1. Let G be a group. Let $H \triangleleft G$ be a normal subgroup of prime index p. Let $a \in H$. Suppose the conjugacy class of a inside$G$ is of size m. Show that the conjugacy class of a inside H is of size either m or m/p.

2. Classify all groups of order 253 up to isomorphism.

3. Either prove or disprove the following statement, with a full justification.If R is an integral domain that is not a field, then the polynomial ring $R[x]$ can neverbe a principal ideal domain.

4. (a) Suppose that A is a complex $n \times n$ matrix with $A^3 = A$. Show that A is diagonalizable.

   (b) Suppose that A is a 2×2 matrix over the field $\mathbb{Q}$ of rational numbers with no non-trivial eigenvectors with entries in $\mathbb{Q}$, and that $A^3 = A$. Show that A is similar over $\mathbb{Q}$ to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

5. Find the number of monic irreducible sextic polynomials in $\mathbb{F}_3[x]$,where $\mathbb{F}_3$ is the field of three elements.

6. Let $\mathbb{Q}$ be the field of rational numbers, and $\mathbb{C}$ the field of complex numbers. Let $\sqrt{2}$ denote the positive square root of 2 in $\mathbb{C}$. Let $\alpha = \sqrt{4 + 3\sqrt{2}}$ denote the positive square root of $4 + 3\sqrt{2}$ in $\mathbb{C}$.

   (a) Determine the minimal polynomial of α.

   (b) Show that $L = \mathbb{Q}(\alpha)$ is not galois over Q.

   (c) Let M be the galois closure of L over $\mathbb{Q}$. What is the order of the galois group G of M over $\mathbb{Q}$?