INTRO TO ALGEBRAIC TOPOLOGY

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7. Assignment due 2018-11-30

7.1. **[1, No. 53.4].** *Given.* Let $q\colon X \to Y$ and $r\colon Y \to Z$ be covering maps. Let $p = r \circ q$. Suppose $r^{-1}(z)$ is finite for each $z \in Z$.

To prove. p is a covering map.

Proof. Let $z \in Z$ and $r^{-1}(z) = \{y_i\}_{i=1}^n$. There's a neighborhood of z evenly covered by (a finite collection) $\{U_i\}$, each open in Y and indexed by $y_i \in U_i$.

For each $i \in \{1, \ldots, n\}$ there's a neighborhood O_i of y_i such that $O_i \cap U_i$ can be evenly covered by (an arbitrary collection) $\{V_{\alpha_i}\}$ indexed by $x_{\alpha_i} \in V_{\alpha_i}$ where $q^{-1}(y_i) = \{x_{\alpha_i}\}$. Since r is an open map, $\cap_{i=1}^n r(O_i \cap U_i)$ is a neighborhood of z.

Because q and r are both single valued,

$$p^{-1}(z) = \sqcup_{i=1}^n \sqcup_{\alpha_i} x_{\alpha_i}.$$

I claim z is evenly covered under $p = r \circ q$ by

$$\sqcup_{i=1}^n \sqcup_{\alpha_i} V_{\alpha_i},$$

observing that

- V_{α_i} is homeomorphic by a restriction of q to $O_i \cap U_i$ that is homeomorphic to $r(O_i \cap U_i)$ containing z.
- The V_{α_i} are disjoint because both
  - for each i, $V_{\alpha_i} \cap V_{\beta_i} = \varnothing$ for all indices $\alpha_i \neq \beta_i$
  - for each $i \neq j$, if V_{β_i} met V_{β_j} nontrivially, then $q\,|_{V_{\beta_i} \cap V_{\beta_j}}$ would be multivalued, which is absurd.

We've shown that $p = r \circ q$ is a covering map $p\colon X \to Z$. □

7.2. **[1, No. 53.6].** *Given.* Let $p\colon E \to B$ be a covering map.

To prove.

- If B is Hausdorff,
- If B is regular, then so is E.
- If B is completely regular, then so is E.
- If B is locally compact Hausdorff, then so is E.
- If B is compact and $p^{-1}(b)$ is finite for each $b \in B$, then E is compact.

Proof. We state the hypothesized topological property assumed for the base space B, and prove that it holds for the total space E.

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Fréchet separation (T_1). Suppose singletons are closed in B. For any $x \in E$, $\{p(x)\}$ is closed in B. There's a neighborhood $U_{p(x)}$ of $p(x)$ with pullback $p^{-1}(U)$ that's partitioned into slices V_α. Check $x \in p^{-1}(x) \cap V_{\alpha'}$ for a unique index α'. Since $\{p(x)\}$ is closed in B, $\{x\} = p^{-1}(x) \cap V_{\alpha'}$ is closed in E.

Hausdorff separation (T_2). Say $x \neq y$ in E. If $p(x) = p(y)$, there's a neighborhood U of $p(x)$ such that $p^{-1}(U)$ is the disjoint union of slices V_α and $x \in V_\beta$ and $y \in V_\gamma$ is a (unique) separation of x and y into disjoint open sets. Else if $p(x) \neq p(y)$, then because B is Hausdorff, there exist disjoint neighborhoods $U_{p(x)}$ and $U_{p(y)}$. Because B is the base of a total cover, there also exist neighborhoods $O_{p(x)}$ and $O_{p(y)}$ with slices $\{V_{\alpha_x}\}$ and $\{V_{\alpha_y}\}$. We obtain a separation of x and y in E with

$$x \in p^{-1}(U_{p(x)} \cap (\sqcup V_{\alpha_x}) \quad \text{and} \quad y \in p^{-1}(U_{p(y)} \cap (\sqcup V_{\alpha_y}).$$

Lemma. Suppose $b \in B$ and U is a neighborhood of b such that $\{V_\alpha\}$ is a partition of $p^{-1}(U)$ into slices. Let C be a closed set of B such that $C \subset U$. Then $p^{-1}(C)$ is closed. Moreover, $p^{-1}(C) \cap V_\alpha$ contains its limit points, and is closed as well.

Regular separation (T_3). Let $w \in E$ and W a neighborhood of w. Then $p(w) \in p(W)$ is open in B. There's a neighborhood O of $p(w)$ evenly covered by p. Let $U = p(W) \cap O$. Since B is regular, there's a neighborhood A of $p(w)$ such that

$$p(w) \in \mathrm{Cl}(A) \subset U.$$

Say $p^{-1}(U)$ has slices $\{V_\alpha\}$. Then $w \in V_{\alpha'} \cap p^{-1}(A) \subset V_{\alpha'} \cap p^{-1}(\mathrm{Cl}(A)) \subset V_{\alpha'}$, by the lemma. We conclude E is regular.

Completely regular separation ($T_{3\frac{1}{2}}$). Let $K \subset E$ be closed and take $x \in E \setminus K$. Then $p(E \setminus K)$ is open in B, with obviously $p(x) \in p(E \setminus K)$. There's a neighborhood U of $p(x)$ contained in $p(E \setminus K)$ (intersecting two open sets if necessary to construct U) such that $p^{-1}(U)$ is partitioned into slices $\{V_\alpha\}$. Now $B \setminus U$ is closed and disjoint from $p(x)$. Because B is completely regular, there exists a map $f\colon B \to I$ such that $f(B \setminus U) = \{0\}$ and $f(p(x)) = 1$. Since $p|_{V_{\alpha'}}$ is a homeomorphism onto U from the unique $V_{\alpha'}$ containing x, we may define a map $g\colon E \to I$ as follows:

$$g(z) = \begin{cases} 0 & \text{if } z \in E \setminus V_{\alpha'} \\ \left(f \circ p|_{V_{\alpha'}}\right)(z) & \text{if } z \in V_{\alpha'}. \end{cases}$$

I claim g can be defined as two piecewise continuous maps (verify) that evaluate to 0 on the closed set $\partial V_{\alpha'}$. By the piecing lemma for maps, g is continuous on E. As desired, $g(x) = 1$. Moreover, because

$$V_{\alpha'} \subset p^{-1}(U) \subset E \setminus K \quad \text{and} \quad U \subset p(E \setminus K),$$

we have $K \subset E \setminus V_{\alpha'}$. Thus $g(K) = \{0\}$. So E is completely regular.

Locally compact Hausdorff. Suppose B is locally compact and Hausdorff. We've shown E is Hausdorff. Now let $x \in E$ and $p(x) \in B$ with U a neighborhood of $p(x)$ evenly covered by $\{V_\alpha\}$. B is locally compact so there's a neighborhood W of $p(x)$ such that $\mathrm{Cl}(W)$ is compact and $p(x) \in \mathrm{Cl}(W) \subset U$. Let $V_{\alpha'}$ be the slice over U containing x. Then $p|_{V_{\alpha'}}$ is a homeomorphism from $V_{\alpha'}$ to U. As $\mathrm{Cl}(W)$ is a compact subspace of U, so too $p|_{V_{\alpha'}}^{-1}(\mathrm{Cl}(W))$ is a compact subspace of $V_{\alpha'}$ containing x. So E is locally compact.

Compact with finite fibers. Suppose B is compact and $p\colon E \to B$ has finite fibers for each $b \in B$. Let $\mathscr{W}$ be an open cover of E. For each $b \in B$, there are finitely many W_i in $\mathscr{W}$ such that the fiber $f^{-1}(b) \in \cup_{i=1}^{n} W_i$. For each $b \in B$, define U_b to be $\cup_1^{n_b} W_i$ containing the fiber over b. Observe $p(E \setminus U_b)$ is closed in B and doesn't contain b. So $\{(p(E \setminus U_b))^c\}_{b \in B}$ is an open cover of compact B. There then exist finitely many b_j such that

$$(p(E \setminus U_{b_1}))^c \cup \cdots \cup (p(E \setminus U_{b_m}))^c = B.$$

Equivalently,

$$\begin{aligned} \varnothing &= \cap_1^m p(E \setminus U_{b_j}) \\ &\supset p(\cap_1^m (E \setminus U_{b_j})) \\ &= p(E \setminus (\cup_1^m U_{b_j})). \end{aligned}$$

So

$$E = \bigcup_{j=1}^{m} U_{b_j} = \bigcup_{j=1}^{m} \bigcup_{i=1}^{n_{b_j}} W_i,$$

which demonstrates $\mathscr{W}$ has a finite subcover. We conclude the total space E is compact. □

7.3. **[1, No. 54.3].** *Given.* Let $p\colon E \to B$ be a covering map, let α and β be paths in B with $\alpha(1) = \beta(0)$, and let $\tilde{\alpha}$ and $\tilde{\beta}$ be liftings of them such that $\tilde{\alpha}(1) = \tilde{\beta}(0)$.

To prove. $\tilde{\alpha} * \tilde{\beta}$ is a lifting of $\alpha * \beta$.

Proof. To verify that the paths $\alpha * \beta = p \circ (\tilde{\alpha} * \tilde{\beta})$, clearly both have the same domain I and codomain B. Moreover, they have the same graph as evinced by

$$(\alpha * \beta)(t) = \begin{cases} \alpha(t), & t \in [0, \frac{1}{2}] \\ \beta(t), & t \in [\frac{1}{2}, 1]; \end{cases}$$

$$p \circ (\tilde{\alpha} * \tilde{\beta})(t) = \begin{cases} (p \circ \tilde{\alpha})(t), & t \in [0, \frac{1}{2}] \\ (p \circ \tilde{\beta})(t), & t \in [\frac{1}{2}, 1]. \end{cases}$$

We've shown one lift of concatenated paths is the concatenation of the lifts of the individual segments. □

7.4. **[1, No. 54.4].** *Given.* Let $p\colon \mathbf{R} \times \mathbf{R}_+ \to \mathbf{R}^2 \setminus \mathbf{0}$ be defined

$$p\colon (\theta, r) \mapsto r \exp(2\pi i\theta).$$

To demonstrate. We parametrize and sketch the lifts of the paths $t \in [0, 1]$

- $f(t) = (2 - t, 0)$
- $g(t) = ((1 + t) \cos 2\pi t, (1 + t) \sin 2\pi t)$
- $h(t) = f * g$.

7.5. **[1, No. 54.5].** *Given.* Let $p\colon \mathbf{R}^2 \to \mathbf{T}^2$ be a covering map defined by

$$p\colon \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \exp(2\pi i x) \\ \exp(2\pi i y) \end{pmatrix}.$$

Consider the path

$$f(t) = \begin{pmatrix} \exp(2\pi i t) \\ \exp(4\pi i t) \end{pmatrix}.$$

To demonstrate.

- Sketch f when $\mathbf{T}^2$ is identified with the doughnut surface D.
- Find a lifting $\bar{f}$ of f to $\mathbf{R}^2$.
- Sketch $\bar{f}$.

7.6. **[1, No. 54.7].** *Given.* The torus $\mathbf{T}^2$.

To prove. Without Seifert van Kampen's theorem, $\pi_1(\mathbf{T}^2) \cong \mathbf{Z}^2$.

Proof. (Adapted from JP May [2, Ch. 1].) Consider $\mathbf{T}^2$ as $S^1 \times S^1 \subset \mathbf{C}$. For each $\begin{pmatrix} m \\ n \end{pmatrix} \in \mathbf{Z}^2$, define a loop $f_{m,n}$ in $\mathbf{T}^2$ by

$$f_{m,n}(s) = \begin{pmatrix} \exp 2\pi i s m \\ \exp 2\pi i s n \end{pmatrix}.$$

It is easy to check $[f_{m,n}][f_{k,\ell}] = [f_{m+k,n+\ell}]$. Define a group homomorphism $i\colon \mathbf{Z}^2 \to \pi_1\left(\mathbf{T}^2, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$ by $i\begin{pmatrix} m \\ n \end{pmatrix} = [f_{m,n}]$. Now to argue i is an isomorphism of groups.

- Consider the cover $p\colon \mathbf{R}^2 \to \mathbf{T}^2$ defined by $p\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \exp 2\pi i x \\ \exp 2\pi i y \end{pmatrix}$.
- For each representative $f_{m,n}$ in each loop class $[f_{m,n}] \in i(\mathbf{Z}^2) \subset \pi_1\left(\mathbf{T}^2, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$, there's a unique lift $\tilde{f}_{m,n}(s) = \begin{pmatrix} sm \\ sn \end{pmatrix}$ such that $\tilde{f}_{m,n}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. (Apply the path lifting lemma.)
- Define $j\colon \pi_1\left(\mathbf{T}^2, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) \to \mathbf{Z}^2$ by $j[f] = \tilde{f}(1)$, a vector in $\mathbf{Z}^2$ as $p(\tilde{f}(1)) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
- $j[f]$ is independent of representative from $[f]$ as if g is path homotopic to f, then $\tilde{g}(1) = \tilde{f}(1)$. (Apply the homotopy lifting lemma.)
- $j[f_{m,n}] = \begin{pmatrix} m \\ n \end{pmatrix}$ from explicit definition of

  $$\tilde{f}_{m,n}(s) = \begin{pmatrix} sm \\ sn \end{pmatrix},$$

  so $j \circ i\colon \mathbf{Z}^2 \to \mathbf{Z}^2$ is the identity.
- If suffices to check that j is one-to-one for then i is a bijective homomorphism of groups.
  - Suppose $j[f] = j[g]$. Then $\tilde{f}(1) = \tilde{g}(1)$. Thus $\tilde{g}^{-1} * \tilde{f}$ is a loop at $\vec{0}$ in $\mathbf{R}^2$. Now $\mathbf{R}^2$ is contractible, therefore $[\tilde{g}^{-1} * \tilde{f}] = [c_{\vec{0}}]$.
  - Applying the induced group homomorphism p_*, $[g^{-1}][f] = [g^{-1} * f] = [c_{\vec{1}}]$, the identity in $\pi_1\left(\mathbf{T}^2, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$.
  - Therefore $[g] = [f]$, and j is one-to-one. □

7.7. **[1, No. 54.8].** *Given.* Let $p\colon E \to B$ be a covering map, with E path connected. Suppose B is simply connected.

To prove. p is a homeomorphism.

For each $b \in B$, let $p(e) = b$. As B is simply connected, $\pi_1(B, b_0) = \{1\}$. As E is path connected, the induced isomorphism of sets

$$\Phi\colon \pi_1(B, b_0)/p_*(\pi_1(E, e_0)) \to p^{-1}(b)$$

shows that the fiber over b is the singleton $\{e\}$. Letting b run through B, we see the fibers of p over each b are all singletons. Thus p is injective. We already knew p was an open surjection as a covering map, and thus we conclude p is a homeomorphism. □

7.8. **[1, No. 71.3].** *Given.* $X \cong S^1$ and $Y \cong S^2$.

To prove. $\pi_1(X \vee Y) = \mathbf{Z}$. (As $\mathbf{Z}$ is abelian, we'll have a canonical isomorphism between any two based fundamental groups on $X \vee Y$.)

Proof. Let $X \cap Y = \{x\}$. Now $\pi_1(X \cap Y, x)$ and $\pi(Y, x)$ are trivial (both spaces $X \cap Y$ and Y are simply connected). By Seifert van Kampen,

$$\pi_1(X \vee Y, x) \cong \frac{\pi_1(X, x) * \pi_1(Y, x)}{\{e\}} \cong \pi_1(X, x) \cong \pi_1(S_1, 1) \cong \mathbf{Z},$$

noting homotopy groups are preserved under homeomorphism. □

References

[1] J. Munkres, *Topology*. Prentice Hall, Inc., 2000.

[2] J. P. May, *A concise course in algebraic topology*. Chicago; London: University of Chicago Press, 1999.