CHAPTER 1

An Introduction to the Homotopy Groups of Spheres

This chapter is intended to be an expository introduction to the rest of the book. We will informally describe the spectral sequences of Adams and Novikov, which are the subject of the remaining chapters. Our aim here is to give a conceptual picture, suppressing as many technical details as possible.

In Section 1 we list some theorems which are classical in the sense that they do *not* require any of the machinery described in this book. These include the Hurewicz theorem 1.1.2, the Freudenthal suspension theorem 1.1.4, the Serre finiteness theorem 1.1.8, the Nishida nilpotence theorem 1.1.9, and the Cohen–Moore–Neisendorfer exponent theorem 1.1.10. They all pertain directly to the homotopy groups of spheres and are not treated elsewhere here. The homotopy groups of the stable orthogonal group SO are given by the Bott periodicity theorem 1.1.11. In 1.1.12 we define the J-homomorphism from $\pi_i(SO(n))$ to $\pi_{n+i}(S^n)$. Its image is given in 1.1.13, and in 1.1.14 we give its cokernel in low dimensions. Most of the former is proved in Section 5.3.

In Section 2 we describe Serre's method of computing homotopy groups using cohomological techniques. In particular, we show how to find the first element of order p in $\pi_*(S^3)$ 1.2.4. Then we explain how these methods were streamlined by Adams to give his celebrated spectral sequence 1.2.10. The next four theorems describe the Hopf invariant one problem. A table showing the Adams spectral sequence at the prime 2 through dimension 45 is given in 1.2.15. In Chapter 2 we give a more detailed account of how the spectral sequence is set up, including a convergence theorem. In Chapter 3 we make many calculations with it at the prime 2.

In 1.2.16 we summarize Adams's method for purposes of comparing it with that of Novikov. The basic idea is to use complex cobordism (1.2.17) in place of ordinary mod (p) cohomology. Fig. 1.2.19 is a table of the Adams–Novikov spectral sequence for comparison with Fig. 1.2.15.

In the next two sections we describe the algebra surrounding the E_2-term of the Adams–Novikov spectral sequence. To this end formal group laws are defined in 1.3.1 and a complete account of the relevant theory is given in Appendix 2. Their connection with complex cobordism is the subject of Quillen's theorem (1.3.4) and is described more fully in Section 4.1. The Adams–Novikov E_2-term is described in terms of formal group law theory (1.3.5) and as an Ext group over a certain Hopf algebra (1.3.6).

The rest of Section 3 is concerned with the Greek letter construction, a method of producing infinite periodic families of elements in the E_2-term and (in favorable cases) in the stable homotopy groups of spheres. The basic definitions are given in

1.3.17 and 1.3.19 and the main algebraic fact required is the Morava–Landweber theorem (1.3.16). Applications to homotopy are given in 1.3.11, 1.3.15, and 1.3.18. The section ends with a discussion of the proofs and possible extensions of these results. This material is discussed more fully in Chapter 5.

In Section 4 we describe the deeper algebraic properties of the E_2-term. We start by introducing BP and defining a Hopf algebroid. The former is a minimal wedge summand of MU localized at a prime. A Hopf algebroid is a generalized Hopf algebra needed to describe the Adams–Novikov E_2-term more conveniently in terms of BP (1.4.2). The algebraic and homological properties of such objects are the subject of Appendix 1.

Next we give the Lazard classification theorem for formal group laws (1.4.3) over an algebraically closed field of characteristic p, which is proved in Section A2.2. Then we come to Morava's point of view. Theorem 1.3.5 describes the Adams–Novikov E_2-term as the cohomology of a certain group G with coefficients in a certain polynomial ring L. $\mathrm{Spec}(L)$ (in the sense of abstract algebraic geometry) is an infinite dimensional affine space on which G acts. The points in $\mathrm{Spec}(L)$ can be thought of as formal group laws and the G-orbits as isomorphism classes, as described in 1.4.3. This orbit structure is described in 1.4.4. For each orbit there is a stabilizer or isotropy subgroup of G called S_n. Its cohomology is related to that of G (1.4.5), and its structure is known. The theory of Morava stabilizer algebras is the algebraic machinery needed to exploit this fact and is the subject of Chapter 6. Our next topic, the chromatic spectral sequence (1.4.8, the subject of Chapter 5), connects the theory above to the Adams–Novikov E_2-term. The Greek letter construction fits into this apparatus very neatly.

Section 5 is about unstable homotopy groups of spheres and is not needed for the rest of the book. Its introduction is self-explanatory.

1. Classical Theorems Old and New

Homotopy groups. The Hurewicz and Freudenthal theorems. Stable stems. The Hopf map. Serre's finiteness theorem. Nishida's nilpotence theorem. Cohen, Moore and Neisendorfer's exponent theorem. Bott periodicity. The J-homomorphism.

We begin by recalling some definitions. The nth homotopy group of a connected space X, $\pi_n(X)$, is the set of homotopy classes of maps from the n-sphere S^n to X. This set has a natural group structure which is abelian for $n \geq 2$.

We now state three classical theorems about homotopy groups of spheres. Proofs can be found, for example, in Spanier [**1**].

1.1.1. Theorem. *$\pi_1(S^1) = \mathbf{Z}$ and $\pi_m(S^1) = 0$ for $m > 1$.* □

1.1.2. Hurewicz's Theorem. *$\pi_n(S^n) = \mathbf{Z}$ and $\pi_m(S^n) = 0$ for $m < n$. A generator of $\pi_n(S^n)$ is the class of the identity map.* □

For the next theorem we need to define the suspension homomorphism $\sigma\colon \pi_m(S^n) \to \pi_{m+1}(S^{n+1})$.

1.1.3. Definition. *The kth suspension $\Sigma^k X$ of a space X is the quotient of $I^k \times X$ obtained by collapsing $\partial I^k \times X$ onto ∂I^k, ∂I^k being the boundary of I^k, the k-dimensional cube. Note that $\Sigma^i \Sigma^j X = \Sigma^{i+j} X$ and $\Sigma^k f\colon \Sigma^k X \to \Sigma^k Y$ is the quotient of $1 \times f\colon I^k \times X \to I^k \times Y$. In particular, given $f\colon S^m \to S^n$ we have $\Sigma f\colon S^{m+1} \to S^{n+1}$, which induces a homomorphism $\pi_m(S^n) \to \pi_{m+1}(S^{n+1})$.* □

1.1.4. Freudenthal Suspension Theorem. *The suspension homomorphism* $\sigma\colon \pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+q})$ *defined above is an isomorphism for* $k < n-1$ *and a surjection for* $k = n-1$. □

1.1.5. Corollary. *The group* $\pi_{n+k}(S^n)$ *depends only on* k *if* $n > k+1$. □

1.1.6. Definition. *The stable k-stem or kth stable homotopy group of spheres* π_k^S *is* $\pi_{n+k}(S^n)$ *for* $n > k+1$. *The groups* $\pi_{n+k}(S^n)$ *are called stable if* $n > k+1$ *and unstable if* $n \le k+1$. *When discussing stable groups we will not make any notational distinction between a map and its suspensions.* □

The subsequent chapters of this book will be concerned with machinery for computing the stable homotopy groups of spheres. Most of the time we will not be concerned with unstable groups. The groups π_k^S are known at least for $k \le 45$. See the tables in Appendix 3, along with Theorem 1.1.13. Here is a table of π_k^S for $k \le 15$:

k	0	1	2	3	4	5	6	7	8
π_k^S	$\mathbf{Z}$	$\mathbf{Z}/(2)$	$\mathbf{Z}/(2)$	$\mathbf{Z}/(24)$	0	0	$\mathbf{Z}/(2)$	$\mathbf{Z}/(240)$	$(\mathbf{Z}/(2))^2$

k	9	10	11	12	13	14	15
π_k^S	$(\mathbf{Z}/2)^3$	$\mathbf{Z}/6$	$\mathbf{Z}/(504)$	0	$\mathbf{Z}/(3)$	$(\mathbf{Z}/(2))^2$	$\mathbf{Z}/(480) \oplus \mathbf{Z}/(2)$

This should convince the reader that the groups do not fall into any obvious pattern. Later in the book, however, we will present evidence of some deep patterns not apparent in such a small amount of data. The nature of these patterns will be discussed later in this chapter.

When homotopy groups were first defined by Hurewicz in 1935 it was hoped that $\pi_{n+k}(S^n) = 0$ for $k > 0$, since this was already known to be the case for $n = 1$ (1.1.1). The first counterexample is worth examining in some detail.

1.1.7. Example. $\pi_3(S^2) = \mathbf{Z}$ generated by the class of the Hopf map $\eta\colon S^3 \to S^2$ defined as follows. Regard S^2 (as Riemann did) as the complex numbers $\mathbf{C}$ with a point at infinity. S^3 is by definition the set of unit vectors in $\mathbf{R}^4 = \mathbf{C}^2$. Hence a point in S^3 is specified by two complex coordinates (z_1, z_2). Define η by

$$\eta(z_1, z_2) = \begin{cases} z_1/z_2 & \text{if } z_2 \ne 0 \\ \infty & \text{if } z_2 = 0. \end{cases}$$

It is easy to verify that η is continuous. The inverse image under η of any point in S^2 is a circle, specifically the set of unit vectors in a complex line through the origin in $\mathbf{C}^2$, the set of all such lines being parameterized by S^2. Closer examination will show that any two of these circles in S^3 are linked. One can use quaternions and Cayley numbers in similar ways to obtain maps $\nu\colon S^7 \to S^4$ and $\sigma\colon S^{15} \to S^8$, respectively. Both of these represent generators of infinite cyclic summands. These three maps (η, ν, and σ) were all discovered by Hopf [**1**] and are therefore known as the Hopf maps.

We will now state some other general theorems of more recent vintage.

1.1.8. Finiteness Theorem (Serre [**3**]). $\pi_{n+k}(S^n)$ *is finite for* $k > 0$ *except when* $n = 2m$, $k = 2m-1$, *and* $\pi_{4m-1}(S^{2m}) = \mathbf{Z} \oplus F_m$, *where* F_m *is finite.* □

The next theorem concerns the ring structure of $\pi_*^S = \bigoplus_{k\geq 0} \pi_k^S$ which is induced by composition as follows. Let $\alpha \in \pi_i^S$ and $\beta \in \pi_j^S$ be represented by $f\colon S^{n+i} \to S^n$ and $g\colon S^{n+i+j} \to S^{n+i}$, respectively, where n is large. Then $\alpha\beta \in \pi_{i+j}^S$ is defined to be the class represented by $f \cdot g\colon S^{n+i+j} \to S^n$. It can be shown that $\beta\alpha = (-1)^{ij}\alpha\beta$, so π_*^S is an anticommutative graded ring.

1.1.9. Nilpotence Theorem (Nishida [**1**]). *Each element $\alpha \in \pi_k^S$ for $k > 0$ is nilpotent, i.e., $\alpha^t = 0$ for some finite t.* □

For the next result recall that 1.1.8 says $\pi_{2i+1+j}(S^{2i+1})$ is a finite abelian group for all $j > 0$.

1.1.10. Exponent Theorem (Cohen, Moore, and Neisendorfer [**1**]). *For $p \geq 5$ the p-component of $\pi_{2i+1+j}(S^{2i+1})$ has exponent p^i, i.e., each element in it has order $\leq p^i$.* □

This result is also true for $p = 3$ (Neisendorfer [**1**]) as well, but is known to be false for $p = 2$. For example, the 2-component of 3-stem is cyclic of order 4 (see Fig. 3.3.18) on S^3 and of order 8 on S^8 (see Fig. 3.3.10). It is also known (Gray [**1**]) to be the best possible, i.e., $\pi_{2i+1+j}(S^{2i+1})$ is known to contain elements of order p^i for certain j.

We now describe an interesting subgroup of π_*^S, the image of the Hopf–Whitehead J-homomorphism, to be defined below. Let $SO(n)$ be the space of $n\times n$ special orthogonal matrices over $\mathbf{R}$ with the standard topology. $SO(n)$ is a subspace of $SO(n+1)$ and we denote $\bigcup_{n>0} SO(n)$ by SO, known as the stable orthogonal group. It can be shown that $\pi_i(SO) = \pi_i(SO(n))$ if $n > i+1$. The following result of Bott is one of the most remarkable in all of topology.

1.1.11. Bott Periodicity Theorem (Bott [**1**]; see also Milnor [**1**]).

$$\pi_i(SO) = \begin{cases} \mathbf{Z} & \text{if } i \equiv -1 \mod 4 \\ \mathbf{Z}/(2) & \text{if } i = 0 \text{ or } 1 \mod 8 \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

We will now define a homomorphism $J\colon \pi_i(SO(n)) \to \pi_{n+i}(S^n)$. Let $\alpha \in \pi_i(SO(n))$ be the class of $f\colon S^i \to SO(n)$. Let D^n be the n-dimensional disc, i.e., the unit ball in $\mathbf{R}^n$. A matrix in $SO(n)$ defines a linear homeomorphism of D^n to itself. We define $\hat{f}\colon S^i \times D^n \to D^n$ by $\hat{f}(x,y) = f(x)(y)$, where $x \in S^i$, $y \in D^n$, and $f(x) \in SO(n)$. Next observe that S^n is the quotient of D^n obtained by collapsing its boundary S^{n-1} to a single point, so there is a map $p\colon D^n \to S^n$, which sends the boundary to the base point. Also observe that S^{n+i}, being homeomorphic to the boundary of $D^{i+1} \times D^n$, is the union of $S^i \times D^n$ and $D^{i+1} \times S^{n-1}$ along their common boundary $S^i \times S^{n-1}$. We define $\tilde{f}\colon S^{n+i} \to S^n$ to be the extension of $p\hat{f}\colon S^i \times D^n \to S^n$ to S^{n+i} which sends the rest of S^{n+i} to the base point in S^n.

1.1.12. Definition. *The Hopf–Whitehead J-homomorphism $J\colon \pi_i(SO(n)) \to \pi_{n+i}(S^n)$ sends the class of $f\colon S^i \to SO(n)$ to the class of $\tilde{f}\colon S^{n+i} \to S^n$ as described above.* □

We leave it to the skeptical reader to verify that the above construction actually gives us a homomorphism.

Note that both $\pi_i(SO(n))$ and $\pi_{n+i}(S^n)$ are stable, i.e., independent of n, if $n > i+1$. Hence we have $J\colon \pi_k(SO) \to \pi_k^S$. We will now describe its image.

1.1.13. THEOREM (Adams [**1**] and Quillen [**1**]). *$J\colon \pi_k(SO) \to \pi_k^S$ is a monomorphism for $k \equiv 0$ or $1 \mod 8$ and $J(\pi_{4k-1}(SO))$ is a cyclic group whose 2-component is $\mathbf{Z}_{(2)}/(8k)$ and whose p-component for $p \geq 3$ is $\mathbf{Z}_{(p)}/(pk)$ if $(p-1) \mid 2k$ and 0 if $(p-1) \nmid 2k$, where $\mathbf{Z}_{(p)}$ denotes the integers localized at p. In dimensions 1, 3, and 7, $\operatorname{im} J$ is generated by the Hopf maps (1.1.7) η, ν, and σ, respectively. If we denote by x_k the generator in dimension $4k-1$, then ηx_{2k} and $\eta^2 x_{2k}$ are the generators of $\operatorname{im} J$ in dimensions $8k$ and $8k+1$, respectively.* □

The image of J is also known to a direct summand; a proof can be found for example at the end of Chapter 19 of Switzer [**1**]. The order of $J(\pi_{4k-1}(SO))$ was determined by Adams up to a factor of two, and he showed that the remaining ambiguity could be resolved by proving the celebrated Adams conjecture, which Quillen and others did. Denote this number by a_k. Its first few values are tabulated here.

k	1	2	3	4	5	6	7	8	9	10
a_k	24	240	504	480	264	65,520	24	16,320	28,728	13,200

The number a_k has interesting number theoretic properties. It is the denominator of $B_k/4k$, where B_k, is the kth Bernoulli number, and it is the greatest common divisor of numbers $n^{t(n)}(n^{2k}-1)$ for $n \in \mathbf{Z}$ and $t(n)$ sufficiently large. See Adams [**1**] and Milnor and Stasheff [**5**] for details.

Having determined $\operatorname{im} J$, one would like to know something systematic about $\operatorname{coker} J$, i.e., something more than its structure through a finite range of dimensions. For the reader's amusement we record some of that structure now.

1.1.14. THEOREM. *In dimensions ≤ 15, the 2-component of $\operatorname{coker} J$ has the following generators, each with order 2:*

$$\eta^2 \in \pi_2^S, \quad \nu^2 \in \pi_6^S, \quad \bar{\nu} \in \pi_8^S, \quad \eta\bar{\nu} = \nu^3 \in \pi_9^S, \quad \mu \in \pi_9^S,$$
$$\eta\mu \in \pi_{10}^S, \quad \sigma^2 \in \pi_{14}^S, \quad \kappa \in \pi_{14}^S \quad \text{and} \quad \eta\kappa \in \pi_{15}^S.$$

(There are relations $\eta^3 = 4\nu$ and $\eta^2\mu = 4x_3$). For $p \geq 3$ the p-component of $\operatorname{coker} J$ has the following generators in dimensions $\leq 3pq-6$ (where $q = 2p-2$), each with order p:

$$\beta_1 \in \pi_{pq-2}^S, \qquad \alpha_1\beta_1 \in \pi_{(p+1)q-3}^S$$

where $\alpha_1 = x_{(p-1)/2} \in \pi_{q-1}^S$ is the first generator of the p-component of $\operatorname{im} J$,

$$\beta_1^2 \in \pi_{2pq-4}^S, \quad \alpha_1\beta_1^2 \in \pi_{(2p+1)q-5}^S, \quad \beta_2 \in \pi_{(2p+1)q-2}^S,$$
$$\alpha_1\beta_2 \in \pi_{(2p+2)q-3}^S, \quad \text{and} \quad \beta_1^3 \in \pi_{3pq-6}^S.$$ □

The proof and the definitions of new elements listed above will be given later in the book, e.g., in Section 4.4.

2. Methods of Computing $\pi_*(S^n)$

Eilenberg–Mac Lane spaces and Serre's method. The Adams spectral sequence. Hopf invariant one theorems. The Adams–Novikov spectral sequence. Tables in low dimensions for $p = 3$.

In this section we will informally discuss three methods of computing homotopy groups of spheres, the spectral sequences of Serre, Adams, and Novikov. A fourth method, the EHP sequence, will be discussed in Section 5. We will not give any proofs and in some cases we will sacrifice precision for conceptual clarity, e.g., in our identification of the E_2-term of the Adams–Novikov spectral sequence.

The Serre spectral sequence (circa 1951) (Serre [**2**]) is included here mainly for historical interest. It was the first systematic method of computing homotopy groups and was a major computational breakthrough. It has been used as late as the 1970s by various authors (Toda [**1**], Oka [**1, 2, 3**]), but computations made with it were greatly clarified by the introduction of the Adams spectral sequence in 1958 in Adams [**3**]. In the Adams spectral sequence the basic mechanism of the Serre spectral sequence information is organized by homological algebra.

For the 2-component of $\pi_*(S^n)$ the Adams spectral sequence is indispensable to this day, but the odd primary calculations were streamlined by the introduction of the Adams–Novikov spectral sequence (Adams–Novikov spectral sequence) in 1967 by Novikov [**1**]. It is the main subject in this book. Its E_2-term contains more information than that of the Adams spectral sequence; i.e., it is a more accurate approximation of stable homotopy and there are fewer differentials in the spectral sequence. Moreover, it has a very rich algebraic structure, as we shall see, largely due to the theorem of Quillen [**2**], which establishes a deep (and still not satisfactorily explained) connection between complex cobordism (the cohomology theory used to define the Adams–Novikov spectral sequence; see below) and the theory of formal group laws. Every major advance in the subject since 1969, especially the work of Jack Morava, has exploited this connection.

We will now describe these three methods in more detail. The starting point for Serre's method is the following classical result.

1.2.1. Theorem. *Let X be a simply connected space with $H_i(X) = 0$ for $i < n$ for some positive integer $n \geq 2$. Then*

(a) (Hurewicz [**1**]). $\pi_n(X) = H_n(X)$.

(b) (Eilenberg and Mac Lane [**2**]). *There is a space $K(\pi, n)$, characterized up to homotopy equivalence by*

$$\pi_i(K(\pi,n)) = \begin{cases} \pi & \text{if } i = n \\ 0 & \text{if } i \neq n. \end{cases}$$

If X is above and $\pi = \pi_n(X)$ then there is a map $f\colon X \to K(\pi, n)$ such that $H_n(f)$ and $\pi_n(f)$ are isomorphisms. □

1.2.2. Corollary. *Let F be the fiber of the map f above. Then*

$$\pi_i(F) = \begin{cases} \pi_i(X) & \text{for } i \geq n+1 \\ 0 & \text{for } i \leq n. \end{cases} \qquad \square$$

In other words, F has the same homotopy groups as X in dimensions above n, so computing $\pi_*(F)$ is as good as computing $\pi_*(X)$. Moreover, $H_*(K(\pi, n))$ is known, so $H_*(F)$ can be computed with the Serre spectral sequence applied to the fibration $F \to X \to K(\pi, n)$.

Once this has been done the entire process can be repeated: let $n' > n$ be the dimension of the first nontrivial homology group of F and let $H_{n'}(F) = \pi'$. Then $\pi_{n'}(F) = \pi_{n'}(X) = \pi'$ is the next nontrivial homotopy group of X. Theorem 1.2.1 applied to F gives a map $f'\colon F \to K(\pi', n')$ with fiber F', and 1.2.2 says

$$\pi_i(F') = \begin{cases} \pi_i(X) & \text{for } i > n' \\ 0 & \text{for } i \leq n'. \end{cases}$$

Then one computes $H_*(F')$ using the Serre spectral sequence and repeats the process.

As long as one can compute the homology of the fiber at each stage, one can compute the next homotopy group of X. In Serre [**3**] a theory was developed which allows one to ignore torsion of order prime to a fixed prime p throughout the calculation if one is only interested in the p-component of $\pi_*(X)$. For example, if $X = S^3$, one uses 1.2.1 to get a map to $K(\mathbf{Z}, 3)$. Then $H_*(F)$ is described by:

1.2.3. Lemma. *If F is the fibre of the map $f\colon S^3 \to K(\mathbf{Z}, 3)$ given by* 1.2.1, *then*

$$H_i(F) = \begin{cases} \mathbf{Z}/(m) & \text{if } i = 2m \text{ and } m > 1 \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

1.2.4. Corollary. *The first p-torsion in $\pi_*(S^3)$ is $\mathbf{Z}/(p)$ in $\pi_{2p}(S^3)$ for any prime p.* $\square$

Proof of 1.2.3. (It is so easy we cannot resist giving it.) We have a fibration

$$\Omega K(\mathbf{Z}, 3) = K(\mathbf{Z}, 2) \to F \to S^3$$

and $H^*(K(\mathbf{Z}, 2)) = H^*(\mathbf{C}P^\infty) = \mathbf{Z}[x]$, where $x \in H^2(\mathbf{C}P^\infty)$ and $\mathbf{C}P^\infty$ is an infinite-dimensional complex projective space. We will look at the Serre spectral sequence for $H^*(F)$ and use the universal coefficient theorem to translate this to the desired description of $H_*(F)$. Let u be the generator of $H^3(S^3)$. Then in the Serre spectral sequence we must have $d_3(x) = \pm u$; otherwise F would not be 3-connected, contradicting 1.1.2. Since d_3 is a derivation we have $d_3(x^n) = \pm nux^{n-1}$. It is easily seen that there can be no more differentials and we get

$$H^i(F) = \begin{cases} \mathbf{Z}/(m) & \text{if } i = 2m+1,\ m > 1 \\ 0 & \text{otherwise} \end{cases}$$

which leads to the desired result. $\square$

If we start with $X = S^n$ the Serre spectral sequence calculations will be much easier for $\pi_{k+n}(S^n)$ for $k < n - 1$. Then all of the computations are in the stable range, i.e., in dimensions less than twice the connectivity of the spaces involved. This means that for a fibration $F \xrightarrow{i} X \xrightarrow{f} K$, the Serre spectral sequence gives a long exact sequence

$$\cdots \to H_j(F) \xrightarrow{i_*} H_j(X) \xrightarrow{f_*} H_j(K) \xrightarrow{d} H_{j-1}(F) \to \cdots, \tag{1.2.5}$$

where d corresponds to Serre spectral sequence differentials. Even if we know $H_*(X)$, $H_*(K)$, and f_*, we still have to deal with the short exact sequence

$$0 \to \operatorname{coker} f_* \to H_*(F) \to \ker f_* \to 0. \tag{1.2.6}$$

It may lead to some ambiguity in $H_*(F)$, which must be resolved by some other means. For example, when computing $\pi_*(S^n)$ for large n one encounters this problem in the 3-component of $\pi_{n+10}(S^n)$ and the 2-component of $\pi_{n+14}(S^n)$. This difficulty is also present in the Adams spectral sequence, where one has the possibility of a nontrivial differential in these dimensions. These differentials were first calculated by Adams [**12**], Liulevicius [**2**], and Shimada and Yamanoshita [**3**] by methods involving secondary cohomology operations and later by Adams and Atiyah [**13**] by methods involving K-theory

The Adams spectral sequence of Adams [**3**] begins with a variation of Serre's method. One works only in the stable range and only on the p-component. Instead of mapping X to $K(\pi, n)$ as in 1.2.1, one maps to $K = \prod_{j>0} K(H^j(X; \mathbf{Z}/(p)), j)$ by a certain map g which induces a surjection in mod (p) cohomology. Let X_1 be the fiber of g. Define spaces X_i and K_i inductively by $K_i = \prod_{j>0} K(H^j(X_i; \mathbf{Z}/(p)), j)$ and X_{i+1} is the fiber of $g\colon X_i \to K_i$ (this map is defined in Section 2.1, where the Adams spectral sequence is discussed in more detail). Since $H^*(g_i)$ is onto, the analog of 1.2.5 is an short exact sequence in the stable range

$$0 \leftarrow H^*(X_i) \leftarrow H^*(K_i) \leftarrow H^*(\Sigma X_{i+1}) \leftarrow 0, \tag{1.2.7}$$

where all cohomology groups are understood to have coefficients $\mathbf{Z}/(p)$. Moreover, $H^*(K_i)$ is a free module over the mod (p) Steenrod algebra A, so if we splice together the short exact sequences of 1.2.7 we get a free A-resolution of $H^*(X)$

$$0 \leftarrow H^*(X) \leftarrow H^*(K) \leftarrow H^*(\Sigma^1 K_1) \leftarrow H^*(\Sigma^2 K_2) \leftarrow \cdots \tag{1.2.8}$$

Each of the fibration $X_{i+1} \to X_i \to K_i$ gives a long exact sequence of homotopy groups. Together these long exact sequences form an exact couple and the associated spectral sequence is the Adams spectral sequence for the p-component of $\pi_*(X)$. If X has finite type, the diagram

$$K \to \Sigma^{-1} K_1 \to \Sigma^{-2} K_2 \to \cdots \tag{1.2.9}$$

(which gives 1.2.8 in cohomology) gives a cochain complex of homotopy groups whose cohomology is $\mathrm{Ext}_A(H^*(X); \mathbf{Z}/(p))$. Hence one gets

1.2.10. Theorem (Adams [**3**]). *There is a spectral sequence converging to the p-component of $\pi_{n+k}(S^n)$ for $k < n-1$ with*

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(\mathbf{Z}/(p), \mathbf{Z}/(p)) =: H^{s,t}(A)$$

and $d_r\colon E_r^{s,t} \to E_r^{s+r,t+r-1}$. Here the groups $E_\infty^{s,t}$ for $t-s=k$ form the associated graded group to a filtration of the p-component of $\pi_{n+k}(S^n)$. □

Computing this E_2-term is hard work, but it is much easier than making similar computations with Serre spectral sequence. The most widely used method today is the spectral sequence of May [**1, 2**] (see Section 3.2). This is a trigraded spectral sequence converging to $H^{**}(A)$, whose E_2-term is the cohomology of a filtered form of the Steenrod algebra. This method was used by Tangora [**1**] to compute $E_2^{s,t}$ for $p = 2$ and $t - s \leq 70$. Most of his table is reproduced here in Fig. A3.1a–c. Computations for odd primes can be found in Nakamura [**2**].

As noted above, the Adams E_2-term is the cohomology of the Steenrod algebra. Hence $E_2^{1,*} = H^1(A)$ is the indecomposables in A. For $p = 2$ one knows that A is generated by Sq^{2^i} for $i \geq 0$; the corresponding elements in $E_2^{1,*}$ are denoted by $h_i \in E_2^{1,2^i}$. For $p > 2$ the generators are the Bockstein β and $\mathcal{P}^{p^i}$ for $i \geq 0$ and the corresponding elements are $a_0 \in E_2^{1,1}$ and $h_i \in E_2^{1,qp^i}$, where $q = 2p-2$.

For $p = 2$ these elements figure in the famous Hopf invariant one problem.

1.2.11. Theorem (Adams [**12**]). *The following statements are equivalent.*

(a) *S^{2^i-1} is parallelizable, i.e., it has $2^i - 1$ globally linearly independent tangent vector fields.*

(b) *There is a division algebra (not necessarily associative) over $\mathbf{R}$ of dimension 2^i.*

(c) *There is a map* $S^{2\cdot 2^i-1} \to S^{2^i}$ *of Hopf invariant one* (*see* 1.5.2).

(d) *There is a* 2*-cell complex* $X = S^{2^i} \cup e^{2^{i+1}}$ [*the cofiber of the map in* (*c*)] *in which the generator of* $H^{2^{i+1}}(X)$ *is the square of the generator of* $H^{2^i}(X)$.

(e) *The element* $h_i \in E_2^{1,2^i}$ *is a permanent cycle in the Adams spectral sequence.* □

Condition (b) is clearly true for $i = 0$, 1, 2 and 3, the division algebras being the reals **R**, the complexes **C**, the quaternions **H** and the Cayley numbers, which are nonassotiative. The problem for $i \geq 4$ is solved by

1.2.12. THEOREM (Adams [**12**]). *The conditions of* 1.2.11 *are false for* $i \geq 4$ *and in the Adams spectral sequence one has* $d_2(h_i) = h_0 h_{i-1}^2 \neq 0$ *for* $i \geq 4$. □

For $i = 4$ the above gives the first nontrivial differential in the Adams spectral sequence. Its target has dimension 14 and is related to the difficulty in Serre's method referred to above.

The analogous results for $p > 2$ are

1.2.13. THEOREM (Liulevicius [**2**] and Shimada and Yamanoshita [**3**]). *The following are equivalent.*

(a) *There is a map* $S^{2p^{i+1}-1} \to \widehat{S}^{2p^i}$ *with Hopf invariant one* (*see* 1.5.3 *for the definition of the Hopf invariant and the space* $\widehat{S}^{2m}$).

(b) *There is a* p*-cell complex* $X = S^{2p^i} \cup e^{4p^i} \cup e^{6p^i} \cup \cdots \cup e^{2p^{i+1}}$ [*the cofiber of the map in* (*a*)] *whose* mod (p) *cohomology is a truncated polynomial algebra on one generator.*

(c) *The element* $h_i \in E_2^{1,qp^i}$ *is a permanent cycle in the Adams spectral sequence.* □

The element h_0 is the first element in the Adams spectral sequence above dimension zero so it is a permanent cycle. The corresponding map in (a) suspends to the element of $\pi_{2p}(S^3)$ given by 1.2.4. For $i \geq 1$ we have

1.2.14. THEOREM (Liulevicius [**2**] and Shimada and Yamanoshita [**3**]). *The conditions of* 1.2.13 *are false for* $i \geq 1$ *and* $d_2(h_i) = a_0 b_{i-1}$, *where* b_{i-1} *is a generator of* E_2^{2,qp^i} (*see Section* 5.2). □

For $i = 1$ the above gives the first nontrivial differential in the Adams spectral sequence for $p > 2$. For $p = 3$ its target is in dimension 10 and was referred to above in our discussion of Serre's method.

FIG. 1.2.15 shows the Adams spectral sequence for $p = 3$ through dimension 45. We present it here mainly for comparison with a similar figure (1.2.19) for the Adams–Novikov spectral sequence. $E_2^{s,t}$ is a $\mathbf{Z}/(p)$ vector space in which each basis element is indicated by a small circle. Fortunately in this range there are just two bigradings [(5,28) and (8,43)] in which there is more than one basis element. The vertical coordinate is s, the cohomological degree, and the horizontal coordinate is $t - s$, the topological dimension. These extra elements appear in the chart to the right of where they should be, and the lines meeting them should be vertical. A d_r is indicated by a line which goes up by r and to the left by 1. The vertical lines represent multiplication by $a_0 \in E_2^{1,1}$ and the vertical arrow in dimension zero indicates that all powers of a_0 are nonzero. This multiplication corresponds to multiplication by p in the corresponding homotopy group. Thus from the figure one can read off $\pi_0 = \mathbf{Z}$, $\pi_{11} = \pi_{45} = \mathbf{Z}/(9)$, $\pi_{23} = \mathbf{Z}/(9) \oplus \mathbf{Z}/(3)$, and $\pi_{35} = \mathbf{Z}/(27)$. Lines that go up 1 and to the right by 3 indicate multiplication by $h_0 \in E_2^{1,4}$, while those that go to the right by 7 indicate the Massey product $\langle h_0, h_0, -\rangle$ (see A1.4.1). The elements a_0 and h_i for $i = 0$, 1, 2 were defined above and the elements $b_0 \in E_2^{2,12}$, $k_0 \in E_2^{2,28}$, and $b_1 \in E_2^{2,36}$ are up to the sign the Massey products $\langle h_0, h_0, h_0\rangle$, $\langle h_0, h_1, h_1\rangle$, and $\langle h_1, h_1, h_1\rangle$, respectively. The unlabeled elements in $E_2^{i,5i-1}$ for $i \geq 2$ (and $h_0 \in E_2^{1,4}$) are related to each other by the Massey product $\langle h_0, a_0, -\rangle$. This accounts for all of the generators except those in $E_2^{3,26}$, $E_2^{7,45}$ and $E_2^{8,50}$, which are too complicated to describe here.

We suggest that the reader take a colored pencil and mark all of the elements which survive to E_∞, i.e., those which are not the source or target of a differential. There are in this range 31 differentials which eliminate about two-thirds of the elements shown.

Now we consider the spectral sequence of Adams and Novikov, which is the main object of interest in this book. Before describing its construction we review the main ideas behind the Adams spectral sequence. They are the following.

1.2.16. Procedure. (i) Use mod (p)-cohomology as a tool to study the p-component of $\pi_*(X)$. (ii) Map X to an appropriate Eilenberg–Mac Lane space K, whose homotopy groups are known. (iii) Use knowledge of $H^*(K)$, i.e., of the Steenrod algebra, to get at the fiber of the map in (ii). (iv) Iterate the above and codify all information in a spectral sequence as in 1.2.10. □

An analogous set of ideas lies behind the Adams–Novikov spectral sequence, with mod p cohomology being replaced by complex cobordism theory. To elaborate, we first remark that "cohomology" in 1.2.16(i) can be replaced by "homology" and 1.2.10 can be reformulated accordingly; the details of this reformulation need not be discussed here. Recall that singular homology is based on the singular chain complex, which is generated by maps of simplices into the space X. Cycles in the chain complex are linear combinations of such maps that fit together in an appropriate way. Hence $H_*(X)$ can be thought of as the group of equivalence classes of maps of certain kinds of simplicial complexes, sometimes called "geometric cycles," into X.

Our point of departure is to replace these geometric cycles by closed complex manifolds. Here we mean "complex" in a very weak sense; the manifold M must be smooth and come equipped with a complex linear structure on its stable normal bundle, i.e., the normal bundle of some embedding of M into a Euclidean space of even codimension. The manifold M need not be analytic or have a complex structure on its tangent bundle, and it may be odd-dimensional.

The appropriate equivalence relation among maps of such manifolds into X is the following.

1.2.17. Definition. *Maps $f_i\colon M \to X$ $(i = 1, 2)$ of n-dimensional complex (in the above sense) manifolds into X are bordant if there is a map $g\colon W \to X$ where W is a complex mainfold with boundary $\partial W = M_1 \cup M_2$ such that $g|M_i = f_i$. (To be correct we should require the restriction to M_2 to respect the complex structure on M_2 opposite to the given one, but we can ignore such details here.)* □

One can then define a graded group $MU_*(X)$, the *complex bordism* of X, analogous to $H_*(X)$. It satisfies all of the Eilenberg–Steenrod axioms except the dimension axiom, i.e., $MU_*(\mathrm{pt})$, is not concentrated in dimension zero. It is by definition

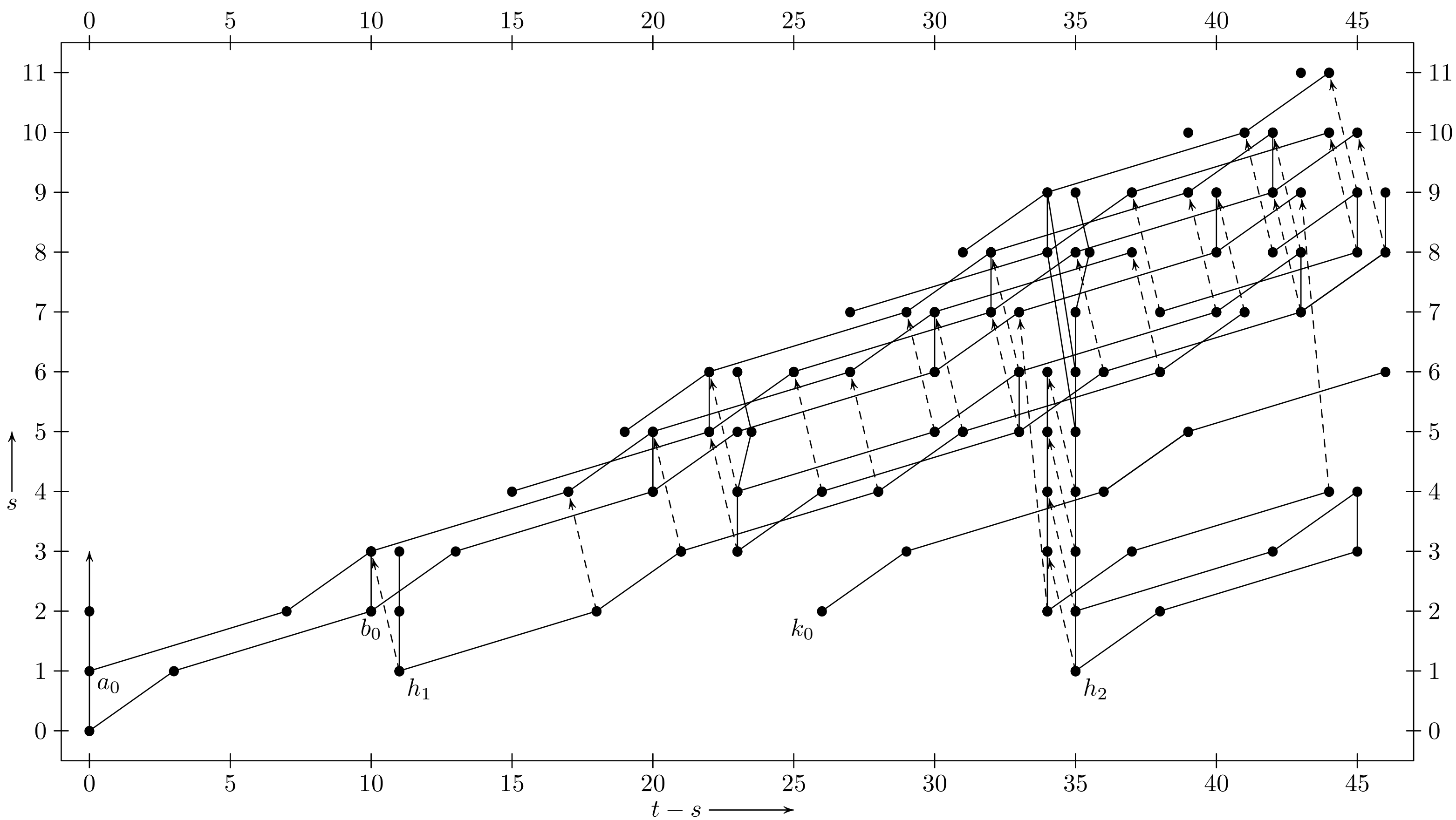

FIGURE 1.2.15. The Adams spectral sequence for $p = 3$, $t - s \leq 45$.

the set of equivalence classes of closed complex manifolds under the relation of 1.2.17 with $X = \mathrm{pt}$, i.e., without any condition on the maps. This set is a ring under disjoint union and Cartesian product and is called the *complex bordism ring*. as are the analogous rings for several other types of manifolds; see Stong [**1**].

1.2.18. THEOREM (Thom [**1**], Milnor [**4**], Novikov [**2**]). *The complex bordism ring,* $MU_*(\mathrm{pt})$*, is* $\mathbf{Z}[x_1, x_2, \ldots]$ *where* $\dim x_i = 2i$. □

Now recall 1.2.16. We have described an analog of (i), i.e., a functor $MU_*(-)$ replacing $H_*(-)$. Now we need to modify (ii) accordingly, e.g., to define analogs of the Eilenberg–Mac Lane spaces. These spaces (or rather the corresponding spectrum MU) are described in Section 4.1. Here we merely remark that Thom's contribution to 1.2.18 was to equate $MU_i(\mathrm{pt})$ with the homotopy groups of certain spaces and that these spaces are the ones we need.

To carry out the analog of 1.2.16(iii) we need to know the complex bordism of these spaces, which is also described (stably) in Section 4.1. The resulting spectral sequence is formally introduced in Section 4.4, using constructions given in Section 2.2. We will not state the analog of 1.2.10 here as it would be too much trouble to develop the necessary notation. However we will give a figure analogous to 1.2.15.

The notation of FIG. 1.2.19 is similar to that of FIG. 1.2.15 with some minor differences. The E_2-term here is not a $\mathbf{Z}/(3)$-vector space. Elements of order > 3 occur in $E_2^{0,0}$ (an infinite cyclic group indicated by a square), and in $E_2^{1,12t}$ and $E_2^{3,48}$, in which a generator of order 3^{k+1} is indicated by a small circle with k parentheses to the right. The names α_t, β_t, and $\beta_{s/t}$ will be explained in the next section. The names α_{3t} refer to elements of order 3 in, rather than generators of, $E_2^{1,12t}$. In $E_2^{3,48}$ the product $\alpha_1\beta_3$ is divisible by 3.

One sees from these two figures that the Adams–Novikov spectral sequence has far fewer differentials than the Adams spectral sequence. The first nontrivial Adams–Novikov differential originates in dimension 34 and leads to the relation $\alpha_1\beta_1^3$ in $\pi_*(S^0)$. It was first established by Toda [**2, 3**].

3. The Adams–Novikov E_2-term, Formal Group Laws, and the Greek Letter Construction

Formal group laws and Qillen's theorem. The Adams–Novikov E_2-term as group cohomology. Alphas, beta and gamma. The Morava–Landweber theorem and higher Greek letters. Generalized Greek letter elements.

In this section we will describe the E_2-term of the Adams–Novikov spectral sequence introduced at the end of the previous section. We begin by defining formal group laws (1.3.1) and describing their connection with complex cobordism (1.3.4). Then we characterize the E_2-term in terms of them (1.3.5 and 1.3.6). Next we describe the Greek letter construction, an algebraic method for producing periodic families of elements in the E_2-term. We conclude by commenting on the problem of representing these elements in $\pi_*(S)$.

Suppose T is a one-dimensional commutative analytic Lie group and we have a local coordinate system in which the identity element is the origin. Then the group operation $T \times T \to T$ can be described locally as a real-valued analytic function of two variables. Let $F(x, y) \in \mathbf{R}[[x, y]]$ be the power series expansion of this function about the origin. Since 0 is the identity element we have

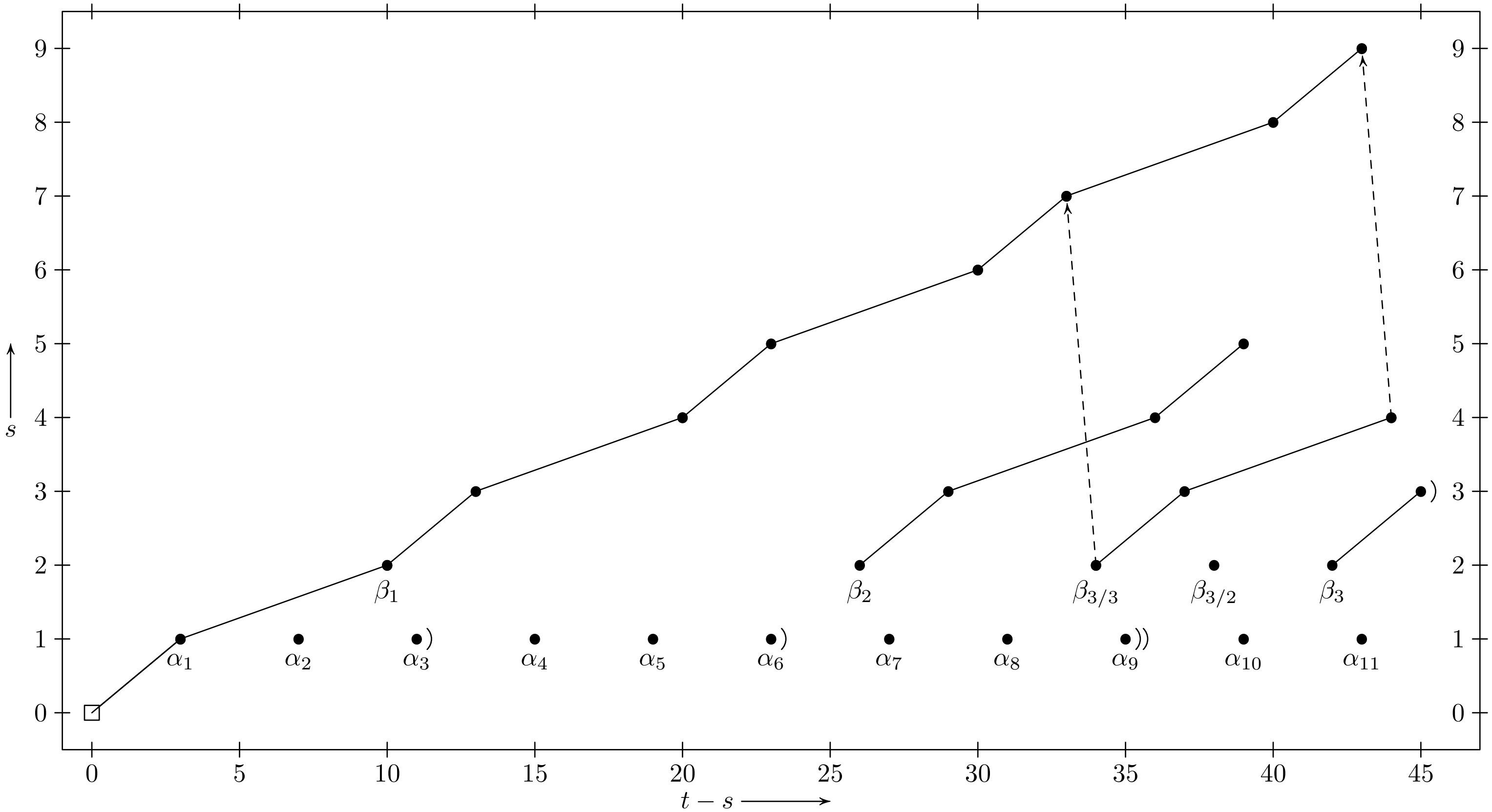

FIGURE 1.2.19. The Adams–Novikov spectral sequence for $p = 3$, $t - s \leq 45$

$F(x,0) = F(0,x) = x$. Commutativity and associativity give $F(x,y) = F(y,x)$ and $F(F(x,y),z) = F(x,F(y,z))$, respectively.

1.3.1. DEFINITION. *A* formal group law *over a commutative ring with unit R is a power series $F(x,y) \in R[[x,y]]$ satisfying the three conditions above.* □

Several remarks are in order. First, the power series in the Lie group will have a positive radius of convergence, but there is no convergence condition in the definition above. Second, there is no need to require the existence of an inverse because it exists automatically. It is a power series $i(x) \in R[[x]]$ satisfying $F(x,i(x)) = 0$; it is an easy exercise to solve this equation for $i(x)$ given F. Third, a rigorous self-contained treatment of the theory of formal group laws is given in Appendix 2.

Note that $F(x,0) = F(0,x) = x$ implies that $F \equiv x + y \mod (x,y)^2$ and that $x + y$ is therefore the simplest example of an formal group law; it is called the *additive* formal group law and is denoted by F_{a}. Another easy example is the *multiplicative* formal group law, $F_{\mathrm{m}} = x + y + rxy$ for $r \in \mathbf{R}$. These two are known to be the only formal group laws which are polynomials. Other examples are given in A2.1.4.

To see what formal group laws have to do with complex cobordism and the Adams–Novikov spectral sequence, consider $MU^*(\mathbf{C}P^\infty)$, the complex cobordism of infinite-dimensional complex projective space. Here $MU^*(-)$ is the cohomology theory dual to the homology theory $MU_*(-)$ (complex bordism) described in Section 2. Like ordinary cohomology it has a cup product and we have

1.3.2. THEOREM. *There is an element $x \in MU^2(\mathbf{C}P^\infty)$ such that*

$$MU^*(\mathbf{C}P^\infty) = MU^*(\mathrm{pt})[[x]]$$

and

$$MU^*(\mathbf{C}P^\infty \times \mathbf{C}P^\infty) = MU^*(\mathrm{pt})[[x \otimes 1, 1 \otimes x]].$$

□

Here $MU^*(\mathrm{pt})$ is the complex cobordism of a point; it differs from $MU_*(\mathrm{pt})$ (described in 1.2.18) only in that its generators are negatively graded. The generator x is closely related to the usual generator of $H^2(\mathbf{C}P^\infty)$, which we also denote by x. The alert reader may have expected $MU^*(\mathbf{C}P^\infty)$ to be a polynomial rather than a power series ring since $H^*(\mathbf{C}P^\infty)$ is traditionally described as $\mathbf{Z}[x]$. However, the latter is really $\mathbf{Z}[[x]]$ since the cohomology of an infinite complex maps onto the inverse limit of the cohomologies of its finite skeleta. [$MU^*(\mathbf{C}P^n)$, like $H^*(\mathbf{C}P^n)$, is a truncated polynomial ring.] Since one usually considers only homogeneous elements in $H^*(\mathbf{C}P^\infty)$, the distinction between $\mathbf{Z}[x]$ and $\mathbf{Z}[[x]]$ is meaningless. However, one can have homogeneous infinite sums in $MU^*(\mathbf{C}P^\infty)$ since the coefficient ring is negatively graded.

Now $\mathbf{C}P^\infty$ is the classifying space for complex line bundles and there is a map $\mu\colon \mathbf{C}P^\infty \times \mathbf{C}P^\infty \to \mathbf{C}P^\infty$ corresponding to the tensor product; in fact, $\mathbf{C}P^\infty$ is known to be a topological abelian group. By 1.3.2 the induced map μ^* in complex cobordism is determined by its behavior on the generator $x \in MU^2(\mathbf{C}P^\infty)$ and one easily proves, using elementary facts about line bundles,

1.3.3. PROPOSITION. *For the tensor product map $\mu\colon \mathbf{C}P^\infty \times \mathbf{C}P^\infty \to \mathbf{C}P^\infty$, $\mu^*(x) = F_U(x \otimes 1, 1 \otimes x) \in MU^*(\mathrm{pt})[[x \otimes 1, 1 \otimes x]]$ is an formal group law over $MU^*(\mathrm{pt})$.* □

A similar statement is true of ordinary cohomology and the formal group law one gets is the additive one; this is a restatement of the fact that the first Chern class of a tensor product of complex line bundles is the sum of the first Chern classes of the factors. One can play the same game with complex K-theory and get a multiplicative formal group law.

$\mathbf{C}P^\infty$ is a good test space for both complex cobordism and K-theory. One can analyze the algebra of operations in both theories by studying their behavior in $\mathbf{C}P^\infty$ (see Adams [**5**]) in the same way that Milnor [**2**] analyzed the mod (2) Steenrod algebra by studying its action on $H^*(\mathbf{R}P^\infty; \mathbf{Z}/(2))$. (See also Steenrod and Epstein [**1**].)

The formal group law of 1.3.3 is not as simple as the ones for ordinary cohomology or K-theory; it is complicated enough to have the following universal property.

1.3.4. THEOREM (Quillen [**2**]). *For any formal group law F over any commutative ring with unit R there is a unique ring homomorphism $\theta\colon MU^*(\mathrm{pt}) \to R$ such that $F(x, y) = \theta F_U(x, y)$.* □

We remark that the existence of such a universal formal group law is a triviality. Simply write $F(x, y) = \sum a_{i,j} x^i y^i$ and let $L = \mathbf{Z}[a_{i,j}]/I$, where I is the ideal generated by the relations among the $a_{i,j}$ imposed by the definition 1.3.1 of an formal group law. Then there is an obvious formal group law over L having the universal property. Determining the explicit structure of L is much harder and was first done by Lazard [**1**]. Quillen's proof of 1.3.4 consisted of showing that Lazard's universal formal group law is isomorphic to the one given by 1.3.3.

Once Quillen's Theorem 1.3.4 is proved, the manifolds used to define complex bordism theory become irrelevant, however pleasant they may be. All of the applications we will consider follow from purely algebraic properties of formal group laws. This leads one to suspect that the spectrum MU can be constructed somehow using formal group law theory and without using complex manifolds or vector bundles. Perhaps the corresponding infinite loop space is the classifying space for some category defined in terms of formal group laws. Infinite loop space theorists, where are you?

We are now just one step away from a description of the Adams–Novikov spectral sequence E_2-term. Let $G = \{f(x) \in \mathbf{Z}[[x]] \mid f(x) \equiv x \mod (x)^2\}$. Here G is a group under composition and acts on the Lazard/complex cobordism ring $L = MU_*(\mathrm{pt})$ as follows. For $g \in G$ define an formal group law F_f over L by $F_g(x, y) = g^{-1} F_U(g(x), g(y))$. By 1.3.4 F_g is induced by a homomorphism $\theta_g\colon L \to L$. Since g is invertible under composition, θ_g is an automorphism and we have a G-action on L.

Note that $g(x)$ defines an isomorphism between F and F_g. In general, isomorphisms between formal group laws are induced by power series $g(x)$ with leading term a unit multiple (not necessarily one) of x. An isomorphism induced by a g in G is said to be *strict*.

1.3.5. THEOREM. *The E_2-term of the Adams–Novikov spectral sequence converging to π_*^S is isomorphic to $H^{**}(G; L)$.* □

There is a difficulty with this statement: since G does not preserve the grading on L, there is no obvious bigrading on $H^*(G; L)$. We need to reformulate in terms of L as a comodule over a certain Hopf algebra B defined as follows.