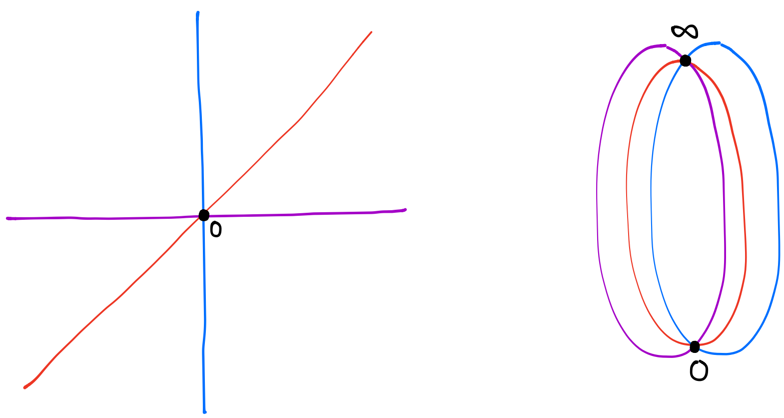
January 2010 Qualifying Exam

1 Let X be the space obtained from $\mathbb{R}^3$ by removing the three coordinate axes. Calculate $\pi_1(X)$ and $H_*(X)$.

First notice that for a line $L \subseteq \mathbb{R}^3$, we have $\mathbb{R}^3 - L \simeq S^3 - S^1$ (the circle S^1 is identified with $L \cup \{\infty\} \subseteq \mathbb{R}^3 \cup \{\infty\} \simeq S^3$). So removing the axes from $\mathbb{R}^3$ is the same as removing three circles (which intersect at two points) from S^3, as shown:

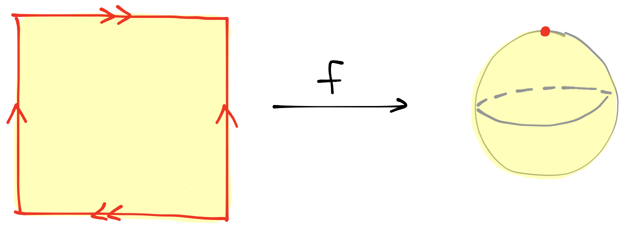


These circles ar homotopic to the wedge of 5 circles, so X is homotopic to $S^3 - \vee^5 S^1$. By taking the wedge point of $\vee^5 S^1$ to be ∞ and using the previous argument, we see that removing the wedge of 5 circles from S^3 is the same as removing 5 parallel lines from $\mathbb{R}^3$. By collapsing $\mathbb{R}^3$ to a plane, we see that X is homotopic to $\mathbb{R}^2$ with 5 points removed, which is homotopic to the wedge of 5 circles.

Thus we can conclude that $\pi_1(X) = F_5$, the free group on 5 generators, and $H_1(X) = \mathbb{Z}^5$.

2 True or False: A continuous map $f : X \to Y$ which induces trivial maps f_* in the (reduced) $\mathbb{Z}$-homology is nullhomotopic. Explain your answer.

False. Let K be the Klein bottle and consider the quotient map $f : K \to S^2$ by the 1-skeleton as shown:



Since K is a nonorientable closed 2-manifold, we have $H_n(K) = 0$ for $n \geq 2$, and we also know $H_1(S^2) = 0$, so $f_* : \tilde{H}_n(K) \to \tilde{H}_n(S^2)$ must be trivial for all n.

But f cannot be nullhomotopic because it induces a nontrivial map in the $\mathbb{Z}_2$ homology. This can be seen as follows: Let C denote the single 2-cell in the CW structure of K above, and let C' be the 2-cell in S^2. In the cellular chain complex, we have $C_2(K; \mathbb{Z}_2) = \langle C \rangle$ and $C_2(S^2; \mathbb{Z}_2) = \langle C' \rangle$, so $f_* : C \mapsto C'$ in the induced chain map.

Since K and S^2 are closed 2-manifolds, we know $H_2(K; \mathbb{Z}_2) = H_2(S^2; \mathbb{Z}_2) = \mathbb{Z}_2$, so $[C]$ and $[C']$ are the generators, and $f_* : [C] \mapsto [C']$ in the induced map on homology, so is nontrivial.

4 Is $S^2 \times S^4$ homotopy equivalent to $\mathbb{CP}^3$? Explain.

No. If $f : S^2 \times S^4 \to \mathbb{CP}^3$ is a homotopy equivalence, then it must induce an isomorphism in the cohomology rings.

<u>Kunneth Formula</u>

If X, Y are CW complexes and $H^k(Y;R)$ is f.g. free R-module for all k, then the map

$$H^*(X;R) \otimes_R H^*(Y;R) \xrightarrow{\sim} H^*(X \times Y; R)$$
$$a \otimes b \longmapsto a \times b$$

is an isomorphism of rings.

We have:

$$H^*(S^2 \times S^4) = H^*(S^2) \otimes H^*(S^4) = \mathbb{Z}[\alpha]/\alpha^2 \otimes \mathbb{Z}[\beta]/\beta^2$$

$$H^*(\mathbb{CP}^3) \simeq \mathbb{Z}[\gamma]/(\gamma^4)$$

$H^*(S^n;\mathbb{Z}) = \mathbb{Z}[\alpha]/\alpha^2$, $|\alpha| = n$

with $|\alpha| = 2$, $|\beta| = 4$, $|\gamma| = 2$

$H^*(\mathbb{CP}^n;\mathbb{Z}) = \mathbb{Z}[\alpha]/\alpha^{n+1}$, $|\alpha| = 2$

Any isomorphism $H^*(\mathbb{CP}^3) \to H^*(S^2 \times S^4)$ must send $\gamma \longmapsto \pm\alpha$. But $\alpha^2 = 0$, while $\gamma^2 \neq 0$, so no such isomorphism exists.

5 Show that

$$H^*(\mathbb{RP}^3;\mathbb{Z}) \cong H^*(\mathbb{RP}^2 \vee S^3;\mathbb{Z})$$

is a ring isomorphism. Is $\mathbb{RP}^3$ homotopy equivalent to $\mathbb{RP}^2 \vee S^3$? Explain.

Using the known ring structures for $H^*(\mathbb{RP}^n)$ and $H^*(S^n)$, we have:

$$\boxed{\begin{array}{l} H^*(\mathbb{RP}^{2k};\mathbb{Z}) = \mathbb{Z}[\alpha]/(2\alpha, \alpha^{k+1}) \\ H^*(\mathbb{RP}^{2k+1};\mathbb{Z}) = \mathbb{Z}[\alpha,\beta]/(2\alpha, \alpha^{k+1}, \beta^2, \alpha\beta) \\ \qquad |\alpha| = 2,\ |\beta| = 2k+1 \end{array}}$$

$$H^*(\mathbb{RP}^3;\mathbb{Z}) = \mathbb{Z}[\alpha,\beta]/(2\alpha, \alpha^3, \beta^2, \alpha\beta)$$

$$H^*(\mathbb{RP}^2 \vee S^3;\mathbb{Z}) = \mathbb{Z}[\gamma]/(2\gamma,\gamma^3) \oplus \mathbb{Z}[\delta]/\delta^2$$

with $|\alpha| = 2$, $|\beta| = 3$, $|\gamma| = 2$, $|\delta| = 3$.

To see that these rings are isomorphic, simply note that $\mathbb{Z}[\gamma] \oplus \mathbb{Z}[\delta] = \mathbb{Z}[\gamma,\delta]/(\gamma\delta)$, since $\gamma\delta \equiv (\gamma \times 0)(0 \times \delta) = 0$.

These spaces are not homotopy equivalent, however, which can be seen by computing the rings with $\mathbb{Z}_2$-coefficients:

$$\boxed{\begin{array}{c} H^*(\mathbb{RP}^n;\mathbb{Z}_2) = \mathbb{Z}_2[\alpha]/(\alpha^{n+1}) \\ |\alpha| = 1 \end{array}}$$

$$\boxed{\begin{array}{c} H^*(S^n;\mathbb{Z}_2) = \mathbb{Z}_2[\alpha]/(\alpha^2) \\ |\alpha| = n \end{array}}$$

$$H^*(\mathbb{RP}^3;\mathbb{Z}_2) = \mathbb{Z}_2[\alpha]/(\alpha^4)$$

$$H^*(\mathbb{RP}^2 \vee S^3;\mathbb{Z}_2) = \mathbb{Z}_2[\gamma]/(\gamma^3) \oplus \mathbb{Z}_2[\delta]/(\delta^2)$$

with $|\alpha| = 1$, $|\gamma| = 1$, $|\delta| = 3$.

6 Let M be a closed, connected, oriented n-manifold and let $f : S^n \to M$ be a continuous map of non-zero degree, i.e., the morphism

$$f_* : H_n(S^n; \mathbb{Z}) \to H_n(M; \mathbb{Z})$$

is non-trivial. Show that M and S^n have the same $\mathbb{Q}$-homology.

$\boxed{H_n(X;\mathbb{Q}) = H_n(X;\mathbb{Z}) \otimes \mathbb{Q}}$

Since $H_k(S^n; \mathbb{Z}) = \mathbb{Z}$ for $k = 0, n$, and is otherwise 0, and $\mathbb{Z} \otimes \mathbb{Q} = \mathbb{Q}$, we have

$$H_k(S^n; \mathbb{Q}) = \begin{cases} \mathbb{Q}, & k = 0, n \\ 0, & \text{otherwise} \end{cases}$$

Since M is connected and orientable, $H_k(M; \mathbb{Z}) = \mathbb{Z}$, $k = 0, n$, so $H_k(M; \mathbb{Q}) = \mathbb{Q}$ for $k = 0, n$.

Since M is closed and orientable, Poincaré duality applies, so we only need to show $H_k(M; \mathbb{Q}) = H^{n-k}(M; \mathbb{Q}) = 0$ for $0 < k < n$.

<u>Poincaré Duality</u>
M closed, R-orientable n-manifold, for all k,
$H^k(M; R) \cong H_{n-k}(M; R)$

Suppose not, and let $\alpha \neq 0$ be an element of $H^k(M; \mathbb{Q})$. Then there must be $\beta \in H^{n-k}(M; \mathbb{Q})$ st $\alpha \cup \beta = \pm [M] \in H^n(M; \mathbb{Q})$.

If M closed connected orientable n-manifold, F a field, then for every nonzero $\alpha \in H^k(M; F)$, there is $\beta \in H^{n-k}(M; F)$ st $\alpha \cup \beta = \pm [M]$.

For $H^k(M; \mathbb{Z})$, the result is for α of infinite order, not a multiple of any other element.

If f^* is the induced map on cohomology, then we have

$$f^*([M]) = f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta) = 0 \cup 0 = 0,$$

which contradicts the fact that $f^*([M])$ is nontrivial.

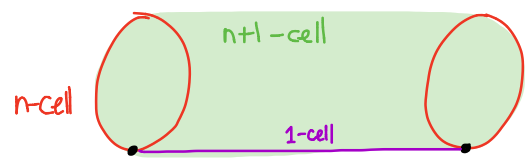
January 2011 Qualifying Exam

1. For a given a sequence of continuous maps $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots$ define the mapping telescope as the quotient space

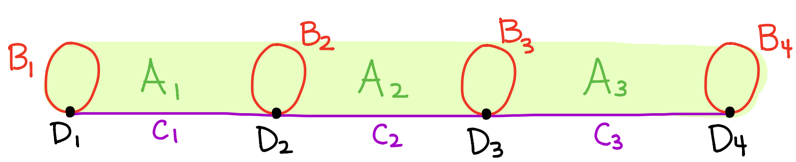
$$M := \left(\bigsqcup_{i \geq 1} X_i \times [0,1] \right) / ((x_i, 1) \sim (f_i(x_i), 0))$$

obtained from the disjoint union of cylinders $X_i \times [0,1]$ via the identification of $(x_i, 1) \in X_i \times \{1\}$ with $(f_i(x_i), 0) \in X_{i+1} \times \{0\}$. Compute the homology groups of M in the case when each X_i is the n-sphere S^n (for $n \geq 1$ fixed) and each $f_i : S^n \to S^n$ is a map of degree i ($i \geq 1$).

First consider the space $S^n \times [0,1]$. This has a CW structure with two 0-cells, one 1-cell, two n-cells, and one n+1-cell (for n=1, three 1-cells).



Now in our mapping telescope, one of the n-cells of each $X_i \times [0,1]$ glues to an n-cell of $X_{i+1} \times [0,1]$ via a degree i map, so we get a cell structure as follows: For each $X_i \times [0,1]$, Let A_i be the n+1-cell, Let B_i be the "left" n-cell, Let C_i be the 1-cell, and D_i the "left" 0-cell.



In the cellular chain complex, the boundary maps are:

$\partial_{n+1} : A_i \longmapsto i B_{i+1} - B_i, \quad \partial_1 : C_i \longmapsto D_{i+1} - D_i$

Since $\ker \partial_{n+1}$ is trivial, we get $H_{n+1}(M)=0$, and since M is connected, we have $H_0(M)=\mathbb{Z}$. The only other potentially nontrivial homologies are in dimension n and 1.

If $n\neq 1$, then $\ker\partial_1=0$ so $H_1(M)=0$, and $\partial_n=0$. If $n=1$, then $\ker\partial_n=\ker\partial_1=\langle B_i\rangle$, so in any case the n-cells B_i generate the n^{th} homology. So we get $H_n(M)=\langle B_i\rangle/\langle iB_{i+1}-B_i\rangle$.

Thus each generator B_i satisfies $B_i=iB_{i+1}$, so the homology becomes the direct limit of the sequence

$$\mathbb{Z}\xrightarrow{1}\mathbb{Z}\xrightarrow{2}\mathbb{Z}\xrightarrow{3}\mathbb{Z}\xrightarrow{4}\mathbb{Z}\xrightarrow{5}\cdots$$

where each copy of $\mathbb{Z}$ is generated by a B_i. This direct limit is $\mathbb{Q}$, so we get $H_n(M)=\mathbb{Q}$.

To conclude, we have

$$H_k(M)=\begin{cases}\mathbb{Z}, & k=0\\ \mathbb{Q}, & k=n\\ 0, & \text{otherwise}\end{cases}$$

4. Show that $\mathbb{RP}^{2n}$ cannot be the boundary of a compact manifold. Can $\mathbb{RP}^{2n+1}$ be a boundary? Explain.

Suppose M is a compact manifold with $\partial M = \mathbb{RP}^{2n}$. Then $\dim M = \dim \mathbb{RP}^{2n} + 1 = 2n+1$. Now consider the manifold $2M$ which results from gluing two copies of M along their common boundary.

A closed manifold of odd dimension has Euler char 0.

$2M$ is a closed, $2n+1$-dimensional manifold, and thus $\chi(2M) = 0$. But we also have $\chi(2M) = 2\chi(M) - \chi(\partial M)$. We conclude that $2\chi(M) = \chi(\partial M) = \chi(\mathbb{RP}^{2n}) = 1$, which is a contradiction.

To show $\mathbb{RP}^{2n-1}$ occurs at the boundary of a compact manifold, first note the following fact: If $p: \tilde{M} \to M$ is a two sheeted covering map, and M is a closed n-manifold (without boundary), then the mapping cylinder $M_p = \tilde{M} \times I \sqcup_p M$ is a compact $n+1$-manifold with boundary $\tilde{M} \times \{0\}$. Thus if we can show that $\mathbb{RP}^{2n-1}$ is a two-sheeted cover of a closed manifold, we are done.

Let $L_{n,m}$ denote the Lens space $S^{2n-1}/\mathbb{Z}_m$ which is the orbit space of $S^{2n-1} \subseteq \mathbb{C}^n$ with $\mathbb{Z}_m$-action generated by $(z_1, \dots, z_n) \mapsto (e^{2\pi i/m} z_1, \dots, e^{2\pi i/m} z_n)$. Then $L_{n,m} \to L_{n,km}$ is a k-sheeted covering space. Since $L_{n,2} = \mathbb{CP}^{2n-1}$, we have $\mathbb{CP}^{2n-1}$ is a two-sheeted cover of $L_{n,4}$.

5. Use cup products to compute the map $H^*(\mathbb{CP}^n;\mathbb{Z}) \to H^*(\mathbb{CP}^n;\mathbb{Z})$ induced by the map $\mathbb{CP}^n \to \mathbb{CP}^n$ that is a quotient of the map $\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ raising each coordinate to the d-th power, $(z_0, \cdots, z_n) \mapsto (z_0^d, \cdots, z_n^d)$, for a fixed integer $d > 0$.

We know $H^*(\mathbb{CP}^n; \mathbb{Z}) = \mathbb{Z}[\alpha]/\alpha^{n+1}$, $|\alpha| = 2$.
So in the induced map $H^* \to H^*$, the image of α determines the entire map.

Viewing $\mathbb{CP}^n$ as ordered $n+1$-tuples $(x_0, \dots, x_n) \in \mathbb{C}^{n+1} - \{0\}$, equivalent up to scaling, we can view the cell structure as follows: there is a single $2k$-cell for $0 \le k \le n$ which occurs as the copy of $\mathbb{C}^k \subseteq \mathbb{CP}^n$ with $x_k \neq 0$, $x_i = 0$ for $i > k$, and $(x_0, \dots, x_{k-1}) \subseteq \mathbb{C}^k$. With this view, our map sends our two-cell to itself via the map

$$(x_0, 1, 0, \dots, 0) \longmapsto (x_0^d, 1, 0, \dots, 0).$$

This map is surjective and d-to-one. Thus the induced chain map take the 2-cell C to dC. The resulting maps in homology and cohomology are thus multiplication by d, so $\alpha \longmapsto d\alpha$.

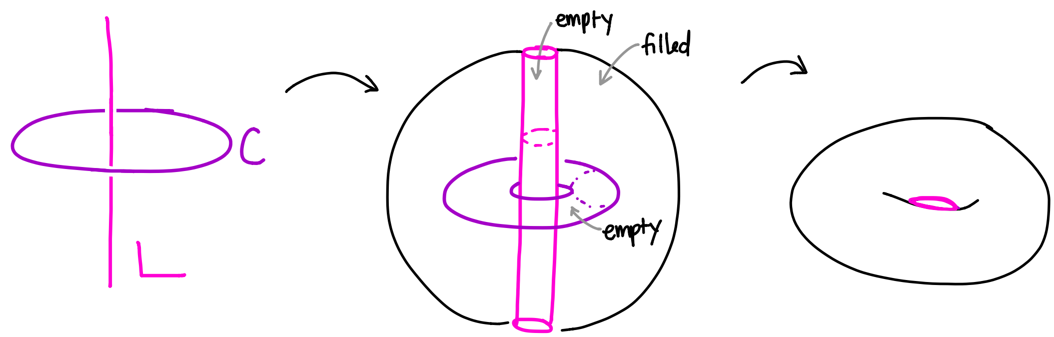
To further support this calculation, note that the map on the $2k$-cell is d^k-to-one, so the induced map on the $2k^{th}$ homology should be multiplication by d^k. This is consistent with the cup product:

$$\alpha^k = \alpha \cup \alpha \cup \cdots \cup \alpha \longmapsto d\alpha \cup d\alpha \cup \cdots d\alpha = d^k \alpha^k.$$

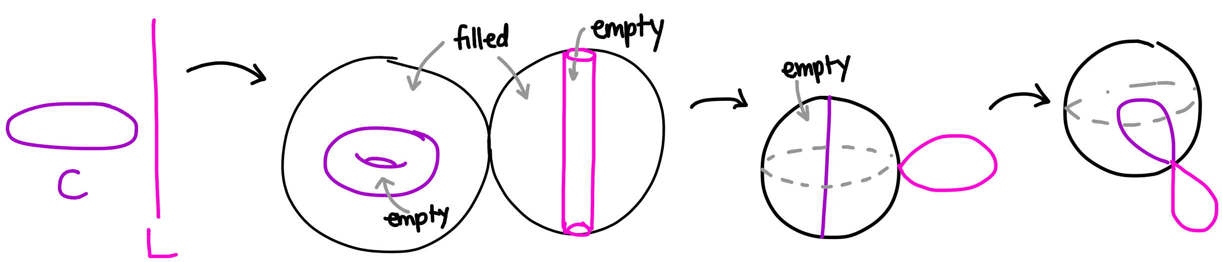
August 2011 Qualifying Exam

1. Let L be a line in $\mathbb{R}^3$. Let C be a round circle in $\mathbb{R}^3$ disjoint from L. Calculate the fundamental group of $\mathbb{R}^3 - (L \cup C)$. Note that there are two cases to consider: one where the line goes through the interior of the circle, and the other where it doesn't. Are the spaces obtained in the two situations homotopy equivalent?

In the case where the line goes through the circle, the space deformation retracts to a torus:



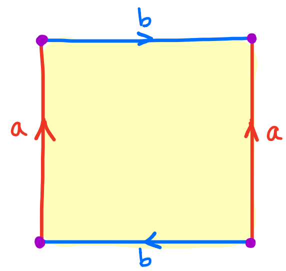
In the other case, note that $\mathbb{R}^3 - L$ deformation retracts to a circle, while $\mathbb{R}^3 - C$ deformation retracts to $S^2 \vee S^1$, so the space in question deformation retracts to $S^2 \vee S^1 \vee S^1$:



Thus in the first case, we have $\pi_1(T^2) = \mathbb{Z}^2$, and in the second case, we have $\pi_1(S^2 \vee S^1 \vee S^1) = 1 * \mathbb{Z} * \mathbb{Z} = \mathbb{Z} * \mathbb{Z}$. The two cases are therefore not homotopy equivalent.

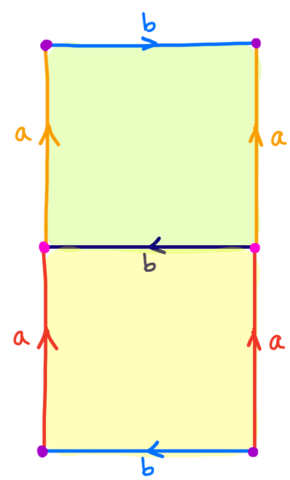
2. (a) Show that the torus T^2 is a two-fold cover of the Klein bottle.
(b) Is it possible to realize the Klein bottle as a two-fold cover of itself?
(c) Find the universal cover of the Klein bottle.

a) Consider the CW structure of the Klein bottle shown:



$\pi_1(K) = \langle a, b \mid aba^{-1}b \rangle$

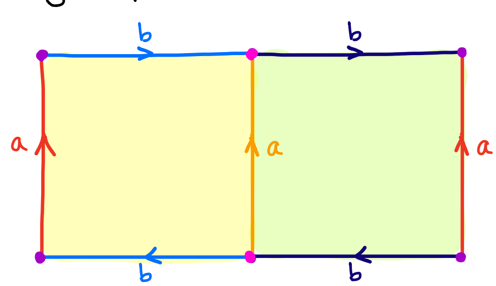
Then T^2 is a covering space of K corresponding to the subgroup $\pi_1(T^2) = \langle a^2, b \rangle$ as shown:



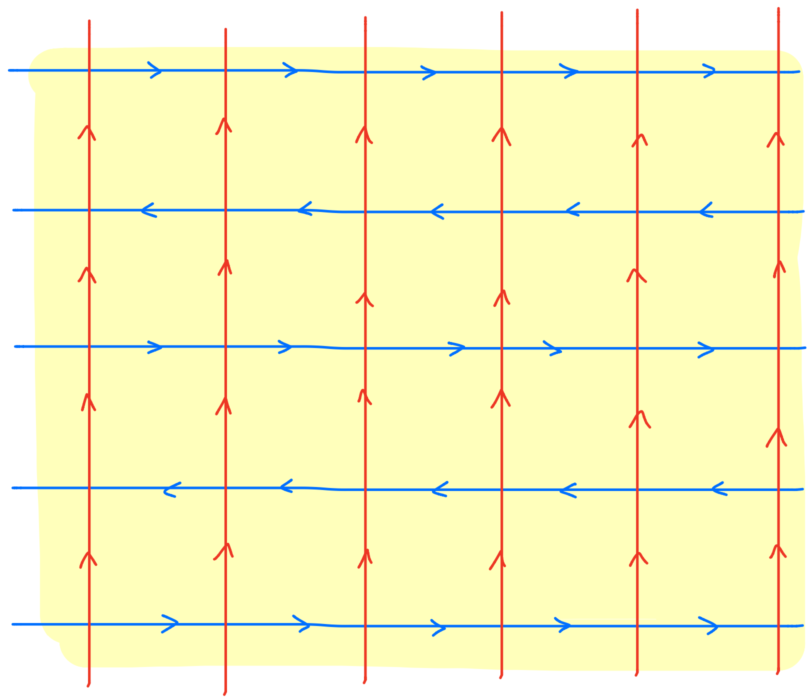
Note that setting $c = a^2$ and using the relation $ab = b^{-1}a$, we get $cb = a^2b = ab^{-1}a = a(a^{-1}b^{-1}a)^{-1}a = ba^2 = bc$

So $\pi_1(T^2) = \langle c, b \mid cb = bc \rangle = \mathbb{Z}^2$ (just to check).

b) Yes, the two-fold cover below corresponds to the index-2 subgroup $\langle a, b^2 \rangle$ as shown:

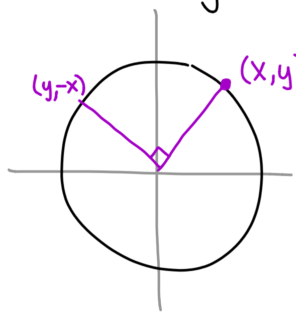


c) The universal cover of the Klein bottle is $\mathbb{R}^2$.
Using that $\mathbb{R}^2$ is the universal cover of T^2, we easily get the covering space $\mathbb{R}^2 \to T^2 \to K$.



3. Is there a continuous map $f : \mathbb{RP}^{2k-1} \to \mathbb{RP}^{2k-1}$ with no fixed points? Explain.

Yes. First consider the case $\mathbb{RP}^1$. If we consider $\mathbb{RP}^1$ as a circle in $\mathbb{R}^2$ with antipodal points identified, then a 90° rotation in $\mathbb{R}^2$ takes antipodal points to antipodal points, and therefore factors through the antipodal map.



In general, the map $(x_1, x_2, \dots, x_{2k}) \mapsto (x_2, -x_1, \dots, x_{2k}, -x_{2k-1})$ also factors through the antipodal map in $S^{2k-1} \subseteq \mathbb{R}^{2k}$, and is therefore a continuous map on $\mathbb{RP}^{2k-1}$ with no fixed points.

Note that this is impossible for $\mathbb{RP}^{2k}$, since an invertible transformation $\mathbb{R}^{2k+1} \to \mathbb{R}^{2k+1}$ has at least one real eigenvalue, which corresponds to a fixed point on the induced map on $\mathbb{RP}^{2k}$.

Given $f: X \to X$ for finite CW cx X, the Lefschetz number $\tau(f)$ is defined to be $\sum_n (-1)^n \operatorname{tr}(f_*: H_n \to H_n)$

If $f \simeq \mathrm{id}$, then $\tau(f) = \chi(X)$

In fact, the Lefschetz fixed point theorem implies that for any space X with homology of a point (modulo torsion), all maps $X \to X$ have a fixed point.

Lefschetz Fixed Point Theorem
If $\tau(f) \neq 0$, f has a fixed point.

4. (a) Let X^n be the cone on $\mathbb{CP}^n$. Show that X^n is a manifold with boundary if and only if $n = 1$.

(b) Show that $\mathbb{CP}^{2n}$ is not the boundary of any manifold.

a) Since $\mathbb{CP}^1 = S^2$ and $CS^2 = \overline{B(0,1)} \subseteq \mathbb{R}^3$, the "if" direction is immediate.

Now note that if M is a n-manifold with boundary such that the cone CM is a manifold with boundary, then CM is n+1-dimensional, and $\partial CM = M$.

So if $X = C\mathbb{CP}^n$ is a manifold with boundary, the LES for the pair $(X, \partial X) = (X, \mathbb{CP}^n)$ gives

$$H_k(X) \to H_k(X, \partial X) \to H_{k-1}(\mathbb{CP}^n) \to H_{k-1}(X)$$

Since cones are contractible, we have $H_k(X) = 0$ for $k > 0$, so we have isomorphisms for every $k > 1$:

$$H_k(X, \partial X) \cong H_{k-1}(\mathbb{CP}^n)$$

A connected manifold is orientable if $\pi_1(M)$ has no subgroup of index 2. In particular, simply connected manifolds are orientable.

Since X is contractible, it is simply connected, hence orientable. Thus we can apply Lefschetz Duality:

$$H_k(X, \partial X) \cong H_{2n+1-k}(X)$$

<u>Lefschetz Duality</u>
M compact R-orientable:
$H^k(M, \partial M; R) \cong H_{n-k}(M; R)$
$H^k(M; R) \cong H_{n-k}(M, \partial M; R)$

Since $H_0(X) = \mathbb{Z}$ and $H_i(X) = 0$ for $i > 0$, we get $H_k(X, \partial X) = \mathbb{Z}$ for $k = 2n+1$, and otherwise 0.

Therefore we get $H_i(\mathbb{CP}^n) = 0$ for $0 < i < 2n$. This is true only for $n = 1$.

b). We just show that $\mathbb{CP}^{2n}$ cannot be the boundary of a compact manifold (since I don't know how to generalize).

Say M is a compact manifold with $\partial M = \mathbb{CP}^{2n}$. Then $\dim M = 2n+1$. Let $2M$ denote the double of M, two copies of M identified by their common boundary. Then $2M$ is a closed $2n+1$-manifold, so $\chi(2M)=0$. We have

$$0 = \chi(2M) = 2\chi(M) - \chi(\partial M)$$

So $\chi(\partial M)$ is even. This contradicts the fact that $\chi(\mathbb{CP}^{2n}) = 2n+1$ (since there is a cell in every even dimension $0, 2, \dots, 2n$).

5. An H–space is a topological space Y with a continuous map

$$m : Y \times Y \to Y$$

having a distinguished element e in Y such that

$$m(e, y) = m(y, e) = y$$

for all y in Y.

Show that the 8–sphere is not homeomorphic to any H–space.

Suppose the contrary. Consider the inclusion $i: S^8 \to S^8 \times S^8$ defined by $x \mapsto (x, e)$. Then $m \circ i = id_{S^8}$.

Now consider the induced maps on cohomology:

$$H^*(S^8) \xrightarrow{m^*} H^*(S^8 \times S^8) \xrightarrow{i^*} H^*(S^8).$$

Using the Künneth formula, we have

$$\mathbb{Z}[\gamma]/\gamma^2 \xrightarrow{m^*} \mathbb{Z}[\alpha, \beta]/\alpha^2, \beta^2 \xrightarrow{i^*} \mathbb{Z}[\alpha]/\alpha^2$$

where each generator has degree 8.

Note that in the middle ring, we have $\alpha\beta = \beta\alpha$, since

In the Künneth Formula, multiplication is defined by

$(a \otimes b)(c \otimes d) = (-1)^{|b||c|}(ac \otimes bd)$

$$\alpha\beta = (\alpha \otimes 1)(1 \otimes \beta) = (-1)^0 (\alpha \otimes \beta) = \alpha \otimes \beta$$

$$\beta\alpha = (1 \otimes \beta)(\alpha \otimes \beta) = (-1)^{16}(\alpha \otimes \beta) = \alpha \otimes \beta$$

Now we have $m^*(\gamma) = c_1\alpha + c_2\beta$ for some $c_i \in \mathbb{Z}$.

Since $i^* \circ m^* = id^*$ and $i^*(\alpha) = \alpha$, $i^*(\beta) = 0$, we get $c_1 = 1$.

The same argument gives $c_2 = 1$. We have

$$(\alpha + \beta)^2 = 2\alpha\beta$$

But this gives $m^*(\gamma)^2 = 2\alpha\beta \neq 0$, which contradicts $\gamma^2 = 0$.

Alternatively, we could use cellular cohomology to show that the induced map m^* must send $\gamma \longmapsto \alpha + \beta$.
To see this, give S^8 a cell structure with a single 0-cell and a single 8-cell. Then $S^8 \times S^8$ has a single 0-cell, two 8-cells, and a single 16-cell.

Let C denote the 8-cell in S^8, and A, B the 8-cells in $S^8 \times S^8$. In the induced chain map $C_8(S^8 \times S^8) \to C_8(S^8)$ we have $m_*(A) = m_*(A \times \{e\}) = C$ and $m_*(B) = C$.
Since α, β, γ are the cochains corresponding to A, B, C, we get that $m^*(\gamma) = \alpha + \beta$.

August 2012 Qualifying Exam

2. Let S_g be the surface of genus g embedded into $\mathbb{R}^3$ in the standard way. Then S_g is the boundary of a compact manifold M in $\mathbb{R}^3$. Let X be the closed 3–manifold obtained by gluing two copies of M together along their boundaries via the identity map. Compute the homology groups $H_*(X;\mathbb{Z})$ and the relative homology groups $H_*(M, S_g;\mathbb{Z})$.

Mayer-Vietoris for $X = M \cup M$ gives a long exact sequence:

$$\cdots \longrightarrow H_n(M)^2 \longrightarrow H_n(X) \longrightarrow H_{n-1}(\partial M) \longrightarrow H_{n-1}(M)^2 \longrightarrow \cdots$$

Using the known homology of orientable surfaces, and that M deformation retracts to a wedge of g circles, we get

$$H_n(M) = \begin{cases} \mathbb{Z}, & n=0 \\ \mathbb{Z}^g, & n=1 \\ 0, & n>1 \end{cases} \qquad H_n(\partial M) = \begin{cases} \mathbb{Z}, & n=0,2 \\ \mathbb{Z}^{2g}, & n=1 \\ 0, & n>2 \end{cases}$$

Thus for $n=3$, the sequence becomes

$$0 \longrightarrow H_3(X) \longrightarrow \mathbb{Z} \longrightarrow 0$$

So $H_3(X) \simeq \mathbb{Z}$. The rest of the sequence becomes

$$0 \longrightarrow H_2(X) \xrightarrow{\partial} \mathbb{Z}^{2g} \xrightarrow{\psi} \mathbb{Z}^g \oplus \mathbb{Z}^g \xrightarrow{\phi} H_1(X) \longrightarrow \mathbb{Z} \longrightarrow \cdots$$

To compute $H_2(X)$, note that since ∂ is injective and by exactness, we have

$$H_2(X) \simeq \operatorname{im}\partial = \ker \psi.$$

Now the map ψ takes a 1-cycle $\alpha \in H_1(\partial M)$ to the pair $(\alpha, -\alpha) \in H_1(M) \oplus H_1(M)$.

To see what this does, consider the deformation retraction of M to $\vee_g S^1$. With the usual cell structure of ∂M with $2g$ 1-cells $a_1, b_1, \dots, a_g, b_g$, the deformation retraction preserves all the a_i's, but collapses all the b_i's.

So a cycle of the form $\sum (c_i a_i + d_i b_i) \in H_1(\partial M)$ maps to $(\sum c_i a_i, -\sum c_i a_i) \in H_1(M) \oplus H_1(M)$, since cycles b_i are trivial in $H_1(M)$

Thus we have $\ker \psi = \langle b_1, \dots, b_g \rangle \in \mathbb{Z}^{2g}$, and we conclude that $H_2(X) = \mathbb{Z}^g$.

To calculate $H_1(X)$, first note that $\chi(X) = 0$, since X is a closed manifold of odd dimension. Let $r = \text{rank}(H_1(X))$. Then we have

$$0 = \chi(X) = \sum (-1)^n \text{rank}(H_n(X)) = -1 + g - r + 1$$

So $r = g$. We show that $H_1(X)$ is torsion-free, to conclude $H_1(X) = \mathbb{Z}^g$. The part of our long exact sequence we consider is

$$\mathbb{Z}^{2g} \xrightarrow{\psi} \mathbb{Z}^g \oplus \mathbb{Z}^g \xrightarrow{\phi} H_1(X) \xrightarrow{\partial} \mathbb{Z} \to \cdots$$

Let α be a 1-cycle in X such that $n\alpha = 0$. Then $\partial\alpha = 0$ since ∂ maps to a free group, so by exactness, $\alpha \in \operatorname{im}\phi$.

Let $(x,y) \in H_1(M) \oplus H_1(M)$ with $\phi(x,y) = \alpha$. By definition of the map ϕ, we have $x + y = \alpha$, so $nx + ny = n\alpha = 0$. Thus $(nx, ny) \in \ker\phi = \operatorname{im}\psi$.

Recall that cycles in $H_1(\partial M)$ have the form $\sum(c_i a_i + d_i b_i)$. If we write a cycle z as $z_1 + z_2$, where $z_1 = \sum c_i a_i$ and $z_2 = \sum d_i b_i$, then the map ψ takes $z \mapsto (z_1, -z_1)$.

Let z be a cycle with $\psi(z) = (nx, ny)$. Then $nx = -ny = z_1$. So $nx = \sum c_i a_i$ for some $c_i \in \mathbb{Z}$, and we must have $n \mid c_i$ for each $i = 1, \dots, g$. Define $c_i' = \frac{c_i}{n}$, and set $z' = \sum c_i' a_i$. Then $\psi(z') = (x,y)$, so $(x,y) \in \ker\phi$, which means $\alpha = x + y = 0 \in H_1(X)$. Thus $H_1(X)$ is torsion-free.

To conclude, we have

$$H_n(X) = \begin{cases} \mathbb{Z}, & n = 0, 3 \\ \mathbb{Z}^g, & n = 1, 2 \\ 0, & \text{otherwise} \end{cases}$$

5. Show that if a connected manifold M is the boundary of a compact manifold, then the Euler characteristic of M is even. Use this to show that $\mathbb{RP}^{2n}$ and $\mathbb{CP}^{2n}$ are not boundaries.

Since $\mathbb{RP}^n = e_0 \cup e_1 \cup e_2 \cup \cdots \cup e_n$ and $\mathbb{CP}^n = e_0 \cup e_2 \cup \cdots \cup e_{2n}$ (where e_i represents a i-cell), we have

$$\chi(\mathbb{RP}^{2n}) = \sum_{k=0}^{2n} (-1)^k = 1 \quad \text{and} \quad \chi(\mathbb{CP}^{2n}) = \sum_{k=0}^{2n} (-1)^{2k} = 2k+1,$$

so we only need to show the first claim.

Method 1:

First note that for a finite long exact sequence of finitely generated groups

$$0 \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to 0$$

we have $\sum_{k=0}^{n} (-1)^k \operatorname{rank} G_k = 0.$

Thus in the long exact sequence for a pair (X,A) where $\dim X = n$, we have

$$0 \to H_n(A) \to H_n(X) \to H_n(X,A) \to \cdots \to H_0(X,A) \to 0$$

so we get

$$\sum_{k=0}^{n} (-1)^k \left[\operatorname{rank} H_k(A) - \operatorname{rank} H_k(X) + \operatorname{rank} H_k(X,A)\right] = 0$$

If we define $\chi(X,A) := \sum_k \operatorname{rank} H_k(X,A) = \chi(X/A) - 1$, we get the formula:

$$\chi(A) + \chi(X,A) = \chi(X)$$

Now suppose M is a connected n-manifold which is the boundary of a compact manifold N. Applying Lefschetz duality, we have for all k:

uses N orientable, oops!

$$H_k(N,M) = H^{n+1-k}(N)$$

By the universal coefficient theorem, $\operatorname{rank} H_k(N) = \operatorname{rank} H^k(N)$ for all k, so we conclude $\operatorname{rank} H_k(N,M) = \operatorname{rank} H_{n+1-k}(N)$ for all k. We have

$$\begin{aligned}
\chi(N,M) &= \sum_{k=0}^{n+1} (-1)^k \operatorname{rank} H_k(N,M) \\
&= \sum_{k=0}^{n+1} (-1)^k \operatorname{rank} H_{n+1-k}(N) \\
&= \sum_{i=0}^{n+1} (-1)^{n+1-i} \operatorname{rank} H_i(N) \\
&= (-1)^{n+1} \chi(N).
\end{aligned}$$

Plugging in to our formula, we have

$$\chi(M) + (-1)^{n+1}\chi(N) = \chi(N)$$

So we conclude $\chi(M) = 2\chi(N)$ if n is even and $\chi(M) = 0$ if n is odd, which finishes the proof.

this is a fact for all closed odd-dimensional manifolds

Method 2

Suppose $M = \partial N$, as before. Denote $2N$ the manifold which results when gluing two copies of N along M. Mayer-Vietoris for $2N = N \cup N$ gives

$$0 \to H_{n+1}(N)^2 \to H_{n+1}(2N) \to H_n(M) \to \cdots \to H_0(2N) \to 0$$

So as before we have

$$\sum_{k=0}^{n+1} (-1)^k \left[\operatorname{rank} H_k(M) - 2 \operatorname{rank} H_k(N) + \operatorname{rank}_k(2N)\right] = 0$$

So we get the formula

$$\chi(M) + \chi(2N) = 2\chi(N).$$

Now recall that closed odd-dimensional manifolds have Euler characteristic 0, so either $\chi(M) = 0$ or $\chi(2N) = 0$ (depending on if M is odd or even dim).

In the first case, we are done. In the second, we get $\chi(M) = 2\chi(N)$, and so $\chi(M)$ is even, which completes the proof.

6. Are $\mathbb{CP}^{n(n+1)/2}$ and $S^2 \times S^4 \times \cdots \times S^{2n}$ homotopy equivalent? Are $S^2 \times S^2$ and $\mathbb{CP}^2 \# \mathbb{CP}^2$ homotopy equivalent?

Let $N = \frac{n(n+1)}{2}$. Then $H^*(\mathbb{CP}^N) = \mathbb{Z}[\alpha]/\alpha^{N+1}$, $|\alpha| = 2$, and by Künneth formula, $H^*(S^2 \times \cdots \times S^{2n}) = \mathbb{Z}[\beta_1, \ldots, \beta_n]/\beta_i^2$ where $|\beta_i| = 2i$. An isomorphism between these rings must send a generator of degree i to a generator of degree i, so in particular, $\beta_1 \mapsto \pm\alpha$. But $\beta_1^2 = 0$ and $(\pm\alpha)^2 \neq 0$, so no such isomorphism exists. Thus $\mathbb{CP}^N$ and $S^2 \times S^4 \times \cdots \times S^{2n}$ cannot be homotopy equivalent.

The reason $S^2 \times S^2$ is not homotopy equivalent to $\mathbb{CP}^2 \# \mathbb{CP}^2$ is more subtle. Künneth gives $H^*(S^2 \times S^2) = \mathbb{Z}[\alpha_1, \alpha_2]/\alpha_i^2$, $|\alpha_i| = 2$. In particular, $\alpha_1 \cup \alpha_2 \neq 0$. We show that while both spaces have second cohomology $\mathbb{Z}^2$, the generators cup to zero in $H^*(\mathbb{CP}^2 \# \mathbb{CP}^2)$.

Let X, Y denote the two space $\mathbb{CP}^2$ with the interior of a 4-ball removed, so $\mathbb{CP}^2 \# \mathbb{CP}^2 = X \cup Y$ and $X \cap Y = S^3$. Mayer-Vietoris gives an exact sequence

$$H^1(S^3) \longrightarrow H^2(\mathbb{CP}^2 \# \mathbb{CP}^2) \xrightarrow{\psi} H^2(X) \oplus H^2(Y) \to H^2(S^3)$$

So since $H^2(S^3) = H^1(S^3) = 0$, we have an isomorphism ψ.

This is probably enough to conclude what we need, but for good measure I'll compute more.

Since $\mathbb{CP}^2$ is orientable, X is a compact orientable manifold with boundary, so applying Lefschetz duality, we have

$$H_k(X, \partial X) \cong H^{k-k}(X)$$

for all k. Since $X/\partial X = \mathbb{CP}^2$, in particular we have

$$H^2(X) = H_2(X, \partial X) = H_2(X/\partial X) = H_2(\mathbb{CP}^2) = \mathbb{Z}.$$

So $H^2(\mathbb{CP}^2 \# \mathbb{CP}^2) \cong H^2(X) \oplus H^2(Y) = \mathbb{Z}^2$.

Now let β_1, β_2 be the generators for $\mathbb{Z}^2$, i.e. β_1 is the generator for $H^2(X)$ and β_2 for $H^2(Y)$. The map ψ sends a coccyle γ in $\mathbb{CP}^2 \# \mathbb{CP}^2$ to the pair $(\gamma|X, \gamma|Y)$. Let $\gamma_i = \psi^{-1}(\beta_i)$. Then $\psi(\gamma_1 \cup \gamma_2) = (\gamma_1|X \cup \gamma_2|X, \gamma_1|Y \cup \gamma_2|Y)$. Since $\gamma_2|X = \gamma_1|Y = 0$, we have $\psi(\gamma_1 \cup \gamma_2) = 0$, i.e. $\gamma_1 \cup \gamma_2 = 0$. This finishes the proof.

August 2014 Qualifying Exam

Algebraic Topology: 751 - 752

Thursday, August 21, 2014

1. (a) Let X be a compact connected subset of S^3 homeomorphic to a 1–dimensional cell complex. Prove that $H_1(S^3 - X)$ is free abelian of the same rank as $H_1(X)$ and that $H_n(S^3 - X) = 0$ for $n > 1$.

   (b) Let $X \subset S^3$ be homeomorphic to the disjoint union of two circles, and let Y be the disjoint union of two disks. Build a space Z by attaching Y to S^3 by identifying ∂Y and X via a homeomorphism. Compute the homology groups of Z.

a) Since X is compact and homeomorphic to a 1-dimensional cell complex, X must be a finite graph, i.e. it is homotopic to a finite wedge of spheres $\vee^n S^1$.

Now $S^3 - X$ is homotopic to $S^3 - \vee^n S^1$, which is homeomorphic to $\mathbb{R}^3 - \cup^n L$, that is $\mathbb{R}^3$ with n nonintersecting lines removed. This can be seen by noticing that $\mathbb{R}^3 = S^3 - \{x_0\}$, and taking the wedge point x_0 as ∞ in $\mathbb{R}^3$.

Now $\mathbb{R}^3 - \cup^n L$ deformation retracts to $\mathbb{R}^2 - \{x_1, \ldots, x_n\}$, which is homotopic to $\vee^n S^1$. Thus we have shown that $S^3 - X$ is homotopic to X. Part a follows immediately.

b) Since $Z = S^3 \cup Y$ and $X = S^3 \cap Y$, we can apply Mayer-Vietons to get a long exact sequence

$$H_n(X) \to H_n(S^3) \oplus H_n(Y) \to H_n(Z) \to H_{n-1}(X) \to H_{n-1}(S^3) \oplus H_{n-1}(Y)$$

Since Z has dimension 3, $H_n(Z) = 0$ for $n > 3$. We also have $H_1(X) = H_1(S^1 \sqcup S^1) = \mathbb{Z} \oplus \mathbb{Z}$, $H_0(X) = \mathbb{Z} \oplus \mathbb{Z}$ (because X has 2 path-components), and $H_n(X) = 0$ for $n > 1$. We know $H_n(Y) = 0$ for $n > 0$, and $H_0(Y) = \mathbb{Z}$.

Starting at $H_3(X)$, our sequence becomes

$$0 \to \mathbb{Z} \to H_3(Z) \to 0 \to 0 \to H_2(Z) \to \mathbb{Z}^2 \to 0$$

So we get $H_3(Z) = \mathbb{Z}$, $H_2(Z) = \mathbb{Z}^2$.

To determine $H_1(Z)$,

3. Let X be a surface of genus 2 and Y a torus with one boundary component (*i.e.* a torus from which an open disk has been removed). Let W be a nonseparating circle in X. Let Z be the space obtained by attaching Y to X by identifying W and ∂Y via a homeomorphism.

   (a) Compute the fundamental group and all the homology groups of Z.

   (b) Show that Z retracts onto the wedge of two circles. Does Z deformation retract onto the wedge of two circles? Rigorously justify your answer.

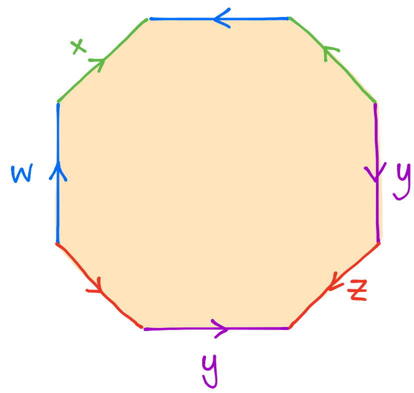
   (c) Show that Z has a 3–fold irregular covering space.

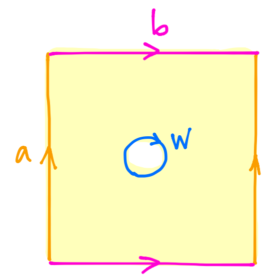
a) First note that if X and Y are topological spaces with a common subspace W, then a retraction (or deformation retraction) of X or Y factors through the quotient map $X \sqcup_W Y$. This follows from the following: Suppose $f_1 : X \to X$, $f_2 : Y \to Y$ are continuous, and denote $f : X \sqcup Y \to X \sqcup Y$ to be both these maps together. If $q : X \sqcup Y \to X \sqcup_W Y$ is the quotient map, then we have a diagram

$$\begin{tikzcd} X \sqcup Y \arrow[r, "f"] \arrow[d, "q"'] & X \sqcup Y \arrow[d, "q"'] \\ X \sqcup_W Y \arrow[r, dashed, "f'"'] & X \sqcup_W Y \end{tikzcd}$$

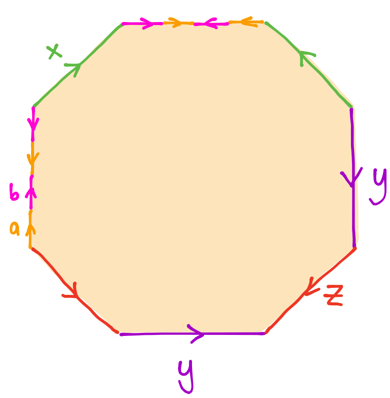
By the universal property of quotients of topological spaces, there is a unique continuous map $f' : X \sqcup_W Y \to X \sqcup_W Y$ such that $q \circ f = f' \circ q$. If f was a retraction or deformation retraction, then so is f'.

Now back to the problem. Our spaces X and Y can be modeled as shown:





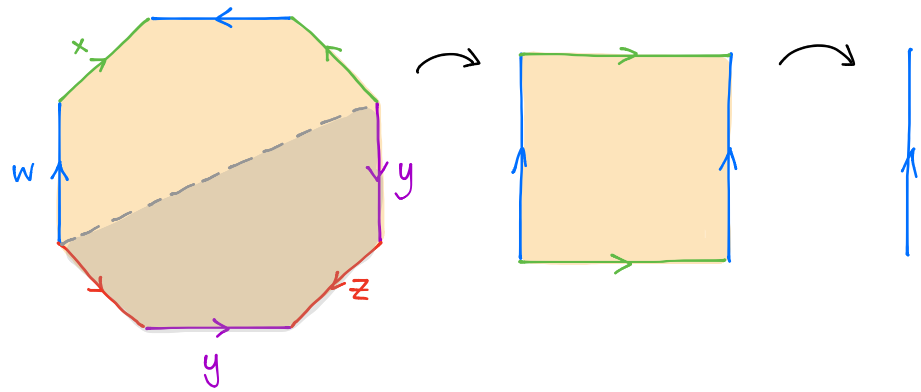
The loop w corresponds to the subspace W in both X and Y. Y deformation retracts to a wedge of circles a and b, and under this deformation retraction, w is sent to the loop $aba^{-1}b^{-1}$. Thus by the previous argument there is a deformation retraction of Z to the space which glues $S^1 \vee S^1$ to X via $w \mapsto aba^{-1}b^{-1}$, as shown.



This space is homotopic to Z, so we compute

$$\pi_1(Z) = \langle a, b, x, y, z \mid [[a,b], x] \cdot [y, z] \rangle$$

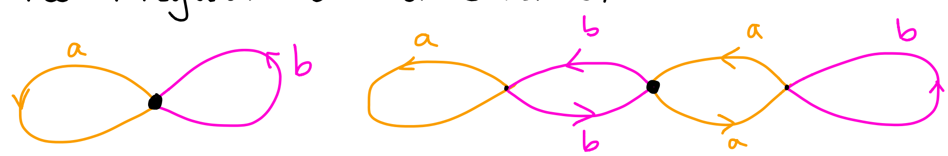
b) Now note that X retracts to W. This can be seen as the composition of a quotient to a torus with a retraction to a circle.



Putting together this retraction of X with the deformation retraction of Y to $S^1 \vee S^1$ gives a retraction of Z to $S^1 \vee S^1$.

Since $\pi_1(S^1 \vee S^1) = F_2$, the free group on 2 generators, there can be no deformation retraction from Z to $S^1 \vee S^1$.

c) Consider the irregular 3-fold cover of $S^1 \vee S^1$ as shown:



Since Y is homotopic to $S^1 \vee S^1$, this corresponds to a cover of Y, which we can extend to a cover of Z.

Algebraically, we have the subgroup of $\pi_1(Z)$:

$$H = \langle a^2, b^2, aba^{-1}, bab^{-1}, x, y, z \rangle$$

which has cosets H, aH, bH, and is nonnormal.
Thus H corresponds to an irregular 3-fold cover of Z.

August 2015 Qualifying Exam

Algebraic Topology: 751 - 752

1. This is a "True or False" question. Justify your answer by proving the statement if "True" or by giving a counterexample if "False":

(a) Any subgroup of a free group is free.

(b) If we write S^3 as the union of two connected non-empty open subsets X and Y, then the intersection $X \cap Y$ is connected.

a) True. Let G be a free group with n generators. Then $G = \pi_1(X)$, where $X = \vee^n S^1$ is a wedge of n circles. Let $H \leq G$ be a subgroup. Then H corresponds to a covering space $\tilde{X}$ of X, that is, there is a covering map $p: \tilde{X} \to X$ such that $p_*(\pi_1(\tilde{X})) = H$. Since $\tilde{X}$ is a covering space of a 1-dimensional CW complex, $\tilde{X}$ is also a 1-dim CW complex, i.e. $\tilde{X}$ is a graph. By collapsing a maximal tree, we can see that $\tilde{X}$ is homotopic to a wedge of circles (not nessessarily finite). This shows that $\pi_1(X) = H$ (since p_* is always injective).

b) True. We show that $H_0(X \cap Y) = \mathbb{Z}$, which implies $X \cap Y$ is connected. Since X, Y are open with $S^3 = X \cup Y$, we can use the Mayer-Vietoris sequence

For any space X, $H_0(X)$ is a direct sum of $\mathbb{Z}$'s, one for each path-component of X.

$$\cdots \to H_1(S^3) \to H_0(X\cap Y) \to H_0(X) \oplus H_0(Y) \to H_0(S^3) \to 0$$

Since X, Y are connected, we have $H_0(X) \cong H_0(Y) \cong \mathbb{Z}$, and we also have $H_1(S^3) = 0$, $H_0(S^3) = \mathbb{Z}$. Thus we have a short exact sequence

$$0 \to H_0(X\cap Y) \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0$$

So $H_0(X\cap Y)$ must have rank 1, i.e. $H_0(X\cap Y) = \mathbb{Z}$.

Alternatively, we could look at the Mayer-Vietoris sequence for reduced homology. This gives

$$\cdots \to \tilde{H}_1(S^3) \to \tilde{H}_0(X\cap Y) \to \tilde{H}_0(X) \oplus \tilde{H}_0(Y) \to \tilde{H}_0(S^3) \to 0$$

Since $\tilde{H}_0(S^3) \cong \tilde{H}_0(X) \cong \tilde{H}_0(Y) = 0$, we get

$$0 \to \tilde{H}_0(X\cap Y) \to 0$$

$\boxed{\tilde{H}_0(X) \oplus \mathbb{Z} = H_0(X)}$

and so $\tilde{H}_0(X\cap Y) = 0$, i.e. $H_0(X\cap Y) = \mathbb{Z}$.

2. (a) Show that the n-sphere admits a nowhere vanishing continuous unit vector field if, and only if, n is odd.

(b) Let $SO(n)$ be the set of orthogonal n-by-n matrices with real coefficients and determinant 1. Show that $SO(n)$ admits a nowhere vanishing continuous vector field.

a) First, suppose n is odd. Write $n=2k-1$, and consider $S^{2k-1} \subseteq \mathbb{R}^{2k}$ as the unit sphere. For each unit vector $x=(x_1, \dots, x_{2k}) \subseteq \mathbb{R}^{2k}$, define $v(x)=(x_2, -x_1, x_4, -x_3, \dots)$. Then $v(x)\cdot x = 0$, so $v(x) \perp x$. Since $\|v(x)\| = \|x\| = 1$, we get a continuous unit vector field, as desired.

Conversely, suppose such a v exists for $S^n \subseteq \mathbb{R}^{n+1}$. Define $f_t : S^n \to \mathbb{R}^{n+1}$ by $f_t(x) = \cos(\pi t)x + \sin(\pi t)v(x)$. Since $f_0(x) = x$ and $f_1(x) = v(x)$, f_t is a homotopy from x to $-x$. We would like to show that f_t is a homotopy of S^n to itself, that is $\|f_t(x)\|=1$ for all $x \in S^n$. Since $x \perp v(x)$, we have

$$
\begin{aligned}
\|f_t(x)\|^2 &= \|\cos(\pi t)x + \sin(\pi t)v(x)\|^2 \\
&= \|\cos(\pi t)x\|^2 + \|\sin(\pi t)v(x)\|^2 \\
&= \cos^2(\pi t)\|x\|^2 + \sin^2(\pi t)\|v(x)\|^2 = 1
\end{aligned}
$$

Thus the antipodal map on S^n is homotopic to the identity, which means it has degree 1. Since the antipodal map has degree $(-1)^{n+1}$, we conclude that n is odd.

3. Given $m > 1$ and integers $l_1, \cdots, l_n$ so that $(l_k, m) = 1$ for all k, define the lens space $L = L_m(l_1, \cdots, l_n)$ to be the orbit space $S^{2n-1}/\mathbb{Z}_m$ of the unit sphere S^{2n-1} with the $\mathbb{Z}_m$-action generated by the rotation:

$$\rho(z_1, \cdots, z_n) = \left(e^{2\pi i l_1/m} z_1, \cdots, e^{2\pi i l_n/m} z_n\right),$$

rotating the j-th $\mathbb{C}$-factor of $\mathbb{C}^n$ by an angle $2\pi i l_j/m$.

(a) Compute the fundamental group of L.

(b) Describe the homotopy type of a continuous map $f : L \to S^1$.

(c) Construct a CW-structure on L and compute the differentials of the resulting cellular chain complex.

(d) Compute the homology of L.

Since S^{2n-1} is a covering space of L which is simply connected, S^{2n-1} is the universal cover of L, and since $L = S^{2n-1}/\mathbb{Z}_m$, we have $\pi_1(L) = \mathbb{Z}_m$.

Suppose $f: L \to S^1$ is continuous. Then f induces a map $f_*: \pi(L) \to \pi_1(S^1)$ which must be trivial since there are no nontrivial homomorphisms $\mathbb{Z}_m \to \mathbb{Z}$. Hence f satisfies the lifting criterion for any covering space of S^1, and so there is a lift $\tilde{f}: L \to \mathbb{R}$ with $p \circ \tilde{f} = f$ (where $p: \mathbb{R} \to S^1$ is a covering map). Since $\mathbb{R}$ is contractible, $\tilde{f}$ is nullhomotopic, and so is f. Therefore any continuous map $L \to S^1$ is nullhomotopic.

Just as with $\mathbb{RP}^{2n-1}$, we can build a cell structure on L with a cell in each dimension, where the attaching maps alternate between "wrapping around" m times and attaching to "either side" of the previous cell.

Lens spaces are a generalization of real projective space: when $m=2$, $L = \mathbb{RP}^{2n-1}$.

So we get the cellular chain complex

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

with $2n$ copies of $\mathbb{Z}$, one for each cell of dimension at most $2n-1$.

The resulting homology is $H_k(L) = \begin{cases} \mathbb{Z}, & k=0, 2n-1 \\ \mathbb{Z}_m, & k \text{ odd}, 0<k<2n-1 \\ 0, & \text{otherwise} \end{cases}$

5. (a) Recall that a cone on a space Y is the space $(Y \times I)/(Y \times \{0\})$. Let X^n be the cone on $\mathbb{CP}^n$. Show that X^n is a manifold with boundary if and only if $n = 1$.

(b) Show that $\mathbb{CP}^{2n}$ is not the boundary of any manifold.

$\boxed{CS^n = D^{n+1}}$

a) Since $\mathbb{CP}^1 = S^2$, we have $X^1 = D^3$, the 3-dimensional disk, so X^1 is a manifold with boundary.

Conversely, suppose X^n is a manifold with boundary ∂X^n. The long exact sequence for the pair $(X^n, \partial X^n)$ gives

January 2015 Qualifying Exam

Algebraic Topology: 751 - 752

January 15, 2015

1. Let S_g be the closed orientable surface of genus g. Show that if a and b are positive, and $f\colon S_b \to S_a$ is a map of non–zero degree such that $f_*\colon \pi_1(S_b) \to \pi_1(S_a)$ is injective, then $b \geq a$.

Let $H = f_*(\pi_1(S_b))$ and $G = \pi_1(S_a)$. Then H corresponds to a covering space $p: \tilde{X} \to S_a$ with $p_*(\pi_1(\tilde{X})) = H$. Since covering maps induce injections on fundamental groups, we know $\pi_1(\tilde{X}) \cong H$.

Now since $f_*(\pi_1(S_b)) = p_*(\pi_1(\tilde{X}))$, we can lift f to a map $\tilde{f}: S_b \to X$ with $p \circ \tilde{f} = f$. Since $p_* \circ \tilde{f}_* = f_*$, we must have $\tilde{f}_*: \pi_1(S_b) \xrightarrow{\sim} \pi_1(\tilde{X})$ an isomorphism.

If M is a connected, noncompact n-manifold, then $H_i(M) = 0$ for $i \geq n$

Now $\tilde{X}$ must be compact and orientable. Otherwise, we would have $H_2(\tilde{X}) = 0$, which implies $f_*: H_2(S_b) \to H_2(S_a)$ must be zero. This contradicts the assumption that $\deg(f)$ is nonzero. Since $\pi_1(\tilde{X}) = \pi_1(S_b)$, we conclude that $\tilde{X} \cong S_b$, since compact orientable surfaces are classified by their fundamental groups.

For X, Y connected oriented m-manifolds, the <u>degree</u> of a continuous map $f: X \to Y$ is defined by

$$f_*([X]) = \deg(f)[Y],$$

where $[X], [Y]$ are the generators of $H_m(X)$, $H_m(Y)$.

Now since S_b is a compact covering space of S_a, it must have finitely many sheets, say n. We must have $\chi(S_b) = n\chi(S_a)$, where χ denotes the Euler characteristic. (This is true for any covering space, since a cell structure on X lifts to one on $\tilde{X}$, with n copies of each cell).

For a finite CW complex X, the Euler characteristic $\chi(X)$ is defined to be the alternating sum $\Sigma_n (-1)^n c_n$, where c_n is the number of n-cells.

Fact: $\chi(X) = \Sigma_n (-1)^n \operatorname{rank} H_n(X)$

However, we also know that a compact orientable surface X of genus g has $\chi(X) = 2-2g$, since $H_0(X) = H_2(X) = \mathbb{Z}$, and $H_1(X) = 2g$. Thus we have $2-2b = n(2-2a)$, which gives $b = n(a-1)+1$. Since $n \geq 1$, we conclude $b \geq a$.