MATH 6270 HOMEWORK 3

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References

[Rie17] Emily Riehl. *Category Theory in Context*. Courier Dover Publications, March 2017. 1
[Rob96] Derek Robinson. *A Course in the Theory of Groups*. Graduate Texts in Mathematics. Springer-Verlag, New York, 2 edition, 1996. 2

1. For p a prime, the *Prüfer p-group* $\mathbb{Z}(p^\infty) = \mathbb{Z}(p^\infty)$ is the direct limit (or *colimit*) of the sequence of abelian groups

$$0 \hookrightarrow \mathbb{Z}/p \hookrightarrow \mathbb{Z}/p^2 \hookrightarrow \mathbb{Z}/p^3 \hookrightarrow \cdots \hookrightarrow \mathbb{Z}/p^n \hookrightarrow \cdots \tag{1.1}$$

where for each $n \in \mathbb{N}$ the quotient group $\mathbb{Z}/p^n$ is "glued into" $\mathbb{Z}/p^{n+1}$ by the pth power map.

Considering generators and relations (as in [Rie17]), we can also describe the Prüfer p-group as

$$\mathbb{Z}/p^\infty := \big\langle \{g_1, g_2, g_3, \ldots\} \mid \{0, g_1 p, g_2 p^2, \ldots\} \big\rangle$$

If H is a subgroup of $\mathbb{Z}(p^\infty)$, then there either exists some $g_M \in H \cap \{g_1, g_2, g_3, \ldots\}$ of maximal exponent or not.

Say g_M of maximal exponent in H does exist. I claim the inclusion of H into $\mathbb{Z}(p^\infty)$ induces

$$0 \longrightarrow H \longrightarrow \mathbb{Z}(p^\infty) \longrightarrow \mathbb{Z}(p^\infty) \longrightarrow 0$$

because $\mathbb{Z}(p^\infty)$ is isomorphic to the colimit of (1.1) with M additional trivial groups tacked onto the front of the sequence, i.e., the colimit of the diagram

$$\underbrace{0 \hookrightarrow \ldots \hookrightarrow 0}_{M+1 \text{ trivial groups}} \hookrightarrow \mathbb{Z}/p \hookrightarrow \mathbb{Z}/p^2 \hookrightarrow \cdots \tag{1.2}$$

This demonstrates that every finitely generated subgroup of $\mathbb{Z}(p^\infty)$ is a proper subgroup of $\mathbb{Z}(p^\infty)$. Hence $\mathbb{Z}(p^\infty)$ is not finitely generated.

Moreover, say g_M of maximal exponent in H does not exist. This with Neumann's theorem implies H has a countably infinite generating set. Then for each $n \in \mathbb{N}$, the subgroup H contains $\mathbb{Z}/p^n$. The universal property of the colimit pushes out a map that $\mathbb{Z}(p^\infty) \hookrightarrow H$, hence knowing also $H \hookrightarrow Z(p^\infty)$, we have an isomorphism of abelian groups.

We have enough information to exhaustively list the subgroups of $\mathbb{Z}(p^\infty)$:

- (isomorphic copies of) each cyclic group of prime power order p^n
- the trivial subgroup 0
- the entire group $\mathbb{Z}(p^\infty)$ itself.

Lastly, fix an arbitrary n and endow $\mathbb{Z}/p^n$ with the standard multiplication. Then $\mathbb{Z}/p^n$ is a finite field. As fields are never closed under products, Birkhoff's theorem implies that $\mathbb{Z}/p^n$ cannot be contained in a variety of subalgebras of $\mathbb{Z}(p^\infty)$. Thence, specifying to the definition of a variety of groups, $\mathbb{Z}/p^n$ cannot be a verbal subgroup of $\mathbb{Z}(p^\infty)$. Because any homomorphic image of a generator $g_n \in \mathbb{Z}/p^n$ has exponent dividing n, any endomorphism φ in $\mathrm{End}(\mathbb{Z}(p^\infty))$ maps g_n back into $\mathbb{Z}/p^n$. So $\mathbb{Z}/p^n$ is fully invariant.

That 0 and $\mathbb{Z}(p^\infty)$ are verbal subgroups follows trivially by taking the empty word and the 1 letter words respectively.

2. If N and K are normal subgroups of a group G, there's a "diagonal" embedding $G/(N \cap K)$ into the product $G/N \times G/K$ that projects onto each component.

Proof. Knowing that $N \cap K \hookrightarrow N$ and $N \cap K \hookrightarrow K$, there is a well defined mapping on cosets of $N \cap K$ in G to cosets of N and K in G respectively, given by choosing the unique sets gN and gK such that $g(N \cap K) \subset gN$ and $g(N \cap K) \subset gK$. Inspecting the definition of the projections π_N and π_K from G to the quotients G/N and G/K, our afore chosen well defined mapping is a homomorphism of groups:

$$d^* : G/(N \cap K) \to G/N \times G/K$$

$$g(N \cap K) \xmapsto{d^*} gN \times gK$$

The image $d^*\left(\frac{G}{N \cap K}\right)$ seen to be onto the arbitrary component G/H for $H = N, K$ by lifting $1 \times \cdots \times gH \times \cdots \times 1$ to $g(\cdots \cap H \cap \cdots)$. The kernel $\ker d^*$ is seen to be trivial by observing if the cosets $g(N \cap K) \subset 1N$ and $g(N \cap K) \subset 1K$ are in the trivial class of the respective cosets of the identity in the components, then $g(N \cap K) \subset 1(N \cap K)$, hence $d^*(g(N \cap K)) = 1N \times 1K$ only if $g \in N \cap K$. □

3. Suppose that $H \triangleleft G$ is a minimal normal subgroup of a finite solvable group G. Minimality and normality of H in G implies that if $Q \operatorname{chr} H$ and $Q \neq H$, then $Q \triangleleft G$, which is absurd. (See chapter 3 pages 87–88 of [Rob96].) Hence H is a characteristically simple group.

Because G is solvable, choose some composition series of G

$$1 = N_0 \triangleleft N_1 \triangleleft \cdots \triangleleft N_\ell = G \quad \text{with abelian subquotients } N_{i+1}/N_i \text{ for all } i < \ell \tag{3.1}$$

Intersecting H with the terms of the composition series of G yields a filtration of H with abelian subquotients, showing that H too is solvable.

Consider that the derived subgroup of H is characteristic in H:

$$[H, H] \operatorname{chr} H$$

By minimality and normality of H in G, either $[H, H] = 1$ or $[H, H] = H$. In the latter case, the derived series of H does not terminate. As the derived series of H has terms contained in every composition series of H with abelian subquotients, we see that H does not have the composition series guaranteed by the solvability of G, which is absurd. Hence $[H, H] = 1$, and H is abelian.

Let p be any prime dividing $|H|$, and let $P < H$ be a Sylow p-subgroup of H. That H is characteristically simple implies that $\operatorname{Aut} H$ is simple.[1] But as H acts transitively on the conjugates of P in H, the lack of interesting automorphisms implies that P is invariant under conjugation (inner automorphisms). Hence $P \triangleleft H \operatorname{chr} G$ implies $P \triangleleft G$. Minimality and normality of H in G forces $P = H$.

Since G is finite, P is finite. Applying proposition 3.3.15.ii in [Rob96], P is the direct product of finite simple groups. Hence P is an elementary abelian p-group.[2]

[1] Colton needs to revise this claim.

[2] I would love to show more that the automorphism group of P is a project special linear group. Clearly $\operatorname{Aut}(P)$ is $\operatorname{Aut}(\mathbb{Z}_{p^n}) \cong \mathbb{Z}_{\varphi(n)}$, which is not simple. Hence $\operatorname{Aut}(P)$ should be a group of linear transformations of some finite dimensional vector space over the field $\mathbb{Z}/p$. But how does one move from the general linear group of a finite geometry to the projective special linear group?