Def: For $X$ a space, $M_1, M_2$ almost complex $n$-mfds, we say $f: M_1 \to X$ and $g: M_2 \to X$ are <u>bordant</u> if $\exists\, W^{n+1} \to X$ s.t. $\partial W = M_1 \amalg M_2$ and $H|_{M_1} = f$, $H|_{M_2} = g$. Define $MU_n X = \{f: M^n \to X\}/\text{bordism}$.

Thom-Pontryagin: $MU_*(pt) \otimes \mathbb{Q} \cong \mathbb{Q}[[\mathbb{C}P^1], [\mathbb{C}P^2], [\mathbb{C}P^3], \dots]$ and $MU_*(pt) \cong \mathbb{Z}[t_2, t_4, t_6, \dots]$, $|t_{2i}| = 2i$.

Note: For $L$ the Lazard ring, there's a noncanonical iso $L \cong \mathbb{Z}[x_2, x_4, \dots]$.

Def: A multiplicative cohomology theory $E^*(-)$ is <u>complex-oriented</u> if for $i: S^2 = \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^\infty$, $i^*: \underset{\substack{\| \\ E^*[[x]]}}{E^*\mathbb{C}P^\infty} \to E^* S^2$ is surjective.

Clearly $i^*: H^p(\mathbb{C}P^\infty; MU^q(pt)) \to H^p(S^2; MU^q(pt))$ is surjective, and since both AHSS collapse,

$$\begin{array}{ccc} H^p(\mathbb{C}P^\infty; MU^q(pt)) & \to & H^p(S^2; MU^q(pt)) \\ \Downarrow & & \Downarrow \\ MU^{p+q}(\mathbb{C}P^\infty) & \to & MU^{p+q}(S^2) \end{array}$$

$MU^*$ is complex-oriented.

Note: This means that for $V$ a $\mathbb{C}^n$-bundle, there's an orientation $u \in MU^{2n}(V, V_0)$.

The choice is not canonical.

Theorem: For every choice of orientation, the FGL corresponds to an isomorphism $L \xrightarrow{\sim} MU_*$.

Def: For $F$ a FGL/$R$, its logarithm is $\log x \in R \otimes \mathbb{Q}[\![x]\!]$ given by $\log x = \int_0^x \frac{dt}{\frac{\partial F}{\partial y}(t,0)}$. We have $\log(F(x,y)) = \log x + \log y$.

Proposition: For every choice of orientation, the FGL has logarithm $\log x = \sum_{n\geq 0} \frac{[\mathbb{C}P^n]}{n+1} x^{n+1} \in MU_* \otimes \mathbb{Q}[\![x]\!]$. (Mischenko)

Proof of theorem: Let $F_u$ be a universal FGL/$L$ and $\varphi : L \to MU_*$ such that $\varphi^* F_u = F$ where $F$ corresponds to our orientation.

$F_u \otimes \mathbb{Q}$ is uniquely determined by its log $\sum_{n\geq 0} \frac{p_n}{n+1} x^{n+1}$. Since $F \otimes \mathbb{Q}$ has log $\sum_{n\geq 0} \frac{[\mathbb{C}P^n]}{n+1} x^{n+1}$, it follows that $\varphi(p_n) = [\mathbb{C}P^n]$ so $\varphi \otimes \mathbb{Q}$ is an iso, and thus $\varphi$ is injective.

Consider for $i \leq j$ the Milnor hypersurfaces $H_{ij} \subset \mathbb{C}P^i \times \mathbb{C}P^j$

$H_{ij} = \{ x_0 y_0 + \dots + x_i y_i = 0 \}$.

Let $f_i : H_{ij} \hookrightarrow \mathbb{C}P^i \times \mathbb{C}P^j \to \mathbb{C}P^i$, $f_j : H_{ij} \hookrightarrow \mathbb{C}P^i \times \mathbb{C}P^j \to \mathbb{C}P^j$,

$\xi_i = f_i^* \gamma_i$, $\xi_j = f_j^* \gamma_j$ and $f : H_{ij} \to \mathbb{C}P^\infty$ s.t. $\xi_i \otimes \xi_j = f^* \gamma$.

Let $\pi : H_{ij} \to pt$ and consider $\pi_*(\xi_i \otimes \xi_j)$:

$$\begin{tikzcd} V(\xi_i \otimes \xi_j) \arrow[r] \arrow[d] & V(\pi_*(\xi_i \otimes \xi_j)) \arrow[d] \\ H_{ij} \arrow[r, "\pi"'] & pt \end{tikzcd}$$

We have $V(\pi_*(\xi_i \otimes \xi_j)) \cong f(H_{ij})$ and

the mapping cylinder makes $[H_{ij}] = [f(H_{ij})]$ in $MU^*$.

Then $[H_{ij}] = \pi_* c_1((\xi_i \otimes \xi_j)) = \pi_* F(c_1(\xi_i), c_1(\xi_j)) = \sum_{n,m} \varphi(a_{nm}) \pi_*(c_1(\xi_i)^n c_1(\xi_j)^m)$.

$$= \sum_{n,m} \varphi(a_{nm}) [\mathbb{C}P^{i-n}][\mathbb{C}P^{j-m}].$$ Hence $\varphi$ hits all $[H_{ij}]$.

Proposition: The $[H_{ij}]$ generate $MU^*$.

Proof: Requires finding certain characteristic numbers of $H_{ij}$.

The spectrum $MU$:

Let $MU(n)$ be the Thom space of the canonical $\gamma_n : V(\gamma_n) \to BU(n)$.

For $i : BU(n) \to BU(n+1)$, $i^* \gamma_{n+1}$ has Thom space $\Sigma^2 MU(n)$, so

$\Sigma^2 MU(n) \to MU(n+1)$ gives a spectrum $MU$.

Fact: $H_*(MU;\mathbb{Z}) \cong \mathbb{Z}[b_1, b_2, \ldots]$ for $|b_i| = 2i$.

The AHSS $\mathrm{Ext}_{MU_*MU}(H_*S^0, H_*MU) \Rightarrow \pi_* MU$ shows that

$\pi_* MU \cong MU_*$ so $MU$ represents complex cobordism.