# BASES, SUBSPACES, AND CLOSED SETS

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## 1. Assignment due 2018-09-10

1.1. **The topology generated by a basis [1, No. 13.5].** We show that if $\mathscr{A}$ is a basis for a topology on $X$, then the topology generated by the $\mathscr{A}$ equals the intersection of all topologies on $X$ that contain $\mathscr{A}$. Further, the same result holds if $\mathscr{A}$ is a subbasis.

1.2. **Incomparable topologies on R [1, No. 13.6].** We show two topologies on the real line are not comparable, namely $\mathbf{R}_\ell$, the lower limit topology generated by half open intervals $[a, b)$, and $\mathbf{R}_K$, the $K$-topology generated by the punctured open intervals $(a, b) \setminus \{n^{-1} : n \in \mathbf{N}\}$.

1.3. **Partially ordering topologies by inclusion [1, No. 13.7].** We determine a partial ordering for the following topologies on $\mathbf{R}$:

- the standard topology
- the $K$-topology
- the finite complement topology
- the upper limit topology
- the topology generated by the rays $(-\infty, a)$

1.4. **Countable bases for topologies on R [1, No. 13.8].**

(a) We show that $\mathscr{B} = \{(a, b) : a < b \text{ and } a, b \in \mathbf{Q}\}$ is a countable basis for the standard topology on $\mathbf{R}$.

(b) We'll demonstrate the collection $\mathscr{C} = \{[a, b) : a < b \text{ and } a, b \in \mathbf{Q}\}$ generates a topology distinct from the lower limit topology.

1.5. **Defining open sets in a subspace [1, No. 16.3].** Consider the set $Y = [-1, 1]$ as a subspace of $\mathbf{R}$. Which of the following sets are open in $Y$? in $\mathbf{R}$?

- $\left\{x : \frac{1}{2} < |x| < 1\right\}$
- $\left\{x : \frac{1}{2} < |x| \leq 1\right\}$
- $\left\{x : \frac{1}{2} \leq |x| < 1\right\}$
- $\left\{x : \frac{1}{2} \leq |x| \leq 1\right\}$
- $\left\{x : 0 < |x| < 1 \text{ and } \frac{1}{x} \notin \mathbf{N}\right\}$

1.6. **Open maps between spaces [1, No. 16.4].** Consider the space $X$ and its subspace $Y$. We'll demonstrate that the projections

$$\pi_1 \colon X \times Y \to X \text{ and } \pi_2 \colon X \times Y \to Y$$

are open maps.

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1.7. **A countable basis for $\mathbf{R}^2$ [1, No. 16.6].** We'll prove that the countable collection

$$\{(a,b)\times(c,d) : a < b \text{ and } c < d \text{ and } a,b,c,d \in \mathbf{Q}\}$$

is a basis for $\mathbf{R}^2$.

1.8. **Endowing a line with subspace topologies [1, No. 16.8].** Consider a straight line $L$ in the plane. We'll describe the topological structure of $L$ endowed with the subspace topology of $\mathbf{R}_\ell \times \mathbf{R}$ and $\mathbf{R}_\ell \times \mathbf{R}_\ell$ respectively.

1.9. **Closed sets in subspaces [1, No. 17.2].** If $A$ is closed in $Y$ and $Y$ is closed in $X$, then $A$ is closed in $X$.

1.10. **[1, No. 17.6].** Let $A$, $B$, and $A_\alpha$ denoted subsets of a space $X$.

(a) If $A \subset B$, then $\overline{A} \subset \overline{B}$.

(b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(c) $\overline{\cup A_\alpha} \supset \cup \overline{A_\alpha}$, and there's an example of proper containment.

1.11. **Finite products of Hausdorff spaces [1, No. 17.11].** The product of two Hausdorff spaces is itself Hausdorff.

1.12. **Characterizing topologies on R [1, No. 17.16].**

(a) We have the following table for the closure of the set $K = \{1/n : n \in \mathbf{N}\}$ in each topology on $\mathbf{R}$.

| which topology on $\mathbf{R}$? | what is $\overline{K}$? |
|---|---|
| standard | |
| $K$-topology | |
| finite complement | |
| upper limit | |
| generated by $(-\infty, a)$ | |

(b) Further, we'll distinguish the above spaces in terms of two separation axioms.

| which topology on $\mathbf{R}$? | Hausdorff? | $T_1$? |
|---|---|---|
| standard | | |
| $K$-topology | | |
| finite complement | | |
| upper limit | | |
| generated by $(-\infty, a)$ | | |

## References

[1] J. R. Munkres, *Topology*, 2nd ed. Hardcover; Prentice Hall, Inc., 2000.