1. Which of these functions is **not** uniformly continuous on $(0,1)$?

    (a) x^2

    (b) $1/x^2$

    (c) $f(x) = 1$ for $x \in (0,1)$, $f(0) = f(1) = 0$

    (d) $\sin(x)$

    (e) $\frac{\sin(x)}{x}$

    Solution: $\boxed{B}$. x^2 is uniformly continuous, because you can extend it so that $f(0) = 0$ and $f(1) = 1$, and then it's continuous on a closed interval, so it is uniformly continuous. The same extension could work for answer choice c, if we let $f(0) = f(1) = 1$. It does not matter that its discontinuous at the end points. Both parts d and e can also be extended to continuous functions on the closed interval. In particular, for part e) it is important to note that $\lim_{x \to 0} \sin(x)/x = 1$. The main issue with answer choice B is that it is unbounded on the open interval! Therefore, it cannot be uniformly continuous.

2. Let s_n be a sequence of real numbers on a bounded set S, where $\liminf s_n \neq \limsup s_n$. Which of the following is not necessarily true?

    (a) $\lim s_n$ does not exist.

    (b) s_n is not Cauchy.

    (c) $\liminf s_n < \limsup s_n$

    (d) There exists a convergent subsequence.

    (e) s_n has an infinite number of dominant terms.

    Solution: $\boxed{E}$ a) is true, because in order for a limit to exist, the $\limsup$ must equal the $\liminf$, and it turns out that that is equal to the limit. Because the sequence can't converge, choice b) is also out, since all Cauchy sequences of real numbers converge. For choice c), notice that in general $\liminf s_n \leq \limsup s_n$, and we have excluded the equal case in the prompt of the question. For choice d), the Bolzano Weierstrass Theorem still holds, since the sequence is defined on a bounded set. e) is not necessarily true. An explicit counterexample is $s_n = 1 - \frac{1}{n}$ for even n, and $s_n = 0$ for odd n.

3. Which of the following is not true about $s_n = \frac{1}{n}$?

   (a) The sequence converges to 0.

   (b) $\lim_{n \to \infty} \sum_{i=1}^{n} s_i = L$, for some finite L.

   (c) $\limsup s_n = 0$.

   (d) The series $\sum (-1)^n s_n$ converges.

   (e) The series $\sum s_n^2$ converges.

   Solution: $\boxed{B}$ This series is known as the harmonic series, which diverges. Choice b) is simply stating the definition of a series sum as the limit of the partial sums. a) is true, since when n goes to infinity, $1/n$ goes to zero. c) is true, since if the limit exists, then $\limsup s_n = \liminf s_n = \lim s_n$. Choice d) is true, because then, we are considering an alternating series, which has a less stringent convergence condition, namely that $\lim |s_n| = 0$, which is satisfied. Choice e) is true, because $1/n^2$ is a convergent p series.

4. Let $\sum a_n$ be a conditionally convergent series. Which of the following is not necessarily true?

   (a) The series converges to some finite L.

   (b) The series sum is independent of order of terms.

   (c) $\sum |a_n|$ diverges.

   (d) $\lim (-1)^n a_n = 0$.

   (e) None of the above. They're all necessarily true.

   Solution: $\boxed{B}$ Part a) is true by definition! If it converges, it converges to something. This does not contradict part b), since the order of terms in a conditionally convergent series needs to be respected! The order of terms as written defines the series sum, despite other sums being possible if the terms are moved around. Part c) is false, because, if $|a_n|$ converges, then the series would be absolutely convergent, not conditionally convergent. Part d) is true, since that is exactly the statement of the alternating series test.

5. Which of the following series converges?

   (a) $\sum \frac{x^n}{n!}$, $\forall x$

   (b) $\sum \frac{1}{n+\sin(n)}$

   (c) $\sum (-1)^n n$

   (d) $\sum \sin(n)$

   (e) $\sum \frac{2^n}{\sqrt{n!}}$

   Solution: $\boxed{A, E}$ Oops, I put two convergent series! Choice a) converges. This can be done with the ratio test, or with the recognition, that a) is the Taylor series for e^x, which is valid everywhere. Choice b) diverges, because $\frac{1}{n+\sin(n)} \geq \frac{1}{n+1}$, which diverges, since that is just the harmonic series with a relabeling. Choice c) does not converge to any particular limit, as the terms go as: $-1, 2, -3, \dots$. For choice d), note that $\lim \sin(n) \neq 0$, so it can't converge. Choice e) converges by the ratio test.

6. Which of the following must be true of a continuous function on (a, b)?

   (a) The function achieves its maximum on (a, b).

   (b) The function is bounded.

   (c) For all Cauchy Sequences s_n on the set (a, b), $f(s_n)$ is also Cauchy.

   (d) If $f(a) = 2$, and $f(b) = 5$, then $f(c) = 3$, for some $c \in (a, b)$.

   (e) None of the above.

   Solution: $\boxed{E}$ None of these are true! Choice a) is not true, by a counterexample. If $a = 0$ and $b = 1$, then $1/x$ is continuous on $(0, 1)$, but does not achieve its maximum. The key here is that the statement only guarantees continuity on the open interval. Part b) is false by the counterexample above. Part c) is also true by the counterexample above. Let $\lim s_n = 0$, and then $f(s_n)$ will diverge, and not be Cauchy. d) is not true, because it doesn't need to be continuous at the end points. The function $f(x) = 5$ except at a, where $f(a) = 2$, is a counterexample.

7. Which of the following is not necessarily true about a uniformly continuous function, f, on $[a, b]$?

   (a) The function is bounded.

   (b) The function achieves its maximum on the set (a, b).

   (c) If $f(a) = 4$ and $f(b) = 6$, then $f'(c) = 2$ for some $c \in (a, b)$.

   (d) The derivative f' is bounded.

   (e) If $f'(a) = 3$, and $f'(b) = 4$, then $f'(c) = 3.5$ for some $c \in (a, b)$.

   Solution: $\boxed{C, E}$ Oops, two of these statements are false. Part a) is true, because any continuous function on a closed interval is bounded. Part b) is also true of any continuous function on a closed interval. Part c) is not necessarily true, since the continuity doesn't necessarily imply differentiability. Part d) is true about uniformly continuous functions. The unboundedness of the derivative is what prevents finding the one size fits all δ for all x. Part e) is not necessarily true, because, we do not know that the derivative is continuous!

8. Find $\lim_{x \to b} \frac{\sqrt{x} - \sqrt{b}}{x - b}$ for $b > 0$.

   (a) ∞

   (b) $\frac{1}{2\sqrt{b}}$

   (c) 0

   (d) $2\sqrt{b}$

   (e) b

   Solution: $\boxed{B}$ If you multiply both top and bottom by $\sqrt{x} + \sqrt{b}$, the numerator becomes $x - b$, and cancels the denominator. Taking the limit results in choice b).

9. Let f be a differentiable function, where all derivatives exist, such that $f(0) = 0$, $f'(0) = 0$, and $|f''(x)| \leq M, \forall x$. Which of the following is not necessarily true?

   (a) $f(1) \leq \frac{M}{2}$

   (b) 0 is neither a maximum nor a minimum.

   (c) $\forall \epsilon > 0, \exists \delta > 0$ s.t. if $x \in (-\delta, \delta), |f(x)| < \epsilon$

   (d) If $\lim s_n = 0$, then $\lim f(s_n) = 0$.

   (e) None of the above.

Solution: $\boxed{B}$ Choice a) is true, since that is just the statement of Taylor's Theorem of Remainder. It says that

$$f(x) = f(0) + f'(0)x + \frac{f''(c)x^2}{2!} \text{ where } c \in [0, x]$$

Notice that $f(0) = f'(0) = 0$, and that the derivative is bounded by M. Then, this states:

$$f(1) \leq 0 + 0(1) + \frac{M(1)^2}{2!}$$
$$\leq \frac{M}{2}$$

Choice b) is false by counterexample. Let $f(x) = x^2$, then 0 is a minimum, and the second derivative is bounded by 2. Choice c) is true, since it is just the statement of continuity at $x = 0$, and being differentiable implies continuity.