BASES, SUBSPACES, AND CLOSED SETS

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1. Assignment due 2018-09-10

1.1. **The topology generated by a basis [1, No. 13.5].** If $\mathscr{B}$ is a basis for a topology on X, then the topology generated by $\mathscr{B}$ equals the intersection of all topologies on X that contain $\mathscr{B}$.

Proof by set inclusion. ($\subset$) Suppose $\mathscr{B}$ is a basis for and $\mathscr{T}_{\mathscr{B}}$ is the topology generated by $\mathscr{B}$, i.e., the set of unions of of families of $B_i \in \mathscr{B}$. Now, if U is open in $\mathscr{T}_{\mathscr{B}}$, then $U = \cup B_i$. If $\mathscr{T}_j$ is any other topology on X such that $\mathscr{T}_j \supset \mathscr{B}$, because $\mathscr{T}_j$ is closed under unions of open sets, we have

$$U = \cup B_i \in \mathscr{T}_j.$$

So $\mathscr{T}_{\mathscr{B}} \subset \mathscr{T}_j$ for all topologies $\mathscr{T}_j \supset \mathscr{B}$ on X. Hence

$$\mathscr{T}_{\mathscr{B}} \subset \bigcap_{\forall \mathscr{T}_j \supset \mathscr{B}} \mathscr{T}_j.$$

($\supset$) Now suppose that the open set

$$V \in \bigcap_{\forall \mathscr{T}_j \supset \mathscr{B}} \mathscr{T}_j.$$

We need to show that $V \in \mathscr{T}_{\mathscr{B}}$. But $V \in \mathscr{T}_j$ for all topologies containing $\mathscr{B}$. Since $\mathscr{T}_{\mathscr{B}} \supset \mathscr{B}$ by construction, $V \in \mathscr{T}_{\mathscr{B}}$. So the topology generated by the basis $\mathscr{T}_{\mathscr{B}}$ is in the intersection of all topologies $\mathscr{T}_j$ containing the basis. $\square$

Further, the same result holds if $\mathscr{A}$ is a subbasis.

Proof by set inclusion. ($\subset$) Let U be an open set in the topology generated by the sub-basis $\mathscr{A}$, denoted $\mathscr{T}_{\mathscr{A}}$. Then U is the union of finite intersections of sets A_i in the sub-basis. Now note that if $\mathscr{T}_j$ is a topology on X and $\mathscr{T}_j \supset \mathscr{A}$, because $\mathscr{T}_j$ is closed under unions and finite intersections, we have $U \in \mathscr{T}_j$. So $\mathscr{T}_{\mathscr{A}}$ for all such topologies on X containing the sub-basis. Whence

$$\mathscr{T}_{\mathscr{A}} \subset \bigcap_{\mathscr{T}_j \supset \mathscr{A}} \mathscr{T}_j.$$

($\supset$) Let $U \in \mathscr{T}_j$ for all topologies $\mathscr{T}_j$ containing $\mathscr{A}$. Clearly $\mathscr{T}_{\mathscr{A}}$ is one such topology. So $U \in \mathscr{T}_{\mathscr{A}}$. Whence $\mathscr{T}_{\mathscr{A}}$ is contained in the intersection of all $\mathscr{T}_j$ containing the sub-basis $\mathscr{A}$. $\square$

1.2. **Incomparable topologies on R [1, No. 13.6].** We show two topologies on the real line are not comparable, namely $\mathscr{T}_\ell$, the lower limit topology generated by half open intervals $[a, b)$, and $\mathscr{T}_K$, the K-topology generated by the punctured open intervals $(a, b) \setminus \{n^{-1} : n \in \mathbf{N}\}$. First, a lemma.

Lemma. The lower limit topology $\mathscr{T}_\ell$ and the K-topology $\mathscr{T}_K$ are strictly finer than the standard topology on $\mathbf{R}$.

Proof sketch. Note the basis of the K-topology contains the basis for the standard topology. For the lower limit topology, verify that every standard basis element (a, b) in $\mathbf{R}$ can be written as the countable union of lower limit basis elements $\left[a + \frac{1}{n}, b\right)$ for $n \in \mathbf{N}$.

Now to show that the lower limit topology and the K-topology are not comparable, we'll show that neither one is a subset of the other.

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Demonstration. Consider $U = [-1, 0) \in \mathscr{T}_\ell$. Is U open in $\mathscr{T}_K$? No, for take $-1 \in U$ and suppose that any basis element B_K of the K-topology contains -1. Then there's an $\varepsilon > 0$ such that $-1 - \varepsilon \in B_K$, whence $B_K \not\subset U$. So U does not have the property that every point in U can be contained in a basis element B_K that is contained in U. Since the B_K are basis element of the K-topology, we conclude that U is not an K-topology open set.

On the other hand, consider $V = (-1, 1) \setminus \{1/n : n \in \mathbf{N}\}$, an open set in $\mathscr{T}_K$. For contradiction, suppose that V is open in the lower limit topology. Then for all points in the open set V there's a basis element of the form $[a, b)$ containing the point and nested in V. Take $0 \in V$ and suppose that B is a lower limit basis element for which $0 \in B \subset V$. Now we have $B = [a, b)$ for some $b > 0$. By the archimedean principle, there exists a natural number $1/n < b$, so $1/n \in B$. But $1/n \notin V$ so $B \not\subset V$, as desired for our contradiction. Whence V is not open in the lower limit topology.

1.3. **Partially ordering topologies [1, No. 13.7].** There's a natural partial ordering (by inclusion) for the following topologies on $\mathbf{R}$:

- the standard topology
- the K-topology
- the finite complement topology
- the upper limit topology
- the topology generated by the rays $(-\infty, a)$

The structure of the poset is given by the Hasse diagram, with the finest topologies at top.

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  upper limit topology
          |
      K topology
          |
   standard topology
    /             \
ray topology     finite complement
                  topology
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To offer brief justification.

1. The upper limit topology contains all standard basis elements (e.g., as infinite unions of the form $(a, b - 1/n]$ for all natural numbers n). Moreover, the upper limit topology contains all K-topology basis elements since $(a, 0]$ is open in the upper limit topology (verify).

2. We have that the K-topology is finer than the standard topology by the above lemma.

3. We know the standard topology contains the ray topology because every ray basis element $(-\infty, a)$ is open in the standard topology.

4. We know the standard topology contains the finite complement topology because every set of real numbers with a finite complement (say, n distinct points) is the union of finitely many open intervals (e.g., $(-\infty, r_1) \cup (r_1, r_2) \cup \cdots \cup (r_n, \infty)$).

5. The finite complement topology and the ray topology are not comparable as

- $(-\infty, a)$ has no finite complement and
- $(-\infty, a) \cup (a, \infty)$ cannot be represented as a union of rays.

1.4. **Countable bases for topologies on R [1, No. 13.8].**

(a) $\mathscr{B} = \{(a, b) : a < b \text{ and } a, b \in \mathbf{Q}\}$ is a countable basis for the standard topology on $\mathbf{R}$.

Proof. Suppose U is open in $\mathbf{R}$ and take $x \in U$. Now, since the open balls are a basis for the standard topology, there's an $\varepsilon > 0$ such that the open ball around x of radius ε satisfies $x \in B_\varepsilon(x) \subset U$. By the archimedean principle, there exists a natural number n and (with help from the division algorithm) a rational number p such that

$$\frac{1}{n} < \frac{\varepsilon}{2} \text{ and } |p - x| < \frac{\varepsilon}{4}.$$

Now the rational ball $(p - 1/n, p + 1/n)$ contains x (one should verify) and is nested in $B_\varepsilon(x)$ (verify). So the rational balls are a basis for $\mathbf{R}$. Further, since the rational open balls are parameterized by two countable sets, they themselves are countable. □

(b) The collection $\mathscr{C} = \{[a, b) : a < b \text{ and } a, b \in \mathbf{Q}\}$ generates a topology distinct from the lower limit topology.

Demonstration. We have $[\pi, b) \in \mathscr{T}_\ell$, the lower limit topology, but $[\pi, b) \notin \mathscr{C}$. Why? Suppose for contradiction that $[\pi, b) \in \mathscr{T}_\ell$. Since $[\pi, b)$ is an open set there's some basis element $[r, s)$ containing the point π where r and s are rational. But since $\pi \neq r$, and since we can't have $\pi < r$ (recall $\pi \in [r, s)$ as a basis element), it must be that $r < \pi$. But then $[r, s) \not\subset [\pi, b)$, our desired contradiction.

1.5. **Defining open sets in a subspace [1, No. 16.3].** Consider the set $Y = [-1, 1]$ as a subspace of $\mathbf{R}$. Which of the following sets are open in Y? in $\mathbf{R}$?

parameters	union of intervals	open in $\mathbf{R}$?	open in Y?
$\left\{x : \frac{1}{2} < \lvert x\rvert < 1\right\}$	$(-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1)$	yes	yes
$\left\{x : \frac{1}{2} < \lvert x\rvert \leq 1\right\}$	$[-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1]$	no	yes
$\left\{x : \frac{1}{2} \leq \lvert x\rvert < 1\right\}$	$(-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1)$	no	no
$\left\{x : \frac{1}{2} \leq \lvert x\rvert \leq 1\right\}$	$[-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$	no	no
$\left\{x : 0 < \lvert x\rvert < 1 \text{ and } \frac{1}{x} \notin \mathbf{N}\right\}$	$(-1, 0) \cup \left(\bigcup_{n=1}^\infty \left(\frac{1}{n+1}, \frac{1}{n}\right)\right)$	yes	yes

1.6. **Open maps between spaces [1, No. 16.4].** Consider a space X and its subspace Y. The projections

$$\pi_1 \colon X \times Y \to X \text{ and } \pi_2 \colon X \times Y \to Y$$

are open maps.

Proof. Suppose $U \times V$ is open in $X \times Y$ (so U and V are open in X and Y respectively). Then $\pi_1(U \times V) = U$ is open in X, and $\pi(U \times V) = V$ is open in Y. Since open sets are mapped to open sets by both projections, we say both projections are *open maps*. □

1.7. **A countable basis for $\mathbf{R}^2$ [1, No. 16.6].** The countable collection

$$\{(a, b) \times (c, d) : a < b \text{ and } c < d \text{ and } a, b, c, d \in \mathbf{Q}\}$$

is a basis for $\mathbf{R}^2$.

Proof. We are to show that for an arbitrary open $U \in \mathbf{R}^2$, for each $x \in U$ there's a rational rectangle containing x and embedded in U. Now, the open balls are a basis for $\mathbf{R}^2$, so we know there exists and $\varepsilon > 0$ such that $x \in B_\varepsilon(x) \subset U$. We need to fit our rational rectangle within $B_\varepsilon(x)$.

Now, consider the vector form of x as $(x_1, x_2)^T$. Choose a rational $r = (r_1, r_2)^T$ such that

$$\frac{\varepsilon}{3\sqrt{2}} > \max\{|x_1 - r_1|, |x_2 - r_2|\}.$$

Then take the rational rectangle (a product of rational open intervals)

$$B_{\mathbf{Q}} = \left(r_1 - \frac{2\varepsilon}{3\sqrt{2}}, r_1 + \frac{2\varepsilon}{3\sqrt{2}}\right) \times \left(r_2 - \frac{2\varepsilon}{3\sqrt{2}}, r_2 + \frac{2\varepsilon}{3\sqrt{2}}\right).$$

I claim that $B_{\mathbf{Q}} \in B_\varepsilon(x)$ since the boundary of $B_{\mathbf{Q}}$ at most coincides with the boundary of $B_\varepsilon(x)$ at one point (verify). By construction $x \in B_{\mathbf{Q}}$, so all together we have $x \in B_{\mathbf{Q}} \subset B_\varepsilon(x) \subset U$. Whence the rational rectangles are a basis for the standard topology on the plane $\mathbf{R}^2$.

1.8. **Endowing a line with subspace topologies [1, No. 16.8].** Consider a straight line L in the plane. We'll describe the topological structure of L endowed with the subspace topology of $\mathbf{R}_\ell \times \mathbf{R}$ and $\mathbf{R}_\ell \times \mathbf{R}_\ell$ respectively (where $\mathbf{R}_\ell$ is the lower limit topological space, and $\mathbf{R}$ is the standard topological space on the real line).

We have two trivial cases to rule out. Consider when $L \subset \mathbf{R} \times \mathbf{R}$ is either $\{x\} \times \mathbf{R}$ or $\mathbf{R} \times \{y\}$. Then L is homoemorphic to the topological space in it's left (respectively right) coordinate.

Now, without loss of generality, suppose L is the diagonal line through the origin.

$$L = \{(t, t) : t \in \mathbf{R}\} \subset \mathbf{R} \times \mathbf{R}.$$

- Endow L with the subspace topology from $\mathbf{R}_\ell \times \mathbf{R}$. We have basis elements B of $\mathbf{R}_\ell \times \mathbf{R}$ of the form $[a, b) \times (c, d)$. Now these basis elements intersect with L to produce sets of the form

$$\{(t, t) : a \leq t < b\} \cup \{(t, t) : c < t < d\}.$$

  More concisely, since we can represent (c, d) as the union of sets of the form $[a, b)$ we see that L is homeomorphic to the lower limit topology. That is, L has the topology generated by half open intervals $L \cap ([a, b) \times [a, b))$.
- On the other hand, endow L with the subspace topology from $\mathbf{R}_\ell \times \mathbf{R}_\ell$. Then immediately we see L has the topology generated by half open intervals $L \cap ([a, b) \times [a, b))$.

1.9. **Closed sets in subspaces [1, No. 17.2].** Suppose that Y is closed in X and A is closed in Y where Y is given the subspace topology. To show that A is closed in X, note that A is closed in Y iff $A = K \cap Y$ where K is closed in X. But since $A \subset Y$, the only[1] X-closed set K for which $K \cap Y = A$ is A. So A is closed in X.

1.10. **[1, No. 17.6].** Let A, B, and A_α for some index α be subsets of a space X.

(a) If $A \subset B$, then $\overline{A} \subset \overline{B}$.

Proof. Suppose that $A \subset B$. Because $B \subset \overline{B}$, we have $A \subset \overline{B}$. Since $\overline{A}$ is contained in the intersection of all closed sets containing A, it follows that $\overline{A} \subset \overline{B}$. □

(b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Proof by set containment. ($\subset$) By the previous argument we have $\overline{A} \subset \overline{A \cup B}$ and further $\overline{B} \subset \overline{A \cup B}$. Whence $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$. ($\supset$) Clearly $\overline{A} \cup \overline{B} \supset A \cup B$, so we only need to consider those limit points p of $A \cup B$ and argue they are contained in either $\overline{A}$ or $\overline{B}$. We show the contrapositive. Suppose that p is not a limit point of A or B. Then there exist open sets U_A and U_B both containing p such that the intersections $U_A \cap A \setminus \{p\}$ and $U_B \cap A \setminus \{p\}$ are both nonempty. Now $p \in U_A \cap U_B$, an open set. We deduce that $(U_A \cap U_B) \cap (A \cup B) \setminus \{p\}$ is nonempty, which suffices to show that p is not a limit point of $A \cup B$. □

(c) $\overline{\cup A_\alpha} \supset \cup \overline{A_\alpha}$, and there's an example of proper containment.

Proof. Let $x \in \cup \overline{A_\alpha}$. If $x \in A_\alpha$ then clearly $x \in \cup A_\alpha \subset \overline{\cup A_\alpha}$. Suppose then that x is a limit point of some A_α. Then every open set $U \ni x$ has nontrivial intersection with $A_\alpha \setminus \{x\}$, say, at a point $y \in U \cap A_\alpha \setminus \{x\}$. Whence $y \in U \cap (\cup A_\alpha) \setminus \{x\}$. We deduce then that x is a limit point of the union $\cup A_\alpha$. So $x \in \overline{\cup A_\alpha}$. □

For an example of proper containment, consider $A_n = (-1 + \frac{1}{n}, 1 - \frac{1}{n})$ in the standard topology on the real line. We have

$$[-1, 1] = \overline{\bigcup A_n} \supsetneq \bigcup \overline{A_n} = (-1, 1).$$

[1]One should verify: Why is A the *only such set*?

1.11. **Finite products of Hausdorff spaces [1, No. 17.11].** The product of two Hausdorff spaces is itself Hausdorff.

Proof. Suppose X and Y are Hausdorff spaces and consider the product space $X \times Y$. We'll show that because $X \times Y$ is component-wise Hausdorff, we have that $X \times Y$ is Hausdorff. So let (x_1, y_1) and (x_2, y_2) be distinct points in the space. Now component-wise, there are disjoint open sets U_1, U_2 in X such that $U_1 \ni x_1, U_2 \ni x_2$ and similarly for Y there are disjoint open sets V_1, V_2 in such that $V_1 \ni y_1, V_2 \ni y_2$. Now in the product topology, we have disjoint open sets

$$U_1 \times V_1 \ni (x_1, y_1) \text{ and } U_2 \times V_2 \ni (x_2, y_2),$$

as desired. □

1.12. **Characterizing topologies on R [1, No. 17.16].**

(a) We have the following table for the closure of the set $K = \{1/n : n \in \mathbf{N}\}$ in each topology on $\mathbf{R}$.

which topology on $\mathbf{R}$?	what is $\overline{K}$?
standard	$K \cup \{0\}$
K-topology	K
finite complement	$\mathbf{R}$
upper limit	K
generated by $(-\infty, a)$	$\mathbf{R}_{\geq 0}$

(b) Further, we'll distinguish the above spaces in terms of two separation axioms. (Recall that every Hausdorff space is T_1.)

which topology on $\mathbf{R}$?	Hausdorff?	T_1?
standard	yes	yes
K-topology	yes	yes
finite complement	no	yes
upper limit	yes	yes
generated by $(-\infty, a)$	no	no

1.13. **Closure operations [1, No. 17.17].** Consider $\mathbf{R}_\ell$ the lower limit topology on the reals and the topology $\mathscr{C} = \{[a, b) : a < b \text{ and } a, b \in \mathbf{Q}\}$.

We have the following table for the closure of two distinct sets in the aforementioned topologies respectively.

	$\mathbf{R}_\ell$	$\mathscr{C}$
$(0, \sqrt{2})$	$[0, \sqrt{2})$	$[0, \sqrt{2}]$
$(\sqrt{2}, 3)$	$[\sqrt{2}, 3)$	$[\sqrt{2}, 3)$

References

[1] J. R. Munkres, *Topology*, 2nd ed. Hardcover; Prentice Hall, Inc., 2000 [Online]. Available: `http://www.worldcat.org/isbn/0131816292`