SOLUTION SET (FUNCTIONS AND LIMITS)

COLTON GRAINGER (MATH 1300)

- Consider the function $f(x) := x^2 + 1$. What is the polynomial describing $f(f(x))$?

  (A) $x^2 + 2$
  (B) $x^4 + x^2 + 1$
  (C) $x^4 + x^2 + 2$
  (D) $x^4 + 2x^2 + 1$
  (E) $x^4 + 2x^2 + 2$

  Answer: Option (E). Explanation from Vipul Naik:

  > We have $f(f(x)) = f(x^2 + 1) = (x^2 + 1)^2 + 1 = x^4 + 2x^2 + 1 + 1$, which simplifies to option (E).
  >
  > Option (A) is $(x^2 + 1) + 1 = x^2 + 2$. The error here is is not squaring $x^2 + 1$.
  >
  > Option (D) is $(x^2 + 1)^2 = x^4 + 2x^2 + 1$. The error here is in forgetting to add 1.
  >
  > Options (B) and (C) are like (D) and (E), with an error in the coefficient of x^2.

- If $f(g(x)) = 5$ and $f(x) = x + 3$ for all real x, then $g(x) =$

  (A) $x - 3$
  (B) $3 - x$
  (C) $\frac{5}{x+3}$
  (D) 2
  (E) 8

  Answer: (D) $g(x) = 2$. Explanation:

  > With the constant function $g(x) = 2$, we evaluate $f(g(x)) = f(2) = 5$.

- For all positive functions f and g of the real variable x, let $\sim$ be a relation defined by

  $$f \sim g \text{ if and only if } \lim_{x \to \infty} \frac{f(x)}{g(x)} = 1.$$

  Which of the following is NOT a consequence of $f \sim g$?

  (A) $f^2 \sim g^2$
  (B) $\sqrt{f} \sim \sqrt{g}$
  (C) $e^f \sim e^g$

(D) $f + g \sim 2g$

(E) $g \sim f$

Answer: (C) $e^f \sim e^g$. Explanation from Charlie Rambo:

> Let's find a counter example to $e^f \sim e^g$. Consider $f(x) = x$ and $g(x) = x + 1$.
> Clearly $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$. But
>
> $$\lim_{x \to \infty} \frac{e^x}{e^{x+1}} = \lim_{x \to \infty} \frac{1}{e} = \frac{1}{e} \neq 1.$$

References.

- Vipul Naik, *Math 152 Week 1*. `https://vipulnaik.com/math-152/`.
- GRE Mathematics Test Form GR0568 and Form GR9367.