DERIVATIVES I (SOLUTIONS)

COLTON GRAINGER (MATH 1300)

Both the quiz and these solutions were written by Vipul Naik [1] at the University of Chicago.

1. Consider the expression $x^2 + t^2 + xt$. What is the derivative of this with respect to x (with t assumed to be a constant)?

   (A) $2x + 2t + x + t$
   (B) $2x + 2t + 1$
   (C) $2x + 2t$
   (D) $2x + t + 1$
   (E) $2x + t$

   Answer: Option (E)

   Explanation: When we differentiate with respect to x, keeping t constant, the x^2 differentiates to $2x$, the t^2 differentiates to 0 (because it is constant) and the xt differentiates to t.

   Note that there's something else called *implicit differentiation* where we do not assume t to be a constant but rather an *unknown function of x*. In that case, we differentiate the functions of t with respect to t and tag along a dt/dx. With that interpretation, the derivative would be $2x + 2t(dt/dx) + x(dt/dx) + t$. However, we're assuming t constant, so $dt/dx = 0$, and so the dt/dx terms vanish and we are just left with $2t + x$.

   The other choices:

   Option (A) is what you get if you just differentiate each thing with respect to its own variable naively: x^2 gives $2x$, t^2 gives $2t$, xt, by the product rule, gives $x + t$. This is *completely wrong* because we are differentiating only with respect to x. (See the note on implicit differentiation right above this).

   Options (B), (C) and (D) are also possible results of incorrect differentiation.

2. Which of the following verbal statements is **not valid as a general rule**?

   (A) The derivative of the sum of two functions is the sum of the derivatives of the functions.
   (B) The derivative of the difference of two functions is the difference of the derivatives of the functions.
   (C) The derivative of a constant times a function is the same constant times the derivative of the function.
   (D) The derivative of the product of two functions is the product of the derivatives of the functions.
   (E) None of the above, i.e., they are all valid as general rules.

   Answer: Option (D)

   Explanation: The correct replacement of option (D) is the product rule for derivatives, which, in words, states that: "the derivative of the product of two functions is the sum of the product of the derivative of the first function with the second function and the product of the first function with the derivative of the second function." If that seems cumbersome to you, feel grateful for the power of algebraic symbols to capture this compactly:

$$(f \cdot g)' = (f' \cdot g) + (f \cdot g')$$

Date: 2018-09-20.
Compiled: 2018-10-24.
Repo: `https://github.com/coltongrainger/pro19ta`.

3. Which of the following statements is **definitely true** about the tangent line to the graph of an everywhere differentiable function f on $\mathbf{R}$ at the point $(a, f(a))$ (Here, "everywhere differentiable" means that the derivative of f is defined and finite for all $x \in \mathbf{R}$)?

   (A) The tangent line intersects the curve at precisely one point, namely $(a, f(a))$.
   (B) The tangent line intersects the x-axis.
   (C) The tangent line intersects the $f(x)$-axis (the y-axis).
   (D) Any line through $(a, f(a))$ other than the tangent line intersects the graph of f at at least one other point.
   (E) None of the above need be true.

   Answer: Option (C)

   Explanation: If the function is differentiable, then the tangent line has finite slope, and hence cannot be vertical. Thus, it is not parallel to the y-axis, and hence must intersect the y-axis.

   The other choices:

   Option (A) is not true, as discussed in class. For instance, for the sin function, the tangent line through any of the peak points is $y = 1$, and passes through all the peak points, hence it intersects the graph infinitely often. We can graphically construct a lot of examples where the tangent line at one point in the graph intersects the graph elsewhere. There are certain classes of functions for which the statement of option (A) *is* true, and we'll talk more about this when we discuss concave up and concave down.

   Option (B) is not true. The tangent line to $(\pi/2, 1)$ for the sin function is $y = 1$ – a horizontal line. Thus, it does not intersect the x-axis. In general, the tangent line does not intersect the x-axis iff it is horizontal, which happens iff the derivative at the point is zero.

4. For a function $f : (0, \infty) \to (0, \infty)$, denote by $f^{(k)}$ the k^{th} derivative of f. Suppose $f(x) := x^r$ with domain $(0, \infty)$, and r a rational number. What is the **precise set of values** of r satisfying the following: there exist a positive integer k (dependent on r) for which $f^{(k)}$ is identically the zero function.

   (A) r should be an integer.
   (B) r should be a nonnegative integer.
   (C) r should be a positive integer.
   (D) r should be a nonnegative rational number.
   (E) r should be a positive rational number.

   Answer: Option (B)

   Explanation: If r is 0, then the derivative of the function is zero. For r a positive integer, the $(r+1)^{th}$ derivative is 0. See also Routine Problem 14 of Homework 3 (Exercise 3.3.64 of the book, Page 129).

   For any other value of r, the power of x keeps going down by 1 each time we differentiate. However, since we didn't start with a nonnegative integer, the power of x never becomes 0, so we keep going down and never stop. For instance, if $r = 5/3$, we have:

   $$f(x) = x^{5/3}, f^{(1)}(x) = (5/3)x^{2/3}, f^{(2)}(x) = (10/9)x^{-1/3}, \ldots$$

   Note that the powers of x in f and its derivatives are $5/3$, $2/3$, $-1/3$, $-4/3$ and so on, going down by 1 each time. Note that going down from $2/3$ to $-1/3$, the power skips right past 0.

References

[1] V. Naik, "Math 152 Course Notes" [Online]. Available: https://vipulnaik.com/math-152/