Controllable Random Graphs and Eigenvectors of Sparse Symmetric Random Matrices

Intellectual Merit Introduction: The objective of this research project is to study the eigenvectors of sparse symmetric random matrices and apply the results to problems on the controllability of random graphs. The controllability of a system, given by the pair (A, b), where A is a matrix and b is a vector, can be reformulated as a problem concerning the orthogonality of b and the eigenvectors of A. For the sake of space, I will state this reformulation as a definition with the understanding that it is equivalent to standard definitions. A pair (A, b) is uncontrollable if(and only if) there exists an eigenvector v of A such that $v^T b = 0$. Thus the eigenvectors of large random matrices are of interest from both a random matrix theory perspective and from the perspective of applications to control theory. This project will initially investigate eigenvectors of sparse symmetric Wigner matrices. Wigner matrices are random matrices such that the upper triangular entries are independent identically distributed random variables. These matrices were first introduced by Eugene Wigner in the 1950s to model properties of heavy nuclei atoms. An Erdős–Rényi graph $G(n, p)$ is a random graph where any two vertices have a probability p of having an edge between them. The adjacency matrix of $G(n, p)$ is thus a Wigner matrix. A sparse random matrix is one where each entry has some increasing probability of being 0. One specific goal of the project is to show that with high probability a system (A, b) is controllable, for a sparse symmetric Wigner matrix A. Finding the conditions on how sparse the matrix can be and for what b this holds will be a major component of the project. Naturally the results will then be extended to show $G(n, p)$ is controllable with high probability for certain conditions on $p \to 0$ as $n \to \infty$.

Broader impact Introduction: Most results on the controllability of random matrices focuses mainly on those which are not sparse. When considered as a model of large scale networks this means an assumption of a possibly unreasonably connected system, such as everyone on a social network being connected to around half of all other people on the network. This project would work to extend results to very sparse systems which could model systems in which there are very low levels of connectivity.

Intellectual Merit, Research Plan: As mentioned before this projection will initially focus on the controllability of certain sparse (A, b). A necessary condition for the pair (A, b) to be controllable is the matrix A have simple spectrum, i.e. no repeated eigenvalues. Tao and Vu showed with high probability standard Wigner matrices have no repeated eigenvalues. O'Rourke and Touri then showed conditions on (A, b) so that the system is controllable with high probability and that $G(n, 1/2)$ is controllable with high probability. In February of this year Luh and Vu showed that with high probability matrices $M_{ij} = \delta_{ij}\xi_{ij}$ with centered subgaussian ξ_{ij} and Bernoulli δ_{ij} with $\mathbb{P}(\delta_{ij} = 1) = p \geq n^{-1+\delta}$ have simple spectrum. They additionally extend this results to $G(n, p)$ for $1 - n^{-1+\delta} \geq p \geq n^{-1+\delta}$. Thus these recent results give the initial goal for this project to show controllability of $G(n, p)$ and pairs (M_{ij}, b) for matrices of this type.

(1) As just stated the first goal of this project will be to show the pair (A, b) is controllable for A as in Luh and Vu, and b as in O'Rourke and Touri. This will rely heavily on Littlewood-Offord theory, concentration of measure, and resolvents as is common in similar areas of random matrix theory.

(2) The next step would be to extend this result to a larger class of matrices than is currently available from Luh and Vu. The expected lower bound on p for simple spectrum is $\frac{\ln n}{n}$. Also, these is no reason to expect the seed variables ξ_{ij} need to be subgaussian, or even have any finite moments. Showing simple spectrum is related to the eigenvectors of the matrix, and thus the motivation for a robust study of eigenvectors.

(3) The goals in **(1)** and **(2)** should be achievable within the three years of funding. A more ambitious goal for this project is to extend the results to a wider class of b. Even in the non-sparse case results are only known for b with a finite number or a very large number of nonzero entires. This is because the more nonzero entries of b the better the bounds Littlewood-Offord theory can give on $\mathbb{P}(v^T b = 0)$ for random v. This is yet another motivation for a more thorough study of eigenvectors of sparse symmetric random matrices.

If a symmetric matrix has simple spectrum then there exists an ordering on the eigenvalues, and a unique(up to sign) unit eigenvector associated to each eigenvalue. In fact these eigenvectors form an othronormal basis of $\mathbb{R}^n$. Thus by ordering the eigenvectors and forming a matrix whose i^{th} column is the i^{th} eigenvector we can defined a distribution on the orthogonal group. I am curious if the group or topological properties of the orthogonal group can be used to gain information about the eigenvectors.

(4) Where the project moves from there will depend on the state of random matrix theory. A natural extension would be to look at other sparse symmetric matrix ensembles without independent identically distributed entries. This could be used to model networks with some notion of closeness where a connection is more likely between close nodes. Of course if simple spectrum for non-symmetric matrices is shown this project could give insights into how to approach showing controllability of nonsymmetric random matrices, such as the adjacency matrix of directed graphs. Showing nonsymmetric random matrices have simple spectrum appears to be extremely difficult so this could be well outside the reach of three years of funding. However if this is already shown within the next two years or so the goals defined in **(1), (2)** and **(3)** could give insight into showing controllability of pairs (A, b) where A is nonsymmetric.

Conclusion: This project will expand results to a wider range of random matrices giving more models for applications. In doing so the project will develop and expand tools for working with eigenvectors of random matrices giving insight into problems for which current techniques fail.

References

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