Week 5.1

a) Starting from the triangle inequality $|a+b| \leq |a| + |b|$, show that: $|a| - |b| \leq |a-b|$

b) Using induction, show that: $|a| - \sum_{i=1}^{n} |b_i| \leq |a - \sum_{i=1}^{n} b_i|$

~~$a, b \in \mathbb{N}$ $\exists n \in \mathbb{N}$ s.t. $an > b$~~

a) $|a| = |a|$

$= |a - b + b|$

$|a| \leq |a-b| + |b|$

$|a| - |b| \leq |a-b|$

$\underset{\sim\sim\sim}{|a| \leq |a-b| + |b|}$

b) $\underline{|a|} - \sum_{i=1}^{k} |b_i| \leq |a - \sum_{i=1}^{k} b_i|$

$|a| - \underbrace{\sum_{i=1}^{k} |b_i| - |b_{k+1}|}_{= \sum_{i=1}^{k+1} |b_i|} \leq \underbrace{|a - \sum_{i=1}^{k} b_i|} - \underbrace{|b_{k+1}|} \leq |a - \underbrace{\sum_{i=1}^{k} b_i - b_{k+1}}_{= \sum_{i=1}^{k} b_i}|$ ✓

Week 5.2

Given $\lim s_n = s$ and $\lim t_n = t$ (and $\underline{t_n \neq 0 \; \forall n}$ and $\underline{t > 0}$), show that $\underline{\lim s_n / t_n = s/t}$.

5.2 $\lim \frac{1}{t_n} = \frac{1}{t}$ $\quad \lim(s_n a_n) = sa$, $\quad a_n = \frac{1}{t_n}$; $a = \frac{1}{t}$

$\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n > N$, $|s_n a_n - \underline{sa}| < \varepsilon$

$|s_n a_n - s_n a + s_n a - sa| = |s_n (a_n - a) + (s_n - s) a|$ Δ ineq.

$\leq |s_n||a_n - a| + |s_n - s||a|$ C-S

$\qquad\quad M_1 \qquad\qquad\qquad \frac{\varepsilon}{2M_2} \quad M_2$

$\exists M_1 > 0$ s.t. $\forall n \in \mathbb{N}$, $|s_n| < M_1$

$\exists M_2 > 0$ s.t. $\forall n \in \mathbb{N}$, $|a_n| < M_2$

$\forall \varepsilon > 0$, $\exists N_1 \in \mathbb{N}$ s.t. $\forall n > N_1$, $|s_n - s| < \varepsilon / 2M_2$

$\forall \varepsilon > 0$, $\exists N_2 \in \mathbb{N}$ s.t. $\forall n > N_2$, $|a_n - a| < \varepsilon / 2M_1$

$|s_n a_n - sa| < M_1 \cdot \frac{\varepsilon}{2M_1} + \frac{\varepsilon}{2M_2} \cdot M_2$ $\qquad \lim s_n a_n = sa$

$< \varepsilon/2 + \varepsilon/2$

$< \varepsilon$ ✓

Week 5.3

Let $s_1 = 1$ and for $n \geq 1$ let $s_{n+1} = \sqrt{s_n + 1}$. Given that $\lim s_n = s$, prove that $s = \frac{1}{2}(1+\sqrt{5})$.

$$\lim s_n = \lim s_{n+1} = s$$

$$\lim s_n = \lim \sqrt{s_n + 1}$$

$$= \sqrt{\lim(s_n) + 1}$$

$$\lim s_n = \underbrace{\sqrt{s+1} = s},$$

$$s+1 = s^2$$

$$0 = s^2 - s - 1$$

$$s = \frac{1 \pm \sqrt{1-4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}$$

$\frac{1-\sqrt{5}}{2}$ ext.

$\frac{1}{2}(1+\sqrt{5}) = s$

Week 6.1

a) Show that $\lim \inf(s_n + t_n) \geq \lim \inf(s_n) + \lim \inf(t_n)$ for bounded sequences s_n and t_n.

b) Invent an example where $\lim \inf(s_n + t_n) > \lim \inf(s_n) + \lim \inf(t_n)$

a) $\inf(s_n + t_n) \geq \inf(s_n) + \inf(t_n)$

$$\lim_{N\to\infty} \inf\{(s_n + t_n) : n > N\} \geq \lim_{N\to\infty} \inf\{s_n : n > N\} + \lim_{N\to\infty} \inf\{t_n : n > N\}$$

$\lim \inf(s_n + t_n) \geq \lim \inf(s_n) + \lim \inf(t_n)$ ✓

b) $s_n = \{2, 1, 2, 1, \ldots\}$ $\lim \inf s_n = 1$

$t_n = \{1, 2, 1, 2, \ldots\}$ $\lim \inf t_n = 1$

$s_n + t_n = \{3, 3, 3, 3, \ldots\}$ $\lim \inf(s_n + t_n) = 3$

$1 + 1 < 3$

Week 6.2

Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n > 1$.

a) Use induction to show $s_n > 1/2$ $\forall n$.

b) Show that s_n is a decreasing sequence.

c) Show that $\lim s_n$ exists and find $\lim s_n = s$.

a) $s_1 = 1 > 1/2$

$s_2 = \frac{1}{3}(1+1) = 2/3 > 1/2$

assume: $s_{n-1} > 1/2$; $\underline{wts}$: $s_n > 1/2$

$s_n = \frac{1}{3}(s_{n-1}+1) > \frac{1}{3}(\frac{1}{2}+1)$ $s_{n-1} > 1/2$

$> \frac{1}{\cancel{3}} \cdot \frac{\cancel{3}}{2}$

$> 1/2$ ✓

b) $\underline{wts}$: $s_{n+1} < s_n$ $\forall n$

$s_n - s_{n+1} > 0$ $\forall n$

$s_n - s_{n+1} = s_n - \frac{1}{3}(s_n + 1)$

$= \frac{2}{3}s_n - \frac{1}{3}$

$> \frac{\cancel{2}}{3} \cdot \frac{1}{\cancel{2}} - 1/3$ since $s_n > 1/2$

> 0

c) s_n bounded above/below $\Rightarrow$ converges

$s_n > 1/2$ $\forall n$) s_n decreasing

$\lim s_{n+1} = \lim s_n = s$

$s_{n+1} = \frac{1}{3}(s_n + 1)$

$\lim(s_{n+1}) = \frac{1}{3}(\lim(s_n)+1)$

$s = \frac{1}{3}(s+1)$

$\boxed{s = 1/2}$

Week 6.3

Find radius of convergence R and exact interval of convergence of the series: $\sum x^{n!}$ $\underline{\text{* ratio test}}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{(n+1)!}}{x^{n!}}\right| = \left|\frac{x^{n!(n+1)}}{x^{n!}}\right| = \left|\left(\frac{x^{(n+1)}}{x}\right)^{n!}\right| = \left|\left(\frac{\cancel{x}\cdot x^n}{\cancel{x}}\right)^{n!}\right| = \left|(x^n)^{n!}\right| \le 1$$

$=?$

$\underbrace{(x^n)^{n!}} \ge \underbrace{x^n}$ for $x \ge 1$, $\sum x^n$ $\underline{\text{diverges}}$ for $|x| \ge 1$

$(x^n)^{n!} \le x^n$ for $x \le 1$, $\sum x^n$ converges for $x < 1$ $\underline{R = 1}$

$((-1)^n)^{n!} = \underline{((-1)^n)^{n!}}$ $\Rightarrow \sum((-1)^n)^{n!}$ does not converge $x \ne +1$

$n!$ is even for $n > 2$

so $\sum((-1)^n)^{n!} = (-1)^{1\cdot 1} + (-1)^{2\cdot 2} + (1)^{\text{even pwr}}$

$\underline{(-1, 1)}$

~~f cont. (a,b)~~

~~a<b ; y between f(a) and f(b)~~

~~∃ x ∈ (a,b) s.t. f(x)=y~~

~~$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$~~

~~iff $\forall x_n$ s.t. $\lim x_n = x_0$, $\lim f(x_n) = f(x_0)$~~

Week 7.2

Show that $\underline{\sin(x) = \cos(x)}$ for some $x \in (0, \pi/2)$

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x, y \in S, |x-y| < \delta$

$\Rightarrow |f(x) - f(y)| < \varepsilon$

$f(x) = \sin(x) - \cos(x)$ $c \in (0, \pi/2)$

$f(0) = \cancel{\sin(0)}^{0} - \cancel{\cos(0)}^{1}$

$f(\pi/2) = \cancel{\sin(\pi/2)}^{1} - \cancel{\cos(\pi/2)}^{0}$

$f(\pi/2) = \underline{1}$

$f(0) = \underline{-1}$

$\exists c \in (0, \pi/2)$ s.t. $f(c) = 0 \Rightarrow \sin(c) = \cos(c)$ ✓

Week 7.1

Prove that if f and g are real-valued functions that are continuous at $x_0 \in \mathbb{R}$, then fg is continuous at x_0 by:

a) ε/δ definition of continuity

b) "no bad sequence" definition of continuity

$$\frac{x\cancel{k}+h^2}{k}$$

a) wts: $|x-x_0| < \delta \Rightarrow |f(x)g(x) - f(x_0)g(x_0)| < \varepsilon$

$$|f(x)g(x) - f(x_0)g(x_0)| \le |f(x)g(x) - f(x)g(x_0)| + |f(x)g(x_0) - f(x_0)g(x_0)|$$
$$\le |f(x)||g(x)-g(x_0)| + |g(x_0)||f(x)-f(x_0)|$$

$f(x)$ cont: $\forall \varepsilon > 0, \exists \delta_1 > 0$ s.t. $|x-x_0| < \delta_1 \Rightarrow |f(x)-f(x_0)| < \frac{\varepsilon}{2|g(x_0)|}$

$g(x)$ cont: $\forall \varepsilon > 0, \exists \delta_2 > 0$, s.t. $|x-x_0| < \delta_2 \Rightarrow |g(x)-g(x_0)| < \varepsilon/2M$

we know $|f(x)|$ is bounded within ε of $f(x_0)$: $|f(x)| < M$ for $|x-x_0| < \delta$

Choose $\delta = \min(\delta_1, \delta_2)$: $|f(x)g(x) - f(x_0)g(x_0)| < M \cdot \frac{\varepsilon}{2M} + \cancel{|g(x_0)|} \cdot \frac{\varepsilon}{2\cancel{|g(x_0)|}}$

$< \varepsilon/2 + \varepsilon/2$

$< \varepsilon$

$|g(x_0)| + 1$

b) f cont. $\lim x_n = x_0 \Rightarrow \lim f(x_n) = f(x_0)$

g cont. $\lim x_n = x_0 \Rightarrow \lim g(x_n) = g(x_0)$ $\Big\} \forall x_n \to x_0$

$\lim(f(x_n) \cdot g(x_n)) = \lim f(x_n) \lim g(x_n) = f(x_0)g(x_0)$ ✓

$x_n \to x_0 \Rightarrow f(x_n)g(x_n) \to f(x_0)g(x_0)$

Week 7.3

Evaluate the following limit without using L'Hôpital's Rule, ~~[illegible]~~ then check using L'Hôpital's rule:

$$\lim_{x\to 0} \frac{\cos(2x) - \cos(x)}{x^2}$$

You may use: $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$; $\cos(2x) = 1 - 2\sin^2(x)$; $\sin(2x) = 2\sin(x)\cos(x)$

$$\frac{\cos(2x)-\cos(x)}{x^2} = \frac{1-2\sin^2(x)-\cos(x)}{x^2} = \frac{-2\sin^2(x)}{x^2} + \frac{1-\cos(x)}{x^2}$$

$\sin^2 + \cos^2 = 1$

$$\frac{(1-\cos(x))}{x^2}\frac{(1+\cos(x))}{(1+\cos(x))} = \frac{1-\cos^2(x)}{x^2(1+\cos(x))} = \frac{\sin^2(x)}{x^2} \cdot \frac{1}{1+\cos(x)}$$

$\lim a^2 = (\lim a)^2$

$$\lim_{x\to 0} \frac{\cos(2x)-\cos(x)}{x^2} = \lim_{x\to 0}\left[-\frac{2\sin^2(x)}{x^2} + \frac{\sin^2(x)}{x^2} \cdot \frac{1}{1+\cos(x)}\right]$$

$$= -2 \cdot \left(\lim_{x\to 0} \frac{\sin(x)}{x}\right)^2 + \left(\lim_{x\to 0}\frac{\sin(x)}{x}\right)^2 \cdot \lim_{x\to 0}\left(\frac{1}{1+\cos(x)}\right)$$

$$= -2(1)^2 + 1^2 \cdot \frac{1}{1+1} = -1.5$$

$$\lim_{x\to 0} \frac{\cos(2x)-\cos(x)}{x^2} = \lim_{x\to 0} \frac{-\sin(2x)\cdot 2 + \sin(x)}{2x} = \lim_{x\to 0} \frac{-2\cos(2x)\cdot 2 + \cos(x)}{2}$$

$$= \frac{-2(1)\cdot 2 + 1}{2} = -\frac{3}{2} = -1.5$$ ✓

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \qquad \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{d}{dx} x^{3/4} = \frac{3}{4} x^{-1/4}$$

Find the derivative with a) limit definition and b) chain rule

a) $$\lim_{h \to 0} \frac{(x+h)^{3/4} - x^{3/4}}{h} \cdot \frac{(x+h)^{3/4} + x^{3/4}}{(x+h)^{3/4} + x^{3/4}}$$

$$\lim_{h \to 0} \frac{(x+h)^{3/2} - x^{3/2}}{h\left((x+h)^{3/4} + x^{3/4}\right)} \cdot \frac{(x+h)^{3/2} + x^{3/2}}{(x+h)^{3/2} + x^{3/2}}$$

$$\lim_{h \to 0} \frac{(x+h)^3 - x^3}{h\left((x+h)^{3/4} + x^{3/4}\right)} \cdot \frac{1}{(x+h)^{3/2} + x^{3/2}}$$

$$\lim_{h \to 0} \frac{\cancel{x^3} + 3x^2\cancel{h} + 3xh^{\cancel{2}} + h^{\cancel{3}\,2} - \cancel{x^3}}{\cancel{h} \sim \sim \sim}$$

$$\frac{3x^2}{2 \cdot x^{3/4} \cdot 2x^{3/2}} = \frac{3}{4} x^{-1/4}$$

b) $f(x) = x^{3/4}$

$f(x)^4 = x^3$

$4 \cdot f(x)^3 \cdot f'(x) = 3x^2$

$$f'(x) = \frac{3}{4} \frac{x^2}{f(x)^3} = \frac{3}{4} \cdot \frac{x^2}{x^{9/4}} = \frac{3}{4} \cdot x^{-1/4}$$

Find $g'(y)$ if:

$g(y) = \arccos y^2$ $\quad x = g(y)$

#6

$x = \arccos y^2$

$f(x) = x$

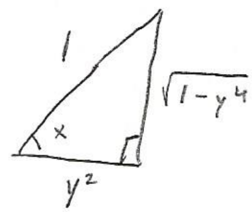
$\cos x = y^2$

$f(x) = x + 3$

$f(x) = y = \sqrt{\cos x}$

$f'(x) = \frac{1}{2}(\cos x)^{-1/2} \cdot (-\sin x)$

$g'(y) = \frac{1}{f'(x)} = \frac{-2\sqrt{\cos x}}{\sin x}$ $\quad x = \arccos y^2$



$= \frac{-2\sqrt{y^2}}{\sqrt{1-y^4}} = \frac{-2y}{\sqrt{1-y^4}}$

$f(x) = f(0) + f(0)'x + \dots \frac{f^n(0)}{n!}x^n$

$f(x) - f(0) - f'(0)x - \dots \frac{f^{n-1}(0)}{(n-1)!}x^{n-1} = \frac{f^n(y)}{n!}x^n$

See video for explanation of Taylor's theorem with remainder

$y \in (0, x)$

Show that for $\ln(1+x)$ remainder goes to 0 at $x = 1$

$\log(1+x) = \ln(1+x)$

$(\ln(x))' = \frac{1}{x}$ $\quad \frac{1}{1+x}, \frac{-1}{(1+x)^2}, \frac{2}{(1+x)^3}, \dots \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$

$f^n(y)$ $\left(\frac{(-1)^{n-1}\cancel{(n-1)!}}{(1+y)^n}\right) \cdot \frac{1}{\cancel{n!}\,n} \cdot \cancel{x^n}$ $\quad 1^n = 1$

$\lim_{n \to \infty} \frac{(-1)^{n-1}}{n} \cdot \left(\frac{1}{1+y}\right)^n$ $\quad y \in (0, x)$

$y \in (0, 1)$

$0 \cdot 0 = 0$

$$S(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \quad , \quad C(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

Gives $\quad S'(x) = C(x) \qquad C'(x) = -S(x)$

w.t.s. $S^2(x) + C^2(x) = 1$

$$(S^2(x) + C^2(x))'$$

$$(S^2(x))' + (C^2(x))' =$$

$$= 2\,S(x)\,S'(x) + 2\,C(x)\,C'(x)$$

$$= 2\,S(x)\,C(x) + 2\,C(x)\,(-S(x))$$

$$= 0$$

$$S(0) = 0 \qquad C(0) = 1$$

$1^2 + 0^2 = \cancel{10} \; 1$ HAHAHA Sebastian is sooooo funny

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$f(x)$ twice. dif. , $f'' < 0$ (a,b) some $x \in (a,b)$ $f'(x) = 0$

$\forall y \in (x,b) \qquad f(y) < f(x)$

$$f'(c) = \frac{f(y) - f(x)}{y - x} \qquad c \in (x,y) \in (x,b)$$

$$\overset{\text{negative}}{f''(d)} = \frac{f'(c) - f'(x)}{\underbrace{c - x}_{\text{positive}}}$$

negative $= f'(c) \cancel{- f'(x)}$

$$\overset{\text{negative}}{f'(c)} = \frac{f(y) - f(x)}{\underbrace{y - x}_{\text{positive}}}$$

negative $= f(y) - f(x)$

$f(x) > f(y)$