October 25th, 20?? pg. 1

Character Tables: Irr G usually represented as a table.

Example: $G = S_3$

Class Representation	()	(1 2)	(1 2 3)	
Class Sizes	1	3	2	
χ_1	1	1	1	trivial character
χ_2	1	-1	1	sign
χ_3	2	0	-1	

- We use the first orthogonality relations to find $\chi_3(12)$, $\chi_3(123)$.

$$0 = \langle \chi_3, \chi_1 \rangle = \cancel{\chi_3(()) \overline{\chi_1(())}} \sum_{g \in S_3} \chi_3(g)\overline{\chi_1(g)}$$

$$\sum_g \chi_3(g)\overline{\chi_1(g)} = 2\cdot 1 + 3\chi_3((12)) + 2\chi_3((123)) = 0 \quad (1)$$

Also, $0 = \langle \chi_3, \chi_2 \rangle = 2 - 3\chi_3((12)) + 2\chi_3((123)) = 0 \quad (2)$

$$4 + 4\chi_3((123)) = 0 \text{ so } \chi_3((123)) = -1$$

and $\chi_3((12)) = 0$.

- Can also use second orthogonality relations: If $g, h \in$ different conjugacy classes, $\sum_{\chi \in Irr G} \chi(g)\overline{\chi(h)} = 0$.

$$0 = 1\cdot 1 + 1\cdot -1 + 2\chi_3((12)) = 0, \text{ so } \chi_3(12) = 0.$$

$$0 = 1\cdot 1 + 1\cdot 1 + 2\cdot \chi_3(123) = 0 \text{ so } \chi_3(123) = -1.$$

Note: $|C_G(g)| = \sum_{\chi \in Irr G} \chi(g)\overline{\chi(g)}$ is determined by any column of the character table.

Hence class sizes of G are determined by the character table.

Characters & Normal Subgroups

Lemma: Let φ be a representation of G w/ character χ. Then $\ker \varphi \trianglelefteq G$ is $\{g \in G \mid \chi(g) = \chi(1)\}$.

Proof: "$\subseteq$" If $g \in \ker \varphi$, $\varphi(g) = 1$ so $\chi(g) = \chi(1)$.

"$\supseteq$" HW.

Convention: We will say the kernel of a character $\ker \chi = \{g \mid \chi(g) = \chi(1)\}$.

The character of χ_i is easily read off from the character table.

- **All** other $N \trianglelefteq G$ are given by intersections of ker χ_i, for $\{\chi_1, \dots, \chi_k\} = Irr G$.

Theorem: For $N \trianglelefteq G$, $N = \bigcap \{\ker \chi_i \mid \chi_i \in Irr G \text{ and } N \le \ker \chi_i\}$.

Proof: Homework.

Note: The character table of G determines (1) normal subgroups, and their sizes: $N \trianglelefteq G$ a union of conjugacy classes, whose sizes follow from 2nd orthogonality relation.

(2) $Z(G)$ is the union of 1-element conjugacy classes.

(3) $G' = \bigcap \{\ker \chi_i \mid \chi_i \in Irr G \text{ and } \chi_i(1) = 1\}$ (so for S_3, $S_3' = \langle (123) \rangle = A_3$)

(and $[G:G'] = \#$ of linear characters of G)

(4) simplicity of G: if all $\ker \chi_i \cap \ker \chi_j = \{1\}$

(5) Solvability: G solvable iff. it has a normal series w/ all factors of prime power order. — not efficient but all can be checked.

(6) Nilpotently of G.

(7) $Irr(G/N)$ $\forall N \trianglelefteq G$. If $N \le \ker \chi_i$, then look @ induced character of $\ker \chi_i / N$: $\hat{\chi}(gN) = \chi(g)$ is a character of G/N.

We can also lift characters from quotients to characters of the group.

$Irr(G/N) = \{\hat{\chi} : \chi \in Irr G, N \le \ker \chi\}$.