MATHEMATICS 23a/E-23a, Fall 2018
Linear Algebra and Real Analysis I
Week 8 (Derivatives, Inverse functions, Taylor series)

Authors: Paul Bamberg and Kate Penner (based on their course MATH S-322)
R scripts by Paul Bamberg
Last modified: November 3, 2018 by Paul Bamberg (fixed proof 8.2)

Reading from Ross

- Chapter 5, sections 28 and 29 (pp.223-240)
- Chapter 5, sections 30 and 31, but only up through section 31.7.
- Chapter 7, section 37 (logarithms and exponentials)

Recorded Lectures

- Lecture 16 (Week 8, Class 1) (watch on October 30 or 31)
- Lecture 17 (Week 8, Class 2) (watch on November 1 or 29)

Proofs to present in section or to a classmate who has done them.

- 8.1 (Ross, pp.233-234, Rolle's Theorem and the Mean Value Theorem)
  - Prove Rolle's Theorem: if f is a continuous function on $[a, b]$ that is differentiable on (a, b) and satisfies $f(a) = f(b)$, then there exists at least one x in (a, b) such that $f'(x) = 0$.
  - Using Rolle's Theorem, prove the Mean Value Theorem: if f is a continuous function on $[a, b]$ that is differentiable on (a, b), then there exists at least one x in (a, b) such that
  $$f'(x) = \frac{f(b) - f(a)}{b - a}$$
- 8.2 Differentiating an inverse function
  Suppose that f is a one-to-one continuous function on open interval I (either strictly increasing or strictly decreasing) Let open interval $J = f(I)$, and define the inverse function $f^{-1} : J \to I$ for which
  $(f^{-1} \circ f)(x) = x$ for $X \in I$; $f \circ f^{-1}(y) = y$ for $y \in J$.
  Let $g = f^{-1}$, and define $y_0 = f(x_0)$.
  Take it as proved that g is continuous at y_0.
  Prove that, if f if differentiable at x_0 and $f'(x_0) \neq 0$, then
  $$\lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} = \frac{1}{f'(x_0)}.$$

Additional proofs(may appear on quiz, students will post pdfs or videos)

- 8.3 (Ross, pp. 228, The Chain Rule – easy special case)
  Assume the following:
  * Function f is differentiable at a.
  * Function g is differentiable at $f(a)$.
  * There is an open interval J containing a on which f is defined and $f(x) \neq f(a)$ (without this restriction, you need the messy Case 2 on page 229).
  * Function g is defined on the open interval $I = f(J)$, which contains $f(a)$.

  Using the sequential definition of a limit, prove that the composite function $g \circ f$ is defined on J and differentiable at a and that

  $$(g \circ f)'(a) = g'(f(a)) \cdot f'(a).$$

- 8.4 Taylor's Theorem with remainder:
  Let f be defined on (a, b) with $a < 0 < b$.
  Suppose that the nth derivative $f^{(n)}$ exists on (a, b).
  Define the remainder

  $$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k.$$

  Prove, by repeated use of Rolle's theorem, that for each $x \neq 0$ in (a, b), there is some y between 0 and x for which

  $$R_n(x) = \frac{f^{(n)}(y)}{n!} x^n.$$

R Scripts

- Script 2.4A-Taylor Series.R
  Topic 1 - Convergence of the Taylor series for the cosine function
  Topic 2 - A function that is not the sum of its Taylor series
  Topic 3 - Illustrating Ross's proof of Taylor series with remainder.

- Script2.4B-LHospital.R Topic 1 - Illustration of proof 6 from Week 8

- Script 2.4C-SampleProblems.R

1 Executive Summary

1.1 The Derivative - Definition and Properties

- A function f is differentiable at some point a if the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists and is finite. It is referred to as $f'(a)$. If a function is differentiable at a point a, then it is continuous at a as well.

- Derivatives, being defined in terms of limits, share many properties with limits. Given two functions f and g, both differentiable at some point a, the following properties hold:
  - scalar multiples: $(cf)'(a) = c \cdot f'(a)$
  - sums of functions: $(f+g)'(a) = f'(a) + g'(a)$
  - Product Rule: $(fg)'(a) = f(a)g'(a) + f'(a)g(a)$
  - Quotient Rule: $(f/g)'(a) = [g(a)f'(a) - f(a)g'(a)]/g^2(a)$ if $g(a) \neq 0$

- The most memorable derivative rule is **The Chain Rule**, which states that if f is differentiable at some point a, and g is differentiable at $f(a)$, then their composite function $g \circ f$ is also differentiable at a, and

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$$

1.2 Increasing and decreasing functions

The terminology is the same as what we used for sequences. It applies to functions whether or not they are differentiable or even continuous.

- A function f is **strictly increasing** on an interval I if $x_1, x_2 \in I$ and $x_1 < x_2 \implies f(x_1) < f(x_2)$
- A function f is **strictly decreasing** on an interval I if $x_1, x_2 \in I$ and $x_1 < x_2 \implies f(x_1) > f(x_2)$
- A function f is **increasing** on an interval I if $x_1, x_2 \in I$ and $x_1 < x_2 \implies f(x_1) \leq f(x_2)$
- A function f is **decreasing** on an interval I if $x_1, x_2 \in I$ and $x_1 < x_2 \implies f(x_1) < f(x_2)$

1.3 Behavior of differentiable functions

These justify our procedures when we are searching for the critical points of a given function. They are the main properties we draw on when reasoning about a function's behavior.

- If f is defined on an open interval, achieves its maximum or minimum at some x_0, and is differentiable there, then $f'(x_0) = 0$.
- Rolle's Theorem. If f is continuous on some interval $[a, b]$ and differentiable on (a, b) with $f(a) = f(b)$, then there exists at least one $x \in (a, b)$ such that $f'(x) = 0$ (Rolle's Theorem).
- Mean Value Theorem. If f is continuous on some interval $[a, b]$ and differentiable on (a, b), then there exists at least one $x \in (a, b)$ such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

- If f is differentiable on (a, b) and $f'(x) = 0\ \forall x \in (a, b)$, then f is a constant function on (a, b).
- If f and g are differentiable functions on (a, b) such that $f' = g'$ on (a, b), then there exists a constant c such that
$\forall x \in (a, b)\ f(x) = g(x) + c$

1.4 Inverse functions and their derivatives

- Review of a corollary of the intermediate value theorem: If function f is continuous and one-to-one on a interval I(which means it must be either strictly increasing or strictly decreasing), then there is a continuous inverse function f^{-1}, whose domain is the interval
$J = f(I)$, such that $f \circ f^{-1}$ and $f^{-1} \circ f$ are both the identity function.
- Not quite a proof: Since $(f \circ f^{-1})(y) = y$, the chain rule states that $f'(f^{-1}(y))(f^{-1})'(y) = y$ and, if $f'(f^{-1}(y)) \neq 0$,

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}.$$

- Example: if $f(x) = \tan x$ with $I = (\frac{-\pi}{2}, \frac{\pi}{2})$, then $f^{-1}(y) = \arctan y$ and

$$(\arctan)'(y) = \frac{1}{(\tan)'(\arctan y)} = \frac{1}{\sec^2(\arctan y)} = \frac{1}{1 + \tan^2(\arctan y)} = \frac{1}{1 + y^2}$$

- The problem: we need to prove that f' is differentiable.

1.5 Defining the logarithm and exponential functions

Define the natural logarithm as an antiderivative:

$$L(y) = \int_1^y \frac{1}{t} dt, \text{ and define } e \text{ so that } \int_1^e \frac{1}{t} dt = 1.$$

¿From this definition it is easy to prove that $L'(y) = \frac{1}{y}$ and not hard to prove that $L(xy) = L(x) + L(y)$.

Now the exponential function can be defined as the inverse function, so that $E(L(y)) = y$. From this definition it follows that $E(x + y) = E(x) + E(y)$ and that $E'(x) = E(x)$.

1.6 L'Hospital's rule

- Suppose that f and g are differentiable functions and that

  $$\lim_{x \to a+} \frac{f'(x)}{g'(x)} = L; \lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = 0; g'(a) < 0.$$

  Then

  $$\lim_{x \to a+} \frac{f(x)}{g(x)} = L.$$

- Replace $x \to a^+$ by $x \to a^-$ or $x \to a$ or $x \to \pm\infty$ and the result is still valid. It is also possible to have $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = \infty$. The restriction to $g'(a) < 0$ is just to make the proof easier; the result is also true if $g'(a) > 0$.

- Once you understand the proof in one special case, the proof in all the other cases is essentially the same.

- Here is the basic strategy: given that

  $$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L,$$

  use the mean value theorem to construct an interval (a, α) on which

  $$|\frac{f(x)}{g(x)} - L| < \epsilon.$$

1.7 Taylor series

- If a function f is *defined* by a convergent power series, i.e.

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \text{ for } |x| < R,$$

then it is easy to show that

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \text{ for } |x| < R.$$

The challenge is to extend this formula to functions that are differentiable many times but that are not defined by power series, like trig functions defined geometrically, or the function $\sqrt{(1+x)}$.

- Taylor's theorem with remainder – version 1
By the mean value theorem, $f(x) - f(0) = f'(y)x$ for some $y \in (0, x)$.

The generalization is that

$$f(x) - f(0) - f'(0)x - \frac{f''(0)}{2!}x^2 - \cdots - \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} = \frac{f^{(n)}(y)}{n!}x^n$$

for some y between 0 and x. It is proved by induction, using Rolle's theorem n times.

- If the right hand side approaches zero in the limit of large n, then the Taylor series converges to the function. This is true if all the derivatives $f^{(n)}$ are bounded by a single constant C. This criterion is sufficient to establish familiar Taylor expansions like
$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$
$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots$

- Taylor's theorem with remainder – version 2
The fundamental theorem of calculus says that $f(x) - f(0) = \int_0^x f'(t)dt$.

The generalization is that

$$f(x) - f(0) - f'(0)x - \frac{f''(0)}{2!}x^2 - \cdots - \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} = \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)dt.$$

It is proved by induction, using integration by parts, but not by us!

- A famous counterexample.
The function $f(x) = e^{-\frac{1}{x}}$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$ has the property that the remainder does not approach a limit of zero. It does not equal the sum of its Taylor series.

2 Lecture Outline

1. The Derivative - Definition and Properties

   A function f is differentiable at some point a if the limit

   $$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

   exists and is finite. It is referred to as $f'(a)$. If a function is differentiable at a point a, then it is continuous at a as well.

2. Derivatives, being defined in terms of limits, share many properties with limits. Given two functions f and g, both differentiable at some point a, the following properties hold:

   - scalar multiples: $(cf)'(a) = c \cdot f'(a)$
   - sums of functions: $(f + g)'(a) = f'(a) + g'(a)$
   - Product Rule: $(fg)'(a) = f(a)g'(a) + f'(a)g(a)$
   - Quotient Rule: $(f/g)'(a) = [g(a)f'(a) - f(a)g'(a)]/g^2(a)$ if $g(a) \neq 0$

3. (Ross, p.226, Sum and Product Rule for Derivatives)
   Consider two functions f and g. Prove that if both functions are differentiable at some point a, then $(f+g)$ and fg are differentiable at a as well, and:

   - $(f+g)'(a) = f'(a) + g'(a)$
   - $(fg)'(a) = f(a)g'(a) + f'(a)g(a)$

4. Proving differentiation formulas by induction

   Using the product rule, which we just proved, show by induction that for the function $f(x) = x^n$

   $f'(x) = nx^{n-1}$ for all $n > 0$.

   Using the product rule, show by induction that for the function $g(x) = x^{-n}$

   $g'(x) = -nx^{-(n+1)}$ for all $n > 0$

5. (Proof 8.3 – Ross, pp. 228, The Chain Rule – easy special case)
Assume the following:

   - Function f is differentiable at a.
   - Function g is differentiable at $f(a)$.
   - There is an open interval J containing a on which f is defined and $f(x) \neq f(a)$ (without this restriction, you need the messy Case 2 on page 229).
   - Function g is defined on the open interval $I = f(J)$, which contains $f(a)$.

   Using the sequential definition of a limit, prove that the composite function $g \circ f$ is defined on J and differentiable at a and that

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a).$$

6. Calculating derivatives by using the chain rule

   Let $f(x) = \sqrt[3]{x}$.

   (a) Calculate $f'(x)$ using the definition of the derivative.

   (b) Calculate $f'(x)$ by applying the chain rule to $(f(x))^3 = x$.

   (c) Using the chain rule, which we just proved, show that for the function $f(x) = x^{m/n}$ with $m, n > 0$

$$f'(x) = \frac{m}{n} x^{\frac{m}{n}-1}.$$

7. Behavior of differentiable functions

   These justify our procedures when we are searching for the critical points of a given function. They are the main properties we draw on when reasoning about a function's behavior.

   - If f is defined on an open interval, achieves its maximum or minimum at some x_0, and is differentiable there, then $f'(x_0) = 0$.
   - Rolle's Theorem. If f is continuous on some interval $[a, b]$ and differentiable on (a, b) with $f(a) = f(b)$, then there exists at least one $x \in (a, b)$ such that $f'(x) = 0$ (Rolle's Theorem).
   - Mean Value Theorem. If f is continuous on some interval $[a, b]$ and differentiable on (a, b), then there exists at least one $x \in (a, b)$ such that
     $$f'(x) = \frac{f(b) - f(a)}{b - a}$$
   - If f is differentiable on (a, b) and $f'(x) = 0 \ \forall x \in (a, b)$, then f is a constant function on (a, b).
   - If f and g are differentiable functions on (a, b) such that $f' = g'$ on (a, b), then there exists a constant c such that
     $\forall x \in (a, b)\ f(x) = g(x) + c$

8. The derivative at a maximum or minimum (Ross, page 232)
Prove that if f is defined on an open interval containing x_0, if f has its maximum of minimum at x_0, and if f is differentiable at x_0, then $f'(x_0) = 0$.

9. (Proof 8.1 – Ross, pp.233-234, Rolle's Theorem and Mean Value Theorem) Prove Rolle's Theorem: if f is a continuous function on $[a, b]$ that is differentiable on (a, b) and satisfies $f(a) = f(b)$, then there exists at least one x in (a, b) such that $f'(x) = 0$.

Using Rolle's Theorem, prove the Mean Value Theorem: if f is a continuous function on $[a, b]$ that is differentiable on (a, b), then there exists at least one x in (a, b) such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

10. Using the Mean Value Theorem

    Suppose f is differentiable on $\mathbb{R}$ and $f(0) = 0$, $f(1) = 1$, and $f(2) = 1$. Show that $f'(x) = 1/2$ for some $x \in (0, 2)$.
    Then, by applying the Intermediate Value Theorem and Rolle's Theorem to $g(x) = f(x) - \frac{1}{4}x$, show that $f'(x) = \frac{1}{4}$ for some $x \in (0, 2)$.

11. Increasing and decreasing functions
    The terminology is the same as what we used for sequences. It applies to functions whether or not they are differentiable or even continuous.

    - A function f is **strictly increasing** on an interval I if $x_1, x_2 \in I$ and $x_1 < x_2 \implies f(x_1) < f(x_2)$
    - A function f is **strictly decreasing** on an interval I if $x_1, x_2 \in I$ and $x_1 < x_2 \implies f(x_1) > f(x_2)$
    - A function f is **increasing** on an interval I if $x_1, x_2 \in I$ and $x_1 < x_2 \implies f(x_1) \leq f(x_2)$
    - A function f is **decreasing** on an interval I if $x_1, x_2 \in I$ and $x_1 < x_2 \implies f(x_1) < f(x_2)$

    Prove that if f is a differentiable function on an interval (a, b) and $f'(x) > 0 \; \forall x \in (a, b)$, then f is strictly increasing.

12. Defining inverse functions

A function $f(x)$, defined on an interval I, has an inverse function $g(y)$ for which $f(g(y)) = y)$ and $g(f(x)) = x)$ only if it is monotone (either increasing or decreasing) on I.
If I is an open interval (a, b) on which f is differentiable, the inverse function is differentiable everywhere on the interval $J = f(I)$ if and only if f is either strictly increasing or strictly decreasing.

(a) Sketch a graph for the case $f(x) = \sin x, I = [\frac{-\pi}{2}, \frac{\pi}{2}]$ to show how the arc sine function is defined.

(b) Although $f(x) = \sin x$ is strictly increasing on the closed interval I, the derivative rule requires an open interval! Show that $g(y) = \arcsin(y)$ is differentiable on the open interval $(-1, 1)$.

13. Proof 8.2 – Differentiating an inverse function
Suppose that f is a one-to-one continuous function on open interval I (either strictly increasing or strictly decreasing) Let open interval $J = f(I)$, and define the inverse function $f^{-1} : J \to I$ for which

$(f^{-1} \circ f)(x) = x$ for $X \in I$; $f \circ f^{-1}(y) = y$ for $y \in J$.

Let $g = f^{-1}$, and define $y_0 = f(x_0)$.
Take it as proved that g is continuous at y_0.

Prove that, if f if differentiable at x_0, then

$$\lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} = \frac{1}{f'(x_0)}.$$

14. Applying the inverse-function rule

    The function $g(y) = \arctan y^2, y \geq 0$ is continuous and strictly increasing, hence invertible.
    Calculate its derivative by finding a formula for the inverse function $f(x)$, which is easy to differentiate, then using the rule for the derivative of an inverse function. You can confirm your answer by using the known derivative of the arctan function.

15. (L'Hospital's Rule; based on Ross, 30.2, but simplified to one special case)

Suppose that f and g are differentiable functions and that

$$\lim_{z \to a+} \frac{f'(z)}{g'(z)} = L; f(a) = 0, g(a) = 0; g'(a) > 0.$$

Choose $x > a$ so that for $a < z \leq x, g(z) > 0$ and $g'(z) > 0$.
(You do not have to prove that this can always be done!)

By applying Rolle's Theorem to $h(z) = f(z)g(x) - g(z)f(x)$,

prove that

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = L.$$

16. Using L'Hospital's rule – tricks of the trade

(a) Conversion to a quotient – evaluate

$$\lim_{x \to 0+} x \log_e x^2.$$

(b) Evaluate

$$\lim_{x \to 0} \frac{xe^x - \sin x}{x^2}$$

both by using L'Hospital's rule and by expansion in a Taylor series.

17. Taylor series

- If a function f is *defined* by a convergent power series, i.e.

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \text{ for } |x| < R,$$

then it is easy to show that

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \text{ for } |x| < R.$$

The challenge is to extend this formula to functions that are differentiable many times but that are not defined by power series, like trig functions defined geometrically, or the function $\sqrt{(1+x)}$.

- Taylor's theorem with remainder
By the mean value theorem, $f(x) - f(0) = f'(y)x$ for some $y \in (0, x)$. The generalization is that

$$f(x) - f(0) - f'(0)x - \frac{f''(0)}{2!}x^2 - \cdots - \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} = \frac{f^{(n)}(y)}{n!}x^n$$

for some y between 0 and x. It is proved by induction, using Rolle's theorem n times.

- If the right hand side approaches zero in the limit of large n, then the Taylor series converges to the function. This is true if, as is often the case, all the derivatives $f^{(n)}$ are bounded by a single constant C.

18. (Proof 8.4 – Ross, page 250; version 1 of Taylor's Theorem with remainder, setting $c = 0$)

    Let f be defined on (a, b) with $a < 0 < b$. Suppose that the nth derivative $f^{(n)}$ exists on (a, b).

    Define the remainder

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k.$$

    Prove, by repeated use of Rolle's theorem, that for each $x \neq 0$ in (a, b), there is some y between 0 and x for which

$$R_n(x) = \frac{f^{(n)}(y)}{n!} x^n.$$

19. Calculating a Taylor series

(a) Derive the Taylor series for the function $f(x) = \cos x$.

(b) Prove that the series converges for all x.

(c) Use an appropriate form of remainder to prove that it converges to the cosine function.

20. A function that is not equal to the sum of its Taylor series

(a) Use L'Hospital's rule to show that for all $n > 0$,

$$\lim_{x \to 0} x^{-n} e^{-\frac{1}{x}} = 0.$$

(b) Show that the function defined by

$$f(x) = 0 \text{ for } x \leq 0, f(x) = e^{-\frac{1}{x}} \text{ for } x > 0$$

is not equal to the sum of its Taylor series.

21. (Ross, pp. 342-343; defining the natural logarithm)

Define

$$L(y) = \int_1^y \frac{1}{t} dt.$$

Prove from this definition the following properties of the natural logarithm:

- $$L'(y) = \frac{1}{y} \text{ for } y \in (0, \infty).$$
- $L(yz) = L(y) + L(z)$ for $y, z \in (0, \infty)$.
- $\lim_{y \to \infty} L(y) = +\infty$.

22. Definition and properties of the exponential function

Denote the function inverse to L by E, i.e.

$$(E(L(y)) = y \text{ for } y \in (0, \infty)$$
$$L(E(x)) = x \text{ for } x \in \mathbb{R}$$

Prove from this definition the following properties of the exponential function E:

- $E'(x) = E(x)$ for $x \in \mathbb{R}$.
- $E(u+v) = E(u)E(v)$ for $u, v \in \mathbb{R}$.

23. Hyperbolic functions, defined by their Taylor series

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots ; \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

- Calculate $\sinh' x$ and $\cosh' x$, and prove that $\cosh^2 x - \sinh^2 x = 1$.
- Use Taylor's theorem to prove that
  $\sinh(a + x) = \sinh a \cosh x + \cosh a \sinh x$.

3 Seminar Topics

Your section instructor will either have emailed a list of topics to prepare or will have posted a signup list of appointments on the Calendar tab of Canvas. Either way, there will be one of the following topics that you should be prepared to present.

Practice your presentation so that it takes about 8 minutes. The text of the presentation will be projected onto a screen so that you need not recopy it. To save time, avoid writing long sentences on the chalkboard. You may use notes, but be discreet about it.

1. (Proof 8.3 – The Chain Rule – easy special case)
   Assume the following:

   - Function f is differentiable at a.
   - Function g is differentiable at $f(a)$.
   - There is an open interval J containing a on which f is defined and $f(x) \neq f(a)$ (without this restriction, you need the messy Case 2 on page 229).
   - Function g is defined on the open interval $I = f(J)$, which contains $f(a)$.

   Using the sequential definition of a limit, prove that the composite function $g \circ f$ is defined on J and differentiable at a and that

   $$(g \circ f)'(a) = g'(f(a)) \cdot f'(a).$$

2. (Proof 8.1 – Rolle's Theorem and the Mean Value Theorem)

   - Prove Rolle's Theorem: if f is a continuous function on $[a, b]$ that is differentiable on (a, b) and satisfies $f(a) = f(b)$, then there exists at least one x in (a, b) such that $f'(x) = 0$.

   - Using Rolle's Theorem, prove the Mean Value Theorem: if f is a continuous function on $[a, b]$ that is differentiable on (a, b), then there exists at least one x in (a, b) such that

     $$f'(x) = \frac{f(b) - f(a)}{b - a}$$

3. (Proof 8.2 – Differentiating an inverse function)
   Suppose that f is a one-to-one continuous function on open interval I (either strictly increasing or strictly decreasing) Let open interval $J = f(I)$, and define the inverse function $f^{-1} : J \to I$ for which

   $(f^{-1} \circ f)(x) = x$ for $X \in I$; $f \circ f^{-1}(y) = y$ for $y \in J$.

   Let $g = f^{-1}$, and define $y_0 = f(x_0)$.
   Take it as proved that g is continuous at y_0.

   Prove that, if f if differentiable at x_0 and $f'(x_0) \neq 0$, then

   $$\lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} = \frac{1}{f'(x_0)}.$$

4. Suppose that function f is *defined* by a convergent power series, i.e.

   $$f(x) = \sum_{k=0}^{\infty} a_k x^k \text{ for } |x| < R,$$

   Prove that in this case the Taylor series formula

   $$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \text{ for } |x| < R$$

   is correct.

5. (Proof 8.4 – Taylor's Theorem with remainder)
   Let f be defined on (a, b) with $a < 0 < b$.
   Suppose that the nth derivative $f^{(n)}$ exists on (a, b).

   Define the remainder

   $$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k.$$

   Prove, by repeated use of Rolle's theorem, that for each $x \neq 0$ in (a, b), there is some y between 0 and x for which

   $$R_n(x) = \frac{f^{(n)}(y)}{n!} x^n.$$

6. (Extra topic) The derivative at a maximum or minimum (Ross, page 232)
   Prove that if f is defined on an open interval containing x_0, if f has its maximum or minimum at x_0, and if f is differentiable at x_0, then $f'(x_0) = 0$.

4 Workshop Problems

1. Proving differentiation rules

   (a) Trig functions

   - Prove that $(\sin x)' = \cos x$ from scratch using the fact that

     $$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

   - Let $f(x) = \csc x$ so that $\sin x f(x) = 1$. Use the product rule to prove that

     $$(\csc x)' = -\csc x \cot x.$$

   (b) Non-integer exponents

   - Negative rational exponent: Let $f(x) = x^{-m/n}$, so that $(f(x))^n x^m = 1$.
     Prove that

     $$f'(x) = \frac{-m}{n} x^{\frac{-m}{n} - 1}.$$

   - Irrational exponent:
     Let p be any real number and define $f(x) = x^p = E(pL(x))$.
     Prove that $f'(x) = px^{p-1}$.

2. MVT, L'Hospital, inverse functions

(a)
- When a local minimum is also a global minimum
  Suppose that f is twice differentiable on (a, b), with $f'' > 0$, and that there exists $x \in (a, b)$ for which $f'(x) = 0$, so that x is a local minimum of f. Consider $y \in (x, b)$. By using the mean value theorem twice, prove that $f(y) > f(x)$. This, along with a similar result for $y \in (a, x)$, establishes that x is also the global minimum of f on (a, b).
- Evaluate the limit

$$\lim_{x \to 0} \frac{1 - \cos x}{e^x - x - 1}$$

  by using L'Hospital's rule, then confirm your answer by expanding both numerator and denominator in a Taylor series.

(b)
- Applying the inverse-function rule
  The function $g(y) = \arcsin \sqrt{y}, 0 < y < 1$ is important in the theory of random walks.
  Calculate its derivative by finding a formula for the inverse function $f(x)$, which is easy to differentiate, then using the rule for the derivative of an inverse function. You can confirm your answer by using the known derivative of the arcsin function.
- Evaluate the limit

$$\lim_{x \to 0} \frac{\csc x - \cot x}{x}.$$

  It takes a little bit of algebraic work to rewrite this in a form to which L'Hospital's rule can be applied.

3. Taylor series

(a) Using the Taylor series for the trig functions
Define functions $S(x)$ and $C(x)$ by the power series

$$S(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots ; C(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

- Calculate $S'(x)$ and $C'(x)$, and prove that $S^2(x) + C^2(x) = 1$.
- Use Taylor's theorem to prove that
$C(a + x) = C(a)C(x) - S(a)S(x)$.

(b) Using the remainder to prove convergence
Define $f(x) = \log_e(1 + x)$ for $x \in (-1, \infty)$.
Using the remainder formula

$$R_n(x) = \frac{f^{(n)}(y)}{n!} x^n$$

prove that

$$\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots .$$

Show that the remainder does not go to zero if you set $x = -1$.

5 Homework

Again, if you do the entire assignment in TeX, you may omit one problem and receive full credit for it.

1. Ross, 28.2
2. Ross, 28.8
3. Ross, 29.12
4. Ross, 29.18
5. Ross, exercises 30-1(d) and 30-2(d). Do these two ways: once by using L'Hospital's rule, once by replacing each function by the first two or three terms of its Taylor series.
6. Ross, 30-4. Use the result to convert exercise 30-5(a) into a problem that involves a limit as $y \to \infty$.
7. One way to define the exponential function is as the sum of its Taylor series: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$.

   Using this definition and Taylor's theorem, prove that $e^{a+x} = e^a e^x$.
8. Ross, exercise 31.5. For part (a), just combine the result of example 3 (whose messy proof you need not study) with the chain rule.
9. Ross, exercise 37.9.