OPTIONAL, THOUGHT PROVOKING

RECITATION SUPPLEMENT

Colton Grainger

WEEK 11 Spring 2019

MATH 1300 Calculus 1

0. Big Picture Reading Material
    - Halmos '70
    - Körner '14

1. Background for Extra Problem Set
    - Priestly '74

2. Extra Problem Set
    - Spivak '15

3. Solutions
    - Spivak '15

BIG PICTURE READING MATERIAL [Hal70]

"How to write Mathematics" by Paul Halmos

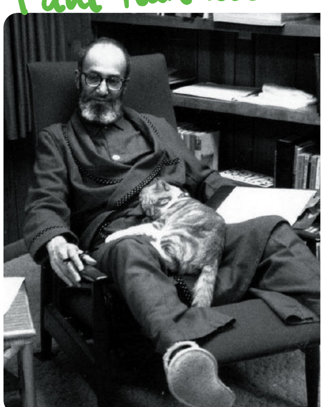
The basic problem in writing mathematics is the same as in writing biology, writing a novel, or writing directions for assembling a harpsichord: the problem is to communicate an idea. To do so, and to do it clearly, you must have something to say, and you must have someone to say it to, you must organize what you want to say, and you must arrange it in the order you want it said in, you must write it, rewrite it, and re-rewrite it several times, and you must be willing to think hard about and work hard on mechanical details such as diction, notation, and punctuation. That's all there is to it.

2. Say something

It might seem unnecessary to insist that in order to say something well you must have something to say, but it's no joke. Much bad writing, mathematical and otherwise, is caused by a violation of that first principle. Just as there are two ways for a sequence not to have a limit (no cluster points or too many), there are two ways for a piece of writing not to have a subject (no ideas or too many).

The first disease is the harder one to catch. It is hard to write many words about nothing, especially in mathematics, but it can be done, and the result is bound to be hard to read. There is a classic crank book by Carl Theodore Heisel [5] that serves as an example. It is full of correctly spelled words strung together in grammatical sentences, but after three decades of looking at it every now and then I still cannot read two consecutive pages and make a one-paragraph abstract of what they say; the reason is, I think, that they don't say anything.

author: Paul Halmos



with cat Roger.

The second disease is very common: there are many books that violate the principle of having something to say by trying to say too many things. Teachers of elementary mathematics in the U.S.A. frequently complain that all calculus books are bad. That is a case in point. Calculus books are bad because there is no such subject as calculus; it is not a subject because it is many subjects. What we call calculus nowadays is the union of a dab of logic and set theory, some axiomatic theory of complete ordered fields, analytic geometry and topology, the latter in both the "general" sense (limits and continuous functions) and the algebraic sense (orientation), real-variable theory properly so called (differentiation), the combinatoric symbol manipulation called formal integration, the first steps of low-dimensional measure theory, some differential geometry, the first steps of the classical analysis of the trigonometric, exponential, and logarithmic functions, and, depending on the space available and the personal inclinations of the author, some cook-book differential equations, elementary mechanics, and a small assortment of applied mathematics. Any one of these is hard to write a good book on; the mixture is impossible.

[Koe14]

"Why is Calculus Hard?" by T. W. Körner

The basic ideas of the calculus, like the basic ideas of the rest of mathematics, are easy (how else would a bunch of apes fresh out of the trees be able to find them?), but calculus requires a lot of work to master (after all, we are just a bunch of apes fresh out of the trees). Here is a list of some of the difficulties facing the reader.

Mathematics is a 'ladder subject'. If you are taught history at school and you pay no attention during the year spent studying Elizabethan England, you will get bad grades for that year, but you will not be at a disadvantage next year when studying the American Civil War. In mathematics, each topic depends on the previous topic and you cannot miss out too much.

The ladder described in this book has many rungs and it will be a very rare reader who starts without any knowledge of the calculus and manages to struggle though to the end. (On the other hand, some readers will be familiar with the topics in the earlier chapters and will, I hope, be able to enjoy the final chapters.) Please do not be discouraged if you cannot understand everything; experience shows that if you struggle hard with a topic, even if unsuccessfully, it will be much easier to deal with when you study it again.[1]

Mathematics involves deferred gratification. Humans are happy to do A in order to obtain B. The are less happy to do A in order to to do B, to do B in order to do C and then to do C in order to obtain D. Results in mathematics frequently require several preliminary steps whose purpose may not be immediately apparent. In Chapter 2 we spend a long time discussing the integral and the fundamental theorem of the calculus. It is only at the end of the chapter that we get our first payback in the form of a solution to an interesting problem.

Mathematics needs practice. I would love to write music like Rossini. My university library contains many books on the theory of music and the art of composing, but I know that, however many books I read, I will not be able to write music. A composer needs to know the properties of musical instruments, and to know the properties of instruments you need to play at least one instrument well. To play an instrument well requires years of practice.

Each stage of mathematics requires fluency in the previous stage and this can only be acquired by hours of practice, working through more or less routine examples. Although this book contains some exercises[2] it contains nowhere near enough. If the reader does not expect to get practice elsewhere, the first volume of *An Analytical Calculus* by A. E. Maxwell [5] provides excellent exercises in a rather less off-putting format than the standard 'door stop' text book. (However, any respectable calculus book will do.)

This is a first look and not a complete story. As I hope to make clear, the calculus presented in this book is not a complete theory, but deals with 'well behaved objects' without giving a test for 'good behaviour'. This does not prevent it from being a very powerful tool for the investigation of the physical world, but is unsatisfactory both from a philosophical and a mathematical point of view. In the final chapter, I discuss the way in which the first university course in analysis resolves these problems. I shall refer to the calculus described in the book as 'the old calculus' and to the calculus as studied in university analysis courses as 'the new calculus' or 'analysis'.[3]

D'Alembert is supposed to have encouraged his students with the cry 'Allez en avant et la foi vous viendra' (push on and faith will come to you). My ideal reader will be prepared to accept my account *on a provisional basis*, but be prepared to begin again from scratch when she meets rigorous analysis.

It is a very bad idea to disbelieve everything that your teachers tell you and a good idea to accept everything that your teachers tell you. However, it is an even better idea to accept that, though most of what you are taught is correct, it is sometimes over-simplified and may occasionally turn out to be mistaken.

The calculus involves new words and symbols. The ideas of the calculus are not arbitrary, but the names given to the new objects and the symbols used are. At the simplest level, the reader will need to recognise the Greek letter δ (pronounced 'delta') and learn a new meaning for the word 'function'. At a higher level, she will need to accept that the names and notations used reflect choices made by many different mathematicians, with many different views of their subject, speaking many languages at many different times over the past 350 years. If we could start with a clean sheet, we would probably make different choices (just as, if we could redesign the standard keyboard, we would probably change the position of the letters). However, we wish to talk with other mathematicians, so we must adopt their language.

author

My British accent. The theory and practice of calculus are international. The teaching of calculus varies widely from country to country, often reflecting the views of some long dead charismatic educationalist or successful textbook writer. In some countries, calculus is routinely taught at school level whilst, in others, it is strictly reserved for university. Several countries use calculus as an academic filter, a coarse filter in those countries with a strong egalitarian tradition, a fine one in those with an elitist bent.[4] Yikes!

I was brought up under a system, very common in twentieth century Western Europe, where calculus was taught as a computational tool in the last two years of school and rigorous calculus was taught in the first year of university. Some of the discussion in the introduction and the final chapter reflects my background, but I do not think this should trouble the reader.

The Apes comment, for example.

A shortage of letters. The calculus covers so many topics that we run into a shortage of letters. Mathematicians have dealt with this partly by introducing new alphabets and fonts giving us A, a, α, $\mathbf{A}$, $\mathbf{a}$, $\aleph$, $\mathcal{A}$, A, a, $\mathbb{A}$, $\mathfrak{A}$, $\mathfrak{a}$, ... Rather than learn a slew of new symbols, I think that my readers will prefer to accept that r will sometimes be an integer[5] and sometimes the radius of a circle, and that the same letter will be used to represent different things in different places.

But it is beautiful. Hill walking is hard work, but the views are splendid and the exercise is invigorating. The calculus is one of the great achievements of mankind and one of the most rewarding to study.

"On Remembering"

Exercise 1.4.8. *(i) Let*

$$f(x) = \frac{1 + x^{1/3}}{1 + \sqrt{1 + x^2}}.$$

Our object is to find $f'(x)$.

Observe that $f(x) = g(x)/h(x)$ with

$$g(x) = 1 + x^{1/3},\ \ h(x) = 1 + \sqrt{1 + x^2}.$$

Now note that $g(x) = a(x) + b(x)$, $h(x) = a(x) + c(x)$ with

$$a(x) = 1,\ \ b(x) = x^{1/3},\ c(x) = \sqrt{1 + x^2}$$

and $c(x) = S\big(u(x)\big)$ with

$$u(x) = 1 + x^2,\ \ S(x) = \sqrt{x}.$$

Now obtain $a'(x)$, $S'(x)$, $u'(x)$, $c'(x)$, $h'(x)$, $b'(x)$, $g'(x)$ and $f'(x)$.

(ii) Write down some more functions along the lines laid out in (i) and find their derivatives.

Easy; Try it?

With practice and experience, the reader will find that she can reduce the number of sub-problems required.

On remembering and understanding. Chess players do not carry around a notebook explaining how the knight moves and bridge players do not clutch a memo reminding them that there are four suits each of 13 cards. In the same way, mathematicians do not consult a 'formula book' for the differentiation rules just given (or for anything else). It is impossible to work on difficult mathematics unless you can work quickly and efficiently through the easy parts.

When a mathematician cannot remember a fact or a formula, her first action is to attempt to re-derive it for herself. If she cannot do this, she concludes that she does not understand the result and looks up *not the result* but its *derivation* and studies the derivation until she is certain that she understands why the result is true. If you understand why a result is true, it is easy to remember it. If you do not understand why a result is true, it is useless to memorise it.[19]

[19] Except the night before an examination.

BACKGROUND FOR EXTRA PROBLEM SET

[Pri79]

§8. Solving Optimization Problems

Where are we now? We have just completed an unavoidable digression from our original theme, which was the solution of optimization problems. As we saw in Chapter 1, an optimization problem leads to the problem of finding the highest (or lowest) point on a certain curve. This, in turn, has led to the study of derivatives, because derivatives cast light on the behavior of a curve. And now, at last, we know how to bypass Fermat's method and use the following rules instead.

(1) $(cf)' = c \cdot f'$ (constant multiples).
(2) $(f + g)' = f' + g'$ (sums).
(3) $(1/g)' = (-1/g^2)g'$ (reciprocals).
(4) $(g^2)' = 2gg'$ (squares).
(5) $(\sqrt{g})' = (1/2\sqrt{g})g'$ (square roots).
(6) $(fg)' = fg' + gf'$ (products).
(7) $(f/g)' = (gf' - fg')/g^2$ (quotients).

The reader should practice using these rules until they have been memorized. Then the taking of derivatives will be quite a routine matter, and the most important step in solving an optimization problem will have been mastered.

We can finally come to grips with the topic to which the title of this chapter alludes. What are the steps leading to the solution of an optimization problem? Basically, there are just two steps. First, translate the problem into the geometric problem of finding the highest (or lowest) point on a certain curve f; and second, find f' and use it as an aid in understanding how the curve f behaves.

The critical points to be found in sketching a curve f are those where the tangent line to the curve is horizontal. [That leads to a definition: *To say that x is a* **critical point** *of f is to say that $f'(x) = 0$.*] Usually, although not always, the function f will attain its optimal value at a critical point.

To verify whether the optimum has been found, make a rough sketch of the curve near each critical point (the second derivative is helpful here) and near each endpoint of the domain.

As we have seen, some curves do not have a highest (or lowest) point. It can be proved, however, that a curve must have such points if it comes from a *continuous* function and if the domain is an interval *containing its endpoints.* This is a deep theorem of analysis, the modern branch of mathematics into which seventeenth-century calculus evolved, and cannot be proved here. The moral for us is to be aware of what a function is doing near the endpoints of its domain, particularly if the domain does not include endpoints. If a continuous curve fails to have a highest (or lowest) point, then by the theorem of analysis the trouble must lie in the behavior of the function near an endpoint missing from its domain.

EXAMPLE 5. Find the highest point on the curve f given by

$$f(x) = 2x + 3,$$

on the domain

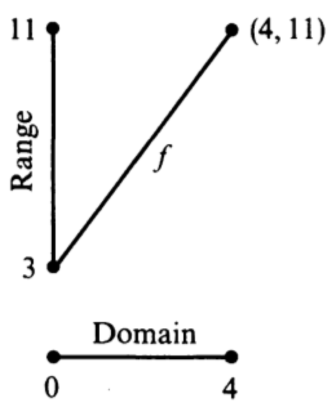
(a) $0 \leq x \leq 4$.
(b) $0 < x < 4$.

Let us look first for all critical points in the domain, that is, all values x for which $f'(x) = 0$. Here we have $f'(x) = 2$, which shows that there are no such values. Since f has no critical points, the principle of analysis mentioned above guarantees that the extreme values of f must occur at the endpoints of the domain. At the endpoint 0, the value of f is 3; at the endpoint 4, the value of f is 11. Therefore,

(a) if the domain is $0 \leq x \leq 4$, then $(4, 11)$ is the highest point on the curve f.
(b) if the domain is $0 < x < 4$, then the curve f contains no highest point.

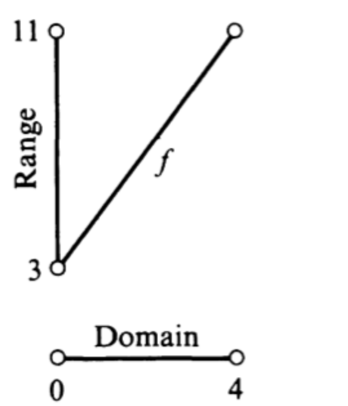
Note that, to draw the conclusions (a) and (b), *we did not have to draw a picture of the curve f*! The reader may wish to draw a picture anyway, to see better what is going on. The expression $2x + 3$ reveals f to be a linear function of slope 2:

(a) Domain: $0 \leq x \leq 4$
Range: $3 \leq y \leq 11$.



The highest point on the curve is (4, 11). The greatest number in the range is 11; the least is 3.

(b) Domain: $0 < x < 4$
Range: $3 < y < 11$.



There is no highest (or lowest) point on the curve, because the range contains no greatest (or least) number.

§9. Summary

Here, in detail, are the steps that have been illustrated above.

Step 1. Algebraic formulation:

(a) See the problem in terms of variables. (The quantity to be optimized is one variable, say y, and you have to find a second variable, say x, on which y depends.)

(b) Write down an algebraic rule f, giving y in terms of x.

(c) Specify the domain of the function f.

Step 2. Geometric analysis:

(a) See the problem as one of finding the highest (or lowest) point on the curve f.

(b) Find the derivative f'. (And find f'' too, if it can be done without much trouble.)

(c) Find the critical points, if any, that lie in the domain of f. (That is, find all values of x in the domain of f that satisfy the equation $f'(x) = 0$.)

(d) Check what happens near the endpoints of the domain.

(e) Using the information of steps 2(c) and 2(d), find the desired highest (or lowest) point on the curve f.

[The second derivative may be helpful in steps 2(d) and 2(e).]

Step 3. Back to everyday life:

(a) Read the problem again, to determine exactly what was called for. (Was it the *first* or *second* coordinate, or *both*, of the *highest* or *lowest* point of the curve that you were seeking?)

(b) Give a direct answer to the question raised in the problem, by writing a complete, concise sentence.

Step 1(c) is easy to forget, and thus deserves emphasis. The domain must be specified; otherwise, steps 2(c) and 2(d) cannot be carried out. Step 3 is also easy to forget. In concentrating on step 2, you can lose sight of your goal and, as a consequence, do unnecessary work. When a problem takes a long time to work, it is a good idea to remind yourself now and then what you are after.

Here is another example to illustrate these steps.

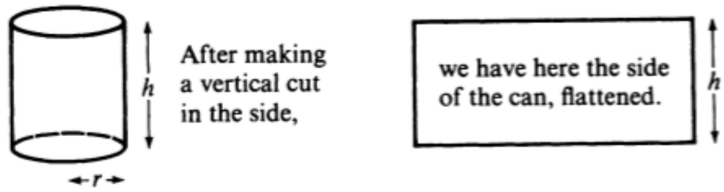
We'll do this on Thurs in recitation.

EXAMPLE 7. An ordinary metal can (shaped like a cylinder) is to be fashioned, using 54π square inches of metal. What choice of radius and height will maximize the volume of the can?

Here, we want to maximize the volume, so let V denote the volume, which is given in terms of the radius r and height h by the formula

$$V = (\text{area of base})(\text{height}) = \pi r^2 h. \tag{7}$$

The rule $V = \pi r^2 h$ gives V in terms of *two* variables. We need to get V in terms of only *one* variable, and this can be done, as follows, by finding a relation between r and h. The picture below shows that the area of the side of the can is given by $2\pi rh$:



The total amount of metal available, 54π square inches, must equal the amount in the side of the can, plus the amount in the circular top and bottom:

$$54\pi = 2\pi rh + 2\pi r^2.$$

This is a relation between r and h. It is easy to solve for h (the reader is asked to do it), and obtain

$$h = \frac{27 - r^2}{r}. \tag{8}$$

Putting equations (7) and (8) together gives

$$\begin{aligned} V &= \pi r^2 \left(\frac{27 - r^2}{r} \right) \\ &= \pi r (27 - r^2) \\ &= 27\pi r - \pi r^3, \end{aligned}$$

which expresses V in terms of r alone. The problem now is to find the value of r that yields the maximal volume V, where

$$V = 27\pi r - \pi r^3, \qquad 0 < r < \sqrt{27}.$$

[The radius r must be less than $\sqrt{27}$. *Reason*: The height h must be positive, so, by equation (8), $27 - r^2$ must be positive.]

r	V	V'
?		0
r	$27\pi r - \pi r^3$	$27\pi - 3\pi r^2$

Let us find critical points. The derivative is given by

$$V' = 27\pi - 3\pi r^2,$$

which is zero when (dividing by 3π)

$$\begin{aligned} 0 &= 9 - r^2, \\ r^2 &= 9, \\ r &= \pm 3. \end{aligned}$$

Since -3 is not in the domain, the only critical point is 3.

We now show that when r is 3, the volume V is maximal. This is easy to see, for the second derivative is given by

$$V'' = -6\pi r,$$

which is (obviously) negative *throughout the domain*. The curve is therefore always bending to its right (or frowning), and hence it must reach its highest point at the place where it has a horizontal tangent line. (At both endpoints of the domain, V tends to zero.)

To maximize the volume, the radius should be 3 inches, and the corresponding height, by equation (8), should be 6 inches. □

PROBLEM SET

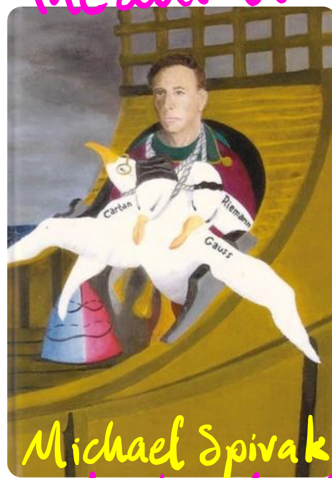
[Spi15]

Try to answer enough problems on the following scale:
10 pts DECENT 20 pts EXCELLENT 30 pts OVERKILL

PROBLEMS

1. For each of the following functions, find the maximum and minimum values on the indicated intervals, by finding the points in the interval where the derivative is 0, and comparing the values at these points with the values at the end points.

the author



1/2 pt. each

(i) $f(x) = x^3 - x^2 - 8x + 1$ on $[-2, 2]$.

(ii) $f(x) = x^5 + x + 1$ on $[-1, 1]$.

(iii) $f(x) = 3x^4 - 8x^3 + 6x^2$ on $[-\frac{1}{2}, \frac{1}{2}]$.

(iv) $f(x) = \dfrac{1}{x^5 + x + 1}$ on $[-\frac{1}{2}, 1]$.

(v) $f(x) = \dfrac{x+1}{x^2+1}$ on $[-1, \frac{1}{2}]$.

(vi) $f(x) = \dfrac{x}{x^2-1}$ on $[0, 5]$.

0 pts. 1½. Bonus. What's the "albatross around my neck" a metaphor from?

3 pts. **2.** Now sketch the graph of each of the functions in Problem 1, and find all local maximum and minimum points.

1/2 pt. each **3.** Sketch the graphs of the following functions.

(i) $f(x) = x + \dfrac{1}{x}$.

(ii) $f(x) = x + \dfrac{3}{x^2}$.

(iii) $f(x) = \dfrac{x^2}{x^2-1}$.

(iv) $f(x) = \dfrac{1}{1+x^2}$.

5. For each of the following functions, find all local maximum and minimum points.

1 pt. (i) $f(x) = \begin{cases} x, & x \neq 3, 5, 7, 9 \\ 5, & x = 3 \\ -3, & x = 5 \\ 9, & x = 7 \\ 7, & x = 9. \end{cases}$

3 pts. (iii) $f(x) = \begin{cases} x, & x \text{ rational} \\ 0, & x \text{ irrational}. \end{cases}$

4 pts. **8.** A straight line is drawn from the point $(0, a)$ to the horizontal axis, and then back to $(1, b)$, as in Figure 23. Prove that the total length is shortest when the angles α and β are equal. (Naturally you must bring a function into the picture: express the length in terms of x, where $(x, 0)$ is the point on the horizontal axis. The dashed line in Figure 23 suggests an alternative geometric proof; in either case the problem can be solved without actually finding the point $(x, 0)$.)

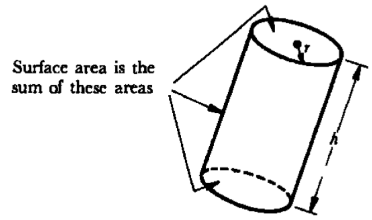


FIGURE 24

2 pts. **9.** Prove that of all rectangles with given perimeter, the square has the greatest area.

1 pt. **10.** Find, among all right circular cylinders of fixed volume V, the one with smallest surface area (counting the areas of the faces at top and bottom, as in Figure 24).

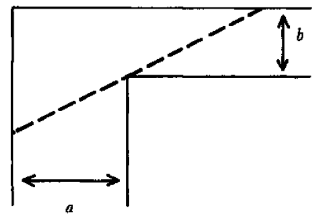


FIGURE 25

2 pts. **11.** A right triangle with hypotenuse of length a is rotated about one of its legs to generate a right circular cone. Find the greatest possible volume of such a cone.

6 pts. **12.** Two hallways, of widths a and b, meet at right angles (Figure 25). What is the greatest possible length of a ladder which can be carried horizontally around the corner?

4 pts. **13.** A garden is to be designed in the shape of a circular sector (Figure 26), with radius R and central angle θ. The garden is to have a fixed area A. For what value of R and θ (in radians) will the length of the fencing around the perimeter be minimized?

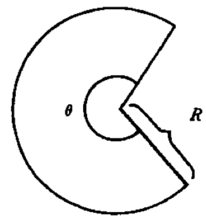


FIGURE 26

1 pt. **14.** Show that the sum of a positive number and its reciprocal is at least 2.

6 pts. **62.** Let $f(x) = x^4 \sin^2 1/x$ for $x \neq 0$, and let $f(0) = 0$ (Figure 32).

(a) Prove that 0 is a local minimum point for f.
(b) Prove that $f'(0) = f''(0) = 0$.

This function thus provides another example to show that Theorem 6 cannot be improved. It also illustrates a subtlety about maxima and minima that often goes unnoticed: a function may not be increasing in any interval to the right of a local minimum point, nor decreasing in any interval to the left.

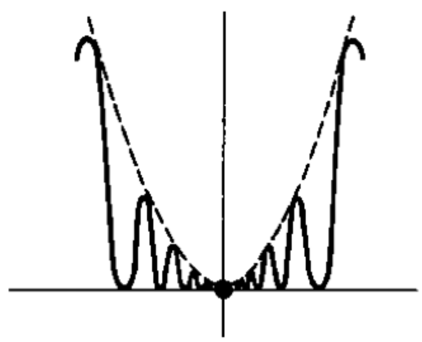


FIGURE 32

SOLUTIONS TO EXTRA PROBLEM SET

Disclaimer: There may be typos.

[Sp 15]

1. (ii) $f'(x) = 5x^4 + 1 = 0$ for no x;
$f(-1) = -1,\ f(1) = 3$;
maximum $= 3$, minimum $= -1$.

(iv) $f'(x) = -\dfrac{(5x^4+1)}{(x^5+x+1)^2} = 0$ for no x;
$f(-1/2) = 32/15,\ f(1) = 1/3$;
maximum $= 32/15$, minimum $= 1/3$.

(Notice that $g(x) = x^5 + x + 1$ is increasing, since $g'(x) = 5x^4 + 1 > 0$ for all x; since $g(-1/2) = 15/32 > 0$, this shows that $g(x) \neq 0$ for all x in $[-1/2, 1]$, so f is differentiable on $[-1/2, 1]$.)

(vi) f is not bounded above or below on $[0, 5]$.

(i) $0 = f'(x) = 3x^2 - 2x - 8$ for $x = 2$ and $x = -\frac{4}{3}$, both of which are in $[-2, 2]$;
$f(-2) = 5,\ f(2) = -11,\ f(-\frac{4}{3}) = \frac{203}{27}$;
maximum $= \frac{203}{27}$, minimum $= -11$.

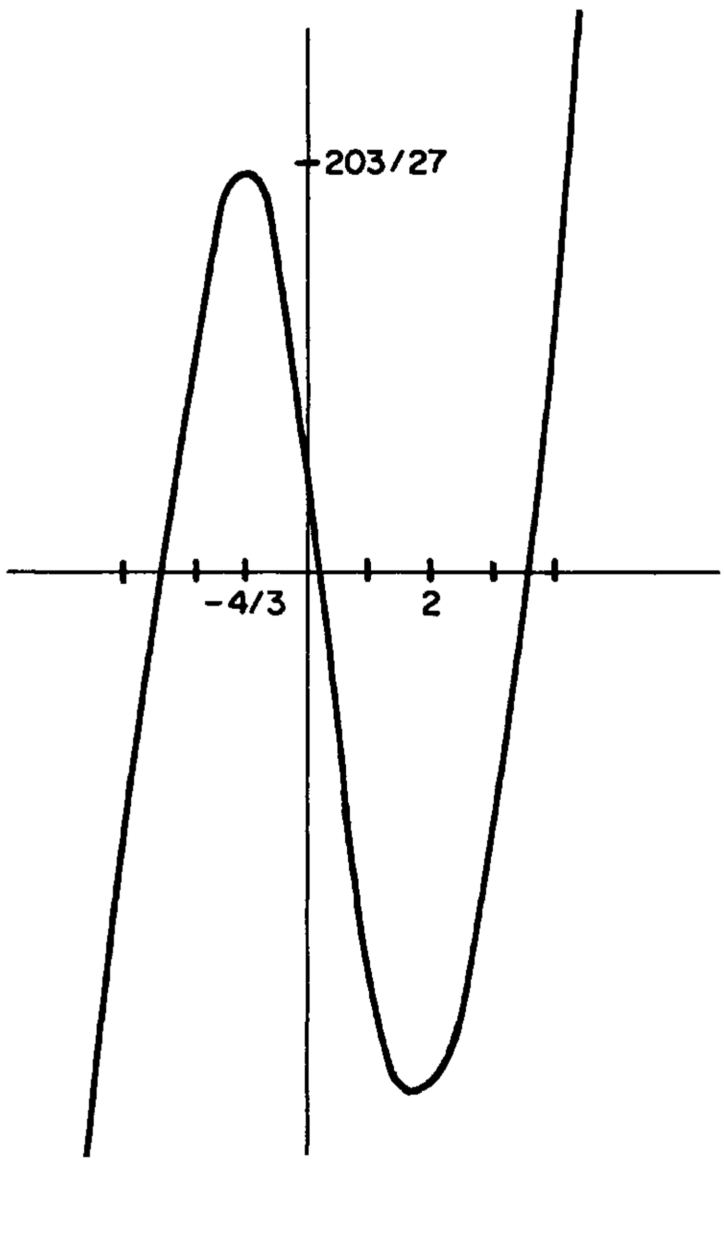
(iii) $0 = f'(x) = 12x^3 - 24x^2 + 12x = 12x(x^2 - 2x + 1)$ for $x = 0$ and $x = 1$, of which only 0 is in $[-\frac{1}{2}, \frac{1}{2}]$;
$f(-\frac{1}{2}) = \frac{43}{16},\ f(\frac{1}{2}) = \frac{11}{16},\ f(0) = 0$;
maximum $= \frac{43}{16}$, minimum $= 0$.

(v) $0 = f'(x) =$

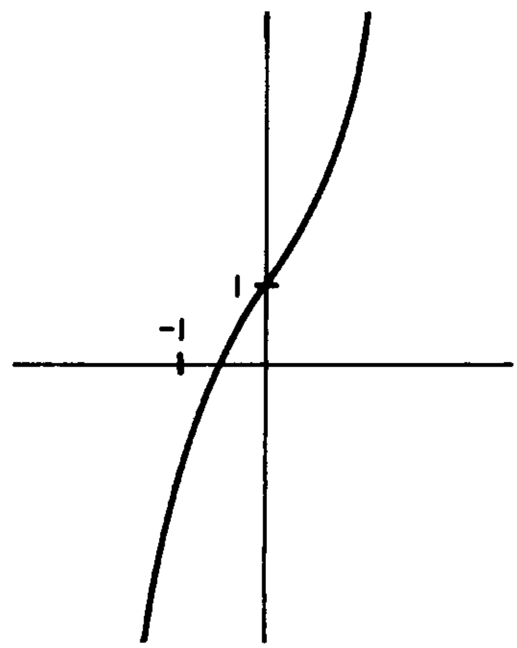
$$\frac{x^2+1-(x+1)2x}{(x^2+1)^2} = \frac{1-2x-x^2}{(x^2+1)^2}$$

for $x = -1 + \sqrt{2}$ and $x = -1 - \sqrt{2}$, of which only $-1 + \sqrt{2}$ is in $[-1, \frac{1}{2}]$;
$f(-1) = 0,\ f(\frac{1}{2}) = \frac{6}{5},\ f(-1+\sqrt{2}) = (1+\sqrt{2})/2$;
maximum $= (1+\sqrt{2})/2$, minimum $= 0$.

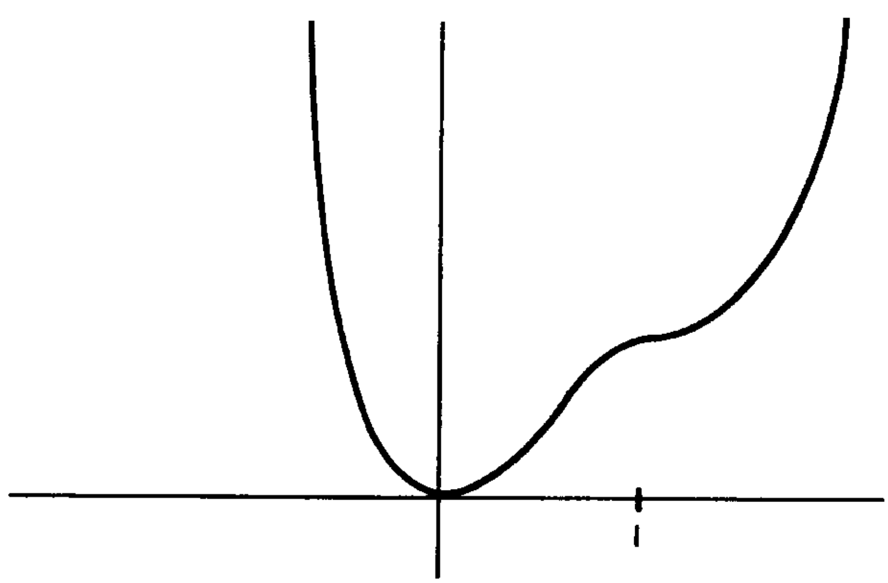
2. (i) $-4/3$ is a local maximum point, and 2 is a local minimum point.



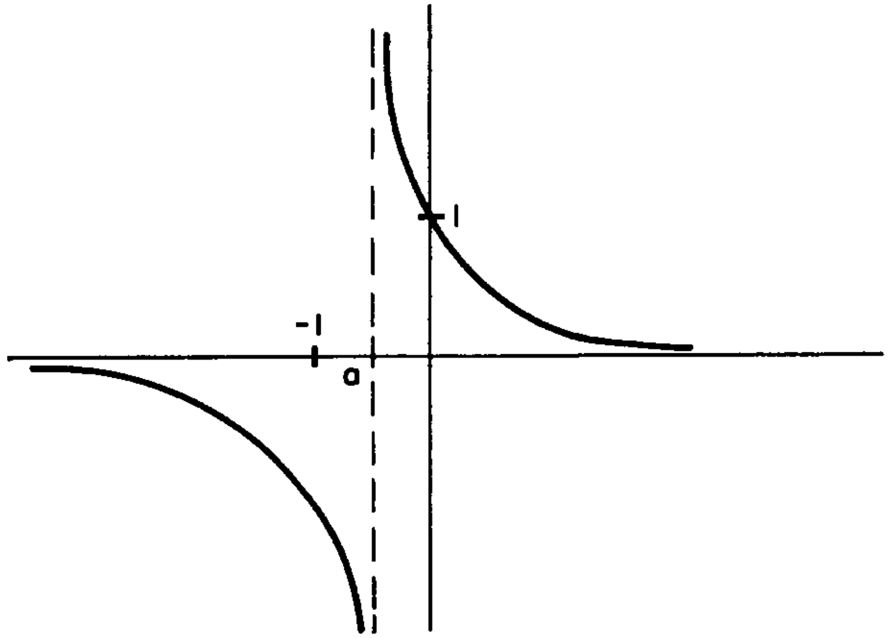
(ii) No local maximum or minimum points.



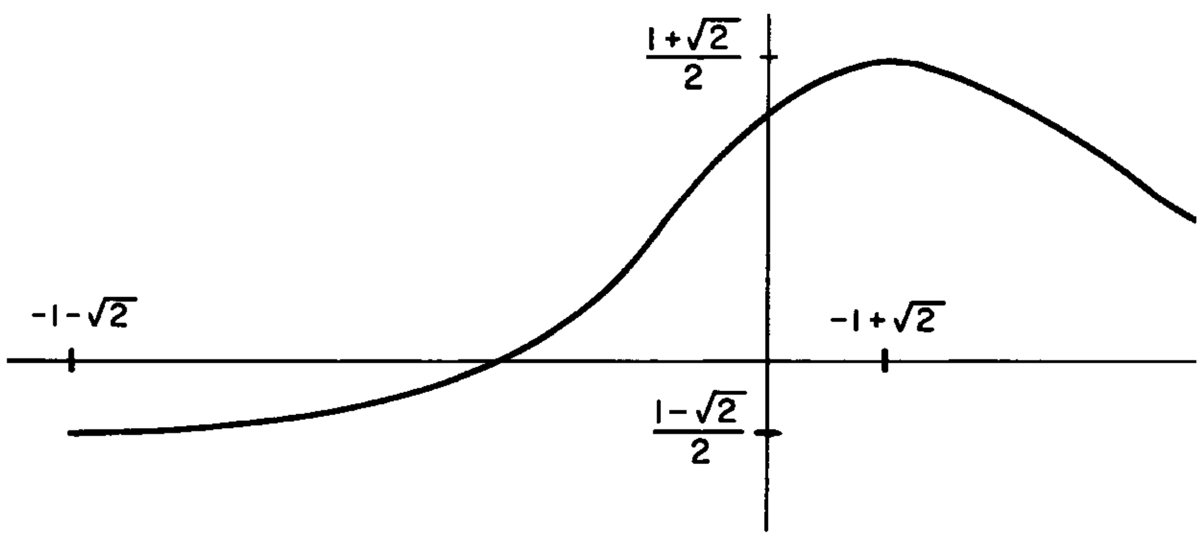
(iii) 0 is a local minimum point, and there are no local maximum points.



(iv) No local maximum or minimum points. In the figure below, a is the unique root of $x^5 + x + 1 = 0$.

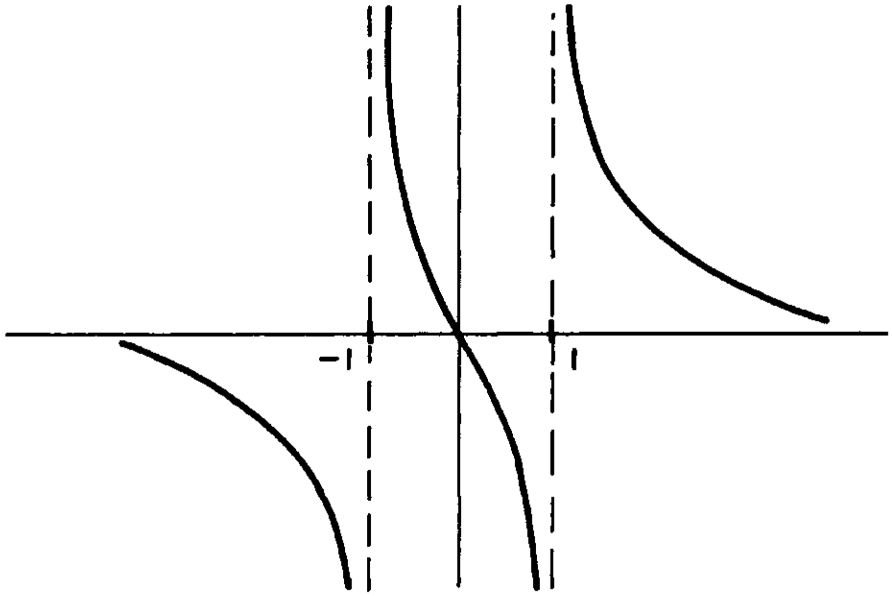


(v) $-1+\sqrt{2}$ is a local maximum point, and $-1-\sqrt{2}$ is a local minimum point.



(vi) No local maximum or minimum points, since

$$f'(x) = -\frac{(1+x^2)}{(x^2-1)^2} < 0 \qquad \text{for } x \neq \pm 1.$$

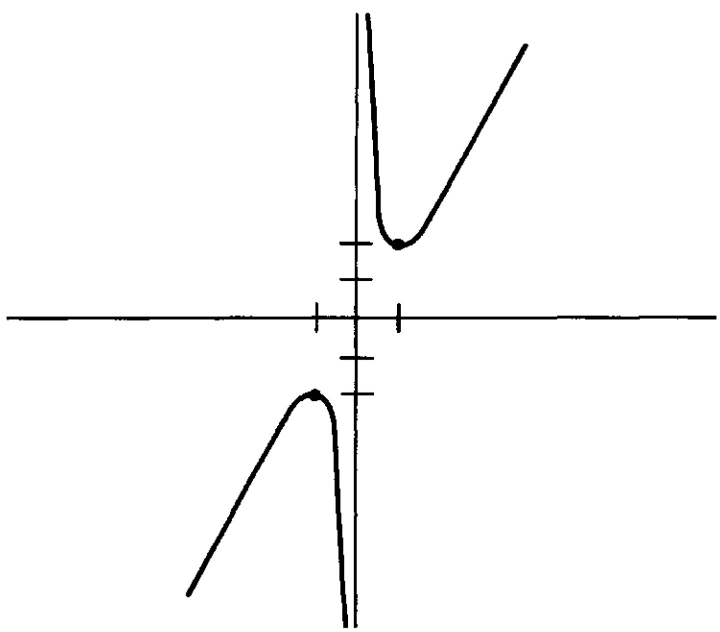


3. (i) f is odd;

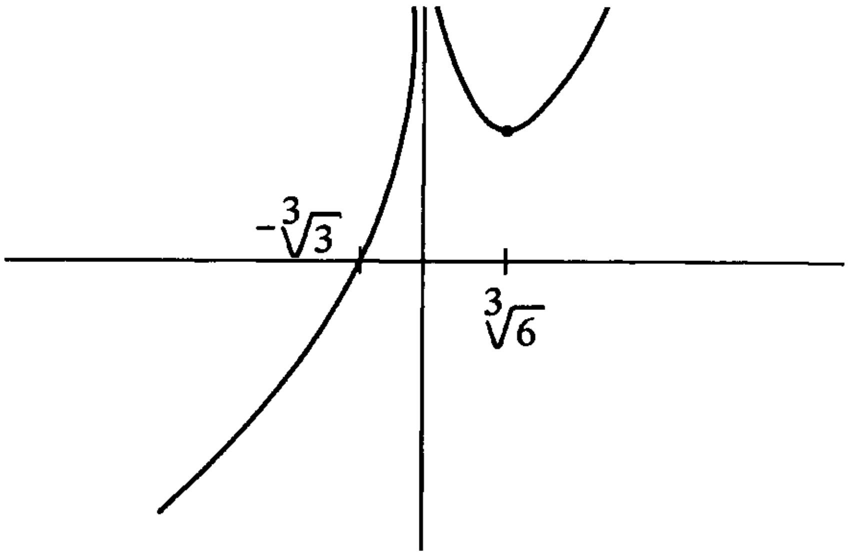
$$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2};$$

$f'(x) = 0$ for $x \neq \pm 1$, $f'(x) > 0$ for $|x| > 1$;

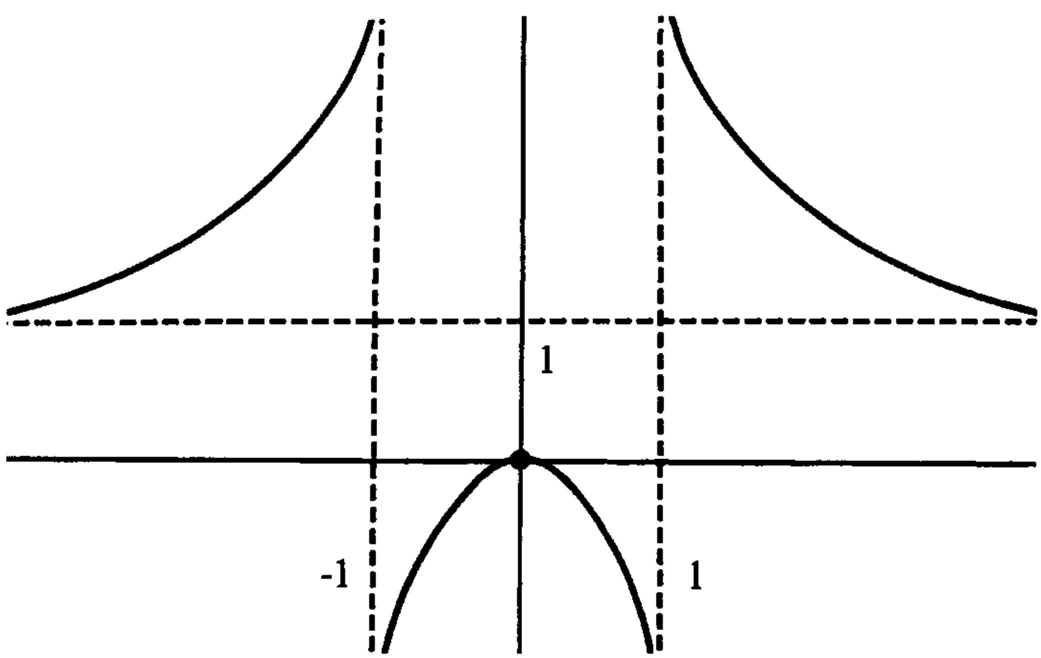
$f(1) = 2$, $f(-1) = -2$.



(ii) $f'(x) = 1 - \dfrac{6}{x^3} = \dfrac{x^3 - 6}{x^3}$;
$f'(x) = 0$ for $x = \sqrt[3]{6}$, $f'(x) > 0$ for $x > \sqrt[3]{6}$ and $x < 0$;
$f(x) = 0$ for $x = -\sqrt[3]{3}$.



(iii) f is even;
$f'(x) = \dfrac{2x(x^2-1) - 2xx^2}{(x^2-1)^2} = \dfrac{-2x}{(x^2-1)^2}$;
$f'(x) = 0$ for $x = 0$, $f'(x) < 0$ for $x > 0$, $f'(x) > 0$ otherwise;
$f(0) = 0$.

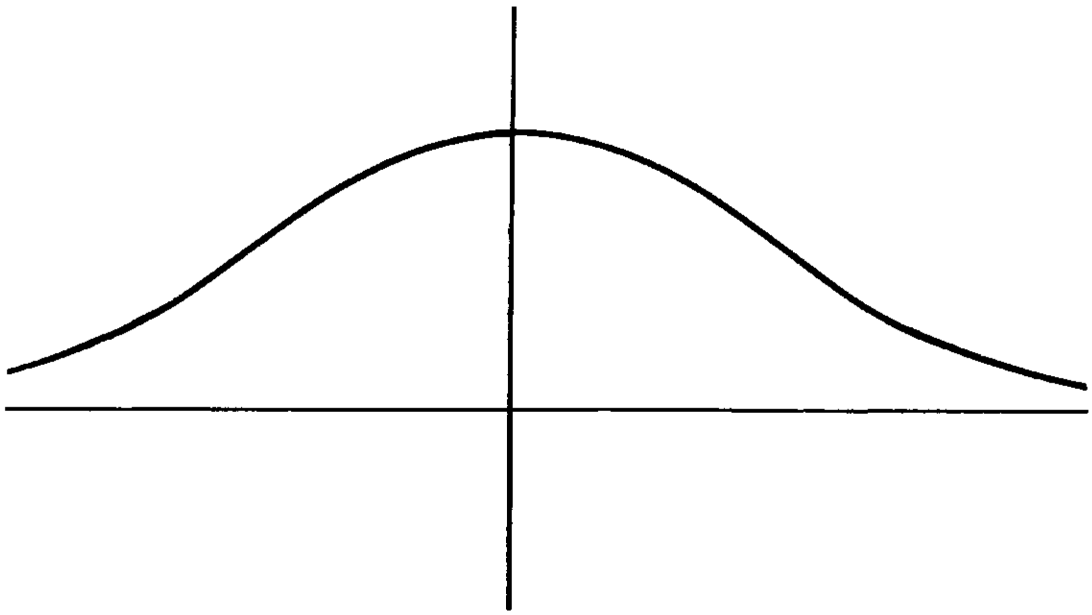


(iv) f is even, $f(x) > 0$ for all x;

$$f'(x) = \frac{-2x}{(1+x^2)^2};$$

$f'(x) = 0$ for $x = 0$, $f'(x) > 0$ for $x < 0$, $f'(x) < 0$ otherwise;

$f(0) = 1$.



5. (i) 3 and 7 are local maximum points, and 5 and 9 are local minimum points.

   (iii) All irrational $x > 0$ are local minimum points, and all irrational $x < 0$ are local maximum points.

Q] What's an irrational number?

A] Try watching https://youtu.be/5sKah3pJnHI.

He's in the video.

Also, James Grime will give a talk on Friday in the MARC.

why does Spivak write "clearly" here?

8. If $f(x)$ is the total length of the path, then

$$f(x) = \sqrt{x^2 + a^2} + \sqrt{(1-x)^2 + b^2}.$$

Convince yourself.

The positive function f clearly has a minimum, since $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = \infty$, and f is differentiable everywhere, so the minimum occurs at a point x with $f'(x) = 0$. Now, $f'(x) = 0$ when

$$\frac{x}{\sqrt{x^2+a^2}} - \frac{(1-x)}{\sqrt{(1-x)^2+b^2}} = 0.$$

You need to draw right triangles, with angles α and β respectively.

This equation says that $\cos\alpha = \cos\beta$.

finish the argument ...

It is also possible to notice that $f(x)$ is equal to the sum of the lengths of the dashed line segment and the line segment from $(x, 0)$ to $(1, b)$. This is shortest when the two line segments lie along a line (because of Problem 4-9(b), if a rigorous reason is required); a little plane geometry shows that this happens when $\alpha = \beta$.

9. If x is the length of one side of a rectangle of perimeter P, then the length of the other side is $(P - 2x)/2$, so the area is

$$A(x) = \frac{x(P - 2x)}{2}.$$

So the rectangle with greatest area occurs when x is the maximum point for f on $(0, P/2)$. Since A is continuous on $[0, P/2]$, and $A(0) = A(P/2) = 0$, and $A(x) > 0$ for x in $(0, P/2)$, the maximum exists. Since A is differentiable on $(0, P/2)$, the minimum point x satisfies

$$\begin{aligned} 0 = A'(x) &= \frac{P - 2x}{2} - x \\ &= \frac{P - 4x}{2}, \end{aligned}$$

so $x = P/4$.

We've seen another way to solve this problem in [Pri79].

10. Let $S(r)$ be the surface area of the right circular cylinder of volume V with radius r. Since

$$V = \pi r^2 h \quad \text{where } h \text{ is the height,}$$

we have $h = V/\pi r^2$, so

$$\begin{aligned} S(r) &= 2\pi r^2 + 2\pi r h \\ &= 2\pi r^2 + \frac{2V}{r}. \end{aligned}$$

We want the minimum point of S on $(0, \infty)$; this exists, since $\lim_{r\to 0} S(r) = \lim_{r\to\infty} S(r) = \infty$. Since S is differentiable on $(0, \infty)$, the minimum point r satisfies

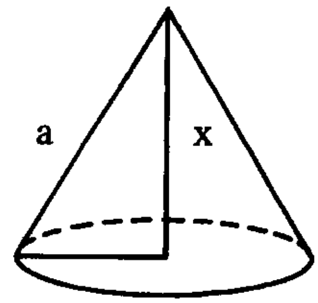
$$\begin{aligned} 0 = S'(r) &= 4\pi r - \frac{2V}{r^2} \\ &= \frac{4\pi r^3 - 2V}{r^2}, \end{aligned}$$

or

$$r = \sqrt[3]{\frac{V}{2\pi}}.$$

11. Let x be the height of the cone. The volume $V(x)$ is given by

$$V(x) = \frac{1}{3}x\pi\left(\sqrt{a^2 - x^2}\right)^2 = \frac{\pi}{3}(a^2x - x^3).$$



So the volume is greatest when

$$0 = V'(x) = \frac{\pi}{3}[a^2 - 3x^2],$$

or $x = a/\sqrt{3}$. For this x we have

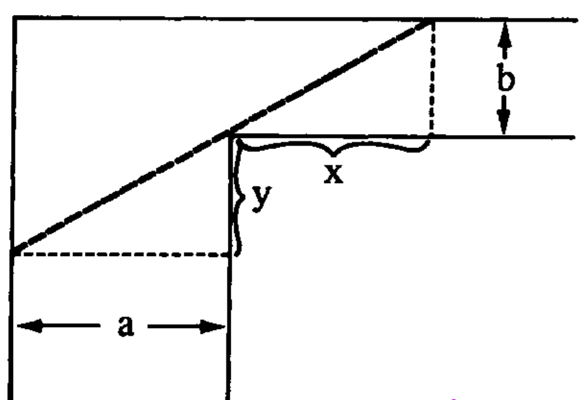
why?

$$V(x) = \frac{\pi}{3}\left(\frac{a^3}{\sqrt{3}} - \frac{a^3}{3\sqrt{3}}\right) = \frac{2\sqrt{3}a^3}{27}.$$

Tricky, but "just do it".

12. In the Figure below we have

$$\frac{b}{x} = \frac{y}{a},$$



so the length of the dashed line is

why?

$$\sqrt{b^2 + x^2} + \sqrt{a^2 + \frac{a^2b^2}{x^2}} = \sqrt{b^2 + x^2} + \frac{a}{x}\sqrt{x^2 + b^2} = \left(1 + \frac{a}{x}\right)\sqrt{x^2 + b^2}.$$

The maximum length of a ladder which can be carried horizontally around the corner is the *minimum* length of this dashed line. This occurs when

why?

$$
\begin{aligned}
0 &= -\frac{a}{x^2}\sqrt{x^2+b^2} + \left(1+\frac{a}{x}\right)\frac{x}{\sqrt{x^2+b^2}} \\
&= \left[-\frac{a}{x^2}(x^2+b^2) + x + a\right]\cdot\frac{1}{\sqrt{x^2+b^2}},
\end{aligned}
$$

or

$$
\begin{aligned}
ax^2 + ab^2 &= x^3 + ax^2, \\
x &= a^{1/3}b^{2/3},
\end{aligned}
$$

and the length is

$$
\begin{aligned}
\left(1+\frac{a^{2/3}}{b^{2/3}}\right)\sqrt{a^{2/3}b^{4/3}+b^2} &= \left(b^{2/3}+a^{2/3}\right)\sqrt{\frac{a^{2/3}b^{4/3}+b^2}{b^{4/3}}} \\
&= \left(b^{2/3}+a^{2/3}\right)^{3/2}.
\end{aligned}
$$

13. If $R(\theta)$ is the appropriate value of R for given θ, we have

$$\frac{\theta}{2}\cdot R(\theta)^2 = A.$$

The perimeter for this θ will have value

$$
\begin{aligned}
P(\theta) &= \theta R(\theta) + 2R(\theta) \\
&= \sqrt{2A}(\theta+2)\cdot\theta^{-1/2}.
\end{aligned}
$$

So the minimum occurs when

$$
\begin{aligned}
0 = P'(\theta) &= \sqrt{2A}\left[\frac{1}{\theta^{1/2}} - \frac{\theta+2}{2\theta^{3/2}}\right] \\
&= \sqrt{2A}\cdot\frac{\theta-2}{2\theta^{3/2}}
\end{aligned}
$$

or $\theta = 2$ radians, and $R = \sqrt{A}$.

14. If

$$f(x) = x + \frac{1}{x} \qquad (x > 0)$$

then

$$f'(x) = 1 - \frac{1}{x^2},$$

which has the minimum value for $x = 1$, with $f(x) = 2$.

62. **(a)** 0 is actually a minimum on all of $\mathbf{R}$, since $f(0) = 0$ and $f(x) \geq 0$ for all x.

(b)

By def$^{\underline{n}}$ of the derivative.

You need $\sin^2(\frac{1}{h}) \leq 1$, so that $\lim_{h\to 0} \sin^2(\frac{1}{h}) \cdot h^3 = 0$.

$$f'(0) = \lim_{h\to 0} \frac{h^4 \sin^2(1/h)}{h} = 0,$$

Also $\lim_{h\to 0} \sin^2(\frac{1}{h}) \cdot h^2 = 0$.

and

$$f'(h) = 4h^3 \sin^2(1/h) - 2h^2 \sin(1/h)\cos(1/h) \quad \text{for } h \neq 0.$$

So

Because $|\sin(\frac{1}{h})\cos(\frac{1}{h})| \leq 1$, $\lim_{h\to 0} h \cdot \sin(\frac{1}{h})\cos(\frac{1}{h}) = 0$.

$$f''(0) = \lim_{h\to 0} \frac{4h^3 \sin^2(1/h) - 2h^2 \sin(1/h)\cos(1/h)}{h}$$
$$= 0.$$

why? Because $\frac{d}{dh}[f(h)] = \frac{d}{dh}\left[h^4 \cdot \sin^2(\frac{1}{h})\right] = \overbrace{\underbrace{4h^3 \sin^2(\tfrac{1}{h})}_{\text{power rule}} + \underbrace{h^4\, 2\sin(\tfrac{1}{h})\cos(\tfrac{1}{h}) \cdot \left(\tfrac{-1}{h^2}\right)}_{\text{chain rule}}}^{\text{product rule}}$.