COMPACTNESS

COLTON GRAINGER (TOPOLOGY MATH 6210)

5. Assignment due 2018-10-26

5.1. **[1, No. 26.1].** Let $\mathscr{T}$ and $\mathscr{T}'$ be two topologies on the set X; suppose that $\mathscr{T} \supset \mathscr{T}'$.

(a) What does compactness of X under one of these topologies imply about compactness under the other?

If $(X, \mathscr{T})$ is compact, then $(X, \mathscr{T}')$ is compact. The converse fails to hold generally.

(b) If X is compact Hausdorff under both $\mathscr{T}$ and $\mathscr{T}'$, then either $\mathscr{T}$ and $\mathscr{T}'$ are equal or they are not comparable.

Suppose $\mathscr{T}' \subset \mathscr{T}$ and both spaces are Hausdorff. The identity $(X, \mathscr{T}') \xrightarrow{\text{id}} (X, \mathscr{T})$ is a continuous bijection. Since the domain is Hausdorff and the codomain in compact, it's a homeomorphism. □

5.2. **[1, No. 26.5].** Let A and B be disjoint compact subspaces of the Hausdorff space X. There exist disjoint open sets U and V containing A and B, respectively.

Proof. Try to cover one at a time! Fix $a \in A$. As X is Hausdorff, we can find pairs $U_{(a,b)}$ and $V_{(a,b)}$ of open sets to separate the fixed point a from each point b as b varies through B.

Now $\{V_{(a,b)}\}$ is an open cover of B. There's a finite set of points $\{b_j\}_1^n$ corresponding to the finite subcover of B by $\{V_{(a,b_j)}\}_1^n$. Moreover the corresponding $U_{(a,b_j)}$ satisfy $a \in \cap_1^n U_{(a,b_j)}$.

Since a was fixed, and we need to vary a, let's save memory of these open sets for later and denote

$$U_a = \bigcap_1^n U_{(a,b_j)} \quad \text{and} \quad V_a = \bigcup_1^n V_{(a,b_j)}.$$

Now let a run through A. We generate an open cover $\{U_a\}$ of A with each set U_a separated from B and V_a. There's then a finite subcover $\{U_{a_k}\}_1^m$ with $\{V_{a_k}\}_1^m$ corresponding. A separation of A and B by open sets is given by

$$\underbrace{\bigcup_1^m U_{a_i}}_{\text{containing } A} \cap \underbrace{\bigcap_1^m V_{a_i}}_{\text{containing } B} = \varnothing$$

as desired. □

5.3. **[1, No. 26.7].** If Y is compact, then the projection $\pi_1 : X \times Y \to X$ is a closed map.

Proof. Let $K \in X \times Y$ be closed. We want to show that $X \setminus \pi(K)$ is X-open.

Let $z \in X \backslash \pi(K)$. Since K is closed and the slice $\pi^{-1}(z)$ is disjoint from K, each point in $\pi^{-1}(z)$ can be separated from K by an open N_α in the product space. Call the union of such $N = \cup N_\alpha$.

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Observe both $N \cap K = \varnothing$ and $N \supset \pi^{-1}(z)$. (Alternatively, the existence of open N follows from the previous exercise, as $\pi^{-1}(z)$ is compact.) By the tube lemma, there's an X-open $U \ni x$ such that $\pi^{-1}(U) \subset N$. By construction, N separates $\pi^{-1}(U)$ from K, so

$$\pi(K) \cap U = \varnothing.$$

So each point in $X \setminus \pi(K)$ can be contained in an open set $U \subset X \setminus \pi(K)$.

We've shown $X \setminus \pi(K)$ is open; therefore $\pi(K)$ is closed. We conclude that π is a closed map. □

5.4. **[1, No. 26.11].** Let X be a compact Hausdorff space. Let $\mathscr{A}$ be a collection of closed connected subsets of X that is simply ordered by proper inclusion. Then

$$Y = \bigcap_{A \in \mathscr{A}} A$$

is connected.

Proof.[1] By way contradiction, suppose C and D separate $\cap_{A \in \mathscr{A}} A$. Note that C and D are closed in X as closed sets in the closed subspace $\cap A$. Since X is Hausdorff with C and D closed (thus compact), we can find disjoint X-open $U \supset C$ and $V \supset D$ separating C and D.

Because the collection $\mathscr{A}$ is a family of compact subsets of the compact space X, and $\cap_{A \in \mathscr{A}} A \subset U \cup V$, we must have

$$\bigcap_{A \in \mathscr{A}} (A \setminus (U \cup V)) = \varnothing.$$

Since the collection $\mathscr{A}$ is linearly ordered, there's a set $B \in \mathscr{A}$ such that $B \subset U \cup V$. But then B is separated by U and V, which is absurd. So the set $\cap A$ is connected. □

5.5. **[1, No. 26.12].** Let $p \colon X \to Y$ be a closed continuous surjective map such that $p^{-1}(\{y\})$ is compact, for each $y \in Y$ (such a map is called a *perfect map*). If Y is compact, then X is compact.

Proof.[2] We lead out with the lemma.

Lemma. In the setup above, if U is an open set containing $p^{-1}(\{y\})$, there is a neighborhood W of y such that $p^{-1}(W)$ is contained in U.

For each $y \in Y$, take an X-open $U_y \supset p^{-1}(\{y\})$. Observe

- $X \setminus U_y$ is closed in X
- $p(X \setminus U_y)$ is closed in Y
- $y \in \underbrace{Y \setminus p(X \setminus U_y)}_{\text{open!}}$

Denote the open set $W_y := Y \setminus p(X \setminus U_y)$. By construction $y \in W_y$ and

$$W_y \text{ is the complement of the image of the complement of } U_y,$$

therefore $p^{-1}(\{y\}) \subset p^{-1}(W_y) \subset U_y$.

Now continuing the proof, let $\{U_\alpha\}$ cover X. Since $p^{-1}(\{y\})$ is compact, there's a finite subcover $\{U_j\}_1^n$ such that

$$p^{-1}(\{y\}) \subset \bigcup_1^n U_j.$$

For each y, obtain an open cover of Y as follows

[1] See also `https://math.stackexchange.com/questions/1238027`
[2] See also `https://math.stackexchange.com/questions/902472/`.

- Set $U_y = \bigcup_1^{n_y} U_j^y$ as a finite subcover of $p^{-1}(\{y\})$.
- Take the open W_y corresponding to U_y from the lemma.

Since there's an open W_y containing each $y \in Y$, the collection $\{W_y\}$ is a open cover of Y. Since Y is compact, there's a finite subcover $\{W_{y_k}\}_1^m$ with

$$p^{-1}(\{y_k\} \subset p^{-1}(W_{y_k}) \subset \bigcup_1^{n_{y_k}} U_j.$$

Now $X = p^{-1}(Y)$, thus

$$X = \bigcup_1^m \underbrace{p^{-1}(W_y)}_{\text{all with finite covers}}$$

demonstrating any $\{U_\alpha\}$ open cover of X contains a finite subcover. □

5.6. **[1, No. 27.3].** In $\mathbf{R}_K$, the K-topology on the real line:

(a) $[0, 1]$ is not compact.

Recall the previous exercise. If $(X, \mathscr{T})$ and $(X, \mathscr{T}')$ are both compact Hausdorff and comparable, then $\mathscr{T} = \mathscr{T}$.

Both the standard topology ST and the K-topology induce a subspace topology on $[0, 1]$. From analysis, we know that $[0, 1]$ is compact Hausdorff in ST. Now $\mathbf{R}_K$ is strictly finer than ST. By the contrapositive of the previous exercise, either $[0, 1]$ must fail to be compact or fail to be Hausdorff. Clearly $[0, 1]$ is Hausdorff in $\mathbf{R}_K$, so $[0, 1]$ fails to be compact. □

(b) $\mathbf{R}_K$ is connected.

Consider $(-\infty, 0)$ and $(0, \infty)$ with their usual topologies as subspaces from the real line. Now I claim both sets are induced with the same relative topology from $\mathbf{R}_K$. We focus on the positive valued interval. The following are equivalent

- U is $\mathbf{R}_K$-open in $(0, \infty)$
- U is $(0, \infty) \cap V$ for V $\mathbf{R}_K$-open in $\mathbf{R}$
- U is $(0, \infty) \cap W$ for W open in the standard topology and $W \not\ni 0$.

Both intervals are connected in the standard topology. Also, the closures

- Cl $((-\infty, 0)) = (\infty, 0]$
- Cl $((0, \infty)) = [0, \infty)$

are connected. Therefore $\mathbf{R}_K = (-\infty, 0] \cap [0, \infty)$ is connected. □

(c) $\mathbf{R}_K$ is not path connected.

We've just seen that $[0, 1]$ is not compact in $\mathbf{R}_K$, so we'll argue for a contradiction. Suppose $p \colon [0, 1] \to \mathbf{R}_K$ is a path from $p(0) = 1$ to $p(1) = 0$ (going down to the origin). But the image of a compact, connected set under a continuous map is compact and connected, and $[0, 1]$ is not compact in $\mathbf{R}_K$. Absurd! □

5.7. **[1, No. 27.4].** A *connected* metric space having more than one point is uncountable.

Proof. Consider two points x and y in the metric space, with a distance function d. Let $t \in (0, 1)$ be a parameter to obtain a contradiction.

If there's no α in the metric space such that $d(x, \alpha) = td(x, y)$, then we have a separation of the space into nonempty open sets $\{\beta : d(x, \alpha) < td(x, y)\}$ and $\{\beta : d(x, \alpha) > td(x, y)\}$. □

5.8. **[1, No. 27.5].** This is a special case of the *Baire category theorem*. Let X be a compact Hausdorff space, let $\{A_n\}$ be a countable collection of closed sets of X. If each A_n has empty interior in X, then the union $\cup A_n$ has empty interior in X.

Given. X a compact Hausdorff space, $\{A_n\}_{\mathbf{N}}$ closed sets with empty interiors.

To prove. $\operatorname{Int}(\cup_{\mathbf{N}} A_n)$ is empty.

Proof. We can find open sets disjoint from closed sets in X like we could find open sets disjoint from points in the proof "if X is a nonempty compact Hausdorff space and X has no isolated points, then X is uncountable."

Explicitly, if K is a closed set in X and U is an open set such that $U \setminus K \neq \varnothing$, then there's an open set $V \subset U$ such that $\operatorname{Cl}(V) \subset U \setminus K$. See [1, p. 176].

We'll show any open V has a point $x \notin \cup_{\mathbf{N}} A_n$. So let V be open. Since $\operatorname{Int}(A_1)$ is empty, we have by the previous argument $V_1 \subset V \setminus A_1$, with $\operatorname{Cl}(V) \subset V \setminus A_1$. We'll define now a nested sequence of closed sets $\operatorname{Cl}(V)_1 \supset \operatorname{Cl}(V)_2 \supset \cdots$ inductively by finding

$$V_n \subset V_{n-1} \setminus A_n \quad \text{with} \quad \operatorname{Cl}(V_n) \subset V_{n-1} \subset \operatorname{Cl}(V_{n-1}).$$

Now since X is compact, there exists a point

$$x \in \underbrace{\bigcap_{\mathbf{N}} \operatorname{Cl}(V_n)}_{\text{contained in } V} \quad \text{such that} \quad x \notin \bigcup_{\mathbf{N}} A_n,$$

which demonstrates the union $\cup A_n$ has trivial interior. □

5.9. **[1, No. 28.2].** $[0, 1]$ is not limit point compact as a subspace of $\mathbf{R}_\ell$.

Demonstration. Consider the open set $\{1\}$ in $\mathbf{R}_\ell$. We have a sequence

$$\sin\left(\frac{n\pi}{(n+1)2}\right) \xrightarrow{\text{ST}} 1$$

that converges in the standard topology, but fails to converge in the lower limit topology $\mathbf{R}_\ell$. □

5.10. **[1, No. 28.3].** Let X be limit point compact.

(a) If $f\colon X \to Y$ is continuous, does it follow that $f(X)$ is limit point compact?

No. Consider the continuous projection of sets in $\mathbf{N} \times \{0, 1\}$ onto $\mathbf{N}$ (where $\{0, 1\}$ is an indiscrete space and $\mathbf{N}$ discrete).

(b) If A is a closed subset of X, does it follow that A is limit point compact?

Yes. Any infinite subset of A has a limit point in X as setup above.

(c) If X is a subspace of the Hausdorff space Z, does it follow that X is closed in Z?

No. Consider the minimal uncountable well-ordered set $S_\Omega \subset \overline{S_\Omega}$.

References

[1] J. R. Munkres, *Topology*, 2nd ed. Hardcover; Prentice Hall, Inc., 2000 [Online]. Available: `http://www.worldcat.org/isbn/0131816292`