DIAGNOSTIC FOR FINAL

COLTON GRAINGER (MATH 6210)

2018-10-10 Midterm.
Question 1A. Recall that the **upper-limit topology** is the topology $\mathscr{U}$ on $\mathbf{R}$ with basis

$$\{(a, b] : a, b \in \mathbf{R}, a \leqslant b\}.$$

i. Determine whether the sets (a, b) and $[a, b)$ are open and/or closed in $(\mathbf{R}, \mathscr{U})$. Explain.
ii. Is $(\mathbf{R}, \mathscr{U})$ connected? Explain.

Question 1B. Let X be a nonempty set. Is X with the finite complement topology connected? Is the answer the same for all sets $X \neq \varnothing$, or do you need to distinguish cases that depend on certain properties of X? Give proof.
Question 2A. Let $\mathscr{N} = \{1/n : n \in \mathbf{N}\} \subset \mathbf{R}$ denote the set of all reciprocals of natural numbers. Let

$$\mathscr{B} = \{(a, b) : a, b \in \mathbf{R}\} \cup \{(a, b) \setminus \mathscr{N} : a, b \in \mathbf{R}\} \cup \{\mathbf{R}, \varnothing\}.$$

Now $\mathscr{B}$ generates a topology on $\mathbf{R}$, denote that topology $\mathscr{T}_{\mathscr{B}}$.

i. What is the closure of $\mathscr{N}$ in $\mathscr{T}_{\mathscr{B}}$?
ii. Is every closed set of the topological space $(\mathbf{R}, \mathscr{T}_{\mathscr{B}})$ closed in the standard topology on $\mathbf{R}$?

Question 2B. Is every closed set of the standard topology on $\mathbf{R}$ also closed in the finite complement topology on $\mathbf{R}$?
Question 3. For $n, m \in \mathbf{Z}_{\geqslant 0}$, prove that $(Y_n, \mathscr{T}_n)$ is homeomorphic to $(Y_m, \mathscr{T}_m)$ iff $n = m$. Let $(Y_n, \mathscr{T}_n)$ is defined as follows.

- Start with the subspace $X_n \subset \mathbf{R}^3$ parametrized by
$$X_n := \{(x, y, z) : (x - (1 + 4j))^2 + y^2 + z^2 = 1 \text{ for some } j = 1, \dots, n\}.$$
- Define the equivalence relation $\sim_n$ on X_n where each point is equivalent to itself and
$$(0, 0, 0) \sim_n (4, 0, 0) \sim_n (8, 0, 0) \sim_n \dots \sim_n (4n, 0, 0).$$
- Let $(Y_n, \mathscr{T}_n)$ be the space of equivalence classes $Y_n := X_n / \sim_n$ endowed with the quotient topology.

Question 4A. Prove or provide a counter-example: The path connected components of a topological space are always closed sets.
Question 4B. Prove or provide a counter-example: The path connected components of a topological space are always open if the space is locally path connected?

2018-11-07 Midterm.
Question 1A. Compute the homotopy classes of maps $S^0 \to S^0$.
Question 1B. Prove that if the space $(X, \mathscr{T})$ is path connected, then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ for arbitrary $x_0, x_1 \in X$.
Question 1C. Prove that the fundamental group of the real line based with the standard topology, based at the origin, is trivial.
Question 2A. Let $(X, \mathscr{T})$ and $(Y, \mathscr{W})$ be two spaces and let $f \colon X \to Y$ be a surjective map. Prove that if X is Lindelöff, or has a countable dense subset, then Y satisfies the same condition.
Question 2B. Prove that if a topological space $(X, \mathscr{T})$ is compact, then it is also limit point compact.

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Question 2C. For the real line $\mathbf{R}$ with the standard topology, a subspace $C \subset \mathbf{R}$ is compact if and only if it's closed and bounded. Is the same true for the real line $\mathbf{R}_\ell$ with the lower limit topology?
Question 3A. Show that every metrizable topological space $(X, \mathscr{T})$ with a countable dense subset S has a countable basis.
Question 3B. Give an example of a topological space with a compact subset that is not closed.

Quotienting.

i. Prove that D^n/S^{n-1} is homeomorphic to S^n.[1, No. 22.D]
ii. Prove that $S^1 \times S^1/[(z,w) \sim (-z,\bar{w})]$ is homeomorphic to the Klein bottle $I^2/[(t,0 \sim (t,1),(0,t) \sim (1,1-t)]$. [1, No. 22.14]

Gluing. Recall: if X and Y are topological spaces, $A \subset Y$, and $f\colon A \to X$ is a continuous map, then the *gluing* (or *attaching*) of Y to X via f is the quotient space

$$X \cup_f Y = X \sqcup Y/[a \sim f(a) : a \in A].$$

i. Prove that, by attaching the n-disk D^n to its copy via the identity map of the boundary sphere S^{n-1}, we obtain a space homeomorphic to S^n. [1, No. 22.M]

Real projective space. Recall $\mathbf{RP}^n$ is the quotient space of S^n by the partition into pairs of antipodal points.

i. Show that $\mathbf{RP}^n$ is canonically homeomorphic to the metric space whose points are lines of $\mathbf{R}^{n+1}$ through the origin, where the angle measure between two lines serves as a metric. Check that angle measure *does* give a metric. (Hint: use homogeneous coordinates $(x_0 : x_1 : \cdots : x_n)$ in $\mathbf{RP}^n$.) [1, No. 23]
ii. Prove that the natural projection $S^n \to \mathbf{RP}^n$ is a covering.

Deformation retractions. Recall: If X is a space and $A \subset X$, then $\rho\colon X \to A$ is a *retraction* if ρ is continuous and $\rho|_A = \mathrm{id}_A$. A retraction $\rho\colon X \to A$ is a *deformation retraction* if its composition $\iota \circ \rho$ with $\iota\colon A \hookrightarrow X$ is homotopic to the identity map id_X on X. Lastly, a continuous map $f\colon X \to Y$ is said to be a *homotopy equivalence* between X and Y if there's a continuous map $g\colon Y \to X$ such that $g \circ f \cong \mathrm{id}_X$ and $f \circ g \cong \mathrm{id}_Y$. Two spaces X and Y are said to be *homotopy equivalent* if there exists a homotopy equivalence between them.

i. Prove that if A is a deformation retraction of X, then A and X are homotopy equivalent. [1, No. 39.D]
ii. Prove that any two deformation retractions of one and the same space are homotopy equivalent. [1, No. 39]
iii. Prove that S^n is a deformation retraction of $\mathbf{R}^{n+1} \setminus \{0\}$. [1, No. 39]

Bouquet of circles. Recall: Given a family of topological space $\{X_\alpha\}$, we may mark a point x_α in each, take the disjoint sum and identify all marked points. The resulting topological space $\vee_\alpha X_\alpha$ is the *bouquet* of $\{X_\alpha\}$. Let B_q denote the *bouquet of* q *circles*. Let $u_1, \ldots, u_q$ be loops in B_q starting at c and parametrizing the q copies of S^1. Denote by α_i the homotopy class of u_i.

i. Prove that a plane with q punctures is homotopy equivalent to the bouquet of q circles. [1, No. 39.4].

Covering maps.

i. Prove that $\mathbf{R} \to S^1 : x \mapsto \exp(2\pi i x)$ is a covering. [1, No. 34.C]
ii. Prove that $\mathbf{C} \to \mathbf{C}\colon z \mapsto \exp(z)$ is a covering. [1, No. 34]
iii. In what sense are the above coverings the same? Define an appropriate equivalence relation. [1, No. 34.2]

References

[1] Y. Viro, O. Ivanov, Y. Netsvetaev, and V. Kharlamov, *Elementary topology*. American Mathematical Soc., 2008 [Online]. Available: `http://www.pdmi.ras.ru/~olegviro/topoman/`