MATH 6270 HOMEWORK 12

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1. That "an equivalence between group extensions E_1 and E_2 of N by G yields an isomorphism $E_1 \xrightarrow{\varphi} E_2$" is a consequence of the five lemma, which we now prove.

$$\begin{tikzcd}
1 \arrow[r] & N \arrow[r] \arrow[d, equal] & E_1 \arrow[r] \arrow[d, "\varphi"] & G \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & N \arrow[r] & E_2 \arrow[r] & G \arrow[r] & 1,
\end{tikzcd}$$

Lemma (Five Lemma [Wei94]). Suppose the following diagram of groups has exact rows.

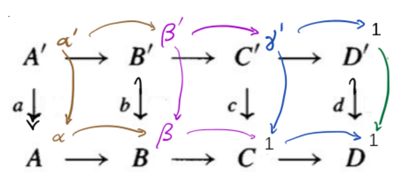
$$\begin{tikzcd}
A' \arrow[r] \arrow[d, "a"] & B' \arrow[r] \arrow[d, "b"] & C' \arrow[r] \arrow[d, "c"] & D' \arrow[r] \arrow[d, "d"] & E' \arrow[d, "e"] \\
A \arrow[r] & B \arrow[r] & C \arrow[r] & D \arrow[r] & E
\end{tikzcd}$$

(a) If a is epic and both b and d are monic, then c is monic.

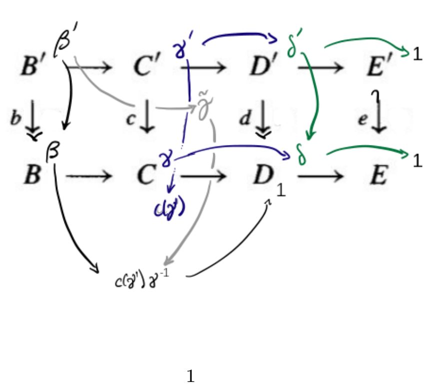
(b) If e is monic and both b and d are epic, then c is epic.

Proof. A diagram chase.

(a) Suppose a is epic, b and d are monic. To show c is monic, let γ' be an element of C' that maps to 1 under c. Say δ' is the image of γ' in D'. Since 1 in C maps to 1 in D, and the right square commutes, that d is monic implies $\delta' = 1$ in D'. Hence, by exactness at C', γ' lifts to β' in B'. Pushing β' down to β, that the center square commutes implies β maps to 1 in C. Hence, by exactness at B, β lifts to α in A, which lifts to α' in A', as a is epic. Because b is monic, that the image of β' is β and the left square commutes implies α' maps to β'. But then exactness at B' implies β' maps to 1 in C'. Since γ' is the image of β', $\gamma' = 1$. Hence c is monic.



(b) Suppose e is monic, b and d are epic. To show c is epic, let γ be an arbitrary element of C. Let δ be the image of γ in D. As d is epic, δ lifts to δ' in D'. Because δ is in the image of C, δ maps to 1 in E. But e is monic and the right square commutes, so δ' must map to 1 in E', and hence exactness at D' yields a lift γ' in C' of δ'. Since the center square commutes, both $c(\gamma')$ and γ map to δ in D, that is, both elements are in the same coset of $\ker(C \to D)$. Because $C \to D$ is a homomorphism, that $c(\gamma') \equiv \gamma \mod \ker(C \to D)$ implies $c(\gamma')\gamma^{-1}$ maps to 1 in D. Exactness at C yields a lift β of $c(\gamma')\gamma^{-1}$, where β is the image of β' under epic b, and $\tilde{\gamma}$ is the image of β' in C'. That the left square commutes implies $\tilde{\gamma}$ pushes down to $c(\gamma')\gamma^{-1}$. But c is a homomorphism, so $\gamma = c\big((\tilde{\gamma})^{-1}\gamma'\big)$. Hence c is epic.



□

2. Non-equivalent extensions may give isomorphic groups. For example, consider $\mathbb{Z}/(9)$ as the following extensions:

$$0 \longrightarrow \mathbb{Z}/(3) \xrightarrow{\cdot 3} \mathbb{Z}/(9) \longrightarrow \mathbb{Z}/(3) \longrightarrow 0\ ,$$

$$0 \longrightarrow \mathbb{Z}/(3) \xrightarrow{\cdot 6} \mathbb{Z}/(9) \longrightarrow \mathbb{Z}/(3) \longrightarrow 0\ .$$

These extensions are not equivalent.

Proof by contradiction. Suppose we had an equivalence

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/(3) \arrow[r, "\cdot 3"] \arrow[d, equal] & \mathbb{Z}/(9) \arrow[r] \arrow[d, "\varphi"] & \mathbb{Z}/(3) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{Z}/(3) \arrow[r, "\cdot 6"] & \mathbb{Z}/(9) \arrow[r] & \mathbb{Z}/(3) \arrow[r] & 0.
\end{tikzcd}$$

That the left square commutes implies, e.g., $\varphi(\bar{3}) = \bar{6}$. That the right square commutes (where both quotient maps are modulo 3) implies the image of $\bar{1}$ under φ lands in the coset $\{\bar{1}, \bar{4}, \bar{7}\}$. But if the φ maps the generator $\bar{1}$ to any of these elements, extending linearly forces $\varphi(\bar{3}) = \bar{3}$, a contradiction. $\square$

At the same time, it's not too hard to modify the quotient map in the second sequence to "untwist" the extension and obtain equivalence:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/(3) \arrow[r, "\cdot 3"] \arrow[d, equal] & \mathbb{Z}/(9) \arrow[r] \arrow[d, "\cdot 2"] & \mathbb{Z}/(3) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{Z}/(3) \arrow[r, "\cdot 6"] & \mathbb{Z}/(9) \arrow[r, "\cdot 2"] & \mathbb{Z}/(3) \arrow[r] & 0.
\end{tikzcd}$$

References

[Wei94] Charles A. Weibel. An Introduction to Homological Algebra, April 1994. 1