PROBLEM SET 2: GROUP THEORY REVIEW

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1. Definitions

Know how to define each of the following 4 concepts in about 5 minutes. (See references in section 3.)

1.
   - group
   - subgroup
   - order of an element
   - order of a group
2.
   - group homomorphism
   - group isomorphism
   - kernel
   - image
3.
   - (universal property of) quotient groups
   - short exact sequence of groups
   - $\mathrm{GL}_n(k)$ where $n \geq 2$ and k is a field
   - $\mathrm{SL}_n(k)$ similarly
4.
   - group action on a set X
   - $\mathrm{Aut}_{\mathrm{Set}}(X)$
   - $\mathrm{Stab}_G A$ where $A \subset X$
   - $\mathrm{Norm}_G A$ similarly
5.
   - normalizer of a *subset* $A \subset G$
   - centralizer of $A \subset G$ similarly
   - conjugacy class of an element
   - normal subgroup
6.
   - orbit of an element
   - index of a subgroup
   - center of a group
   - the group operation on G/H when H is normal
7.
   - the symmetric group S_n
   - permutation representation
   - integer partition
   - cycle type
8.
   - discriminant Δ for S_n acting on $k[x_1, \ldots, x_n]$
   - sign of a permutation
   - the alternating group A_n
   - simple group
9.
   - normal series
   - composition series
   - abelian subquotient
   - (universal property of) the commutator subgroup

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10.
- group of automorphisms $\mathrm{Aut}(G)$ of a group G
- group of outer automorphisms (similarly)
- semi-direct product
- split exact sequence

11.
- derived series
- lower central series
- upper central series
- nilpotence class

12.
- p-group
- Sylow p-subgroup
- n_{p_i} for primes p_i dividing $|G|$
- elementary abelian group

13.
- free group functor $\mathsf{Set} \to \mathsf{Grp}$
- presentation of a group
- free abelian group functor $\mathsf{Set} \to \mathsf{Ab}$
- abelianization functor $\mathsf{Grp} \to \mathsf{Ab}$

2. Main results

Know how to make quick arguments in support of the following. (Again, see references in section 3.)

1. Counting
   - State and prove Lagrange's theorem.
   - State and prove Cayley's theorem
   - State Cauchy's theorem.
   - State and prove the Orbit-Stabilizer theorem.
   - State and justify the class equation.
   - State Burnside's counting lemma.
2. Sylow theory
   - State the Sylow theorems.
   - List 5 techniques for group classification from Sylow theory.
   - Prove there is no simple group of order 120.
   - Prove that if G is a group of order pq, with p and q distinct primes, then G is not simple.
3. Abstruse results
   - State Feit-Thompson's theorem.
   - State Burnside's theorem.
   - State P. Hall's theorem.
   - State Nielsen–Schreier's theorem
4. Abelian groups
   - State the FTFGAG.
   - Argue that abelian groups are exactly $\mathbf{Z}$-modules.
   - Use the FTFGAG to obtain a projective resolution for a finitely generated abelian group G.
   - Classify all abelian groups of order 1500.
5. Free groups
   - State and prove a universal property for free groups.
   - State and prove a universal property for the commutator subgroup.
   - Prove that a free abelian group is a free group if and only if it is cyclic.

- Prove that the cyclic group of order 6 is the group defined by the generators a, b and the relations $a^2 = b^2 = a^{-1}b^{-1}ab = e$.
- Let F be a free group and let N be the subgroup generated by the set $\{x^n : x \in F, n \text{ a fixed integer}\}$. Prove that $N \triangleleft F$.

6. Nilpotence
    - Give at least 4 equivalent criteria to recognize a nilpotent group.
    - Prove that a maximal subgroup of a finite nilpotent group has prime index.
7. Frattini's argument
    - State Frattini's argument.

3. Reading

Be familiar with the following.

- Tanaka and Conrad describe semi-direct products as split exact sequences.
- Rotman proves a few technical results in group theory using module theory.
- For help with memorization, Peter has copied down relevant definitions and theorems from Dummit and Foote; his outline is included for the sake of completeness.

source	pages	summary
Rock, 2018	13 pages	outline of Dummit and Foote, ch. 1–6
Tanaka, 2014	5 pages	semidirect products are split short exact sequences
Conrad, 2005	18 pages	splitting of short exact sequences for groups
Rotman, 1995	ch. 7, pp. 154–171	extensions, automorphism groups, and semidirect products
Rotman, 1995	ch. 10, pp. 307–320	abelian groups, proof of the FTFGAG