SOLUTIONS TO ATTENDANCE QUIZ (WEEK 9)

COLTON GRAINGER (MATH 1300)

This was an **ungraded** quiz given on 2019-03-07. I didn't collect statistics, but here's my impression (on a 0 to 4 scale with 4/4 high).

problem number	actual class performance	problem difficulty
1	1/4	1/4
2	0/4	4/4
3	1/4	2/4
4	1/4	3/4

1. Definitions

Let f be a function and A a set of numbers contained in the domain of f. (If you like, you may assume A is an interval of real numbers, like (a, b) or $(-\infty, \infty)$ or $[a, b]$. However, these definitions hold for *any set* of numbers A in the domain of f.)

1. A point x_{max} in A is a **maximum point** for f on A if
$$f(x_{\text{max}}) \geq f(x) \quad \text{for every } x \text{ in } A.$$
   i. The number $f(x_{\text{max}})$ is called the **maximum value** of f on A.
   ii. We also say that f "has its maximum value on A at x_{max}".
2. A particular number y_{above} is a **bound** for f on A if
$$y_{\text{above}} \geq |f(x)| \quad \text{for every } x \text{ in } A.$$
3. A point x_{local} in A is a **local maximum point** for f on A if:

   There is a positive number $\delta > 0$ such that x_{local} is a maximum point for f on the set
$$\{\text{points in } A \text{ whose distance to } x_{\text{local}} \text{ is less than } \delta\}.$$

2. Short answer

1. What part(s) of which definition(s) above should be modified to instead define a **minimum point** and the **minimum value** for a function f on A?

   Correct answer. In definition 1, amend "max" to "min", "maximum" to "minimum", and "$\geq$" to "$\leq$".

   Other answers. Please actually *write* "$\leq$". It is *not* sufficient to say that the inequalities "flip". For example, is the "flip" of the inequality
$$\rho \geq 0$$
   the inequality
$$\rho \leq 0 \quad \text{or} \quad \rho < 0?$$

Date: 2019-03-14.

I empathize[1] with the casual attitude, but it is worth knowing why a *strict* inequality differs from a general inequality.

2. List five numbers, in decreasing order, that bound the function $\arctan\colon \mathbf{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$.

   Example answer. Let's convince ourselves that the absolute value $|\arctan x|$ is strictly less than $\pi/2$ for all real numbers x.

   - Well, consider some angle $-\pi/2 < \theta < \pi/2$ and suppose that we let this angle θ gradually increase and decrease.
   - Then $\tan\theta$ will gradually range between $-\infty$ and ∞, because *tangent measures the slope of a ray* that makes an angle of θ with the x-axis.
   - Because arctan is the inverse function to tan, if we let the point $-\infty < x < \infty$ gradually increase and decrease, then $\arctan x$ should gradually increase and decrease from just above $-\pi/2$ to just below $\pi/2$.

   Taking absolute values, we know $|\arctan x| < \pi/2$. Then *any number* greater than or equal to $\pi/2$ is a bound for the arctan function. Here's a list of such numbers in decreasing order.

   - $\frac{\pi}{2} + 5$
   - $\frac{\pi}{2} + 4$
   - ...
   - $\frac{\pi}{2} + 1$
   - $\frac{\pi}{2}$.

3. Suppose the function

$$f(x) := \begin{cases} -x^2 & \text{if } -1 < x < 1 \\ 0 & \text{if } x \leq -1 \text{ or } x \geq 1 \end{cases}$$

   represents force (N) in the positive x-direction on the head of a pendulum when the pendulum is displaced by x (cm) from equilibrium. If there is a local maximum point for f on $\mathbf{R}$, find it; else, find a local minimum point for f on $\mathbf{R}$.

   Correct answer. $x = 0$ gives a local maximum point, and in fact *any* point $x \leq -1$ or $1 \leq x$ is a local maximum point.

   This problem was rendered trivial by a **typo**. The restoring force on a pendulum for small displacements is *not* $-x^2$, but actually (where $k > 0$)

$$g(x) := \begin{cases} -kx & \text{if } -1 < x < 1 \\ 0 & \text{if } x \leq -1 \text{ or } 1 \leq x \end{cases}$$

   The function $g(x)$ has neither a local minimum point nor a local maximum point.

4. Suppose f and g are increasing functions from $\mathbf{R}$ to $\mathbf{R}$. (Recall that a function g is **increasing** on an interval if $g(a) < g(b)$ whenever a and b are two numbers in the interval with $a < b$.) Which of the following functions is *not* guaranteed to be an increasing function from $\mathbf{R}$ to $\mathbf{R}$?

   (A) $f + g$
   (B) $f \cdot g$
   (C) $f \circ g$
   (D) All of the above, i.e., none of them is guaranteed to be increasing.
   (E) None of the above, i.e., they are all guaranteed to be increasing.

[1] "Modern man, writes Foucault, was born in a welter of regulations: meticulous rules and subrules, fussy inspections, 'the supervision of the smallest fragment of life [...] in the context of the school, the barracks, the hospital or the workshop.'" [1, Ch. 9]

Correct answer. (B)

Explanation.[2] The problem with option (B) arises when one or both functions take negative values. For instance, consider the case $f(x) := x$ and $g(x) := x$. Both are increasing functions on all of $\mathbf{R}$. However, the pointwise product is the function $x \mapsto x^2$, which is a decreasing function for negative x. Formally, the issue is that we cannot multiply inequalities of the form $A < B$ and $C < D$ unless we are guaranteed to be working with positive numbers.

3. References

As usual, question 4 is from Naik [2] and the definitions are from Spivak [3].

[1] J. G. Merquior, *Foucault*. University of California Press, 1987.

[2] V. Naik, "Math 152 Course Notes," 2012.

[3] M. Spivak, *Calculus*, 3rd ed. Publish or Perish, Inc., 1994.

[2]From Vipul Naik's Math 152 quiz solutions.