, and the second derivative is bounded by 2. Choice c) is true, since it is just the statement of continuity at $x = 0$, and being differentiable implies continuity.

MATHEMATICS E-23a, Fall 2017
Quiz #2 Practice Questions
November 2017

These questions were written by the course assistants in Fall 2015. They are quite skillfully done.

A handwritten document that also includes the answers is in the file Quiz 2 Review.pdf.

1. (Inspired by Week 5, group problems #1)

   (a) Starting from the triangle inequality $|a+b| \le |a| + |b|$, show that

   $$|a| - |b| \le |a - b|.$$

   (b) Using induction, show that:

   $$|a| - \sum_{i=1}^{n} |b_i| \le |a - \sum_{i=1}^{n} b_i|.$$

2. (Inspired by Week 5, group problems #2)
   Given $\lim s_n = s$ and $\lim t_n = t$ (and $t_n \neq 0 \; \forall n$ and $t > 0$), show that

$$\lim \frac{s_n}{t_n} = \frac{s}{t}.$$

3. (Inspired by Week 5, group problems #3)
   Let $s_1 = 1$ and for $n \geq 1$ let $s_{n+1} = \sqrt{s_n + 1}$.

   Given that $\lim s_n = s$, prove that

$$s = \frac{1}{2}(1 + \sqrt{5}).$$

4. (Inspired by Week 6, group problems #1)

   (a) Show that $\liminf(s_n + t_n) \geq \liminf s_n + \liminf t_n$ for bounded sequences s_n and t_n.

   (b) Invent an example where $\liminf(s_n + t_n) > \liminf s_n + \liminf t_n$.

5. (Inspired by Week 6, group problems #2)
   Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n \geq 1$.

   (a) Use induction to show that $s_n > \frac{1}{2}\ \forall n$.

   (b) Show that s_n is a decreasing sequence.

   (c) Show that $\lim s_n$ exists and find $\lim s_n = s$.

6. (Inspired by Week 6, group problems #3)
   Fine the radius of convergence R and the exact interval of convergence of the series

$$\sum x^{n!}.$$

7. (Inspired by Week 7, group problems #1)
   Prove that if f and g are real-valued functions that are continuous at $x_0 \in \mathbb{R}$, then fg is continuous at x_0 by

   (a) ϵ/δ definition of continuity.

   (b) "no bad sequence" definition of continuity.

8. (Inspired by Week 7, group problems #2)
   Show that $\sin x = \cos x$ for some $x \in (0, \frac{\pi}{2})$.

9. (Inspired by Week 7, group problems #3)
   Evaluate the following limit without using L'Hospital's Rule, then check using L'Hospital's Rule:

$$\lim_{x \to 0} \frac{\cos 2x - \cos x}{x^2}.$$

   You may use $\lim_{x\to 0} \frac{\sin x}{x} = 1$; $\cos 2x = 1 - 2\sin^2 x$;$\sin 2x = 2 \sin x \cos x$.

10. (Inspired by Week 8, group problems #1)
Let
$$f(x) = x^{\frac{3}{4}}.$$
Find the derivative $f'(x)$

(a) using the definition of the derivative as a limit.

(b) by rising both sides to the 4th power and using the chain rule.

11. (Inspired by Week 8, group problems #2)
Let $g(y) = \arccos y^2$.

Find $g'(y)$ by finding and differentiating the inverse function $y = f(x)$.

You can check your answer by using the chain rule and the derivative of the arccos function.

12. (Inspired by Week 8, group problems #3)
    Construct the Taylor series for the function $f(x) = \ln(1 + x)$, and use Taylor's theorem with remiander to show that the series converges to the function for $x \leq 1$.

Week 5.1

a) Starting from the triangle inequality $|a+b| \leq |a| + |b|$, show that: $|a| - |b| \leq |a-b|$

b) Using induction, show that: $|a| - \sum_{i=1}^{n} |b_i| \leq |a - \sum_{i=1}^{n} b_i|$

~~$a, b \in \mathbb{N}$ $\exists n \in \mathbb{N}$ s.t. $an > b$~~

a) $|a| = |a|$ $\qquad$ $|a| \leq |a-b| + |b|$

$= |a - b + b|$

$|a| \leq |a-b| + |b|$

$|a| - |b| \leq |a-b|$

b) $|a| - \sum_{i=1}^{k} |b_i| \leq |a - \sum_{i=1}^{k} b_i|$

$$|a| - \underbrace{\sum_{i=1}^{k} |b_i| - |b_{k+1}|}_{= \sum_{i=1}^{k+1} |b_i|} \leq |a - \sum_{i=1}^{k} b_i| - |b_{k+1}| \leq |a - \underbrace{\sum_{i=1}^{k} b_i - b_{k+1}}_{= \sum_{i=1}^{k} b_i}| \quad \checkmark$$

Week 5.2

Given $\lim s_n = s$ and $\lim t_n = t$ (and $t_n \neq 0 \ \forall n$ and $t > 0$), show that $\lim s_n/t_n = s/t$.

5.2 $\lim \frac{1}{t_n} = \frac{1}{t}$ $\qquad$ $\lim (s_n a_n) = sa$, $\quad a_n = \frac{1}{t_n}$; $a = \frac{1}{t}$

$\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n > N$, $|s_n a_n - sa| < \varepsilon$

$|s_n a_n - s_n a + s_n a - sa| = |s_n(a_n - a) + (s_n - s)a|$ $\quad$ Δ ineq.

$\leq |s_n||a_n - a| + |s_n - s||a|$ $\quad$ C-S

$M_1 \qquad \frac{\varepsilon}{2M_2} \qquad M_2$

$\exists M_1 > 0$ s.t. $\forall n \in \mathbb{N}$, $|s_n| < M_1$

$\exists M_2 > 0$ s.t. $\forall n \in \mathbb{N}$, $|a_n| < M_2$

$\forall \varepsilon > 0$, $\exists N_1 \in \mathbb{N}$ s.t. $\forall n > N_1$, $|s_n - s| < \varepsilon/2M_2$

$\forall \varepsilon > 0$, $\exists N_2 \in \mathbb{N}$ s.t. $\forall n > N_2$, $|a_n - a| < \varepsilon/2M_1$

$|s_n a_n - sa| < M_1 \cdot \frac{\varepsilon}{2M_1} + \frac{\varepsilon}{2M_2} \cdot M_2$ $\qquad$ $\lim s_n a_n = sa$

$< \varepsilon/2 + \varepsilon/2$

$< \varepsilon$ $\checkmark$

Week 5.3

Let $s_1 = 1$ and for $n \geq 1$ let $s_{n+1} = \sqrt{s_n + 1}$. Given that $\lim s_n = s$, prove that $s = \frac{1}{2}(1+\sqrt{5})$.

$$\lim s_n = \lim s_{n+1} = s$$

$$\lim s_n = \lim \sqrt{s_n + 1}$$

$$= \sqrt{\lim(s_n) + 1}$$

$$\lim s_n = \sqrt{s+1} = s,$$

$$s+1 = s^2$$

$$0 = s^2 - s - 1$$

$$s = \frac{1 \pm \sqrt{1-4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}$$

$\frac{1-\sqrt{5}}{2}$ ext.

$\frac{1}{2}(1+\sqrt{5}) = s$

Week 6.1

a) Show that $\liminf(s_n + t_n) \geq \liminf(s_n) + \liminf(t_n)$ for bounded sequences s_n and t_n.

b) Invent an example where $\liminf(s_n + t_n) > \liminf(s_n) + \liminf(t_n)$

a) $\inf(s_n + t_n) \geq \inf(s_n) + \inf(t_n)$

$$\lim_{N\to\infty} \inf\{(s_n + t_n) : n > N\} \geq \lim_{N\to\infty} \inf\{s_n : n > N\} + \lim_{N\to\infty} \inf\{t_n : n > N\}$$

$$\liminf(s_n + t_n) \geq \liminf(s_n) + \liminf(t_n) \checkmark$$

b) $s_n = \{2, 1, 2, 1, \ldots\}$ $\liminf s_n = 1$

$t_n = \{1, 2, 1, 2, \ldots\}$ $\liminf t_n = 1$

$s_n + t_n = \{3, 3, 3, 3, \ldots\}$ $\liminf(s_n + t_n) = 3$

$1 + 1 < 3$

Week 6.2

Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n+1)$ for $n > 1$.

a) Use induction to show $s_n > 1/2 \ \forall n$.

b) Show that s_n is a decreasing sequence.

c) Show that $\lim s_n$ exists and find $\lim s_n = s$.

a) $s_1 = 1 > 1/2$

$s_2 = \frac{1}{3}(1+1) = 2/3 > 1/2$

assume: $s_{n-1} > 1/2$; wts: $s_n > 1/2$

$s_n = \frac{1}{3}(s_{n-1}+1) > \frac{1}{3}(\frac{1}{2}+1)$ $s_{n-1} > 1/2$

$> \frac{1}{\cancel{3}} \frac{\cancel{3}}{2}$

$> 1/2$ ✓

b) wts: $s_{n+1} < s_n \ \forall n$

$s_n - s_{n+1} > 0 \ \forall n$

$s_n - s_{n+1} = s_n - \frac{1}{3}(s_n+1)$

$= \frac{2}{3}s_n - \frac{1}{3}$

$> \frac{\cancel{2}}{3} \cdot \frac{1}{\cancel{2}} - 1/3$ since $s_n > 1/2$

> 0

c) s_n bounded above/below $\Rightarrow$ converges

$s_n > 1/2 \ \forall n$) s_n decreasing

$\lim s_{n+1} = \lim s_n = s$

$s_{n+1} = \frac{1}{3}(s_n+1)$

$\lim (s_{n+1}) = \frac{1}{3}(\lim(s_n)+1)$

$s = \frac{1}{3}(s+1)$

$\boxed{s = 1/2}$

Week 6.3

Find radius of convergence R and exact interval of convergence of the series: $\sum x^{n!}$ *ratio test

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{(n+1)!}}{x^{n!}}\right| = \left|\frac{x^{n!(n+1)}}{x^{n!}}\right| = \left|\left(\frac{x^{(n+1)}}{x}\right)^{n!}\right| = \left|\left(\frac{\cancel{x}\cdot x^n}{\cancel{x}}\right)^{n!}\right| = \left|(x^n)^{n!}\right| \underset{=?}{\leq} 1$$

$(x^n)^{n!} \geq x^n$ for $x \geq 1$, $\sum x^n$ diverges for $|x| \geq 1$

$(x^n)^{n!} \leq x^n$ for $x \leq 1$, $\sum x^n$ converges for $x < 1$ $R = 1$

$((-1)^n)^{n!} = ((-1)^n)^{n!} \Rightarrow \sum((-1)^n)^{n!}$ does not converge $x \neq +1$

$n!$ is even for $n > 2$

so $\sum((-1)^n)^{n!} = (-1)^{1\cdot 1} + (-1)^{2\cdot 2} + (1)^{\text{even pwr}}$

$(-1, 1)$

~~f cont. (a,b) $a<b$; y between $f(a)$ and $f(b)$ $\exists x \in (a,b)$ s.t. $f(x)=y$~~

~~$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|x-x_0| < \delta \Rightarrow |f(x)-f(x_0)| < \varepsilon$~~

~~iff $\forall x_n$ s.t. $\lim x_n = x_0$, $\lim f(x_n) = f(x_0)$~~

Week 7.2

Show that $\sin(x) = \cos(x)$ for some $x \in (0, \pi/2)$

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x, y \in S, |x-y| < \delta \Rightarrow |f(x)-f(y)| < \varepsilon$

$f(x) = \sin(x) - \cos(x)$ $c \in (0, \pi/2)$

$f(0) = \cancel{\sin(0)}^{0} - \cancel{\cos(0)}^{1}$

$f(\pi/2) = \cancel{\sin(\pi/2)}^{1} - \cancel{\cos(\pi/2)}^{0}$

$f(\pi/2) = 1$

$f(0) = -1$

$\exists c \in (0, \pi/2)$ s.t. $f(c) = 0 \Rightarrow \sin(c) = \cos(c)$ ✓

Week 7.1

Prove that if f and g are real-valued functions that are continuous at $x_0 \in \mathbb{R}$, then fg is continuous at x_0 by:

a) ε/δ definition of continuity

b) "no bad sequence" definition of continuity

$$\frac{x\cancel{k} + h^2}{k}$$

a) wts: $|x - x_0| < \delta \Rightarrow |f(x)g(x) - f(x_0)g(x_0)| < \varepsilon$

$$|f(x)g(x) - f(x_0)g(x_0)| \le |f(x)g(x) - f(x)g(x_0)| + |f(x)g(x_0) - f(x_0)g(x_0)|$$
$$\le |f(x)|\,|g(x) - g(x_0)| + |g(x_0)|\,|f(x) - f(x_0)|$$

$f(x)$ cont: $\forall \varepsilon > 0, \exists \delta_1 > 0$ s.t. $|x - x_0| < \delta_1 \Rightarrow |f(x) - f(x_0)| < \frac{\varepsilon}{2|g(x_0)|}$

$g(x)$ cont: $\forall \varepsilon > 0, \exists \delta_2 > 0$, s.t. $|x - x_0| < \delta_2 \Rightarrow |g(x) - g(x_0)| < \varepsilon/2M$

we know $|f(x)|$ is bounded within ε of $f(x_0)$: $|f(x)| < M$ for $|x - x_0| < \delta$

Choose $\delta = \min(\delta_1, \delta_2)$: $|f(x)g(x) - f(x_0)g(x_0)| < M \cdot \frac{\varepsilon}{2M} + \cancel{|g(x_0)|} \cdot \frac{\varepsilon}{2\cancel{|g(x_0)|}}$

$< \varepsilon/2 + \varepsilon/2$

$< \varepsilon$

$|g(x_0)| + 1$

b) f cont. $\lim x_n = x_0 \Rightarrow \lim f(x_n) = f(x_0)$

g cont. $\lim x_n = x_0 \Rightarrow \lim g(x_n) = g(x_0)$ $\Big\}\ \forall x_n \to x_0$

$\lim(f(x_n) \cdot g(x_n)) = \lim f(x_n) \lim g(x_n) = f(x_0)g(x_0)$ ✓

$x_n \to x_0 \Rightarrow f(x_n)g(x_n) \to f(x_0)g(x_0)$

Week 7.3

Evaluate the following limit without using L'Hôpital's Rule, ~~[illegible]~~ then check using L'Hôpital's rule:

$$\lim_{x\to 0} \frac{\cos(2x) - \cos(x)}{x^2}$$

You may use: $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$; $\cos(2x) = 1 - 2\sin^2(x)$; $\sin(2x) = 2\sin(x)\cos(x)$

$$\frac{\cos(2x) - \cos(x)}{x^2} = \frac{1 - 2\sin^2(x) - \cos(x)}{x^2} = \frac{-2\sin^2(x)}{x^2} + \frac{1 - \cos(x)}{x^2}$$

$\sin^2 + \cos^2 = 1$

$$\frac{(1 - \cos(x))}{x^2}\frac{(1 + \cos(x))}{(1 + \cos(x))} = \frac{1 - \cos^2(x)}{x^2(1 + \cos(x))} = \frac{\sin^2(x)}{x^2} \cdot \frac{1}{1 + \cos(x)}$$

$\lim a^2 = (\lim a)^2$

$$\lim_{x\to 0} \frac{\cos(2x) - \cos(x)}{x^2} = \lim_{x\to 0}\left[-\frac{2\sin^2(x)}{x^2} + \frac{\sin^2(x)}{x^2} \cdot \frac{1}{1 + \cos(x)}\right]$$

$$= -2 \cdot \left(\lim_{x\to 0} \frac{\sin(x)}{x}\right)^2 + \left(\lim_{x\to 0} \frac{\sin(x)}{x}\right)^2 \cdot \lim_{x\to 0}\left(\frac{1}{1 + \cos(x)}\right)$$

$$= -2(1)^2 + 1^2 \cdot \frac{1}{1+1} = -1.5$$

$$\lim_{x\to 0} \frac{\cos(2x) - \cos(x)}{x^2} = \lim_{x\to 0} \frac{-\sin(2x) \cdot 2 + \sin(x)}{2x} = \lim_{x\to 0} \frac{-2\cos(2x) \cdot 2 + \cos(x)}{2}$$

$$= \frac{-2(1) \cdot 2 + 1}{2} = -\frac{3}{2} = -1.5 \checkmark$$

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \qquad \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{d}{dx} x^{3/4} = \frac{3}{4} x^{-1/4}$$

Find the derivative with a) limit definition and b) chain rule

a) $$\lim_{h \to 0} \frac{(x+h)^{3/4} - x^{3/4}}{h} \cdot \frac{(x+h)^{3/4} + x^{3/4}}{(x+h)^{3/4} + x^{3/4}}$$

$$\lim_{h \to 0} \frac{(x+h)^{3/2} - x^{3/2}}{h\left((x+h)^{3/4} + x^{3/4}\right)} \cdot \frac{(x+h)^{3/2} + x^{3/2}}{(x+h)^{3/2} + x^{3/2}}$$

$$\lim_{h \to 0} \frac{(x+h)^3 - x^3}{h\left((x+h)^{3/4} + x^{3/4}\right)} \cdot \frac{1}{(x+h)^{3/2} + x^{3/2}}$$

$$\lim_{h \to 0} \frac{\cancel{x^3} + 3x^2\cancel{h} + 3xh^{\cancel{2}} + h^{\cancel{3}\,2} - \cancel{x^3}}{\cancel{h}}$$

$$\frac{3x^2}{2 \cdot x^{3/4} \cdot 2x^{3/2}} = \frac{3}{4} x^{-1/4}$$

b) $$f(x) = x^{3/4}$$

$$f(x)^4 = x^3$$

$$4 \cdot f(x)^3 \cdot f'(x) = 3x^2$$

$$f'(x) = \frac{3}{4} \frac{x^2}{f(x)^3} = \frac{3}{4} \cdot \frac{x^2}{x^{9/4}} = \frac{3}{4} \cdot x^{-1/4}$$

Find $g'(y)$ if:

$g(y) = \arccos y^2 \qquad x = g(y)$

&c

$x = \arccos y^2$

$f(x) = x$

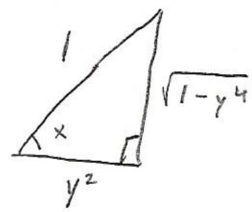
$\cos x = y^2$

$f(x) = x+3$

$f(x) = y = \sqrt{\cos x}$

$f'(x) = \frac{1}{2}(\cos x)^{-1/2} \cdot (-\sin x)$

$g'(y) = \frac{1}{f'(x)} = \frac{-2\sqrt{\cos x}}{\sin x} \qquad x = \arccos y^2$



$= \frac{-2\sqrt{y^2}}{\sqrt{1-y^4}} = \frac{-2y}{\sqrt{1-y^4}}$

$f(x) = f(0) + f(0)'x + \dots \frac{f^n(0)}{n!}x^n$

$f(x) - f(0) - f'(0)x \cancel{+} \dots \frac{f^{n-1}(0)}{(n-1)!}x^{n-1} = \frac{f^n(y)}{n!}x^n$

See video for explanation of Taylor's theorem with remainder

$y \in (0, x)$

$\log(1+x) = \ln(1+x)$

Show that for $\ln(1+x)$ remainder goes to 0 at $x=1$

$(\ln(x))' = \frac{1}{x}$ $\qquad \frac{1}{1+x}, \frac{-1}{(1+x)^2}, \frac{2}{(1+x)^3}, \dots \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$

$f^n(y) \quad \left(\frac{(-1)^{n-1}\cancel{(n-1)!}}{(1+y)^n}\right) \cdot \frac{1}{\cancel{n!}} \cdot \cancel{x^n} \qquad 1^n = 1$

n

$\lim_{n\to\infty} \frac{(-1)^{n-1}}{n} \cdot \left(\frac{1}{1+y}\right)^n \qquad y \in (0, x)$

$y \in (0, 1)$

$0 \cdot 0 = 0$

$S(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$, $C(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$

Gives $S'(x) = C(x)$ $C'(x) = -S(x)$

w.t.s. $S^2(x) + C^2(x) = 1$

$(S^2(x) + C^2(x))'$

$(S^2(x))' + (C^2(x))' =$

$= 2\, S(x)\, S'(x) + 2\, C(x)\, C'(x)$

$= 2\, S(x)\, C(x) + 2\, C(x)\, (-S(x))$

$= 0$

$S(0) = 0$ $C(0) = 1$

$1^2 + 0^2 = \cancel{10} \; 1$ HAHAHA Sebastian is sooooo funny

---

$f(x)$ twice. dif. , $f'' < 0$ (a,b) some $x \in (a,b)$ $f'(x) = 0$

$\forall y \in (x,b)$ $f(y) < f(x)$

$$f'(c) = \frac{f(y) - f(x)}{y - x} \qquad c \in (x,y) \in (x,b)$$

$$\overset{\text{negative}}{f''(d)} = \frac{f'(c) - f'(x)}{\underbrace{c - x}_{\text{positive}}}$$

negative $= f'(c) \cancel{- f'(x)}$

$$\overset{\text{negative}}{f'(c)} = \frac{f(y) - f(x)}{\underbrace{y - x}_{\text{positive}}}$$

negative $= f(y) - f(x)$

$f(x) > f(y)$

With thanks to Kate Penner for her Fall 2015 review session.

Week 9

1. Axioms of topology
   - Empty set and closed set are open.
   - Finite/infinite union of open sets is open.
   - Finite intersection of open sets is open.
2. Website topology
3. Prove axioms of topology from definitions
4. Vocabulary related to sets
   - Open set
   - Closed set
   - Boundary
   - Closure
   - Interior
5. Redefining convergence topologically in $\mathbb{R}^n$
6. Hausdorff space (Proof 9.1: $\mathbb{R}$ is Hausdorff)
7. Prove convergence in $\mathbb{R}^n$
8. Proof 9.2: Infinite triangle inequality
9. Solve differential equations using diagonalization

Week 10

1. Limit of a sequence in $\mathbb{R}$
2. Limit of a function $f : \mathbb{R}^m \to \mathbb{R}^n$
3. *Skill:* Show a limit exists or does not exist
   - Exists: try using polar coordinates (only depends on θ)
   - DNE: depends on angle of approach (θ in polar coordinates), or two paths disagree
4. Continuity in $\mathbb{R}^n$ (Proof 10.1: f is continuous iff every sequence $\vec{x}_n \to \vec{x}_0$ is good)
5. Compact sets: closed and bounded
6. Bounded: wholly contained within a ball centered at the origin
7. Bolzano-Weierstrass: on a compact set, any sequence has a convergent subsequence
8. Proof 10.2: A continuous function on a compact set has and achieves its supremum

9. Nested Compact Set Theorem: If you have a decreasing sequence of compact sets $U_1 \supseteq U_2 \supseteq U_3 \cdots$, the infinite intersection is **not** empty

10. Heine-Borel: On a compact set, any open cover has a finite subcover

11. Directional derivatives
$$\nabla_{\vec{v}} f(\vec{a}) = \lim_{h \to 0} \frac{f(\vec{a} + h\vec{v}) - f(\vec{a})}{h}$$

12. Partial derivatives

13. Gradient vector: column of partial derivatives

14. Jacobian matrix ($f : \mathbb{R}^n \to \mathbb{R}$):
$$\begin{bmatrix} D_1 f & \cdots & D_n f \end{bmatrix}$$

15. : Linear approximation: $f(\vec{a} + h\vec{v}) = f(\vec{a}) + [Jf(\vec{a})](h\vec{v})$

Week 11

1. Use remainder $\to 0$ to prove a derivative (Proof 11.1: product rule)

2. Prove differentiability with remainder technique

3. Proof 11.2: Derivative = Jacobian

4. When Jacobian exists but function is not differentiable (Not differentiable: linearity of the derivative breaks down)

5. Matrix derivatives with the chain rule (e.g. inverse and squaring functions)

6. Newton's Method (use and relation to tangent line approximation)

7. Inverse function theorem: if f is strictly increasing/decreasing, there exists a local inverse g
$$g'(y_0) = \frac{1}{f'(g(y_0))} \quad \to \quad [Dg(\vec{y})] = [Df(g(\vec{y}))]^{-1}$$

8. Uses of the inverse function theorem

Week 12

1. Implicit function theorem and use

2. Manifolds: particularly well-behaved smooth curves/surfaces in an arbitrary number of dimensions

3. 3 ways to describe manifolds
   - Graph
   - Locus function (F = 0)

- Parametrization

4. Smooth: locally the graph of a C^1 function (or, for locus functions, $[DF]$ is onto)
5. Parametrizations should be one-to-one and onto
6. Tangent **space**: $\dot{x} = [Dg(\vec{z})]\dot{y}$
7. Tangent **plane**: $\vec{x} - \vec{a} = [Dg(\vec{z})](\vec{y} - \vec{b})$
8. ker $[DF(\vec{c})]$ or img $[D\gamma]$ gives a basis for the tangent space
9. Unconstrained critical points: Find by setting all partials equal to zero, and classify by using the Hessian matrix
10. Constricted optimization
    - $\vec{c}$ is a critical point of F restricted to M iff$Df = \lambda_1 DF_1 + \lambda_2 DF_2 + ... + \lambda_k DF_k$
    - Apply Lagrange Multipliers
    - f is the function we want to maximize
    - F is our constraint (written as a locus), which forms a manifold
    - Or parametrize the manifold using a parametrization γ and consider critical points of $f(\gamma)$.

1 Multiple Choice and True/False

If a true-false statement is false, devise an explicit counterexample.

1. In a website consisting of six pages numbered 1 through 6, $\{12\}$ and $\{256\}$ are defined to be open. Using our standard website topology, which of the following sets is not necessarily open?
   (a) $\{123456\}$
   (b) $\emptyset$
   (c) $\{2\}$
   (d) $\{156\}$
   (e) $\{1256\}$
2. *True or false:* The intersection of any number, finite or infinite, of open sets is open.
3. *True or false:* From any sequence $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}, ...$ s.t. $\forall n\ \mathbf{x_n} \in \mathbb{R}^3$, we can extract a convergent subsequence.
4. M is a three-dimensional manifold in $\mathbb{R}^7$. Which of the following is a way to describe M? **Choose all that apply.**
   (a) a parametrization $\gamma : \mathbb{R}^4 \to \mathbb{R}^3$
   (b) a parametrization $\gamma : \mathbb{R}^3 \to \mathbb{R}^7$
   (c) a parametrization $\gamma : \mathbb{R}^7 \to \mathbb{R}^4$
   (d) the graph of a function $g : \mathbb{R}^3 \to \mathbb{R}^7$
   (e) the graph of a function $g : \mathbb{R}^3 \to \mathbb{R}^4$
   (f) the graph of a function $g : \mathbb{R}^4 \to \mathbb{R}^7$
   (g) the set of points at which a function $F : \mathbb{R}^7 \to \mathbb{R}^4$ is zero
   (h) the set of points at which a function $F : \mathbb{R}^7 \to \mathbb{R}^3$ is zero
   (i) the set of points at which a function $F : \mathbb{R}^4 \to \mathbb{R}^7$ is zero

2 Problems

1. To protect your beloved rose bushes, you decide to hire Heine-Borel Fence Construction Company to build a fence along the front of your yard, forming a one-dimensional fence from location $x = 0$, on the left edge of your lawn, to location $x = 1$, on the right edge of your lawn. However, your cantankerous neighbor forbids you from touching his lawn, so the fence is not allowed to actually reach $x = 1$. (The left side of your lawn buts up against your driveway, so you don't care if the fence actually reaches $x = 0$ or even goes slightly past.) Also, for whatever reason, Heine-Borel Fence Co. really likes building fences on open intervals.
   a. Heine-Borel Construction proposes the following construction worker scheme. Construction worker 0 will build a fence on the interval $(-0.01, 0.6)$. Then, for $k \geq 1$, worker k will build a fence on the interval $\left(1 - \frac{1}{2^k}, 1 - \frac{1}{2^{k+1}}\right)$. Does this proposed scheme result in an open cover of the interval $[0, 1)$? Why or why not?
   b. Heine-Borel Construction decides to offer you a second construction option. Construction worker 0 will build a fence on the interval $(-0.01, 0.6)$. For $k \geq 1$, guard k will cover the interval $\left(1 - \frac{1}{2^k}, 1 - \frac{1}{2^{k+2}}\right)$. Explain why this scheme would enable Heine-Borel Construction to charge you an infinite amount of money to build the fence, even if each construction worker receives a finite salary.

c. You would really rather not spend an infinite amount of money building this fence. Propose a way to modify *your instructions* to Heine-Borel Fence Co. so that you only have to hire a finite number of construction workers.

2. The following limit does not exist:

$$\lim_{\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}} \frac{x^2 - y^2}{x^2 + y^2}$$

a. Using polar coordinates, show that this limit does not exist.

b. Using the "no bad sequence" definition, show that this limit does not exist.

3. The differential equation $\ddot{x} + 4\dot{x} + 3x = 0$ probably represents something interesting in physics and economics. Find a matrix A such that

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = A \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

and then solve to determine $x(t)$ for initial conditions $\begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

4. Preparing for finals, the Harvard libraries have asked you to help them develop a new policy for library hours during finals period. They have realized that, in order to give their librarians enough rest, they need to close Widener and Lamont for a combined 20 hours. They want to minimize student frustration in the process. If Lamont is closed for x hours, student frustration will increase by $5x^2$. If Widener is closed for y hours, student frustration will increase by $10y$.

(a) Use a parameter t to parametrize the manifold $x + y = 20$. Then determine how to minimize the student frustration function $f = 5x^2 + 10y$ given this constraint.

(b) Use Lagrange multipliers to solve the same problem of minimizing the student frustration function $f = 5x^2 + 10y$ subject to the constraint $x + y = 20$.

5. Barnum and Bailey are planning to expand their Troupe of Remarkably Trained Pigs (which actually existed, apparently). Barnum trains his x pigs more efficiently than Bailey trains his y pigs, so the two agree that the number of pigs that each trains should increase as follows:

$$\dot{x} = 4x + 2y$$

$$\dot{y} = 2x + y$$

(a) Diagonalize the matrix $A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$ to express A as PDP^{-1}.

(b) At time 0, Barnum has 2 pigs and Bailey has 4. Exponentiate A to find formulas for x and y in terms of t.

6. The locus function

$$F \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x^2 + y^2 - z^2 \\ 2x + y + 4z + 8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

determines a one-dimensional manifold in $\mathbb{R}^3$. As it happens, the point $\vec{c} = \begin{pmatrix} 4 \\ 0 \\ -4 \end{pmatrix}$ lies on this manifold.

(a) Near this point, an implicit function g can express x and y in terms of z. Determine $g'(-4)$.

(b) Using $g'(-4)$, which you just found, approximate the x- and y-coordinates of a point lying on the manifold for which $z = -3.98$.

7. You want to guess the time t, measured in years from now, at which Barden College's endowment will reach a whopping \$90 megabucks. You happen to know that the endowment size is s given by $s(t) = t^4 - 9t^3 + 10$. You guess that Barden's endowment will reach \$90 megabucks in around 10 years from now. Use one iteration of Newton's Method on the equation $t^4 - 9t^3 + 10 = 90$ to refine your estimate.

Name: ______________________________

Section: ____________________________

MATHEMATICS 23a/E-23a, Fall 2018
Final Examination
Monday, December 17, 2018

You may omit one multiple-choice question in Part I, one question in Part III, and one proof in Part IV.

You may use a calculator, but only for arithmetic, perhaps in support of Newton's method.

No other aids or references are allowed.

Problem	Answer	Points	Score
$I-1$		2	
$I-2$		2	
$I-3$		2	
$I-4$		2	
$I-5$		2	
$I-6$		2	
II	$--$	6	
$III-1$	$--$	5	
$III-2$	$--$	5	
$III-3$	$--$	5	
$III-4$	$--$	5	
$III-5$	$--$	5	
$III-6$	$--$	5	
$III-7$	$--$	5	
$III-8$	$--$	5	
$IV-1$	$--$	5	
$IV-2$	$--$	5	
$IV-3$	$--$	5	
Marked omit(s)	$--$	1	
Total		61	

Part I. Answer five of the six multiple-choice questions. Transcribe your answers onto page 1, and mark an X in the score box on page 1 to indicate which question you have omitted.

If you answer all six questions, the last one will be ignored, and you will lose the extra-credit point for marking omitted questions!

1. Consider set $X = \{123456\}$ where a finite topology is being constructed. The sets $\{12\}$, $\{34\}$ and $\{15\}$ are open. What are all the open sets, if this is a valid topology?

   (a) $\{12\}, \{34\}, \{1234\}, \{15\}, \{1345\}, \{125\}, \{12345\}$

   (b) $\{12\}, \{34\}, \{1234\}, \{15\}, \{1345\}, \{125\}, \{12345\}, \phi, X$

   (c) $\{12\}, \{34\}, \{1234\}, \{15\}, \{1345\}, \{125\}, \{12345\}, \{1\}, \phi, X$

   (d) $\{12\}, \{34\}, \{1234\}, \{15\}, \{1345\}, \{125\}, \{12345\}, \{1\}, \{134\}, \phi, X$

   (e) $\{12\}, \{34\}, \{1234\}, \{15\}, \{1345\}, \{125\}, \{12345\}, \{1\}, \{134\}, \{2\}, \phi, X$

2. Which one of these functions is not continuous at the origin? For all these functions $f\begin{pmatrix}0\\0\end{pmatrix} = 0$.

   (a) $x^2y/(x^2 + y^2)$

   (b) $(x^2y + xy^2)/(x^2 + y^2)$

   (c) $xy^2/(x + y)$

   (d) $x^3y/(x^2 + y^2)^{\frac{3}{2}}$

   (e) $x/(x + y)$

3. A manifold M is described by a parameterization function $\gamma : \mathbb{R}^5 \to \mathbb{R}^{43}$. The locus function's domain and codomain are:

   (a) $F : \mathbb{R}^{43} \to \mathbb{R}^{38}$

   (b) $F : \mathbb{R}^{38} \to \mathbb{R}^5$

   (c) $F : \mathbb{R}^{43} \to \mathbb{R}^5$

   (d) $F : \mathbb{R}^5 \to \mathbb{R}^{38}$

   (e) $F : \mathbb{R}^{43} \to \mathbb{R}$

4. Which of these is true about a continuous function on an open set U?

   (a) There exists only one good sequence at all points on U

   (b) $\forall \mathbf{x_0} \in U, \forall \epsilon > 0, \exists \delta > 0$ s.t. $\forall \mathbf{x} \in U, |\mathbf{x} - \mathbf{x_0}| < \delta \implies |f(\mathbf{x}) - f(\mathbf{x_0})| < \epsilon$

   (c) $\forall \epsilon > 0, \exists \delta > 0$ s.t. $, \forall \mathbf{x}, \mathbf{x_0} \in U, |\mathbf{x} - \mathbf{x_0}| < \delta \implies |f(\mathbf{x}) - f(\mathbf{x_0})| < \epsilon$

   (d) The function is bounded on U

   (e) The function is differentiable on U

5. Which of these is not true about a function f that has continuous partial derivatives on an open set U?

   (a) The function achieves its maximum on the set U

   (b) The function is continuous on the set U

   (c) The function is C^1 on the set U

   (d) The derivative is a linear function of the direction

   (e) The directional derivative in the direction $\vec{v}$, where $\vec{v}$ is a unit vector, at point $\mathbf{c}$ is given by $[Df\mathbf{c}]\vec{v}$

6. What are all the critical points of the function $f\begin{pmatrix} y \\ x \end{pmatrix} = xy^2 + y^3 - 4x$? Note the change of order of the variables in the argument of the function!

   (a) The critical points are $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ and are both saddle points.

   (b) The critical points are $\begin{pmatrix} -2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and are both saddle points.

   (c) The critical points are $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ and are both maximums.

   (d) The critical points are $\begin{pmatrix} -2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and are both maximums.

   (e) The critical points are $\begin{pmatrix} -2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and are both minimums.

Part II(6 points, 2 per false statement) Of the following statements, more than three are false. Choose any three of the false statements and explain why they are false. For full credit you must both comment on what is wrong with the statement and also cite an explicit counterexample. Just ignore the true statements.

Example:

Statement: "Any two unequal nonzero vectors in $\mathbb{R}^2$ span $\mathbb{R}^2$."

Answer: "False: the vectors could be linearly dependent, like $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}2\\2\end{bmatrix}$"

1. For a compact set X, if there exists an infinite set of open sets, $\{U_i\}$, such that:

$$X \in \bigcup_{i=1}^{\infty} U_i$$

then there exists some finite subset of this set of sets, $\{V_i\} \subset \{U_i\}$, such that:

$$X \in \bigcup_{i=1}^{m} V_i$$

2. If $\mathbf{c}$ is a constrained critical point of function $f : \mathbb{R}^n \to \mathbb{R}$ on a k dimensional manifold M in $\mathbb{R}^n$ described by a locus function $\mathbf{F}$, then $\ker[Df(\mathbf{c})] \subset \ker[D\mathbf{F}(\mathbf{c})]$.

3. Given a k dimensional manifold M in $\mathbb{R}^n$, specified by locus function $\mathbf{F}(\mathbf{z})$, where $[D\mathbf{F}(\mathbf{z})]$ is onto, any ordering of the variables in $\mathbf{z}$ will result in the existence of an implicit function that expresses the first (first in order of variables of $\mathbf{z}$) $n-k$ passive variables in terms of the next k active variables.

4. A continuous real valued function defined on a compact set has its maximum on that set.

5. A function f is differentiable at $\mathbf{a}$ if:

$$\lim_{\vec{h}\to 0} \frac{1}{|\vec{h}|}(f(\mathbf{a}+\vec{h}) - f(\mathbf{a}) - [Df(\mathbf{a})]\vec{h}) = 0$$

6. If a real valued differentiable function f at a point $\mathbf{c}$ has directional derivatives on a set of basis vectors equal to zero, then $\mathbf{c}$ is a critical point of f.

7. Consider the sequence of open sets $X_1, X_2, X_3, ...$ where $X_1 \supset X_2 \supset X_3 \supset ....$

$$\bigcap_{k=1}^{\infty} X_k \neq \phi$$

8. When trying to solve a system of equations, that are set equal to zero, using Newton's method, any initial guess is guaranteed to eventually superconverge to one of the roots of the system.

Part III. Answer seven of the eight questions. Mark an X in the score box on page 1 to indicate which question you have omitted.

1. (Inspired by Week 9, group problems 3)

   Applying Newton's second law of motion to a mass of 1 attached to a spring of spring constant 4, and a damping force that is equal to 4 times the velocity leads us to the following differential equation

   $$\ddot{x} = -4\dot{x} - 4x$$

   Solve this differential equation for $x(t)$ given $x(0) = 4$, and $\dot{x}(0) = 0$. (Hint: To get started let $\vec{w} = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$, and write the differential equation in the form $\dot{\vec{w}} = A\vec{w}$)

2. (Inspired by Week 10, group problems 3)

(a) Let $f\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^2 - 1$. Evaluate the Jacobian Matrix at $f\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and use it to find the best affine approximation to $\begin{pmatrix} 1 \\ 0.1 \end{pmatrix}$.

(b) Let $g\begin{pmatrix} r \\ t \end{pmatrix} = \begin{pmatrix} r\cos t \\ r\sin t \end{pmatrix}$. Evaluate the Jacobian Matrix at $g\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and find r and h such that $g\begin{pmatrix} 1+r \\ h \end{pmatrix} = \begin{pmatrix} 1 \\ 0.1 \end{pmatrix}$.

3. (Inspired by Week 11, group problems 1)

Consider the functions $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2y \\ y + x^2 \\ 3x^2 + 2y \end{pmatrix}$ and $g\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} \sqrt{vw} \\ v + w \end{pmatrix}$.

(a) Calculate the derivative of $f \circ g\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ by using the chain rule.

(b) Calculate the derivative of $f \circ g\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ by explicitly calculating a formula for $f \circ g$, and then taking the derivative of that composition function.

4. (Inspired by Week 12, group problems 2)

For your Freshman Seminar Class, "What Would Life be Like on a Deserted Island?" your term project is to analyze the question that the seminar is named after as the culmination of a semester's worth of insightful discussion and watching "Lost". You have come to a perfect model of how a group of survivors of a plane crash ought to behave. This deserted island has polar bears that can be hunted, guns, for no good reason, and unchartered territory. There are 40 survivors, and they will do one of three tasks, hunt polar bears, x, use the guns to defend the **deserted** island (just cause there may be other peple on the island, oxymoronic? Don't think too hard about this one.) y, or to explore the rest of the island, z. You are constrained by the number of survivors, so $x+y+z=40$. Let people's happiness be given by

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3x + y^3 + \frac{3}{2}z^2$$

As you can see, the survivors really like guns, and seem to be more concerned with exploring the dangerous island than getting food.

(a) Use Lagrange Multipliers to find the constrained critical points.

(b) This plane can also be parameterized by $\gamma\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 40-s-t \\ s \\ t \end{pmatrix}$. Use this parameterization function to find the constrained critical points, and classify this critical point.

5. (Inspired by a proof or problem from lecture or homework)

Manifold M is described by the following parameterization function near the point $\mathbf{c} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

$$\gamma \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} uv + v^2 \\ \sqrt{uv} \\ \sqrt{v} \end{pmatrix}$$

(a) Use this parameterization function to find a basis for the tangent space at the point $\mathbf{c}$ which corresponds to the set of parameters $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(b) Find a locus function that also describes this manifold near $\mathbf{c}$.

(c) Confirm your answer to a) by also using the locus function to find a basis for the tangent space $T_{\mathbf{c}}M$.

6. (Inspired by a proof or problem from lecture or homework)

   Patrick is considering how many hours, x, he should devote to sleeping and how many hours, y he should devote to studying for his Math 23 exam. He is under a couple of constraints. His mom demands that he exercises everyday, spends time for a good dinner with friends, and that he leaves time for talking to them. His parents demand that these three activities ought to total 8 hours of his day. Therefore, the total number of hours he can devote to sleeping and studying is 16 hours, namely, $x + y = 16$. However, Patrick is also in dire need of studying for math, and believes that his math time is worth more than sleeping. Therefore, he imposes the constraint that $x^2 + y = 36$. Namely, he should not sleep more than 6 hours a day.

   (a) Use one iteration of Newton's method with an initial guess of 5.5 hours of sleep and 10.5 hours of studying to solve for the number of hours Patrick will sleep.

   (b) Solve the system of equations exactly for how many hours Patrick will sleep. Note: This is not a good way to spend your days prior to the exam!

7. (Inspired by a proof or problem from lecture or homework)

   Prove that if a sequence $\mathbf{a_1}, \mathbf{a_2}...$ in $\mathbb{R}^n$ converges, then all of its coordinates are bounded.

8. (Inspired by a proof or problem from lecture or homework)

   Better Lawn Sprinkler Co. has been hired to install sprinklers to cover your entire 1 dimensional lawn. These sprinklers can only sprinkle water on an open set, for unknown reasons. Your lawn is the interval, $[0, 1)$. You have lost the boundary of the right half of your lawn due to a dispute with your neighbor, and therefore he decided to build a fence right on the boundary of your lawn. The manager of Better Lawn Sprinkler Co. is going to propose several plans of sprinkler installation.

   (a) The manager is feeling sneaky today. Invent an infinite open cover of this interval, where there does exist an finite subcover, only if you can find it, after reviewing your Math 23 notes.

   (b) The manager is feeling maliciously evil. Invent an infinite open cover of this interval, where no finite subcover exists, and the manager has succeeded in scamming you of all your money.

   (c) How can you modify your lawn, so that the manager will always fail to force you to buy an infinite number of sprinklers?

Part V. Do two of the three proofs. Mark an X in the score box on page 1 to indicate which proof you have omitted.

1. (Proof 9.2)

   Starting from the triangle inequality for two vectors, prove the triangle inequality for n vectors, then prove the "infinite triangle inequality" for $\mathbb{R}^n$

   $$|\sum_{i=1}^{\infty} \vec{\mathbf{a_i}}| \leq \sum_{i=1}^{\infty} |\vec{\mathbf{a_i}}|$$

   under the assumption that the infinite series on the right is convergent, which in turn implies that the infinite series of vectors on the left is convergent.

2. (Proof 11.2)

Using the mean value theorem, prove that if a function $f : \mathbb{R}^2 \to \mathbb{R}$ has partial derivatives $D_1 f$ and $D_2 f$ that are continuous at $\mathbf{a}$, it is differentiable at $\mathbf{a}$ and its derivative is the Jacobian matrix $\begin{bmatrix} D_1 f(\mathbf{a}) & D_2 f(\mathbf{a}) \end{bmatrix}$.

3. (Proof 12.3)

   Let M be a manifold known by a real-valued C^1 function $F(\mathbf{x}) = 0$, where F goes from an open subset U of $\mathbb{R}^n$ to $\mathbb{R}$ and $[\mathbf{D}F(\mathbf{x})]$ is nowhere zero. Let $f : U \to \mathbb{R}$ be a C^1 function.

   Prove that $\mathbf{c} \in M$ is a critical point of f restricted to M if and only if there exists a Lagrange multiplier λ such that $[\mathbf{D}f(\mathbf{c})] = \lambda[\mathbf{D}F(\mathbf{c})]$.

Name: ______________________________

Section: ____________________________

MATHEMATICS 23a/E-23a, Fall 2018
Final Examination
Monday, December 17, 2018

You may omit one multiple-choice question in Part I, one question in Part III, and one proof in Part IV.

You may use a calculator, but only for arithmetic, perhaps in support of Newton's method.

No other aids or references are allowed.

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$I-1$		2	
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$I-3$		2	
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$III-3$	$--$	5	
$III-4$	$--$	5	
$III-5$	$--$	5	
$III-6$	$--$	5	
$III-7$	$--$	5	
$III-8$	$--$	5	
$IV-1$	$--$	5	
$IV-2$	$--$	5	
$IV-3$	$--$	5	
Marked omit(s)	$--$	1	
Total		61	

Part I. Answer five of the six multiple-choice questions. Transcribe your answers onto page 1, and mark an X in the score box on page 1 to indicate which question you have omitted.

If you answer all six questions, the last one will be ignored, and you will lose the extra-credit point for marking omitted questions!

1. Consider set $X = \{123456\}$ where a finite topology is being constructed. The sets $\{12\}$, $\{34\}$ and $\{15\}$ are open. What are all the open sets, if this is a valid topology?

   (a) $\{12\}, \{34\}, \{1234\}, \{15\}, \{1345\}, \{125\}, \{12345\}$

   (b) $\{12\}, \{34\}, \{1234\}, \{15\}, \{1345\}, \{125\}, \{12345\}, \phi, X$

   (c) $\{12\}, \{34\}, \{1234\}, \{15\}, \{1345\}, \{125\}, \{12345\}, \{1\}, \phi, X$

   (d) $\{12\}, \{34\}, \{1234\}, \{15\}, \{1345\}, \{125\}, \{12345\}, \{1\}, \{134\}, \phi, X$

   (e) $\{12\}, \{34\}, \{1234\}, \{15\}, \{1345\}, \{125\}, \{12345\}, \{1\}, \{134\}, \{2\}, \phi, X$

   Solution: $\boxed{D}$ Here, we have used the three axioms of topology governing open sets. The empty and entire set are open. The union of open sets are open, and the intersection of open sets are open. Answer choice D includes all the possible open sets.

2. Which one of these functions is not continuous at the origin? For all these functions $f\begin{pmatrix}0\\0\end{pmatrix} = 0$.

   (a) $x^2y/(x^2+y^2)$

   (b) $(x^2y+xy^2)/(x^2+y^2)$

   (c) $xy^2/(x+y)$

   (d) $x^3y/(x^2+y^2)^{\frac{3}{2}}$

   (e) $x/(x+y)$

   Solution: $\boxed{E}$ The best approach here is to consider an approach in polar coordinates to the origin. Let $y = r\sin\theta$ and $x = r\cos\theta$, and take the limit as $r \to 0$. If our answer is independent of θ, then our limit is not sequence dependent, and we only have to check that the limit agrees with the defined function value. Using this technique on answer choice E, we obtain:

   $$\lim_{r\to 0} \frac{r\cos\theta}{r\cos\theta + r\sin\theta} = \frac{\cos\theta}{\cos\theta + \sin\theta}$$

   This is dependent on the θ of approach! For all other answer choices, we are left with a r in the numerator which will make the function go to zero as $r \to 0$.

3. A manifold M is described by a parameterization function $\gamma : \mathbb{R}^5 \to \mathbb{R}^{43}$. The locus function's domain and codomain are:

   (a) $F : \mathbb{R}^{43} \to \mathbb{R}^{38}$

   (b) $F : \mathbb{R}^{38} \to \mathbb{R}^5$

   (c) $F : \mathbb{R}^{43} \to \mathbb{R}^5$

   (d) $F : \mathbb{R}^5 \to \mathbb{R}^{38}$

   (e) $F : \mathbb{R}^{43} \to \mathbb{R}$

   Solution: $\boxed{A}$ Because this manifold is described by 5 parameters, it is a 5 dimensional manifold. It lives in $\mathbb{R}^{43}$, because that is the number of output dimensions of the parameterization function. Therefore, the locus function goes from the entire space, $\mathbb{R}^{43}$ and imposes 38 constraints $\mathbb{R}^{38}$ to get a 5 dimensional manifold, so the locus function goes from $\mathbb{R}^{43}$ to $\mathbb{R}^{38}$.

4. Which of these is true about a continuous function on an open set U?

   (a) There exists only one good sequence at all points on U

   (b) $\forall \mathbf{x_0} \in U, \forall \epsilon > 0, \exists \delta > 0$ s.t. $\forall \mathbf{x} \in U, |\mathbf{x} - \mathbf{x_0}| < \delta \implies |f(\mathbf{x}) - f(\mathbf{x_0})| < \epsilon$

   (c) $\forall \epsilon > 0, \exists \delta > 0$ s.t. $, \forall \mathbf{x}, \mathbf{x_0} \in U, |\mathbf{x} - \mathbf{x_0}| < \delta \implies |f(\mathbf{x}) - f(\mathbf{x_0})| < \epsilon$

   (d) The function is bounded on U

   (e) The function is differentiable on U

   Solution: $\boxed{B}$ This is precisely the $\epsilon - \delta$ definition of continuity on a set U. A is not true, because **all** sequences must be good at the points in U. C is not true, because that has the stronger constraint that a given δ has to work for all points. This is the statement of uniform continuity. D is not correct. The function need not be bounded, because it could diverge at the boundary and still be continuous. If it was required to be continuous on the closed interval, then it would be bounded. E is also incorrect, because continuity does not imply differentiability, however the converse is true.

5. Which of these is not true about a function f that has continuous partial derivatives on an open set U?

   (a) The function achieves its maximum on the set U
   (b) The function is continuous on the set U
   (c) The function is C^1 on the set U
   (d) The derivative is a linear function of the direction
   (e) The directional derivative in the direction $\vec{v}$, where $\vec{v}$ is a unit vector, at point $\mathbf{c}$ is given by $[Df\mathbf{c}]\vec{v}$

   Solution: $\boxed{A}$ This would be true, if the set was closed! B is true, because continuous partial derivatives imply differentiability which implies continuous. C is true, because continuous partial derivatives is the definition of C^1. D and E are both true as consequences of differentiability.

6. What are all the critical points of the function $f\begin{pmatrix} y \\ x \end{pmatrix} = xy^2 + y^3 - 4x$? Note the change of order of the variables in the argument of the function!

   (a) The critical points are $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ and are both saddle points.
   (b) The critical points are $\begin{pmatrix} -2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and are both saddle points.
   (c) The critical points are $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ and are both maximums.
   (d) The critical points are $\begin{pmatrix} -2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and are both maximums.
   (e) The critical points are $\begin{pmatrix} -2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and are both minimums.

   Solution: $\boxed{A}$ To find the critical points, we set the partial derivative equal to zero. We obtain $y^2 - 4 = 0$ and $2xy + 3y^2 = 0$. From this, we get $y = 2, x = -3$ and $y = -2, x = 3$. To classify these critical points, we find the Hessian Matrix, which is:

$$\begin{bmatrix} 0 & 2y \\ 2y & 2x + 6y \end{bmatrix}$$

   For both critical points, the determinant of the Hessian matrix is negative, implying eigenvalues of different sign, hence a saddle point.

Part II(6 points, 2 per false statement) Of the following statements, more than three are false. Choose any three of the false statements and explain why they are false. For full credit you must both comment on what is wrong with the statement and also cite an explicit counterexample. Just ignore the true statements.

Example:

Statement: "Any two unequal nonzero vectors in $\mathbb{R}^2$ span $\mathbb{R}^2$."

Answer: "False: the vectors could be linearly dependent, like $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}2\\2\end{bmatrix}$"

1. For a compact set X, if there exists an infinite set of open sets, $\{U_i\}$, such that:

$$X \in \bigcup_{i=1}^{\infty} U_i$$

then there exists some finite subset of this set of sets, $\{V_i\} \subset \{U_i\}$, such that:

$$X \in \bigcup_{i=1}^{m} V_i$$

2. If $\mathbf{c}$ is a constrained critical point of function $f : \mathbb{R}^n \to \mathbb{R}$ on a k dimensional manifold M in $\mathbb{R}^n$ described by a locus function $\mathbf{F}$, then $\ker[Df(\mathbf{c})] \subset \ker[D\mathbf{F}(\mathbf{c})]$.

False: An unconstrained critical point that happened to coincidentally lie on the manifold we were constrained to, would have $\dim(\ker[Df(\mathbf{c})]) > \dim(\ker[DF(\mathbf{c})])$, because an unconstrained critical point has $\mathbb{R}^n$ as its kernel.

An explicit counterexample can be maximizing the function $f\begin{pmatrix}x\\y\end{pmatrix} = xe^{-x}$ on the manifold $F\begin{pmatrix}x\\y\end{pmatrix} = x^2 + y^2 - 1$. Notice that at the critical point $\mathbf{c} = \begin{pmatrix}1\\1\end{pmatrix}$, $[DF(\mathbf{c})] = \begin{bmatrix}2 & 2\end{bmatrix}$, and $[Df(\mathbf{c})] = \begin{bmatrix}0 & 0\end{bmatrix}$. This way, while $[DF(\mathbf{c})]$ has a 1 dimensional kernel, $[Df(\mathbf{c})]$ has a 2 dimensional kernel!

3. Given a k dimensional manifold M in $\mathbb{R}^n$, specified by locus function $\mathbf{F}(\mathbf{z})$, where $[D\mathbf{F}(\mathbf{z})]$ is onto, any ordering of the variables in $\mathbf{z}$ will result in the existence of an implicit function that expresses the first (first in order of variables of $\mathbf{z}$) $n-k$ passive variables in terms of the next k active variables.

   False: There were many examples on the homework where certain passive variables could be described by some active variables, but not in another permutation. The requirement is that whatever the ordering of the variables, the matrix A must be invertible for it to be valid!

4. A continuous real valued function defined on a compact set has its maximum on that set.

5. A function f is differentiable at $\mathbf{a}$ if:

$$\lim_{\vec{h}\to 0} \frac{1}{|\vec{h}|}(f(\mathbf{a}+\vec{h}) - f(\mathbf{a}) - [Df(\mathbf{a})]\vec{h}) = 0$$

6. If a real valued differentiable function f at a point $\mathbf{c}$ has directional derivatives on a set of basis vectors equal to zero, then $\mathbf{c}$ is a critical point of f.

7. Consider the sequence of open sets $X_1, X_2, X_3, ...$ where $X_1 \supset X_2 \supset X_3 \supset ....$

$$\bigcap_{k=1}^{\infty} X_k \neq \phi$$

   False: The nested compact set theorem, requires these sets to be compact not open! An explicit counterexample is the sequence of sets, $(0,1), (0,\frac{1}{2}), (0,\frac{1}{3}), ...$ The infinite intersection is empty!

8. When trying to solve a system of equations, that are set equal to zero, using Newton's method, any initial guess is guaranteed to eventually superconverge to one of the roots of the system.

   False: A guess where the derivative is zero is an invalid guess, because the derivative matrix cannot be inverted, and Newton's method will not work. Graphically in the 1 variable case, this is equivalent to a horizontal line that'll never touch the x axis.

Part III. Answer seven of the eight questions. Mark an X in the score box on page 1 to indicate which question you have omitted.

1. (Inspired by Week 9, group problems 3)

   Applying Newton's second law of motion to a mass of 1 attached to a spring of spring constant 4, and a damping force that is equal to 4 times the velocity leads us to the following differential equation

$$\ddot{x} = -4\dot{x} - 4x$$

   Solve this differential equation for $x(t)$ given $x(0) = 4$, and $\dot{x}(0) = 0$. (Hint: To get started let $\vec{w} = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$, and write the differential equation in the form $\dot{\vec{w}} = A\vec{w}$)

   Solution: Letting $\vec{w}$ equal the suggestion, we can write:

$$\dot{\vec{w}} = A\vec{w}$$

   where,

$$A = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}$$

   We know that the solution to this given a set of initial conditions, $\vec{w}_0$, is $\vec{w} = e^{At}\vec{w}_0$. To exponentiate the matrix A, we need to find the eigenvalues and eigenvectors. Now, we use Axler's method to attempt to find the eigenvalues and eigenvectors of this matrix. We use an initial vector $\vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We form our Axler's matrix, and obtain:

$$\begin{bmatrix} 1 & 0 & -4 \\ 0 & -4 & 16 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -4 \end{bmatrix}$$

   Therefore, we obtain that:

$$\begin{aligned} A^2\vec{w} + 4A\vec{w} + 4\vec{w} &= 0 \\ (A + 2I)^2 &= 0 \end{aligned}$$

   We notice that $N = A + 2I$ is nilpotent, and no eigenbasis exists! However, we do know that $A = N - 2I$, and therefore:

$$
\begin{aligned}
e^{At} &= e^{(N-2I)t} \\
&= e^{Nt}e^{-2It}
\end{aligned}
$$

To exponentiate a nilpotent matrix, we use the Series Definition of the exponential, and we obtain:

$$
e^{At} = (I + Nt + \frac{N^2t^2}{2!} + ...) \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-2t} \end{bmatrix}
$$

However, because the matrix N is nilpotent, then $N^2 = 0$, and all further powers are zero, so we simply get:

$$
e^{At} = \begin{bmatrix} 2t+1 & t \\ -4t & -2t+1 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-2t} \end{bmatrix} = \begin{bmatrix} (1+2t)e^{-2t} & te^{-2t} \\ -4te^{-2t} & (-2t+1)e^{-2t} \end{bmatrix}
$$

Therefore, we obtain:

$$
\vec{w} = \begin{bmatrix} (1+2t)e^{-2t} & te^{-2t} \\ -4te^{-2t} & (-2t+1)e^{-2t} \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix}
$$

Finally, we know that the first component of $\vec{w}$ is $x(t)$, and we get:

$$
\boxed{x(t) = 4e^{-2t} + 8te^{-2t}}
$$

2. (Inspired by Week 10, group problems 3)

(a) Let $f\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^2 - 1$. Evaluate the Jacobian Matrix at $f\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and use it to find the best affine approximation to $\begin{pmatrix} 1 \\ 0.1 \end{pmatrix}$.

(b) Let $g\begin{pmatrix} r \\ t \end{pmatrix} = \begin{pmatrix} r\cos t \\ r\sin t \end{pmatrix}$. Evaluate the Jacobian Matrix at $g\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and find r and h such that $g\begin{pmatrix} 1+r \\ h \end{pmatrix} = \begin{pmatrix} 1 \\ 0.1 \end{pmatrix}$.

Solution: a) First, we evaluate the Jacobian Matrix and obtain:

$$\boxed{[Df\begin{pmatrix} 1 \\ 0 \end{pmatrix}] = \begin{bmatrix} 2 & 0 \end{bmatrix}}$$

To find the best affine approximation, we obtain:

$$f\begin{pmatrix} 1 \\ 0.1 \end{pmatrix} \approx f\begin{pmatrix} 1 \\ 0 \end{pmatrix} + [Df\begin{pmatrix} 1 \\ 0 \end{pmatrix}]\begin{bmatrix} 0 \\ 0.1 \end{bmatrix}$$

$$\boxed{f\begin{pmatrix} 1 \\ 0.1 \end{pmatrix} \approx 0}$$

b) Now, we evaluate the Jacobian matrix and obtain:

$$\boxed{[Dg\begin{pmatrix} 1 \\ 0 \end{pmatrix}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

Now, we use the affine approximation to obtain:

$$\begin{bmatrix} 1 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} r \\ h \end{bmatrix}$$

From this, we obtain $\boxed{r = 0}$ and $\boxed{h = 0.1}$

3. (Inspired by Week 11, group problems 1)

Consider the functions $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2y \\ y+x^2 \\ 3x^2+2y \end{pmatrix}$ and $g\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} \sqrt{vw} \\ v+w \end{pmatrix}$.

(a) Calculate the derivative of $f \circ g\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ by using the chain rule.

(b) Calculate the derivative of $f \circ g\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ by explicitly calculating a formula for $f \circ g$, and then taking the derivative of that composition function.

Solution: a) First, we use the chain rule, which states:

$$\begin{aligned} [D(f \circ g)\begin{pmatrix} v \\ w \end{pmatrix}] &= [Df(g\begin{pmatrix} v \\ w \end{pmatrix})][Dg\begin{pmatrix} v \\ w \end{pmatrix}] \\ &= \begin{bmatrix} 2\sqrt{vw}(v+w) & vw \\ 2\sqrt{vw} & 1 \\ 6\sqrt{vw} & 2 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{w}}{2\sqrt{v}} & \frac{\sqrt{v}}{2\sqrt{w}} \\ 1 & 1 \end{bmatrix} \end{aligned}$$

Now, we substitute the point $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$, and we get:

$$[D(f \circ g)\begin{pmatrix} 2 \\ 2 \end{pmatrix}] = \begin{bmatrix} 16 & 4 \\ 4 & 1 \\ 12 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 1 \end{bmatrix}$$

$$\boxed{[D(f \circ g)\begin{pmatrix} 2 \\ 2 \end{pmatrix}] = \begin{bmatrix} 12 & 12 \\ 3 & 3 \\ 8 & 8 \end{bmatrix}}$$

b) Now, we explicitly calculate a formula for $f \circ g$. We obtain:

$$f \circ g\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} v^2w + vw^2 \\ v + w + vw \\ 3vw + 2v + 2w \end{pmatrix}$$

Now, we calculate the derivative, and obtain:

$$[D(f \circ g)\begin{pmatrix} v \\ w \end{pmatrix}] = \begin{bmatrix} 2vw + w^2 & v^2 + 2vw \\ 1 + w & 1 + v \\ 3w + 2 & 3v + 2 \end{bmatrix}$$

Now, we plug in our point, and obtain:

$$\boxed{[D(f \circ g)\begin{pmatrix} 2 \\ 2 \end{pmatrix}] = \begin{bmatrix} 12 & 12 \\ 3 & 3 \\ 8 & 8 \end{bmatrix}}$$

4. (Inspired by Week 12, group problems 2)

For your Freshman Seminar Class, "What Would Life be Like on a Deserted Island?" your term project is to analyze the question that the seminar is named after as the culmination of a semester's worth of insightful discussion and watching "Lost". You have come to a perfect model of how a group of survivors of a plane crash ought to behave. This deserted island has polar bears that can be hunted, guns, for no good reason, and unchartered territory. There are 40 survivors, and they will do one of three tasks, hunt polar bears, x, use the guns to defend the **deserted** island (just cause there may be other peple on the island, oxymoronic? Don't think too hard about this one.) y, or to explore the rest of the island, z. You are constrained by the number of survivors, so $x+y+z=40$. Let people's happiness be given by

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3x + y^3 + \frac{3}{2}z^2$$

As you can see, the survivors really like guns, and seem to be more concerned with exploring the dangerous island than getting food.

(a) Use Lagrange Multipliers to find the constrained critical points.

(b) This plane can also be parameterized by $\gamma\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 40-s-t \\ s \\ t \end{pmatrix}$. Use this parameterization function to find the constrained critical points, and classify this critical point.

Solution: Our only constraint arises from the number of people on the island, and is given by the locus function:

$$F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x + y + z - 40$$

We first find $[DF\begin{pmatrix} x \\ y \\ z \end{pmatrix}]$, and $[Df\begin{pmatrix} x \\ y \\ z \end{pmatrix}]$. We obtain:

$$[DF\begin{pmatrix} x \\ y \\ z \end{pmatrix}] = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

$$[Df\begin{pmatrix} x \\ y \\ z \end{pmatrix}] = \begin{bmatrix} 3 & 3y^2 & 3z \end{bmatrix}$$

Now, using a Lagrange Multiplier, we obtain that:

$$\begin{bmatrix} 3 & 3y^2 & 3z \end{bmatrix} = \lambda \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

Therefore, $\lambda = 3$, and our resulting equations are:

$$\begin{aligned} 3y^2 &= 3 \\ 3z &= 3 \\ x + y + z &= 40 \end{aligned}$$

From these, we obtain, $\boxed{x = 38, y = 1, z = 1}$

b) To use the parameterization function to find the constrained critical points, we find the unconstrained critical points of $f \circ \gamma$. We obtain:

$$\begin{aligned} f \circ \gamma \begin{pmatrix} s \\ t \end{pmatrix} &= 3(40 - s - t) + s^3 + \frac{3}{2}t^2 \\ &= 120 - 3s - 3t + s^3 + \frac{3}{2}t^2 \\ [D(f \circ \gamma) \begin{pmatrix} s \\ t \end{pmatrix}] &= \begin{bmatrix} -3 + 3s^2 & -3 + 3t \end{bmatrix} \end{aligned}$$

Setting these partials equal to zero, we obtain $s = 1$ and $t = 1$, which correspond to $\boxed{x = 38, y = 1, z = 1}$, which agrees with our previous results. To classify this critical point, we find the Hessian Matrix. We obtain:

$$H = \begin{bmatrix} 6s & 0 \\ 0 & 3 \end{bmatrix}$$

The determinant of this matrix at $s = 1$, $t = 1$ is 18, which is positive, so this is a $\boxed{\textit{minimum}}$.

5. (Inspired by a proof or problem from lecture or homework)

Manifold M is described by the following parameterization function near the point $\mathbf{c} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

$$\gamma \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} uv + v^2 \\ \sqrt{uv} \\ \sqrt{v} \end{pmatrix}$$

(a) Use this parameterization function to find a basis for the tangent space at the point $\mathbf{c}$ which corresponds to the set of parameters $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(b) Find a locus function that also describes this manifold near $\mathbf{c}$.

(c) Confirm your answer to a) by also using the locus function to find a basis for the tangent space $T_{\mathbf{c}}M$.

Solution: a) We know that $T_{\mathbf{c}}M = \text{img}[D\gamma \begin{pmatrix} 1 \\ 1 \end{pmatrix}]$. Therefore, we need to find the image of the derivative matrix. First, we find the derivative matrix, and obtain:

$$[D\gamma \begin{pmatrix} u \\ v \end{pmatrix}] = \begin{bmatrix} v & u + 2v \\ \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} \\ 0 & \frac{1}{2\sqrt{v}} \end{bmatrix}$$

$$[D\gamma \begin{pmatrix} 1 \\ 1 \end{pmatrix}] = \begin{bmatrix} 1 & 3 \\ \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$$

Therefore, a basis for the tangent space, consists of $\boxed{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 1 \\ 1 \end{bmatrix}}$

b) A locus function is a function that equals zero. We notice that:

$$\boxed{F \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x - y^2 - z^4}$$

c) We know that $T_{\mathbf{c}}M = \ker[DF \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}]$. First, we find the derivative of the locus function. We obtain:

$$[DF\begin{pmatrix}x\\y\\z\end{pmatrix}] = \begin{bmatrix}1 & -2y & -4z^3\end{bmatrix}$$

$$[DF\begin{pmatrix}2\\1\\1\end{pmatrix}] = \begin{bmatrix}1 & -2 & -4\end{bmatrix}$$

To find the vectors in the kernel, we use the vectors $\begin{bmatrix}a\\1\\0\end{bmatrix}, \begin{bmatrix}b\\0\\1\end{bmatrix}$. Acting this on the matrix $[DF\begin{pmatrix}2\\1\\1\end{pmatrix}]$, we get $a = 2$, and $b = 4$. Therefore, we have a basis for the tangent space, $\boxed{\begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}4\\0\\1\end{bmatrix}}$

Remark: We ought to check that our results for a) and c) are consistent. Therefore, let's make sure that the normal vector to the plane they define is the same (so that it's the same plane!). For our vectors in a), we get:

$$\begin{bmatrix}2\\1\\0\end{bmatrix} \times \begin{bmatrix}6\\1\\1\end{bmatrix} = \begin{bmatrix}1\\-2\\-4\end{bmatrix}$$

For our vectors in b), we get:

$$\begin{bmatrix}2\\1\\0\end{bmatrix} \times \begin{bmatrix}4\\0\\1\end{bmatrix} = \begin{bmatrix}1\\-2\\-4\end{bmatrix}$$

Yes, this is consistent!

6. (Inspired by a proof or problem from lecture or homework)

Patrick is considering how many hours, x, he should devote to sleeping and how many hours, y he should devote to studying for his Math 23 exam. He is under a couple of constraints. His mom demands that he exercises everyday, spends time for a good dinner with friends, and that he leaves time for talking to them. His parents demand that these three activities ought to total 8 hours of his day. Therefore, the total number of hours he can devote to sleeping and studying is 16 hours, namely, $x + y = 16$. However, Patrick is also in dire need of studying for math, and believes that his math time is worth more than sleeping. Therefore, he imposes the constraint that $x^2 + y = 36$. Namely, he should not sleep more than 6 hours a day.

(a) Use one iteration of Newton's method with an initial guess of 5.5 hours of sleep and 10.5 hours of studying to solve for the number of hours Patrick will sleep.

(b) Solve the system of equations exactly for how many hours Patrick will sleep. Note: This is not a good way to spend your days prior to the exam!

Solution: a) The function that we are trying to find the zeros of is a vector valued function:

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y - 16 \\ x^2 + y - 36 \end{pmatrix}$$

To use the formula for Newton's method, we need to calculate the function value at our initial guess, and the derivative at our initial guess. We obtain:

$$f\begin{pmatrix} 5.5 \\ 10.5 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{19}{4} \end{pmatrix}$$

$$[Df\begin{pmatrix} 5.5 \\ 10.5 \end{pmatrix}] = \begin{bmatrix} 1 & 1 \\ 11 & 1 \end{bmatrix}$$

$$[Df\begin{pmatrix} 5.5 \\ 10.5 \end{pmatrix}]^{-1} = \begin{bmatrix} -0.1 & 0.1 \\ 1.1 & -0.1 \end{bmatrix}$$

Now, applying the formula for Newton's method, we obtain:

$$\mathbf{a_1} = \begin{bmatrix} 5.5 \\ 10.5 \end{bmatrix} - \begin{bmatrix} -0.1 & 0.1 \\ 1.1 & -0.1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{19}{4} \end{bmatrix}$$

$$\mathbf{a_1} = \begin{pmatrix} 5.025 \\ 10.975 \end{pmatrix}$$

Patrick will get $\boxed{5.025}$ hours of sleep.

b) Solving this exactly, we can solve for y in the first equation, and substitute into the second equation to obtain:

$$x^2 - x - 20 = 0$$
$$(x-5)(x+4) = 0$$

Because the amount of sleep you get must be positive, we obtain $x = 5$, so Patrick needs $\boxed{5}$ hours of sleep, which is very close to our Newton's method estimate.

7. (Inspired by a proof or problem from lecture or homework)

   Prove that if a sequence $\mathbf{a_1}, \mathbf{a_2}...$ in $\mathbb{R}^n$ converges, then all of its coordinates are bounded.

   Solution: Week 9 Proof 2 states that if a sequence of vectors is convergent, then the corresponding sequences of coordinates are also convergent. Using this fact, we will prove that a given coordinate is bounded, and the proof can be repeated for all n coordinates. Assume the sequence converges to some limit point $\mathbf{b}$. This means that for a given coordinate i,

$$\forall \epsilon > 0, \exists N \text{ s.t. } \forall n > N, |a_{n,i} - b_i| < \epsilon$$

   Choosing some $\epsilon = \epsilon_0$, and finding the guaranteed corresponding $N = N_0$, we get that all the coordinates of vectors beyond N_0 are bounded by $(b_i - \epsilon_0, b_i + \epsilon_0)$. Therefore, we can find the maximum of this set as follows:

$$M = \max(a_{1,i}, a_{2,i}, ..., a_{N_0,i}, b_i + \epsilon_0)$$

   and the corresponding minimum:

$$m = \min(a_{1,i}, a_{2,i}, ..., a_{N_0,i}, b_i - \epsilon_0)$$

   Hence, coordinate i is bounded, and this can be repeated for all the coordinates.

8. (Inspired by a proof or problem from lecture or homework)

Better Lawn Sprinkler Co. has been hired to install sprinklers to cover your entire 1 dimensional lawn. These sprinklers can only sprinkle water on an open set, for unknown reasons. Your lawn is the interval, $[0, 1)$. You have lost the boundary of the right half of your lawn due to a dispute with your neighbor, and therefore he decided to build a fence right on the boundary of your lawn. The manager of Better Lawn Sprinkler Co. is going to propose several plans of sprinkler installation.

(a) The manager is feeling sneaky today. Invent an infinite open cover of this interval, where there does exist an finite subcover, only if you can find it, after reviewing your Math 23 notes.

(b) The manager is feeling maliciously evil. Invent an infinite open cover of this interval, where no finite subcover exists, and the manager has succeeded in scamming you of all your money.

(c) How can you modify your lawn, so that the manager will always fail to force you to buy an infinite number of sprinklers?

Solution: a) Consider the open cover consisting of the sets defined by:

$$S_0 = (-0.1, 0.1)$$
$$S_k = (\frac{0.4}{2^k}, \frac{2.4}{2^k}), \text{ where } k \in \mathbb{N}$$

This open cover has a finite subcover, because there will be overlap between S_0 and all the S_k beyond a certain k.

b) Consider the open cover consisting of the sets defined by:

$$S_k = (-\frac{1}{k}, 1 - \frac{1}{k})$$

Here, you need all the sets in order to cover every point infinitesimally close to the right side boundary.

c) Declare war on your neighbor, knock down your neighbor's wall and claim it as your territory! Once, you do this, your lawn will be the interval $[0, 1]$, so it will be compact (closed and bounded), and thus the Heine-Borel Theorem guarantees the existence of a finite subcover for any open cover.

Remark: There can be many different correct answers for a and b!

Part V. Do two of the three proofs. Mark an X in the score box on page 1 to indicate which proof you have omitted.

1. (Proof 9.2)

   Starting from the triangle inequality for two vectors, prove the triangle inequality for n vectors, then prove the "infinite triangle inequality" for $\mathbb{R}^n$

   $$|\sum_{i=1}^{\infty} \vec{\mathbf{a_i}}| \leq \sum_{i=1}^{\infty} |\vec{\mathbf{a_i}}|$$

   under the assumption that the infinite series on the right is convergent, which in turn implies that the infinite series of vectors on the left is convergent.

2. (Proof 11.2)

Using the mean value theorem, prove that if a function $f : \mathbb{R}^2 \to \mathbb{R}$ has partial derivatives $D_1 f$ and $D_2 f$ that are continuous at $\mathbf{a}$, it is differentiable at $\mathbf{a}$ and its derivative is the Jacobian matrix $\begin{bmatrix} D_1 f(\mathbf{a}) & D_2 f(\mathbf{a}) \end{bmatrix}$.

3. (Proof 12.3)

   Let M be a manifold known by a real-valued C^1 function $F(\mathbf{x}) = 0$, where F goes from an open subset U of $\mathbb{R}^n$ to $\mathbb{R}$ and $[\mathbf{D}F(\mathbf{x})]$ is nowhere zero. Let $f : U \to \mathbb{R}$ be a C^1 function.

   Prove that $\mathbf{c} \in M$ is a critical point of f restricted to M if and only if there exists a Lagrange multiplier λ such that $[\mathbf{D}f(\mathbf{c})] = \lambda[\mathbf{D}F(\mathbf{c})]$.