**Name:** ______________________________

**Section:** ____________________________

MATHEMATICS 23a/E-23a, Fall 2018
Final Examination
Monday, December 17, 2018

You may omit one multiple-choice question in Part I, one question in Part III, and one proof in Part IV.

You may use a calculator, but only for arithmetic, perhaps in support of Newton's method.

No other aids or references are allowed.

| Problem | Answer | Points | Score |
|---|---|---|---|
| $I-1$ | | 2 | |
| $I-2$ | | 2 | |
| $I-3$ | | 2 | |
| $I-4$ | | 2 | |
| $I-5$ | | 2 | |
| $I-6$ | | 2 | |
| $II$ | $--$ | 6 | |
| $III-1$ | $--$ | 5 | |
| $III-2$ | $--$ | 5 | |
| $III-3$ | $--$ | 5 | |
| $III-4$ | $--$ | 5 | |
| $III-5$ | $--$ | 5 | |
| $III-6$ | $--$ | 5 | |
| $III-7$ | $--$ | 5 | |
| $III-8$ | $--$ | 5 | |
| $IV-1$ | $--$ | 5 | |
| $IV-2$ | $--$ | 5 | |
| $IV-3$ | $--$ | 5 | |
| Marked omit(s) | $--$ | 1 | |
| Total | | 61 | |

Part I. Answer five of the six multiple-choice questions. Transcribe your answers onto page 1, and mark an X in the score box on page 1 to indicate which question you have omitted.

If you answer all six questions, the last one will be ignored, and you will lose the extra-credit point for marking omitted questions!

1. Consider set $X = \{123456\}$ where a finite topology is being constructed. The sets $\{12\}$, $\{34\}$ and $\{15\}$ are open. What are all the open sets, if this is a valid topology?

   (a) $\{12\}, \{34\}, \{1234\}, \{15\}, \{1345\}, \{125\}, \{12345\}$

   (b) $\{12\}, \{34\}, \{1234\}, \{15\}, \{1345\}, \{125\}, \{12345\}, \phi, X$

   (c) $\{12\}, \{34\}, \{1234\}, \{15\}, \{1345\}, \{125\}, \{12345\}, \{1\}, \phi, X$

   (d) $\{12\}, \{34\}, \{1234\}, \{15\}, \{1345\}, \{125\}, \{12345\}, \{1\}, \{134\}, \phi, X$

   (e) $\{12\}, \{34\}, \{1234\}, \{15\}, \{1345\}, \{125\}, \{12345\}, \{1\}, \{134\}, \{2\}, \phi, X$

2. Which one of these functions is not continuous at the origin? For all these functions $f\begin{pmatrix}0\\0\end{pmatrix} = 0$.

   (a) $x^2y/(x^2+y^2)$

   (b) $(x^2y+xy^2)/(x^2+y^2)$

   (c) $xy^2/(x+y)$

   (d) $x^3y/(x^2+y^2)^{\frac{3}{2}}$

   (e) $x/(x+y)$

3. A manifold M is described by a parameterization function $\gamma : \mathbb{R}^5 \to \mathbb{R}^{43}$. The locus function's domain and codomain are:

   (a) $F : \mathbb{R}^{43} \to \mathbb{R}^{38}$

   (b) $F : \mathbb{R}^{38} \to \mathbb{R}^{5}$

   (c) $F : \mathbb{R}^{43} \to \mathbb{R}^{5}$

   (d) $F : \mathbb{R}^{5} \to \mathbb{R}^{38}$

   (e) $F : \mathbb{R}^{43} \to \mathbb{R}$

4. Which of these is true about a continuous function on an open set U?

   (a) There exists only one good sequence at all points on U

   (b) $\forall \mathbf{x_0} \in U, \forall \epsilon > 0, \exists \delta > 0$ s.t. $\forall \mathbf{x} \in U, |\mathbf{x}-\mathbf{x_0}| < \delta \implies |f(\mathbf{x})-f(\mathbf{x_0})| < \epsilon$

   (c) $\forall \epsilon > 0, \exists \delta > 0$ s.t. $, \forall \mathbf{x}, \mathbf{x_0} \in U, |\mathbf{x} - \mathbf{x_0}| < \delta \implies |f(\mathbf{x}) - f(\mathbf{x_0})| < \epsilon$

   (d) The function is bounded on U

   (e) The function is differentiable on U

5. Which of these is not true about a function $f$ that has continuous partial derivatives on an open set U?

   (a) The function achieves its maximum on the set U

   (b) The function is continuous on the set U

   (c) The function is $C^1$ on the set U

   (d) The derivative is a linear function of the direction

   (e) The directional derivative in the direction $\vec{v}$, where $\vec{v}$ is a unit vector, at point $\mathbf{c}$ is given by $[Df\mathbf{c}]\vec{v}$

6. What are all the critical points of the function $f\begin{pmatrix} y \\ x \end{pmatrix} = xy^2 + y^3 - 4x$? Note the change of order of the variables in the argument of the function!

   (a) The critical points are $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ and are both saddle points.

   (b) The critical points are $\begin{pmatrix} -2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and are both saddle points.

   (c) The critical points are $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ and are both maximums.

   (d) The critical points are $\begin{pmatrix} -2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and are both maximums.

   (e) The critical points are $\begin{pmatrix} -2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and are both minimums.

Part II(6 points, 2 per false statement) Of the following statements, more than three are false. Choose any three of the false statements and explain why they are false. For full credit you must both comment on what is wrong with the statement and also cite an explicit counterexample. Just ignore the true statements.

Example:

Statement: "Any two unequal nonzero vectors in $\mathbb{R}^2$ span $\mathbb{R}^2$."

Answer: "False: the vectors could be linearly dependent, like $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$"

1. For a compact set X, if there exists an infinite set of open sets, $\{U_i\}$, such that:

$$X \in \bigcup_{i=1}^{\infty} U_i$$

then there exists some finite subset of this set of sets, $\{V_i\} \subset \{U_i\}$, such that:

$$X \in \bigcup_{i=1}^{m} V_i$$

2. If $\mathbf{c}$ is a constrained critical point of function $f : \mathbb{R}^n \to \mathbb{R}$ on a k dimensional manifold M in $\mathbb{R}^n$ described by a locus function $\mathbf{F}$, then $\ker[Df(\mathbf{c})] \subset \ker[D\mathbf{F}(\mathbf{c})]$.

3. Given a k dimensional manifold M in $\mathbb{R}^n$, specified by locus function $\mathbf{F}(\mathbf{z})$, where $[D\mathbf{F}(\mathbf{z})]$ is onto, any ordering of the variables in $\mathbf{z}$ will result in the existence of an implicit function that expresses the first (first in order of variables of $\mathbf{z}$) $n-k$ passive variables in terms of the next $k$ active variables.

4. A continuous real valued function defined on a compact set has its maximum on that set.

5. A function $f$ is differentiable at $\mathbf{a}$ if:

$$\lim_{\vec{h}\to 0} \frac{1}{|\vec{h}|}(f(\mathbf{a}+\vec{h}) - f(\mathbf{a}) - [Df(\mathbf{a})]\vec{h}) = 0$$

6. If a real valued differentiable function $f$ at a point $\mathbf{c}$ has directional derivatives on a set of basis vectors equal to zero, then $\mathbf{c}$ is a critical point of $f$.

7. Consider the sequence of open sets $X_1, X_2, X_3, ...$ where $X_1 \supset X_2 \supset X_3 \supset ....$

$$\bigcap_{k=1}^{\infty} X_k \neq \phi$$

8. When trying to solve a system of equations, that are set equal to zero, using Newton's method, any initial guess is guaranteed to eventually superconverge to one of the roots of the system.

Part III. Answer seven of the eight questions. Mark an X in the score box on page 1 to indicate which question you have omitted.

1. (Inspired by Week 9, group problems 3)

   Applying Newton's second law of motion to a mass of 1 attached to a spring of spring constant 4, and a damping force that is equal to 4 times the velocity leads us to the following differential equation

$$\ddot{x} = -4\dot{x} - 4x$$

   Solve this differential equation for $x(t)$ given $x(0) = 4$, and $\dot{x}(0) = 0$. (Hint: To get started let $\vec{w} = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$, and write the differential equation in the form $\dot{\vec{w}} = A\vec{w}$)

2. (Inspired by Week 10, group problems 3)

(a) Let $f\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^2 - 1$. Evaluate the Jacobian Matrix at $f\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and use it to find the best affine approximation to $\begin{pmatrix} 1 \\ 0.1 \end{pmatrix}$.

(b) Let $g\begin{pmatrix} r \\ t \end{pmatrix} = \begin{pmatrix} r\cos t \\ r\sin t \end{pmatrix}$. Evaluate the Jacobian Matrix at $g\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and find r and h such that $g\begin{pmatrix} 1+r \\ h \end{pmatrix} = \begin{pmatrix} 1 \\ 0.1 \end{pmatrix}$.

3. (Inspired by Week 11, group problems 1)

Consider the functions $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2y \\ y + x^2 \\ 3x^2 + 2y \end{pmatrix}$ and $g\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} \sqrt{vw} \\ v + w \end{pmatrix}$.

(a) Calculate the derivative of $f \circ g \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ by using the chain rule.

(b) Calculate the derivative of $f \circ g \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ by explicitly calculating a formula for $f \circ g$, and then taking the derivative of that composition function.

4. (Inspired by Week 12, group problems 2)

For your Freshman Seminar Class, "What Would Life be Like on a Deserted Island?" your term project is to analyze the question that the seminar is named after as the culmination of a semester's worth of insightful discussion and watching "Lost". You have come to a perfect model of how a group of survivors of a plane crash ought to behave. This deserted island has polar bears that can be hunted, guns, for no good reason, and unchartered territory. There are 40 survivors, and they will do one of three tasks, hunt polar bears, $x$, use the guns to defend the **deserted** island (just cause there may be other peple on the island, oxymoronic? Don't think too hard about this one.) $y$, or to explore the rest of the island, $z$. You are constrained by the number of survivors, so $x+y+z=40$. Let people's happiness be given by

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3x + y^3 + \frac{3}{2}z^2$$

As you can see, the survivors really like guns, and seem to be more concerned with exploring the dangerous island than getting food.

(a) Use Lagrange Multipliers to find the constrained critical points.

(b) This plane can also be parameterized by $\gamma\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 40-s-t \\ s \\ t \end{pmatrix}$. Use this parameterization function to find the constrained critical points, and classify this critical point.

5. (Inspired by a proof or problem from lecture or homework)

Manifold M is described by the following parameterization function near the point $\mathbf{c} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

$$\gamma \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} uv + v^2 \\ \sqrt{uv} \\ \sqrt{v} \end{pmatrix}$$

(a) Use this parameterization function to find a basis for the tangent space at the point $\mathbf{c}$ which corresponds to the set of parameters $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(b) Find a locus function that also describes this manifold near $\mathbf{c}$.

(c) Confirm your answer to a) by also using the locus function to find a basis for the tangent space $T_{\mathbf{c}}M$.

6. (Inspired by a proof or problem from lecture or homework)

Patrick is considering how many hours, $x$, he should devote to sleeping and how many hours, $y$ he should devote to studying for his Math 23 exam. He is under a couple of constraints. His mom demands that he exercises everyday, spends time for a good dinner with friends, and that he leaves time for talking to them. His parents demand that these three activities ought to total 8 hours of his day. Therefore, the total number of hours he can devote to sleeping and studying is 16 hours, namely, $x + y = 16$. However, Patrick is also in dire need of studying for math, and believes that his math time is worth more than sleeping. Therefore, he imposes the constraint that $x^2 + y = 36$. Namely, he should not sleep more than 6 hours a day.

(a) Use one iteration of Newton's method with an initial guess of 5.5 hours of sleep and 10.5 hours of studying to solve for the number of hours Patrick will sleep.

(b) Solve the system of equations exactly for how many hours Patrick will sleep. Note: This is not a good way to spend your days prior to the exam!

7. (Inspired by a proof or problem from lecture or homework)

   Prove that if a sequence $\mathbf{a_1}, \mathbf{a_2}...$ in $\mathbb{R}^n$ converges, then all of its coordinates are bounded.

8. (Inspired by a proof or problem from lecture or homework)

   Better Lawn Sprinkler Co. has been hired to install sprinklers to cover your entire 1 dimensional lawn. These sprinklers can only sprinkle water on an open set, for unknown reasons. Your lawn is the interval, $[0, 1)$. You have lost the boundary of the right half of your lawn due to a dispute with your neighbor, and therefore he decided to build a fence right on the boundary of your lawn. The manager of Better Lawn Sprinkler Co. is going to propose several plans of sprinkler installation.

   (a) The manager is feeling sneaky today. Invent an infinite open cover of this interval, where there does exist an finite subcover, only if you can find it, after reviewing your Math 23 notes.

   (b) The manager is feeling maliciously evil. Invent an infinite open cover of this interval, where no finite subcover exists, and the manager has succeeded in scamming you of all your money.

   (c) How can you modify your lawn, so that the manager will always fail to force you to buy an infinite number of sprinklers?

Part V. Do two of the three proofs. Mark an X in the score box on page 1 to indicate which proof you have omitted.

1. (Proof 9.2)

   Starting from the triangle inequality for two vectors, prove the triangle inequality for $n$ vectors, then prove the "infinite triangle inequality" for $\mathbb{R}^n$

$$|\sum_{i=1}^{\infty} \vec{\mathbf{a_i}}| \leq \sum_{i=1}^{\infty} |\vec{\mathbf{a_i}}|$$

   under the assumption that the infinite series on the right is convergent, which in turn implies that the infinite series of vectors on the left is convergent.

2. (Proof 11.2)

   Using the mean value theorem, prove that if a function $f : \mathbb{R}^2 \to \mathbb{R}$ has partial derivatives $D_1 f$ and $D_2 f$ that are continuous at $\mathbf{a}$, it is differentiable at $\mathbf{a}$ and its derivative is the Jacobian matrix $\begin{bmatrix} D_1 f(\mathbf{a}) & D_2 f(\mathbf{a}) \end{bmatrix}$.

3. (Proof 12.3)

   Let $M$ be a manifold known by a real-valued $C^1$ function $F(\mathbf{x}) = 0$, where $F$ goes from an open subset $U$ of $\mathbb{R}^n$ to $\mathbb{R}$ and $[\mathbf{D}F(\mathbf{x})]$ is nowhere zero. Let $f : U \to \mathbb{R}$ be a $C^1$ function.

   Prove that $\mathbf{c} \in M$ is a critical point of $f$ restricted to $M$ if and only if there exists a Lagrange multiplier $\lambda$ such that $[\mathbf{D}f(\mathbf{c})] = \lambda[\mathbf{D}F(\mathbf{c})]$.