MATH 6270 HOMEWORK 4

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1. *To prove.*[1] For all $i \in \mathbb{N}$, the derived subgroup $G^{(i)}$ of an arbitrary group G is contained in 2^ith subgroup $\gamma_{2^i}(G)$ of the lower central series for G.

Proof. □

2. *Given.* Let $\mathrm{GL}(n, p)$ be the multiplicative group of invertible $n \times n$ matrices with entries in the finite field of prime order p. Let $\mathrm{U}(n, p)$ be the subgroup of unit upper triangular matrices.

Proposition.

(a) $\mathrm{U}(n, p)$ is nilpotent of class $n - 1$.

(b) $\mathrm{U}(n, p)$ is a Sylow p-subgroup of $\mathrm{GL}(n, p)$.

(c) Every finite p-group embeds into some $\mathrm{U}(n, p)$.

Proof. □

3. *To prove.* If G is a finite group with Sylow[2] subgroup P such that $N_G(P) \leq H \leq G$, then H is self-normalizing in G.

Proof. □

4. *Given.* The generalized dihedral group $\mathrm{Dih}(A)$ is the data of

- an abelian group A (or, if you like, a $\mathbb{Z}$-module A),
- the embedding $\mathbb{Z}_2 \xrightarrow{\alpha} \mathrm{Aut}(A)$ defined by $0 \mapsto \mathrm{id}(-)$ and $1 \mapsto (-)^{-1}$,
- the semidirect product $\mathbb{Z}_2 \ltimes_\alpha A$.

Proposition. For the Prüfer p-group $A = \mathbb{Z}(2^\infty)$, the generalized dihedral group $\mathrm{Dih}(A)$

- is not nilpotent, but
- each $H \in \mathrm{Dih}(A)$ has normalizer $N_{\mathrm{Dih}(A)}(H)$ strictly larger than H.

Proof. □

References

[Rob96] Derek Robinson. *A Course in the Theory of Groups*. Graduate Texts in Mathematics. Springer-Verlag, New York, 2 edition, 1996. 1

[1] See sections 5.1.11 and 5.1.12 in [Rob96].

[2] Hint: Use Sylow's theorem to obtain a transitive action of G on its Sylow p-subgroups, then apply an argument like Frattini's.