CONTINUITY AND COMMON SPACES

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2. Assignment due 2018-09-24

2.1. Continuity by nested closures. Let $(X, \mathscr{T})$ and $(Y, \mathscr{W})$ be topological spaces. Consider a function $f\colon X \to Y$. The following are equivalent.

(a) f is continuous.
(b) $f(\overline{A}) \subset \overline{f(A)}$ for all sets $A \subset X$ in domain.
(c) $\overline{f^{-1}(B)} \subset f^{-1}\left(\overline{B}\right)$ for all sets $B \subset Y$ in the codomain.

Proof sketch. (a) implies (c) implies (b) implies (a). I had proved each implication on paper but misplaced my notes. I may revise this proof in a latter assignment.

2.2. Continuity by nested interiors. Consider the same spaces $(X, \mathscr{T})$ and $(Y, \mathscr{W})$ again with a function $f\colon X \to Y$. The following are also equivalent.

(a) f is continuous.
(b) $f^{-1}((B)^\circ) \subset \left(f^{-1}(B)\right)^\circ$ for all sets $B \subset Y$ in the codomain.

Proof sketch. (a) implies (b) implies (a). Same case as above.

2.3. Homeomorphic subspaces of the real line [1, No. 18.5]. The subspace (a, b) of $\mathbf{R}$ is homeomorphic with $(0, 1)$.

Proof. Let $f\colon (a, b) \to (0, 1)$ with $x \mapsto \frac{x-a}{b-a}$. We'll show that f is (i) bijective, (ii) continuous, and (iii) open. Whence f will be a homeomorphism.

(i) Let $f^{-1}\colon (0, 1) \to (a, b)$ be defined by $y \mapsto (b - a)y + a$. Then $f \circ f^{-1} = \mathrm{id}_{(0,1)}$ and $f^{-1} \circ f = \mathrm{id}_{(a,b)}$, so f has a two-sided inverse, and therefore f is a bijection.

(ii) Take an open interval $B_\epsilon(y) \subset (0, 1)$. Now

$$f^{-1}(B\epsilon(y)) = B_\delta(f^{-1}(y)) \subset (a, b) \text{ for } \delta = (b - a)\epsilon$$

is open too.

(iii) Suppose $B_{\epsilon'}(x)$ is open in (a, b). Then $f(B_{\epsilon'}(x)) = B_\delta(f(x)) \subset (0, 1)$ for $\delta' = \frac{\epsilon'}{b-a}$ is open in $(0, 1)$. □

Also, the subspace $[a, b]$ (of $\mathbf{R}$) is homeomorphic with $[0, 1]$.

Proof. Extend f above defined to $g\colon [0, 1] \to [a, b]$ with the same rule of assignment $x \mapsto \frac{x-a}{b-a}$.

(i) g^{-1} exists as the extension of f^{-1} and is the two sided inverse of g, whence g is a bijection.

(ii) Suppose U is open in $[0, 1]$, then $U = W \cap [0, 1]$ for W $\mathbf{R}_{std}$-open. So $f^{-1}(U) = f^{-1}(W) \cap [a, b]$, which is open in $[a, b]$ iff $f^{-1}(W)$ is open in $\mathbf{R}_{std}$. Take any open interval $B_\epsilon(y) \subset W$. Then for a radius $\delta = (b - a)\epsilon$ and a point $x = (b - a)y + a$ we have

$$f^{-1}(B_\epsilon(y)) = B_\delta(z) \subset f^{-1}(W).$$

Hence $f^{-1}(W)$ is the union of open intervals in $\mathbf{R}_{std}$ and therefore open itself.

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(iii) Suppose B is a basis element of the subspace $[a,b]$, then $B = B_{\epsilon'}(x) \cap [a,b]$ and $f(B) = f(B_{\epsilon'}(x)) \cap f([a,b]) = B_{\delta'}(f(x)) \cap [0,1]$ is open in $[0,1]$ for $\delta' = \frac{\epsilon'}{b-a}$. □

2.4. **Continuous at a single point [1, No. 18.6].** The function $f\colon \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbf{Q}, \\ -x^2 & \text{else} \end{cases}$$

is continuous at a single point in the domain, namely 0.

Proof. Let $x \neq 0$ be a real number in the domain. The image $f(x)$ is nested in the open interval $B_\epsilon(f(x))$ with $\epsilon = x^2 > 0$. Now consider any open ball $B_\delta(x)$ in the domain. Both $\mathbf{R} \setminus \mathbf{Q}$ and $\mathbf{Q}$ are dense in $\mathbf{R}$, so there's a point $p \in \mathbf{Q}$ and $\alpha \in \mathbf{R} \setminus \mathbf{Q}$ such that $p, \alpha \in B_\delta(x)$. Whence we have that the open interval $f(B_\delta(x))$ contains both positive and negative points, whereas $B_\epsilon(f(x))$ contains either only positive points, if we choose $x \in \mathbf{Q}$, or only negative points, if we choose $x \in \mathbf{R} \setminus \mathbf{Q}$. Hence $f(B_\delta(x) \not\subset B_\epsilon(f(x))$. Therefore f is not continuous at $x \neq 0$.

If $x = 0$ then for any $\epsilon > 0$ there's a $\delta = \sqrt{\epsilon}$ such that $F(B_\delta(0)) \subset B_\epsilon(0)$, so f is continuous at $x = 0$. □

2.5. **Left and right continuity [1, No. 18.7].**

(a) Suppose that $f\colon \mathbf{R} \to \mathbf{R}$ is right continuous[1] at each point $a \in \mathbf{R}$. Then f is continuous when consider as a function from $\mathbf{R}_\ell$ to $\mathbf{R}$.

(b) We exhaustively list the functions

- which are continuous from $\mathbf{R}$ to $\mathbf{R}_\ell$, also
- which are continuous from $\mathbf{R}_\ell$ to itself.

For (a) note $\mathbf{R}_\ell$ is finer than $\mathbf{R}$ (here $\mathbf{R}_\ell$ denotes the topological space on the real numbers endowed with the lower limit topology), so $C_{\mathbf{R}}(\mathbf{R}_\ell) \subset C_{\mathbf{R}}(\mathbf{R})$ (where $C_D(Y)$ denotes the set of continuous functions out of some space D into the space Y).

Now I claim $C_{\mathbf{R}}(\mathbf{R}_\ell)$ is the set of constant functions.

Proof. ($\subset$) For suppose f is a constant function. Then for all $\mathbf{R}_\ell$-open U, either $f^{-1}(U) = \varnothing$ or $f^{-1}(U) = \mathbf{R}$, both open in $\mathbf{R}$. So $f \in C_{\mathbf{R}}(\mathbf{R}_\ell)$.

($\supset$) In the other direction, suppose that $f \in C_{\mathbf{R}}(\mathbf{R}_\ell)$ with f not constant. Then WLOG there exist $a < b$ such that $f(a) < f(b)$. Let

$$C = \{c \in (a,b) : f(c) \in (f(a), f(b))\}.$$

Since $f \in C_{\mathbf{R}}(\mathbf{R})$ we have the intermediate value property, therefore C is a non-empty bounded set of real numbers. Now let $c = \inf C$. I claim $f^{-1}([f(c), f(b)))$ is not open in $\mathbf{R}$. Indeed $c \in f{-}1([f(c), f(b)))$ but for all $\epsilon > 0$ the open interval $B_\epsilon(c) \not\subset f^{-1}([f(c), f(b)))$. Why? Because if $c - \epsilon/2 \in f^{-1}([f(c), f(b)))$, then

$$f(c - \epsilon/2) \in [f(c), f(b)) \subset (f(a), f(b))$$

[1] See discussion from "Definition of limit of function on topological spaces", `https://math.stackexchange.com/questions/835978`. Mathematics Stack Exchange. Retrieved September 23, 2018.

`mle:` Let be (A,τ),(C,ζ) two topological spaces, $f \in C^E$, with $E \subseteq A$, and x_0 an accumulation point of E, a point $l \in C$ is *limit* of f as x approaches x_0 if $\forall Y \in \mathcal{U}_{(C,\zeta)}(l)(\exists X \in \mathcal{U}_{(A,\tau)}(x_0)(f((X - \{x_0\}) \cap E) \subseteq Y))$ where $\mathcal{U}_{(A,\tau)}(x_0)$ is neighbourhood system for x_0 and $\mathcal{U}_{(C,\zeta)}(l)$ is neighbourhood system for l.

`Daniel Fischer:` Another way to put the definition is to say that $\tilde{f}\colon E \cup \{x_0\} \to C$ [is continuous when] defined by

$$\tilde{f}(x) = \begin{cases} f(x), & x \in E \setminus \{x_0\} \\ l, & x = x_0. \end{cases}$$

Proof. Suppose f is right continuous at all points in its domain. That is, suppose $\lim_{x \to a^+} f(x) = f(a)$ for all $a \in \mathbf{R}$. Then for any point $a \in \mathbf{R}$ and any neighborhood Y in the local neighborhood system $\mathcal{U}_{(\mathbf{R},\mathscr{T}_{std})}(f(a))$ about $f(a)$ there's a neighborhood X in the neighborhood system $\mathcal{U}_{(\mathbf{R},\mathscr{T}_\ell)}(a)$ surrounding a (where $\mathscr{T}_\ell$ denotes the lower limit topology) such that $f(X \setminus \{a\}) \subset Y$. Now since $f(\{a\}) \subset Y$, we know $f(x) = f(X \setminus \{a\} \cup \{a\}) \subset Y$. So $f\colon (\mathbf{R}, \mathscr{T}_\ell) \to (\mathbf{R}, \mathscr{T}_{std})$ is continuous at every point in its domain, therefore *continuous*.

but $c - \epsilon/2 < c = \inf C$, a contradiction. So if f is not constant, then f is not continuous from $\mathbf{R} \to \mathbf{R}_\ell$. □

I conjecture (without proof) $f\colon \mathbf{R} \to \mathbf{R}_\ell$ is continuous iff f is piecewise defined as $\mathbf{R}_{std}$ continuous and monotonically increasing on each set of any partition of $\mathbf{R}$ into countably many half open intervals.

2.6. **Maps into the order topology [1, No. 18.8].** Let Y be an ordered set in the order topology. Let $f, g\colon X \to Y$ be continuous maps.

(a) The set $\{x : f(x) \leq g(x)\}$ is closed in X.
(b) The function $h\colon X \to Y$ defined by $h(x) = \min\{f(x), g(x)\}$ is continuous.

Proof. To show (a) that $K = \{x : f(x) \leq g(x)\}$ is closed in X, note that if for all $x \in X$ we have that $f(x) \leq g(x)$, then $K = X$ and we're done. So suppose that for some $x \in X$ it's the case that $f(x) > g(x)$. Now we'll argue that K contains all of its limit points.

For contradiction, suppose K has a limit point k such that $k \notin K$ (that is, $f(k) > g(x)$). Since k is a limit point of K for all open sets V in X (including the pullbacks $f^{-1}(U)$ and $f^{-1}(U)$ of some open $U \in Y$) we that

$$V \cap (K \setminus \{k\}) \neq \varnothing.$$

Take a X basis element $B \ni f(k)$ such that if B contains no points[2] c such that $f(c) \leq g(c)$. Then $f^{-1}(B)$ is X open and yet

$$f^{-1}(B) \cap (K \setminus \{k\}) \subset \{x : f(x) \in B\} \cap \{x\colon f(x) \leq g(x)\} = \varnothing$$

since if $f(x) \in B$ then $f(x) > g(x)$. Therefore if k is a limit point of K, we must have that $k \in K$. So K is closed.

To show (b) that h is continuous, take any $a \in X$. We will demonstrate that for all Y-open basis sets $B(h(a))$ there's a X open U such that $h(U) \subset B(h(a))$. Note that if $h(a) = f(a) = g(a)$ we are done, as there exist open $U_f \ni a$ and $U_g \ni a$ satisfying $f(U_f) \subset B(h(a))$ and $g(U_g) \subset B(h(a))$, and $a \in U = U_f \cap U_g$ will satisfy $h(U) \subset B(h(a))$. So without loss of generality, suppose that $h(a) = g(a) < f(a)$ and consider $B(h(a)) = B(g(a))$. Now, because g is continuous at a, there's an open set $U_g \cap a$ such that $g(U_g) \subset B(g(a))$. Take the open set (its the complement of a set we just proved is closed)

$$V = \{x\colon g(x) < f(x)\}$$

and we have the desired open set $U_g \cap B$ such that

$$h(U_g \cap V) = g(U_g \cap V) \subset g(U_g) \subset B(g(a))$$

where the first equality is by definition of h and choice of V. □

2.7. **Maps out of the product topology [1, No. 18.12].** Let $F\colon \mathbf{R} \times \mathbf{R}$ be defined by

$$F(x \times y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } x \times y \neq 0 \times 0, \\ 0 & \text{else.} \end{cases}$$

(a) F is continuous in each variable separately.
(b) We compute the restriction $g\colon \mathbf{R} \to \mathbf{R}$ defined by $g(x) = F(x \times x)$.
(c) F is not continuous.

Demonstration. Without loss of generality, we'll show that F is continuous in x (since F is symmetric in x and y by relabelling). Fix $y = y_0$ and restrict $F|_{\mathbf{R}\times\{y_0\}}$ to the horizontal line through y_0. That $F|_{\mathbf{R}\times\{y_0\}}$ is continuous for $(x, y_0) \neq (0, 0)$ is clear, for it's an sum, product, and quotient of continuous functions. Now suppose that $(x, y_0) = (0, 0)$. Then, consider any $\mathbf{R}$-basis element $B_\epsilon(0) \ni 0$ for $\epsilon > 0$. The open set $\mathbf{R}$ maps into $B_\epsilon(0)$ since (with $y_0 = 0$) the restriction $F|_{\mathbf{R}\times\{0\}}(x) = 0$ for all $x \in \mathbf{R}$. We've shown (a) that F is continuous in each variable separately.

Now let $g\colon \mathbf{R} \to \mathbf{R}$ be defined as the composition $g = F \circ h$ where

$$\mathbf{R} \xrightarrow{h} \mathbf{R}^2 \xrightarrow{F} \mathbf{R}$$

[2] Verify B is nonempty, consider $f(k) \in B$.

and $h\colon x \mapsto (x, x)$. Then

$$g(x) = \begin{cases} \frac{1}{2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

This shows (b).

Lastly, for (c), given g is *not* continuous from $\mathbf{R}$ to $\mathbf{R}$, yet h is continuous, we must have that F is not continuous. □

2.8. **Projections and quotient maps [1, No. 22.3].** Let $\pi_1 : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be the projection on the first coordinate, and consider the subspace $A = \{x \times y : x \geq 0 \text{ or } y = 0\}$. The restriction $q\colon A \to \mathbf{R}$ obtained by $q = \pi_1|_A$ is a quotient map that is neither open nor closed.

Proof. Let the map $q\colon A \to \mathbf{R}$ be defined by the composition $q(z) = (\pi_1 \circ \iota)(z)$ where π_1 is the first coordinate projection of $\mathbf{R}^2$ onto $\mathbf{R}$ and ι is the inclusion map of A into $\mathbf{R}^2$.

To show that q is a quotient map, note that q is clearly surjective. Further, if U is open in $\mathbf{R}$, then $\pi_1^{-1}(U)$ is open in $\mathbf{R}^2$, whence in A we have the open set $q^{-1}(U) = A \cap \pi^{-1}(U)$. Lastly, suppose that $q^{-1}(U)$ is open in A, and consider

$$x \in (\pi_1 \circ \iota)(q^{-1}(U)) = U.$$

Now $q^{-1}(x) \subset q^{-1}(U)$, so for all $z \in q^{-1}(x)$ there's an $\epsilon_z > 0$ such that

$$q^{-1}(U) \supset B_z = B_{\epsilon_z}(z) \cap A \text{ with } B_{\epsilon_z}(z) \text{ an open set in } \mathbf{R}^2.$$

Assuming[3] is open, we have that $\pi_1(B_{\epsilon_z}(z)) \supset \pi_1(\iota(q^{-1}(U))) = U$ and $\pi_1(B_{\epsilon_z}(z)) \ni x$. So U is open, as desired. Because q is a *strongly continuous*, surjective function, we say that q is a *quotient map*.

Now to show that q is neither open nor closed. To see that q is not open, consider the A-open set $U = B_\epsilon(0 \times 1) \cap A$ where $0 < \epsilon < 1$. Then $q(U) = [0, 1)$, which is not $\mathbf{R}$-open. To see that q is not closed, consider the closed set

$$K = \left\{(x, y) : y \leq \frac{1}{1 - x} \text{ and } 1 < x \leq 2\right\}.$$

Its image $q(K) = \pi_1(K) = (1, 2]$ is not closed.

2.9. **Equivalence relations and quotient spaces [1, No. 22.4].**

(a) Define an equivalence relation $\sim$ on the plane $\mathbf{R}^2$ by

$$x_0 \times y_0 \sim x_1 \times y_1 \text{ iff } x_0 + y_0^2 = x_1 + y_1^2.$$

I claim the real line $\mathbf{R}$ is homeomorphic to this equivalence relation's corresponding quotient space $\mathbf{R}^2/\sim$.

That is, the sets in $\mathbf{R}^2/$ are exactly the levels curves $x + y^2 = c$, i.e., left opening parabolas where for each $c \in \mathbf{R}$ the unique parabola on the level curve has the directrix $x = c - \frac{1}{2}$ and the focus $(c + 1/2, 0) \in \mathbf{R}^2$. A homeomorphism is apparent—map the level curves satisfying $x + y^2 = c$ to the point c on the real line.

(b) Now redefine the previous equivalence relation $\sim$ (again on the plane) by

$$x_0 \times y_0 \sim x_1 \times y_1 \text{ iff } x_0^2 + y_0^2 = x_1^2 + y_1^2.$$

In this case, the sets in the quotient space are the level curves $x^2 + y^2 = c$, that is, circles about the origin of radius $\sqrt{c}$. Now $\mathbf{R}/\sim$ is homeomorphic to the half open ray $[0, \infty)$, where such a homeomorphic map can be obtained by mapping the level curves of radius $\sqrt{c}$ to $c \in [0, \infty)$.

[3] I'm not convinced. For example, I'll try to show that π_1 is *not closed* (and therefore, maybe, also not open?) in the next section, so this logic seems a bit circuitous.

2.10. **Restricting to open maps [1, No. 22.5].** Suppose that $p\colon X \to Y$ is an open map. If A is open in X, then the map $q\colon A \to p(A)$ obtained by restricting p is an open map.

Proof. We want to show that if U is open in the subspace A then $q(U)$ is open in the subspace $p(A)$. So let U be open in A. Then $U = W \cap A$ is open in X (since W is open in X by definition of a subspace and since A is open in X by assumption). Then $p(W \cap A) = p(W) \cap p(A)$ is Y-open (its a finite intersection of open sets) and thus $q(U) = p(W \cap A)$ is Y-open. □

2.11. **Quotienting out K in the K-topology [1, No. 22.6].** Let Y be the quotient space obtained from $\mathbf{R}_K$ by collapsing the set K to point, with $p\colon \mathbf{R}_K \to Y$ as the corresponding quotient map.

(a) Y is T_1 but not Hausdorff.
(b) The map $p \times p\colon \mathbf{R}_K \times \mathbf{R}_K \to Y \times Y$ is not a quotient map.[4]

Proof. TODO.

References

[1] J. R. Munkres, *Topology*, 2nd ed. Hardcover; Prentice Hall, Inc., 2000 [Online]. Available: `http://www.worldcat.org/isbn/0131816292`

[4]The diagonal is not closed in $Y \times Y$, but its inverse image is closed in $\mathbf{R}_K \times \mathbf{R}_K$.