MATH 6270 HOMEWORK 2

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Note (Conventions and Symbols).

- We work with concrete categories C, i.e., categories for which there exists a faithful functor $F\colon \mathsf{C} \hookrightarrow \mathsf{Set}$ that is injective from the set[1] of morphisms $\mathsf{Mor}(\mathsf{C})$ in C to the set of functions $\mathsf{Map}(\mathsf{Set})$ in Set).

$\lessdot$

2. Given. Let $F(X)$, $F(Y)$ be free groups over the sets X, Y.

To prove. If $F(X) \cong F(Y)$ are isomorphic as groups, then the sets $X \cong Y$ have the same cardinality.

Proof. We will argue that the free group functor $F\colon \mathsf{Set} \to \mathsf{Grp}$ is a fully faithful functor, in order to see that the isomorphism $F(X) \xrightarrow{\cong} F(Y)$ is induced by a unique bijection $X \xrightarrow{\cong} Y$. See figure.

$\square$

3. Given. Consider a group presentation $G = \langle X \mid R \rangle$ with strictly more generators than relations: $|X| > |R|$.

To prove. G is infinite.

Proof. Define $Q = R \cup \{\text{simple commutators } [x, y] \text{for all } x, y \in X\}$. The normal closure of Q includes the commutator subgroup of $\langle X \rangle$, which yields a quotient map $\langle X \mid R \rangle \twoheadrightarrow \langle X \mid Q \rangle$ onto an abelian group. Let Z denote the set of relations in Q which are not simple commutators. From our hypothesis, Z has cardinality strictly less than X. In the free abelian group $\mathbb{Z}\{X\}$, the relations in Z produce a underdetermined system of $\mathbb{Z}$-linear relations on the generators in X. In the vector space $\mathbb{Z}\{X\} \otimes \mathbb{Q}$, the dimension of the subspace spanned by the solutions to these relations has strictly positive codimension. Thence the quotient of $\mathbb{Z}\{X\} \otimes \mathbb{Q}$ is infinite. Taking fibers over both projections yields the proposition. $\square$

4. Given. Let $G = \langle \{a, b\} : \{a^n, b^m\} \rangle$ be a group presentation with the exponents $m, n \geq 2$.

To prove. G is infinite.

Proof. Consider the free group $F = F(\{a, b\})$ on two generators. Let F act on itself by right translation. I claim the orbit of the normal subgroup $N = \langle \{w(a^n, b^m)^g : g \in F\} \rangle$ is infinite.

Recall two cosets $N\alpha$ and $N\beta$ coincide if and only if $\alpha\beta^{-1} \in N$.

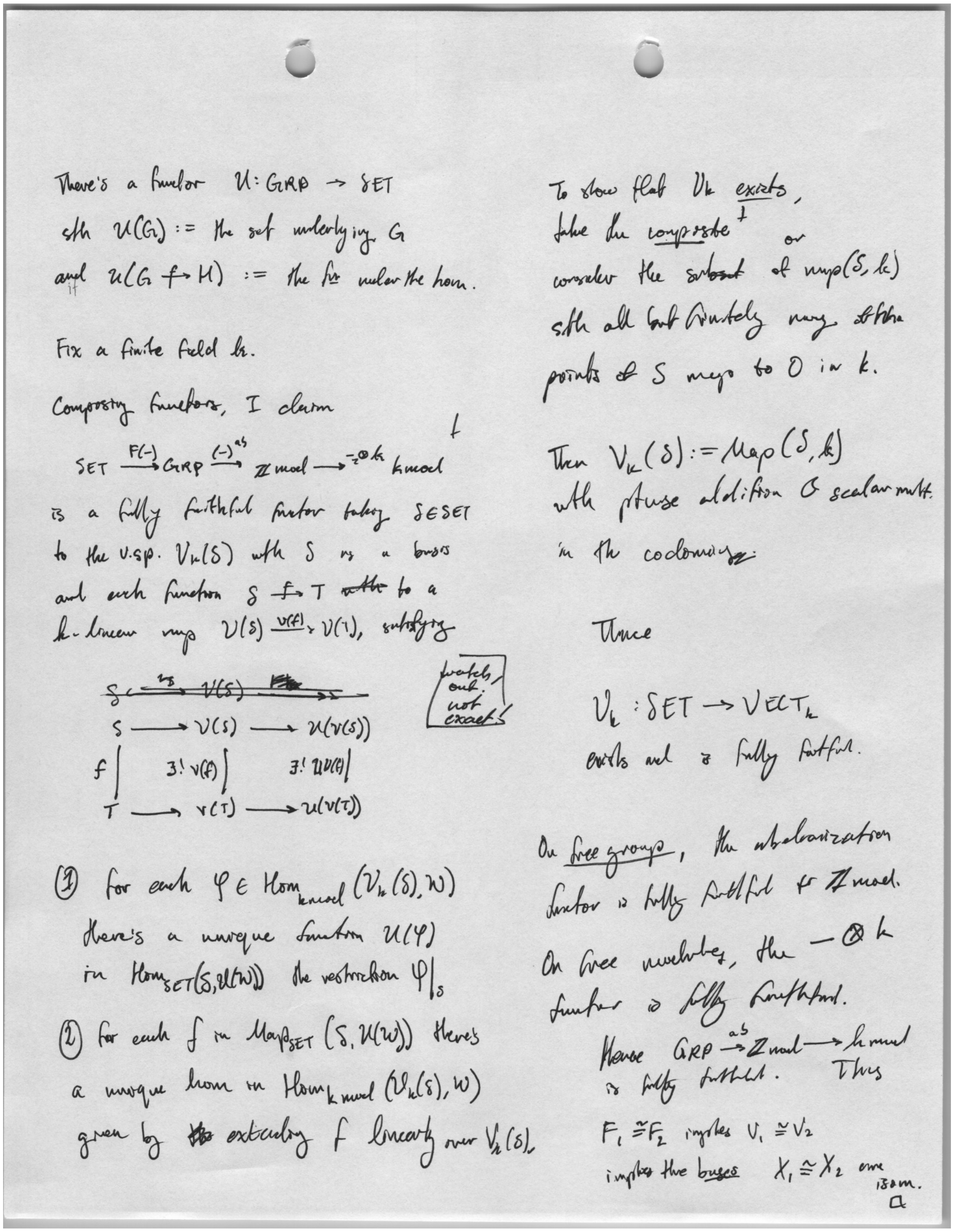
Let $\alpha = [a, b]^\nu := (aba^{-1}b^{-1})^\nu$ and $\beta = [a, b]^\mu$ for $\mu, \nu \in \mathbb{Z}$. Let $w \in N[a, b]^\mu$ be an arbitrary (non-empty) reduced word in the coset $N[a, b]^\mu$. We show that $\alpha\beta^{-1} \in N$ if and only if $\mu = \nu$.

Say $\mu \neq \nu$ and let $\bar{w}$ be a (nonempty) reduced word in N. Note $\alpha\beta^{-1}$ is not empty. Now, because $\bar{w}$ is a conjugate of some word $w(a^n, b^m)$ over the letters a^n and b^m, either the first or last letter of $\bar{w}$ has exponent greater in magnitude than 1, where $n, m > 1$. However, both first and last letters of the reduced word $(aba^{-1}b^{-1})^\nu(aba^{-1}b^{-1})^{-\mu}$ has exponent no larger in magnitude than 1.

We have shown the representative reduced words $\alpha\beta^{-1} \neq \bar{w}$ are not equal, hence $\alpha\beta^{-1} \notin N$. If $\mu - \nu = 0$, we have the empty word $\alpha\beta^{-1} = \alpha\alpha^{-1} \in N$.

We have shown there are countably many cosets of N in F. Hence the quotient F/N is not finite. $\square$

[1]At the risk of being evil, I'm requiring this collection of morphisms to be a set. This is about as much set theory as I'm prepared to handle. See also `https://en.wikipedia.org/wiki/Concrete_category`.



There's a functor $\mathcal{U}: \mathrm{GRP} \to \mathrm{SET}$
s.th. $\mathcal{U}(G) :=$ the set underlying G
and $\mathcal{U}(G \xrightarrow{f} H) :=$ the fn under the hom.

Fix a finite field k.

Composing functors, I claim

$$\mathrm{SET} \xrightarrow{F(-)} \mathrm{GRP} \xrightarrow{(-)^{ab}} \mathbb{Z}\,\mathrm{mod} \xrightarrow{-\otimes_{\mathbb{Z}} k} k\,\mathrm{mod}$$

is a fully faithful functor taking $S \in \mathrm{SET}$ to the v.sp. $V_k(S)$ with S as a basis and each function $S \xrightarrow{f} T$ ~~with~~ to a k-linear map $V(S) \xrightarrow{V(f)} V(T)$, satisfying

$$\begin{array}{ccccc} S & \longrightarrow & V(S) & \longrightarrow & \mathcal{U}(V(S)) \\ f \downarrow & & \exists!\, V(f) \downarrow & & \exists!\, \mathcal{U}V(f) \downarrow \\ T & \longrightarrow & V(T) & \longrightarrow & \mathcal{U}(V(T)) \end{array}$$

$\boxed{\text{watch out. not exact!}}$

(1) for each $\varphi \in \mathrm{Hom}_{k\,\mathrm{mod}}(V_k(S), W)$ there's a unique function $\mathcal{U}(\varphi)$ in $\mathrm{Hom}_{\mathrm{SET}}(S, \mathcal{U}(W))$ the restriction $\varphi|_S$

(2) for each f in $\mathrm{Map}_{\mathrm{SET}}(S, \mathcal{U}(W))$ there's a unique hom in $\mathrm{Hom}_{k\,\mathrm{mod}}(V_k(S), W)$ given by extending f linearly over $V_k(S)$.

To show that V_k exists, take the composite or consider the subset of $\mathrm{map}(S, k)$ s.th. all but finitely many of the points of S maps to 0 in k.

Then $V_k(S) := \mathrm{Map}(S, k)$ with ptwise addition & scalar mult. in the codomain.

Thus

$V_k : \mathrm{SET} \to \mathrm{VECT}_k$ exists and is fully faithful.

On free groups, the abelianization functor is fully faithful to $\mathbb{Z}$ mod.

On free modules, the $-\otimes k$ functor is fully faithful.

Hence $\mathrm{GRP} \xrightarrow{ab} \mathbb{Z}\,\mathrm{mod} \to k\,\mathrm{mod}$ is fully faithful. Thus

$F_1 \cong F_2$ implies $V_1 \cong V_2$ implies the bases $X_1 \cong X_2$ are isom.
$\square$

FIGURE 1. Sketch of proof

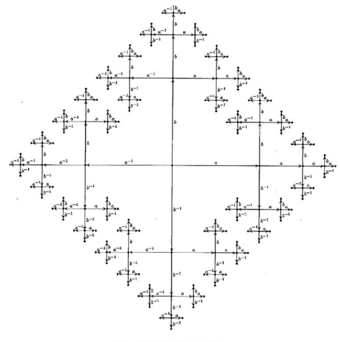


FIGURE 2. Cayley diagram for $F(\{a, b\})$