# 1 Multiple Choice

There **will not** be any "choose all that apply" questions on the exam, to the best of our knowledge; they are just included on this review for practice and discussion.

1. Which of the following sets are countably infinite? (**Choose all that apply.**)
   (a) All points in $\mathbb{R}^3$.
   (b) All points contained within the unit ball in $\mathbb{R}^3$ (i.e. all points $(x, y, z)$ s.t. $x^2 + y^2 + z^2 \leq 1$).
   (c) All points in $\mathbb{R}^3$ whose coordinates are rational numbers.
   (d) All infinite sequences whose elements are all integers.
   (e) All prime numbers.

2. For what values of $x$ does the power series
$$\sum_{n=1}^{\infty} \frac{n(x+2)^n}{5^{n-1}}$$
   converge?
   (a) Only for $x = -2$.
   (b) For all $x \in \mathbb{R}$.
   (c) For all $x \in (-7, 3)$.
   (d) For all $x \in [-7, 3]$.
   (e) For all $x \in [-7, 3)$.

3. Which of the following **sequences** converge as $n \to \infty$? (**Choose all that apply.**)
   (a) $\frac{\sin(n)}{n}$
   (b) $\frac{2^n}{n!}$
   (c) $\sqrt{n+1} - \sqrt{n}$
   (d) $\frac{1}{n}$
   (e) $\sum_{i=1}^{n} \frac{1}{i}$

# 2 Problems

1. First, prove that $\lim_{n\to\infty} n^a = +\infty$ for $a > 0$. Then, prove that $\lim_{n\to\infty} \frac{1}{n^a} = 0$ for $a > 0$.

2. Use a least number proof to show that
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$
   (For practice, you could also do an inductive proof to show the same result.)

3. Prove that the supremum of a set $S$ is unique.

4. (Ross 17.12) Let $f$ be a continuous real-valued function with domain $(a, b)$. Show that if $f(r) = 0$ for every rational number $r \in (a, b)$, then $f(x) = 0 \ \forall x \in (a, b)$.

5. Using the Archimedean Property, prove that there are no infinite elements (larger than all the natural numbers–you could also say larger than all real numbers and get the same conclusion) in the real numbers. (This shows that infinity is not a real number!)

6. Consider the function $f(x)$ defined on the interval $[0, 2]$ and given by

$$f(x) = \begin{cases} x & 0 \le x \le 1 \\ x - 2 & 1 < x \le 2 \end{cases}$$

   (a) Show (using the limit definition of the derivative) that for every point in the interval $(0, 1) \cup (1, 2)$, $\frac{d}{dx} f(x) = 1$. Also explain why $f'(x)$ is undefined at $x = 1$.

   (b) Certainly, $f(0) = 0$ and $f(2) = 0$, but we just saw that there is no $x \in (0, 2)$ satisfying $f'(x) = 0$. Is this a contradiction to Rolle's Theorem?

   (c) In order to try to solve the Rolle's Theorem issue in part (b), we could try building a new function $g(x)$ given by

$$g(x) = \begin{cases} 0 & x \le 0 \\ x - 2 & x > 0 \end{cases}$$

   Again we see that $g(0) = 0$ and $g(2) = 0$. And this time, the function is continuous on $(0, 2)$. But again, there's no $x \in (0, 2)$ such that $g'(x) = 0$. This time, is there a contradiction with Rolle's Theorem?

   (d) Let's consider one more function $h(x)$ defined by

$$h(x) = \begin{cases} x & x \le 1 \\ 1 & x > 1 \end{cases}$$

   $f(2) = 1$ and $f(0) = 0$, so the average slope between $x = 0$ and $x = 2$ is $\frac{f(2)-f(0)}{2-0} = \frac{1}{2}$. Clearly, we haven't met the conditions for Rolle's Theorem, since $f(0)$ doesn't equal $f(2)$. But there is no $x \in (0, 2)$ (or anywhere in $\mathbb{R}$, for that matter) for which $f'(x) = \frac{1}{2}$, even though $h(x)$ is continuous everywhere! Is this a contradiction to the Mean Value Theorem?

7. Use the $h \to 0$ limit definition of the derivative and some fun facts about $\frac{\sin h}{h}$ to prove that $\frac{d}{dx} \sin(2x) = 2\cos(2x)$.

8. Continuity and uniform continuity

   (a) What is the core difference between continuity and uniform continuity? (How would a formal proof of uniform continuity differ from a formal proof of continuity?)

   (b) Using the definition of uniform continuity, prove that $f(x) = x^2$ is uniformly continuous on the interval $(0, 1)$.

   (c) Prove that $f(x) = x^2$ is uniformly continuous on the interval $[0, 1]$. You may assume that $x^2$ is continuous on this interval.