Topology Seminar: FORMAL GROUP LAWS I

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This was prepared for University of Colorado's Topology Seminar (Chromatic Homotopy Theory) in Fall 2019. Please send questions, comments, complaints, and corrections to colton.grainger@colorado.edu. Any mistakes are my own.

Contents

Note (Wikipedia Biography). Michel Paul Lazard (5 December 1924 – 15 September 1987) was a French mathematician who worked in the theory of Lie groups in the context of p-adic analysis. His work "took on a life of its own" in the hands of Daniel Quillen in the late 20th century. Quillen's discovery, that "a ring Lazard used to classify formal group laws " was isomorphic to an "important ring in topology", lead to the subject of chromatic homotopy theory.

Lazard's self-contained treatise on one-dimensional formal groups (of which formal group laws are a "coordinate dependent" manifestation) also birthed the field of p-divisible groups.

His major contributions include:

(1) The classification of p-adic Lie groups: every p-adic Lie group is a closed subgroup of $\mathrm{GL}_n(Z_p)$
(2) The classification of (1-dimensional commutative) formal groups.
(3) That the universal formal group law coefficient ring (Lazard's universal ring) is a polynomial ring.
(4) The concept of "analyseurs" (which have been reinvented by J. Peter May under the name operads).

$\blacktriangleleft$

Definition 0.1. A *formal group law* (FGL) over a ring $R \in \mathsf{CRing}$ is a *formal power series*

$$F(x, y) = \sum_{\substack{n \geq 0 \\ i+j=n}} c^i_j x^i y_j \in R[[x, y]]$$

that *formally satisfies* the axioms for a commutative group operation with 0 as the identity element.

Equivalently, $F \in R[[x, y]]$ is a formal group law if and only if

F is local: $F(x, 0) = x \in R[[x]]$
F is symmetric: $F(x, y) = F(y, x) \in R[[x, y]]$
F has unique iterated brackets: $F(F(x, y), z) = F(x, F(y, z)) \in R[[x, y, z]]$
x **has an inverse:** There is a power series $m(x) \in R[[x]]$ such that $m(0) = 0$ and $F(x, m(x)) = 0$.

In the last two conditions, we need to substitute power series into another. This leads to nonsense if the power series involved have nonzero constant terms, since we have no notion of convergence in the ring R. (Consider substituting 1 for x and y, then working with the coefficient $F(1, 1) = \sum c^i_j$ in the 0th degree. Absurd.) However, if the constant terms are zero, then there is no problem in expanding everything out formally. [Str11]

Example 0.2 (Concrete Examples from Neil Strickland)**.**

(1) The simplest example is
$$F(x, y) = x + y.$$
This is called the additive FGL. It can be defined over any ring R.
(2) If $u \in R$ then we can take
$$F(x, y) = x + y + uxy$$
so that
$$1 + u(x +_F y) = (1 + ux)(1 + uy).$$

In the case $u = 1$, this is called the multiplicative FGL. It can again be defined over any ring R. This FGL should be reminiscent of "changing coordinates" to obtain an additive group law from a one that was multiplicative, e.g., say the multiplicative product in the ring R is

$$G(a, b) = ab$$

. Then change coordinates so that $a = 1 + x$, $b = 1 + y$, and obtain the FGL

$$F = 1 + G \quad \text{such that} \quad F(x, y) = x + y + xy.$$

(3) If c is an invertible element of R then we can define

$$F(x, y) = \frac{x + y}{1 + \frac{xy}{c^2}}$$

We call this the Lorentz FGL; it is the formula for relativistic addition of parallel velocities, where c is the speed of light. We are implicitly using the fact that $(1 + \frac{xy}{c^2})$ is invertible in $R[[x, y]]$, with inverse

$$\sum_{k \geq 0} \left(\frac{-xy}{c^2} \right)^2.$$

◀

We now prove some basic lemmas, as practice in the use of formal power series.

Lemma 0.3 (Terms of small order). *A* FGL *is of the form*

$$x +_F y = x + y + \textit{higher order terms.}$$

That is, if F is an FGL*, then* $F(x, y) = x + y \pmod{xy}$*.*

Proof. Let $F(x, y) = \sum_{i,j \geq 0} c^i_j x^i y_j$ for some coefficients $c^i_j \in \mathbb{R}$ (image an infinite array). So that $x +_F 0 = x$, we must have $c^i_0 = 0$ except for $c^1_0 = 1$. So that $x +_F y = y +_F x$, we must have $c^0_j = 0$ except for $c^0_1 = 1$. Hence

$$F(x, y) = x + y + xy \sum_{i,j > 0} c^i_j x^{i-1} y_{j-1}.$$

□

Imagining an infinite array of coefficients, we have used that "F is local" to determine the only interesting coefficients are past the 0th column and row. Taking this "interesting" subarray helps justify the indexing conventions when we impose the following grading:

$$\deg c^i_j = i + j - 1$$

That "F is symmetric in its arguments" just forces the coefficients in the array to satisfy $c^i_j = c^j_i$. That "F has unique iterated brackets" alludes to the machinations necessary to prove (Michel) Lazard's theorem. historical application

Example 0.4 (deRham cohomology algebra)**.** Recall that a graded algebra over $\mathbb{R}$ is a pair $((A^k)_{k \in \mathbb{Z}}, m)$ where (A^k) is a collection of R-vector spaces A_k, and m is a linear map $m\colon (\oplus_k A^k) \otimes (\oplus_k A^k) \to \oplus_k A^k$ such that m maps $A^k \otimes A^l$ to A^{k+l}. Another example is the polynomial ring in one generator, $\mathbb{R}[x]$, where A^k is the vector space of homogeneous degree k polynomials. A variant is $\mathbb{R}[y]$, where A^{2k} is the vector space of homogeneous degree k polynomials, and A^{odd} is zero. (So y is in degree 2.) Depending on your taste, $\mathbb{R}[y]$ is isomorphic to the deRham cohomology ring of $\mathbb{CP}^\infty$. If your taste is different, then the *power series ring* $\mathbb{R}[[y]]$ is rather isomorphic to the deRham cohomology ring of $\mathbb{CP}^\infty$. ◀

1. Complex Co-bordism

Suppose E is a generalized cohomology theory, and X is a paracompact space.

Definition 1.1 (E-orientation)**.** A vector bundle $V \xrightarrow{p} X$ over a paracompact space X is E-orientable if there exists x in the n-th degree of the cohomology ring $E^n(\mathrm{Th}(X))$ of the *Thom Space* $\mathrm{Th}(X)$ of X TODO

2. Formal Group Laws

(6) In algebraic topology, one can consider a number of complex-orientable generalised cohomology theories. Such a theory assigns to each space X a graded ring EX, subject to various axioms. If L is a complex line bundle over X, one can define an Euler class e(L) EX, which is a useful invariant of L. There is a formal group law F over E(point) such that e(L M) = F(e(L), e(M)). In the case of ordinary cohomology, we get the additive FGL. In the case of complex K-theory, we get the multiplicative FGL. In the case of complex cobordism, we get Lazard's universal FGL (Quillen's theorem). This is the start of a very deep relationship between formal groups and the algebraic aspects of stable homotopy theory.

References

[Str11] Neil Strickland. Formal Groups. January 2011. 1