MATHEMATICS 23a/E-23a, Fall 2018
Linear Algebra and Real Analysis I
Fortnight 12 (Tangent spaces, Critical points, Lagrange multipliers)

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The seminar and workshop happen during Reading Period.
Fortunately, the final exam is not until Dec. 17

Reading

- Hubbard, Section 3.2 (Tangent spaces)
- Hubbard, Section 3.6 (Critical points)
- Hubbard, Section 3.7 through page 354 (constrained critical points)

Recorded Lectures

- Lecture 25 (Fortnight 12, Class 2) (watch on December 4 or 5)
- Lecture 26 (Fortnight 12, Class 3) (watch on December 6 or 7)

Proofs to present in section or to a classmate who has done them.

- 12.1(Hubbard, theorems 3.6.3 and 3.7.1) Let $U \in \mathbb{R}^n$ be an open subset and let $f : U \to \mathbb{R}$ be a C^1 (continuously differentiable) function.
  First prove, using a familiar theorem from single-variable calculus, that if $\mathbf{x}_0 \in U$ is an extremum, then $[\mathbf{D}f(\mathbf{x}_0)] = [0]$.
  Then prove that if $M \subset \mathbb{R}^n$ is a k-dimensional manifold, and $\mathbf{c} \in M \cap U$ is a local extremum of f restricted to M, then $T_{\mathbf{c}}M \subset \ker[\mathbf{D}f(\mathbf{c})]$.

- 12.2(Special case of Hubbard, theorem 3.7.5) Let M be a manifold known by a real-valued C^1 function $F(\mathbf{x}) = 0$, where F goes from an open subset U of $\mathbb{R}^n$ to $\mathbb{R}$ and $[\mathbf{D}F(\mathbf{x})]$ is nowhere zero. Let $f : U \to \mathbb{R}$ be a C^1 function.

  Prove that $\mathbf{c} \in M$ is a critical point of f restricted to M if and only if there exists a Lagrange multiplier λ such that $[\mathbf{D}f(\mathbf{c})] = \lambda[\mathbf{D}F(\mathbf{c})]$.

R Scripts

- Script 3.4A-ImplicitFunction.R
  Topic 1 - Three variables, one constraint
  Topic 2 - Three variables, two constraints
- Script 3.4B-Manifolds2D.R
  Topic 1 - A one-dimensional submanifold of $\mathbb{R}^2$ – the unit circle
  Topic 2 - Interesting examples from the textbook
  Topic 3 - Parametrized curves in $\mathbb{R}^2$
  Topic 4 - A two-dimensional manifold in $\mathbb{R}^2$
  Topic 5 - A zero-dimensional manifold in$\mathbb{R}^2$
- Script 3.4C-Manifolds3D.R
  Topic 1 - A manifold as a function graph
  Topic 2 - Graphing a parametrized manifold
  Topic 3 - Graphing a manifold that is specified as a locus
- Script 3.4D-CriticalPoints
  Topic 1 - Behavior near a maximum or minimum
  Topic 2 - Behavior near a saddle point
- Script 3.5A-LagrangeMultiplier.R
  Topic 1 - Constrained critical points in $\mathbb{R}^2$

1 Executive Summary

1.1 Using the implicit function theorem

Start with an open subset $U \subset \mathbb{R}^n$ and a C^1 function $\mathbf{F} : U \to \mathbb{R}^{n-k}$. Consider the "locus," $M \cap U$, the set of solutions of the equation $\mathbf{F}(\mathbf{z}) = 0$.
If $[\mathbf{DF}(\mathbf{z})]$ is onto (surjective) for every $\mathbf{z} \in M \cap U$, then $M \cap U$ is a smooth k-dimensional manifold embedded in $\mathbb{R}^n$.
Proof: the implicit function theorem says precisely this. The statement that $[\mathbf{DF}(\mathbf{z})]$ is onto guarantees the differentiability of the implicitly defined function.
If $[\mathbf{DF}(\mathbf{z})]$ does not exist or fails to be onto, perhaps even just at a single point, the locus is not a manifold. We use the notation $M \cap U$ because $\mathbf{F}$ may define just part of a larger manifold M that cannot be described as the locus as a single function. To say that M itself is a manifold, we have to find an appropriate U and $\mathbf{F}$ for every point $\mathbf{z}$ in the manifold.

1.2 Parametrizing a manifold

For a k-dimensional submanifold of $\mathbb{R}^n$, the parametrization function is $\gamma : U \to M$, where $U \subset \mathbb{R}^k$ is an open set. The variables in $\mathbb{R}^k$ are called "parameters." The parametrization function γ must be C^1 , one-to-one, and onto M. In other words, we want γ to give us the entire manifold. Finding a local parametrization that gives part of the manifold is of no particular interest, because there is, by definition, a function graph that does that.

An additional requirement: The derivative of the parametrization function is one-to-one for all parameter values. This requirement guarantees that the columns of the the Jacobian matrix $[\mathbf{D}\gamma]$ are linearly independent.

1.3 Tangent space as graph, kernel, or image

Locally, a k-dimensional submanifold M of $\mathbb{R}^n$ is the graph of a function
$\mathbf{g} : \mathbb{R}^k \to \mathbb{R}^{n-k}$. The derivative of $\mathbf{g}$, $[\mathbf{Dg}(\mathbf{b})]$, is an $(n-k) \times k$ matrix that converts a vector of increments to the k active variables, $\dot{\mathbf{y}}$, into a vector of increments to the $n-k$ passive variables, $\dot{\mathbf{x}}$. That is, $\dot{\mathbf{x}} = [\mathbf{Dg}(\mathbf{b})](\dot{\mathbf{y}})$
A point $\mathbf{c}$ of M is specified by the active variables $\mathbf{b}$ and the accompanying passive variables $\mathbf{a}$. The tangent space $T_M(\mathbf{c})$ is the *graph* of this derivative. It is a k-dimensional subspace of $\mathbb{R}^n$.
The k-dimensional manifold M can also be specified as the locus of the equation $\mathbf{F}(\mathbf{z}) = 0$, for $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^{n-k}$ The tangent space $T_{\mathbf{c}}M$ is the *kernel* of the linear transformation $[\mathbf{DF}(\mathbf{c})]$.
Finally, the manifold M can also be described as the image of a parametrization function $\gamma : U \subset \mathbb{R}^k \to \mathbb{R}^n$,

In this case any point of M is the image of some point $\mathbf{u}$ in the parameter space, and the tangent space is $T_{\gamma(\mathbf{u})}M = \operatorname{Img}\,[\mathbf{D}\gamma(\mathbf{u})]$. Whether specified as graph, kernel, or image, the tangent space $T_{\mathbf{c}}M$ is the same! It contains the increment vectors that lead from $\mathbf{c}$ to nearby points that are "almost on the manifold."

1.4 Critical points

Suppose that function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at point $\mathbf{x}_0$ and that the derivative $[Df(\mathbf{x}_0)]$ is not zero. Then there exists a vector $\vec{\mathbf{v}}$ for which the directional derivative is not zero, the function $g(t) = f(\mathbf{x}_0 + t\vec{\mathbf{v}} - f(\mathbf{x}_0)$ has a nonzero derivative at $t = 0$, and, even if we just consider points that lie on a line through $\mathbf{x}_0$ with direction vector $\vec{\mathbf{v}}$, the function f cannot have a maximum or minimum at $\mathbf{x}_0$. So in searching for a a maximum or minimum of f at points where it is differentiable, we need to consider only "critical points" where $[Df(\mathbf{x}_0] = 0$.

A critical point is not necessarily a maximum or minimum, but for $f : \mathbb{R}^n \to \mathbb{R}$ there is a useful test that generalizes the second-derivative test of single-variable calculus. The proof relies on sections 3.3-3.5 of Hubbard, which we are skipping.

Form the "Hessian matrix" of second partial derivatives (Hubbard, p. 348), evaluated at the critical point x of interest.

$$H_{i,j}(\mathbf{x}) = D_i D_j f(\mathbf{x}).$$

H is a symmetric matrix. If it has a basis of eigenvectors and none of the eigenvalues are zero, we can classify the critical point.

If H has a basis of eigenvectors, all with positive eigenvalues, the critical point is a minimum.

If H has a basis of eigenvectors, all with negative eigenvalues, the critical point is a maximum.

If H has a basis of eigenvectors, some with positive eigenvalues and some with negative eigenvalues, the critical point is a saddle: it is neither a maximum or a minimum.

1.5 Constrained critical points

These are of great important in physics, economics, and other areas to which mathematics is applied.

Consider a point $\mathbf{c}$ on manifold M where the function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable. Perhaps f has a maximum or minimum at c when its value is compared to the value at nearby points on M, even though there are points not on M where f is larger or smaller. . In that case we should not consider all increment vectors, but only those increment vectors $\vec{\mathbf{v}}$ that lie in the tangent space to the manifold. The derivative $[\mathbf{D}f(\mathbf{c})]$ does not have to be the zero linear transformation, but it has to give zero when applied to any increment that lies in the tangent space $T_{\mathbf{c}}M$, or

$T_{\mathbf{c}}M \subset \mathrm{Ker}[\mathbf{D}f(\mathbf{c})]$.

When manifold M is specified as the locus where some function $\mathbf{F} = 0$, there is an ingenious way of finding constrained critical points by using "Lagrange multipliers,"but not this week!

1.6 Constrained critical points - three approaches

We have proved the following:
If $M \subset \mathbb{R}^n$ is a k-dimensional manifold, and $\mathbf{c} \in M \cap U$ is a local extremum of f restricted to M, then $T_{\mathbf{c}}M \subset \ker[\mathbf{D}f(\mathbf{c})]$.

Corresponding to each of the three ways that we can "know" the manifold M, there is a technique for finding the critical points of f restricted to M.

- Manifold as a graph
  Near the critical point, the passive variables $\mathbf{x}$ are a function $\mathbf{g}(\mathbf{y}$ of the active variables $\mathbf{y}$. Define the graph-making function

  $$\tilde{\mathbf{g}}(\mathbf{y}) = \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}$$

  Now $f(\mathbf{g}(\mathbf{y})$ specifies values of f only at points on the manifold. Just search for unconstrained critical points of this function by setting $[Df \circ \tilde{\mathbf{g}}(\mathbf{y})] = 0$. This approach works well if you can represent the entire manifold as a single function graph.

- Parametrized manifold
  Points on the manifold are specified by a parametrization $\gamma(\mathbf{u})$.
  Now $f(\gamma(\mathbf{u}))$ specifies values of f only at points on the manifold. Just search for unconstrained critical points of this function by setting $[Df \circ \gamma(\mathbf{u})] = 0$. This approach works well if you can parametrize the entire manifold.

- Manifold specified by constraints
  Points on the manifold all satisfy the constraints $\mathbf{F}(\mathbf{x}) = 0$.
  In this case we know that
  $T_{\mathbf{c}}M = \mathrm{Ker}[\mathbf{DF}(\mathbf{c})]$, so the rule for a critical point becomes
  $\mathrm{Ker}[\mathbf{DF}(\mathbf{c})] \subset \mathrm{Ker}[\mathbf{D}f(\mathbf{c})]$.

  If there is just a single constraint $F(\mathbf{x}) = 0$, both derivative matrices consist of just a single row, and we can represent the condition for a critical point as Ker $\alpha \subset$ Ker β.
  Suppose that $\vec{\mathbf{v}} \in \ker\ \alpha$ and that $\beta = \lambda\alpha$. The quantity λ is called a Lagrange multiplier. Then by linearity, $[\mathbf{D}f(\mathbf{c})]\vec{\mathbf{v}} = \beta\vec{\mathbf{v}} = \lambda\alpha\vec{\mathbf{v}} = 0$.

  So $[\mathbf{D}f(\mathbf{c})]\vec{\mathbf{v}} = 0$ for any vector in the tangent space of $F = 0$, and we have a constrained critical point.

  It is not quite so obvious that the condition $\beta = \lambda\alpha$ is necessary as well as sufficient. We will need to do a proof by contradiction (proof 12.3).

2 Lecture Notes

1. Parametrizing a manifold

   It the spring term we will need to integrate over manifolds in order to evaluate the line integrals and surface integrals that are crucial in physics. The only practical way to do this is by parametrizing the manifold as the image of an open set in $\mathbb{R}^k$. The variables in $\mathbb{R}^k$ are called "parameters," and the functions on the manifold that give the parameter values are called "coordinate functions."

   What parameters do geographers use for the surface of the Earth?

   Here are the strict requirements for the parametrization of a k-dimensional manifold M in $\mathbb{R}^n$, as given in Hubbard, Definition 3.1.18.

   - The parametrization function is $\gamma : U \to M$, where $U \subset \mathbb{R}^k$ is an open set.

     What problem with differentiability would arise if U were not open?

   - The parametrization function γ is C^1 , one-to-one, and onto M. In other words, we want γ to give us the entire manifold. Finding a local parametrization that gives part of the manifold is of no particular interest, because there is, by definition, a function graph that does that.
   - The derivative of the parametrization function is one-to-one for all parameter values.

     If $k < n$, does $[\mathbf{D}\gamma]$ have more rows or more columns? Could it be onto in this case?

     What does this requirement say about the columns of $[\mathbf{D}\gamma]$?

     If $k = 1$ (a smooth curve), how can $[\mathbf{D}\gamma]$ fail to be one-to-one?

2. Examples of parametrizations, with some problems

   - A circle, traced out by a particle that goes around in 2π seconds, parametrized by time. The notation used is standard in physics texts.

$$\vec{\mathbf{r}}(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}, 0 \le t < 2\pi$$

     What requirement is violated, and why is there no way to fix the problem?

   - Polar coordinates to parametrize the manifold $\mathbb{R}^2$.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}, 0 \le \theta < 2\pi, r \ge 0$$

     Where is the requirement that $[\mathbf{D}\gamma]$ is one-to-one not met? What is the related problem with the coordinate function that assigns θ to each point in the plane?

   - Part of the unit sphere (a surface), parametrized by longitude θ and latitude ϕ. The choice of ϕ is standard in geography but not in physics.

$$\vec{\mathbf{r}}\begin{pmatrix} \theta \\ \phi \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ \sin\phi \end{pmatrix}, 0 < \theta < \pi, 0 < \phi < \frac{\pi}{2}$$

What part of the sphere is the manifold given by this parametrization?

What problems occur if we try to use this parametrization for the entire sphere?

3. Graph of a derivative

The concept of the tangent line to the graph of a function $f : \mathbb{R} \to \mathbb{R}$ is familiar from single-variable calculus. In general, a tangent line is just an affine subset, not a vector space. Similarly, the tangent plane to the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$ is an affine subset but not a vector space.

In either case, if the graph happened to pass through the origin, the tangent line or plane would be a vector space.

Locally, a k-dimensional submanifold M of $\mathbb{R}^n$ is the graph of a function $\mathbf{g} : \mathbb{R}^k \to \mathbb{R}^{n-k}$. We can denote the vector of k active variables by $\mathbf{b}$ and the vector of $n-k$ passive variables by $\mathbf{a} = \mathbf{g}(\mathbf{b})$.

For consistency with the implicit function theorem, we put the active variables last, so a point of the manifold is

$$\mathbf{c} = \begin{pmatrix} \mathbf{g}(\mathbf{b}) \\ \mathbf{b} \end{pmatrix}$$

The derivative of the function $\mathbf{g}$ for active variables $\mathbf{b}$, $[\mathbf{Dg}(\mathbf{b})]$,
is an $(n-k) \times k$ matrix. It converts a vector of increments to the k active variables, $\dot{\mathbf{y}}$, into a vector of increments to the $n-k$ passive variables, $\dot{\mathbf{x}}$. That is, $\dot{\mathbf{x}} = [\mathbf{Dg}(\mathbf{b})](\dot{\mathbf{y}})$

Previously we used $\vec{\mathbf{h}}$ to represent the vector of increments on which a derivative acts, but now we have adopted Hubbard's convention of using $\dot{\mathbf{y}}$ for the active "input" to the derivative and $\dot{\mathbf{x}}$ for the passive "output."

Prove that the graph of the derivative is a k-dimensional subspace of $\mathbb{R}^n$. This subspace is called the **tangent space**.

4. Tangent space as a graph (Hubbard Definition 3.2.1)

   Given a point on a manifold, represent the manifold locally as a function graph. Then the tangent space is the graph of the derivative of this function.

   It is true, but not immediately obvious from the definition, that the tangent space comes out the same if we make a different choice of active and passive variables. Try a special case: for example, the circle of radius 5 at the point $x = 3, y = 4$.

   What happens if you express x as a function of y?

   What happens if you express y as a function of x?

   Consider point $\mathbf{c}$ on a manifold M. Call the active variables $\mathbf{b}$ and the passive variables $\mathbf{a}$. The manifold is locally the graph of $\mathbf{x} = \mathbf{g}(\mathbf{y})$.

   Distinguish carefully:

   $\dot{\mathbf{x}} = [\mathbf{Dg}(\mathbf{b})]\dot{\mathbf{y}}$ is the equation whose graph is the tangent **space**.

   $\mathbf{x} - \mathbf{a} = [\mathbf{Dg}(\mathbf{b})](\mathbf{y} - \mathbf{b})$is the equation of the tangent **plane**.

   Example: The graph of $x = y^2 + z^2$ is a paraboloid, centered on the x axis.

   Find the equation of the tangent space at $x = 5, y = 2, z = 1$.

   Find the equation of the tangent plane at $x = 5, y = 2, z = 1$.

5. The equal-dimension lemma

   If subspaces V and W both have dimension k and $V \subset W$, then $V = W$.

   Proof: Choose k basis vectors for V. Since $V \subset W$, these vectors are also k independent vectors in W and therefore also form a basis for W. Thus any vector in W is a linear combination of these vectors and is also a vector in V, i.e., $W \subset V$. But $V \subset W$ and $W \subset V$ means $V = W$.

6. Tangent space as a kernel

   Suppose that a k-dimensional manifold M is specified as the locus of the equation $\mathbf{F}(\mathbf{z}) = 0$ for a function $\mathbf{F}$ whose domain is an open subset of $\mathbb{R}^n$ and whose codomain is $\mathbb{R}^{n-k}$. The implicit function theorem says that such a function is a satisfactory way to describe a manifold at any point $\mathbf{c}$ where $[\mathbf{DF}(\mathbf{c})]$ is onto.

   To prove: the tangent space $T_{\mathbf{c}}M$ is the kernel of the linear transformation $[\mathbf{DF}(\mathbf{c})]$.

   The proof (your proof 12.1) that this method of finding the tangent space is equivalent to the definition is on pp. 310-311 in Hubbard.

   - Write the matrix $[\mathbf{DF}(\mathbf{c})]$ as $[A|B]$, with the columns that correspond to passive variables coming first. Given that $[\mathbf{DF}(\mathbf{c})]$ is onto, the square matrix A will be invertible.
   - The dimension of the image of $[\mathbf{DF}(\mathbf{c})]$ is $n-k$. So the dimension of Ker $[\mathbf{DF}(\mathbf{c})]$, by the rank-nullity theorem, is $n-(n-k) = k$.
   - The derivative of the function $\mathbf{g}(\mathbf{b})$ that expresses the passive variables $\mathbf{a}$ locally in terms of the active variables $\mathbf{b}$ is given by the implicit function theorem:
     $[\mathbf{Dg}(\mathbf{b})] = -A^{-1}B$.

   - A vector in the tangent space can be written as $\dot{\mathbf{z}} = \begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{y}} \end{bmatrix}$, where $\dot{\mathbf{x}}$ and $\dot{\mathbf{y}}$ are increments to passive and active variables respectively. Suppose that $\begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{y}} \end{pmatrix}$ is an element of Ker $[\mathbf{DF}(\mathbf{c})]$.
     Thus $[A|B]\begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{y}} \end{pmatrix} = A\dot{\mathbf{x}} + B\dot{\mathbf{y}} = 0$.
     So $\dot{\mathbf{x}} = -A^{-1}B\dot{\mathbf{y}} = [\mathbf{Dg}(\mathbf{b})]\dot{\mathbf{y}}$.
     We have established that Ker $[\mathbf{DF}(\mathbf{c})]$ is a subspace $T_{\mathbf{c}}M$. But both these spaces have dimension k, so they are equal.

7. Tangent space as a kernel (Proof 12.1 - Hubbard Theorem 3.2.4)

   Suppose that $U \subset \mathbb{R}^n$ is an open subset, $\mathbf{F} : U \to \mathbb{R}^{n-k}$ is a C^1 mapping, and manifold M can be described as the set of points that satisfy $\mathbf{F}(\mathbf{z}) = 0$. Use the implicit function theorem to show that if $[\mathbf{DF}(\mathbf{c})]$ is onto for $\mathbf{c} \in M$, then the tangent space $T_{\mathbf{c}}M$ is the kernel of $[\mathbf{DF}(\mathbf{c})]$. You may assume that the variables have been numbered so that when you row-reduce $[\mathbf{DF}(\mathbf{c})]$, the first $n - k$ columns are pivotal.

8. Tangent space as an image (Hubbard, Proposition 3.2.7)

   Let $U \subset \mathbb{R}^k$ be open, and let $\gamma : U \to \mathbb{R}^n$ be a parametrization of manifold M. Show that

$$T_{\gamma(\mathbf{u})} M = \mathrm{Img}[\mathbf{D}\gamma(\mathbf{u})].$$

9. Summary - three ways to characterize the tangent space of manifold M

   - If M is represented as a function graph $\mathbf{x} = \mathbf{g}(\mathbf{y})$, the tangent space $T_{\mathbf{c}}M$ at $\mathbf{c} = \begin{pmatrix} \mathbf{g}(\mathbf{b}) \\ \mathbf{b} \end{pmatrix}$ is the **graph** of its derivative and $\dot{\mathbf{x}} = [\mathbf{Dg}(\mathbf{b})]\dot{\mathbf{y}}$.
   - If M is represented as the locus of $\mathbf{F}(\mathbf{z}) = \mathbf{0}$, the tangent space at $\mathbf{z} = \mathbf{c}$ is the **kernel** of the derivative of $\mathbf{F}$, $T_{\mathbf{c}}M=$ Ker $[\mathbf{DF}(\mathbf{c})]$, and $[\mathbf{DF}(\mathbf{c})] \begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{y}} \end{bmatrix} = \mathbf{0}$.
   - If M is represented by a parametrization function $\gamma : U \to \mathbb{R}^n$, the the tangent space at $\mathbf{c} = \gamma(\mathbf{u})$ is the **image** of the derivative of γ: $T_{\gamma(\mathbf{u})}M = \mathrm{Img}[\mathbf{D}\gamma(\mathbf{u})]$.

10. Exploring a manifold

A cometary-exploration robot is fortunate enough to land on an ellipsoidal comet whose surface is described by the equation

$$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 9.$$

Its landing point is $x = 2, y = 4, z = 3$.

- Prove that the surface of the comet is a smooth manifold.
- The controllers of the robot want it to move to a nearby point on the surface where $y = 4.02, z = 3.06$. Use the implicit function theorem to determine the approximate x coordinate of this point.
  (Check: $1.98^2 + 4.02^2/4 + 3.06^2/9 = 9.0009$.)
- Find a basis for the tangent space at the landing point.
- Find the equation of the tangent plane at the landing point.
  (Check: $4(1.98) + 2(4.02) + (2/3)(3.06) = 18$.)

11. Manifolds – keeping track of dimension

    Assume that, at the top level, there are nine categories $x_1, x_2, \cdots, x_9$ in the Federal budget. They must satisfy four constraints:

    - One simply fixes the total dollar amount.
    - One comes from your political advisors – it makes the budget looks good to likely voters in swing states.
    - One comes form Congress - it guarantees that everyone can have his or her earmarks.
    - One comes from the Justice Department – it guarantees compliance with all laws.

    These four constraints together define a function $\mathbf{F}$ whose derivative is onto for budgets that satisfy the constraints. The acceptable budgets, for which $\mathbf{F}(\mathbf{x}) = 0$, form a k-dimensional submanifold M of $\mathbb{R}^n$.
    Specify the dimension of the domain and codomain for

    (a) A function $\mathbf{g}$ that specifies that passive variables in terms of the active variables.
    (b) The function $\mathbf{F}$ that specifies the constraints.
    (c) A parametrization function γ that generates a valid budget from a set of parameters.

    For each alternative, specify the shape of the matrix that represents the derivative of the relevant function and explain how, given a valid budget $\mathbf{c}$, it could be used to find a basis for the tangent space $T_{\mathbf{c}}M$.

12. Necessary condition for an extremum (maximum or minimum)

    Recall the rule for a function of one variable:

    Let U be an open interval and $f : U \to \mathbb{R}$ a differentiable function. If x_0 is an extremum, then $f'(x_0) = 0$.

    In $\mathbb{R}^n$ the necessary condition for a differentiable function to have an extremum is the same.

    If $U \subset \mathbb{R}^n$ is open and $f : U \to \mathbb{R}$ is a differentiable function, then at an extremum $\mathbf{x}_0$ of f, $[\mathbf{D}f(\mathbf{x}_0)] = \mathbf{0}$.

    Define the function $g(t) = f(\mathbf{x}_0 + t\mathbf{e}_i)$.

    By the chain rule, $g'(0) = [\mathbf{D}f(\mathbf{x}_0)]\mathbf{e}_i = D_i f(\mathbf{x}_0)$.

    Since $g(t)$ has an extremum at $t = 0$, $g'(0) = D_i f(\mathbf{x}_0) = 0$. Illustrate this argument with a diagram for $\mathbb{R}^2$.

    Therefore each partial derivative of f is zero. Since f is differentiable, its derivative is zero at the extremum.

13. Finding critical points and extrema

    If $U \subset \mathbb{R}^n$ is open and $f : U \to \mathbb{R}$ is a differentiable function,
    a *critical point* of f is any point where its derivative $[\mathbf{D}f]$ vanishes.

    We just proved that an extremum of f must be a critical point. The converse is not true: it is common to have a critical point that is not an extremum.

    Show that for $f(x) = x^3$, $x = 0$ is a critical point but not an extremum.

    Example: $f\begin{pmatrix} x \\ y \end{pmatrix} = xy$.

    Evaluate $[\mathbf{D}f]$ at the origin.

    Find points near the origin where $f > 0$ and where $f < 0$.

14. Finding and classifying critical points

For a function defined on an open subset of $\mathbb{R}^n$, finding and classifying critical points is just an exercise in differentiation and algebra: evaluate all the partial derivatives, set them equal to zero, and try to solve the resulting set of (generally nonlinear) equations.

Here is a test for a critical point of a C^2 function to be a maximum or a minimum that generalizes what you learned in single-variable calculus. The proof relies on sections 3.3-3.5 of Hubbard, which we are skipping.

Form the "Hessian matrix" of second partial derivatives (Hubbard, p. 348), evaluated at the critical point x of interest.

$$H_{i,j}(\mathbf{x}) = D_i D_j f(\mathbf{x}).$$

H is a symmetric matrix. If it has a basis of eigenvectors and none of the eigenvalues are zero, we can classify the critical point.

If H has a basis of eigenvectors, all with positive eigenvalues, the critical point is a minimum.

If H has a basis of eigenvectors, all with negative eigenvalues, the critical point is a maximum.

If H has a basis of eigenvectors, some with positive eigenvalues and some with negative eigenvalues, the critical point is a saddle: it is neither a maximum or a minimum.

Why this test works (not quite a proof):

Suppose first that $f(\vec{\mathbf{x}}) = f(\vec{\mathbf{0}}) + \frac{1}{2}\lambda_1 x_1^2 + \frac{1}{2}\lambda_2 x_2^2 + \cdots + \frac{1}{2}\lambda_n x_n^2$.

Calculate $[\mathbf{D}f(\vec{\mathbf{0}})]$ and H.

Show that $\vec{\mathbf{0}}$ is a critical point and that the test is correct in this case.

Now suppose that all the eigenvalues are real and distinct.

Why is H symmetric?

Prove that any two eigenvectors $\vec{\mathbf{v}}_i$ and $\vec{\mathbf{v}}_j$ are orthogonal.

So we can make an orthonormal basis of eigenvectors. Relative to this basis, the situation is the same as on the preceding page.

Shortcomings of this argument:

- It assumes that the critical point is at the origin.
- It does not deal with the case where there are fewer than n eigenvalues.
- It applies only to quadratic functions, and we have not shown that any C^2 function can be approximated by such a function.
- It ignores the case where an eigenvalue is zero.

Quick test in $\mathbb{R}^2$.

Here $\det H = \lambda_1\lambda_2$. What can we conclude if

- $\det H < 0$?
- $\det H > 0$?
- $\det H = 0$?

15. Critical points

$f\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}x^2 + \frac{1}{3}y^3 - xy$

Calculate the partial derivatives as functions of x and y, and show that the only critical points are $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Calculate the Hessian matrix H and evaluate it numerically at each critical point to get matrices H_0 and H_1.

Find the eigenvalues of H_0 and classify the critical point at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Find the eigenvalues of H_1 and classify the critical point at $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

16. (Proof 12.2 – Hubbard, theorems 3.6.3 and 3.7.1)
Let $U \in \mathbb{R}^n$ be an open subset and let $f : U \to \mathbb{R}$ be a C^1 (continuously differentiable) function.
First prove, using a familiar theorem from single-variable calculus, that if $\mathbf{x}_0 \in U$ is an extremum, then $[\mathbf{D}f(\mathbf{x}_0)] = [0]$.
Then prove that if $M \subset \mathbb{R}^n$ is a k-dimensional manifold, and $\mathbf{c} \in M \cap U$ is a local extremum of f restricted to M, then $T_{\mathbf{c}}M \subset \ker[\mathbf{D}f(\mathbf{c})]$.

17. Constrained critical points - three approaches

We have proved the following:
If $M \subset \mathbb{R}^n$ is a k-dimensional manifold, and $\mathbf{c} \in M \cap U$ is a local extremum of f restricted to M, then $T_{\mathbf{c}}M \subset \mathrm{Ker}[\mathbf{D}f(\mathbf{c})]$.

Corresponding to each of the three ways that we can "know" the manifold M, there is a technique for finding the critical points of f restricted to M.

- Manifold as a graph
  Near the critical point, the passive variables $\mathbf{x}$ are a function $\mathbf{g}(\mathbf{y})$ of the active variables $\mathbf{y}$. Define the graph-making function

  $$\tilde{\mathbf{g}}(\mathbf{y}) = \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}$$

  Now $f(\tilde{\mathbf{g}}(\mathbf{y}))$ specifies values of f only at points on the manifold. Just search for unconstrained critical points of this function by setting $[Df \circ \tilde{\mathbf{g}}(\mathbf{y})] = 0$. This approach works well if you can represent the entire manifold as a single function graph.

- Parametrized manifold
  Points on the manifold are specified by a parametrization $\gamma(\mathbf{u})$.
  Now $f(\gamma(\mathbf{u}))$ specifies values of f only at points on the manifold. Just search for unconstrained critical points of this function by setting $[Df \circ \gamma(\mathbf{u})] = 0$. This approach works well if you can parametrize the entire manifold.

- Manifold specified by constraints
  Points on the manifold all satisfy the constraints $\mathbf{F}(\mathbf{x}) = 0$.
  In this case we know that
  $T_{\mathbf{c}}M = \mathrm{Ker}[\mathbf{DF}(\mathbf{c})]$, so the rule for a critical point becomes
  $\mathrm{Ker}[\mathbf{DF}(\mathbf{c})] \subset \mathrm{Ker}[\mathbf{D}f(\mathbf{c})]$.

  If there is just a single constraint $F(\mathbf{x}) = 0$, both derivative matrices consist of just a single row, and we can represent the condition for a critical point as $\mathrm{Ker}\ \alpha \subset \mathrm{Ker}\ \beta$.
  Suppose that $\vec{\mathbf{v}} \in \ker \alpha$ and that $\beta = \lambda\alpha$. The quantity λ is called a Lagrange multiplier. Then by linearity, $[\mathbf{D}f(\mathbf{c})]\vec{\mathbf{v}} = \beta\vec{\mathbf{v}} = \lambda\alpha\vec{\mathbf{v}} = 0$.

  So $[\mathbf{D}f(\mathbf{c})]\vec{\mathbf{v}} = 0$ for any vector in the tangent space of $F = 0$, and we have a constrained critical point.

  It is not quite so obvious that the condition $\beta = \lambda\alpha$ is necessary as well as sufficient. We will need to do a proof by contradiction (proof 12.3).

18. (Hubbard, theorem 3.7.5 - proof 12.3 is the special case where $m = 1$)
Let M be a manifold known by a real-valued C^1 function $\vec{\mathbf{F}}(\mathbf{x}) = \mathbf{0}$, where $\vec{\mathbf{F}}$ goes from an open subset U of $\mathbb{R}^n$ to $\mathbb{R}^m$ and $[\mathbf{DF}(\mathbf{x})]$ is onto.
Let $f : U \to \mathbb{R}$ be a C^1 function.

Prove that $\mathbf{c} \in M$ is a critical point of f restricted to M if and only if there exist m Lagrange multipliers $\lambda_1, \cdots, \lambda_m$ such that
$[\mathbf{D}f(\mathbf{c})] = \lambda_1[\mathbf{D}F_1(\mathbf{c})] + \cdots + \lambda_m[\mathbf{D}F_m(\mathbf{c})]$.

19. Two approaches to an elementary maximization problem

    Farmer Brown wants to build a rectangular pigpen using fencing of total length 20 meters. One side of the pigpen is his barn, so the width x (side parallel to the barn) and depth y (other two sides) are constrained to lie on the manifold $x + 2y = 20$.

    What choice of x maximizes the area $f\begin{pmatrix} x \\ y \end{pmatrix} = xy$?

    - Solve the problem by elementary methods, using y as the active variable.
    - Solve the problem by using Lagrange multipliers.

20. Using a parametrization

What rectangle inscribed in the ellipse $x^2+4y^2 = 4$ has the greatest perimeter $4(x + y)$?

Solve the problem by using the parametrization

$$\begin{pmatrix} x \\ y \end{pmatrix} = \vec{\gamma}(t) = \begin{pmatrix} 2\cos t \\ \sin t \end{pmatrix},$$

then get the same solution by using Lagrange multipliers.

21. (This problem is equivalent to the derivation of the "Boltzmann factor" $e^{-\frac{E}{kT}}$ that you may have heard of in a chemistry or physics course, but I have rewritten it so that there is no mention of entropy or probability.)

You have taken over from the Postal Service the task of sorting mail in Cambridge. You have 7 million pieces of mail to sort, and you must decide how to divide it among your three "sortation centers." Center 1, an abandoned post office, is rent-free. Center 2, rented from UPS, charges 1 kilobuck for every million pieces of mail that you sort there, Center 3, rented from Harvard, charges 2 kilobucks for every million pieces of mail that you sort there. You are willing to pay 4 kilobucks of rent.

So your constraints are $x_1 + x_2 + x_3 = 7$ and $x_2 + 2x_3 = 4$.

The total effort required to do the sorting is

$$f\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 \log x_1 + x_2 \log x_2 + x_3 \log x_3 \text{ (natural logarithms).}$$

Using two Lagrange multipliers, find the values of x_1, x_2, x_3 that minimize f while satisfying the two constraints. Hint: if you let $t = \frac{x_2}{x_1}$ you can reduce the problem to solving a quadratic equation for t.

3 Seminar Topics (Dec. 6 or 7)

Your section instructor will either have emailed a list of topics to prepare or will have posted a signup list of appointments on the Calendar tab of Canvas. Either way, there will be one of the following topics that you should be prepared to present.

Practice your presentation so that it takes about 8 minutes. The text of the presentation will be projected onto a screen so that you need not recopy it. To save time, avoid writing long sentences on the chalkboard. You may use notes, but be discreet about it.

1. The unit circle is a simple example of a smooth 1-dimensional manifold in $\mathbb{R}^2$. Explain how to describe in it three different ways:

   - As a union of function graphs (the definition).
   - As the locus defined by an equation $F\begin{pmatrix} x \\ y \end{pmatrix} = 0$ for which $[DF]$ is onto at every point on the manifold.
   - As the image of a parametrization function where the parameter is an angle θ.

2. Let M be a k-dimensional submanifold of $\mathbb{R}^n$, and let $\mathbf{z}$ be a point of the manifold, some of whose components are "active" variables $\mathbf{y}$ and the rest of which are "passive" variables $\mathbf{x}$.

   Specify the dimension of the domain and codomain for

   (a) A function $\mathbf{g}$ that specifies the passive variables in terms of the active variables.

   (b) A function $\mathbf{F}$ that specifies $n-k$ constraints that are satisfied by points on the manifold.

   (c) A parametrization function γ that generates points on the manifold from a set of parameters.

   For each alternative, specify the shape of the matrix that represents the derivative of the relevant function and explain how, given a point $\mathbf{c}$ on the manifold, it could be used to find a basis for the tangent space $T_{\mathbf{c}}M$.

3. Suppose that $U \subset \mathbb{R}^n$ is an open subset, $\mathbf{F} : U \to \mathbb{R}^{n-k}$ is a C^1 mapping, and manifold M can be described as the set of points that satisfy $\mathbf{F}(\mathbf{z}) = 0$. Use the implicit function theorem to show that if $[\mathbf{DF}(\mathbf{c})]$ is onto for $\mathbf{c} \in M$, then the tangent space $T_{\mathbf{c}}M$ is the kernel of $[\mathbf{DF}(\mathbf{c})]$. You may assume that the variables have been numbered so that when you row-reduce $[\mathbf{DF}(\mathbf{c})]$, the first $n - k$ columns are pivotal.

4. (Proof 12.1) Let $U \in \mathbb{R}^n$ be an open subset and let $f : U \to \mathbb{R}$ be a C^1 (continuously differentiable) function.
   First prove, using a familiar theorem from single-variable calculus, that if $\mathbf{x}_0 \in U$ is an extremum, then $[\mathbf{D}f(\mathbf{x}_0)] = [0]$.
   Then prove that if $M \subset \mathbb{R}^n$ is a k-dimensional manifold, and $\mathbf{c} \in M \cap U$ is a local extremum of f restricted to M, then $T_{\mathbf{c}}M \subset \ker[\mathbf{D}f(\mathbf{c})]$.

5. (Proof 12.2) Let M be a manifold known by a real-valued C^1 function $F(\mathbf{x}) = 0$, where F goes from an open subset U of $\mathbb{R}^n$ to $\mathbb{R}$ and $[\mathbf{D}F(\mathbf{x})]$ is nowhere zero. Let $f : U \to \mathbb{R}$ be a C^1 function.

   Prove that $\mathbf{c} \in M$ is a critical point of f restricted to M if and only if there exists a Lagrange multiplier λ such that $[\mathbf{D}f(\mathbf{c})] = \lambda[\mathbf{D}F(\mathbf{c})]$.

4 Workshop Problems (Dec. 6 or 7)

If your group has R skills, choose 4a or 4b as your third problem

1. Implicitly defined functions

   (a) The nonlinear equation $\mathbf{F}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x^2+y^2+z^2-3\\x^2+z^2-2\end{pmatrix} = 0$ implicitly determines x and y as a function of z. The first equation describes a sphere of radius 3, the second describes a cylinder of radius 2 whose axis is the y-axis. The intersection is a circle in the plane $y = 1$.

   Near the point $x = 1, y = 1, z = 1$, there is a function that expresses the two passive variables x and y in terms of the active variable z. $\mathbf{g}(z) = \begin{pmatrix}\sqrt{2-z^2}\\1\end{pmatrix}$.

   Calculate $\mathbf{g}'(z)$ and determine the numerical value of $\mathbf{g}'(1)$

   Then get the same answer without using the function $\mathbf{g}$ by forming the Jacobian matrix $[\mathbf{DF}]$ evaluating it at $x = y = z = 1$, and using the implicit function theorem to determine $\mathbf{g}'(z) = -A^{-1}[B]$.

   (b) Dean Smith is working on a budget in which he will allocate x to the library, y to pay raises, and z to the Houses. He is constrained.

   The Library Committee, happy to see anyone get more funds as long as the library does even better, insists that $x^2 - y^2 - z^2 = 1$.

   The Faculty Council, content to see the Houses do well as long as other areas benefit equally, recommends that $x + y - 2z = 1$.

   To comply with these constraints, the dean tries $x = 3, y = 2, z = 2$.

   Given the constraints, x and y are determined by an implicitly defined function $\begin{pmatrix}x\\y\end{pmatrix} = \mathbf{g}(z)$.

   Use the implicit function theorem to calculate $\mathbf{g}'(2)$, and use it to find approximate values of x and y if z increased to 2.1.

2. Critical points

   (a) i. Find the one and only critical point of $f\begin{pmatrix} x \\ y \end{pmatrix} = 4x^2 + \frac{1}{2}y^2 + \frac{8}{x^2 y}$ on the square $\frac{1}{4} \le x \le 4, \frac{1}{4} \le y \le 4$.

      ii. Use second derivatives (the Hessian matrix) to determine whether this critical point is a maximum, minimum, or neither.

   (b) The function $F\begin{pmatrix} x \\ y \end{pmatrix} = x^2 y - 3xy + \frac{1}{2}x^2 + y^2$ has three critical points, two of which lie on the line $x = y$. Find each and use the Hessian matrix to classify it as maximum, minimum, or saddle point.

3. Lagrange Multipliers

   (a) Example with a two-dimensional manifold.
   At what point on the sphere $x^2 + y^2 + z^2 = 7$ does the function xy^2z^4 have a maximum?
   A useful trick with this sort of function is again to take the logarithm – a monotone function, so it has the same critical points.
   $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \log x + 2\log y + 4\log z$ is the function to be maximized.
   $F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x^2 + y^2 + z^2 - 7$ is the constraint.
   Set $[\mathbf{D}f] = \lambda[\mathbf{D}F]$ and solve.

   (b) Reversing the function to be optimized and the constraint
   You are building office space and have contracted to supply 10 units of space, half at the end of year 1 and half at the end of year 2. Because large building projects are inefficient, the cost of building x units of space is x^2. However, if you produce more than 5 units in the first year, you can rent out the excess space during the second year at 8 units of money per unit of space.It might be optimal to do something like producing $x = 6$ in the first year, renting out 1 unit for a year, and producing only $y = 4$ in the second year.

      i. Write down the function of x and y to be minimized and the constraint, then use a Lagrange multiplier to find the optimal amount to produce during the first year.

      ii. When you use Lagrange multipliers, the cost function and the constraint are almost interchangeable. Invent a problem that involves maximizing a linear function and that has the same solution as the original problem.

4. Manifolds and tangent spaces, investigated with help from R

   (a) Manifold M is known by the equation

   $$F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = xz - y^2 = 0 \text{ near the point } \mathbf{c} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}.$$

   It can also be described parametrically by

   $$\gamma\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} s^2 \\ st^2 \\ t^4 \end{pmatrix} \text{ near } s = 2, t = 1.$$

      i. Use the parametrization to find a basis for the tangent space $T_{\mathbf{c}}M$.
      ii. Use the function F to confirm that your basis vectors are indeed in the tangent space $T_{\mathbf{c}}M$.
      iii. Use the parametrization to do a wireframe plot of the parametrized manifold near $s = 2, t = 1$. See script 3.4C, topic 2.

   (b) (Hubbard, Example 3.1.14) $\mathbf{F}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} z_3 \\ z_3 - z_1 z_2 \end{pmatrix}$

   Construct $[\mathbf{DF}]$. It has two rows.

   Find the point for which $[\mathbf{DF}]$ is not onto. Use R to find points on the manifold near this point, and try to figure out what is going on. See the end of script 3.4C for an example of how to find points on a 1-dimensional manifold in $\mathbb{R}^3$.

5 Homework - due on Tuesday, December 11

Although all of these problems except the last one were designed so that they could be done with pencil and paper, it makes sense to do a lot of them in R, and the Week 12 scripts provide good models. For each problem that you choose to do in R, include a "see my script" reference in the paper version. Put all your R solutions into a single script, and upload it to the homework dropbox on the week 12 page.

When you use R, you will probably want to include some graphs that are not required by the statement of the problem.

1. Pat and Terry are in charge of properties for the world premiere of the student-written opera "Goldfinger" at Dunster House. In the climactic scene the anti-hero takes the large gold brick that he has made by melting down chalices that he stole from the Vatican Museum and places it in a safety deposit box in a Swiss bank while singing the aria "Papal gold, now rest in peace."

   The gold brick is supposed to have length $x = 8$, height $y = 2$, and width $z = 4$. With these dimensions in mind, Pat and Terry have spent their entire budget on 112 square inches of gold foil and 64 cubic inches of an alloy that melts at 70 degrees Celsius. They plan to fabricate the brick by melting the alloy in a microwave oven and casting it in a sand mold.

   Alas, the student mailboxes that they have borrowed to simulate safety-deposit boxes turn out to be not quite 4 inches wide. Fortunately, the equation

$$\mathbf{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} xyz - 64 \\ xy + xz + yz - 56 \end{pmatrix} = 0$$

   specifies x and y implicitly in terms of z.

   (a) Use the implicit function theorem to find $[D\mathbf{g}(4)]$, where $\mathbf{g}$ is the function that specifies $\begin{pmatrix} x \\ y \end{pmatrix}$ in terms of z, and find the approximate dimensions of a brick with the same volume and surface area as the original but with a width of only 3.9 inches.

   (b) Show that if the original dimensions had been $x = 2, y = 2, z = 16$, then the constraints of volume 64, surface area 136 specify y and z in terms of x but fail to specify x and y in terms of z.

   (c) Show that if the original brick had been a cube with $x = y = z = 4$, then, with the constraints of volume 64, surface area 96, we cannot show the existence of any implicit function. In fact there is no implicit function, but our theorem does not prove that fact. This happens because this cube has minimum surface area for the given volume.

2. (Physics version)In four-dimensional spacetime, a surface is specified as the intersection of the hypersphere $x^2 + y^2 + z^2 = t^2 - 2$ and the hyperplane $3x + 2y + z - 2t = 2$.

   (Economics version)A resource is consumed at rate t to manufacture goods at rates x, y, and z, and production is constrained by the equation $x^2 + y^2 + z^2 = t^2 - 2$.

   Furthermore, the expense of extracting the resource is met by selling the goods, so that $2t = 3x + 2y + z - 2$.

   In either case, we have a manifold that is the locus of

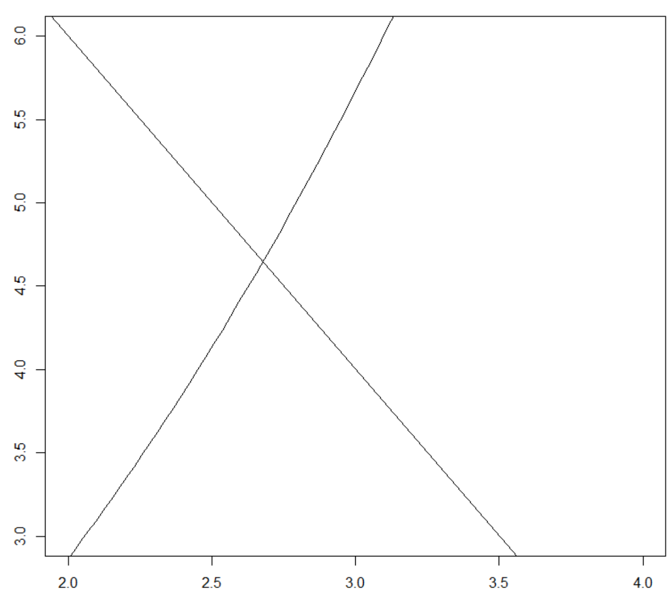
$$\mathbf{F}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} x^2 + y^2 + z^2 - t^2 + 2 \\ 3x + 2y + z - 2t - 2 \end{pmatrix} = 0.$$

   (a) Show that this surface is a smooth 2-dimensional manifold.

   (b) One point on the manifold is $x = 1, y = 2, z = 3, t = 4$. Near this point the manifold is the graph of a function $\mathbf{g}$ that expresses x and y as functions of z and t. Using the implicit function theorem, determine $[\mathbf{Dg}]$ at the point $z = 3, t = 4$.

3. Consider the manifold specified by the parametrization
   $\mathbf{g}(t) = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t + e^t \\ t + e^{2t} \end{pmatrix}, -\infty < t < \infty.$

   Find where it intersects the line $2x+y = 10$. You can get an initial estimate by using the graph below (generated in R), then use Newton's method to improve the estimate.



4. Manifold X, a hyperboloid, can be parametrized as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \gamma \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \sec u \\ \tan u \cos v \\ \tan u \sin v \end{pmatrix}$$

If you use R, you can do a wireframe plot the same way that the sphere was plotted in script 3.4C, topic 2.

   (a) Find the coordinates of the point $\mathbf{c}$ on this manifold for which $u = \frac{\pi}{4}, v = \frac{\pi}{2}$.

   (b) Find the equation of the tangent space $T_{\mathbf{c}}X$ as the image of $[\mathbf{D}\gamma \begin{pmatrix} \frac{\pi}{4} \\ \frac{\pi}{2} \end{pmatrix}]$.

   (c) Find an equation $F \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$ that describes the same manifold near $\mathbf{c}$, and find the equation of the tangent space $T_{\mathbf{c}}X$ as the kernel of $[\mathbf{DF}(\mathbf{c})]$.

   (d) Find an equation $x = g \begin{pmatrix} y \\ z \end{pmatrix}$ that describes the same manifold near $\mathbf{c}$, and find the equation of the tangent space $T_{\mathbf{c}}X$ as the graph of $[\mathbf{Dg} \begin{pmatrix} 0 \\ 1 \end{pmatrix}]$.

5. Hubbard, Exercise 3.6.2. This is the only problem of this genre on the homework that can be done with pencil and paper, but you must be prepared to do one like it on the final exam!

   The final problem, which requires R, is only for graduate-credit students.

6. Here is another function that has one maximum, one minimum, and two saddle points, for all of which x and y are less than 3 in magnitude.

$$f \begin{pmatrix} x \\ y \end{pmatrix} = x^3 - y^3 + 2xy - 5x + 6y.$$

   Locate and classify all four critical points using R, in the manner of script 3.4D. A good first step is to plot contour lines with x and y ranging from -3 to 3. If you do

```
contour(x,y,z, nlevels = 20)
```

   you will learn enough to start zooming in on all four critical points.

   An alternative, more traditional, approach is to take advantage of the fact that the function f is a polynomial. If you set both partial derivatives equal to zero, you can eliminate either x or y from the resulting equations, then find approximate solutions by plotting a graph of the resulting fourth-degree polynomial in x or y.