MATHEMATICS E-23a, Fall 2017
Quiz #2 Practice Questions
November 2017

These questions were written by the course assistants in Fall 2015. They are quite skillfully done.

A handwritten document that also includes the answers is in the file Quiz 2 Review.pdf.

1. (Inspired by Week 5, group problems #1)

   (a) Starting from the triangle inequality $|a+b| \leq |a| + |b|$, show that

   $$|a| - |b| \leq |a - b|.$$

   (b) Using induction, show that:

   $$|a| - \sum_{i=1}^{n} |b_i| \leq |a - \sum_{i=1}^{n} b_i|.$$

2. (Inspired by Week 5, group problems #2)
   Given $\lim s_n = s$ and $\lim t_n = t$ (and $t_n \neq 0\ \forall n$ and $t > 0$), show that

$$\lim \frac{s_n}{t_n} = \frac{s}{t}.$$

3. (Inspired by Week 5, group problems #3)
   Let $s_1 = 1$ and for $n \geq 1$ let $s_{n+1} = \sqrt{s_n + 1}$.

   Given that $\lim s_n = s$, prove that

$$s = \frac{1}{2}(1 + \sqrt{5}).$$

4. (Inspired by Week 6, group problems #1)

   (a) Show that $\liminf(s_n + t_n) \geq \liminf s_n + \liminf t_n$ for bounded sequences s_n and t_n.

   (b) Invent an example where $\liminf(s_n + t_n) > \liminf s_n + \liminf t_n$.

5. (Inspired by Week 6, group problems #2)
   Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n \geq 1$.

   (a) Use induction to show that $s_n > \frac{1}{2}$ $\forall n$.

   (b) Show that s_n is a decreasing sequence.

   (c) Show that $\lim s_n$ exists and find $\lim s_n = s$.

6. (Inspired by Week 6, group problems #3)
   Fine the radius of convergence R and the exact interval of convergence of the series

$$\sum x^{n!}.$$

7. (Inspired by Week 7, group problems #1)
   Prove that if f and g are real-valued functions that are continuous at $x_0 \in \mathbb{R}$, then fg is continuous at x_0 by

   (a) ϵ/δ definition of continuity.

   (b) "no bad sequence" definition of continuity.

8. (Inspired by Week 7, group problems #2)
   Show that $\sin x = \cos x$ for some $x \in (0, \frac{\pi}{2})$.

9. (Inspired by Week 7, group problems #3)
Evaluate the following limit without using L'Hospital's Rule, then check using L'Hospital's Rule:

$$\lim_{x \to 0} \frac{\cos 2x - \cos x}{x^2}.$$

You may use $\lim_{x \to 0} \frac{\sin x}{x} = 1$; $\cos 2x = 1 - 2\sin^2 x$;$\sin 2x = 2 \sin x \cos x$.

10. (Inspired by Week 8, group problems #1)
Let

$$f(x) = x^{\frac{3}{4}}.$$

Find the derivative $f'(x)$

(a) using the definition of the derivative as a limit.

(b) by rising both sides to the 4th power and using the chain rule.

11. (Inspired by Week 8, group problems #2)
Let $g(y) = \arccos y^2$.

Find $g'(y)$ by finding and differentiating the inverse function $y = f(x)$.

You can check your answer by using the chain rule and the derivative of the arccos function.

12. (Inspired by Week 8, group problems #3)
    Construct the Taylor series for the function $f(x) = \ln(1 + x)$, and use Taylor's theorem with remiander to show that the series converges to the function for $x \leq 1$.