DERIVATIVES II (SOLUTIONS)

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True-false questions from [1]; multiple choices questions from [2].

1. If $f(x)$ is a differentiable function, then $f(x)$ is a continuous function.
   - TRUE
   - FALSE

   Answer: TRUE

   Explanation: The derivative of $f(x)$ at $x = a$ is defined as $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ or equivalently $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$. Assume $f(x)$ is differentiable at $x = a$. Then

   $$\lim_{x\to a}(f(x) - f(a)) = \left(\lim_{x\to a}(x-a)\right)\left(\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\right) = \lim_{x\to a}(x-a)\cdot f'(a) = 0$$

   (verify the last equality on your own). We deduce that $\lim_{x\to a}(f(x) - f(a)) = 0$, hence $\lim_{x\to a} f(x) = f(a)$, hence f is continuous at the point a.

2. If g is differentiable at $x = a$ and f is differentiable at $x = g(a)$, then $f \circ g$ is differentiable at $x = a$.
   - TRUE
   - FALSE

   Answer: TRUE

   Explanation: Apply the chain rule! Let $h(x) = f(g(x))$, then $h'(x) = f'(g(x))g'(x)$. Can we evaluate $h'(a)$? Yes, since g is differentiable at $x = a$, the derivative $g'(a)$ exists. Moreover, f is differentiable at $x = g(a)$, so $f'(g(a))$ also exists. We conclude that $h'(a) = f'(g(a))g'(a)$ exists, and so $h = f \circ g$ is differentiable at $x = a$.

3. If $f''(c) = 0$, then $f(x)$ has an inflection point at $x = c$.
   - TRUE
   - FALSE

   Answer: FALSE

   Explanation: Recall that an inflection point is defined as a point at which a function changes concavity. That $f''(c) = 0$ is a necessary — yet not sufficient — condition for to have an inflection point at $(c, f(c))$. Do consider some counter examples here. Take the polynomial $f(x) = k$ for some fixed constant $k \in \mathbf{R}$. Then $f''(x) = 0$ for all x, but none of these points correspond to points of inflection, for f is constant.

4. True or false: The following function is differentiable at $x = 0$, $f(x) := \begin{cases} x+1, & x \le 0 \\ 1-x^2, & x > 0. \end{cases}$
   - TRUE
   - FALSE

   Answer: FALSE

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Repo: `https://github.com/coltongrainger/pro19ta`.

Explanation: Check for continuity with left and right sided limits:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(1-x^2) = 1 = \lim_{x\to 0^-}(x+1) = \lim_{x\to 0^-} f(x).$$

So $\lim_{x\to 0} f(x) = 1 = f(0)$. We've shown that f is continuous at $x = 0$. Is f differentiable at $x = 0$? Consider the limit of difference quotient $\frac{f(h)-f(0)}{h}$ as $h \to 0$ from the left and right.

$$\text{from the left} \quad \lim_{h\to 0^-} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^-} \frac{h+1-1}{h} = 1;$$

$$\text{from the right} \quad \lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^+} \frac{1-h^2-1}{h} = 0.$$

The left and right limits approach 1 and 0, respectively. Graphically, we'd have a sharp cusp at $x = 0$ Therefore f is not differentiable at $x = 0$. (Note the f behaves linearly when $x \le 0$ and quadratically when $x \ge 0$.)

5. Suppose f is a function defined on a closed interval $[a, c]$. Suppose that the left-hand derivative of f at c exists and equals ℓ. Which of the following implications is **true in general**?

   (A) If $f(x) < f(c)$ for all $a \le x < c$, then $\ell < 0$.
   (B) If $f(x) \le f(c)$ for all $a \le x < c$, then $\ell \le 0$.
   (C) If $f(x) < f(c)$ for all $a \le x < c$, then $\ell > 0$.
   (D) If $f(x) \le f(c)$ for all $a \le x < c$, then $\ell \ge 0$.
   (E) None of the above is true in general.

   Answer: Option (D)

   Explanation: If $f(x) \le f(c)$ for all $a \le x < c$, then all difference quotients from the left are nonnegative. The limiting value, which is the left-hand derivative, is thus also nonnegative.

   The other choices: Options (A) and (B) predict the wrong sign. Option (C) is incorrect because even though the difference quotients are all strictly positive, their limiting value could be 0. For instance, $\sin x$ on $[0, \pi/2]$ or x^3 on $[-1, 0]$. It is likely that the people who chose option (B) made a sign computation error.

6. Suppose f and g are increasing functions from $\mathbf{R}$ to $\mathbf{R}$. Which of the following functions is *not* guaranteed to be an increasing function from $\mathbf{R}$ to $\mathbf{R}$?

   (A) $f + g$
   (B) $f \cdot g$
   (C) $f \circ g$
   (D) All of the above, i.e., none of them is guaranteed to be increasing.
   (E) None of the above, i.e., they are all guaranteed to be increasing.

   Answer: Option (B)

   Explanation: The problem with option (B) arises when one or both functions take negative values. For instance, consider the case $f(x) := x$ and $g(x) := x$. Both are increasing functions on all of $\mathbf{R}$. However, the pointwise product is the function $x \mapsto x^2$, which is a decreasing function for negative x. Formally, the issue is that we cannot multiply inequalities of the form $A < B$ and $C < D$ unless we are guaranteed to be working with positive numbers.

References

[1] L. Roberson, "Math 1300 Exam Materials," CU Boulder, Oct-2018 [Online]. Available: `https://math.colorado.edu/math1300/1300exams.html`

[2] V. Naik, "Math 152 Course Notes" [Online]. Available: `https://vipulnaik.com/math-152/`