1. Prove that the "no bad sequence" definition of continuity holds iff the ϵ-δ definition holds.

   (a) Pro tip: I'd usually use the ϵ-δ definition to show continuity and the "no bad sequence" definition to show discontinuity. It's often not too bad to come up with a bad sequence for showing discontinuity, but showing that something is continuous by the "no bad sequence" criterion is often not too fun.

   (b) Using the "no bad sequence" definition, show that $f(x) = x + 1$ is convergent for any $x \in (0, 1)$.

   (c) Using the ϵ-δ definition, show that $f(x) = x + 1$ is convergent for any $x \in (0, 1)$.

   (d) Using the "no bad sequence" definition, show the following sequence (defined on [0,2]) is divergent at $x = 1$:

   $$f(x) = \begin{cases} x+1 & 0 \leq x \leq 1 \\ x & 1 < x \leq 2 \end{cases}$$

   (e) Using the ϵ-δ definition, show the following sequence (defined on [0,2]) is divergent at $x = 1$:

   $$f(x) = \begin{cases} x+1 & 0 \leq x \leq 1 \\ x & 1 < x \leq 2 \end{cases}$$

2. Continuity and uniform continuity

   (a) To prove that $f(x)$ is continuous **at a particular** x_0:
   $\forall \epsilon > 0, \exists \delta > 0$ s.t. $\forall x, |x - x_0| < \delta \to |f(x) - f(x_0)| < \epsilon$

   (b) To prove that $f(x)$ is continuous **everywhere**:
   $\forall x_0, \forall \epsilon > 0, \exists \delta > 0$ s.t. $\forall x, |x - x_0| < \delta \to |f(x) - f(x_0)| < \epsilon$

   (c) To prove that $f(x)$ is **uniformly continuous**:
   $\forall \epsilon > 0, \exists \delta > 0$ s.t. $\forall x, \forall x_0, |x - x_0| < \delta \to |f(x) - f(x_0)| < \epsilon$

   (d) Proof 7.2: If a function f is continuous on a closed interval, it is uniformly continuous on that interval.

   (e) It is a **sufficient but not necessary** criterion for uniform continuity of f on (a, b) that f be differentiable on (a, b), with f' bounded on (a, b).

   (f) Prove that $f(x) = \frac{1}{x}$ is **not uniformly continuous** on $(0, \infty)$.

   (g) Prove that $\sqrt{x}$ is **uniformly continuous** on $[0, \infty)$. (You may use that, as part of a pset problem, you prove(d) that it is continuous on $[0, \infty)$.) Hint: Break the domain into two (overlapping) pieces, [0,2] and $[1, \infty)$.

   (h) Ross 18.6: Prove that $x = \cos(x)$ for some $x \in (0\frac{pi}{2})$.

3. Prove that a function $f(x) : \mathbb{R} \to \mathbb{R}$ is injective iff it is either strictly increasing or strictly decreasing.

4. Ross 17.12: Let f be a continuous real-valued function with domain (a, b). Show that if $f(r) = 0$ for each rational number $r \in (a, b)$, then $f(x) = 0$ $\forall x \in (a, b)$.