1 Major Concepts

1. Matrices and one-sided inverses

   (a) Determine which of these matrices is/are invertible and, where possible, find their inverses. For the invertible matrix/matrices, confirm that the inverse you calculate is both a left and a right inverse.

   $$A = \begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 2 \end{bmatrix}$$

   (b) Matrix D is not invertible, but it does have a one-sided inverse. Which does it have—a left inverse or a right inverse? Find a left/right inverse, and explain why an other-side inverse cannot exist.

   $$D = \begin{bmatrix} 4 & 3 \end{bmatrix}$$

   (c) Similarly, matrix E is not invertible, but it does have a one-sided inverse. Which does it have—a left inverse or a right inverse? Find a left/right inverse, and explain why an other-side inverse cannot exist.

   $$E = \begin{bmatrix} 7 \\ 2 \\ 4 \end{bmatrix}$$

2. Categories, retractions, and sections: Consider a category in which our objects are finite sets A and B and we have an arrow $f : A \to B$.

   (a) Draw an example A, B, and f where A has more elements than B and f has neither a section or a retraction.

   (b) Draw an example A, B, and f where A has more elements than B and f has a section. Can f have a retraction?

   (c) Draw an example A, B, and f where A has fewer elements than B and f has neither a section or a retraction.

   (d) Draw an example A, B, and f where A has fewer elements than B and f has a retraction. Can f have a section?

3. Summation notation

   (a) If $f(i)$ is some expression involving i, then

   $$\sum_{i=1}^{n} f(i) = f(1) + f(2) + \cdots + f(n)$$

   (b) Calculate $\sum_{i=0}^{3} \frac{i^2+1}{i+1}$ and $\sum_{i=1}^{3} \left(\sum_{j=0}^{i} j \right)$.

   (c) Given that matrix A is 2×5, matrix B is 5×3, and matrix C is 3×4, give an expression in summation notation for $(ABC)_{2,3}$. Do this by considering $(AB)C$.

4. Markov processes

   You are playing a board game where you can move your piece to a red, yellow, or blue square, at each round. Here are the rules for the game:

   - If you are currently on a red square, you toss a coin, if you end up with head you move to a yellow square, if you end up with tail, you move to another red square.
   - If you are on a yellow square, you roll a die. If you get 1, you move to a red square. If you get 2,3,or,4 you move to another yellow square. If you get 5 or 6, you move to a blue square.
   - If you are currently on a blue square, you roll a die. If the number you get is even, you stay on blue.If it is odd, you move to a red square.

   (a) Write down the Markov matrix that represents the probabilities for moving from one square to another.

   (b) If you start on a yellow square, after 4 rounds of playing, what is the probability that your piece moves to a red square?

5. More field axioms: Prove $(-1)(-1) = 1$.

6. Matrices as functions: Here's an example for the derivative function as a matrix: The domain and co-domain have bases :x^0, x^1, x^2, x^3
   With this convention, the elements of the matrix are the coefficients of each corresponding basis element. With the 4*4 matrix below, we can differentiate all polynomials of degree four or less.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   For instance, the polynomial $1 + 2x + 3x^2 + 4x^3$can be represented by a column vector, with the entries being the coefficients for the powers of x:

$$v = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

   Now,we need to differentiate our polynomial:
   • In the language of functions: the derivative function, takes in our polynomial, and acts on it.
   • In the language of matrices: matrix A acts on vector b. (We should multiply matrix A by vector b).

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 12 & 0 \end{bmatrix}$$

   We can read the resulting vector as the polynomial:

$$2 + 6x + 12x^2$$

   Our matrix has indeed accomplished differentiation!

7. Hints for question 2 of homework:
   - For part a), pay attention to the definition of surjectivity:
   $\forall c \in C, \ \exists b \in B \ s.t. \ f(b) = c$
   -Similarly, for part b), pay attention to the definition of injectivity:
   if $h(a_i) = h(a_j)$ then, $a_i = a_j$
   Here, it is easier to use the contrapositive of the statement, so we want to show that if $a_i \neq a_j$ then, $h(a_i) \neq h(a_j)$.