

MORE LIMITS (SOLUTIONS)

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- We call a function f left continuous on an open interval I if, for all $a \in I$, $\lim_{x \rightarrow a^-} f(x) = f(a)$. Which of the following is an example of a function that is left continuous but not continuous on $(0, 1)$?

- (A) $f(x) := \begin{cases} x, & 0 < x \leq 1/2 \\ 2x, & 1/2 < x < 1 \end{cases}$
- (B) $f(x) := \begin{cases} x, & 0 < x < 1/2 \\ 2x, & 1/2 \leq x < 1 \end{cases}$
- (C) $f(x) := \begin{cases} x, & 0 < x \leq 1/2 \\ 2x - (1/2), & 1/2 < x < 1 \end{cases}$
- (D) $f(x) := \begin{cases} x, & 0 < x < 1/2 \\ 2x - (1/2), & 1/2 \leq x < 1 \end{cases}$
- (E) All of the above

Answer: Option (A). *Explanation* from Vipul Naik [1]: Note that in all four cases, the two pieces of the function are continuous. Thus, the relevant questions are: (i) do the two definitions agree at the point where the definition changes (in all four cases here, $1/2$)? and (ii) is the point (in all cases, $1/2$) where the definition changes included in the left or the right piece?

For options (C) and (D), the definitions on the left and right piece agree at $1/2$. Namely the function x and $2x - (1/2)$ both take the value $1/2$ at the domain point $1/2$. Thus, options (C) and (D) both define continuous functions (in fact, the same continuous function). This leaves options (A) and (B). For these, the left definition x and the right definition $2x$ do not match at $1/2$: the former gives $1/2$ and the latter gives 1 . In other words, the function has a jump discontinuity at $1/2$. Thus, (ii) becomes relevant: is $1/2$ included in the left or the right definition? For option (A), $1/2$ is included in the left definition, so $f(1/2) = 1/2 = \lim_{x \rightarrow 1/2^-} f(x)$. On the other hand, $\lim_{x \rightarrow 1/2^+} f(x) = 1$. Thus, the f in option (A) is left continuous but not right continuous. For option (B), $1/2$ is included in the right definition, so $f(1/2) = 1$ and f is right continuous but not left continuous at $1/2$.

- Suppose f and g are functions $(0, 1)$ to $(0, 1)$ that are both left continuous on $(0, 1)$. Which of the following is *not* guaranteed to be left continuous on $(0, 1)$?
- (A) $f + g$, i.e., the function $x \mapsto f(x) + g(x)$
- (B) $f - g$, i.e., the function $x \mapsto f(x) - g(x)$
- (C) $f \cdot g$, i.e., the function $x \mapsto f(x)g(x)$
- (D) $f \circ g$, i.e., the function $x \mapsto f(g(x))$
- (E) None of the above, i.e., they are all guaranteed to be left continuous functions

Answer: Option (D). *Explanation* again, from Vipul Naik [1]: We need to construct an explicit example, but we first need to do some theoretical thinking to motivate the right example. The full reasoning is given below.

Motivation for example: Left hand limits split under addition, subtraction and multiplication, so options (A)-(C) are guaranteed to be left continuous, and are thus false. This leaves the option $f \circ g$ for consideration. Let us look at this in more detail. For $c \in (0, 1)$, we want to know whether: $\lim_{x \rightarrow c^-} f(g(x)) \stackrel{?}{=} f(g(c))$. We do know, by assumption, that, as x approaches c from the left, $g(x)$ approaches $g(c)$. However, we do not know whether $g(x)$ approaches $g(c)$ from the left or the right or in oscillatory fashion. If we could somehow guarantee that $g(x)$ approaches $g(c)$ from the left, then

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we would obtain that the above limit holds. However, the given data does not guarantee this, so (D) is false. We need to construct an example where g is *not* an increasing function. In fact, we will try to pick g as a decreasing function, so that when x approaches c from the left, $g(x)$ approaches $g(c)$ from the right. As a result, when we compose with f , the roles of left and right get switched. Further, we need to construct f so that it is left continuous but not right continuous.

Explanation with example: Consider the case where, say: $f(x) := \begin{cases} 1/3, & 0 < x \leq 1/2 \\ 2/3, & 1/2 < x < 1 \end{cases}$ and $g(x) := 1 - x$. Note that both functions have range a subset of $(0, 1)$. Composing, we obtain that: $f(g(x)) = \begin{cases} 2/3, & 0 < x < 1/2 \\ 1/3, & 1/2 \leq x < 1 \end{cases}$. Note that f is left continuous but not right continuous at $1/2$, whereas $f \circ g$ is right continuous but not left continuous at $1/2$.

- Consider the function

$$f(x) := \begin{cases} x, & x \text{ rational} \\ 1/x, & x \text{ irrational} \end{cases}$$

What is the set of all points at which f is continuous?

- (A) $\{0, 1\}$
- (B) $\{-1, 1\}$
- (C) $\{-1, 0\}$
- (D) $\{-1, 0, 1\}$
- (E) f is continuous everywhere

Answer: Option (B). *Explanation* one more time from [1]: In this interesting example, instead of a *left* versus *right* split, we are splitting the domain into rationals and irrationals. For the overall limit to exist at c , we need that: (i) the limit for the function as defined for rationals exists at c , (ii) the limit for the function as defined for irrationals exists at c , and (iii) the two limits are equal. Note that regardless of whether the point c is rational or irrational, we need *both* the rational domain limit and the irrational domain limit to exist and be equal at c . This is because rational numbers are surrounded by irrational numbers and vice versa – both rational numbers and irrational numbers are dense in the reals – hence at any point, we care about the limits restricted to the rationals as well as the irrationals. The limit for rationals exists for all c and equals the value c . The limit for irrationals exists for all $c \neq 0$ and equals the value $1/c$. For these two numbers to be equal, we need $c = 1/c$. Solving, we get $c^2 = 1$ so $c = \pm 1$.

- Define the base e of the “natural” exponential function. Hint: The derivative of every exponential function of the form $f(x) := a^x$ with $a > 0$ is equal to a multiple of itself $f'(x) = \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h} = a^x \lim_{h \rightarrow 0} \frac{a^h - 1}{h}$.
- (A) $e = \lim_{h \rightarrow 0} e^h$
 - (B) e is the number that satisfies $\log(1) = e$
 - (C) $e = \lim_{h \rightarrow 0} \frac{e^h}{h}$
 - (D) e is the number that satisfies $e^{x+y} = e^x e^y$ for all $x, y \in \mathbf{R}$
 - (E) e is the number that satisfies $\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$

Answer: Option (E). *Explanation* from Stephen Leduc [2]: If we can find the value of a such that $\frac{a^h - 1}{h} = 1$, then $f'(x) = f(x)$. So let's define e such that

$$\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1.$$

Options (A), (B) and (C) are fallacious. Option (D) is not unique to e .

REFERENCES

- [1] V. Naik, “Math 152 Course Notes” [Online]. Available: <https://vipulnaik.com/math-152/>
- [2] S. A. Leduc and P. R. Firm, *Cracking the GRE math subject test*. Random House, 2010 [Online]. Available: <http://www.worldcat.org/isbn/9780375429729>