MATH 6270 HOMEWORK 1

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Note (Conventions and Symbols).

- Let $\mathbb{N}$ denote the inductive set of *natural numbers*, i.e., the positive integers $\{1, 2, 3, \ldots\}$.
- Let $A \subset B$ and $H < G$ denote inclusions (not necessarily proper) in the categories Set (sets and functions) and Grp (group and homomorphisms) respectively.
- Let us work in the full subcategory FinGrp of finite groups, i.e., let us assume for the exercises below

  if G is a group, then G is *finite*.

  - Observe FinGrp injects faithfully into Grp.
- When a (finite) group G has a *group action* on a set S, we may construct the following groups and sets.[1]
  - For each $g \in G$, let
  $$\operatorname{Fix} g := \operatorname{Fix}_S g := \{x \in S : x \cdot g = x\} \tag{0.1}$$
  be the set of points of S *fixed* by g.
  - For each $x \in S$, let
  $$\operatorname{Orb} x := \operatorname{Orb}_G x := \{x \cdot g : g \in G\} \tag{0.2}$$
  be the set of points of S in the *orbit* of x under the action of G.
  - For each $x \in S$, let
  $$\operatorname{Stab} x := \operatorname{Stab}_G x := \{g \in G : x \cdot g = x\} \tag{0.3}$$
  be the subgroup of G that *stabilizes* x.
- For a subgroup $H < G$ of the (finite) group G, let
  $$\operatorname{core} H := \operatorname{core}_G H := \bigcap_{g \in G} H^g$$
  be the *normal core* of H in G.
  - Observe $\operatorname{core}_G H$ is the kernel of the induced map $G \hookrightarrow \operatorname{Sym}\{Hg : g \in G\}$ afforded by the regular action of G on the cosets of H in G.

$\blacktriangleleft$

1. Given. Let G be a group of order $2m$ for odd $m \in \mathbb{N}$.

To prove. There is a normal subgroup $H \triangleleft G$ of index 2 and size $|H| = m$.

Proof. Since 2 is a prime dividing the order of the group, $2 \mid |G|$, Cauchy's theorem implies there exists an $a \in G$ such that $\langle a \rangle_G$ is a cyclic subgroup of G of order 2 and index m. Let

$$\rho \colon G \hookrightarrow \operatorname{Sym}|G|$$

be the embedding afforded by the right regular action G on itself (where $\rho_g \colon G \to G$ is an automorphism of sets for each $g \in G$). The image of $\langle a \rangle_G$ in $\operatorname{Sym}|G|$ is $\{\mathrm{id}, \rho_a\}$, where $\rho_a^2 = \rho_{a^2} = \mathrm{id}$. Therefore the least common multiple[2] of the cycle decomposition of ρ_a is 2, and hence ρ_a is the product of disjoint simple transpositions—but how many? To see ρ_a is exactly the product of m disjoint simple transpositions, consider the fundamental counting principle for the restriction of the regular action to the subgroup $\langle a \rangle$ of G:

$$\left|\operatorname{Stab}_{\langle a \rangle} g\right| \cdot \left|\operatorname{Orb}_{\langle a \rangle} g\right| = 2$$

[1] Notation from `https://groupprops.subwiki.org/wiki/Burnside%27s_lemma`.

[2] See Rotman, 1995, exercises 1.1.10 through 1.1.12.

Because

$$\operatorname{Stab}_{\langle a\rangle} g = \{\alpha \in \langle a\rangle : g \cdot \alpha = g\} = \{\alpha \in \langle a\rangle : \alpha = 1\}$$

is trivial, we deduce each $g \in G$ belongs to an orbit of size 2. Because G is partitioned by m of these 2-orbits, ρ_a is seen to be the disjoint union of m simple transpositions. As m is odd, ρ_a has odd sign, which is enough to conclude that the composed group homomorphism

$$G \xrightarrow{\rho} \operatorname{Sym}|G| \xrightarrow{\operatorname{sgn}} C_2$$

is surjective from G onto the cyclic group of order 2. The kernel of this homomorphism is a normal subgroup of G of index 2 and order m (by the universal property of quotient groups). ☐

2. (Cauchy-Frobenius lemma) Given. Let G be a finite group acting on a finite set S with $n \in \mathbb{N}$ orbits.

To prove.

$$\sum_{g\in G} |\operatorname{Fix} g| = \sum_{x\in S} |\operatorname{Stab} x| = n|G|. \tag{2.1}$$

Therefore the number of orbits, i.e., the number of blocks in the partition of S induced by the group action, can be computed by

$$n = \frac{1}{|G|} \sum_{g\in G} |\operatorname{Fix} g|. \tag{2.2}$$

Proof. [3] Consider the set

$$\mathscr{F} = \{(g, x) \in G \times S : x \cdot g = x\}, \tag{2.3}$$

a subset of the finite Cartesian product $G \times S$. Being finite, $|\mathscr{F}|$ may be computed by counting singletons, or equivalently by summing the cardinalities of slices while indexing over either G or S. Therefore

$$\sum_{g\in G} |\{x \in S : x \cdot g = x\}| = |F| = \sum_{x\in S} |\{g \in G : x \cdot g = x\}|. \tag{2.4}$$

By definition (0.1), $\operatorname{Fix}_S g = \{x \in S : x \cdot g = x\}$. By definition (0.3), $\operatorname{Stab}_G x = \{g \in G : x \cdot g = x\}$. Taking the previous three observations in stride,

$$\sum_{g\in G} |\operatorname{Fix}_S g| = \sum_{x\in S} |\operatorname{Stab}_G x|.$$

By the fundamental counting principle, for each $x \in S$, $|G|/|\operatorname{Orb}_G x| = |\operatorname{Stab}_G x|$. Therefore

$$\sum_{g\in G} |\operatorname{Fix}_S g| = |G| \sum_{x\in S} \frac{1}{|\operatorname{Orb}_G x|}.$$

Observing that term $\sum_{x\in S} \frac{1}{|\operatorname{Orb}_G x|}$ in the right hand side of the above equality is the number n of disjoint orbits in S induced by the action of G, the first proposition (2.1) follows. Dividing by $|G|$ yields (2.2). ☐

3. Given. Let G be a finite group, p be a prime, and $\operatorname{Syl}_p G$ the set of all Sylow p-subgroups of G.

To prove.

(a) The subgroup $O_p(G)$ defined by

$$O_p(G) := \bigcap \operatorname{Syl}_p G \tag{3.1}$$

is *characteristic* in G.

(b) Moreover, $O_p(G)$ is the *unique largest normal p-subgroup* of G. That is,

$$\text{if } P \lhd G \text{ is a normal } p\text{-subgroup, then } P \lhd O_p(G). \tag{3.2}$$

Proof. Sylow's theorem implies G acts transitively (by conjugation) on $\operatorname{Syl}_p G$. Hence if $P_{\text{syl}} \in \operatorname{Syl}_p G$, then

$$O_p(G) = \bigcap_{g\in G} g^{-1} P_{\text{syl}} g = \operatorname{core}_G P_{\text{syl}}.$$

(a) Because automorphisms of G preserve the cardinalities of subgroups of G, the collection of Sylow p-subgroups (defined by their cardinality) are automorphism-conjugate. Because the p-core $O_p(G)$ is contained in each of the automorphism conjugate Sylow p-subgroups, $O_p(G)$ is invariant under automorphisms of G, hence characteristic.

[3] This proof is more or less from Week 6, Fall 2018, Algebra I, as taught by Prof. Nathaniel Thiem.

(b) Let $P \triangleleft G$ be a normal p-subgroup. Sylow's theorem implies P is a subgroup of some conjugate of P_{syl}. Because P is invariant under inner automorphisms of G, we see that P is contained in each conjugate of P_{syl}. Thence $P < O_p(G)$. Moreover, because normal subgroups satisfy a *transfer condition*,[4] that $P \triangleleft G$ and $P < O_p(G)$ implies $P = P \cap O_p(G) \triangleleft O_p(G)$.

$\square$

4. Given. Let G be a finite group with k conjugacy classes. Let $a_1, \ldots, a_k$ denote the orders of the centralizers of elements from the distinct classes.

To prove.

(a) The a_j produce a "finite sum of distinct unit fractions"[5] equal to unity:

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_k} = 1. \tag{4.1}$$

(b) The finite groups with 3 conjugacy classes or less are exactly those listed:

- Only the trivial group has exactly 1 conjugacy class.
- Only the group $\mathbb{Z}/2\mathbb{Z}$ has exactly 2 conjugacy classes.
- Only the group S_3 has exactly 3 conjugacy classes.

Proof. Let G act on itself by conjugation. The orbits in G under the conjugation action partition G into a disjoint union of conjugacy classes $G(x_1), \ldots, G(x_n)$ for some representative points $x_i \in G$. Since G is finite, its order is computed by applying the fundamental counting principle and summing over the disjoint union:

$$|G| = \sum_{i=1}^{n} |G : C_G(x_i)|.$$

(a) Dividing the above equation by $|G|$ yields (4.1). Suppose the a_i have been arranged into an integer partition of $|G|$,

$$a_k \geq \cdots \geq a_2 \geq a_1 \quad \text{with} \quad a_k + \cdots + a_2 + a_1 = |G|.$$

The constraint (4.1) forces

$$1/a_k \leq 1/k$$

(or else $1 = k/a_k > k/k = 1$, which is absurd). By Lagrange's theorem, we must also have $a_i || G|$. I claim these constraints have only finitely many integer solutions for the a_i. Since isomorphisms preserve conjugacy classes, we deduce there are only finitely many finite groups (up to isomorphism) with k conjugacy classes.

(b)
- Only the trivial group has exactly 1 conjugacy class, as the identity is in its own class.
- The only integer solution for a_1 and a_2 given the above constraints is $a_1 = 2$ and $a_2 = 2$. Hence the identity is centralized by exactly 2 elements, itself and some g. But then g is centralized by itself and the identity. So any group with 2 conjugacy classes is isomorphic to the (unique) group with two elements $\mathbb{Z}/2\mathbb{Z}$.
- Suppose G is a group with 3 conjugacy classes. If G is abelian, then $G = C_3$. Else say G is not abelian. Suppose $N \in \mathbb{N}$ is a natural number with 3 positive divisors such that each of these positive divisors is less than N and the sum of these divisors is N. Then $N = 6 = 3 + 2 + 1$. Hence $1 = 1/2 + 1/3 + 1/6$ gives one integer solution for a_1, a_2 and a_3. Because the conjugation action defines an automorphism from G to G, conjugate elements have the same order. Now S_3 has 1 element of order 1, 2 elements of order 3, and 3 elements of order 2. Any other finite group with conjugacy classes of size $1, 2$ and 3 only would necessarily be isomorphic to S_3 by order considerations for elements within the conjugacy classes.

$\square$

[4] Defined here: `https://groupprops.subwiki.org/wiki/Transfer_condition`.

[5] Quoted from `https://en.wikipedia.org/wiki/Egyptian_fraction`.