ODDS AND ENDS: GROUP THEORY

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Abstract. This is an abbreviated (and highly personalized) study guide I've written to review 10 weeks of group theory as covered in MATH 6130, taught by Nat Thiem at the University of Colorado during Fall 2018. I have included material from Hiro Lee Tanaka's MATH 122 problem sets as taught during Fall 2014 and 2017. *Mistakes are my own.* Please send corrections to `colton.grainger@colorado.edu`.

1. Group theory

1.1. **Words . . .** Know how to define each of the following 4 concepts in about 5 minutes.

1.
   - group
   - subgroup
   - order of an element
   - order of a group
2.
   - group homomorphism
   - group isomorphism
   - kernel
   - image
3.
   - (universal property of) quotient groups
   - short exact sequence of groups
   - $\mathrm{GL}_n(k)$ where $n \geq 2$ and k is a field
   - $\mathrm{SL}_n(k)$ similarly
4.
   - group action on a set X
   - $\mathrm{Aut}_{\mathrm{Set}}(X)$
   - $\mathrm{Stab}_G A$ where $A \subset X$
   - $\mathrm{Norm}_G A$ similarly
5.
   - normalizer of a *subset* $A \subset G$
   - centralizer of $A \subset G$ similarly
   - conjugacy class of an element
   - normal subgroup
6.
   - orbit of an element
   - index of a subgroup
   - center of a group
   - the group operation on G/H when H is normal
7.
   - the symmetric group S_n
   - permutation representation
   - integer partition
   - cycle type
8.
   - discriminant Δ for S_n acting on $k[x_1, \ldots, x_n]$
   - sign of a permutation
   - the alternating group A_n

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- simple group

9.
   - normal series
   - composition series
   - abelian subquotient
   - (universal property of) the commutator subgroup

10.
   - group of automorphisms $\mathrm{Aut}(G)$ of a group G
   - group of outer automorphisms (similarly)
   - semi-direct product
   - split exact sequence

11.
   - derived series
   - lower central series
   - upper central series
   - nilpotence class

12.
   - p-group
   - Sylow p-subgroup
   - n_{p_i} for primes p_i dividing $|G|$
   - elementary abelian group

13.
   - free group functor $\mathsf{Set} \to \mathsf{Grp}$
   - presentation of a group
   - free abelian group functor $\mathsf{Set} \to \mathsf{Ab}$
   - abelianization functor $\mathsf{Grp} \to \mathsf{Ab}$

1.2. **Actions . . .** Know how to sketch quick arguments in response to the following.

1. Common groups
   - Describe S_n in detail for $n = 3, \ldots 6$.
   - Give two presentations of D_{2n}.
   - Define Q_8, prove that it is possible to inject Q_8 into S_7, then prove that it is *not* possible to inject Q_8 into S_n for $n < 7$.
   - Describe A_n in detail for $n = 4, 5, 6$.
   - Describe the automorphisms of a cyclic group Z_m.
2. Counting
   - State and prove Lagrange's theorem.
   - State Cauchy's theorem.
   - State and prove the Orbit-Stabilizer theorem.
   - State and justify the class equation.
   - State Burnside's counting lemma.
3. Sylow theory
   - State the Sylow theorems.
   - Prove that if G is a group of order pq, with p and q distinct primes, then G is not simple.
   - List 5 techniques for group classification from Sylow theory.
   - Prove there is no simple group of order 120.
4. Abstruse results
   - State Feit-Thompson's theorem.
   - State Burnside's theorem.
   - State P. Hall's theorem.
5. Abelian groups

- State the FTFGAG.
- Classify all abelian groups of order 1500.
- Prove a universal property for the commutator subgroup.
- Argue that abelian groups are exactly $\mathbf{Z}$-modules.

6. Free groups
   - Prove a universal property for free groups.
   - Let F be a free group and let N be the subgroup generated by the set $\{x^n : x \in F, n \text{ a fixed integer}\}$. Show that $N \triangleleft F$.
   - Prove that a free abelian group is a free group if and only if it is cyclic.
   - Show that the cyclic group of order 6 is the group defined by the generators a, b and the relations $a^2 = b^2 = a^{-1}b^{-1}ab = e$.
7. Nilpotence
   - Give at least 4 equivalent criteria to recognize a nilpotent group.
   - Prove that a maximal subgroup of a finite nilpotent group has prime index.
8. Frattini's argument
   - State Frattini's argument.

1.3. **Revisions . . .** Work out and revise solutions to the following (perhaps previously assigned) exercises.

1. (Bruhat decomposition) Prove that the group of general linear $n \times n$ matrices over the field $\mathbf{F}$ can be written as
$$GL_n(\mathbf{F}) = \bigsqcup_{\omega \in S_n} B\omega B$$
where B is the set of upper triangular matrices.

   Sketch. Let B be the subgroup of upper triangular matrices in $GL_n(\mathbf{F})$. We want to choose double coset representatives from the set
$$B\backslash GL_n(\mathbf{F})/B = \{BgB | g \in G\}.$$

   Note that elementary matrices in B perform row operations on elements $g \in GL_n(\mathbf{F})$. So, given an arbitrary $g \in G$, we choose $b \in B$ to maximize the number of leading 0's of the product bg.

   But bg might not be in upper triangular (generally, echelon) form, so find a permutation (matrix) $\omega \in S_n \leq GL_n(\mathbf{F})$ such that $\omega bg \in B$. So $\omega bg = b' \in B$. But then $g = b^{-1}\omega^{-1}b' \in B\omega^{-1}B$, the double coset labelled by an element of the symmetric group.

   Are $B\omega B$ and $B\sigma B$ the same if $\omega, \sigma \in S_n$ and $\omega \neq \sigma$?

   Suppose $B\omega B$ and $B\nu B$ are equal where $\omega, \nu \in S_n$. Then there are upper triangular matrices $a, b \in B$ such that $\nu = a\omega b$, so $\omega^{-1}a^{-1}\nu \in B$. Here's the idea: the diagonal entries of an upper triangular matrix are unique. Hence if $\omega^{-1}a^{-1}\nu \in B$, then $\omega^{-1}\nu \in B$. Moreover, $\omega^{-1}\nu \in S_n \cap B = \{I\}$ (why?) so $\omega = \nu$.
2. As a corollary to the orbit stabilizer theorem, justify: Let G be a finite group and K a subgroup of G. (i) The number of elements in the conjugacy class of $x \in G$ is $|G : C_G(x)|$, which divides $|G|$; (ii) the number of subgroups of G conjugate to K is $|G : N_G(K)|$, which divides $|G|$; (iii) if $G(x_1), \ldots, G(x_n)$ are the distinct conjugacy classes then $|G| = \sum_{i=1}^{n} |G : C_G(x_i)|$. [1, Ch. II.4]
   i. The set G is partitioned into disjoint orbits under the conjugation group action of G on its elements. By orbit-stabilizer, then, summing the index of each conjugacy class in G, we have $|G|$.
   ii. G also acts by conjugation on the set of its subgroups. Now $N_G(K)$ is a subgroup of G, so by Lagrange's theorem, $|N_G(K)|$ divides $|G|$. Therefore $|G : N_G(K)|$ divides $|G|$.
   iii. G is the disjoint union of such conjugacy classes.

3. Classify the thirteen isomorphism types of groups of order 56. *See.* [2, No. 5.5.7]

4. Prove that $G = \langle x, y : x^3 = y^3 = (xy)^3 = 1 \rangle$ is an infinite group as follows.

   Let p be a prime congruent to $1 \mod 3$ and let G_p be the non-abelian group of order $3p$. Let $a, b \in G_p$ with $|a| = p$ and $|b| = 3$. [2, No. 6.3.14] Verify that both ab and ab^2 have order 3. Show that G_p is a homomorphic image of G. Argue that G is therefore an infinite group, as there are infinitely many primes $p \equiv 1 \mod 3$.

   i. Let K and L be groups, let n be a positive integer, let $\rho\colon K \to S_n$ be a homomorphism and let H be the direct product of n copies of L. From [2, No. 5.1.8], we constructed an injective homomorphism ψ from S_n into $\operatorname{Aut}(H)$ by letting the elements of S_n permute the n factors of H. The compositions $\psi \circ \rho$ is a homomorphism from G into $\operatorname{Aut}(H)$. The *wreath product* of L by K is the semidirect product $H \rtimes_\psi K$ with respect to this homomorphism and is denoted by $L \wr K$. Note this wreath product depends on the choice of permutation representation ρ of K — if none is given explicitly, then φ is assumed to be the left regular representation of K. [2, No. 5.5.23]

   (a) Assume K and L are finite groups and ρ is the left regular representation of K. We find $|L \wr K|$ in terms of $|K|$ and $|L|$.

   (b) Let p be a prime, let $K = L = Z_p$. Suppose ρ is the left regular representation of K. Then $Z_p \wr Z_p$ is a non-abelian subgroup of order p^{p+1} and is isomorphic to a Sylow p-subgroup of S_{p^2}. [The p copies of Z_p whose direct products makes up H may be represented by p disjoint p-cycles; these are cyclically permuted by K.]

5. Let p be a prime, let P be a p-subgroup of the finite group G, let N be a normal subgroup of G whose order is relatively prime to p, and let $\bar{G} = G/N$. With Frattini's argument, prove $N_{\bar{G}}\left(\bar{P}\right) = \overline{N_G(P)}$. In addition, prove $N_{\bar{G}}\left(\bar{P}\right) = \overline{N_G(P)}$. [2, No. 6.1.20]

6. Write a program to compute the number of conjugacy classes in S_n. What is the largest n for which your program will work? `http://abstract.pugetsound.edu/aata/actions-exercises-programming.html`

1.4. **Reading ...**

- Aluffi II.3, the category Grp [3]
- Aluffi II.4, group homomorphisms
- Aluffi II.5, free groups
- Aluffi II.6, subgroups
- Aluffi II.7, quotient groups
- Aluffi II.8, canonical decomposition
- Hungerford I.5, isomorphism theorems [1]
- Hungerford, classification of groups of small order
- Aluffi II.9, group actions
- `https://gowers.wordpress.com/2011/11/06/group-actions-i/`
- `https://gowers.wordpress.com/2011/11/09/group-actions-ii-the-orbit-stabilizer-theorem/`
- `https://kconrad.math.uconn.edu/blurbs/grouptheory/subgpseries1.pdf`
- `https://kconrad.math.uconn.edu/blurbs/grouptheory/splittinggp.pdf`
- `https://www.math.columbia.edu/~bayer/S09/ModernAlgebra/semidirect.pdf`
- Lang, isomorphism theorems for groups, then for modules [4]
- `https://www.math.uconn.edu/~kconrad/blurbs/grouptheory/sylowpf.pdf`
- `http://abstract.pugetsound.edu/aata/sylow-sage-exercises.html`
- Dummit and Foote 5.3, techniques for groups of medium order [2]
- Dummit and Foote 6.1, p-groups, nilpotent groups, and solvable groups
- (TODO: group theory for physics, e.g., Lie theory, at the NCAR library.)

References

[1] T. Hungerford, *Algebra.* Springer, 1974.

[2] D. Dummit and R. Foote, *Abstract algebra.* Prentice Hall, 2004.

[3] P. Aluffi and American Mathematical Society., *Algebra : Chapter 0.* American Mathematical Society, 2016.

[4] S. Lang, *Algebra.* 2002.