INTEGRATION (SOLUTIONS)

COLTON GRAINGER (MATH 1300)

Answers due to Vipul Naik [1]. Happy thanksgiving!

1. Consider the function(s) $[0,1] \to \mathbf{R}$. **Identify the function**s for which the integral (using upper sums and lower sums) is not defined.

   (A) $f_1(x) := \begin{cases} 0, & 0 \leqslant x < 1/2 \\ 1, & 1/2 \leqslant x \leqslant 1 \end{cases}$.
   (B) $f_2(x) := \begin{cases} 0, & x \neq 0 \text{ and } 1/x \text{ is an integer} \\ 1, & \text{otherwise} \end{cases}$.
   (C) $f_3(x) := \begin{cases} 0, & x \text{ rational} \\ 1, & x \text{ irrational} \end{cases}$
   (D) All of the above
   (E) None of the above

Answer: Option (C)

Explanation: For option (C), the lower sum for any partition is 0 and the upper sum is 1. Thus, the integral is not well-defined.

For option (A), the function is piecewise continuous with only jump discontinuities, hence the integral is well-defined: in fact, it is $1/2$.

For (B), the integral is zero. We can see this by noting that the points where the function is 0 are all isolated points, so if in our partition the intervals surrounding each of these points is small enough, we can make the upper sums tend to zero. (This is hard to see. You should, however, be able to easily see that (A) has an integral and (C) does not. This forces the answer to be (C)).

2. Suppose $a < b$. Recall that a *regular partition* into n parts of $[a, b]$ is a partition $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ where $x_i - x_{i-1} = (b-a)/n$ for all $1 \leqslant i \leqslant n$. A partition P_1 is said to be a *finer partition* than a partition P_2 if the set of points of P_1 contains the set of points of P_2. Which of the following is a **necessary and sufficient condition** for the regular partition into m parts to be a *finer partition* than the regular partition into n parts? (Note: We'll assume that any partition is finer than itself).

   (A) $m \leqslant n$
   (B) $n \leqslant m$
   (C) m divides n (i.e., n is a multiple of m)
   (D) n divides m (i.e., m is a multiple of n)
   (E) m is a power of n

Answer: Option (D)

Explanation: If n divides m, then the partition into m pieces is obtained by further subdividing the partition into m parts, with each part divided into n/m parts.

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The other choices: Option (B) is a *necessary* condition but is not a sufficient condition. For instance, the regular partition of $[0, 1]$ into two parts corresponds to $\{0, 1/2, 1\}$ and the partition into three parts corresponds to $\{0, 1/3, 2/3, 1\}$. These partitions are incomparable, i.e., neither is finer than the other.

Action point: If you chose option (B), please make sure you understand the distinction between the options. Also review the concept of regular partitions and finer partition till you find this question obvious.

3. *Consumption smoothing (or, advice for Thanksgiving)*: A certain measure of happiness is found to be a logarithmic function of consumption, i.e., the happiness level H of a person is found to be of the form $H = a + b \ln C$ where C is the person's current consumption level, and a and b are positive constants independent of the consumption level. The person has a certain total consumption C_{tot} to be split within two years, year 1 and year 2, i.e., $C_{tot} = C_1 + C_2$. Thus, the person's happiness level in year 1 is $H_1 = a + b \ln C_1$ and the person's happiness level in year 2 is $H_2 = a + b \ln C_2$. How would the person choose to split consumption between the two years to maximize average happiness across the years?

   (A) All the consumption in either one year
   (B) Equal amount of consumption in the two years
   (C) Consume twice as much in one year as in the other year
   (D) Consumption in the two years is in the ratio $a : b$

Answer: Option (B)

Explanation: Basically, happiness is logarithmic in consumption, so if consumption is unequal, then it can be distributed from the higher consumption year to the lower consumption year. The *fractional* loss in the higher consumption year is lower than the *fractional* gain in the lower consumption year. The nature of logarithms means that the *absolute* loss in the higher consumption year is lower than the *absolute* gain in the lower consumption year. The process continues till consumption in both years is exactly equal.

We can also do this formally. We are basically using the fact that the logarithm function is concave down.

[1] V. Naik, "Math 152 Course Notes" [Online]. Available: `https://vipulnaik.com/math-152/`