https://math.stackexchange.com/questions/729907/why-is-pi-so-close-to-3

https://math.stackexchange.com/questions/1540947/prove-that-pi3-using-geometry

Noli tangere circulos meos.
Do not disturb my circles

The area of a regular octagon is 22–√ times its squared circumradius. Thus, considering a regular octagon inscribed in a circle is enough to prove that π>22–√.
Similarly, the area of a dodecagon is 3 times its squared circumradius, proving that π>3.
https://en.wikipedia.org/wiki/Dodecagon

The Pythagorean theorem deals with squares constructed on the three sides of a right triangle. What about semicircles instead? Prove that the area of the semicircle with hypotenuse as diameter is equal to the sum of the areas of the semicircles on the other two sides. Hint. Use the equation A = 1[r2 as an aid in finding the areas of the semicircles. Then use the Pythagorean theorem.

https://www.youtube.com/watch?v=2vnqSwWAn34 (earthquakes)
https://www.youtube.com/watch?v=AvFNCNOyZeE (passing through square hole)
https://www.youtube.com/watch?v=lubGnk0UZt0 (cones)

http://www.math.ucla.edu/~vsv/resource/general/integral%20approximations%20to%20pi.pdf (integral approximations)

(d) How could you convince someone that n is less than 4? Hint. Consider a square circumscribed about the unit circle. What is the area of the square? (e) How could you convince someone that 1t exceeds 3? (A certain state legislature once considered seriously passing a law declaring that 1t was equal to 3 in that state. What are the objections to such a law?)

1.4 The Real Numbers

[Gow08]

A famous discovery of the ancient Greeks, often attributed, despite very inadequate evidence, to the school of PYTHAGORAS [VI.1], was that the square root of 2 is not a rational number. That is, there is no fraction p/q such that $(p/q)^2 = 2$. The Pythagorean theorem about right-angled triangles (which was probably known at least a thousand years before Pythagoras) tells us that if a square has sides of length 1, then the length of its diagonal is $\sqrt{2}$. Consequently, there are lengths that cannot be measured by rational numbers.

This argument seems to give strong practical reasons for extending our number system still further. However, such a conclusion can be resisted: after all, we cannot make any measurements with infinite precision, so in practice we round off to a certain number of decimal places, and as soon as we have done so we have presented our measurement as a rational number. (This point is discussed more fully in NUMERICAL ANALYSIS [IV.20].)

Nevertheless, the *theoretical* arguments for going beyond the rational numbers are irresistible. If we want to solve polynomial equations, take LOGARITHMS [III.25 §4], do trigonometry, or work with the GAUSSIAN DISTRIBUTION [III.73 §5], to give just four examples from an almost endless list, then irrational numbers will appear everywhere we look. They are not used directly for the purposes of measurement, but they are needed if we want to reason theoretically about the physical world by describing it mathematically. This necessarily involves a certain amount of idealization: it is far more convenient to say that the length of the diagonal of a unit square is $\sqrt{2}$ than it is to talk about what would be observed, and with what degree of certainty, if one tried to measure this length as accurately as possible.

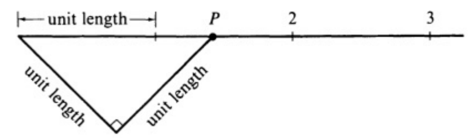
The real numbers can be thought of as the set of all numbers with a finite or infinite decimal expansion. In the latter case, they are defined not directly but by a process of successive approximation. For example, the squares of the numbers 1, 1.4, 1.41, 1.414, 1.4142, 1.41421, ..., get as close as you like to 2, if you go far enough along the sequence, which is what we mean by saying that the square root of 2 is the infinite decimal 1.41421....

The set of all real numbers is denoted $\mathbb{R}$.

Pythagoreans want to explain everything by numbers. Trouble starts when one tries to explain the simplest elements of geometry by numbers. How does one account for points on a line in terms of numbers? This appears easy at first, but the appearance is deceptive. On a line segment a *unit length* is first chosen, and then to each ratio of integers is associated a point, in a natural way that is now familiar to every schoolchild. The ratio $\frac{3}{4}$, for example, names the point obtained by dividing the unit length into 4 equal parts and then taking 3 of them. At first, it appears that every point on the line can be named in this way, by using ratios of integers, or *rational numbers*.

[Pri 79]

The Pythagoreans, however, discovered to their distress that there was a certain point P that could be accounted for by *no rational number whatever*!



Consider the point P situated on the line as indicated above. The number associated with P would measure the length of the hypotenuse of a right triangle whose legs each have a length of one unit. Therefore, by the Pythagorean theorem, *the square of the number associated with P must be equal to* 2. The shock was felt when somehow, out of the Pythagorean school, around 500 B.C., came the following remarkable statement and proof.

Theorem. *There is no rational number whose square is* 2.

PROOF. (The proof uses the indirect, or *reductio ad absurdum* method, which consists in showing that an absurd conclusion results from supposing the theorem false.) Suppose that the theorem stated above is false, i.e., suppose there *is* a rational number whose square is 2. Then, by canceling out any common factors in the numerator and denominator, we should have a rational number a/b *in lowest terms* whose square is 2. We should then have integers a and b satisfying:

$$a \text{ and } b \text{ have no common divisor;} \tag{4}$$

$$\frac{a^2}{b^2} = 2. \tag{5}$$

From equation (5) it follows that

$$a \text{ is even.} \tag{6}$$

Reason: If a were an odd number, then a^2 would be odd; yet equation (5) tells us that $a^2 = 2b^2$, showing that a^2 is even, being twice another integer.

Since we know that a is even, we know that a must be equal to twice some other integer. Calling this other integer k, we then have $a = 2k$, or $a^2 = 4k^2$, so that equation (5) becomes

$$\frac{4k^2}{b^2} = 2, \tag{7}$$

where k and b are integers. From equation (7) it follows that

$$b \text{ is even.} \tag{8}$$

Reason: If b were an odd number, then b^2 would be odd, yet equation (7) tells us (when it is solved for b) that $b^2 = 2k^2$, showing that b^2 is even.

The absurd conclusion is evident when statements (4), (6), and (8) are compared. This shows that an absurd conclusion is a consequence of assuming the theorem false. Therefore, the theorem must be true, Q.E.D.

The Greeks evidently saw no way to explain the point P by a number, since they could conceive of no "number" other than a rational number. Today, most students have no qualms over associating the point P with the number defined by a *never-ending* decimal expansion beginning

$$1.414\ldots.$$

[Koe14]

2.1 Area

Faced with a new mathematical object, we often ask 'how is it defined?' and 'how do we calculate it?'. What happens when we ask these questions about area?

It is easy to answer the second question. To find the area A of a figure, we take a sheet of graph paper divided into little squares of side s and carefully trace A onto the paper. We then count the number N of squares which lie entirely within A and the number of squares M which contain some portion of A. Then

$$Ms^2 \geq \text{area } A \geq Ns^2$$

and the difference $Ms^2 - Ns^2$ can be made as small as we please by taking s sufficiently small.

The first question is harder and many thoughtful mathematicians have chosen to answer, in effect, that the area of A is what we measure by the procedure of the previous paragraph. This answer raises the unpleasant possibility that there might be figures that do not have area either because placing A in a different way on the graph paper and choosing different values of s would produce incompatible estimates or because there do not exist choices of s which make $Ms^2 - Ns^2$ arbitrarily small.

However, it is possible, by careful book-keeping, to show that, if we use the definition just given, all polygons have area and that, if we divide a polygon into several smaller polygons, the sum of the areas of the smaller polygons equals the area of the original polygon. We shall mutter our usual incantation 'well behaved' and assume both that every well behaved region has area and that, if we divide a well behaved region into several smaller well behaved regions, the sum of the areas of the smaller well behaved regions equals the area of the original region.

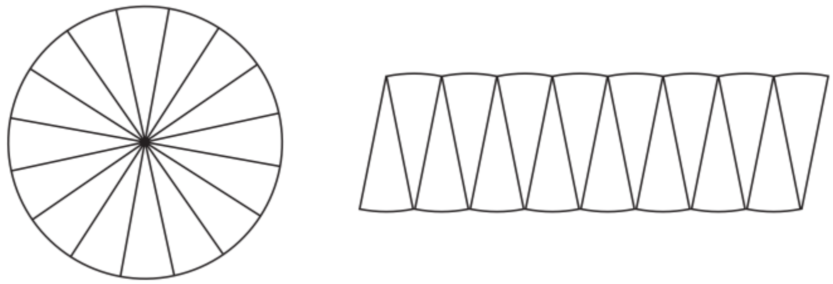


Figure 2.1 Slices of cake

I cannot resist deviating slightly from the main line of our argument to give a very informal geometrical treatment of the well-known constant π. (The ideas go back to the Ancient Greeks. Archimedes gave a full and rigorous exposition of the matter.) Let D be a disc of radius 1 (that is to say, the figure bounded by a circle of radius 1). Suppose that $u > 0$. By definition, if we place D on the graph paper considered above with s sufficiently small, the number N of squares which lie entirely within D and the number of squares M which contain some portion of D will satisfy

$$Ms^2 \geq \text{area } D \geq Ns^2 \text{ and } u \geq Ms^2 - Ns^2 \geq 0. \qquad \bigstar$$

Now look at our system through a magnifying lens, so that the disc has radius r and the graph paper is composed of squares with sides of length rs. Since nothing else has changed,

$$Mr^2s^2 \geq \text{area magnified disc} \geq Nr^2s^2. \qquad \bigstar\bigstar$$

Multiplying the relations in $\bigstar$ by r^2, we have

$$Mr^2s^2 \geq r^2 \text{ area } D \geq Nr^2s^2 \text{ and } ur^2 \geq Mr^2s^2 - Nr^2s^2 \geq 0. \qquad \bigstar\bigstar\bigstar$$

Combining $\bigstar\bigstar$ and $\bigstar\bigstar\bigstar$, we get

$$|\text{area magnified disc} - r^2 \text{ area } D| \leq ur^2.$$

Since u can be chosen as small as we want,

$$\text{area magnified disc} = r^2 \text{ area } D.$$

If we write π for the area of D, we see that the area of a disc of radius r is πr^2.

We can do more. If we take our disc of radius r and cut it up into $2n$ equal sectors, then we can rearrange them as shown in Figure 2.1 to form a figure which looks very much like a rectangle R with one side of length r and the adjacent side of length half the length of the perimeter of the disc. The area of R is the area of the disc, so

$$\begin{aligned} r \times \text{length perimeter disc}/2 &\approx \text{product of lengths of adjacent sides of } R \\ &= \text{area } R \approx \text{area disc} = \pi r^2 \end{aligned}$$

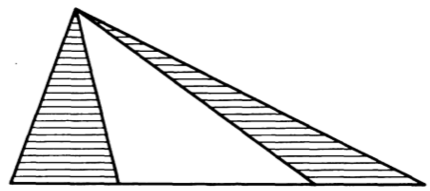
and, assuming that the approximation gets better and better as n increases,

$$r \times \text{length of the perimeter disc}/2 = \pi r^2,$$

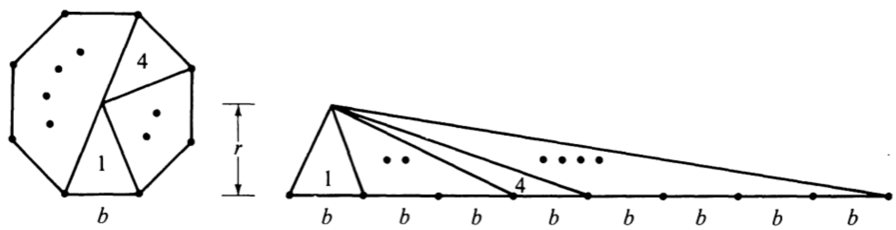
so the length of the perimeter of a circle of radius r is $2\pi r$.

[Pvi 79]

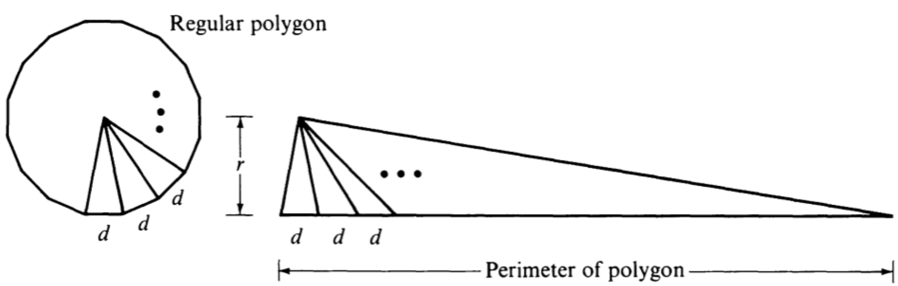
12. Consider the figure below, where the two triangles have a vertex in common and the lengths of their bases are equal. Prove that the triangles have the same area.



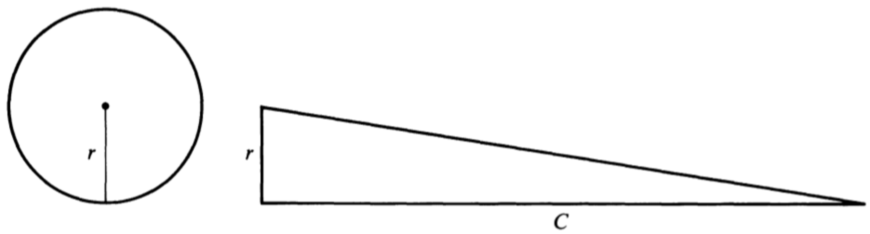
13. Here is a problem that intrigued the Greeks. Given a figure, construct a *triangle* whose area is the same as that of the figure. (For example, given the lunes of problem 11, Hippocrates found a triangle of the same size.) Do this for a *regular octagon.* (*Regular* means all sides have the same length and all angles made by adjacent sides are equal.) *Hint.* Stare at the figure below, and use the result of the preceding problem. (Both figures can be thought of as being made up of eight triangles.)



14. By a *regular polygon of n sides* is meant a figure in the plane bounded by n equal sides with n equal angles. (Problem 13 dealt with a regular polygon of 8 sides.) Let r denote the perpendicular distance from the center of a regular polygon to a side. Show that the area of a regular polygon is equal to the area of a triangle whose height is r and whose base is equal in length to the perimeter of the polygon. *Hint.* Stare at the figure below, and use the same reasoning as you did in the preceding problem.



15. (*Here the reader is asked simply to make a guess after considering the evidence.*) Keep in mind that the equality of areas of the regular polygon and the corresponding triangle pictured in problem 14 holds, no matter how many sides the polygon has. This equality of areas holds for a polygon of a billion sides, for instance. Keeping this in mind, stare at the two figures below. One is a circle of radius r, and the other is a triangle of height r whose base is equal in length to the circumference of the circle.



Now make a guess as to which of the following is true:
(a) The area of the circle exceeds the area of the triangle.
(b) The area of the triangle exceeds that of the circle.
(c) The area of the triangle equals the area of the circle.

16. (*For more ambitious students*) The amount and the type of reasoning which constitute an irrefutable argument in mathematics have never been fixed. Some things that the seventeenth century took as *obvious* (i.e., requiring no proof) the twentieth century and also the ancient Greeks accepted only after a careful demonstration from basic principles had been given. If you believe that the statement in part (c) of problem 15 is "obviously" true, then you are in the good company of some of the keenest minds of the seventeenth century. They would reason that equality between areas of polygons and triangles carries over "in the limit", a circle being regarded as the limit of polygons that approach it more and more closely.

    On the other hand, you may feel that the statement (c) requires a clear proof, because you have only made an educated guess that it is true. If so, then you are at home with Archimedes and with most twentieth-century mathematicians who would think so too. Archimedes proved 15(c) by showing that 15(a) leads to a contradiction, as does 15(b). Can you?

17. The statement in (c) of problem 15 is true. Using it, and letting C stand for the circumference of a circle of radius r, prove that
(a) $\pi r^2 = \frac{1}{2}Cr$.
(b) $C = 2\pi r$.

1. Convert between degrees and radians:

   (a) $180° = \pi$ (b) $135° = \frac{3\pi}{4}$ (c) $\frac{3\pi}{2} = 270°$

   (d) $30° = \pi/6$ (e) $\frac{\pi}{4} = 90°$ (f) $\frac{11\pi}{6} = 330°$

2. Review the unit circle:

   (a) $\sin(3\pi) = 0$ (b) $\cos\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ (c) $\tan\left(\frac{7\pi}{6}\right) = \frac{\sin(\pi/6)}{\cos(\pi/6)} = \frac{1/2}{\sqrt{3}/2}$

   (d) $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$ (e) $\sec\left(\frac{5\pi}{3}\right) = \left[\cos\left(\frac{-\pi}{3}\right)\right]^{-1} = 2$ (f) $\tan\left(-\frac{3\pi}{2}\right) =$ not defined

   what's the domain of tangent?

   $\cos(-3\pi/2) = 0$

3. Review the inverse trig functions:

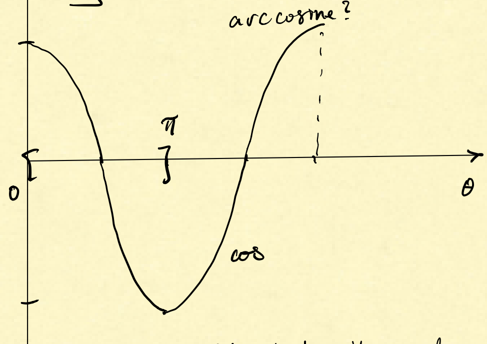
   (a) $\arcsin\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$ (b) $\arccos(-1) = \pi$ (c) $\tan^{-1}\left(\sqrt{3}\right) = \pi/3$

   (d) $\arctan(-1) = -\frac{\pi}{2}$ (e) $\sec^{-1}(2) = \frac{5\pi}{3}$ (f) $\arccos\left(\frac{3}{2}\right) =$ also not def'd

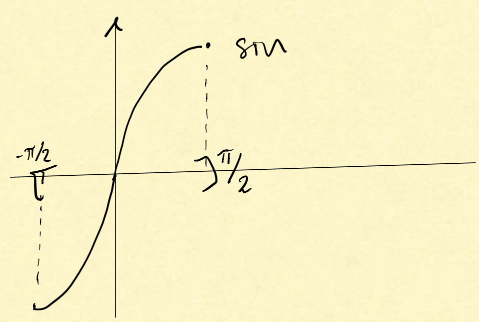
   domain of arccos

4. Review trig identities: match each expression on the left with all expressions on the right that produce a trig identity.

Q) What's the codomain of arccosine?



Q) What's the codomain of arcsin?



Begin class with

$(1, \theta)$ mapping to $(\cos\theta, \sin\theta)$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\sin 2\theta = 2\cos\theta\sin\theta$

get the double angles

and $\cos^2\theta + \sin^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$

Now we will apply our knowledge of basic trigonometric functions (and inverse trigonometric functions) to some real-world problems. You will need these skills later in the semester.

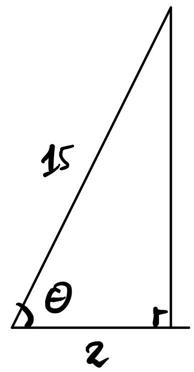
5. A 15-foot long ladder is leaning against a wall with its base 2 feet from the wall. Find the angle the ladder makes with the floor. Use a calculator to compute the answer in radians and in degrees.

$\cos\theta = \frac{2}{15}$

hence $\theta = \arccos\frac{2}{15}$

$= 1.437$ rads

$\approx 89^\circ$



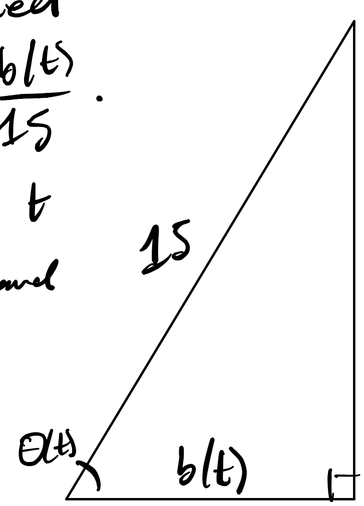
6. The bottom of the ladder in the previous problem starts to slide away from the wall at the constant rate of 1 foot per second.

When will the ladder make a 60° angle with the ground?

Let $b : [2, \infty) \longrightarrow \mathbb{R}$ be defined $b(t) = 2 + t$. Then $\cos\theta(t) = \frac{b(t)}{15}$.

We desire t such that $\theta(t) = \pi/3$. So find t

$\frac{1}{2} = \cos\left(\frac{\pi}{3}\right) = \frac{2+t}{15}$, which is linear, and gives $t = \frac{15}{2} - 2 = \frac{11}{2}$.



Then $\theta(t) = \cos\frac{b(t)}{15}$

we want t s.th $\theta(t) = \pi/3$

so $(\cos\pi/3)15 = b(t) = 2+t$

or $t = (15\cos\frac{\pi}{3}) - 2$.

wrong idea

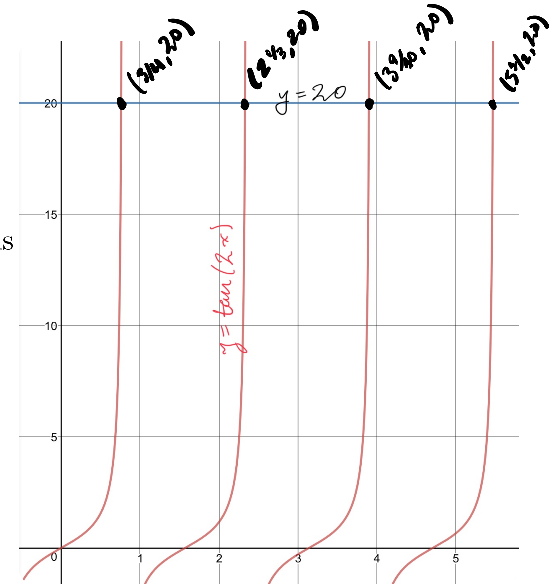
Practice solving some trigonometric equations:

7. Find all solutions to the equation $2\sin x + 1 = 0$

$\sin x = \frac{-1}{2}$ when . . .

$$\left\{ \frac{7\pi}{6} + 2\pi k, \ \frac{-\pi}{6} + 2\pi k \ : \ k \in \mathbb{Z} \right\}$$

8. Use technology to help you find at least two approximate solutions



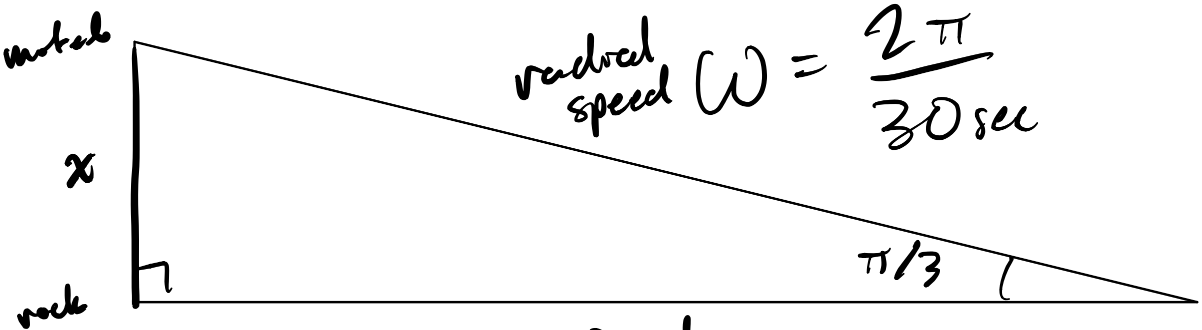
Q| How many sol^ns exist?

IDEA| Points of intersection for the graphs

$y = \tan 2x$ and $y = 20$

More practice with applications:

9. On the shore sits Sea Lion Rock. A lighthouse stands off-shore, 100 yards east of Sea Lion Rock. Due north of Sea Lion Rock is the exclusive See Sea Lion Motel. The lighthouse light rotates twice a minute. If the beam of light from the lighthouse takes 5 seconds to travel along the shore from Sea Lion Rock to the motel, how far is the motel from the rock?



radial speed $\omega = \frac{2\pi}{30\,\text{sec}}$

Hence the angle swept out in 5s

is $5\,\text{sec} \cdot \omega = \frac{2\pi}{6}$

$= \frac{\pi}{3}$.

Now $\tan\frac{\pi}{3} = \frac{x}{100\,\text{yds}}$

so $100\sqrt{3} = x$ (yards!)

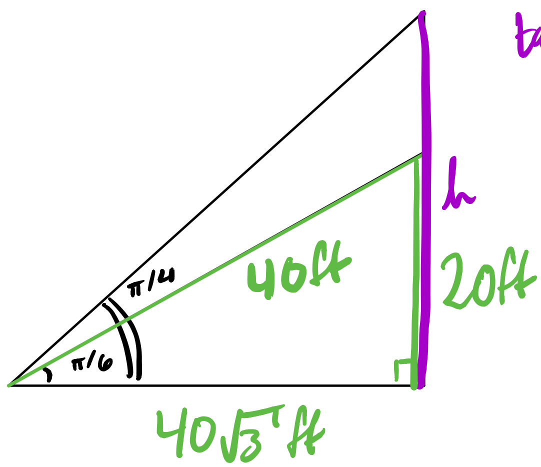
10. While staring out the window, you notice

    a) that it is a warm and sunny day,

    b) that your line of sight to the top of a nearby tree makes an angle of 45 degrees above the horizontal, and

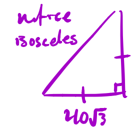
    c) that your line of sight to the base of the tree makes an angle of 30 degrees below the horizontal.

    Taking advantage of a), you go outside and measure that it is 40 feet from the building to the base of the tree. How tall is the tree?



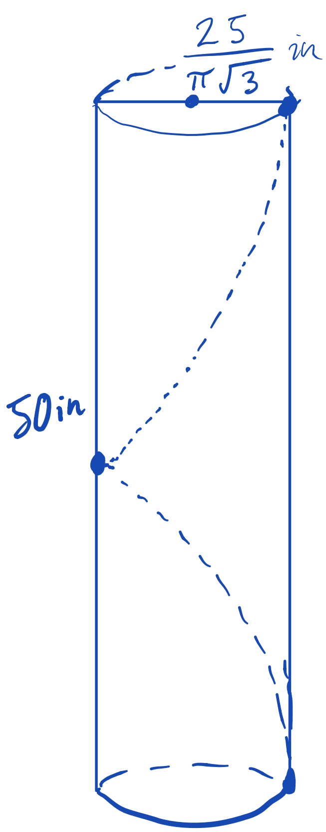
$\tan\frac{\pi}{4} = \frac{h}{40\sqrt{3}\,ft}$, so $h = \tan\frac{\pi}{4}\, 40\sqrt{3}\,ft$
$= 40\sqrt{3}\,ft$

notice isosceles

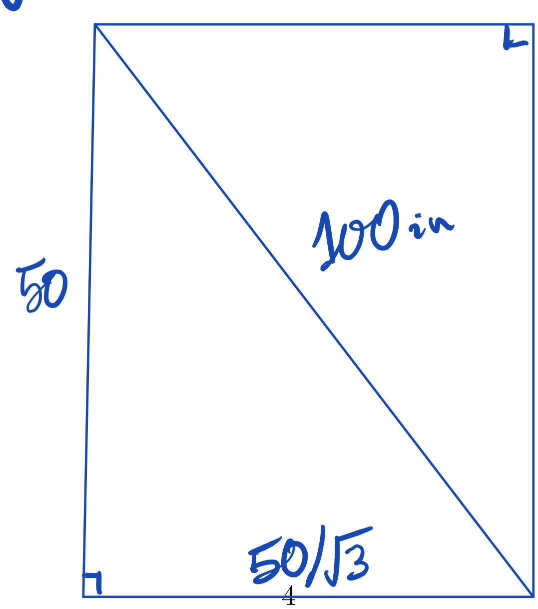


We conclude the height of the tree is $(40\sqrt{3} - 20)$ ft.

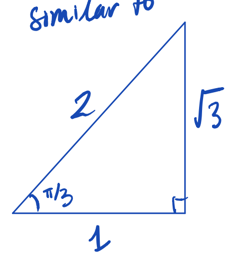
11. The red stripe on a barber pole makes one complete revolution around the pole. If the pole is 50 inches tall and has the amazingly precise radius of $\frac{25}{\pi\sqrt{3}}$ inches, what angle does the stripe make with base of the pole, and how long is the stripe? (Assume the stripe is a thin line.)



By flattening the pole,



Notice similar to



14 Polynomials

Polynomials are one of the simplest and most familiar classes of functions and they find wide use in applied mathematics. A degree-n polynomial

$$p_n(x) = a_0 + a_1 x + \cdots + a_n x^n$$

is defined by its $n+1$ coefficients $a_0, \dots, a_n \in \mathbb{C}$ (with $a_n \neq 0$). Addition of two polynomials is carried out by adding the corresponding coefficients. Thus, if $q_n(x) = b_0 + b_1 x + \cdots + b_n x^n$ then $p_n(x) + q_n(x) = a_0 + b_0 + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n$. Multiplication is carried out by expanding the product term by term and collecting like powers of x:

$$\begin{aligned} p_n(x) q_n(x) = a_0 b_0 &+ (a_0 b_1 + a_1 b_0)x + \cdots \\ &+ (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0)x^n. \end{aligned}$$

The coefficient of x^n, $\sum_{i=0}^{n} a_i b_{n-i}$, is the *convolution* of the vectors $a = [a_0, a_1, \dots, a_n]^{\mathrm{T}}$ and $b = [b_0, b_1, \dots, b_n]^{\mathrm{T}}$. Polynomial division is also possible. Dividing p_n by q_m with $m \leqslant n$ results in

$$p_n(x) = q_m(x) g(x) + r(x), \tag{4}$$

where the quotient g and remainder r are polynomials and the degree of r is less than that of q_m.

The *fundamental theorem of algebra* says that a degree-n polynomial p_n has a *root* in $\mathbb{C}$; that is, there exists $z_1 \in \mathbb{C}$ such that $p_n(z_1) = 0$. If we take $q_m(x) = x - z_1$ in (4) then we have $p_n(x) = (x - z_1)g(x) + r(x)$, where $\deg r < 1$, so r is a constant. But setting $x = z_1$ we see that $0 = p_n(z_1) = r$, so $p_n(x) = (x - z_1)g(x)$ and g clearly has degree $n - 1$. Repeating this argument inductively on g, we end up with a factorization $p_n(x) = (x - z_1)(x - z_2) \cdots (x - z_n)$, which shows that p_n has n roots in $\mathbb{C}$ (not necessarily distinct). If the coefficients of p_n are real it does not follow that the roots are real, and indeed there may be no real roots at all, as the polynomial $x^2 + 1$ shows; however, nonreal roots must occur in complex conjugate pairs $x_j \pm \mathrm{i} y_j$.

Three basic problems associated with polynomials are as follows.

Evaluation: given the polynomial (specified by its coefficients), find its value at a given point. A standard way of doing this is HORNER'S METHOD [I.4 §6].

Interpolation: given the values of a degree-n polynomial at a set of $n+1$ distinct points, find its coefficients. This can be done by various INTERPOLATION SCHEMES [I.3 §3.1].

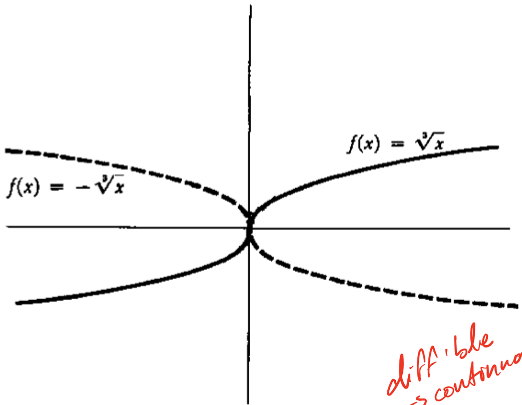


FIGURE 14

The function $f(x) = \sqrt[3]{x}$, although not differentiable at 0, is at least a little better behaved than this. The quotient

$$\frac{f(h)-f(0)}{h} = \frac{\sqrt[3]{h}}{h} = \frac{h^{1/3}}{h} = \frac{1}{h^{2/3}} = \frac{1}{(\sqrt[3]{h})^2}$$

simply becomes arbitrarily large as h goes to 0. Sometimes one says that f has an "infinite" derivative at 0. Geometrically this means that the graph of f has a "tangent line" which is parallel to the vertical axis (Figure 14). Of course, $f(x) = -\sqrt[3]{x}$ has the same geometric property, but one would say that f has a derivative of "negative infinity" at 0.

diff'ble $\Rightarrow$ continuous

Remember that differentiability is supposed to be an improvement over mere continuity. This idea is supported by the many examples of functions which are continuous, but not differentiable; however, one important point remains to be noted:

THEOREM 1 If f is differentiable at a, then f is continuous at a.

continuous $\not\Rightarrow$ diff'ble

PROOF

$$\begin{aligned}\lim_{h\to 0} f(a+h) - f(a) &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \cdot h \\ &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \cdot \lim_{h\to 0} h \\ &= f'(a)\cdot 0 \\ &= 0.\end{aligned}$$

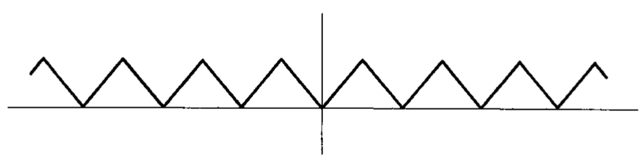
As we pointed out in Chapter 5, the equation $\lim_{h\to 0} f(a+h) - f(a) = 0$ is equivalent to $\lim_{x\to a} f(x) = f(a)$; thus f is continuous at a. ▮

$h : [-\frac{1}{2}, \frac{1}{2}] \longrightarrow \mathbb{R}$ by $h(x) = |x|$.

$h_1 : \mathbb{R} \longrightarrow \mathbb{R}$ is the periodic extension of h, s.th $h_1(x) = h(x)$ and $h_1(x+1) = h_1(x)$.

It is very important to remember Theorem 1, and just as important to remember that the converse is not true. A differentiable function is continuous, but a continuous function need not be differentiable (keep in mind the function $f(x) = |x|$, and you will never forget which statement is true and which false).

The continuous functions examined so far have been differentiable at all points with at most one exception, but it is easy to give examples of continuous functions which are not differentiable at several points, even an infinite number (Figure 15). Actually, one can do much worse than this. There is a function which is *continuous*



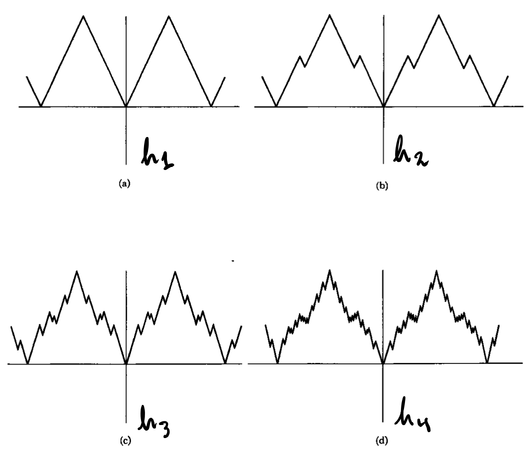


FIGURE 16

everywhere and *differentiable nowhere!* Unfortunately, the definition of this function will be inaccessible to us until Chapter 24, and I have been unable to persuade the artist to draw it (consider carefully what the graph should look like and you will sympathize with her point of view). It is possible to draw some rough approximations to the graph, however; several successively better approximations are shown in Figure 16.

$h_n : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $h_n(x) = \dfrac{h_1(nx)}{2^n}$.

$g(x) := h_1(x) + h_2(x) + h_3(x) + \ldots$

[Spi94]

Composition of mappings

Often one wishes to apply, consecutively, more than one mapping. This is known as *composition.*

It is often easier to understand a composition if one writes it in diagram form; $(f \circ g) : A \to D$ can be written

$$A \xrightarrow[g]{} B \subset C \xrightarrow[f]{} D.$$

Definition 0.4.12 (Composition). If $f : C \to D$ and $g : A \to B$ are two mappings with $B \subset C$, then the *composition* $(f \circ g) : A \to D$ is the mapping given by

$$(f \circ g)(x) = f\big(g(x)\big). \qquad 0.4.5$$

Note that for the composition $f \circ g$ to make sense, the codomain of g must be contained in the domain of f

Example 0.4.13 (Composition of "the father of" and "the mother of"). Consider the following two mappings from the set of persons to the set of persons (alive or dead): F, "the father of," and M, "the mother of." Composing these gives:

$F \circ M$ (the father of the mother of = maternal grandfather of)

$M \circ F$ (the mother of the father of = paternal grandmother of).

It is clear in this case that composition is associative:

$$F \circ (F \circ M) = (F \circ F) \circ M. \qquad 0.4.6$$

The father of David's maternal grandfather is the same person as the paternal grandfather of David's mother. Of course, it is not commutative: the "father of the mother" is not the "mother of the father.") $\triangle$

Example 0.4.14 (Composition of two functions). If $f(x) = x - 1$, and $g(x) = x^2$, then

$$(f \circ g)(x) = f\big(g(x)\big) = x^2 - 1. \quad \triangle \qquad 0.4.7$$

[HH15]

chain rule motivation

example

$$\begin{tikzcd} \mathbb{R} \arrow[r, "S"] \arrow[dr, "f"'] & \mathbb{R} \arrow[d, "\sin"] \\ & \mathbb{R} \end{tikzcd}$$

$i=1$

Differentiating the most interesting functions obviously requires a formula for $(f \circ g)'(x)$ in terms of f' and g'. To ensure that $f \circ g$ be differentiable at a, one reasonable hypothesis would seem to be that g be differentiable at a. Since the behavior of $f \circ g$ near a depends on the behavior of f near $g(a)$ (not near a), it also seems reasonable to assume that f is differentiable at $g(a)$. Indeed we shall prove that if g is differentiable at a and f is differentiable at $g(a)$, then $f \circ g$ is differentiable at a, and

$$(f \circ g)'(a) = f'(g(a)) \cdot g'(a).$$

This extremely important formula is called the *Chain Rule*, presumable because a composition of functions might be called a "chain" of functions. Notice that $(f \circ g)'$ is practically the product of f' and g', but not quite: f' must be evaluated at $g(a)$ and g' at a. Before attempting to prove this theorem we will try a few applications. Suppose

$$f(x) = \sin x^2.$$

Let us, temporarily, use S to denote the ("squaring") function $S(x) = x^2$. Then

$$f = \sin \circ S.$$

Therefore we have

$$\begin{aligned} f'(x) &= \sin'(S(x)) \cdot S'(x) \\ &= \cos x^2 \cdot 2x. \end{aligned}$$

Quite a different result is obtained if

$$f(x) = \sin^2 x.$$

In this case

$$f = S \circ \sin,$$

so

$$\begin{aligned} f'(x) &= S'(\sin x) \cdot \sin'(x) \\ &= 2 \sin x \cdot \cos x. \end{aligned}$$

Notice that this agrees (as it should) with the result obtained by writing $f = \sin \cdot \sin$ and using the product formula.

Although we have invented a special symbol, S, to name the "squaring" function, it does not take much practice to do problems like this without bothering to write down special symbols for functions, and without even bothering to write down the particular composition which f is—one soon becomes accustomed to taking f apart in one's head. The following differentiations may be used as practice for such mental gymnastics—if you find it necessary to work a few out on paper, by all means do so, but try to develop the knack of writing f' immediately after seeing

A function like

$$f(x) = \sin^2 x^2 = [\sin x^2]^2,$$

which is the composition of three functions,

$$f = S \circ \sin \circ S,$$

can also be differentiated by the Chain Rule. It is only necessary to remember that a triple composition $f \circ g \circ h$ means $(f \circ g) \circ h$ or $f \circ (g \circ h)$. Thus if

$$f(x) = \sin^2 x^2$$

we can write

$$\begin{aligned} f &= (S \circ \sin) \circ S, \\ f &= S \circ (\sin \circ S). \end{aligned}$$

The derivative of either expression can be found by applying the Chain Rule twice; the only doubtful point is whether the two expressions lead to equally simple calculations. As a matter of fact, as any experienced differentiator knows, it is much better to use the second:

$$\boxed{f = S \circ (\sin \circ S).}$$

We can now write down $f'(x)$ in one fell swoop. To begin with, note that the first function to be differentiated is S, so the formula for $f'(x)$ begins

$$f'(x) = 2(\qquad) \cdot \underline{\qquad\qquad}.$$

Inside the parentheses we must put $\sin x^2$, the value at x of the second function, $\sin \circ S$. Thus we begin by writing

$$f'(x) = 2 \sin x^2 \cdot \underline{\qquad\qquad}$$

(the parentheses weren't really necessary, after all). We must now multiply this much of the answer by the derivative of $\sin \circ S$ at x; this part is easy—it involves a composition of two functions, which we already know how to handle. We obtain, for the final answer,

$$f'(x) = 2 \sin x^2 \cdot \cos x^2 \cdot 2x.$$

[Spi94]

Background content: Prior to doing this project, students should have a working knowledge of the following:

1. The derivatives of polynomials, exponential functions, and the trig functions.
2. The sum, difference, product, and quotient rules for derivatives and the chain rule.

Philosophy behind this project: This goal of this project is to further solidify the students' understanding of the chain rule. It begins by asking the students to compute the derivative of a polynomial using three different methods. It then increases the complexity of the functions the students have to differentiate to include compositions involving trig and exponential functions. The forth problem requires to use the chain rule abstractly and compute the derivative at a point using a table of values and a graph. Finally, the last problems involve using the chain rule for applications and interpreting their answer.

Learning goals:

1. Compute derivatives using the chain rule.
2. Use tables and graphs to compute the derivatives at a point.
3. Interpret the meaning of the derivative in the context of a problem.

Implementation notes: Most student will have time to work through this entire project. It is important for the class to have enough time to attempt part (a) of problem 6. If time is limited, skip problem 5.

PROBLEM 1: This problem has the students compute the derivative of a function three ways, using the solely the power rule, the power and product rules, and the power and chain rules. As part of the discussion for answering part (d), ask the students what the advantages and disadvantages each method has for this problem and for similar problems.

BREAK: Discuss parts (b) and (d) of Problem 1. For a composition of functions $f(g(x)$, use the notion of the “outside” function and the “inside” function to reference f and g respectively.

PROBLEM 2: Students often get confused about how to use the given identities to rewrite the function in part (a) and to answer part (c). Using words like “replace” or '“substitute” instead of “use the identity” can clarify what the problem is asking for.

PROBLEM 3: Reference the functions involved as the “outside”, “middle”, and “inside” functions. This is a good problem to go over together on the board. Indicate which piece of the derivative can from which function.

PROBLEM 4: Note these types of problems, ones that involve using the chain rule abstractly and getting the required values from a table or a graph, are common on exams.

PROBLEM 5: Give the students a general formula for the exponential model $P = P_0e^{rt}$ (can use $P = P_0(1 + r)^t$, but the other tests the chain rule better).

PROBLEM 6: The goal of this problem is to have the students interpret the derivative in the context of a problem. It is insufficient for students to state the meaning of say $f'(x)$ is “the derivative of $f(x)$” or “the slope of the tangent line of $f(x)$”. They must express what the derivative tells us about the named variables.

Wrap-up: Leave 3-5 minutes for a wrap-up. Discuss the students' meanings of the derivatives in Problem 6. Be sure to go over the expectations for these types of questions.

1. Let $f(x) = (3x^2 + 1)^2$. We are going to find the derivative of $f(x)$ in three ways and then compare the answers.

   (a) Algebraically multiply out the expression for $f(x)$ and then take the derivative.

   Solution:
   $$f(x) = (3x^2 + 1)^2 = 9x^4 + 6x^2 + 1 \text{ so } f'(x) = 36x^3 + 12x$$

   $f(x) = 9x^4 + 6x^2 + 1$ $\qquad \frac{df}{dx} = 36x^3 + 12x$

   (b) View $f(x)$ as a product of two functions, $f(x) = (3x^2 + 1)(3x^2 + 1)$ and use the product rule to find $f'(x)$.

   $\frac{df}{dx} = \frac{du}{dx} \cdot v + u\frac{dv}{dx} = 2(6x)(3x^2+1) = 36x^3 + 12x$

   Solution: Let $u = 3x^2 + 1$ and $v = 3x^2 = 1$, then $(uv)' = u'v + v'u = (6x)(3x^2 + 1) + (6x)(3x^2 + 1) = 36x^3 + 12x$

   (c) Apply the chain rule directly to the expression $f(x) = (3x^2 + 1)^2$

   $\frac{df}{dx} = 2(3x^2+1)(6x) = 36x^3 + 12x$. Why?

   Solution: $f'(x) = 2(3x^2 + 1)(6x) = 36x^3 + 12x$

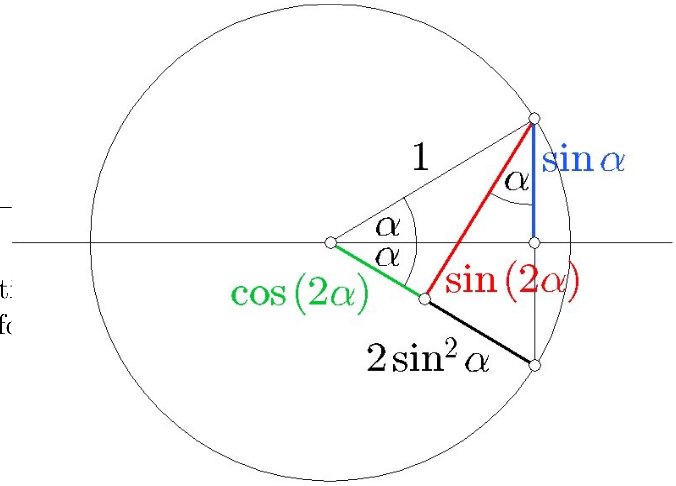
   Let $h(x) = 3x^2 + 1$ and $g(y) = y^2$. Then $f(x) = g(h(x))$, so $\frac{df}{dx} = \frac{dg}{dy} \cdot \frac{dh}{dx}$

   (d) Are your answers in parts a, b, and c the same? Why or why not?

   Solution: All the answers are the same because it doesn't matter which method you use to take a derivative. If done correctly, they should all give the same answer.

What are the advantages / disadvantages in applying the chain rule?

Inside, outside, declare $(f \circ g)(x) = f(g(x))$.



2. Let $f(x) = \sin(2x)$. We are going to find the derivat answers. You will need the double angle formulas f

- $\sin 2x = 2 \sin x \cos x$
- $\cos 2x = \cos^2 x - \sin^2 x$

(a) Rewrite $\sin(2x)$ using the double-angle formula, then apply the product rule to find $f'(x)$.

$\sin(2x) = 2 \sin x \cos x$ Hence $2(\cos^2 x - \sin^2 x) = 2\cos 2x$

Solution: $f(x) = \sin 2x = 2\sin(x)\cos(x)$, by the sine double angle-formula. Let $u = 2\sin(x)$ and $v = \cos(x)$, then $f'(x) = 2\cos(x)\cos(x) - 2\sin(x)\sin(x) = 2(\cos^2(x) - \sin^2(x))$

(b) Apply the chain rule directly to the expression $f(x) = \sin(2x)$ to find its derivative a second way.

$\cos(2x) \cdot 2$

Solution: $f'(x) = \cos(2x) \cdot 2 = 2\cos(2x)$

(c) Are your answers in parts a and b the same? Why or why not?

Solution: Part (a) gives $= f'(x) = 2(\cos^2(x) - \sin^2(x))$, by the cosine double-angle formula, $f'(x) = 2\cos(2x)$. The two answers are the same because it doesn't matter which method you use to take a derivative. If done correctly, they should all give the same answer.

$g : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$

$x \longmapsto \sqrt{x}$

$y \longmapsto f(y)$

$z \longmapsto z^3$

3. Suppose f is differentiable and that $g(x) = (f(\sqrt{x}))^3$.

(a) Calculate $g'(x)$ (your answer will include f and f').

Solution: Applying the chain rule twice we get $g'(x) = 3(f(\sqrt{x}))^2 f'(\sqrt{x}) \frac{1}{2\sqrt{x}}$.

(b) If $f(2) = 1$ and $f'(2) = -2$, calculate $g'(4)$.

Solution: $g'(4) = 3(f(\sqrt{4}))^2 f'(\sqrt{4}) \frac{1}{2\sqrt{4}} = 3(f(2))^2 f'(2)\frac{1}{4} = 3 \cdot 1^2 \cdot -2 \cdot \frac{1}{4} = -\frac{3}{2}$.

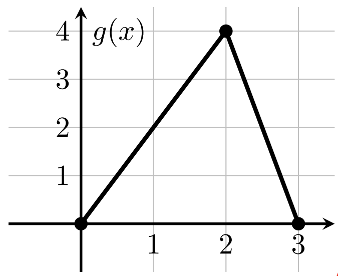
what's the domain of f?

4. Let $f(x)$ and $g(x)$ be two functions. Values of $f(x)$ and $f'(x)$ are given in the table below and the graph of $g(x)$ is as shown.

x	1	2	3
$f(x)$	3	2	1
$f'(x)$	4	5	6

g 2 4 0

g' 2 NaN -4 when



(a) Let $h(x) = g(f(x))$. Find $h'(3)$.

$h'(x) = g'(f(x))\, f'(x)$

Solution: $h'(3) = g'(f(3)) \cdot f'(3) = g'(1) \cdot f'(3) = 2 \cdot 6 = 12$

(b) Let $k(x) = f(g(x))$. Find $k'(1)$.

$k'(x) = f'(g(x))\, g'(x)$

Solution: $k'(1) = f'(g(1)) \cdot g'(1) = f'(2) \cdot g'(1) = 5 \cdot 2 = 10.$

5. The US population on July 1 of 2010 was 309.33 million. The population was 311.59 million on July 1 of 2011.

(a) Find an exponential model $p(t)$ to fit this data. Let $t = 0$ on July 1, 2010.

Solution: We're looking for a function of the form $p(t) = Ae^{kt}$, in millions of people. Substituting $p(0) = 309.33$, we see that $A = 309.33$. Substituting $p(1) = 311.59$ gives $311.59 = 309.33 \cdot e^k$. Solving gives $e^k = 311.59/309.33$, so $k = \ln(311.59/309.33) \approx .00728$. This is an annual growth rate of .728%. Our model is $p(t) = 309.33 \cdot e^{.00728t}$.

(b) Use your model to estimate the US population on November 1 of 2013.

Solution: Substituting $t = 3.33$ into $p(t) = 309.33e^{.00728t}$ gives $p(3) \approx 316.92$ million people. The actual value was approximately 316.98 million.

(c) Find $p'(3)$. Interpret the meaning of this number, including units.

Solution: First take the derivative: $p'(t) = .00728 \cdot 309.33e^{.00728t}$. Substituting $t = 3$ gives $p'(3) = .00728 \cdot 309.33e^{.00728 \cdot 3} \approx 2.3$. This means that on July 1 in the year 2013 the rate of change of the US population was approximately 2.3 million people per year.

Observe $P(0) = 309.33$ mil. Call this P_0.
Then let $P(t) = P_0 e^{rt}$. Then $P(1) = P_0 e^{r}$, so $e^r = \frac{P(1)}{P(0)}$.
Thence $r \neq \log_{10}\left(\frac{P(1)}{P(0)}\right) = 0.00316$!

$309.33\left(\frac{311.59}{309.33}\right)^t \stackrel{?}{=} 311.59^t$

$\stackrel{?}{=} 309.33\, e^{\ln(311.59/309.33)t}$

"dimensional analysis"

6. Chains, Inc. is in the business of making and selling chains. Let $c(t)$ be the number of miles of chain produced after t hours of production. Let $p(c)$ be the profit as a function of the number of miles of chain produced, and let $q(t)$ be the profit as a function of the number of hours of production.

   (a) Suppose the company can produce three miles of chain per hour, and suppose their profit on the chains is \$4000 per mile of chain. Find each of the following (include units).

   $c(t) = 3t$ miles

   $c'(t) = 3$ miles/hour

$$\begin{tikzcd} \text{time (hours)} \arrow[r, "c"] \arrow[dr, "q"'] & \text{chain length (miles)} \arrow[d, "p"] \\ & \text{profit (USD)} \end{tikzcd}$$

   Meaning of $c'(t)$: 3 feet of chain are produced per hour.

   $p(c) = 4000c$ dollars (USD)

   $p'(c) = 4000$ dollars/mile

   Meaning of $p'(c)$: 4000 dollars of profit are earned per mile of chain produced.

   $q(t) = 12000t$ dollars

   $q'(t) = 12000$ dollars/hour

   Meaning of $q'(t)$: 12000 dollars of profit are earned per hour of production.

   How does $q'(t)$ relate to $p'(c)$ and $c'(t)$?
   Solution: By the chain rule $q'(t) = p'(c(t))c'(t)$. So $q'(t) = 4000$ dollars/mile $\cdot 3$ miles/hour $= 12000$ dollars/hours.

   (b) In this part, the production and profit functions are no longer linear. Instead $p(c)$ is modeled by the formula $p(c) = 100 - 100\cos\left(\frac{\pi}{38}c\right)$ (where p is measured in thousands of dollars and c is measured in miles of chain), and $c(t)$ is defined numerically below:

t (in hours)	2	4	6	8	10
c (in miles)	6	14	24	38	52

   Estimate $q'(4)$ and $q'(8)$. What conclusions should you draw about production?

   Solution: First note that $p'(c) = \frac{100\pi}{38}\sin\left(\frac{\pi}{38}c\right)$.
   Using the chain rule, $q'(4) = p'(c(4))c'(4)$. Estimating numerically, $c'(4) \approx \frac{24-6}{6-2} = \frac{18}{4}$. So $q'(4) \approx \frac{100\pi}{38}\sin\left(\frac{\pi}{38}\cdot 14\right)\cdot\frac{18}{4} \approx 34.07$ thousand dollars/hour. This means that after 4 hours of production, the profit increases a rate of about \$34,000 dollars per added hour of production. We should keep the factory running. Similarly, we find that $q'(8) = p'(c(8))c'(8) \approx \frac{100\pi}{38}\sin\left(\frac{\pi}{38}\cdot 38\right)\cdot\frac{28}{4} \approx 0$. The profit is no longer increasing as we increase the number of hours of production. We should determine if this is a maximum, and possibly shut down the factory after 8 hours.

Polya, 1945
How to Solve It

6. Four phases. Trying to find the solution, we may repeatedly change our point of view, our way of looking at the problem. We have to shift our position again and again. Our conception of the problem is likely to be rather incomplete when we start the work; our outlook is different when we have made some progress; it is again different when we have almost obtained the solution.

In order to group conveniently the questions and suggestions of our list, we shall distinguish four phases of the work. First, we have to *understand* the problem; we have to see clearly what is required. Second, we have to see how the various items are connected, how the unknown is linked to the data, in order to obtain the idea of the solution, to make a *plan*. Third, we *carry out* our plan. Fourth, we *look back* at the completed solution, we review and discuss it.

Each of these phases has its importance. It may happen that a student hits upon an exceptionally bright idea and jumping all preparations blurts out with the solution. Such lucky ideas, of course, are most desirable, but something very undesirable and unfortunate may result if the student leaves out any of the four phases without having a good idea. The worst may happen if the student embarks upon computations or constructions without having *understood* the problem. It is generally useless to carry out details without having seen the main connection, or having made a sort of *plan*. Many mistakes can be avoided if, carrying out his plan, the student *checks each step*. Some of the best effects may be lost if the student fails to reexamine and to *reconsider* the completed solution.

7. Understanding the problem. It is foolish to answer a question that you do not understand. It is sad to work for an end that you do not desire. Such foolish and sad things often happen, in and out of school, but the teacher should try to prevent them from happening in his class. The student should understand the problem. But he should not only understand it, he should also desire its solution. If the student is lacking in understanding or in interest, it is not always his fault; the problem should be well chosen, not too difficult and not too easy, natural and interesting, and some time should be allowed for natural and interesting presentation.

First of all, the verbal statement of the problem must be understood. The teacher can check this, up to a certain extent; he asks the student to repeat the statement, and the student should be able to state the problem fluently. The student should also be able to point out the principal parts of the problem, the unknown, the data, the condition. Hence, the teacher can seldom afford to miss the questions: *What is the unknown? What are the data? What is the condition?*

The student should consider the principal parts of the problem attentively, repeatedly, and from various sides. If there is a figure connected with the problem he should *draw a figure* and point out on it the unknown and the data. If it is necessary to give names to these objects he should *introduce suitable notation;* devoting some attention to the appropriate choice of signs, he is obliged to consider the objects for which the signs have to be chosen. There is another question which may be useful in this preparatory stage provided that we do not expect a definitive answer but just a provisional answer, a guess: *Is it possible to satisfy the condition?*

conceive and exploit a good idea.

20. A rate problem. *Water is flowing into a conical vessel at the rate* r. *The vessel has the shape of a right circular cone, with horizontal base, the vertex pointing downwards; the radius of the base is* a, *the altitude of the*

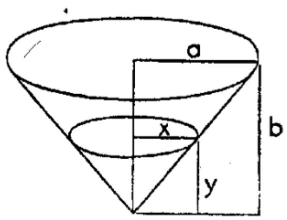


FIG. 6

cone b. *Find the rate at which the surface is rising when the depth of the water is* y. *Finally, obtain the numerical value of the unknown supposing that* a = 4 *ft.*, b = 3 *ft.*, r = 2 *cu. ft. per minute, and* y = 1 *ft.*

The students are supposed to know the simplest rules of differentiation and the notion of "rate of change."

"*What are the data?*"

"The radius of the base of the cone $a = 4$ ft., the altitude of the cone $b = 3$ ft., the rate at which the water is flowing into the vessel $r = 2$ cu. ft. per minute, and the depth of the water at a certain moment, $y = 1$ ft."

"Correct. The statement of the problem seems to suggest that you should disregard, provisionally, the numerical values, work with the letters, express the unknown in terms of a, b, r, y and only finally, after having obtained the expression of the unknown in letters, substitute the numerical values. I would follow this suggestion. Now, *what is the unknown?*"

"The rate at which the surface is rising when the depth of the water is y."

"What is that? Could you say it in other terms?"

"The rate at which the depth of the water is increasing."

"What is that? *Could you restate it still differently?*"

"The rate of change of the depth of the water."

"That is right, the rate of change of y. But what is the rate of change? *Go back to the definition.*"

"The derivative is the rate of change of a function."

"Correct. Now, is y a function? As we said before, we disregard the numerical value of y. Can you imagine that y changes?"

"Yes, y, the depth of the water, increases as the time goes by."

"Thus, y is a function of what?"

"Of the time t."

"Good. *Introduce suitable notation.* How would you write the 'rate of change of y' in mathematical symbols?"

"$\frac{dy}{dt}$"

"Good. Thus, this is your unknown. You have to express it in terms of a, b, r, y. By the way, one of these data is a 'rate.' Which one?"

"r is the rate at which water is flowing into the vessel."

"What is that? Could you say it in other terms?"

"r is the rate of change of the volume of the water in the vessel."

"What is that? *Could you restate it still differently?* How would you write it in *suitable notation?*"

"$r = \frac{dV}{dt}$."

"What is V?"

"The volume of the water in the vessel at the time t."

"Good. Thus, you have to express $\frac{dy}{dt}$ in terms of a, b, $\frac{dV}{dt}$, y. How will you do it?"

. . . . .

"*If you cannot solve the proposed problem try to solve first some related problem.* If you do not see yet the connection between $\frac{dy}{dt}$ and the data, try to bring in some simpler connection that could serve as a stepping stone."

. . . . .

"Do you not see that there are other connections? For instance, are y and V independent of each other?"

"No. When y increases, V must increase too."

"Thus, there is a connection. What is the connection?"

"Well, V is the volume of a cone of which the altitude is y. But I do not know yet the radius of the base."

"You may consider it, nevertheless. Call it something, say x."

"$V = \frac{\pi x^2 y}{3}$."

"Correct. Now, what about x? Is it independent of y?"

"No. When the depth of the water, y, increases the radius of the free surface, x, increases too."

"Thus, there is a connection. What is the connection?"

"Of course, similar triangles.

$$x : y = a : b."$$

"One more connection, you see. I would not miss profiting from it. Do not forget, you wished to know the connection between V and y."

"I have

$$x = \frac{ay}{b}$$

$$V = \frac{\pi a^2 y^3}{3b^2}."$$

"Very good. This looks like a stepping stone, does it not? But you should not forget your goal. *What is the unknown?*"

"Well, $\frac{dy}{dt}$."

"You have to find a connection between $\frac{dy}{dt}$, $\frac{dV}{dt}$, and other quantities. And here you have one between y, V, and other quantities. What to do?"

"Differentiate! Of course!

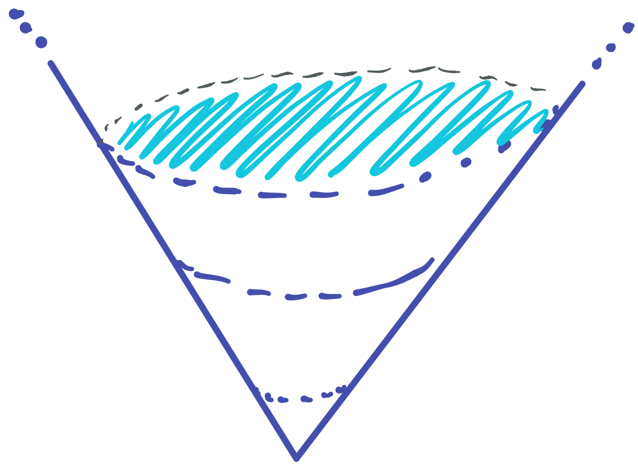
$$\frac{dV}{dt} = \frac{\pi a^2 y^2}{b^2} \frac{dy}{dt}.$$

Here it is."

"Fine! And what about the numerical values?"

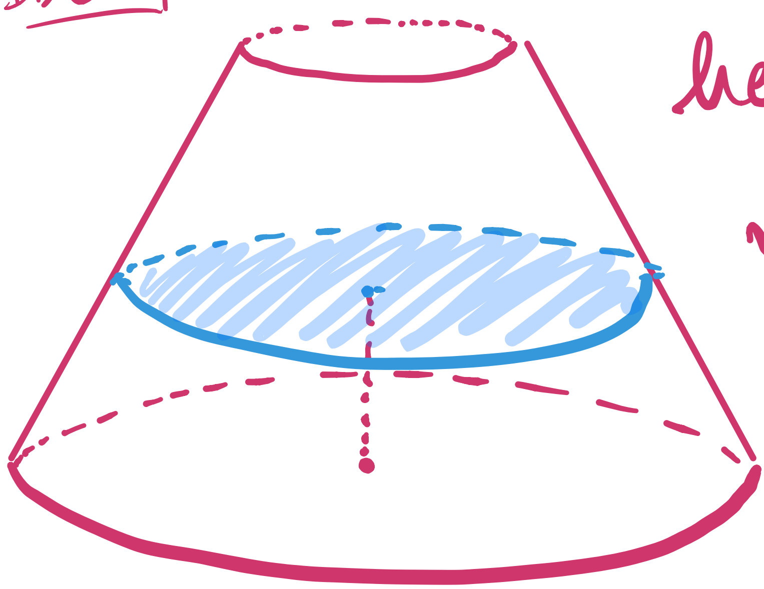
"If $a = 4$, $b = 3$, $\frac{dV}{dt} = r = 2$, $y = 1$, then

$$2 = \frac{\pi \times 16 \times 1}{9} \frac{dy}{dt}."$$



How is dh/dt related to time for an infinite cone?

IDEA 1

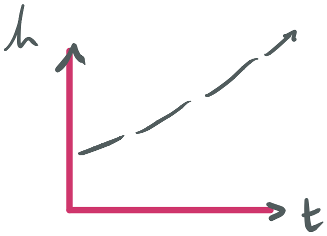


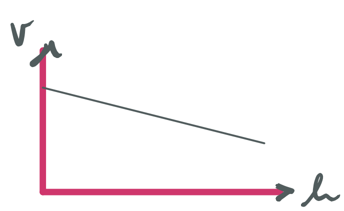
radius $r(h)$

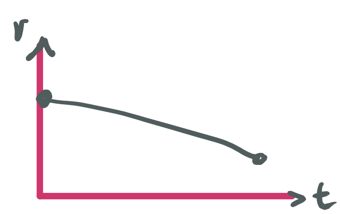
height $h(t)$

what's $\frac{dr}{dt}$?

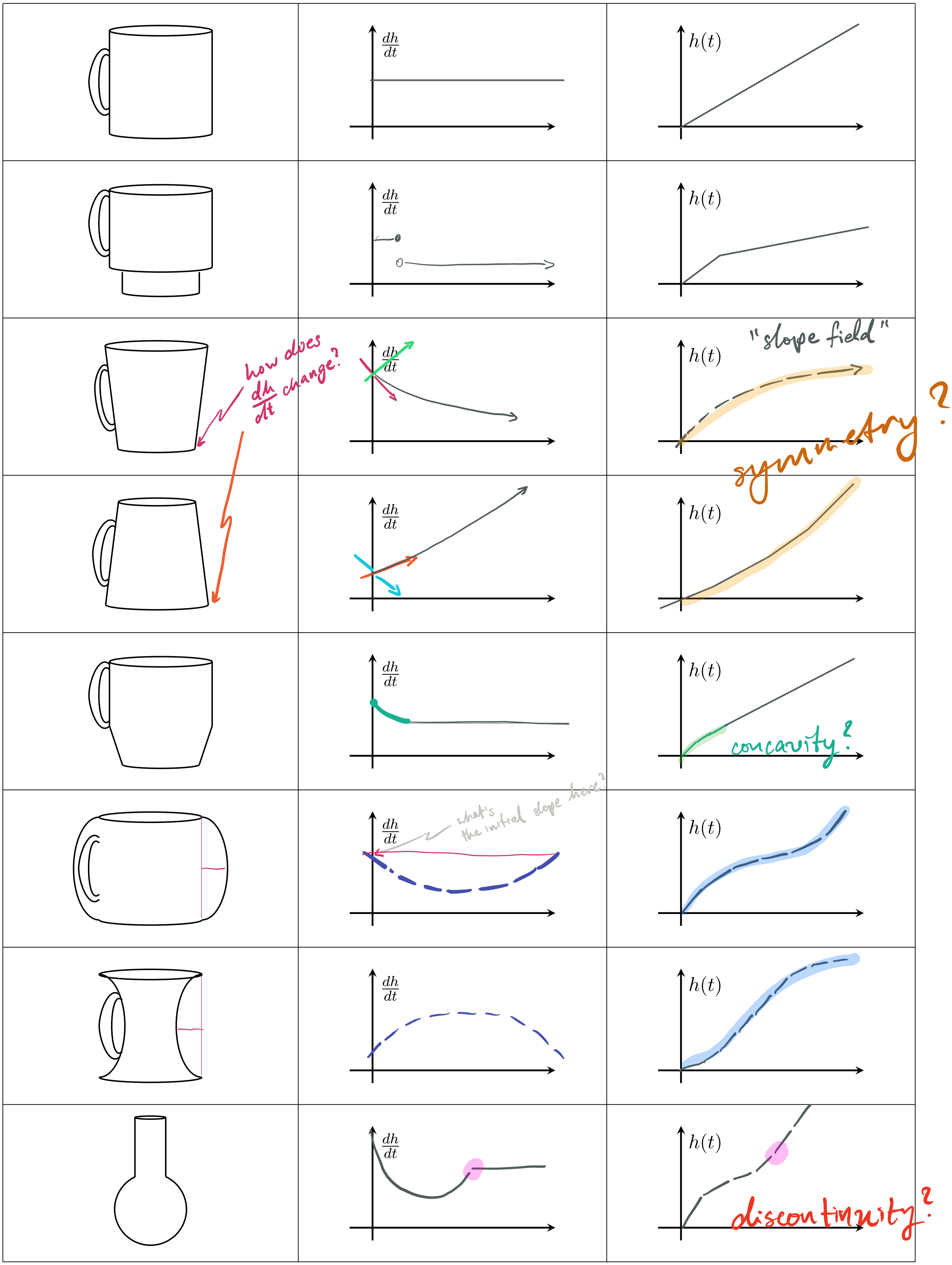
$$\frac{dr}{dt} = \frac{d}{dt}[r(h(t))]$$







Coffee is being poured at a constant rate into coffee cups. Give a rough graph of the rate of change of the height of the coffee in the cup as a function of time and a rough graph of the height of the coffee in the cup as a function of time.



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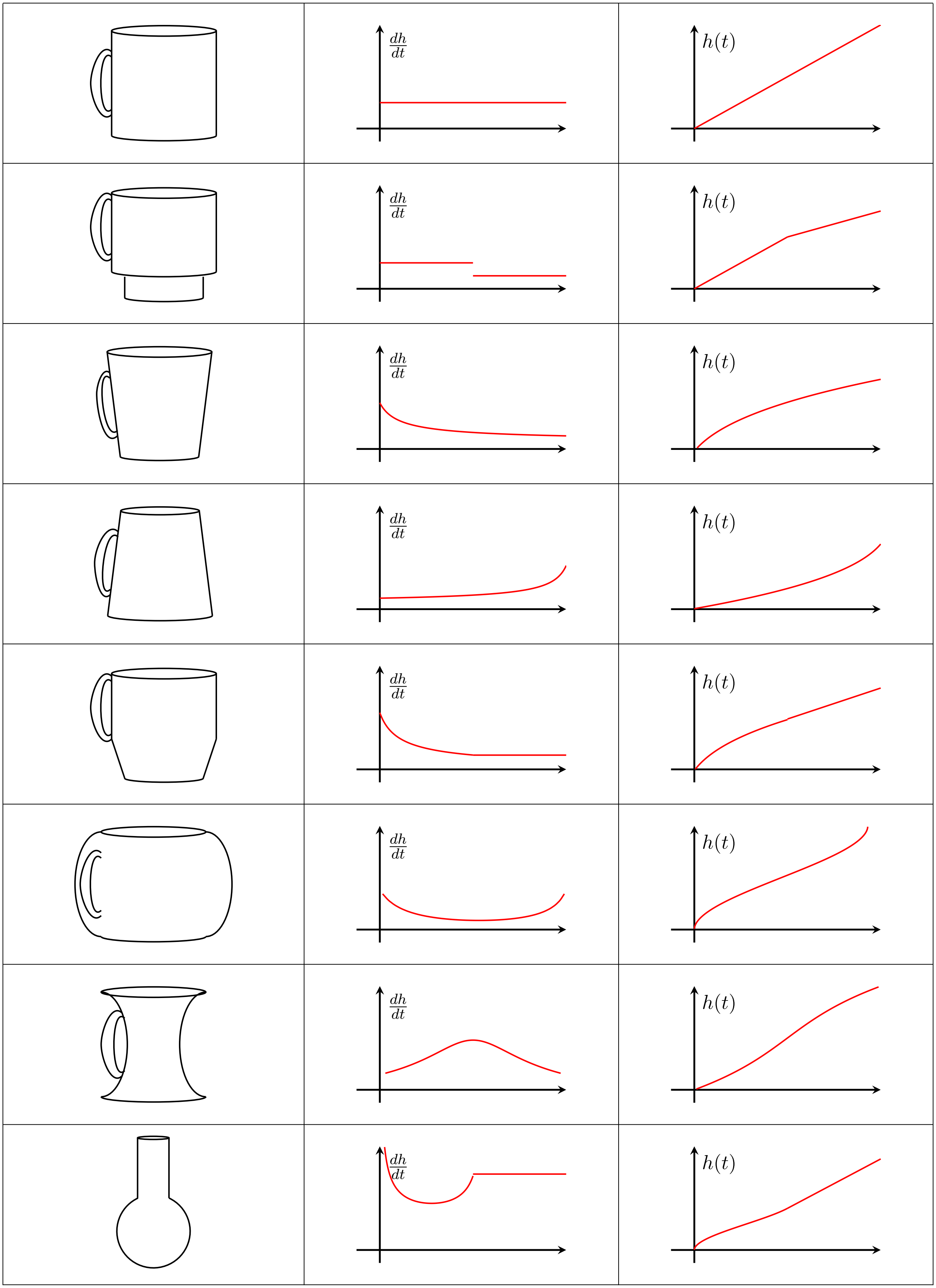
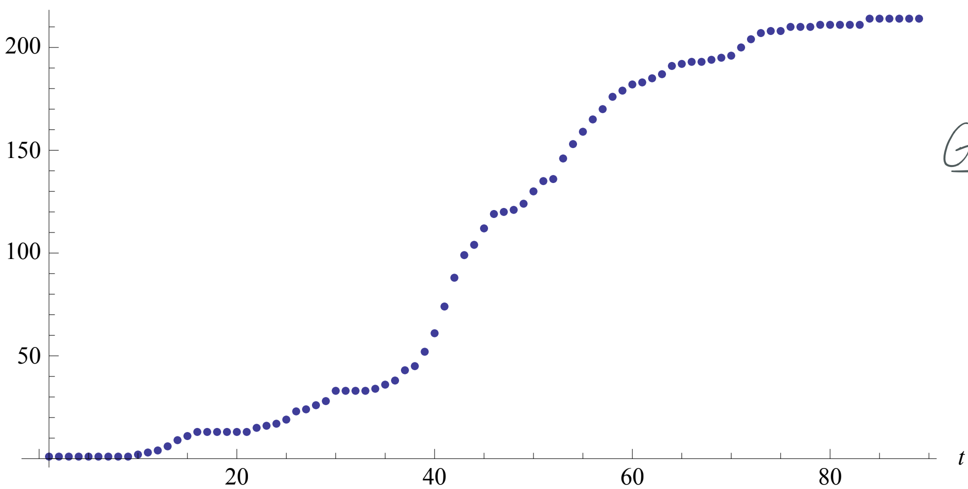


Modeling with an inverse trig function – an Ebola outbreak in 1995

Goal: To study a model for deaths due to an Ebola outbreak and to use this model to discover a connection between the values of a function and the area under the function's derivative.

In 1995 there was a 90-day-long outbreak of Ebola in the Democratic Republic of Congo (DRC). The points below are a plot of a function $N(t)$ which represents the total number of deaths from the beginning of the outbreak to the end of day t.



Given. t : time (days) since outbreak

$N(t)$: discrete # of deaths since

1. (a) Translate the equation $N(22) = 15$ into an explanatory English sentence.

[0, 22) deaths was cumulative 15

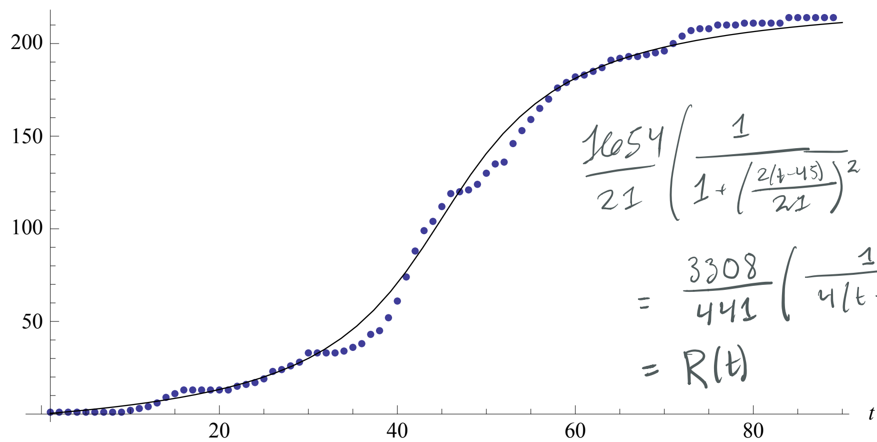
(b) The data shows that $13 = N(16) = N(17) = N(18) = N(19) = N(20) = N(21)$. What can you conclude from this data? (Give your answer in a complete English sentence.)

[16, 21) deaths was cumulative 0.

An important way to analyze data is to find a function that models the data–which means that the graph of the function closely fits the data points. The function

$$D(t) = \frac{1654}{21}\left(\arctan\left(\frac{2(t-45)}{21}\right) + \arctan\left(\frac{30}{7}\right)\right)$$

is a good model for the 1995 Ebola data as the below graph shows:



$$\frac{1654}{21}\left(\frac{1}{1+\left(\frac{2(t-45)}{21}\right)^2}\cdot\left(\frac{2}{21}\right)\right)$$

$$= \frac{3308}{441}\left(\frac{1}{4(t-45)^2+1}\right)$$

$$= R(t)$$

2. How "well" does the *mathematical model* $D(t)$ for the number of deaths represent the *actual* cumulative death count $N(t)$? When does the model least accurately reflect the data? When do we see the largest discrepancy between the rate of change of the model and the rate of change of the actual data?

3. Assuming, for the moment, that the actual cumulative death count is given by the above function $D(t)$, show that the instantaneous death rate $R(t)$, in deaths per day, is given by the formula
$$R(t) = \frac{3308}{441 + 4(t-45)^2}.$$

2) • Logistic growth?

• Least squares?

- linear (point-wise)

• other models?

- how to compare

On May 6, 1995, CDC was notified by health authorities and the U.S. Embassy in Zaire of an outbreak of viral hemorrhagic fever (VHF)-like illness in Kikwit, Zaire (1995 population: 400,000), a city located 240 miles east of Kinshasa. The World Health Organization and CDC were invited by the Government of Zaire to participate in an investigation of the outbreak. This report summarizes preliminary findings from this ongoing investigation.

On April 4, a hospital laboratory technician in Kikwit had onset of fever and bloody diarrhea. On April 10 and 11, he underwent surgery for a suspected perforated bowel. Beginning April 14, medical personnel employed in the hospital to which he had been admitted in Kikwit developed similar symptoms. One of the ill persons was transferred to a hospital in Mosango (75 miles west of Kikwit). On approximately April 20, persons in Mosango who had provided care for this patient had onset of similar symptoms.

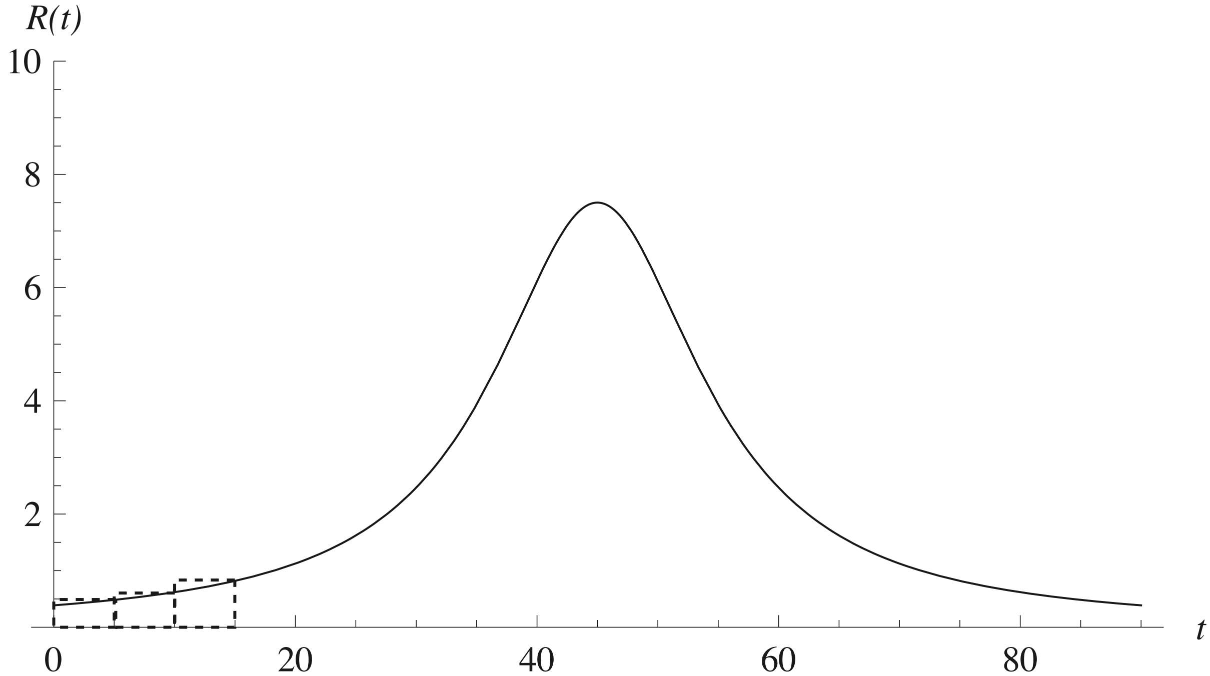
On May 9, blood samples from 14 acutely ill persons arrived at CDC and were processed in the biosafety level 4 laboratory; analyses included testing for Ebola antigen and Ebola antibody by enzyme-linked immunosorbent assay, and reverse transcription-polymerase chain reaction (RT-PCR) for viral RNA. Samples from all 14 persons were positive by at least one of these tests; 11 were positive for Ebola antigen, two were positive for antibodies, and 12 were positive by RT-PCR. Further sequencing of the virus glycoprotein gene revealed that the virus is closely related to the Ebola virus isolated during an outbreak of VHF in Zaire in 1976 (1).

As of May 17, the investigation has identified 93 suspected cases of VHF in Zaire, of which 86 (92%) have been fatal. Public health investigators are now actively seeking cases and contacts in Kikwit and the surrounding area. In addition, active surveillance for possible cases of VHF has been implemented at 13 clinics in Kikwit and 15 remote sites within a 150-mile radius of Kikwit. Educational and quarantine measures have been implemented to prevent further spread of disease. Reported by: M Musong, MD, Minister of Health, Kinshasa, T Muyembe, MD, Univ of Kinshasa; Dr. Kibasa, MD, Kikwit General Hospital, Kikwit, Zaire. World Health Organization, Geneva. Div of Viral and Rickettsial Diseases, and Div of Quarantine, National Center for Infectious Diseases; International Health Program Office, CDC.

Editorial Note

Editorial Note: Ebola virus and Marburg virus are the two known members of the filovirus family. Ebola viruses were first isolated from humans during concurrent outbreaks of VHF in northern Zaire (1) and southern Sudan (2) in 1976. An earlier outbreak of VHF caused by Marburg virus occurred in Marburg, Germany, in 1967 when laboratory workers were exposed to infected tissue from monkeys imported from Uganda (3). Two subtypes of Ebola virus -- Ebola-Sudan and Ebola-Zaire -- previously have been associated with disease in humans (4). In 1994, a single case of infection from a newly described Ebola virus occurred in a person in Cote d'Ivoire. In 1989, an outbreak among monkeys imported into the United States from the Philippines was caused by another Ebola virus (5) but was not associated with human disease.

The function $R(t)$ that you found in problem 3 is sketched on the axes below. (Note the bell shape!!)



Note that we've dashed in a rectangle over each of the first three intervals of length 5, on the above t axis. The height of each rectangle is just the value of the function $R(t)$ at the right endpoint of the interval in question.

4. Continue the process of drawing rectangles over each of the above subintervals of length 5, on the t axis, with the height of each rectangle being the value of $R(t)$ at the rightmost edge of the rectangle.

painful & dis-interested?

5. Let $T(t_0)$ denote the *total area* of the rectangles you've sketched in, between $t = 0$ and $t = t_0$. Compute $T(20)$, $T(40)$, $T(60)$, and $T(80)$. (Recall that each rectangle has base length 5, and height given by the value of $R(t)$ at the right endpoint of the rectangle.) (The total areas you are considering here are called *Riemann sums*; more on these later in class.)

We, for example, want to compute the area of each panel, width 5 days, and height $R(t)$.

6. Compare the above numbers $T(20)$, $T(40)$, $T(60)$, and $T(80)$, to the numbers $D(20)$, $D(40)$, $D(60)$, and $D(80)$ you get by plugging in the appropriate t-values into the formula for $D(t)$ above. Do you see a correspondence between these sequences of numbers? Do you have any idea why this correspondence should be true?

   (This correspondence amounts to a HUGE theorem, called the Fundamental Theorem of Calculus, which we'll discuss in class later.)

   Why do you think the values of $T(t)$ that you calculated in problem 5 start off higher than corresponding values of $D(t)$, and then later $D(t)$ catches up?

Relation to cups? "Numerical integration"

Computation to recognize $T(t)$ and $D(t)$ connected?

Derivative of $\arctan(x)$

Let's use our formula for the derivative of an inverse function to find the derivative of the inverse of the tangent function: $y = \tan^{-1} x = \arctan x$.

We simplify the equation by taking the tangent of both sides:

$$\begin{aligned} y &= \tan^{-1} x \\ \tan y &= \tan(\tan^{-1} x) \\ \tan y &= x \end{aligned}$$

To get an idea what to expect, we start by graphing the tangent function (see Figure 1). The function $\tan(x)$ is defined for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. It's graph extends from negative infinity to positive infinity.

If we reflect the graph of $\tan x$ across the line $y = x$ we get the graph of $y = \arctan x$ (Figure 2). Note that the function $\arctan x$ is defined for all values of x from minus infinity to infinity, and $\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$.

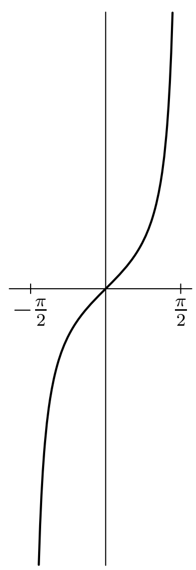


Figure 1: Graph of the tangent function.

You may know that:

$$\begin{aligned} \frac{d}{dy} \tan y &= \frac{d}{dy} \frac{\sin y}{\cos y} \\ &\vdots \\ &= \frac{1}{\cos^2 y} \\ &= \sec^2 y \end{aligned}$$

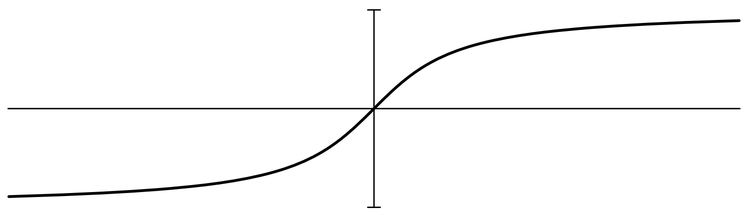


Figure 2: Graph of $\tan^{-1} x$.

(If you haven't seen this before, it's good exercise to use the quotient rule to verify it!)

We can now use implicit differentiation to take the derivative of both sides of our original equation to get:

$$
\begin{array}{rrcl}
 & \tan y & = & x \\
 & \dfrac{d}{dx}(\tan(y)) & = & \dfrac{d}{dx}x \\
\text{(Chain Rule)} & \dfrac{d}{dy}(\tan(y))\dfrac{dy}{dx} & = & 1 \\
 & \left(\dfrac{1}{\cos^2(y)}\right)\dfrac{dy}{dx} & = & 1 \\
 & \dfrac{dy}{dx} & = & \cos^2(y)
\end{array}
$$

Or equivalently, $y' = \cos^2 y$. Unfortunately, we want the derivative as a function of x, not of y. We must now plug in the original formula for y, which was $y = \tan^{-1} x$, to get $y' = \cos^2(\arctan(x))$. This is a correct answer but it can be simplified tremendously. We'll use some geometry to simplify it.

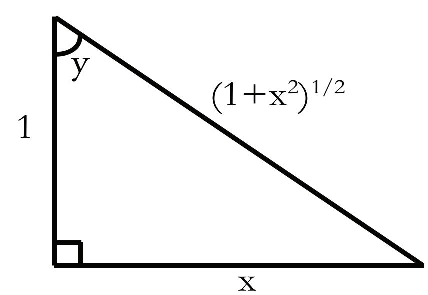


Figure 3: Triangle with angles and lengths corresponding to those in the example.

In this triangle, $\tan(y) = x$ so $y = \arctan(x)$. The Pythagorean theorem tells us the length of the hypotenuse:

$$h = \sqrt{1+x^2}$$

and we can now compute:

$$\cos(y) = \frac{1}{\sqrt{1+x^2}}.$$

From this, we get:

$$\cos^2(y) = \left(\frac{1}{\sqrt{1+x^2}}\right)^2 = \frac{1}{1+x^2}$$

so:

$$\frac{dy}{dx} = \frac{1}{1+x^2}.$$

In other words,

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}.$$

TODO define! Absolute max as y sth $f(x) \leq y$.
Unbounded: pick some number.

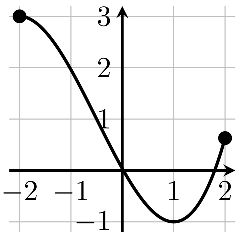
The Extreme Value Theorem

What does it take to be sure a function has an absolute minimum and an absolute maximum on a given domain?

"phat paintbrush" Q| Must have a horizontal asym.

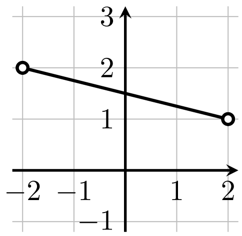
I. Samples – Study these sample functions and their descriptions and fill in the blanks.

a)



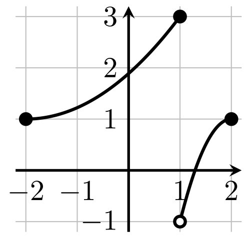
A continuous function with an absolute maximum of 3 at $x =$ -2 and an absolute minimum of -1 at $x =$ 1. Domain: [-2,2]

b)



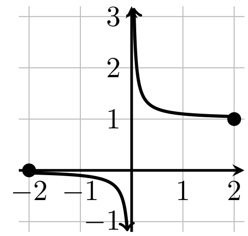
A continuous function with no absolute maximum and no absolute minimum. Domain: (-2,2)

c)



A discontinuous function with an absolute maximum of 3 at $x =$ 1 and no absolute minimum. Domain: [-2,2]

d)



An unbounded discontinuous function with no absolute maximum and no absolute minimum. Domain: $[-2,2] \setminus \{0\}$

$= [-2,0) \cup (0,2]$

In sample c), there is no absolute minimum because:

Suppose there was. Because f is increasing on $(1,2]$, we could find a lesser value for f. ※

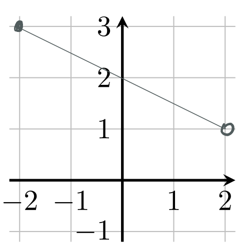
In sample d), there is no absolute maximum because:

There is an asymptote, and so f is unbounded.

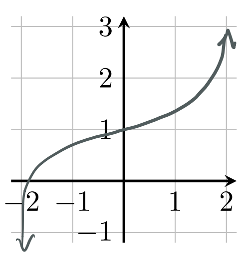
II. Examples – if possible, create graphs of functions satisfying each description

has to blow-up!

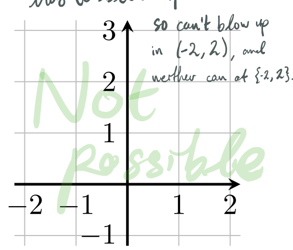
$\frac{3}{1+x^2}$



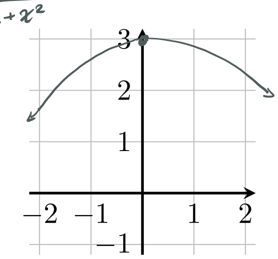
A continuous function with an absolute maximum of 3 and no absolute minimum. Domain: $[-2, 2)$



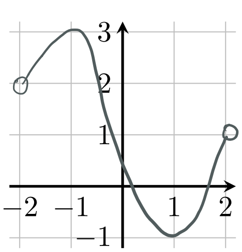
A continuous unbounded function with no absolute maximum and no absolute minimum. Domain: $(-2, 2)$



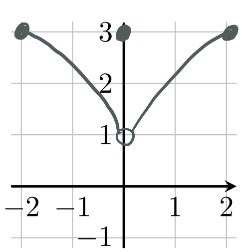
A continuous unbounded function with no absolute maximum and no absolute minimum. Domain: $[-2, 2]$



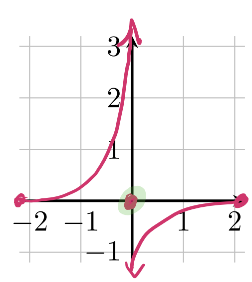
A bounded continuous function with an absolute maximum of 3 and no absolute minimum. Domain: $(-\infty, \infty)$



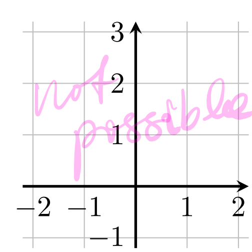
A continuous function with an absolute maximum of 3 and an absolute minimum of -1. Domain: $(-2, 2)$



A function with an absolute maximum of 3 and no absolute minimum. Domain: $[-2, 2]$



A function with no absolute maximum and no absolute minimum. Domain: $[-2, 2]$



A continuous function with no absolute maximum and no absolute minimum. Domain: $[-2, 2]$

"a y-value for every x-value."

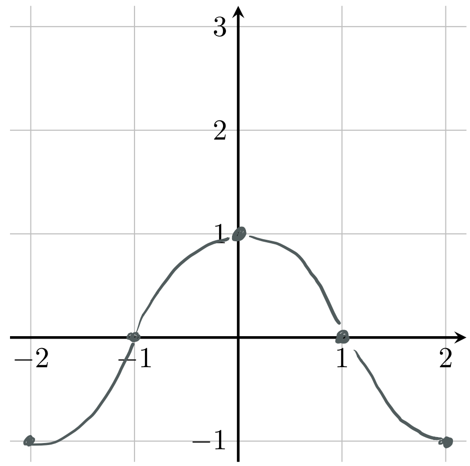
$f(x) := \frac{1}{x}$

$f : (-\infty, 0) \cup (0, \infty)$

III. Theorem: (Extreme Value Theorem) If f is continuous on a closed interval $[a, b]$, then f must attain an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in the interval $[a, b]$.

IV. Draw a continuous function with domain $[-2, 2]$.

Please write out hypotheses, conclusion!



"show me you can do it!"

Does it have an absolute maximum and absolute minimum?

Yes, $\{(-2,-1), (2,-1)\}$ is the set of abs. min coords, and $\{(0,1)\}$ is the set of abs max coords

Check the functions drawn by your classmates. Do all their examples also have absolute maxima and absolute minima? Explain!

explain! communicate!
why are you thinking about your graphs!

Why does sample b) on the top of the previous page not contradict the Extreme Value Theorem?

Why does sample c) on the top of the previous page not contradict the Extreme Value Theorem?

Does the function $f(x) = 5 + 54x - 2x^3$ have an absolute maximum and an absolute minimum on the interval $[0, 4]$? Why or why not? If so, how would you go about finding the absolute maximum and absolute minimum?

Goal: To collect information about the first and second derivatives of a function, then use this information to graph the function without using technology.

1. Consider the function $f(x) = 3x^4 - 8x^3 + 6x^2$. $= (3x^2 - 8x + 6)x^2$ $\frac{+8 \pm \sqrt{64-70}}{6} = \frac{8 \pm \sqrt{-12}}{6} = \frac{4}{3} \pm \frac{\sqrt{-3}}{3}$

   (a) Determine the open intervals on which the function is increasing/decreasing.

   $f'(x) = 12x^3 - 24x^2 + 12x$
   $= 12x(x^2 - 2x + 1)$
   $= 12x(x-1)(x-1)$

   $f'(-1) = -12 \cdot 4 = -48$
   $f'(\frac{1}{2}) = 6 \cdot \frac{1}{4} = \frac{3}{2}$
   $f'(2) = 24$

   $f' \leq 0$ when $x < 0$
   $f' \geq 0$ when $x \geq 0$
   $f' = 0$ at $x = 0, 1$

   (b) Find the local maxima and local minima of $f(x)$, if any. Be sure to find the critical points, classify them using either the first or second derivative test, then substitute the x-values into $f(x)$ to find the local mininum/maximum values.

   $x = 0$ is a local min (also global)

   (c) Find the inflection points of the function, if any. Be sure to find where the second derivative is zero, use a sign chart to determine whether or not the second derivative changes, then substitute the x-values into $f(x)$ to find the y-value at each inflection point.

   $f''(x) = 36x^2 - 48x + 12$
   $= 12(3x^2 - 4x + 1)$
   $= 12 \cdot 3(x-1)(x-\frac{1}{3})$

   $\frac{4 \pm \sqrt{16-12}}{6} = \frac{2}{3} \pm \frac{2}{6} = 1, \frac{1}{3}$

   $f'' > 0$ when $x < \frac{1}{3}$
   $f'' < 0$ when $\frac{1}{3} < x < 1$
   $f'' > 0$ when $x > 1$

(d) Plot the local extrema and the inflection points on the graph. Transfer the information from parts (a) and (b) to the number lines for $f'(x)$ and $f''(x)$. Sketch the graph of the function $f(x) = 3x^4 - 8x^3 + 6x^2$, using all of the information.

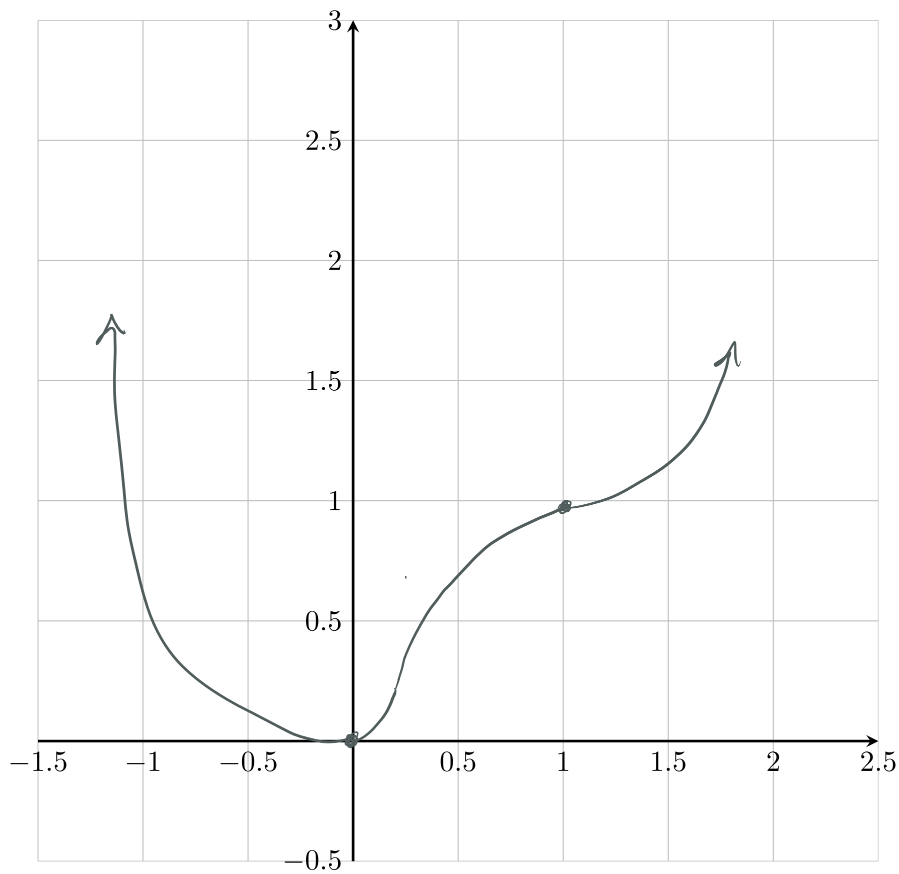
$f'(-1) = -12 \cdot 4 = -48$
$f'(\frac{1}{2}) = 6 \cdot \frac{1}{4} = \frac{3}{2}$
$f'(2) = 24$

$f' \leq 0$ when $x < 0$
$f' \geq 0$ when $x \geq 0$
$f' = 0$ at $x = 0, 1$

$f'' > 0$ when $x < \frac{1}{3}$

$f'' < 0$ when $\frac{1}{3} < x < 1$

$f'' > 0$ when $x > 1$



$f'(x)$: decreasing | increasing | increasing

$f''(x)$: convex | concave | convex

(e) Now use your graphing calculator to get the graph of $y = f(x)$ on this domain, and compare it to the graph you just drew. How well did you do?

2. Using the same process as in the previous problem, graph $f(x) = x^{\frac{1}{3}}(x+4)$ on the next page.

fn	domain	roots	signs

$f = x^{1/3}(x+4)$

$= x^{4/3} + 4x^{1/3}$

$\mathbb{R}$, $-4, 0$

$f > 0$ when $x < -4$

$f < 0$ when $-4 < x < 0$

$f > 0$ when $0 < x$

$f' = \frac{4}{3}x^{1/3} + \frac{4}{3}x^{-2/3}$

$= \frac{4}{3}(1+x)x^{-2/3}$

$\mathbb{R}\setminus\{0\}$, -1

$f' < 0$ when $x < -1$

$f' > 0$ when $-1 < x < 0$

$f' > 0$ when $0 < x$

$f'' = \frac{4}{3}\left(\frac{1}{3}x^{-2/3} - \frac{2}{3}x^{-5/3}\right)$

$\mathbb{R}\setminus\{0\}$, 2

$= \frac{4}{9}\left(x^{-2/3} - 2x^{-5/3}\right)$

$= \frac{4}{9}\left(x^{-5/3}\right)\left(x-2\right)$

$= \frac{4(x-2)}{9x^{5/3}}$

$f'' > 0$ when $x < 0$

$f'' < 0$ when $0 < x < 2$

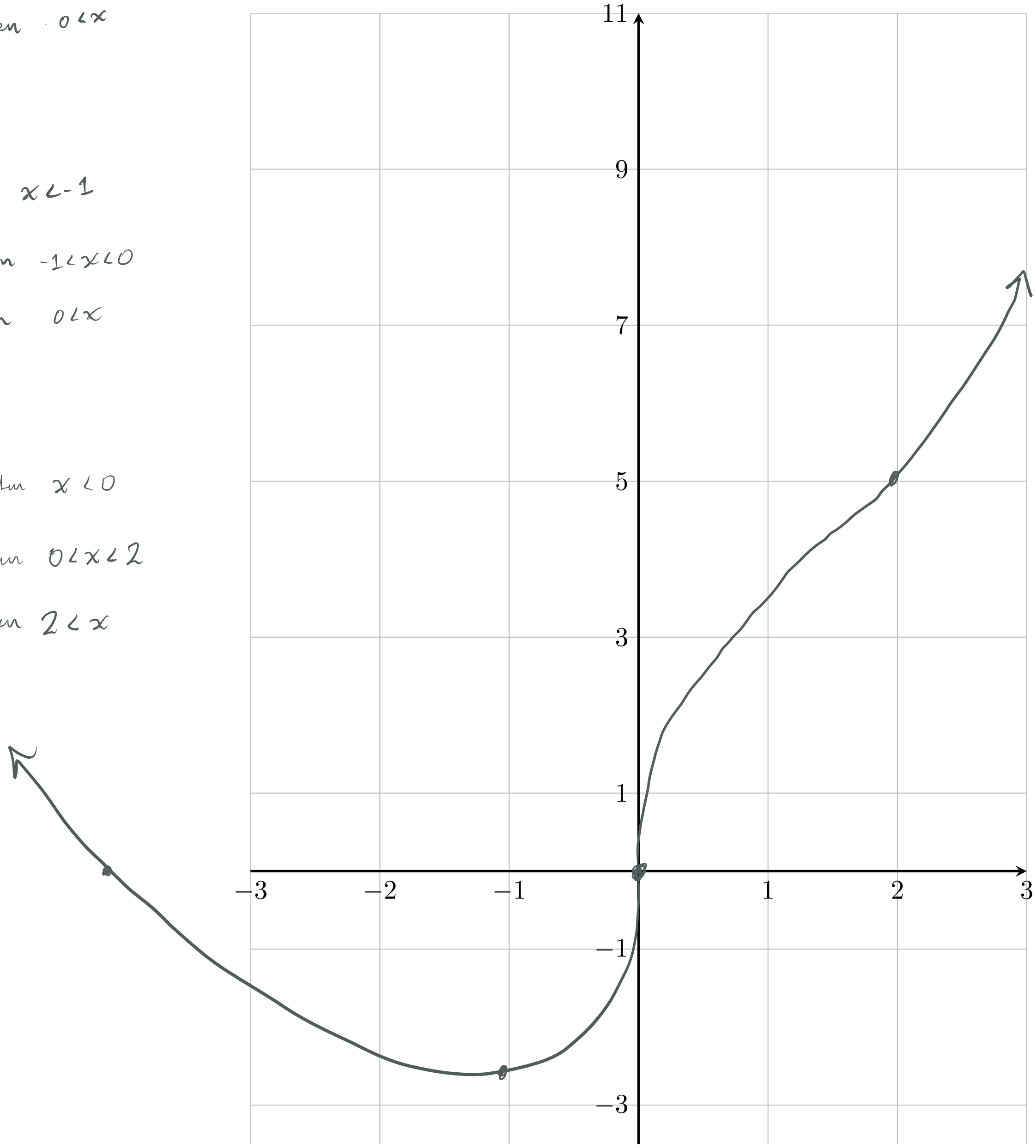
$f'' > 0$ when $2 < x$

$f > 0$ when $x < -4$
$f < 0$ when $-4 < x < 0$
$f > 0$ when $0 < x$

$f' < 0$ when $x < -1$
$f' > 0$ when $-1 < x < 0$
$f' > 0$ when $0 < x$

$f'' > 0$ when $x < 0$
$f'' < 0$ when $0 < x < 2$
$f'' > 0$ when $2 < x$

Graph of $f(x) = x^{\frac{1}{3}}(x+4)$



$f'(x)$: decreasing | increase | increase

$f''(x)$: convex | concave | convex

PROJECT SOLUTIONS À LA SAGEMATH (WEEK 10)

COLTON GRAINGER (MATH 1300)

One goal today is to be able to plot *with and without* Sage[1]. You will need *some sort* of computer to access `sagecell.sagemath.org`.

(If you have trouble, these links are all posted at `quamash.net/math1300`.) You can also scan this QR code.

FIGURE 1. `https://sagecell.sagemath.org/?q=ilvouj`

1. Sage commands to get started

To define a *symbolic* function.

```
f(x) = x^3 + 4*x^2 - 2*x + 1
```

To differentiate $f(x)$ *with respect to* x, i.e., to find $\frac{\mathrm{d}f}{\mathrm{d}x}$.

```
diff(f(x), x)
```

To differentiate twice, i.e., to find $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$.

```
diff(f(x), x, 2)
```

To differentiate $xy + \sin(x^2) + e^{-x}$ with respect to x (notice the **asterisk** for multiplication).

```
diff(x*y + sin(x^2) + e^(-x), x)
```

To plot a parabola on the domain $-1 < x < 1$.

```
plot(x^2)
```

To plot $g(x) = x^2 \sin(1/x)$ on the domain $[-5, 5]$.

```
g(x) = x^2 * sin(1/x)
plot(g(x), -5, 5)
```

Date: 2019-03-20.

[1]Sage is a family of free open-source mathematics software packages, first released in 2005 by William Stein (among others) at the University of Washington.

To plot the function $h(x) = |x|\sin(1/x)$ and its first derivative $\frac{dh}{dx}$ on the domain $[-2, 2]$.

```
h(x) = abs(x)*sin(1/x)
hprime(x)= diff(h(x), x)
plot( h(x),-2,2 ) + plot( hprime(x),-2,2, color='red' )
```

Notice that I defined `hprime(x)` as the *symbolic* derivative of `h(x)`. In the last line, I *concatenated* the `plot` command so that Sage will *superimpose* the plots. That is, `plot(h) + plot(diff(h,x))` tells Sage to plot h, then to plot $\frac{dh}{dx}$ in the same pane.

2. Project solutions

Please *make a complete attempt* at problem 1 before looking at this solution.

Figure 2. `https://sagecell.sagemath.org/?q=kcogeo`

Again, *make a complete attempt* at problem 2 before looking at this solution.

Figure 3. `https://sagecell.sagemath.org/?q=xezxqf`

3. (Optional) Reading material

If you like Sage, I have posted a Gregory Bard's *Sage for Undergraduates* along with two other guides at `quamash.net/math1300`. Bard's introduction (pages 1–20) and access to `sagecell.sagemath.org` is legit the best place to start.

4. Enjoy your Spring Break!

ATTENDANCE QUIZ (WEEK 10)

COLTON GRAINGER (MATH 1300)

Your name (print clearly in capital letters): ____

This is an **ungraded** quiz that will count for attendance; it is due by the end of recitation.

1. Definitions and comments

1. A **zero** (or a **root**) of a function f is

$$\text{a point } r \text{ such that } f(r) = 0.$$

   i. The term *root* is traditionally[1] used for the study of polynomial functions.
   ii. For example, if r is a root of the polynomial function p (that is, if $p(r) = 0$), then

$$p(x) = (x - r)q(x) \quad \text{for some polynomial function } q.$$

2. A **critical point** of a function f is

$$\text{a point } a \text{ such that } f'(a) = 0.$$

   i. The number $f(a)$ itself is called a **critical value** of f.

3. A **point of local extremum** of a function f is

$$\text{a point of either local maximum or minimum.}$$

   i. If b is a point of local extremum, we say $f(b)$ is a **local extreme value**.

4. An **inflection point** of a function f is

$$\text{a point } c \text{ such that the tangent line to } f \text{ at } (c, f(c)) \text{ crosses the graph of } f.$$

   i. In order for c to be an inflection point of a function f, it is necessary that f'' should have different signs to the left and right of c.
   ii. For example (see figure), $\sqrt{1/3}$ and $-\sqrt{1/3}$ are inflection points of $f(x) := 1/(1 + x^2)$.

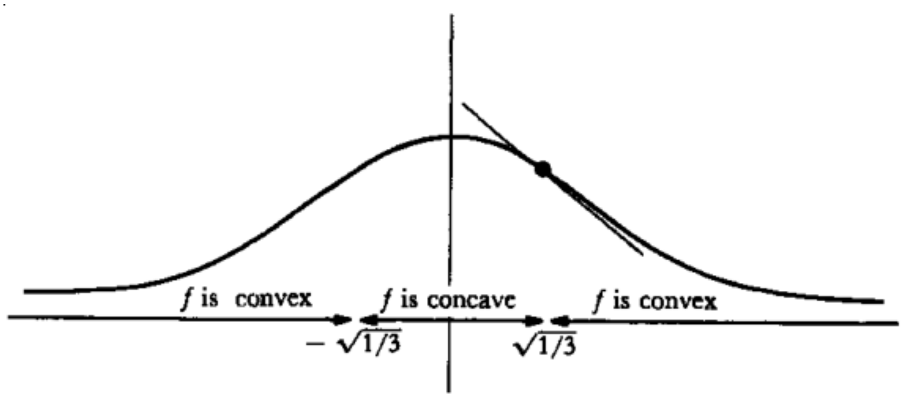


Figure 1. Inflection points

Date: 2019-03-17.

[1] "The **roote quadrat** of the whole number, is the desired distance or line Hypothenusal." (Digges, *Pantom*, 1571).

2. Multiple Choice

1. Consider the function $f\colon \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^3(x-1)^4(x-2)^2$. Which of the following **is true**?

   (A) 0, 1, and 2 are all critical points and all of them are points of local extrema.
   (B) 0, 1, and 2 are all critical points, but only 0 is a point of local extremum.
   (C) 0, 1, and 2 are all critical points, but only 1 and 2 are points of local extrema.
   (D) 0, 1, and 2 are all critical points, and none of them is a point of local extremum.
   (E) 1 and 2 are the only critical points.

   Your answer: C

2. Say $f\colon \mathbf{R} \to \mathbf{R}$ is a differentiable function. What information **can not** be determined from **only** the first derivative f'?

   (A) roots of the function
   (B) critical points
   (C) points of inflection
   (D) local extreme values
   (E) neither (D) nor (A)

   Your answer: E

3. Say $f\colon \mathbf{R} \to \mathbf{R}$ is a twice-differentiable function. What information **can** be determined from **only** the second derivative f''?

   (A) intervals where the function is positive or negative
   (B) intervals where the function increases and decreases
   (C) intervals where the function is concave up and concave down
   (D) y-values of horizontal asymptotes
   (E) all of the above

   Your answer: C

4. Suppose f and g are continuously differentiable functions on $\mathbf{R}$. Suppose f and g are both concave up. Which of the following is **always true**?

   (A) $f+g$ is concave up.
   (B) $f-g$ is concave up.
   (C) $f \cdot g$ is concave up.
   (D) $f \circ g$ is concave up.
   (E) All of the above.

   Your answer: A

3. References

The definitions are from Spivak [1]. The questions are adapted from Naik's Math-152 notes [2].

[1] M. Spivak, *Calculus*, 3rd ed. Publish or Perish, Inc., 1994.

[2] V. Naik, "Math 152 Course Notes," 2012.

1. A company is producing three types of food in closed cylindrical cans with the dimensions given in the table below. Fill in the blanks in the table.

Type	Radius (in)	Height (in)	Volume (in^3)	Surface Area (in^2)
Chicken Noodle Soup	1.37	3.68	~ 21.69	43.47
Condensed Milk	1.50	3.07	21.70	~ 43.471
Almonds	1.63	2.60	21.70	43.32

The company wants to redesign their cans to minimize the amount of aluminum needed. The cans will still need to hold 12 oz each. Use that 12 oz is approximately 21.7 in^3.

(a) Write the equation for the surface area A of a can in terms of the radius r.

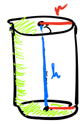
$$S(r) = 2(\pi r^2) + h \cdot 2\pi r$$
$$= 2\pi(r^2 + hr)$$
$$= 2\pi\left(r^2 + \frac{V}{\pi r}\right)$$

(b) What is the domain of this function?

$S : (0, \infty) \to \mathbb{R}$ or $S : \mathbb{R}_{>0} \to \mathbb{R}$

(c) Minimize this function. Justify your answer.

EX] We determine the minimal surface area for a right cylinder of volume V.

Demo. Let  r and h be the radius and height of our cylinder.

Since $V = h \cdot \pi r^2$, we know $h = \frac{V}{\pi r^2}$ and then the S.A. is

$$S(r) = 2(\pi r^2) + h \cdot 2\pi r \qquad \text{(computing S.A. from pre-calculus)}$$
$$= 2\pi(r^2 + hr)$$
$$= 2\pi\left(r^2 + \frac{V}{\pi r}\right) \qquad \left(\text{where } h = \frac{V}{\pi r^2}\right)$$

We want to minimize $S(r)$ w.r.t. r, so differentiate w.r.t. r and find critical points.

$$\frac{dS}{dr} = 2\pi\left(2r - \frac{V}{\pi r^2}\right) \qquad (V \text{ and } \pi \text{ are constants w.r.t. } r)$$
$$= 2\pi\left(r\left(2 - \frac{V}{\pi r^3}\right)\right)$$

Q] For which $R \geq 0$ does $\left.\frac{dS}{dr}\right|_{r=R} = 0$?

A] Because $\frac{dS}{dr}$ has rational factors r and $\frac{2\pi r^3 - V}{\pi r^3}$,
only $R = 0$ and $2\pi R^3 = V$ will produce $S'(R) = 0$.

As "applied mathematicians," we throw out $R = 0$. WHY?

Check then $\lim_{r \to 0^+} S(r) = \lim_{r \to 0^+} 2\pi\left(r^2 + \frac{V}{\pi r}\right) = 2\pi \lim_{r \to 0^+} (r^2)\left(1 + \frac{V}{\pi r^3}\right)$ diverges to $+\infty$.

Then will test $R^3 = \frac{V}{2\pi} \Rightarrow R = \sqrt[3]{\frac{V}{2\pi}}$.

$$S\left(\sqrt[3]{\frac{V}{2\pi}}\right) = 2\pi\left(\left(\frac{V}{2\pi}\right)^{2/3} + \frac{V}{\pi\sqrt[3]{\frac{V}{2\pi}}}\right)$$

$=$

(d) What does your answer to part (c) mean in the context of the problem?

Because $\lim_{r \to \infty} S(r) = +\infty$, and $\lim_{r \to 0^+} S(r) = +\infty$,

we've the following observation: $R = \sqrt[3]{\frac{V}{\pi r^2}} = 1.511$ is a global minimum

(e) What are the dimensions of the can the company should make?

1.51×3.02

radius $\times$ height

2. League of Legends is a multiplayer online video game. One aspect of the game involves battling other players. A player's *Effective Health* when defending against physical damage is given by

$$E = h + \frac{ha}{100},$$

where h is an indicator of the player's *Health* and a is an indicator of the player's *Armor*. Players can purchase more Health and Armor with gold coins. Health costs 2.5 coins per unit and Armor costs 18 coins per unit. Assume a player has 2669 gold coins. What is the maximum Effective Health the player can achieve?

(a) Assume the player will spend all of their coins. Write the equation for the player's Effective Health in terms of Health h. Do NOT round any decimals!

Let $2.5h + 18a = 2669$ be a constraint. Then $2669 - 2.5h = 18a$

So $a = \frac{2669 - 2.5h}{18}$. Whence $E(h) = h + \frac{h}{100}\left(\frac{2669-2.5h}{18}\right)$.

Now $0 \leq h \leq 1067.6$, but $h \in \mathbb{N}$. So we have

$E : [0, 1067.6] \longrightarrow \mathbb{R}_{\geq 0}$, but since E is a deg 2 polynom,

E is cts., diff'ble, and so by the EVT

(b) What is the domain of this function?

(c) Maximize this function. Do NOT round any decimals! Justify your answer.

$$E(h) = h + \frac{h}{100}\left(\frac{2669 - 2.5h}{18}\right).$$

$$= h + \frac{2669h - 2.5h^2}{1800}$$

$$E'(h) = 1 + \frac{1}{1800}(2669 - 5h)$$

$$E'(h) = 0 \quad \text{iff} \quad -1800 = 2669 - 5h$$

$$\text{iff} \quad h = 893.8$$

h	E
0	0
893	1109.5525
894	$1109.55\overline{33}$
1067.6	1067.6

(d) Assume that players cannot buy partial units of health or armor. Is the this maximum actually achievable? Explain.

No, $E : [0, 1067] \cap \mathbb{Z} \longrightarrow \mathbb{R}_{>0}$. Then we require $h \in \mathbb{Z}$.

Testing, $E(893) < E(894)$. Since regardless, the endpts have a lessor value, we conclude 894 is the maxim. value

(e) What is the maximum Effective Health this player can achieve? Explain.

Cunningham, Chris. Optimization Problems That Today's Students Might Actually Encounter? Mathematics Educators Stack Exchange, matheducators.stackexchange.com/questions/1550/optimization-problems-that-todays-students-might-actually-encounter/1561.

Names: ______________________________ ______________________________

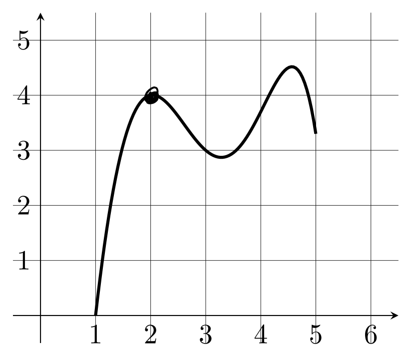
______________________________ ______________________________

Exit Ticket

1. Describe the difference between a continuous function and a discrete function. Graph examples of each.

continuum countable

2. A student would like to maximize a discrete function f that is only defined on $x = 1, 2, 3, 4, 5$. They found that f can be modeled by a continuous function g. That is, $f(a) = g(a)$ when $a = 1, 2, 3, 4, 5$. Below is the graph of g.



we note here f' has many (3) roots. Whence we've to test $\sim \frac{1}{2}$ radius from each critical point.

If the student wants to find the absolute maximum value of f, is it enough to find the absolute maximum value of g? Explain.

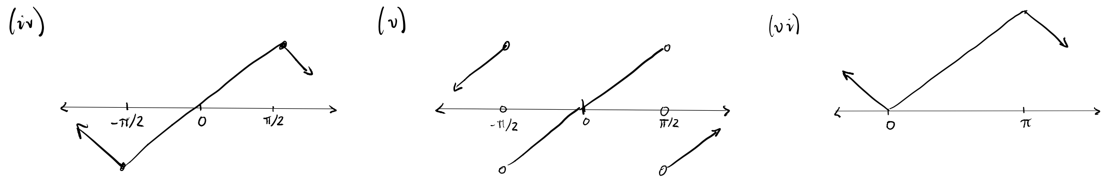
No.

3. Describe how one could find the absolute maximum of a discrete function f that is modeled by a continuous function g.

p.242, ch.7. prob.28 Each of the following expressions is of the form $f^{-1}(f(x))$:

(i) $\arctan(\tan x)$
(ii) $\arccos(\cos x)$
(iii) $\arcsin(\sin x)$

(a) Match each of them with their graph below



p.243, ch.7, pb.30 What is wrong with the following "argument"?

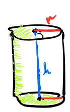
> If $f(x) = \arctan(\tan x)$, then by the chain rule
>
> $$f'(x) = \frac{1}{1+\tan^2 x} \cdot \sec^2 x = \frac{1}{\sec^2 x} \cdot \sec^2 x = 1.$$
>
> Since $f'(x) = 1$ for all x, it follows that $f(x) = x$.
>
> Therefore $f(x) = \arctan(\tan x) = x$ for all x.

~~TODO~~

~~Email out pages 95-101 (history)~~

~~pgs 29-31 (Körner) & admonishment to derive.~~

EX] We determine the minimal surface area for a right cylinder of volume V.

Demo. Let  r and h be the radius and height of our cylinder.

Since $V = h \cdot \pi r^2$, we know $h = \frac{V}{\pi r^2}$ and then the S.A. is

$$\begin{aligned} S(r) &= 2(\pi r^2) + h \cdot 2\pi r && \text{(computing S.A. from pre-calculus)} \\ &= 2\pi(r^2 + hr) \\ &= 2\pi\left(r^2 + \frac{V}{\pi r}\right) && \left(\text{where } h = \frac{V}{\pi r^2}\right) \end{aligned}$$

We want to minimize $S(r)$ w.r.t. r, so differentiate w.r.t. r and find critical points.

$$\begin{aligned} \frac{dS}{dr} &= 2\pi\left(2r - \frac{V}{\pi r^2}\right) && (V \text{ and } \pi \text{ are constants w.r.t. } r) \\ &= 2\pi\left(r\left(2 - \frac{V}{\pi r^3}\right)\right) \end{aligned}$$

Q] For which $R \geq 0$ does $\left.\frac{dS}{dr}\right|_{r=R} = 0$?

A] Because $\frac{dS}{dr}$ has rational factors r and $\frac{2\pi r^3 - V}{\pi r^3}$,),
only $R = 0$ and $2\pi R^3 = V$ will produce $S'(R) = 0$.

WARNING: $h \neq \pi r^2 / V$. Else ...

$$\begin{aligned} S(r) &= 2(\pi r^2) + h \cdot 2\pi r && \text{(computing S.A. from pre-calculus)} \\ &= 2\pi(r^2 + hr) \\ &= 2\pi\left(r^2 + \frac{\pi}{V} r^3\right) && \left(\text{where } h = \frac{\pi r^2}{V}\right) \end{aligned}$$

We want to minimize $S(r)$ w.r.t. r, so differentiate w.r.t. r and find critical points.

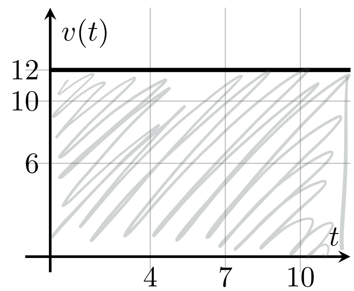
$$\begin{aligned} \frac{dS}{dr} &= 2\pi\left(2r + \frac{\pi}{V} 3r^2\right) \\ &= 2\pi\left(r\left(2 + \frac{3\pi}{V} r\right)\right) \\ &= 2\pi\left(r\left(\frac{V}{3\pi} \cdot \frac{3\pi}{V}\right)\left(2 + \frac{3\pi}{V} r\right)\right) \\ &= \frac{6\pi^2}{V}\left(r\left(r + \frac{2V}{3\pi}\right)\right), \end{aligned}$$

Because $\frac{dS}{dr}$ is quadratic with linear factors $r - 0$ and $r - \left(\frac{-2V}{3\pi}\right)$,
we have 0 and $\frac{-2V}{3\pi}$ as roots

As "applied mathematicians," we throw out $R = 0$. WHY?

Check then $\lim_{r \to 0^+} S(r) = \lim_{r \to 0^+} 2\pi \left(r^2 + \frac{V}{\pi r} \right) = 2\pi \lim_{r \to 0^+} (r^2)\left(1 + \frac{V}{\pi r^3}\right)$ diverges to $+\infty$.

1. A girl is running at a velocity of 12 feet per second for 10 seconds, as shown in the velocity graph below.



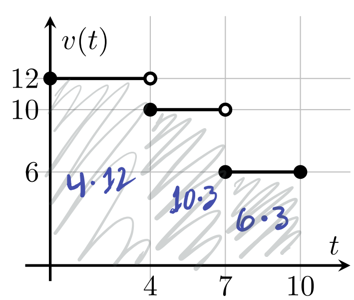
How far does she travel during this time?

$12 \cdot 10 = 120$ ft

This distance can be depicted graphically as a rectangle. Shade such a rectangle and explain why it gives the distance.

area of rectangle

2. Now the girl changes her velocity as she runs. Her velocity graph is approximately as shown:



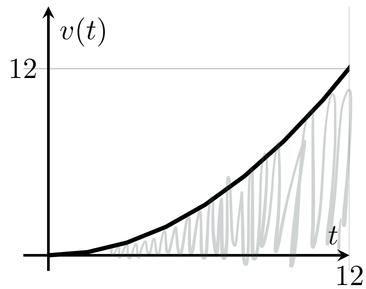
$48 + 30 + 18 = 96$

How far does she travel this time?

96

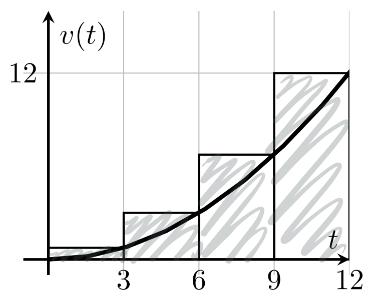
Q] What's your plan of attack!? A] Still using it!

3. This time she starts off slowly and speeds up.



units again!

The velocity is given by $v(t) = \frac{t^2}{12}$ (time in seconds, velocity in ft/sec). We can no longer exactly find the distance travelled using areas of rectangles. But we can estimate it using areas of rectangles.



$\left(v(3)+v(6)+v(9)+v(12)\right)\cdot 3\text{ft}$

$\frac{3^2+6^2+9^2+12^2}{12}\cdot 3\text{ft}$

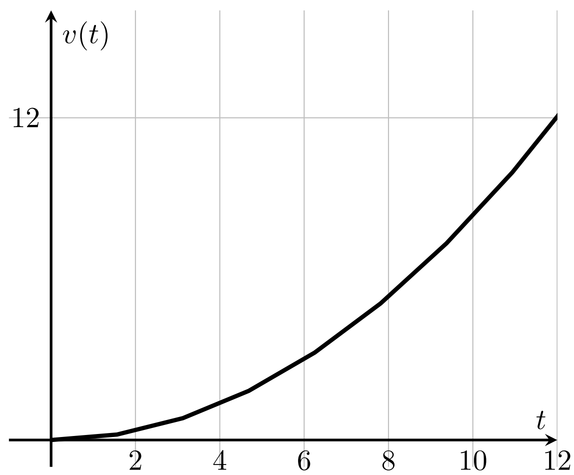
Find her velocity at time $t = 3, 6, 9, 12$ and use it to estimate her distance travelled in the first 12 seconds. In your answer to this problem **use fractions, not decimals.**

$\frac{270}{4}$ ft

4. Now, for the same velocity function $v(t) = \frac{t^2}{12}$, get a better estimate of how far she travelled using $n = 6$ rectangles. Draw a graph showing the areas, and use their areas to estimate her distance travelled in the first 12 seconds. Again **use fractions, not decimals.**

for k in range(6):

$2 \cdot v(2+2 \cdot k)$



5. Now we will estimate the area when there are $n = 37$ rectangles. **Use fractions, not decimals.**

   (a) Width of each rectangle:

   $\frac{12}{37} = \Delta x$

   (b) List of right-hand endpoint of each rectangle:

   $\frac{12}{37}$, $\frac{24}{37}$, $\frac{36}{37}$, ..., $\frac{432}{37}$, 12

   (c) List of heights of each rectangle:

   ________, ________, ________, ..., ________, 12

   (d) List of areas of rectangles:

   ________, ________, ________, ..., ________, ________

   (e) Sum of all areas:

   ________ + ________ + ________ + ... + ________ + ________

6. Now we will figure out the estimate when there are an arbitrary number of rectangles, or n rectangles.

   (a) Width of each rectangle:

   $\frac{12}{n}$

   (b) List of right-hand endpoint of each rectangle:

   $\frac{12}{n}$, $2\frac{12}{n}$, $3\frac{12}{n}$, ..., $(n-1)\frac{12}{n}$, $n\frac{12}{n}$

   (c) List of heights of each rectangle:

   ________, ________, ________, ..., ________, ________

   (d) List of areas of rectangles:

   $\frac{12}{n}\Big[\left(\frac{12}{n}\right)^2/12$, $\cdots$, ________, ..., $\left((n-1)\frac{12}{n}\right)^2/12$, $\frac{12^2}{12}/12\Big]$

   (e) Sum of all areas:

   TODO ________ + ________ + ________ + ... + ________ + ________

boo! coeff

7. Manipulate the sum algebraically until it is of the form:

$$\boxed{stuff}\,(1 + 4 + 9 + ... + n^2).$$

$$\frac{12}{n}\sum_{k=1}^{n}\left(\frac{k\Delta n}{12}\right)^2 = \frac{12}{n}\sum_{k=1}^{n}\frac{1}{12^2}\cdot k^2\cdot\left(\frac{12}{n}\right)^2$$

$$= \frac{12}{n^3}\sum_{k=1}^{n} k^2$$

8. Simplify further by substituting $1 + 4 + 9 + ... + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ into your answer above. Check that it gives the same answer for $n = 6$ that you got in problem 4.

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

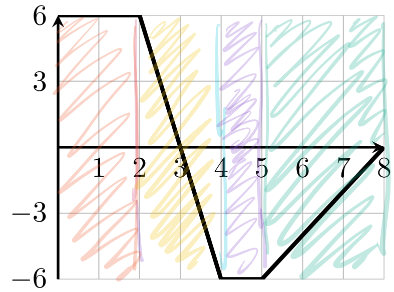
$$\frac{12}{n^3}\left(\sum_{k=1}^{n} k^2\right) = \frac{12}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

$$= \frac{2n(n+1)(2n+1)}{n^3}$$

9. As n approaches infinity we find her exact distance travelled (the exact area under the curve). Take the limit as n goes to infinity for your answer to the previous problem.

Notice that you just found the area inside a region with a curved edge!

1. Review and Warm-up: The graph of f is shown below. Calculate exactly each of the definite integrals that follow.

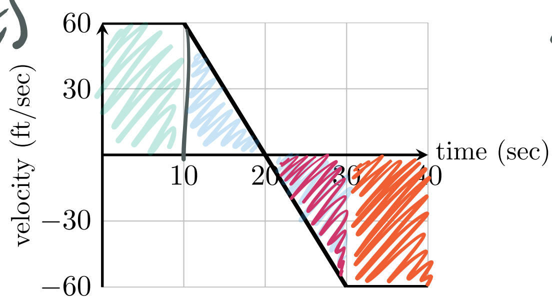


want to express concavity

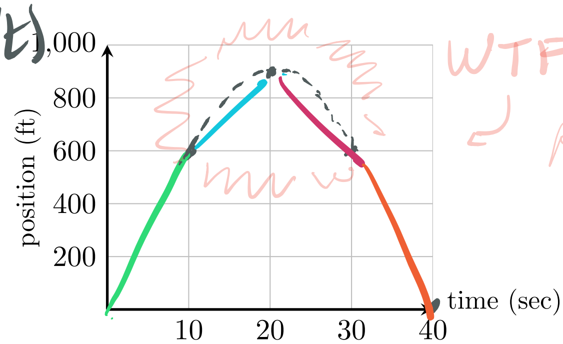
(a) $\int_0^2 f(x)\,dx =$ 12
(b) $\int_0^3 f(x)\,dx =$ 15
(c) $\int_4^5 f(x)\,dx =$ -6
(d) $\int_5^8 f(x)\,dx =$ -9
(e) $\int_2^4 f(x)\,dx =$ 0
(f) $\int_0^8 f(x)\,dx =$ -3

2. Let $s(t)$ be the position, in feet, of a car along a straight east/west highway at time t seconds. Positive values of s indicate eastward displacement of the car from home, and negative values indicate westward displacement. At $t = 0$ the car is at home. Let $v(t)$ represent the velocity of this same car, in feet per second, at time t seconds (see graph below).

v(t)



s(t)



WTF

parabola, no?

smoothing

(a) Write a definite integral representing each of the following:

$s(10) = \int_0^{10} v(t)\,dt$

$s(30) = \int_0^{30} v(t)\,dt$

iii $s(t) = \int_0^{t} v(x)\,dx$

Now use these integrals and the velocity graph to help you fill in the chart below:

t	0	10	20	30	40
$s(t)$	0	600	900	600	0

(b) Use these values to help you plot the position function.

(c) Fill in the chart below:

Definite integral of velocity	Change in position
$\int_0^{10} v(t)dt =$ 600 ft	$s(10) - s(0) =$ 600f
$\int_{10}^{20} v(t)dt =$ 300 ft	$s(20) - s(10) =$ "
$\int_0^{40} v(t)dt =$ 0 ft	$s(40) - s(0) =$ "

(d) Why do these two columns give the same answers?

3. The following data is from the U.S. Bureau of Economic Analysis. It shows the rate of change $r(t)$ (in dollars per month) of per capita personal income, where t is the number of months after January 1, 2012.

RT Problem!

NO!

t (months)	0	2	4	6	8	10	~~12~~
$r(t)$ (dollars per month)	154	17	10	79	278	-432	~~290~~

total change vs. net change!

Use left-hand Riemann sums to estimate the total change in personal income during 2012.

$2 \cdot (396) = 792$? no!

$\boxed{2 \cdot 106}$

4. A can of soda is put into a refrigerator to cool. The temperature of the soda is given by $F(t)$. The **rate** at which the temperature of the soda is changing is given by

$$F'(t) = f(t) = \boxed{-25e^{-2t}} \text{ (in degrees Fahrenheit per hour)}$$

F°/hr

(a) Find the rate at which the soda is cooling after 0, 1, and 2 hours. Then use this information to estimate the temperature of the soda after 3 hours if the soda was 60°F when it was placed in the refrigerator.

$f(t) = -25 \cdot e^{-2t}$

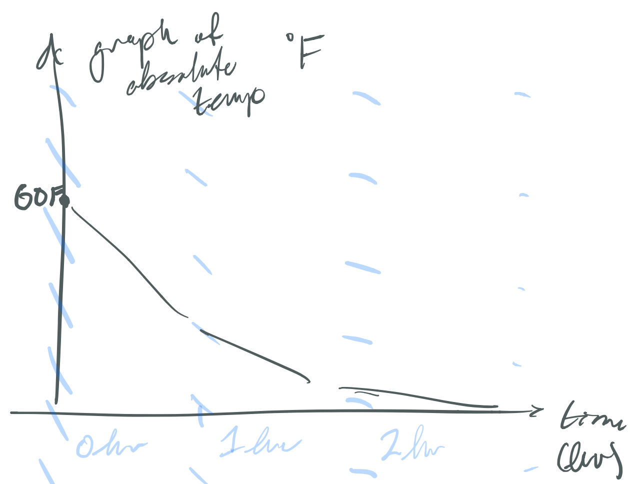
$f(0) = -25$

$f(1) = \frac{-25}{e^2}$

$f(2) = \frac{-25}{e^4}$

$-25 \cdot \left(1 + \frac{1}{e^2} + \frac{1}{e^4}\right)$

$\approx -25 \cdot \frac{73}{64}$ (°F)



(b) Now we will find the exact temperature of the soda after three hours have passed.

i. Find an antiderivative of $f(t)$, that is, find a function $F(t)$ such that $F'(t) = f(t) = -25e^{-2t}$.
(Hint: take a couple of derivatives of $f(t)$ and try to find a pattern.)

VERIFY this is the antiderivative!

$$\int_0^3 -25e^{-2t}\,dt = -25\int_0^3 e^{-2t}\,dt \qquad u=-2t,\ du=-2dt \qquad dt=\frac{du}{-2} \qquad t=3,\ u=-6;\ t=0,\ u=0$$

$$= \frac{-25}{-2}\int_0^{-6} e^u\,du = \frac{25}{2}\cdot\int_0^{-6} e^u\,du = -\frac{25}{2}\int_{-6}^{0} e^u\,du$$

$$= \frac{-25}{2}\left[e^0 - e^{-6}\right] \approx -12.5 \quad \text{what!}$$

ii. The Fundamental Theorem of Calculus (the Evaluation Theorem) tells us that $\int_a^b f(t)\,dt = F(b) - F(a)$. Use this theorem and the function you found in the last step to find the temperature of the soda after 3 hours have passed.

what! $\approx$ order of magnitude $\times 2$

(c) Why do you think your estimate in part (a) is so far off?

$$\left|\frac{d^2f}{dt^2}\right| \gg 1 \text{ for } t \text{ near } 0,$$

hence large error over $[0,1]$.

AREA²! Mouth, lung
- work, energy, solid of revolution, moment of inertia,

what do you mean by area?

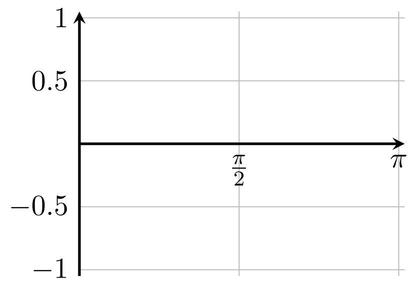
- "integrals" / "areas"

5. (a) Write an integral which represents the area between $f(x) = x^4$ and the x-axis, between $x = 0$ and $x = 2$.

   (b) Evaluate this integral using the Fundamental Theorem of Calculus (the Evaluation Theorem).

6. (a) Using the Fundamental Theorem of Calculus (as in the last problem), evaluate $\int_0^{\pi} \cos(x)\, dx$.

   (b) Show the area represented by the integral in part (a) on the graph.



7. Evaluate $\int_0^1 \frac{1}{1+x^2}\, dx$ without using technology.

OPTIONAL, THOUGHT PROVOKING

RECITATION SUPPLEMENT

Colton Grainger

WEEK 11 Spring 2019

MATH 1300 Calculus 1

0. Big Picture Reading Material
    - Halmos '70
    - Körner '14
1. Background for Extra Problem Set
    - Priestly '74
2. Extra Problem Set
    - Spivak '15
3. Solutions
    - Spivak '15

BIG PICTURE READING MATERIAL

[Hal70]

"How to write Mathematics" by Paul Halmos

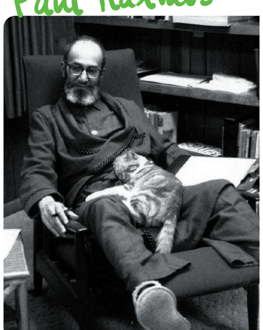
The basic problem in writing mathematics is the same as in writing biology, writing a novel, or writing directions for assembling a harpsichord: the problem is to communicate an idea. To do so, and to do it clearly, you must have something to say, and you must have someone to say it to, you must organize what you want to say, and you must arrange it in the order you want it said in, you must write it, rewrite it, and re-rewrite it several times, and you must be willing to think hard about and work hard on mechanical details such as diction, notation, and punctuation. That's all there is to it.

2. Say something

It might seem unnecessary to insist that in order to say something well you must have something to say, but it's no joke. Much bad writing, mathematical and otherwise, is caused by a violation of that first principle. Just as there are two ways for a sequence not to have a limit (no cluster points or too many), there are two ways for a piece of writing not to have a subject (no ideas or too many).

The first disease is the harder one to catch. It is hard to write many words about nothing, especially in mathematics, but it can be done, and the result is bound to be hard to read. There is a classic crank book by Carl Theodore Heisel [5] that serves as an example. It is full of correctly spelled words strung together in grammatical sentences, but after three decades of looking at it every now and then I still cannot read two consecutive pages and make a one-paragraph abstract of what they say; the reason is, I think, that they don't say anything.

author: Paul Halmos



with cat Roger.

The second disease is very common: there are many books that violate the principle of having something to say by trying to say too many things. Teachers of elementary mathematics in the U.S.A. frequently complain that all calculus books are bad. That is a case in point. Calculus books are bad because there is no such subject as calculus; it is not a subject because it is many subjects. What we call calculus nowadays is the union of a dab of logic and set theory, some axiomatic theory of complete ordered fields, analytic geometry and topology, the latter in both the "general" sense (limits and continuous functions) and the algebraic sense (orientation), real-variable theory properly so called (differentiation), the combinatoric symbol manipulation called formal integration, the first steps of low-dimensional measure theory, some differential geometry, the first steps of the classical analysis of the trigonometric, exponential, and logarithmic functions, and, depending on the space available and the personal inclinations of the author, some cook-book differential equations, elementary mechanics, and a small assortment of applied mathematics. Any one of these is hard to write a good book on; the mixture is impossible.

[Koe14]

"Why is Calculus Hard?" by T. W. Körner

The basic ideas of the calculus, like the basic ideas of the rest of mathematics, are easy (how else would a bunch of apes fresh out of the trees be able to find them?), but calculus requires a lot of work to master (after all, we are just a bunch of apes fresh out of the trees). Here is a list of some of the difficulties facing the reader.

Mathematics is a 'ladder subject'. If you are taught history at school and you pay no attention during the year spent studying Elizabethan England, you will get bad grades for that year, but you will not be at a disadvantage next year when studying the American Civil War. In mathematics, each topic depends on the previous topic and you cannot miss out too much.

The ladder described in this book has many rungs and it will be a very rare reader who starts without any knowledge of the calculus and manages to struggle though to the end. (On the other hand, some readers will be familiar with the topics in the earlier chapters and will, I hope, be able to enjoy the final chapters.) Please do not be discouraged if you cannot understand everything; experience shows that if you struggle hard with a topic, even if unsuccessfully, it will be much easier to deal with when you study it again.[1]

Mathematics involves deferred gratification. Humans are happy to do A in order to obtain B. The are less happy to do A in order to to do B, to do B in order to do C and then to do C in order to obtain D. Results in mathematics frequently require several preliminary steps whose purpose may not be immediately apparent. In Chapter 2 we spend a long time discussing the integral and the fundamental theorem of the calculus. It is only at the end of the chapter that we get our first payback in the form of a solution to an interesting problem.

Mathematics needs practice. I would love to write music like Rossini. My university library contains many books on the theory of music and the art of composing, but I know that, however many books I read, I will not be able to write music. A composer needs to know the properties of musical instruments, and to know the properties of instruments you need to play at least one instrument well. To play an instrument well requires years of practice.

Each stage of mathematics requires fluency in the previous stage and this can only be acquired by hours of practice, working through more or less routine examples. Although this book contains some exercises[2] it contains nowhere near enough. If the reader does not expect to get practice elsewhere, the first volume of *An Analytical Calculus* by A. E. Maxwell [5] provides excellent exercises in a rather less off-putting format than the standard 'door stop' text book. (However, any respectable calculus book will do.)

This is a first look and not a complete story. As I hope to make clear, the calculus presented in this book is not a complete theory, but deals with 'well behaved objects' without giving a test for 'good behaviour'. This does not prevent it from being a very powerful tool for the investigation of the physical world, but is unsatisfactory both from a philosophical and a mathematical point of view. In the final chapter, I discuss the way in which the first university course in analysis resolves these problems. I shall refer to the calculus described in the book as 'the old calculus' and to the calculus as studied in university analysis courses as 'the new calculus' or 'analysis'.[3]

D'Alembert is supposed to have encouraged his students with the cry 'Allez en avant et la foi vous viendra' (push on and faith will come to you). My ideal reader will be prepared to accept my account *on a provisional basis*, but be prepared to begin again from scratch when she meets rigorous analysis.

It is a very bad idea to disbelieve everything that your teachers tell you and a good idea to accept everything that your teachers tell you. However, it is an even better idea to accept that, though most of what you are taught is correct, it is sometimes over-simplified and may occasionally turn out to be mistaken.

The calculus involves new words and symbols. The ideas of the calculus are not arbitrary, but the names given to the new objects and the symbols used are. At the simplest level, the reader will need to recognise the Greek letter δ (pronounced 'delta') and learn a new meaning for the word 'function'. At a higher level, she will need to accept that the names and notations used reflect choices made by many different mathematicians, with many different views of their subject, speaking many languages at many different times over the past 350 years. If we could start with a clean sheet, we would probably make different choices (just as, if we could redesign the standard keyboard, we would probably change the position of the letters). However, we wish to talk with other mathematicians, so we must adopt their language.

author

My British accent. The theory and practice of calculus are international. The teaching of calculus varies widely from country to country, often reflecting the views of some long dead charismatic educationalist or successful textbook writer. In some countries, calculus is routinely taught at school level whilst, in others, it is strictly reserved for university. Several countries use calculus as an academic filter, a coarse filter in those countries with a strong egalitarian tradition, a fine one in those with an elitist bent.[4] Yikes!

I was brought up under a system, very common in twentieth century Western Europe, where calculus was taught as a computational tool in the last two years of school and rigorous calculus was taught in the first year of university. Some of the discussion in the introduction and the final chapter reflects my background, but I do not think this should trouble the reader.

The Apes comment, for example.

A shortage of letters. The calculus covers so many topics that we run into a shortage of letters. Mathematicians have dealt with this partly by introducing new alphabets and fonts giving us A, a, α, $\mathbf{A}$, $\mathbf{a}$, $\aleph$, $\mathcal{A}$, A, a, $\mathbb{A}$, $\mathfrak{A}$, $\mathfrak{a}$, ... Rather than learn a slew of new symbols, I think that my readers will prefer to accept that r will sometimes be an integer[5] and sometimes the radius of a circle, and that the same letter will be used to represent different things in different places.

But it is beautiful. Hill walking is hard work, but the views are splendid and the exercise is invigorating. The calculus is one of the great achievements of mankind and one of the most rewarding to study.

"On Remembering"

Exercise 1.4.8. *(i) Let*

$$f(x) = \frac{1 + x^{1/3}}{1 + \sqrt{1 + x^2}}.$$

Our object is to find $f'(x)$.

Observe that $f(x) = g(x)/h(x)$ with

$$g(x) = 1 + x^{1/3},\ \ h(x) = 1 + \sqrt{1 + x^2}.$$

Now note that $g(x) = a(x) + b(x)$, $h(x) = a(x) + c(x)$ with

$$a(x) = 1,\ \ b(x) = x^{1/3},\ c(x) = \sqrt{1 + x^2}$$

and $c(x) = S\big(u(x)\big)$ with

$$u(x) = 1 + x^2,\ \ S(x) = \sqrt{x}.$$

Now obtain $a'(x)$, $S'(x)$, $u'(x)$, $c'(x)$, $h'(x)$, $b'(x)$, $g'(x)$ and $f'(x)$.

(ii) Write down some more functions along the lines laid out in (i) and find their derivatives.

Easy; Try it?

With practice and experience, the reader will find that she can reduce the number of sub-problems required.

On remembering and understanding. Chess players do not carry around a notebook explaining how the knight moves and bridge players do not clutch a memo reminding them that there are four suits each of 13 cards. In the same way, mathematicians do not consult a 'formula book' for the differentiation rules just given (or for anything else). It is impossible to work on difficult mathematics unless you can work quickly and efficiently through the easy parts.

When a mathematician cannot remember a fact or a formula, her first action is to attempt to re-derive it for herself. If she cannot do this, she concludes that she does not understand the result and looks up *not the result* but its *derivation* and studies the derivation until she is certain that she understands why the result is true. If you understand why a result is true, it is easy to remember it. If you do not understand why a result is true, it is useless to memorise it.[19]

[19] Except the night before an examination.

BACKGROUND FOR EXTRA PROBLEM SET

[Pri79]

§8. Solving Optimization Problems

Where are we now? We have just completed an unavoidable digression from our original theme, which was the solution of optimization problems. As we saw in Chapter 1, an optimization problem leads to the problem of finding the highest (or lowest) point on a certain curve. This, in turn, has led to the study of derivatives, because derivatives cast light on the behavior of a curve. And now, at last, we know how to bypass Fermat's method and use the following rules instead.

(1) $(cf)' = c \cdot f'$ (constant multiples).
(2) $(f + g)' = f' + g'$ (sums).
(3) $(1/g)' = (-1/g^2)g'$ (reciprocals).
(4) $(g^2)' = 2gg'$ (squares).
(5) $(\sqrt{g})' = (1/2\sqrt{g})g'$ (square roots).
(6) $(fg)' = fg' + gf'$ (products).
(7) $(f/g)' = (gf' - fg')/g^2$ (quotients).

The reader should practice using these rules until they have been memorized. Then the taking of derivatives will be quite a routine matter, and the most important step in solving an optimization problem will have been mastered.

We can finally come to grips with the topic to which the title of this chapter alludes. What are the steps leading to the solution of an optimization problem? Basically, there are just two steps. First, translate the problem into the geometric problem of finding the highest (or lowest) point on a certain curve f; and second, find f' and use it as an aid in understanding how the curve f behaves.

The critical points to be found in sketching a curve f are those where the tangent line to the curve is horizontal. [That leads to a definition: *To say that x is a* **critical point** *of f is to say that $f'(x) = 0$.*] Usually, although not always, the function f will attain its optimal value at a critical point.

To verify whether the optimum has been found, make a rough sketch of the curve near each critical point (the second derivative is helpful here) and near each endpoint of the domain.

As we have seen, some curves do not have a highest (or lowest) point. It can be proved, however, that a curve must have such points if it comes from a *continuous* function and if the domain is an interval *containing its endpoints.* This is a deep theorem of analysis, the modern branch of mathematics into which seventeenth-century calculus evolved, and cannot be proved here. The moral for us is to be aware of what a function is doing near the endpoints of its domain, particularly if the domain does not include endpoints. If a continuous curve fails to have a highest (or lowest) point, then by the theorem of analysis the trouble must lie in the behavior of the function near an endpoint missing from its domain.

EXAMPLE 5. Find the highest point on the curve f given by

$$f(x) = 2x + 3,$$

on the domain

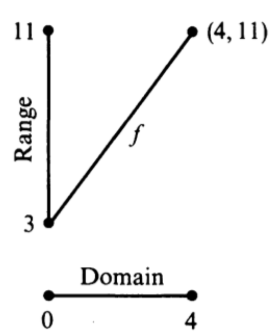
(a) $0 \leq x \leq 4$.
(b) $0 < x < 4$.

Let us look first for all critical points in the domain, that is, all values x for which $f'(x) = 0$. Here we have $f'(x) = 2$, which shows that there are no such values. Since f has no critical points, the principle of analysis mentioned above guarantees that the extreme values of f must occur at the endpoints of the domain. At the endpoint 0, the value of f is 3; at the endpoint 4, the value of f is 11. Therefore,

(a) if the domain is $0 \leq x \leq 4$, then $(4, 11)$ is the highest point on the curve f.
(b) if the domain is $0 < x < 4$, then the curve f contains no highest point.

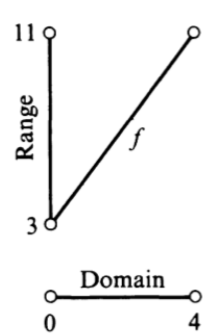
Note that, to draw the conclusions (a) and (b), *we did not have to draw a picture of the curve f!* The reader may wish to draw a picture anyway, to see better what is going on. The expression $2x + 3$ reveals f to be a linear function of slope 2:

(a) Domain: $0 \leq x \leq 4$
Range: $3 \leq y \leq 11$.



The highest point on the curve is $(4, 11)$. The greatest number in the range is 11; the least is 3.

(b) Domain: $0 < x < 4$
Range: $3 < y < 11$.



There is no highest (or lowest) point on the curve, because the range contains no greatest (or least) number.

§9. Summary

Here, in detail, are the steps that have been illustrated above.

Step 1. Algebraic formulation:

(a) See the problem in terms of variables. (The quantity to be optimized is one variable, say y, and you have to find a second variable, say x, on which y depends.)

(b) Write down an algebraic rule f, giving y in terms of x.

(c) Specify the domain of the function f.

Step 2. Geometric analysis:

(a) See the problem as one of finding the highest (or lowest) point on the curve f.

(b) Find the derivative f'. (And find f'' too, if it can be done without much trouble.)

(c) Find the critical points, if any, that lie in the domain of f. (That is, find all values of x in the domain of f that satisfy the equation $f'(x) = 0$.)

(d) Check what happens near the endpoints of the domain.

(e) Using the information of steps 2(c) and 2(d), find the desired highest (or lowest) point on the curve f.

[The second derivative may be helpful in steps 2(d) and 2(e).]

Step 3. Back to everyday life:

(a) Read the problem again, to determine exactly what was called for. (Was it the *first* or *second* coordinate, or *both*, of the *highest* or *lowest* point of the curve that you were seeking?)

(b) Give a direct answer to the question raised in the problem, by writing a complete, concise sentence.

Step 1(c) is easy to forget, and thus deserves emphasis. The domain must be specified; otherwise, steps 2(c) and 2(d) cannot be carried out. Step 3 is also easy to forget. In concentrating on step 2, you can lose sight of your goal and, as a consequence, do unnecessary work. When a problem takes a long time to work, it is a good idea to remind yourself now and then what you are after.

Here is another example to illustrate these steps.

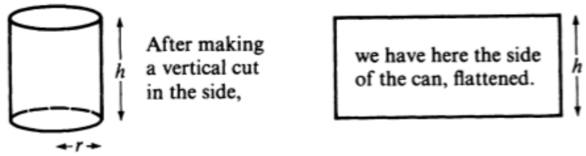
We'll do this on Thurs in recitation.

EXAMPLE 7. An ordinary metal can (shaped like a cylinder) is to be fashioned, using 54π square inches of metal. What choice of radius and height will maximize the volume of the can?

Here, we want to maximize the volume, so let V denote the volume, which is given in terms of the radius r and height h by the formula

$$\begin{aligned} V &= (\text{area of base})(\text{height}) \\ &= \pi r^2 h. \end{aligned} \tag{7}$$

The rule $V = \pi r^2 h$ gives V in terms of *two* variables. We need to get V in terms of only *one* variable, and this can be done, as follows, by finding a relation between r and h. The picture below shows that the area of the side of the can is given by $2\pi rh$:



The total amount of metal available, 54π square inches, must equal the amount in the side of the can, plus the amount in the circular top and bottom:

$$54\pi = 2\pi rh + 2\pi r^2.$$

This is a relation between r and h. It is easy to solve for h (the reader is asked to do it), and obtain

$$h = \frac{27 - r^2}{r}. \tag{8}$$

Putting equations (7) and (8) together gives

$$\begin{aligned} V &= \pi r^2 \left(\frac{27 - r^2}{r}\right) \\ &= \pi r(27 - r^2) \\ &= 27\pi r - \pi r^3, \end{aligned}$$

which expresses V in terms of r alone. The problem now is to find the value of r that yields the maximal volume V, where

$$V = 27\pi r - \pi r^3, \qquad 0 < r < \sqrt{27}.$$

[The radius r must be less than $\sqrt{27}$. *Reason*: The height h must be positive, so, by equation (8), $27 - r^2$ must be positive.]

r	V	V'
?		0
r	$27\pi r - \pi r^3$	$27\pi - 3\pi r^2$

Let us find critical points. The derivative is given by

$$V' = 27\pi - 3\pi r^2,$$

which is zero when (dividing by 3π)

$$\begin{aligned} 0 &= 9 - r^2, \\ r^2 &= 9, \\ r &= \pm 3. \end{aligned}$$

Since -3 is not in the domain, the only critical point is 3.

We now show that when r is 3, the volume V is maximal. This is easy to see, for the second derivative is given by

$$V'' = -6\pi r,$$

which is (obviously) negative *throughout the domain*. The curve is therefore always bending to its right (or frowning), and hence it must reach its highest point at the place where it has a horizontal tangent line. (At both endpoints of the domain, V tends to zero.)

To maximize the volume, the radius should be 3 inches, and the corresponding height, by equation (8), should be 6 inches. □

PROBLEM SET

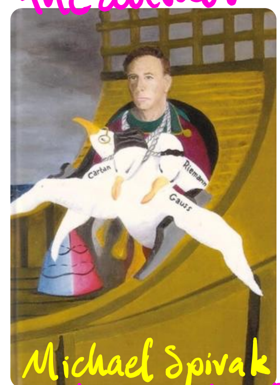
[Spi:15]

Try to answer enough problems on the following scale:
10 pts DECENT   20 pts EXCELLENT   30 pts OVERKILL

PROBLEMS

1. For each of the following functions, find the maximum and minimum values on the indicated intervals, by finding the points in the interval where the derivative is 0, and comparing the values at these points with the values at the end points.

the author



½ pt. each

(i) $f(x) = x^3 - x^2 - 8x + 1$ on $[-2, 2]$.

(ii) $f(x) = x^5 + x + 1$ on $[-1, 1]$.

(iii) $f(x) = 3x^4 - 8x^3 + 6x^2$ on $[-\frac{1}{2}, \frac{1}{2}]$.

(iv) $f(x) = \dfrac{1}{x^5 + x + 1}$ on $[-\frac{1}{2}, 1]$.

(v) $f(x) = \dfrac{x+1}{x^2+1}$ on $[-1, \frac{1}{2}]$.

(vi) $f(x) = \dfrac{x}{x^2-1}$ on $[0, 5]$.

0 pts. 1½. Bonus. What's the "albatross around my neck" a metaphor from?

3 pts. **2.** Now sketch the graph of each of the functions in Problem 1, and find all local maximum and minimum points.

½ pt. each **3.** Sketch the graphs of the following functions.

(i) $f(x) = x + \dfrac{1}{x}$.

(ii) $f(x) = x + \dfrac{3}{x^2}$.

(iii) $f(x) = \dfrac{x^2}{x^2-1}$.

(iv) $f(x) = \dfrac{1}{1+x^2}$.

5. For each of the following functions, find all local maximum and minimum points.

1 pt. (i) $f(x) = \begin{cases} x, & x \neq 3, 5, 7, 9 \\ 5, & x = 3 \\ -3, & x = 5 \\ 9, & x = 7 \\ 7, & x = 9. \end{cases}$

3 pts. (iii) $f(x) = \begin{cases} x, & x \text{ rational} \\ 0, & x \text{ irrational.} \end{cases}$

4 pts. **8.** A straight line is drawn from the point $(0, a)$ to the horizontal axis, and then back to $(1, b)$, as in Figure 23. Prove that the total length is shortest when the angles α and β are equal. (Naturally you must bring a function into the picture: express the length in terms of x, where $(x, 0)$ is the point on the horizontal axis. The dashed line in Figure 23 suggests an alternative geometric proof; in either case the problem can be solved without actually finding the point $(x, 0)$.)

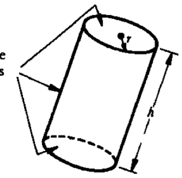


FIGURE 24

2 pts. **9.** Prove that of all rectangles with given perimeter, the square has the greatest area.

1 pt. **10.** Find, among all right circular cylinders of fixed volume V, the one with smallest surface area (counting the areas of the faces at top and bottom, as in Figure 24).

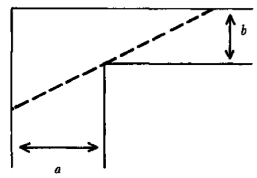


FIGURE 25

2 pts. **11.** A right triangle with hypotenuse of length a is rotated about one of its legs to generate a right circular cone. Find the greatest possible volume of such a cone.

6 pts. **12.** Two hallways, of widths a and b, meet at right angles (Figure 25). What is the greatest possible length of a ladder which can be carried horizontally around the corner?

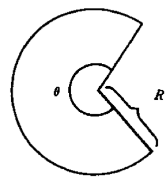


FIGURE 26

4 pts. **13.** A garden is to be designed in the shape of a circular sector (Figure 26), with radius R and central angle θ. The garden is to have a fixed area A. For what value of R and θ (in radians) will the length of the fencing around the perimeter be minimized?

1 pt. **14.** Show that the sum of a positive number and its reciprocal is at least 2.

6 pts. **62.** Let $f(x) = x^4 \sin^2 1/x$ for $x \neq 0$, and let $f(0) = 0$ (Figure 32).

(a) Prove that 0 is a local minimum point for f.
(b) Prove that $f'(0) = f''(0) = 0$.

This function thus provides another example to show that Theorem 6 cannot be improved. It also illustrates a subtlety about maxima and minima that often goes unnoticed: a function may not be increasing in any interval to the right of a local minimum point, nor decreasing in any interval to the left.

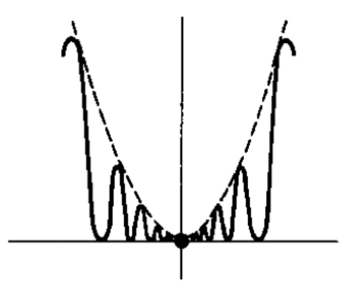


FIGURE 32

SOLUTIONS TO EXTRA PROBLEM SET

Disclaimer: There may be typos.

[Sp 15]

1. (ii) $f'(x) = 5x^4 + 1 = 0$ for no x;
$f(-1) = -1$, $f(1) = 3$;
maximum $= 3$, minimum $= -1$.

(iv) $f'(x) = -\dfrac{(5x^4+1)}{(x^5+x+1)^2} = 0$ for no x;
$f(-1/2) = 32/15$, $f(1) = 1/3$;
maximum $= 32/15$, minimum $= 1/3$.

(Notice that $g(x) = x^5 + x + 1$ is increasing, since $g'(x) = 5x^4 + 1 > 0$ for all x; since $g(-1/2) = 15/32 > 0$, this shows that $g(x) \neq 0$ for all x in $[-1/2, 1]$, so f is differentiable on $[-1/2, 1]$.)

(vi) f is not bounded above or below on $[0, 5]$.

(i) $0 = f'(x) = 3x^2 - 2x - 8$ for $x = 2$ and $x = -\frac{4}{3}$, both of which are in $[-2, 2]$;
$f(-2) = 5$, $f(2) = -11$, $f(-\frac{4}{3}) = \frac{203}{27}$;
maximum $= \frac{203}{27}$, minimum $= -11$.

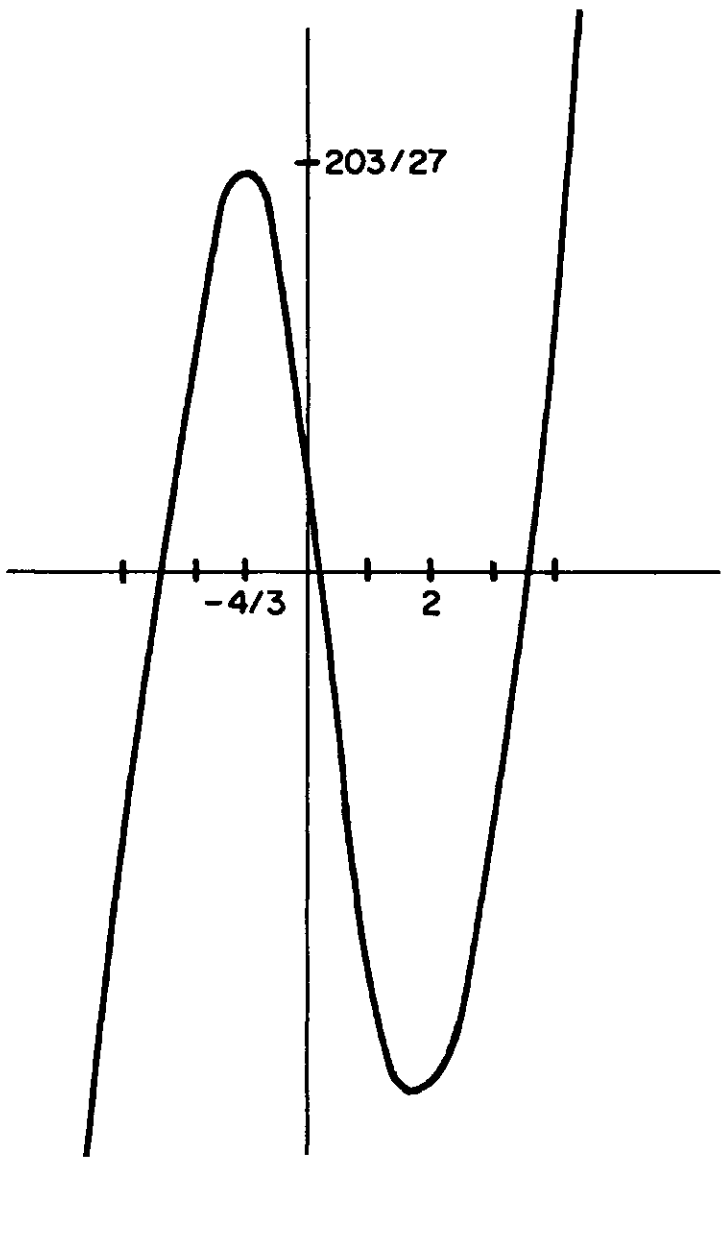
(iii) $0 = f'(x) = 12x^3 - 24x^2 + 12x = 12x(x^2 - 2x + 1)$ for $x = 0$ and $x = 1$, of which only 0 is in $[-\frac{1}{2}, \frac{1}{2}]$;
$f(-\frac{1}{2}) = \frac{43}{16}$, $f(\frac{1}{2}) = \frac{11}{16}$, $f(0) = 0$;
maximum $= \frac{43}{16}$, minimum $= 0$.

(v) $0 = f'(x) =$

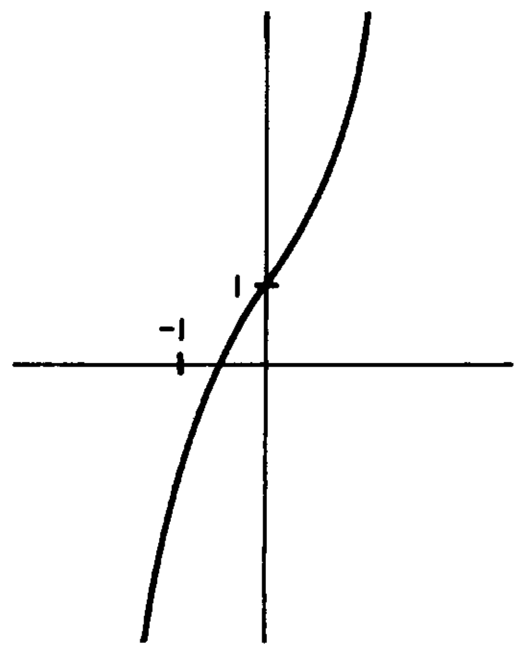
$$\frac{x^2 + 1 - (x+1)2x}{(x^2+1)^2} = \frac{1 - 2x - x^2}{(x^2+1)^2}$$

for $x = -1 + \sqrt{2}$ and $x = -1 - \sqrt{2}$, of which only $-1 + \sqrt{2}$ is in $[-1, \frac{1}{2}]$;
$f(-1) = 0$, $f(\frac{1}{2}) = \frac{6}{5}$, $f(-1 + \sqrt{2}) = (1 + \sqrt{2})/2$;
maximum $= (1 + \sqrt{2})/2$, minimum $= 0$.

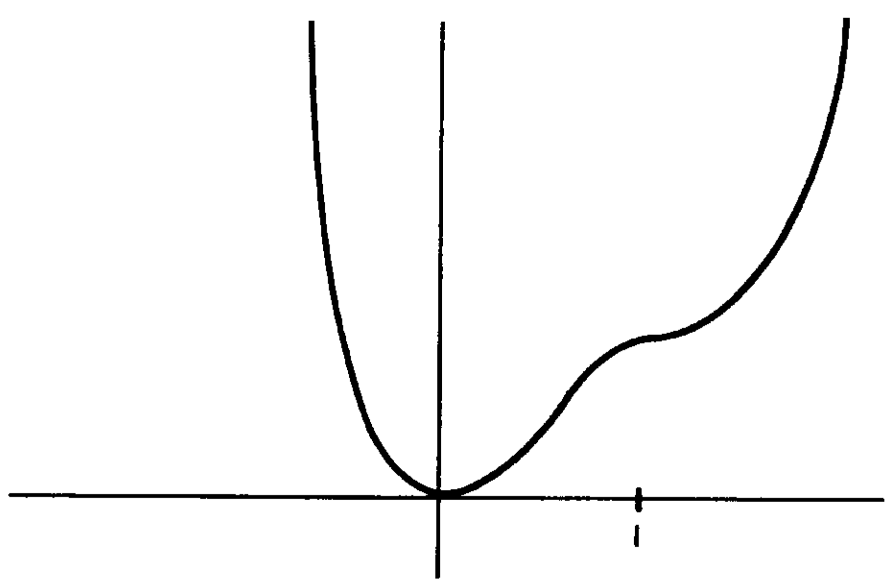
2. (i) $-4/3$ is a local maximum point, and 2 is a local minimum point.



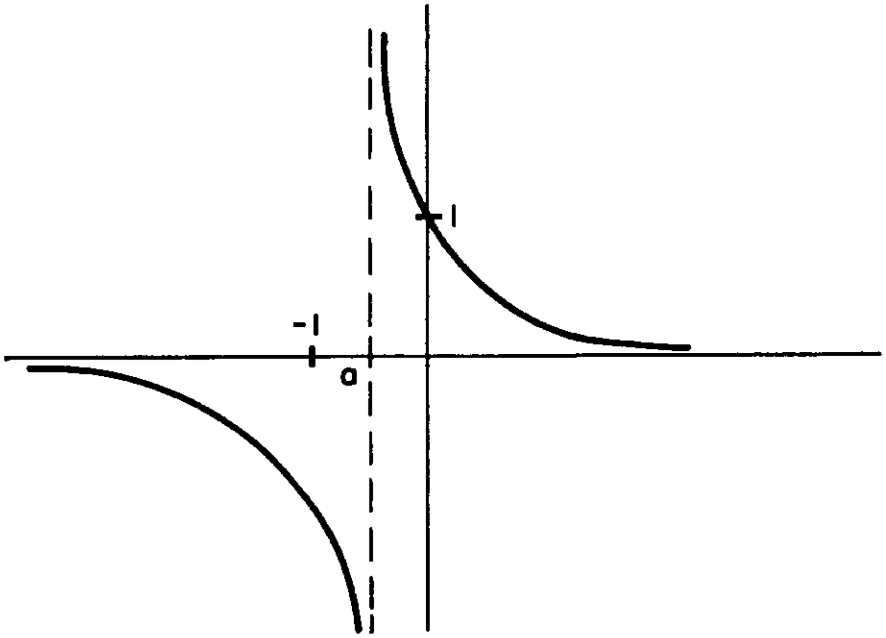
(ii) No local maximum or minimum points.



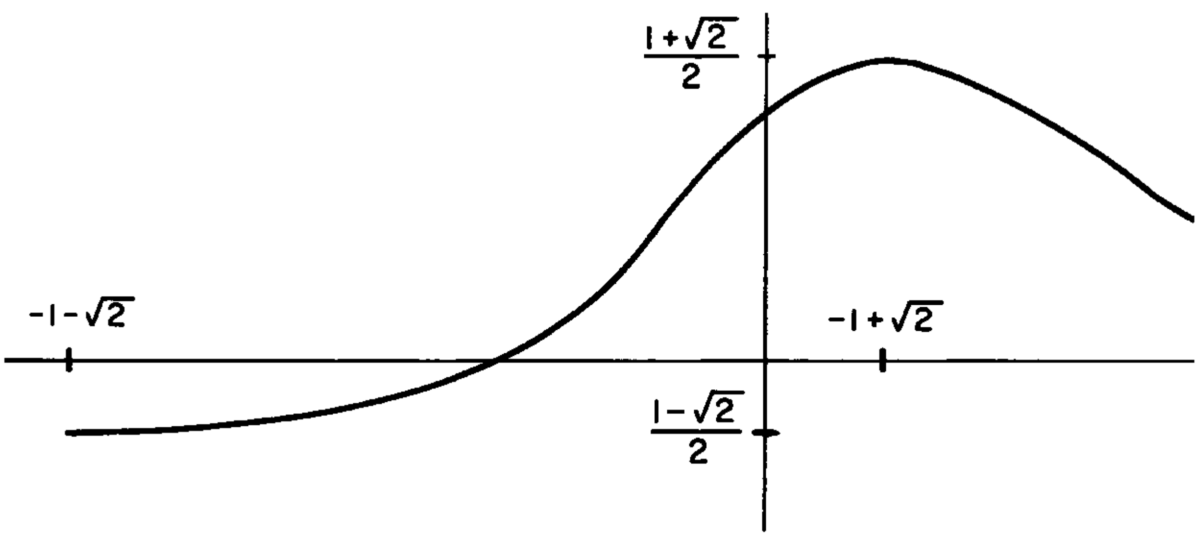
(iii) 0 is a local minimum point, and there are no local maximum points.



(iv) No local maximum or minimum points. In the figure below, a is the unique root of $x^5 + x + 1 = 0$.

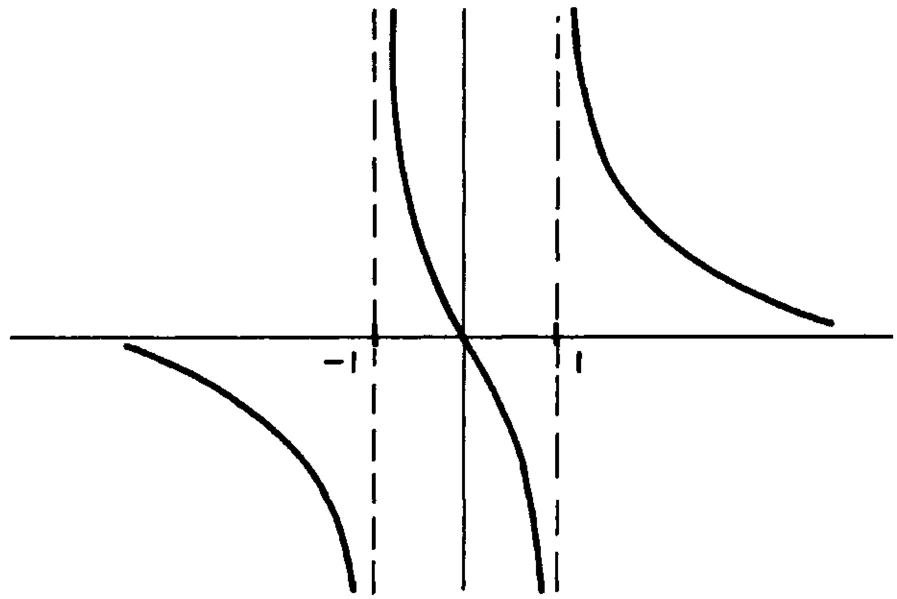


(v) $-1+\sqrt{2}$ is a local maximum point, and $-1-\sqrt{2}$ is a local minimum point.



(vi) No local maximum or minimum points, since

$$f'(x) = -\frac{(1+x^2)}{(x^2-1)^2} < 0 \qquad \text{for } x \neq \pm 1.$$

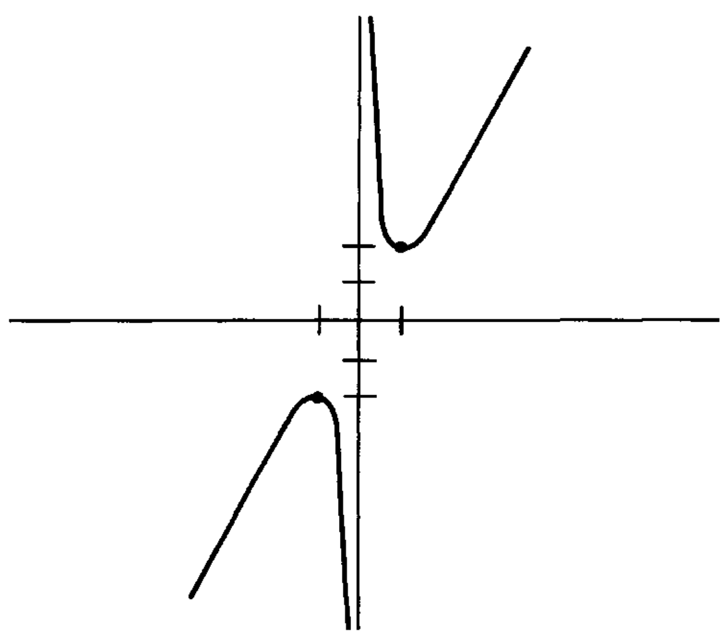


3. (i) f is odd;

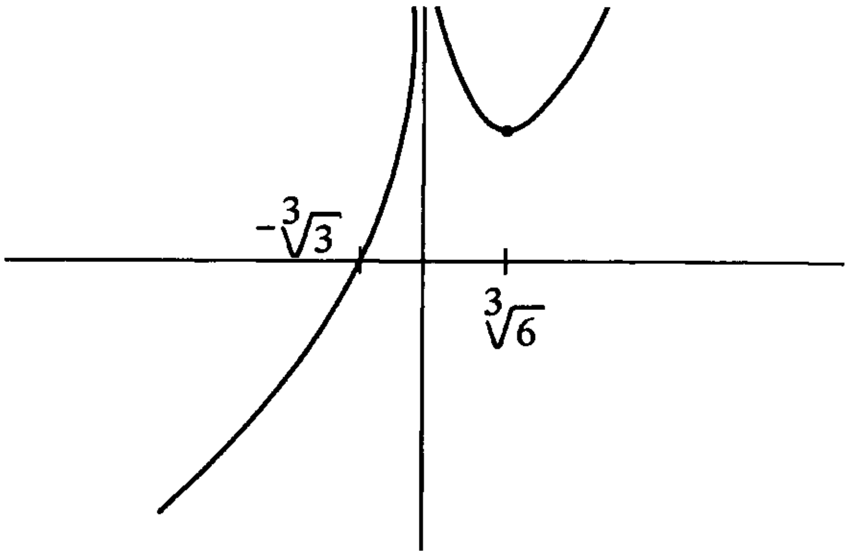
$$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2};$$

$f'(x) = 0$ for $x \neq \pm 1$, $f'(x) > 0$ for $|x| > 1$;

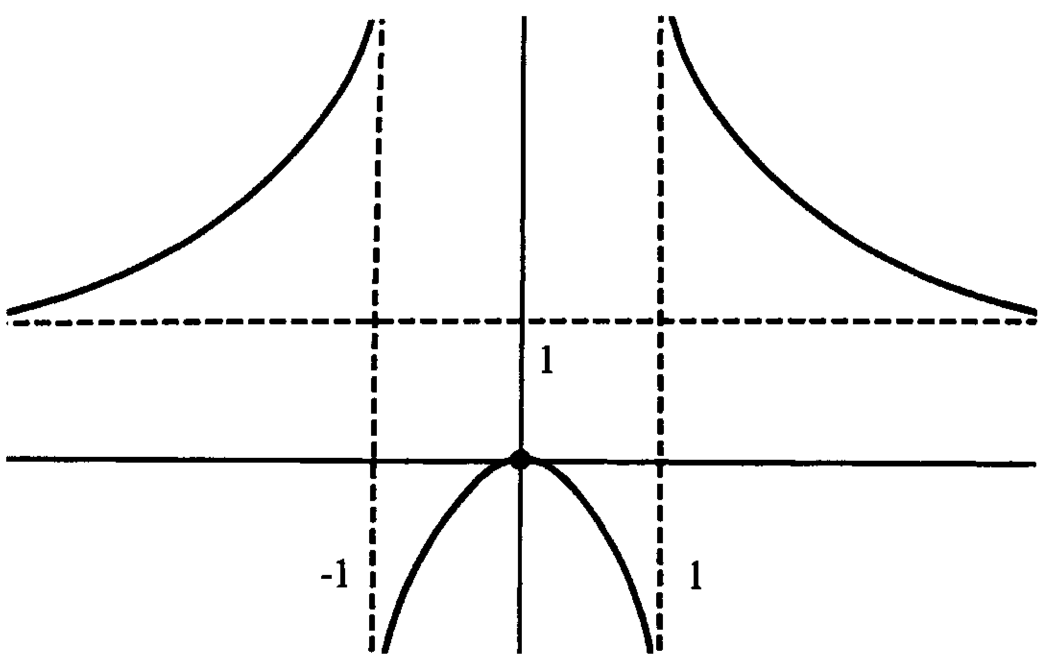
$f(1) = 2$, $f(-1) = -2$.



(ii) $f'(x) = 1 - \dfrac{6}{x^3} = \dfrac{x^3 - 6}{x^3}$;
$f'(x) = 0$ for $x = \sqrt[3]{6}$, $f'(x) > 0$ for $x > \sqrt[3]{6}$ and $x < 0$;
$f(x) = 0$ for $x = -\sqrt[3]{3}$.



(iii) f is even;
$f'(x) = \dfrac{2x(x^2-1) - 2xx^2}{(x^2-1)^2} = \dfrac{-2x}{(x^2-1)^2}$;
$f'(x) = 0$ for $x = 0$, $f'(x) < 0$ for $x > 0$, $f'(x) > 0$ otherwise;
$f(0) = 0$.

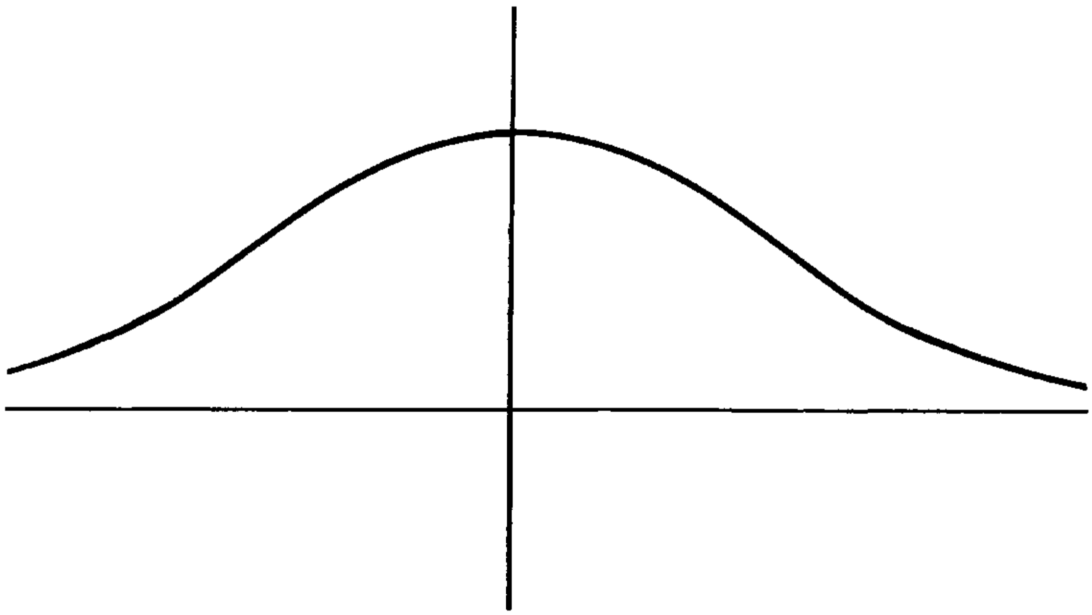


(iv) f is even, $f(x) > 0$ for all x;

$f'(x) = \dfrac{-2x}{(1+x^2)^2}$;

$f'(x) = 0$ for $x = 0$, $f'(x) > 0$ for $x < 0$, $f'(x) < 0$ otherwise;

$f(0) = 1$.



5. (i) 3 and 7 are local maximum points, and 5 and 9 are local minimum points.

(iii) All irrational $x > 0$ are local minimum points, and all irrational $x < 0$ are local maximum points.

Q] What's an irrational number?

A] Try watching https://youtu.be/5sKah3pJnHI.

He's in the video.

Also, James Grime will give a talk on Friday in the MARC.

8. If $f(x)$ is the total length of the path, then

$$f(x) = \sqrt{x^2 + a^2} + \sqrt{(1-x)^2 + b^2}.$$

Convince yourself.

Why does Spivak write "clearly" here?

The positive function f clearly has a minimum, since $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = \infty$, and f is differentiable everywhere, so the minimum occurs at a point x with $f'(x) = 0$. Now, $f'(x) = 0$ when

$$\frac{x}{\sqrt{x^2+a^2}} - \frac{(1-x)}{\sqrt{(1-x)^2+b^2}} = 0.$$

You need to draw right triangles, with angles α and β respectively.

This equation says that $\cos\alpha = \cos\beta$.

finish the argument ...

It is also possible to notice that $f(x)$ is equal to the sum of the lengths of the dashed line segment and the line segment from $(x, 0)$ to $(1, b)$. This is shortest when the two line segments lie along a line (because of Problem 4-9(b), if a rigorous reason is required); a little plane geometry shows that this happens when $\alpha = \beta$.

9. If x is the length of one side of a rectangle of perimeter P, then the length of the other side is $(P - 2x)/2$, so the area is

$$A(x) = \frac{x(P - 2x)}{2}.$$

So the rectangle with greatest area occurs when x is the maximum point for f on $(0, P/2)$. Since A is continuous on $[0, P/2]$, and $A(0) = A(P/2) = 0$, and $A(x) > 0$ for x in $(0, P/2)$, the maximum exists. Since A is differentiable on $(0, P/2)$, the minimum point x satisfies

$$\begin{aligned} 0 = A'(x) &= \frac{P - 2x}{2} - x \\ &= \frac{P - 4x}{2}, \end{aligned}$$

so $x = P/4$.

We've seen another way to solve this problem in [Pri79].

10. Let $S(r)$ be the surface area of the right circular cylinder of volume V with radius r. Since

$$V = \pi r^2 h \quad \text{where } h \text{ is the height,}$$

we have $h = V/\pi r^2$, so

$$\begin{aligned} S(r) &= 2\pi r^2 + 2\pi r h \\ &= 2\pi r^2 + \frac{2V}{r}. \end{aligned}$$

We want the minimum point of S on $(0, \infty)$; this exists, since $\lim\limits_{r\to 0} S(r) = \lim\limits_{r\to\infty} S(r) = \infty$. Since S is differentiable on $(0, \infty)$, the minimum point r satisfies

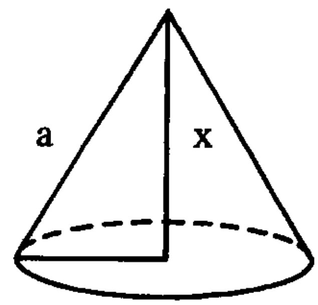
$$\begin{aligned} 0 = S'(r) &= 4\pi r - \frac{2V}{r^2} \\ &= \frac{4\pi r^3 - 2V}{r^2}, \end{aligned}$$

or

$$r = \sqrt[3]{\frac{V}{2\pi}}.$$

11. Let x be the height of the cone. The volume $V(x)$ is given by

$$V(x) = \frac{1}{3}x\pi\left(\sqrt{a^2 - x^2}\right)^2 = \frac{\pi}{3}(a^2x - x^3).$$



So the volume is greatest when

$$0 = V'(x) = \frac{\pi}{3}[a^2 - 3x^2],$$

or $x = a/\sqrt{3}$. For this x we have

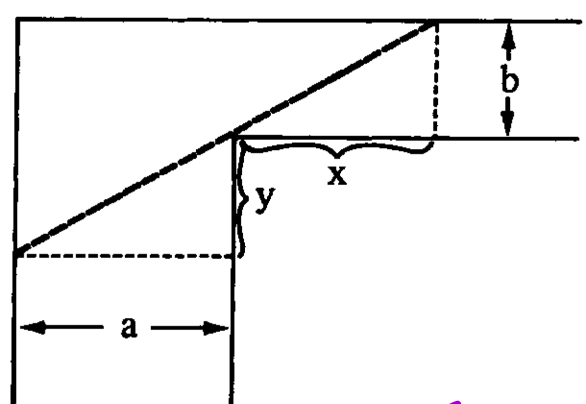
why?

$$V(x) = \frac{\pi}{3}\left(\frac{a^3}{\sqrt{3}} - \frac{a^3}{3\sqrt{3}}\right) = \frac{2\sqrt{3}a^3}{27}.$$

Tricky, but "just do it".

12. In the Figure below we have

$$\frac{b}{x} = \frac{y}{a},$$



so the length of the dashed line is

why?

$$\sqrt{b^2 + x^2} + \sqrt{a^2 + \frac{a^2b^2}{x^2}} = \sqrt{b^2 + x^2} + \frac{a}{x}\sqrt{x^2 + b^2} = \left(1 + \frac{a}{x}\right)\sqrt{x^2 + b^2}.$$

The maximum length of a ladder which can be carried horizontally around the corner is the *minimum* length of this dashed line. This occurs when

why?

$$0 = -\frac{a}{x^2}\sqrt{x^2+b^2} + \left(1+\frac{a}{x}\right)\frac{x}{\sqrt{x^2+b^2}}$$
$$= \left[-\frac{a}{x^2}(x^2+b^2) + x + a\right] \cdot \frac{1}{\sqrt{x^2+b^2}},$$

or

$$ax^2 + ab^2 = x^3 + ax^2,$$
$$x = a^{1/3}b^{2/3},$$

and the length is

$$\left(1+\frac{a^{2/3}}{b^{2/3}}\right)\sqrt{a^{2/3}b^{4/3}+b^2} = \left(b^{2/3}+a^{2/3}\right)\sqrt{\frac{a^{2/3}b^{4/3}+b^2}{b^{4/3}}}$$
$$= \left(b^{2/3}+a^{2/3}\right)^{3/2}.$$

13. If $R(\theta)$ is the appropriate value of R for given θ, we have

$$\frac{\theta}{2} \cdot R(\theta)^2 = A.$$

The perimeter for this θ will have value

$$P(\theta) = \theta R(\theta) + 2R(\theta)$$
$$= \sqrt{2A}(\theta+2) \cdot \theta^{-1/2}.$$

So the minimum occurs when

$$0 = P'(\theta) = \sqrt{2A}\left[\frac{1}{\theta^{1/2}} - \frac{\theta+2}{2\theta^{3/2}}\right]$$
$$= \sqrt{2A} \cdot \frac{\theta-2}{2\theta^{3/2}}$$

or $\theta = 2$ radians, and $R = \sqrt{A}$.

14. If

$$f(x) = x + \frac{1}{x} \qquad (x > 0)$$

then

$$f'(x) = 1 - \frac{1}{x^2},$$

which has the minimum value for $x = 1$, with $f(x) = 2$.

62. **(a)** 0 is actually a minimum on all of $\mathbf{R}$, since $f(0) = 0$ and $f(x) \geq 0$ for all x.

By def^{n} of the derivative.

You need $\sin^2(\frac{1}{h}) \leq 1$, so that $\lim_{h\to 0} \sin^2(\frac{1}{h}) \cdot h^3 = 0$.

(b)

$$f'(0) = \lim_{h\to 0} \frac{h^4 \sin^2(1/h)}{h} = 0,$$

Also $\lim_{h\to 0} \sin^2(\frac{1}{h}) \cdot h^2 = 0$.

and

$$f'(h) = 4h^3 \sin^2(1/h) - 2h^2 \sin(1/h)\cos(1/h) \quad \text{for } h \neq 0.$$

So

Because $|\sin(\frac{1}{h})\cos(\frac{1}{h})| \leq 1$, $\lim_{h\to 0} h \cdot \sin(\frac{1}{h})\cos(\frac{1}{h}) = 0$.

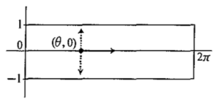
$$f''(0) = \lim_{h\to 0} \frac{4h^3 \sin^2(1/h) - 2h^2 \sin(1/h)\cos(1/h)}{h}$$
$$= 0.$$

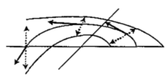
why? Because $\frac{d}{dh}[f(h)] = \frac{d}{dh}[h^4 \cdot \sin^2(\frac{1}{h})] = \overbrace{\underbrace{4h^3 \sin^2(\tfrac{1}{h})}_{\text{power rule}} + \underbrace{h^4\, 2\sin(\tfrac{1}{h})\cos(\tfrac{1}{h}) \cdot (\tfrac{-1}{h^2})}_{\text{chain rule}}}^{\text{product rule}}$.

There is another example where we can prove that no appropriate homeomorphism $T(M, i) \to M \times \mathbb{R}^2$ exists, without appealing to a hard theorem of topology. The map i will just be the inclusion $M \to \mathbb{R}^3$ where M is a Möbius strip, to be precise, the particular subset of $\mathbb{R}^3$ defined in Chapter 1—M is the image of the map $f\colon [0, 2\pi] \times (-1, 1) \to \mathbb{R}^3$ defined by

$$f(\theta, t) = \left(2\cos\theta + t\cos\tfrac{\theta}{2}\cos\theta,\ 2\sin\theta + t\cos\tfrac{\theta}{2}\sin\theta,\ t\sin\tfrac{\theta}{2}\right).$$

At each point $p = (2\cos\theta, 2\sin\theta, 0)$ of M, the vector





$$v_p = (-2\sin\theta, 2\cos\theta, 0)_p = f_*\big((1,0)_{(\theta,0)}\big)$$

is a tangent vector. The same is true for all multiples of $f_*\big((0,1)_{(\theta,0)}\big)$, shown as dashed arrows in the picture. Notice that

$$\begin{aligned} f_*\big((0,1)_{(0,0)}\big) &= [Df(0,0)(0,1)]_{(2,0,0)} \\ &= \left[\frac{\partial f}{\partial t}(0,0)\right]_{(2,0,0)} = (1,0,0)_{(2,0,0)}, \end{aligned}$$

while

$$f_*\big((0,1)_{(2\pi,0)}\big) = \left[\frac{\partial f}{\partial t}(2\pi,0)\right]_{(2,0,0)} = (-1,0,0)_{(2,0,0)}.$$

This means that we can never pick non-zero dashed vectors *continuously* on the set of all points $(2\cos\theta, 2\sin\theta, 0)$: If we could, then each vector would be

$$f_*\big((0,\lambda(\theta))_{(\theta,0)}\big)$$

for some continuous function $\lambda\colon [0, 2\pi] \to \mathbb{R}$. This function would have to be non-zero everywhere and also satisfy $\lambda(2\pi) = -\lambda(0)$, which it can't (by an easy theorem of topology). The impossibility of choosing non-zero dashed vectors continuously clearly shows that there is no way to map $T(M, i)$, fibre by fibre,

As examples, here are three of the fundamental results of calculus; they are called – by me, at least – the three **Interval Theorems**, because they all concern an arbitrary continuous function defined on a closed, bounded interval.

THEOREM 1.1. *(Intermediate Value Theorem) Let $f : [a, b] \to \mathbb{R}$ be a continuous function defined on a closed, bounded interval. Suppose that $f(a) < 0$ and $f(b) > 0$. Then there exists c with $a < c < b$ such that $f(c) = 0$.*

THEOREM 1.2. *(Extreme Value Theorem) Let $f : [a, b] \to \mathbb{R}$ be a continuous function defined on a closed, bonuded interval. Then f is bounded and assumes its maximum and minimum values: there are numbers $m \leq M$ such that*
a) For all $x \in [a, b]$, $m \leq f(x) \leq M$.
b) There exists at least one $x \in [a, b]$ such that $f(x) = m$.
c) There exists at least one $x \in [a, b]$ such that $f(x) = M$.

THEOREM 1.3. *(Uniform Continuity and Integrability) Let $f : [a, b] \to \mathbb{R}$ be a continuous function defined on a closed, bounded interval. Then:*
a) f is uniformly continuous.[1]
b) f is integrable: $\int_a^b f$ exists and is finite.

Except for the part about uniform continuity, these theorems are familiar results from freshman calculus. **Their proofs, however, are not.** Most freshman calculus texts like to give at least *some* proofs, so it is often the case that these three theorems are used to prove even more famous theorems in the course, e.g. the

1.3. Why do we not do calculus on $\mathbb{Q}$?

To paraphrase the title question, why do we *want* to use $\mathbb{R}$ to do calculus? Is there something stopping us from doing calculus over, say, $\mathbb{Q}$?

The answer to the second question is **no**: we can define limits, continuity, derivatives and so forth for functions $f : \mathbb{Q} \to \mathbb{Q}$ *exactly* as is done for real functions. The most routine results carry over with no change: it is still true, for instance, that sums and products of continuous functions are continuous. However most of the big theorems – especially, the **Interval Theorems** – become false over $\mathbb{Q}$.

For $a, b \in \mathbb{Q}$, let $[a, b]_{\mathbb{Q}} = \{x \in \mathbb{Q} \mid a \leq x \leq b\}$.

EXAMPLE 1.1. *Consider the function $f : [0, 2]_{\mathbb{Q}} \to \mathbb{Q}$ given by $f(x) = -1$ if $x^2 < 2$ and $f(x) = 1$ if $x^2 > 2$. Note that we do not need to define $f(x)$ at $x = \pm\sqrt{2}$, because by the result of the previous section these are not rational numbers. Then f is continuous – in fact it is differentiable and has identically zero derivative. But $f(0) = -1 < 0$, $f(2) = 1 > 0$, and there is no $c \in [0, 2]_{\mathbb{Q}}$ such that $f(c) = 0$. Thus the Intermediate Value Theorem fails over $\mathbb{Q}$.*

EXAMPLE 1.2. *Consider the function: $f : [0, 2]_{\mathbb{Q}} \to \mathbb{Q}$ given by $f(x) = \frac{1}{x^2-2}$. Again, this function is well-defined at all points of $[0, 2]_{\mathbb{Q}}$ because $\sqrt{2}$ is not a rational number. It is also a continuous function. However it is* not *bounded above: by taking rational numbers which are arbitrarily close to $\sqrt{2}$, $x^2 - 2$ becomes arbitrarily small and thus $f(x)$ becomes arbitarily large.*[3] *In particular, f certainly does not attain a maximum value. Thus the Extreme Value Theorem fails over $\mathbb{Q}$.*

4.2. Proof of the Mean Value Theorem.

We will deduce the Mean Value Theorem from the (as yet unproven) Extreme Value Theorem. However, it is convenient to first establish a special case.

THEOREM 5.17. *(Rolle's Theorem) Let $f : [a, b] \to \mathbb{R}$. We suppose:*
(i) f is continuous on $[a, b]$.
(ii) f is differentiable on (a, b).
(iii) $f(a) = f(b)$.
Then there exists c with $a < c < b$ and $f'(c) = 0$.

So we need to back up a bit and give a definition of $\int_a^b f$. As you probably know, the general idea is to construe $\int_a^b f$ as the result of some kind of limiting process, wherein we divide $[a, b]$ into subintervals and take the sum of the areas of certain rectangles which approximate the function f at various points of the interval (**Riemann sums**). As usual in freshman calculus, reasonably careful definitions appear in the textbook somewhere, but with so little context and development that (almost) no actual freshman calculus student can really appreciate them.

But wait! Before plunging into the details of this limiting process, let's take a more **axiomatic approach**: given that we want $\int_a^b f$ to represent the area under $y = f(x)$, what properties should it satisfy? Here are some reasonable ones.

(I1) If $f = C$ is a constant function, then $\int_a^b C = C(b - a)$.
(I2) If $f_1(x) \leq f_2(x)$ for all $x \in [a, b]$, then $\int_a^b f_1 \leq \int_a^b f_2$.
(I3) If $a \leq c \leq b$, then $\int_a^b f = \int_a^c f + \int_b^c f$.

Exercise 1.1: Show (I1) implies: for any $f : [a, b] \to \mathbb{R}$ and any $c \in [a, b]$, $\int_c^c f = 0$.

It turns out that these three axioms already imply many of the other properties we want an integral to have. Even more, there is essentially only one way to define $\int_a^b f$ so as to satisfy (I1) through (I3).

If this business of "integrable functions" seems abstruse, then on the first pass just imagine that $\mathcal{R}[a, b]$ is precisely the set of all continuous functions $f : [a, b] \to \mathbb{R}$.

Now we have the following extremely important result.

THEOREM 8.1. *(Fundamental Theorem of Calculus) Let $f \in \mathcal{R}[a, b]$ be any integrable function. For $x \in [a, b]$, define $\mathcal{F}(x) = \int_a^x f$. Then:*
a) The function $\mathcal{F} : [a, b] \to \mathbb{R}$ is continuous at every $c \in [a, b]$.
b) If f is continuous at $c \in [a, b]$, then $\mathcal{F}$ is differentiable at c, and $\mathcal{F}'(c) = f(c)$.
c) If f is continuous and F is any antiderivative of f – i.e., a function $F : [a, b] \to \mathbb{R}$ such that $F'(x) = f(x)$ for all $x \in [a, b]$, then $\int_a^b f = F(b) - F(a)$.

integrable function $f : [0, 1] \to \mathbb{R}$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f.$$

Now observe that our limit can be recognized as a special case of this:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{k^2 + n^2} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{n^2}{k^2 + n^2} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{(\frac{k}{n})^2 + 1} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right),$$

where $f(x) = \frac{1}{x^2+1}$. Thus the limit is

$$\int_0^1 \frac{dx}{x^2 + 1} = \arctan 1 - \arctan 0 = \frac{\pi}{2}.$$

3. Approximate Integration

Despite the emphasis on integration (more precisely, antidifferentiation!) techniques in a typical freshman calculus class, it is a dirty secret of the trade that in practice many functions you wish to integrate do not have an "elementary" antiderivative, i.e., one that can be written in (finitely many) terms of the elementary functions one learns about in precalculus mathematics. Thus one wants methods for evaluating definite integrals *other* than the Fundamental Theorem of Calculus. In practice it would often be sufficient to *approximate* $\int_a^b f$ rather than know it exactly.[2]

THEOREM 9.4. *(Endpoint Approximation Theorem) Let $f : [a, b] \to \mathbb{R}$ be differentiable with bounded derivative: there is $M \geq 0$ such that $|f'(x)| \leq M$ for all $x \in [a, b]$. For $n \in \mathbb{Z}^+$, let $L_n(f)$ be the* ***left endpoint Riemann sum*** *obtained by dividing $[a, b]$ into n equally spaced subintervals: thus for $0 \leq i \leq n - 1$, $x_i{}^* = a + i\left(\frac{b-a}{n}\right)$ and $L_n(f) = \sum_{i=0}^{n-1} f(x_i{}^*)\frac{b-a}{n}$. Then*

$$\left| \int_a^b f - L_n(f) \right| \leq \left(\frac{(b-a)^2 M}{2} \right) \frac{1}{n}$$

21. Prove that if h is continuous, f and g are differentiable, and

$$F(x) = \int_{f(x)}^{g(x)} h(t)\,dt,$$

then $F'(x) = h(g(x)) \cdot g'(x) - h(f(x)) \cdot f'(x)$. Hint: Try to reduce this to the two cases you can already handle, with a constant either as the lower or the upper limit of integration.

***15.** A function f is **periodic**, with **period** a, if $f(x+a) = f(x)$ for all x.

(a) If f is periodic with period a and integrable on $[0, a]$, show that

$$\int_0^a f = \int_b^{b+a} f \quad \text{for all } b.$$

(b) Find a function f such that f is not periodic, but f' is. Hint: Choose a periodic g for which it can be guaranteed that $f(x) = \int_0^x g$ is not periodic.

(c) Suppose that f' is periodic with period a. Prove that f is periodic if and only if $f(a) = f(0)$.

1. According to Newton's law of cooling, the rate of change of the temperature of an object is proportional to the difference in temperature between the object and its surroundings. A forensic scientist enters a crime scene at 5:00 pm and discovers a cup of tea at temperature 40°C. At 5:30 pm its temperature is only 30°C. Giving all details of the mathematical methodology employed and assumptions made, estimate the time at which the tea was made.

2. Determine the half-life of Thorium-234 if a sample of 5 grams is reduced to 4 grams in one week. What amount of Thorium is left after three months?

Section 1. These integrals are fundamental. If you can't do these then you should consult a text book, colleague or supervisor and learn the answers by rote.

1) 1	3) $\frac{1}{x}$	5) $\sin x$	7) $\sec^2 x$
2) $2x$	4) e^x	6) $\cos x$	8) $\sec x \tan x$

Section 2. A little pre- or post-processing required.

9) x^2	13) $\sin 3x$	17) $(x+2)(x+3)$
10) e^{5x}	14) $\sqrt{x^5}$	18) e^{5x+3}
11) $\sqrt{x}$	15) $(1+x)^{\frac{1}{4}}$	19) $(x^3 - x^5)/\sqrt{x}$
12) x^{-1}	16) x^{1066}	20) $\frac{2+x}{(1+x)^2}$ [Hint: $2 + x = 1 + x + 1$]

Section 3. An obvious theme (I hope!)

21) $2x(3+x^2)$	24) $xe^{x^2/2}$	27) $\tan x = \frac{\sin\ x}{\cos\ x}$
22) $2x \sin\ x^2$	25) $x(1-x^2)^{1/2}$	28) $\cot x$
23) $\cos x \sin^3 x$	26) $\sec^2 x \tan x$	