Differential Geometry Preliminary Exam Notes

This is a list of most of the definitions, theorems, and propositions contained within *Introduction to Smooth Manifolds* by John M. Lee as well as some extra useful ones. All the numbered theorems within this set of notes should correspond with the actual numbering of the theorems within the book.

Also, in these notes we will conform to the Einstein summation convention.

Important things that really, really need to be remembered will be highlighted in blue. Things to keep in mind are highlighted in yellow, important definitions are highlighted in green, and good examples are highlighted in orange.

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C. Appendix C Review of Calculus

Definition. Let V, W be finite-dimensional vector spaces. If $U \subset V$ is an open subset and $a \in U$, a map $F : U \to W$ is said to be ***differentiable at a*** if there exists a linear map $L : V \to W$ such that

$$\lim_{v \to 0} \frac{|F(a+v) - F(a) - Lv|}{|v|} = 0.$$

If F is differentiable at a, the linear map L satisfying this condition is denoted $DF(a)$ and is called the ***total derivative of F at a***. This condition may also be written as

$$F(a+v) = F(a) + DF(a)v + R(v)$$

where $R(v)$ satisfies $|R(v)|/|v| \to 0$ as $v \to 0$.

Proposition C.3 (The Chain Rule for Total Derivatives). Suppose V, W, X are finite-dimensional vector spaces, $U \subset V$ ad $\widetilde{U} \subset W$ are open subsets, and $F : U \to \widetilde{U}$ and $G : \widetilde{U} \to X$ are maps. If F is differentiable at $a \in U$ and G is differentiable at $F(a) \in \widetilde{U}$, then $G \circ F$ is differentiable at a, and

$$D(G \circ F)(a) = DG(F(a)) \circ DF(a).$$

Definition. Suppose $U \subset \mathbb{R}^n$ is open and $f : U \to \mathbb{R}$ is a real-valued function. For any $a = (a^1, \dots, a^n) \in U$ and any $j \in \{1, \dots, n\}$, the **j^{th} partial derivative of f at a** is defined to be

$$\frac{\partial f}{\partial x^j}(a) = \lim_{h \to 0} \frac{f(a + he_j) = f(a)}{h},$$

where e_j is the j^{th} elementary basis vector for $\mathbb{R}^n$.

Definition. For vector-valued functions $F : U \to \mathbb{R}^m$, we can write the coordinates of $F(x)$ as $F(x) = (F^1(x), \dots, F^m(x))$. These m functions are called the ***component functions of F***. The matrix of partial derivatives of the component functions given by

$$J = \left(\frac{\partial F^i}{\partial x^j} \right)$$

is called the ***Jacobian matrix of F***.

Definition. If $F : U \to \mathbb{R}^m$ is a function for which the partial derivative exists at each point in U and the functions $\partial F^i / \partial x^j : U \to \mathbb{R}$ are all continuous, then F is said to be of ***class C^1*** or ***continuously differentiable***. The ***second-order partial derivatives*** can be obtained by

$$\frac{\partial^2 F^i}{\partial x^k \partial x^j} = \frac{\partial}{\partial x^k} \left(\frac{\partial F^i}{\partial x^j} \right)$$

and continuing in this way we can obtain the ***partial derivatives of F of order k***. A function is function $F : U \to \mathbb{R}^m$ is said to be of ***class C^k*** if all the partial derivatives of F of order less than or equal to k exist and are continuous functions on U. A function is called ***smooth*** if it is of class C^∞.

Definition. If U and V are open subsets of Euclidean spaces, a function $F : U \to V$ is called a ***diffeomorphism*** if it is smooth and bijective and its inverse function is also smooth.

Proposition C.4. Suppose $U \subseteq \mathbb{R}^n$ and $V \subseteq R^m$ are open subsets and $F : U \to V$ is a diffeomorphism. Then $m = n$, and for each $a \in U$, the total derivative $DF(a)$ is invertible, with $DF(a)^{-1} = D(F^{-1})(F(a))$.

Proposition C.6 (Equality of Mixed Partial Derivatives). If U is an open subset of $\mathbb{R}^n$ and $F : U \to \mathbb{R}^m$ is a function of class C^2, then the mixed second-order partial derivatives of F do not depend on the order of differentiation:

$$\frac{\partial^2 F^i}{\partial x^j \partial x^k} = \frac{\partial^2 F^i}{\partial x^k \partial x^j}.$$

Corollary. If $F : U \to \mathbb{R}^m$ is smooth, then the mixed partial derivatives of F of any order are independent of the order of differentiation.

Proposition C.8. Let $U \subseteq \mathbb{R}^n$ be open, and suppose $F : U \to \mathbb{R}^m$ is differentiable at $a \in U$. Then all of the partial derivatives of F at a exist, and $DF(a)$ is the linear map whose matrix is the Jacobian of F at a:

$$DF(a) = \left(\frac{\partial F^j}{\partial x^i}(a)\right).$$

Proposition C.10. Let $U \subseteq \mathbb{R}^n$ be open. If $F : U \to \mathbb{R}^m$ is of class C^1, then it is differentiable at each point of U.

Corollary C.11 (The Chain Rule for Partial Derivatives). Let $U \subseteq \mathbb{R}^n$ and $\widetilde{U} \subseteq \mathbb{R}^m$ be open subsets, and let $x = (x^1, \dots, x^n)$ denote the standard coordinates on U, and $y = (y^1, \dots, y^m)$ those on $\widetilde{U}$.

(a) A composition of C^1 function $F : U \to \widetilde{U}$ and $G : \widetilde{U} \to \mathbb{R}^p$ is again of class C^1, with partial derivatives given by

$$\frac{\partial (G^i \circ F)}{\partial x^j}(x) = \sum_{k=1}^{m} \frac{\partial G^i}{\partial y^k}(F(x)) \frac{\partial F^k}{\partial x^j}(x).$$

(b) If F and G are smooth, then $G \circ F$ is smooth.

Definition. Suppose $f : U \to \mathbb{R}$ is a smooth real-valued function on an open subset $U \subseteq \mathbb{R}^n$, and $a \in U$. For each vector $\mathbf{v} \in \mathbb{R}^n$, we define the ***directional derivative of f in the direction of v at a*** to be the number

$$D_{\mathbf{v}} f(a) = \left.\frac{d}{dt}\right|_{t=0} \frac{\partial f}{\partial x^i}(a) = Df(a)\mathbf{v}$$

Theorem C.34 (Inverse Function Theorem). Suppose U and V are open subsets of $\mathbb{R}^n$, and $F : U \to V$ is a smooth function. If $DF(a)$ is invertible at some point $a \in U$

Definition. Let X be a metric space. A map $G : X \to X$ is said to be a ***contraction*** if there is a constant $\lambda \in (0, 1)$ such that $d(G(x), G(y)) \le \lambda\, d(x, y)$ for all $x, y \in X$. A ***fixed point*** of a map $G : X \to X$ is a point $x \in X$ such that $G(x) = x$.

Lemma C.35 (Contraction Lemma). Let X be a nonempty complete metric space. Every contraction $G : X \to X$ has a unique fixed point.

Corollary C.36. uppose $U \subseteq \mathbb{R}^n$ is an open subset, and $F : U \to \mathbb{R}^n$ is a smooth function whose Jacobian determinant is nonzero at every point in U.

(a) F is an open map.

(b) If F is injective, then $F : U \to F(U)$ is a diffeomorphism.

Theorem C.37 (Implicit Function Theorem). Let $U \subseteq \mathbb{R}^n \times \mathbb{R}^k$ be an open subset, and let $(x, y) = (x^1, \ldots, x^n, y^1, \ldots, y^k)$ denote the standard coordinates on U. Suppose $\Phi : U \to \mathbb{R}^k$ is a smooth function, $(a, b) \in U$, and $c \in \Phi(a, b)$. If the $k \times k$ matrix

$$\left(\frac{\partial \Phi^i}{\partial y^j}(a, b)\right)$$

is nonsingular, then there exist neighborhoods $v_0 \subseteq \mathbb{R}^n$ of a and $W_0 \subseteq \mathbb{R}^k$ of b and a smooth function $F : V_0 \to \mathbb{W}_0$ such that $\Phi^{-1}(c) \cap (V_0 \times W_0)$ is the graph of F, that is, $\Phi(x, y) = c$ for $(x, y) \in V_0 \times W_0$ if and only if $y = F(x)$.

Note: This theorem is very, very useful for any proofs involving maps between smooth charts that have sufficiently "nice" local coordinate expressions. You 100% need this theorem for the prelim exam.

1. Smooth Manifolds

When dealing with smooth manifolds, the idea that you want to keep in mind is that, locally, they all are supposed to look like a section of $\mathbb{R}^n$. As such, a good chunk of Differential Geometry deals with the ways in which we may extend our understanding of $\mathbb{R}^n$ to generalized differentiable manifolds. There is actually a very nice theorem (called the Whitney Embedding Theorem which will be discussed in a later section) that tells us that all smooth manifolds of dimension n can be embedded in a space of dimension $\mathbb{R}^{2n+1}$. But that is a topic that will not be covered until the section on Sard's Theorem. For now, we will work on how, exactly, we define smooth manifolds, and give ourselves some tools for doing Calculus on them.

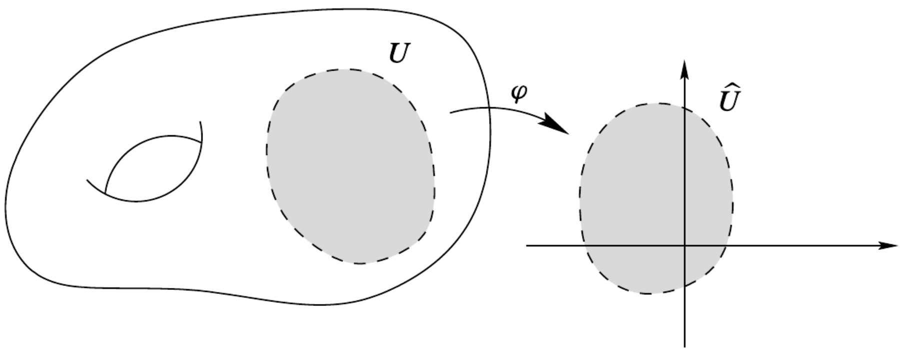
- Definition of a Smooth Manifold -

Definition. We say that a topological space M is a ***topological n-manifold*** if

- M is a Hausdorff space.
- M is second-countable.
- M is locally Euclidean of dimension n. That is, every point $p \in M$ has a neighborhood that is homeomorphic to $\mathbb{R}^n$ for a fixed n. This means that for each $p \in M$ we can find
  - an open neighborhood $U \subseteq M$ around p,
  - an open neighborhood $\widehat{U} \subseteq \mathbb{R}^n$, and
  - a homeomorphism $\varphi : U \to \widehat{U}$.

Theorem 1.2 (Topological Invariance of Dimension). A non-empty topological n-manifold cannot be homeomorphic to a topological m-manifold unless $m = n$.

Definition. Let M be a topological n-manifold. A ***coordinate chart*** on M is an ordered pair (U, φ), where U is an open subset of M (called the ***coordinate domain***) and $\varphi : U \to \widehat{U}$ is a homeomorphism from U to an open subset $\widehat{U} = \varphi(U) \subseteq \mathbb{R}^n$. If $\varphi(p) = 0$ for $p \in U$, then we say that the chart is ***centered at p***.



Proposition 1.11. Let M be a topological manifold.

(a) M is locally path connected.

(b) M is connected if and only if it is path-connected.

(c) The components of M are the same as its path components.

(d) M has countably many components, each of which is an open subset of M and a connected topological manifold.

Proposition 1.12. Every topological manifold is locally compact.

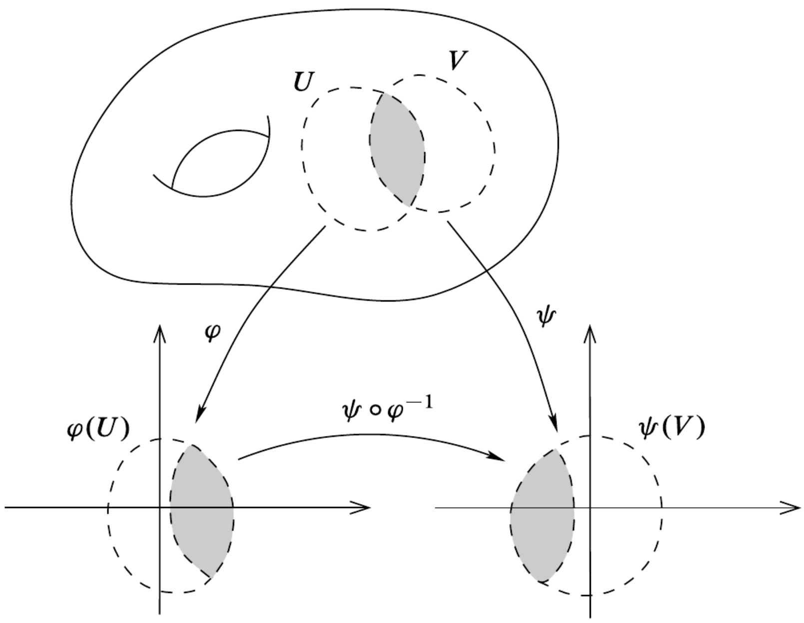
Definition. Let M be a topological space. A collection $\mathcal{X}$ of subsets of M is said to be ***locally finite*** if each point of M has a neighborhood that intersects at most finitely many members of $\mathcal{X}$.

Definition. Given a cover $\mathcal{U}$ pf M, another cover $\mathcal{V}$ is called a ***refinement of $\mathcal{U}$*** if for each $V \in \mathcal{V}$ there is some $U \in \mathcal{U}$ such that $V \subseteq U$. We say that M is ***paracompact*** if every open cover of M admits an open, locally finite refinement.

Theorem 1.15 (Manifolds are Paracompact). Every topological manifold is paracompact. In fact, given a topological manifold M, an open cover $\mathcal{X}$ of M, and any basis $\mathcal{B}$ for the topology on M, there exists a countable, locally finite open refinement of $\mathcal{X}$ consisting of elements of $\mathcal{B}$.

Definition. If U and V are open subsets of Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, a function $F : U \to V$ is said to be ***smooth*** if each of its component functions has continuous partial derivatives of all orders. If in addition F is bijective and has a smooth inverse map, it is called a ***diffeomorphism***.

Definition. Let M be a topological n-manifold. If (U, φ), (V, ψ) are two charts such that $U \cap V \neq \varnothing$, the composite map $\psi \circ \varphi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is called the ***transition map from φ to ψ***. It is a composition of homeomophsims, and is therefore itself a homeomorphism. Two charts (U, φ) and (V, ψ) are said to be ***smoothly compatible*** if either $U \cap V = \varnothing$ or the transition map $\psi \circ \varphi^{-1}$ is a diffeomorphism.



Definition. An ***atlas*** for a manifold M is a collection of charts whose domains cover M. An atlas $\mathcal{A}$ is called a ***smooth atlas*** if any two charts in $\mathcal{A}$ are smoothly compatible.

Note: Given two particular charts (U, φ) and (V, ψ), it is often easiest to show that they are smoothly compatible by verifying that $\psi \circ \varphi^{-1}$ is smooth and injective with nonsingular Jacobian at each point, and appealing to Corollary C.36.

Definition. A smooth atlas $\mathcal{A}$ in M is ***maximal*** if it is not properly contained in any larger smooth atlas.

Definition. If M is a topological manifold, a ***smooth structure on M*** is a maximal smooth atlas. A ***smooth manifold*** is a pair $(M, \mathcal{A})$, where M is a topological manifold and $\mathcal{A}$ is a smooth structure on M.

Theorem 1.17. Let M be a topological manifold.

(a) Every smooth atlas $\mathcal{A}$ for M is contained in a unique maximal smooth atlas, called the ***smooth structure determined by $\mathcal{A}$***.

(b) Two smooth atlases for M determine the same smooth structure if and only if their union is a smooth atlas.

Definition. If M is a smoothm manifold, any chart (U, φ) contained in th given maximal smooth atlas is called a ***smooth chart***, and the corresponding coordinate map φ is called a ***smooth coordinate map***.

- Examples of Smooth Manifolds -

Example 1.22 (Euclidean Spaces). For each nonnegative integer n, the Euclidean space $\mathbb{R}^n$ is a smooth n-manifold with the smooth structure determined by the atlas consisting of the single chart $(\mathbb{R}^n, \mathrm{Id}_{\mathbb{R}}^n)$. We call this the ***standard smooth structure on $\mathbb{R}^n$*** and the resulting coordinate map ***standard coordinates***. Unless we explicitly specify otherwise, we always use this smooth structure on $\mathbb{R}^n$. With respect to this smooth structure, the smooth coordinate charts for $\mathbb{R}^n$ are exactly those charts (U, φ) such that φ is a diffeomorphism from U to another open subset $\widehat{U} \subseteq \mathbb{R}^n$.

Example 1.25 (Spaces of Matrices). Let $M(m \times n, \mathbb{R})$ denote the set of $m \times n$ matrices with real entries. Because it is a real vector space of dimension mn under matrix addition and scalar multiplication, $M(m \times n, \mathbb{R})$ is a smooth mn-dimensional manifold. Similarly, the space $M(m \times n, \mathbb{C})$ of $m \times n$ complex matrices is a vector space of dimension $2mn$ over $\mathbb{R}$, and thus a smooth manifold of dimension $2mn$.

Example 1.26 (Open Submanifolds). Let U be any open subset of $\mathbb{R}^n$. Then U is a topological n-manifold, and the single chart (U, Id_U) defines a smooth structure on U.

More generally, let M be a smooth n-manifold and let $U \subseteq M$ be any open subset. Define an atlas on U by

$$\mathcal{A}_U = \{\text{smooth charts } (V, \varphi) \text{ for } M \text{ such that } V \subseteq U\}$$

Every point $p \in U$ is contained in the domain of some chart (W, φ) for M; if we set $V = W \cap U$, then $(V, \varphi|_V)$ is a chart in $\mathcal{A}_U$ whose domain contains p. It is then easy to verify that $\mathcal{A}_U$ is a maximal smooth atlas for U, and thus any open subset of M is itself a smooth n-manifold in a natural way. Endowed with this smooth structure, we call any open subset an ***open submanifold of M*** (indeed, this will turn out to be motivation for how we define an *Embedded Submanifold* later on in chapter 5).

Example 1.27 (The General Linear Group). The general linear group $GL(n(\mathbb{R})$ is the set of $n \times n$ matrices with real entries. It is a smooth n^2-dimensional manifold because it is an open subset of the

n^2-dimensional vector space $M_n(\mathbb{R})$, namely the set where the (continuous) determinant function is nonzero.

Example 1.31 (Spheres). We know that the n-sphere $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ is a topological n-manifold. We put a smooth structure on $\mathbb{S}^n$ as follows. For each $i = 1..n+1$, let $(U_i^{\pm}.\varphi_i^{\pm})$ denote the graph coordinate charts given by

$$U_i^{\pm} = \{(x^1, \ldots, x^{n+1} \in \mathbb{R}^{n+1} \ : \ \pm x^i > 0\}$$

and

$$\varphi_i^{\pm}(x^1, \ldots, x^{n+1}) = (x^1, \ldots, \widehat{x}^i, \ldots x^{n+1}).$$

For any distinct indices i and j, the transition map $\varphi_i^{\pm} \circ (\varphi^{\pm})^{-1}$ is easily computed. In the case where $i < j$, we get

$$\varphi_i^p m \circ (\varphi_j^{\pm})^{-1}(u^1, \ldots, u^n) = \left(u^1, \ldots, \widehat{u}^1, \ldots, \pm\sqrt{1 = |u|^2}, \ldots, u^n\right)$$

and a similar formula holds when $i > j$. When $i = j$, $\varphi_i^+ \circ (\varphi_i^-)^{-1} = \varphi_i^- \circ (\varphi_i^+)^{-1} = \mathrm{Id}_{\mathbb{B}^n}$. Thus the collection of charts $\{(\mathbb{U}_i^{\pm}, \varphi_i^{\pm})\}$ is a smooth atlas, as so defines a smooth structure on $\mathbb{S}^n$. We call this its ***standard smooth structure***.

Example 1.32 (Level Sets). We can actually generalize the preceding example in the following way. Suppose $U \subseteq \mathbb{R}^n$ is an open subset and $\Phi : U \to \mathbb{R}$ is a smooth function. For any $c \in \mathbb{R}$, the set $\Phi^{-1}(c)$ is called a ***level set of*** $\boldsymbol{\Phi}$. Choose some $c \in \mathbb{R}$, let $M = \Phi^{-1}(c)$, and suppose it happens that the total derivative $D\Phi(a)$ is nonzero for each $a \in \Phi^{-1}(c)$. Because $D\Phi(a)$ is a row matrix whose entries are the partial derivatives $\left(\frac{\partial \Phi}{\partial x^1}(a), \ldots, \frac{\partial \Phi}{\partial x^n}(a)\right)$, for each $a \in M$ there is some i such that $\frac{\partial \Phi}{\partial x^i}(a) \neq 0$. It follow from the Implicit Function Theorem, that there is a neighborhood U_0 of a such that $M \cap U_0$ can be expressed as the graphi of an equation of the form

$$x^i = f(x^1, \ldots, \widehat{x}^i, \ldots, x^n),$$

for some smooth real-valued function f defined on an open subset of $\mathbb{R}^{n-1}$. Therefore, arguing just as in the case of the n-sphere, we wee that M is a topological manifold of dimension $(n-1)$, and has a smooth structure such that each of the graph coordinate charts associated with a choice of f as above is a smooth chart.

Example 1.33 (Projective Spaces). The n-dimensional real projective space $\mathbb{RP}^n$ is a topological n-manifold when given coordinate charts as in Example 1.5 of the Lee book. If we assume for convenience that $i > j$, then we can get

$$\varphi_j \circ \varphi_i^{-1}(u^1, \ldots, u^n) = \left(\frac{u^1}{u^j}, \ldots, \frac{u^{j-1}}{u^j}, \frac{u^{j+1}}{u^j}, \ldots, \frac{u^{i-1}}{u^j}, \frac{1}{u^j}, \frac{u^{i+1}}{u^j}, \ldots, \frac{u^n}{u^j}\right),$$

which is a diffeomorphism form $\varphi_i(U_i \cap U_j)$ to $\varphi_j(U_i \cap U_j)$.

Example 1.34 (Smooth Product Manifolds). If $M_1, \ldots, M_k$ are smooth manifolds of dimensions $n_1, \ldots, n_k$, repectively, then the product space $M_1 \times \cdots \times M_k$ is a topological manifold of dimension $n_1 + \cdots + n_k$, with charts of the form $(U_1 \times \cdots \times U_k, \varphi_1 \times \cdots \times \varphi_k)$. Any two such charts are smoothly compatible because

$$(\psi_1 \times \cdots \times \psi) \circ (\varphi \times \cdots \times \varphi_k)^{-1} = (\psi_1 \circ \varphi_1^{-1}) \times \cdots \times (\psi \circ \varphi_k^{-1}),$$

which is a smooth map. This defines a natural smooth manifold structure on the product, called the ***product smooth manifold structure***.

Lemma 1.35 (Smooth Manifold Chart Lemma). Let M be a set, and suppose we are given a collection $\{U_\alpha\}$ of subsets of M together with maps $\varphi_\alpha : U\alpha \to \mathbb{R}^n$, such that the following properties are satisfied:

(i) For each α, φ_α is a bijection between U_α and an open subset $\varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$.

(ii) For each α and β, the sets $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$ are open in $\mathbb{R}^n$.

(iii) Whenever $U_\alpha \cap U_\beta \neq \varnothing$, the map $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is smooth,

(iv) Countably many U_α cover M.

(v) Whenever p, q are distinct points in M, either there exists some U_α containing both p and q or there exist disjoint sets U_α, U_β with $p \in U_\alpha$ and $q \in U_\beta$.

Then M has a unique smooth manifold structure such that each $(U_\alpha, \varphi_\alpha)$ is a smooth chart.

- Manifolds With Boundary -

Definition. We define the ***closed n-dimensional upper half-space*** $\mathbb{H}^n \subseteq \mathbb{R}^n$, defined as

$$\mathbb{H}^n = \{(x^1, \dots, x^n) \in \mathbb{R}^n \ : x^n \geq 0\}.$$

We will use the notations $\mathrm{Int}(\mathbb{H}^n)$ and $\partial\mathbb{H}^n$ to denote the interior an boundary of $\mathbb{H}^n$. When $n > 0$, this means

$$\begin{aligned} \mathrm{Int}(\mathbb{H}^n) &= \{(x^1, \dots, x^n) \in \mathbb{R}^n \ : \ x^n > 0\}, \\ \partial\mathbb{H}^n &= \{(x^1, \dots, x^n) \in \mathbb{R}^n \ : \ x^n = 0\}. \end{aligned}$$

Definition. An ***n-dimensional topological manifold with boundary*** is a second-countable Hausdorff space M in which every point has a neighborhood homeomorphic to either an open subset of $\mathbb{R}^n$ or to an open subset of $\mathbb{H}^n$. Charts on M are defined in the obvious way, and a chart (U, φ) will be called an ***interior chart*** if $\varphi(U)$ is homeomorphic to an open subset of $\mathbb{R}^n$, and a ***boundary chart*** if $\varphi(U)$ is homeomorphic to an open subset of $\mathbb{H}^n$ such that $\varphi(U) \cap \partial\mathbb{H}^n \neq \varnothing$.

Definition. A point $p \in M$ is called an ***interior point of M*** if it is in the domain of some interior chart, and a ***boundary point of M*** if it is in the domain of a boundary chart that sends p to $\partial\mathbb{H}^n$.

Theorem 1.38. Let M be a topological n-manifold with boundary.

(a) $\mathrm{Int}(M)$ is an open subset of M and a topological n-manifold without boundary.

(b) ∂M is a closed subset of M and a topological $(n-1)$-manifold without boundary.

(c) M is a topological manifold if and only if $\partial M = \varnothing$.

(d) If $n = 0$, then $\partial M = 0$ and M is a 0-manifold.

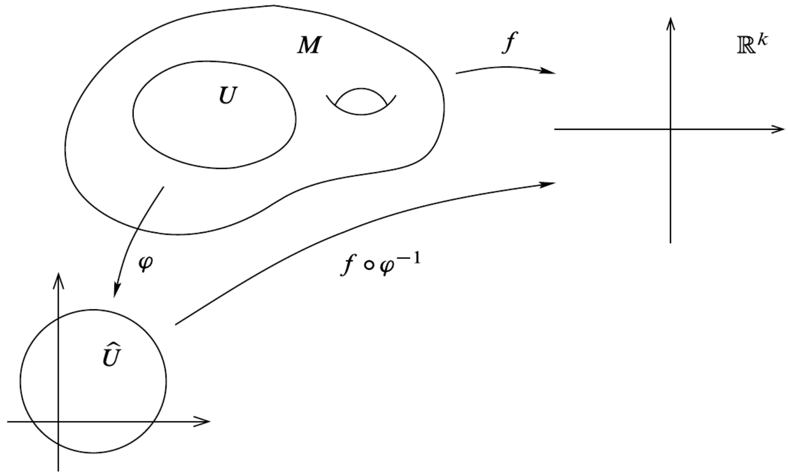
Theorem 1.46 (Smooth Invariance of the Boundary). Suppose M is a smooth manifold with boundary and $p \in M$. If there is some smooth chart (U, φ) for M such that $\varphi(U) \subseteq \mathbb{H}^n$ and $\varphi(p) \in \partial\mathbb{H}^n$, then the same is true for every smooth chart whose domain contains p.

2. Smooth Maps

The entire point of defining smooth structures on manifolds was to enable us to define smooth maps between manifolds. In this chapter, we discuss how to tell if two smooth manifolds are "essentially the same" by defining the notion of a diffeomorphism.

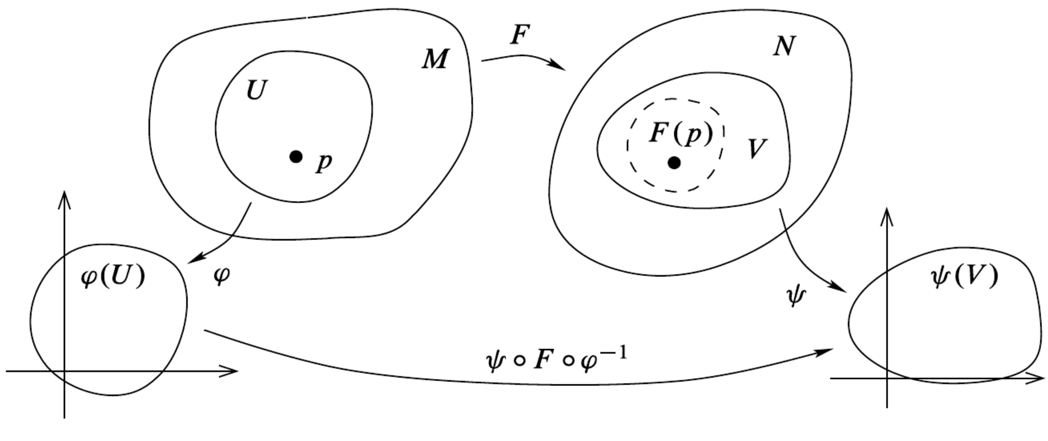
- Smooth Functions and Smooth Maps -

Definition. Suppose that M is a smooth n-manifold, k is a nonnegative integer, and $f : M \to \mathbb{R}^k$ is any function. We say that f is a ***smooth function*** if for every $p \in M$, there exists a smooth chart (U, φ) for M whose domain contains p and such that the composite function $f \circ \varphi^{-1}$ is smooth on the open set $\widehat{U} = \varphi(U) \subseteq \mathbb{R}^n$.



Definition. Given a function $f : M \to \mathbb{R}^k$ and a chart (U, φ) for M, the function $\widehat{f} : \varphi(U) \to \mathbb{R}^k$ defined by $\widehat{f}(x) = f \circ \varphi^{-1}(x)$ is called the ***coordinate representation of f***.

Definition. Let $M,\ N$ be smooth manifolds, and let $F : M \to N$ be any map. We say that F is a ***smooth map*** if for every $p \in M$, there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subseteq V$ and the composite map $\psi \circ \varphi^{-1}$ is smooth from $\varphi(U)$ to $\psi(V)$.



Proposition 2.4. Every smooth map is continuous

Proposition 2.5 (Equivalent Characterizations of Smoothness). Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is a map. Then F is smooth if and only if either of the following conditions is satisfied:

(a) For every $p \in M$, there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $U \cap F^{-1}(V)$ is open in M and the composite map $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U \cap F^{-1}(V))$ to $\psi(V)$.

(b) F is continuous and there exist smooth atlases $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ for M and N, respectively, such that for each α and β, $\psi_\beta \circ F \circ \varphi_\alpha^{-1}$ is a smooth map from $\varphi_\alpha(U_\alpha \cap F^{-1}(V_\beta))$ to $\psi_\beta(V_\beta)$.

Proposition 2.6 (Smoothness is Local). Let M and N be smooth manifolds with or without boundary, and let $F : M \to N$ be a map.

(a) If every point $p \in M$ has a neighborhood U such that the restriction $F|_U$ is smooth, then F is smooth.

(b) Conversely, if F is smooth, then its restriction to every open subset is smooth.

Corollary 2.8 (Gluing Lemma for Smooth Maps). Let M and N be smooth manifolds with or without boundary, and let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of M. Suppose that for each $\alpha \in A$, we are given a smooth map $F_\alpha : U_\alpha \to N$ such that the maps agree on overlaps: $F_\alpha|_{U_\alpha \cap U_\beta} = F_\beta|_{U_\alpha \cap U_\beta}$ for all α and β. Then there exists a unique smooth map $F : M \to N$ such that $F|_{U_\alpha} = F_\alpha$ for each $\alpha \in A$.

Proposition 2.10. Let M, N, and P be smooth manifolds with or without boundary.

(a) Every constant map $c : M \to n$ is smooth.

(b) The identity map of M is smooth.

(c) If $C \subseteq M$ is an open submanifold with or without boundary, then the inclusion map $U \hookrightarrow M$ is smooth.

(d) If $F : M \to N$ and $G : N \to P$ are smooth, then so is $G \circ F : M \to P$.

Proposition 2.12. Suppose $M_1, \ldots, M_k$ and N are smooth manifolds with or without boundary, such that at most one of $M_1, \ldots, M_k$ has nonempty boundary. For each i, let $\pi_I : M_1 \times \cdots \times M_k \to M_i$ denote the projection onto the M_i factor. A map $F : N \to M_1 \times \cdots \times M_k$ is smooth if and only if each of the component maps $F_i = \pi_i \circ F : N \to M_i$ is smooth.

Definition. If M and N are smooth manifolds with or without boundary, a ***diffeomorphism from M to N*** is a smooth bijective map $F : M \to N$ that has a smooth inverse.

Theorem 2.18 (Diffeomorphism Invariance of Dimension). A nonempty smooth manifold of dimension m cannot be diffeomorphic to an n-dimensional smooth manifold unless $m = n$

Theorem 2.19 (Diffeomorphism Invariance of Boundary). Suppose M and N are smooth manifolds with boundary and $F : M \to N$ is a diffeomorphism. Then $F(\partial M) = \partial N$, and F restricts to a diffeomorphism from $\text{Int}(M)$ to $\text{Int}(N)$.

- Partitions of Unity -

Lemma 2.20. The functions $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} e^{-\frac{1}{t}}, & t > 0, \\ 0. & t \leq 0, \end{cases}$$

is smooth.

Lemma 2.21. Given any real numbers r_1 and r_2 such that $r_1 < r_2$, there exists a smooth function $h : \mathbb{R} \to \mathbb{R}$ such that $h(t) \equiv 1$ for $t \leq r_1$, $0 < h(t) < 1$ for $r_1 < t < r_2$, and $h(t) \equiv 0$ for $t \geq r_2$.

Proof. Let f be the function of the previous lemma, and set

$$h(t) = \frac{f(r_2 - t)}{f(r_2 - t) + f(t - r_1)}.$$

Then the denominator is always positive since $r_2 \neq r_1$ and either $r_2 - t$ or $t - r_1$ will always be positive, and the rest follows form the properties of f. ♣

Lemma 2.22. Given any positive real numbers $r_1 < r_2$, there is a smooth function $H : \mathbb{R}^n \to \mathbb{R}$ such that $H \equiv 1$ on $\overline{B}_{r_1}(0)$, $0 < H(x) < 1$ for all $x \in B_{r_2}(0) \backslash \overline{B}_{r_1}(0)$, and $H \equiv 0$ on $\mathbb{R}^n \backslash B_{r_2}(0)$.

Definition. The function H constructed in the previous lemma is an example of a ***smooth bump function***, a smooth real-valued function that is equal to 1 on a specified set and is zero outside a specified neighborhood of that set.

Definition. If f is any real-valued or vector-valued function on a topological space M, the ***support of f***, denoted $\operatorname{supp}(f)$, is the closure of the set of points where f is nonzero:

$$\operatorname{supp}(f) = \overline{\{p \in M \ : \ f(p) \neq 0\}}$$

A function f is said to be ***compactly supported*** if $\operatorname{supp}(f)$ is a compact set.

Definition. Suppose that M is a topological space, and let $\mathcal{X} = (X_\alpha)_{\alpha \in A}$ be an arbitrary open cover of M. A ***partition of unity subordinate to $\boldsymbol{X}$*** is an indexed family $(\psi_\alpha)_{\alpha \in A}$ of continuous functions $\psi_\alpha : M \to \mathbb{R}$ with the following properties:

(i) $0 \leq \psi_\alpha(x) \leq 1$ for all $\alpha \in A$ and all $x \in M$.

(ii) $\operatorname{supp}(\psi_\alpha) \subseteq X_\alpha$ for each $\alpha \in A$.

(iii) The family of supports $(\operatorname{supp}(\psi_\alpha))_{\alpha \in A}$ is locally finite.

(iv) $\sum_{\alpha \in A} \psi_\alpha(x) = 1$ for all $x \in M$.

If M is a smooth manifold with or without boundary, a ***smooth partition of unity*** is one for which each of the unctions ψ_α is smooth.

Theorem 2.23 (Existence of Partitions of Unity). Suppose M is a smooth manifold with or without boundary, and $\mathcal{X} = (X_\alpha)_{\alpha \in A}$ is any indexed open cover of M. Then there exists a smooth partition of unity subordinate to $\mathcal{X}$.

Definition. If M is a topological space, $A \subseteq M$ is a closed subset, and $U \subseteq M$ is an open subset containing A, a continuous function $\psi : M \to \mathbb{R}$ is called a ***bump function for A supported in U*** if $0 \leq \psi \leq 1$ on M, $\psi \equiv 1$ on A, and $\operatorname{supp}(\psi) \subseteq U$.

Proposition 2.25 (Existence of Smooth Bump Functions). Let M be a smooth manifold with or without boundary. For any closed subset $A \subseteq M$ and any open subset U containing A, there exists a smooth bump function for A supported in U.

Definition. Suppose that M and N are smooth manifolds with or without boundary, and $A \subseteq M$ is an arbitrary subset. We say that a map $F : A \to N$ is ***smooth on A*** if it has a smooth extension in a neighborhood of each point: that is, if for every $p \in A$ there is an open subset $W \subseteq M$ containing p an a smooth map $\widetilde{F} : W \to N$ whose restriction to $W \cap A$ agrees with F.

Lemma 2.26 (Extension Lemma for Smooth Functions). Suppose M is a smooth manifold with or without boundary, $A \subseteq M$ a closed subset, and $f : M \to \mathbb{R}^k$ is a smooth function. For any open subset U containing A, there exists a smooth function $\widetilde{f} : M \to \mathbb{R}^k$ such that $\widetilde{f}_A = f$ and $\operatorname{supp}(\widetilde{f}) \subseteq U$.

Definition. Suppose that M is a topological space. An ***exhaustion function for M*** is a continuous function $f : M \to \mathbb{R}$ with the property that the set $f^{-1}((-\infty, c])$ (called a ***sublevel set of f***) is compact for each $c \in \mathbb{R}$. For an example consider the function $f : \mathbb{R}^n \to \mathbb{R} : x \mapsto |x|^2$.

Proposition 2.28 (Existence of Smooth Exhaustion Functions). Every smooth manifold with or without boundary admits a smooth, positive exhaustion function.

Theorem 2.29 (Level Sets of Smooth Functions). Let M be a smooth manifold. If K is any closed subset of M, there is a smooth, nonnegative function $f : M \to \mathbb{R}$ such that $f^{-1}(0) = K$.

3. Tangent Vectors

One of the very central ideas that we have in ordinary calculus is the idea of *linear approximation*. In ordinary calculus, we approximated things by funding the tangent line to some curve at a specified point. However, to make sense of this notion for smooth manifolds, we need to talk about the idea of a *tangent space to a manifold at a point*. The basic principles of these objects were introduced to us back in Calculus III when we learned about tangent vectors to curves and tangent planes to surfaces, so we will use this as the starting position for our discussion of geometric tangent vectors in $\mathbb{R}^n$.

Definition. Given a point $a \in \mathbb{R}^n$, let us define the ***geometric tangent space to*** $\mathbb{R}^n$ ***at a***, denoted by $\mathbb{R}^n_a$, to be the set $\{a\} \times \mathbb{R}^n$. A ***geometric tangent vector*** in $\mathbb{R}^n$ is an element of $\mathbb{R}^n_a$ for some $a \in \mathbb{R}^n$.

Definition. Given any geometric tangent vector $v_a \in \mathbb{R}^n_a$, we can define a map $D_v|_a : C^\infty(\mathbb{R}^n) \to \mathbb{R}$, which takes the directional derivative in the direction v at a:

$$D_v|_a f = D_v f(a) = \left.\frac{d}{dt}\right|_{t=0} f(a + tv).$$

If $v_a = v^i e_i|_a$ in terms of the standard basis, then by the chain rule, we can write

$$D_v|_a f = v^i \frac{\partial f}{\partial x^i}(a).$$

Definition. A map $w : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is called a ***derivation at a*** if it is linear over $\mathbb{R}$ and satisfies the Leibniz rule:

$$w(fg) = f(a)wg + g(a)wf.$$

We will denote the set of all derivations of $C^\infty(\mathbb{R}^n)$ at a by $T_a\mathbb{R}^n$.

Lemma 3.1 (Properties of Derivations). Suppose $a \in \mathbb{R}^n$, $w \in T_a\mathbb{R}^n$, and $f, g \in C^\infty(\mathbb{R}^n)$.

(a) If f is a constant function, then $wf = 0$.

(b) If $f(a) = g(a) = 0$ then $w(fg) = 0$.

Proposition 3.2. Let $a \in \mathbb{R}^n$

(a) For each geometric tangent vector $v_a \in \mathbb{R}^n_a$, the map $D_v|_a : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is a derivation at a.

(b) The map $v_a \mapsto D_v|_a$ is an isomorphism form $\mathbb{R}^n_a$ onto $T_a\mathbb{R}^n$.

Corollary 3.3. For any $a \in \mathbb{R}^n$, the n derivations

$$\left.\frac{\partial}{\partial x^1}\right|_a, \ldots, \left.\frac{\partial}{\partial x^n}\right|_a \quad \text{defined by} \quad \left.\frac{\partial}{\partial x^i}\right|_a f = \frac{\partial f}{\partial x^i}(a)$$

form a basis for $T_a\mathbb{R}^n$, which therefore has dimension n.

With this machinery out of the way, we are ready to generalize our notion of a tangent space to an arbitrary manifold.

Definition. Let M be a smooth manifold with or without boundary, and let p be a point of M. A linear map $v : C^\infty(M) \to \mathbb{R}$ is called a ***derivation at p*** if it satisfies

$$v(fg) = f(p)vg + g(p)vf \quad \text{for all } f, g \in C^\infty(M)$$

The set of all derivations of $C^\infty(M)$ at p, denoted by T_pM, is a vector space called the ***tangent space to M at p***. An element of T_pM is called a ***tangent vector at p***.

Lemma 3.4 (Properties of Tangent Vectors on Manifolds). Suppose M is a smooth manifold with or without boundary, $p \in M$, $v \in T_pM$, and $f, g \in C^\infty(M)$.

(a) If f is a constant function, then $vf = 0$.

(b) If $f(p) = g(p) = 0$, then $v(fg) = 0$.

Definition. If M and N are smooth manifolds with or without boundary and $F : M \to N$ is a smooth map, for each $p \in M$ we define a map

$$dF_p : T_pM \to T_pN,$$

called the ***differential of F at p***, as follows. Given $v \in T_pM$, we let $dF_p(b)$ be the derivation at $F(p)$ that acts on $f \in C^\infty(N)$ by the rule

$$dF_p(v)(f) = v(f \circ F).$$

Proposition 3.6 (Properties of Differentials). Let M, N, and P be smooth manifolds with or without boundary, let $F : M \to N$ and $G : N \to P$ be smooth maps, and let $p \in M$.

(a) $dF_p : T_pM \to T_{F(p)}N$ is linear.

(b) $d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_pM | ra T_{G \circ F(p)}P$

(c) $d(\mathrm{Id}|_M)_p = Id_{T_pM} : T_pM \to T_pM$.

(d) If F is a diffeomorphism, then $dF_p : T_pM \to T_pN$ is an isomorphism and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.

Proposition 3.8. Let M be a smooth manifold with or without boundary, $p \in M$, and $v \in T_pM$. If $f, g \in C^\infty(M)$ agree on some neighborhood of p, then $vf = vg$.

Proposition 3.9 (The Tangent Space to an Open Submanifold). Let M be a smooth manifold with or without boundary, let $U \subseteq M$ be an open subset and let $\iota : U \hookrightarrow M$ be the inclusion map. For every $p \in U$, the differential $d\iota_p : T_pU \to T_pM$ is an isomorphism.

Proposition 3.10 (Dimension of the Tangent Space). If M is an n-dimensional smooth manifold, then for each $p \in M$, the tangent space T_PM is an n-dimensional vector space.

Lemma 3.11. Let $\iota\mathbb{H}^n \hookrightarrow \mathbb{R}^n$ denote the inclusion map. For any $a \in \partial\mathbb{H}^n$, the differential $d\iota_a : T_a\mathbb{H}^n \to T_a\mathbb{R}^n$ is an isomorphism.

Proposition 3.12 (Dimension of Tangent Spaces on a Manifold with Boundary). Suppose M is an n-dimensional smooth manifold with boundary. For each $p \in M$, T_pM is an n-dimensional vector space.

Definition. Suppose V is a finite-dimensional vector space and $a \in V$. For any vector $v \in V$, we define a map $D_v|_a : C^\infty(V) \to \mathbb{R}$ by

$$D_v|_a f = \left.\frac{d}{dt}\right|_{t=0} f(a + tv).$$

Proposition 3.13 (The Tangent Space to a Vector Space). Suppose V is a finite-dimensional vector space with its standard smooth manifold structure. For each point $a \in V$, the $v \mapsto D_v|_a$ as defined above is a canonical isomorphism from V to T_aV, such that for any linear map $L : V \to W$, the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\cong"] \arrow[d, "L"'] & T_aV \arrow[d, "dL_a"] \\ W \arrow[r, "\cong"] & T_{L_a}W \end{tikzcd}.$$

Proposition 3.14 (The Tangent Space to a Product Manifold). Let $M_1, \ldots, M_k$ be smooth manifolds, and for each j, let $\pi_j : M_1 \times \cdots \times M_k \to M_j$ be the projection map onto the M_j factor. For any point $p = (p_1, \ldots, p_k) \in M_1 \times \cdots \times M_k$, the map

$$\alpha : T_p(M_1 \times \cdots \times M_k) \to T_{p_1} \oplus \cdots \oplus T_{p_k} M_k$$

defined by

$$\alpha(v) = (d(\pi_1)_p(v), \ldots, d(\pi_k)_p(v))$$

is an isomorphism. The same is true if one of the spaces M_i is a smooth manifold with boundary.

- Computation in Coordinates -

Proposition 3.15. Let M be a smooth n-manifold with or without boundary, and let $p \in M$. Then T_pM is an n-dimensional vector space, and for any smooth chart, $(U, (x^i))$ (where (x^i) is just the coordinate representation of some chart map φ) containing p, the coordinate vectors $\frac{\partial}{\partial x^1}|_p, \ldots, \frac{\partial}{\partial x^n}|_p$ form a basis for T_pM.

Definition. Any tangent vector $v \in T_pM$ can be written uniquely as a linear combination

$$v = v^i \left.\frac{\partial}{\partial x^i}\right|_p.$$

The ordered basis $\left(\frac{\partial}{\partial x^i}\big|_p\right)$ is called a ***coordinate basis for T_pM***, and the numbers $(v^1, \ldots, v^n)$ are called the ***components of v*** with respect to the coordinate basis.

Definition. In the case where $F : U \to V$ is a smooth map between $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ with U and V open subsets of Euclidean spaces with coordinates $x^1, \ldots, x^n)$ and $(y^1, \ldots, y^m)$ respectively. For any $p \in U$, we can use the chian rule to compute that the action of dF_p on a typical basis vector as follows:

$$dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) f = \left(\frac{\partial F^j}{\partial x^i}(p) \left.\frac{\partial}{\partial y^j}\right|_{F(p)}\right) f.$$

Thus

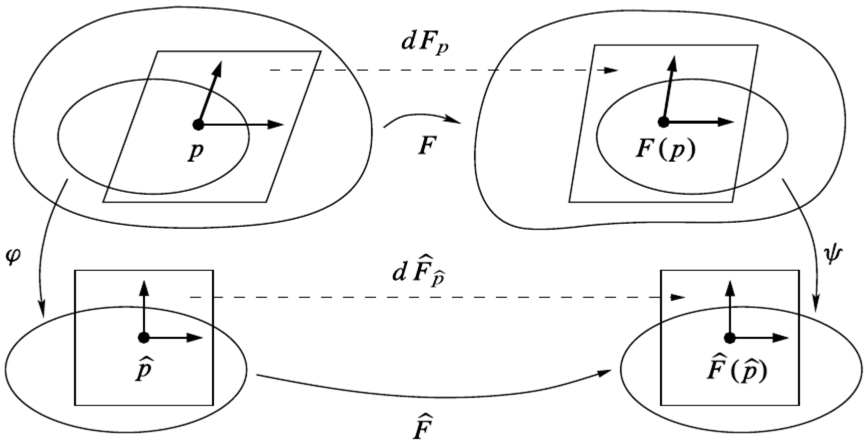
$$dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \frac{\partial F^j}{\partial x^i}(p) \left.\frac{\partial}{\partial y^j}\right|_{F(p)}.$$

And this gives us that the matrix of dF_p in terms of the coordinate bases is

$$\begin{pmatrix} \frac{\partial F^1}{\partial x^1}(p) & \cdots & \frac{\partial F^1}{\partial x^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F^m}{\partial x^1}(p) & \cdots & \frac{\partial F^m}{\partial x^n}(p) \end{pmatrix}.$$

Definition. For a general smooth map $F : M \to N$ between smooth manifolds with or without boundary with coordinate charts (U, φ) for M containing some point p and (V, ψ) for N containing $F(p)$, we obtain the coordinate representation $\widehat{F} = \psi \circ F \circ \varphi^{-1} : \varphi(U \cap F^{-1}(V)) \to \psi(V)$. Let $\widehat{p} = \varphi(p)$ denote the coordinate representation of p. By the computation above $d\widehat{F}_{\widehat{p}}$ is represented with respect to the standard coordinate bases by the Jacobian matrix of $\widehat{F}$ at $\widehat{p}$. Using the fact that $F \circ \varphi^{-1} = \psi \circ \widehat{F}$, we compute that

$$dF_p \left(\left. \frac{\partial}{\partial x^i} \right|_p \right) = \frac{\partial \widehat{F}^j}{\partial x^i}(\widehat{p}) \left. \frac{\partial}{\partial y^j} \right|_{F(p)}.$$



Note: In the differential geometry literature, the differential is sometimes called the ***tangent map***, the ***total derivative***, or simply the ***derivative of F***. It can also sometimes be called the ***(pointwise) pushforward***. Different authors denote it using symbols such as

$$F'(p), \quad DF, \quad DF(p), \quad F_*, \quad TF, \quad T_pF.$$

However, we will stick to the notation dF_p.

Definition. Suppose (U, φ) and (V, ψ) are two smooth charts on M, and $p \in U \cap V$. Let us denote the coordinate functions of φ by (x^i) and those of ψ by $(\widetilde{x}^i)$. Then any tangent vector at p can be represented with respect to either basis $\left(\left. \frac{\partial}{\partial x^i} \right|_p \right)$ or $\left(\left. \frac{\partial}{\partial \widetilde{x}^i} \right|_p \right)$. Using our transition map and the definition of coordinate vectors, we can compute that

$$\left. \frac{\partial}{\partial x^i} \right|_p = \frac{\partial \widetilde{x}^j}{\partial x^i}(\widehat{p}) \left. \frac{\partial}{\partial \widetilde{x}^j} \right|_p.$$

Example 3.16 (Polar Coordinates). The transition map between polar coordinates and standard coordinates in suitable open subsets of the plane is given by $(x, y) = (r\cos(\theta), r\sin(\theta))$. Let $p = (r, \theta) =$

$(2, \pi/2)$, and let $v \in T_p\mathbb{R}^2$ be the tangent vector whose polar coordinate representation is

$$v = 3\,\frac{\partial}{\partial r}\bigg|_p = \frac{\partial}{\partial \theta}\bigg|_p .$$

Applying the rule in the above definition, we get that

$$\begin{aligned}
\frac{\partial}{\partial r}\bigg|_p &= \cos\left(\frac{\pi}{2}\right)\frac{\partial}{\partial x}\bigg|_p + \sin\left(\frac{\pi}{2}\right)\frac{\partial}{\partial y}\bigg|_p = \frac{\partial}{\partial y}\bigg|_p ,\\
\frac{\partial}{\partial \theta}\bigg|_p &= -2\sin\left(\frac{\pi}{2}\right)\frac{\partial}{\partial x}\bigg|_p + 2\cos\left(\frac{\pi}{2}\right)\frac{\partial}{\partial y}\bigg|_p = -2\,\frac{\partial}{\partial x}\bigg|_p ,
\end{aligned}$$

and thus v has the following coordinate representation in standard coordinates:

$$v = 3\,\frac{\partial}{\partial y}\bigg|_p + 2\,\frac{\partial}{\partial x}\bigg|_p .$$

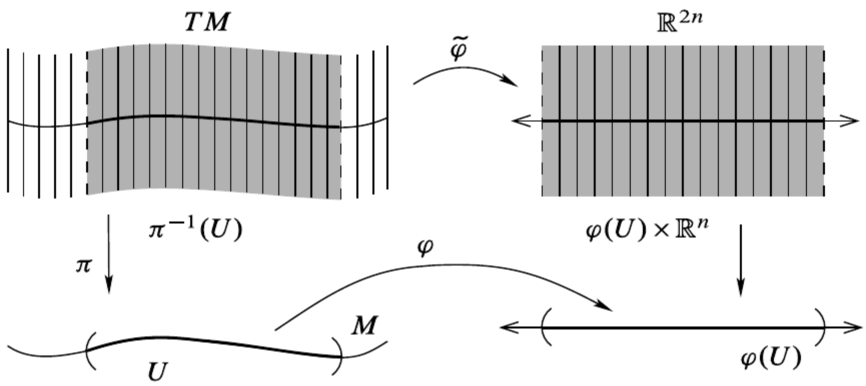
- The Tangent Bundle -

Definition. Given a smooth manifold M with or without boundary, we define the ***tangent bundle of M***, denoted by TM, to be the disjoint union of the tangent spaces at all points of M:

$$TM = \bigsqcup_{p \in M} T_pM.$$

Note: We usually write an element of this disjoint union as an ordered pair (p, v) with $p \in M$ and $v \in T_pM$, and the tangent bundle comes equipped with a natural projection map $\pi : TM \to M : (p, v) \mapsto p$.

Proposition 3.18. For any smooth n-manifold M, the tangent bundle TM has a natural topology and a smooth structure that make it into a $2n$-dimensional smooth manifold. With respect to this structure, the projection $\pi : TM \to M$ is smooth.



Proposition 3.20. If M is a smooth n-manifold with or without boundary, and M can be covered by a single smooth chart, then TM is diffoemorphic to $M \times \mathbb{R}^n$.

Proposition 3.21. If $F : M \to N$ is a smooth map, then its global differential $dF : TM \to TN$ is a smooth map.

Corollary 3.22 (Properties of the Global Differential). Suppose $F : M \to N$ and $G : N \to P$ are smooth maps

(a) $d(G \circ F) = dG \circ dF$.

(b) $d(\mathrm{Id}_M) = \mathrm{Id}_{TM}$

(c) If F is a diffeomorphism, then $dF : TM \to TN$ is also a diffeomorphism, and $(dF)^{-1} = d(F^{-1})$.

Definition. If M is a manifold, a ***curve in M*** is a continuous map $\gamma : J \to M$, where $J \subseteq \mathbb{R}$ is an interval.

Definition. Given a smooth curve $\gamma : J \to M$, and $t_0 \in J$, we define the ***velocity of γ at t_0***, denoted by $\gamma'(t_0)$, to be the vector

$$\gamma'(t_0) = d\gamma \left(\left. \frac{d}{dt} \right|_{t_0} \right) \in T_{\gamma(t_0)} M,$$

where $\left.\frac{d}{dt}\right|_{t_0}$ is the standard coordinate basis vector in $T_{t_0}\mathbb{R}$.

Definition. Given a smooth chart $(U, (x^i))$ with $\gamma(t_0) \in U$, we can write the coordinate representation of γ as $\gamma(t) = (\gamma^1(t), \dots, \gamma^n(t))$. Then the coordinate formula for the differential yields

$$\gamma'(t_0) = \frac{d\gamma^i}{dt}(t_0) \left. \frac{\partial}{\partial x^i} \right|_{\gamma(t_0)}.$$

Proposition 3.23. Suppose M is a smooth manifold with or without boundary and $p \in M$. Every $v \in T_pM$ is the velocity vector of some smooth curve in M.

Proposition 3.24 (The Velocity of a Composite Curve). Let $F : M \to N$ be a smooth map, and let $\gamma : J \to M$ be a smooth curve. For any $t_0 \in J$, the velocity at $t = t_0$ of the composite curve $F \circ \gamma : J \to N$ is given by

$$(F \circ \gamma)'(t_0) = dF(\gamma'(t_0)).$$

Corollary 3.25 (Computing the Differential Using a Velocity Vector). Suppose $F : M \to N$ is a smooth map, $p \in M$, and $v \in T_pM$. Then

$$dF_p(v) = (F \circ \gamma)'(0)$$

for any smooth curve $\gamma : J \to M$ such that $0 \in J$, $\gamma(0) = p$, and $\gamma'(0) = v$.

4. Submersions, Immersions, and Embeddings

We now have a way of giving the "best linear approximation" of a map near a given point on a manifold However, we can learn a great deal about a map by studying the properties of its differential, and that's exactly what we do in this section. Many of the properties of smooth submersions and smooth immersions that we study in this chapter will form the basis for our understanding of submanifolds which we will explore in the next chapter.

- Maps of Constant Rank -

Definition. Suppose M and N are smooth manifolds with or without boundary. Given a smooth map $F : M \to N$ and a point $p \in M$, we define the ***rank of F at p*** to be the rank of the linear map $dF_p : T_pM \to T_{F(p)}N$; it is the rank of the Jacobian matrix of F in any smooth chart. If F has rank r at every point, we say that it has ***constant rank***, and write $\text{rank}(F) = r$.

Note: $\text{rank}(F) \leq \min\{\dim(M), \dim(N)\}$

Definition. If $\text{rank}(dF_p) = \min\{\dim(M), \dim(N)\}$ then we way that ***F has full rank at p***, and if F has full rank everywhere, we say that ***F has full rank***.

Definition. $F : M \to N$ is called a ***smooth submersion*** if its differential is surjective at each point ($\text{rank}(F) = \dim(N)$) and is called a ***smooth immersion*** if its differential is injective at each point ($\text{rank}(F) = \dim(M)$).

Proposition 4.1. Suppose $F : M \to N$ is a smooth map and $p \in M$. If dF_p is surjective, then p has a neighborhood U such that $F|_U$ is a submersion. If dF_p is injective, then p has a neighborhood U such that $F|_U$ is an immersion.

Definition. If M and N are smooth manifolds with or without boundary, a map $F : M \to N$ is called a ***local diffeomorphism*** if every point $p \in M$ has a neighborhood U such that $F(U)$ is open in N and $F|_U : U \to F(U)$ is a diffeomorphism.

Theorem 4.5 (Inverse Function Theorem for Manifolds). Suppose M and N are sooth manifolds, and $F : M \to N$ is a smooth map. If $p \in M$ is a point such that dF_p is invertible, then there are connected neighborhoods U_0 of p and $V_)$ of $F(p)$ such that $F|_{U_0} : U_0 \to V_)$ is a diffeomorphism.

Proposition 4.8. Suppose M and N are smooth manifolds (without boundary), and $F : M \to N$ is a map.

(a) F is a local diffeomorphism if and only if it is both a smooth immersion and a smooth submersion.

(b) If $\dim(M) = \dim(N)$ and F is either a smooth immersion or a smooth submersion, then it is a local diffeomorphism.

Theorem 4.12 (Rank Theorem). Suppose M and N are smooth manifolds of dimensions m and n, respectively, and $F : M \to N$ is a smooth map with constant rank r. For each $p \in M$ there exist smooth charts (U, φ) for M centered at p and (V, ψ) for N centered at $F(p)$ such that $F(U) \subseteq V$, in which F

has a coordinate representation of the form

$$\widehat{F}(x^1, \dots, x^r, x^{r+1}, \dots x^m) = (x^1, \dots, x^r, 0, \dots, 0).$$

In particular, if F is a smooth submersion, this becomes

$$\widehat{F}(x^1, \dots, x^n, x^{n+1}, \dots x^m) = (x^1, \dots, x^n),$$

and if F is a smooth immersion, it is

$$\widehat{F}(x^1, \dots, x^m) = (x^1, \dots, x^m, 0, \dots, 0).$$

Corollary 4.13. Let M and N be smooth manifolds, let $F : M \to N$ be a smooth map, and suppose M is connected. Then the following are equivalent:

(a) For each $p \in M$ there exist smooth charts containing p and $F(p)$ in which the coordinate representation of F is linear.

(b) F has constant rank.

Theorem 4.14 (Global Rank Theorem). Let M and N be smooth manifolds, and suppose $F : M \to N$ is a smooth map of constant rank.

(a) If F is surjective, then it is a smooth submersion.

(b) If F is injective, then it is a smooth immersion.

(c) If F is bijective, then it is a diffeomorphism.

- Embeddings and Submersions -

Definition. If M and N are smooth manifolds with or without boundary, a ***smooth embedding of M into N*** is a smooth immersion $F : M \to N$ that is also a topological embedding (i.e. a homeomorphism onto its image in the subspace topology).

Example 4.18 (A Smooth Topological Embedding). The map $\gamma : \mathbb{R} \to \mathbb{R}^2$ given by $\gamma(t) = (t^3, 0)$ is a smooth map and a topological embedding, but it is not a smooth embedding because $\gamma'(0) = 0$.

Example 4.19 (The Figure-Eight Curve). Consider the curve $\beta : (-\pi, \pi) \to \mathbb{R}^2 : t \to (\sin(2t), \sin(t))$. The image is a set that looks like a figure-eight in the plane. It is easy to see that β is an injective smooth immersion because $\beta'(t)$ never vanishes; but it is not a topological embedding, because its image is compact in the subspace topology, while its domain is not.

Definition. If X and Y are topological spaces, a map $F : X \to Y$ is said to be ***proper*** if for every compact set $K \subseteq Y$, the preimage $F^{-1}(K)$ is compact

Proposition 4.22. Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is an injective smooth immersion. If any of the following holds, then F is a smooth embedding.

(a) F is an open or closed map

(b) F is a proper map.

(c) M is compact.

(d) M has empty boundary an ddim$(M) = \dim(N)$.

Theorem 4.25 (Local Embedding Theorem). Suppose M an dN are smooth manifolds with or without boundary, and $F : M \to N$ is a smooth map. Then F is a smooth immersion if and only if every point in M has a neighborhood $U \subseteq M$ such that $F|_U : U \to N$ is a smooth embedding.

Definition. If $\pi : M \to N$ is any continuous map, a ***section of π*** is a continuous right inverse for π, i.e., a continuous map $\sigma : N \to M$ such that $\pi \circ \sigma = \mathrm{Id}_N$.

$$\begin{tikzcd} M \arrow[d, "\pi"'] \\ N \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$

A ***local section of π*** is a continuous map $\sigma : Y \to M$ defined on some open subset $U \subseteq N$ and satisfying the analogous relation $\pi \circ \sigma = \mathrm{Id}_U$.

Theorem 4.26 (Local Section Theorem). Suppose M and N are smooth manifolds and $\pi : M \to N$ is a smooth map. Then π is a smooth submersion if an only if every point of M is in the image of a smooth local section of π.

Definition. If $\pi : X \to Y$ is a continuous map, we say π is a ***topological submersion*** if ever point of X is in the image of a (continuous) local section of π.

Theorem 4.28 (Properties of Smooth Submersions). Let M and N be smooth manifolds, and suppose $\pi : M \to N$ is a smooth submersion. Then π is an open map, and if it is surjective, it is a quotient map.

Theorem 4.29 (Characteristic Property of Surjective Smooth Submersions). Suppose M and N are smooth manifolds, and $\pi : M \to N$ is a surjective smooth submersion. For any smooth manifold P with or without boundary, a map $F : N \to P$ is smooth if and only if $F \circ \pi$ is smooth:

$$\begin{tikzcd} M \arrow[d, "\pi"'] \arrow[dr, "F\circ\pi"] & \\ N \arrow[r, "F"'] & P. \end{tikzcd}$$

Theorem 4.30 (Passing Smoothly to the Quotient). Suppose M and N are smooth manifolds, and $\pi : M \to N$ is a surjective smooth submersion. If P is a smooth manifold with or without boundary and $F : M \to P$ is a smooth map that is constant on the fibers of π, then there exists a unique smooth map $\widetilde{F} : N \to P$ such that $\widetilde{F} \circ \pi = F$.

Theorem 4.31 (Uniqueness of Smooth Quotients). Suppose M, N_1, and N_2 are smooth manifolds, and $\pi_1 : M \to N_1$ and $\pi_2 : M \to N_2$ are surjective smooth submersions that are constant on each other's fibers. Then there exists a unique diffeomorphism $FM_1 \to N_2$ such that $F \circ \pi_1 = \pi_2$

$$\begin{tikzcd} & M \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ N_1 \arrow[rr, dashed, "F"'] & & N_2 \end{tikzcd}$$

5. Submanifolds

The bulk of this chapter will be focused on the most important type of submanifolds: embedded submanifolds. There have the subspace topology inherited from their containing manifold, and turn out to be exactly the images of smooth embeddings. The other type of submanifold we will discuss, immersed submanifolds, are slightly less nice objects since they are not required to have the subspace topology, but as you would expect, they appear as the images of injective immersions.

- Embedded and Immersed Submanifolds -

Definition. An ***embedded submanifold of M*** is a subset $S \subseteq M$ that is a manifold in the subspace topology, endowed with a smooth structure with respect to which the inclusion map $S \hookrightarrow M$ is a smooth embedding. Embedded submanifolds are also called ***regular submanifolds*** by some.

Definition. If S ins an embedded submanifold of M, the ***codimension of S in M*** is given by $\dim(M) - \dim(S)$. M is called the ***ambient manifold*** for S, and an embedded submanifold of codimension one is called an ***embedded hypersurface***.

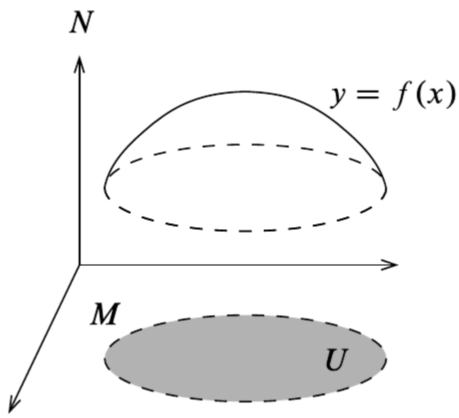
Proposition 5.1 (Open Submanifolds). Suppose M is a smooth manifold. The embedded submanifolds of codimension 0 in M are exactly the open submanifolds.

Proposition 5.2 (Images of Embeddings as Submanifolds). Suppose M is a smooth manifold with or without boundary, N is a smooth manifold, and $F : N \to M$ is a smooth embedding. Let $S = F(N)$. With the subspace topology, S is a topological manifold, and it has a unique smooth structure making it into an embedded submanifold of M with the property that F is a diffeomorphism onto its image.

Proposition 5.4 (Graphs as Submanifolds). Suppose M is a smooth m-manifold (without boundary), N is a smooth n-manifold with or without boundary, $U \subseteq M$ is open, and $f : U \to N$ is a smooth map. Let $\Gamma(f) \subseteq M \times N$ denote the graph of f:

$$\Gamma(f) = \{(x, y) \in M \times N \ : \ x \in U, y = f(x)\}.$$

Then $\Gamma(f)$ is an embedded m-dimensional submanifold of $M \times N$.



Definition. An embedded submanifold $S \subseteq M$ is said to be ***properly embedded*** if the inclusion $S \hookrightarrow M$ is a proper map.

Proposition 5.5. Suppose M is a smooth manifold with or without boundaryand $S \subseteq M$ is an embedded submanifold. Then S is properly embedded if and only if it is a closed subset of M.

Corollary 5.6. Every compact embedded submanifold is properly embedded.

Definition. If Y is an open subset of $\mathbb{R}^n$ and $k \in \{0, \dots, n\}$, a ***k-dimensional slice of U*** (or simply a ***k-slice***) is any subset of the form

$$S = \{(x^1, \dots, x^k, x^{k+1}, \dots, x^n) \in U \ : \ x^{k+1} = c^{k+1}, \dots, x^n = c^n\}$$

for some constants $c^{k+1}, \dots, c^n$.

Definition. Let M be a smooth n-manifold, and let (U, φ) be a smooth chart on M. If S is a subset of U such that $\varphi(S)$ is a k-slice of $\varphi(U)$, the we say that ***S is a k-slice of U***.

Definition. Given a subset $S \subseteq M$ and a nonnegative integer k, we say that S satisfies the ***local k-slice condition*** if each point of S is contained in the domain of a smooth chart (U, φ) for M such that $S \cap U$ is a single k-slice in U. Any such chart is called a ***slice chart for S in M***, and the corresponding coordinates $(x^1, \dots, x^n)$ are called ***slice coordinates***.

Theorem 5.8 (Local Slice Criterion for Embedded Submanifolds). Let M be a smooth n-manifold. If $S \subseteq M$ is an embedded k-dimensional submanifold, then S satisfies the local k-slice condition, then with the subspace topology, S is a topological manifold of dimension k, and it has a smooth structure making it into a k-dimensional embedded submanifold of M.

Example 5.9 (Spheres as Submanifolds). For any $n \geq 0$, $\mathbb{S}^n$ is an embedded submanifold of $\mathbb{R}^{n+1}$, because it is locally the graph of a smooth function: as is shown in Example 1.4 of Lee, the intersection of $\mathbb{S}^n$ with the open subset $\{x \ : \ x^i > 0\}$ is the graph of the smooth function

$$x^i = f(x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^{n+1}),$$

where $f : \mathbb{B}^n \to \mathbb{R}$ is given by $f(u) = \sqrt{1 - |u|^2}$. Similarly, the intersection of $\mathbb{S}^n$ with $\{x \ : \ x^i < 0\}$ is the graph of $-f$. Since every point in $\mathbb{S}^n$ is in one of these sets, $\mathbb{S}^n$ satisfies the local n-slice condition and is thus an embedded submanifold of $\mathbb{R}^{n+1}$. The smooth structure thus induced on $\mathbb{S}^n$ is the same as the one we defined in Chapter 1: in fact, the coordinates for $\mathbb{S}^n$ determined by these slice charts are exactly the graph coordinates defined in Example 1.31 of Lee.

Theorem 5.11. If M is a smooth n-manifold with boundary, then with the subspace topology, ∂M is a topological $(n-1)$-dimensional manifold (without boundary), and has a smooth structure such that it is a properly embedded submanifold of M.

Definition. If $\Phi : M \to N$ is any map and c is any point of N, we call the set $\Phi^{-1}(c)$ a ***level set of Φ***.

Theorem 5.12 (Constant-Rank Level Set Theorem). Let M and N be smooth manifolds, and let $\Phi : M \to N$ be a smooth map with constant rank r. Each level set of Φ is a properly embedded submanifold of codimension r in M.

Corollary 5.13 (Submersion Level Set Theorem). If M and N are smooth manifolds and $\Phi : M \to N$ is a smooth submersion, then each level set of Φ is a properly embedded submanifold whose codimension is equal to the dimension of N.

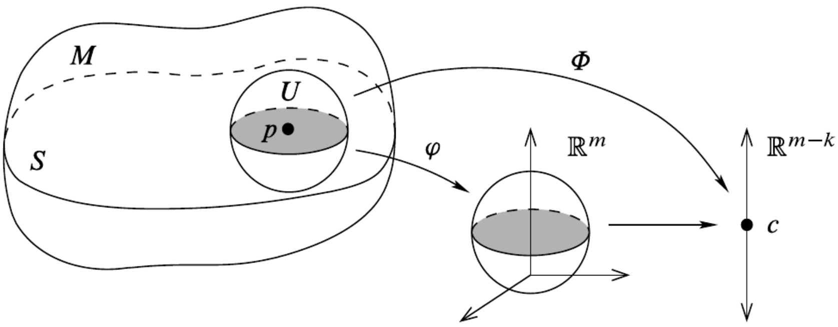
Definition. If $\Phi : M \to N$ is a smooth map, a point $p \in M$ is said to be a ***regular point of* Φ** if $d\Phi_p : T_pM \to T_{\Phi(p)}N$ is surjective; it is a ***critical point of* Φ** otherwise.

Definition. A point $c \in N$ is said to be a ***regular value of* Φ** if every point of the level set $\Phi^{-1}(c)$ is a regular point, and a***critical value*** otherwise. In particular, if $\Phi^{-1}(c) = \varnothing$, then c is a regular value. Finally, a level set $\Phi^{-1}(c)$ is called a ***regular level set*** if c is a regular value of Φ.

Corollary 5.14 (Regular Level Set Theorem). Every regular level set of a smooth map between smooth manifolds is a properly embedded submanifold whose codimension is equal to the dimension of the codomain.

Example 5.15 (Spheres). Now we can give a much easier proof that $\mathbb{S}^n$ is an embedded submanifold of $\mathbb{R}^{n+1}$. The sphere is a regular level set of the smooth function $f : \mathbb{R}^{n+1} \to R$ given by $f(x) = |x|^2$, since $df_x(v) = 2\sum_i x^i v^i$, which is surjective except at the origin.

Proposition 5.16. Let S be a subset of a smooth m-manifold M. Then S is an embedded k-submanifold of M if and only if every point of S has a neighborhood U in M such that $U \cap S$ is a level set of a smooth submersion $\Phi : U \to \mathbb{R}^{m-k}$.



Definition. If $S \subseteq M$ is an embedded submanifold, a smooth map $\Phi : M \to N$ such that S is a regular level set of Φ is called a ***defining map for S***. More generally, if U is an open subset of M and $\Phi : U \to N$ is a smooth map such that $S \cap U$ is a regular level set of Φ, the Φ is called a ***local defining map for S***.

Definition. An ***immersed submanifold of M*** is a subset $S \subseteq M$ endowed with a topology with respect to which it is a topological manifold (without boundary), and a smooth structure with respect to which the inclusion map $S \hookrightarrow M$ is a smooth immersion. As for embedded submanifolds, we define the ***codimension of S in M*** to be $\dim(M) - \dim(S)$.

Proposition 5.18 (Images of Immersions as Submanifolds). Suppose M is a smooth manifold with or without boundary, N is a smooth manifold, an d$F : N \to M$ is an injective smooth immersion. Let $S = F(N)$. Then S has a unique topology and smooth structure such that it is a smooth submanifold of M and such that $F : N \to S$ is a diffeomorphism onto its image.

Proposition 5.21. Suppose M is a smooth manifold with or without boundary, and $S \subseteq M$ is an immersed submanifold. If any of the following holds, then S is embedded.

(a) S has codimension 0 in M.

(b) The inclusion map $S \subseteq M$ is proper.

(c) S is compact.

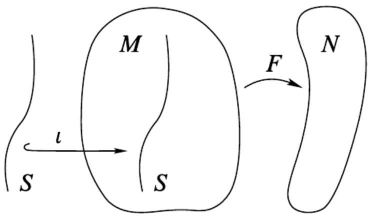
Proposition 5.22 (Immersed Submanifolds Are Locally Embedded). If M is a smooth manifold with or without boundary, and $S \subseteq M$ is an immersed submanifold, then for each $p \in S$ there exists a neighborhood U of p in S that is an embedded submanifold of M.

Definition. Suppose $X \subseteq M$ is an immersed k-dimensional submanifold. A ***local parameterization of S*** is a continuous map $X : U \to M$ whose domain is an open subset $U \subseteq \mathbb{R}^k$, whose image is an open subset of S, and which, considered as a map into S, is a homeomorphism onto its image. It is called a ***smooth local parameterization*** if it is a diffeomorphism onto its image. If the image of X is all of S, it is called a ***global parameterization***.

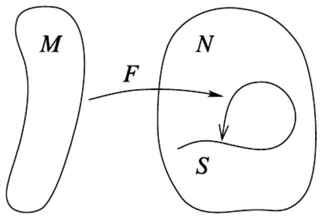
Proposition 5.23. Suppose M is a smooth manifold with or without boundary, $S \subseteq M$ is an immersed k-submanifold, $\iota : S \hookrightarrow M$ is the inclusion map, and U is an open subset of $\mathbb{R}^k$. A map $X : U \to M$ is a smooth local parameterization of S if and only if there is a smooth coordinate chart (V, φ) for S such that $X = \iota \circ \varphi^{-1}$. Therefore, every point of S is the image of some local parametrization.

Theorem 5.27 (Restricting the Domain of a Smooth Map). If M and N are smooth manifold with or without boundary, $F : M \to N$ is a smooth map, and $S \subseteq M$ is an immersed or embedded submanifold, then $F|_S : S \to N$ is smooth.

Theorem 5.29 (Restricting the Codomain of a Smooth Map). Suppose M is a smooth manifold (without boundary), $S \subseteq M$ is an immersed submanifold, and $F : N \to M$ is a smooth map whose image is contained in S. If F is continuous as a map from N to S, then $F : N \to S$ is smooth.



Restricting the Domain



Restricting the Codomain

Corollary 5.30 (Embedded Case). Let M be a smooth manifold and $S \subseteq M$ be an embedded submanifold. Then every smooth map $F : N \to M$ whose image is contained in S is also a smooth map from N to S.

Theorem 5.31. Suppose M is a smooth manifold and $S \subseteq M$ is an embedded submanifold. The subspace topology on S and the smooth structure described in Theorem 5.8 are the only topology and smooth structure with respect to which S is an embedded or immersed submanifold.

Theorem 5.32. Suppose M is a smooth manifold and $S \subseteq M$ is an immersed submanifold. For the given topology on S, there is only one smooth structure making S into an immersed submanifold.

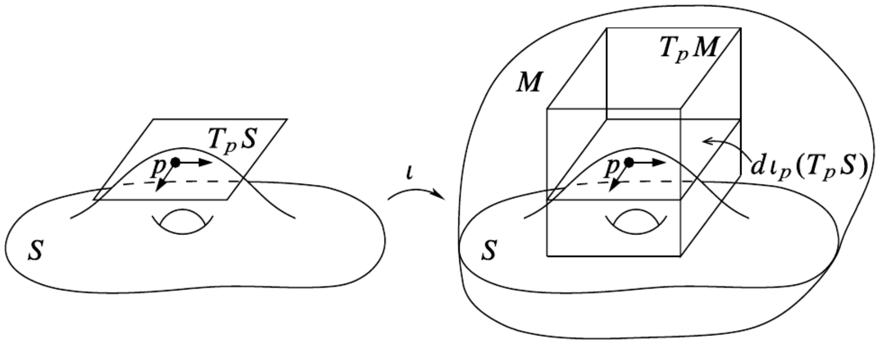
Lemma 5.34 (Extension Lemma for Functions on Submanifolds). Suppose M is a smooth manifold, $S \subseteq M$ is a smooth submanifold, and $f \in C^\infty(S)$.

(a) If S is embedded, then there exists a neighborhood U of S in M and smooth function $\widetilde{f} \in C^\infty(U)$ such that $\widetilde{f}|_S = f$.

(b) If S is properly embedded, then the neighborhood U in part (a) can be taken to be all of M.

- Tangent Space to a Submanifold -

Definition. Let M be a smooth manifold with or without boundary, and let $S \subseteq M$ be an immersed or embedded submanifold. Since the inclusion map $\iota : S \hookrightarrow M$ is a smooth immersion, at each point $p \in S$ we have an injective linear map $d\iota_p : T_pS \to T_pM$. We adopt the convention of identifying T_pS with its image under this map, thereby thinking of T_pS as a certain linear subspace of T_pM.



Proposition 5.35. Suppose M is a smooth manifold with or without boundary, $S \subseteq M$ is an immersed or embedded submanifold, and $p \in S$. A vector $v \in T_pM$ is in T_pS if and only if there is a smooth curve $\gamma : J \to M$ whose image is contained in S, and which is also smooth as a map into S, such that $0 \in J, \gamma(0) = p$, and $\gamma'(0) = v$.

Proposition 5.37. Suppose M is a smoth manifold, $S \subseteq M$ is an embedded submanifold, and $p \in S$. As a subspace of T_pM, the tangent space T_pS is characterized by

$$T_pS = \{v \in T_pM \ : \ vf = 0 \text{ whenever } f \in C^\infty(M) \text{ and } f|_S = 0\}.$$

Proposition 5.38. Suppose M is a smooth manifold and $S \subseteq M$ is an embedded submanifold. If $\Phi : U \to N$ is any local defining map for S, then $T_pS = \ker(d\Phi_p) : T_pM \to T_{\Phi(p)}N$ for each $p \in S \cap U$.

Corollary 5.39. Suppose $S \subseteq M$ is a level set of a smooth submersion $\Phi = (\Phi^1, \dots, \Phi^k) : M \to \mathbb{R}^k$. A vector $v \in T_pM$ is tangent to S if and only if $v\Phi^1 = \cdots = v\Phi^k = 0$.

Note: In general there are several good strategies for showing $S \subseteq M$ is *not* an immersed. Here are some useful facts to keep in mind:

(a) If S is immersed, then T_pS is a linear subspace of T_pM of the same dimension at every $p \in S$

(b) Every $p \in S$ must be the image of some local parameterization $X : U \to S$.

(c) Every $v \in T_pS$ is the velocity vector of some curve in S.

(d) Every tangent vector of S annihilates every smooth function on M that is constant on S.

6. Sard's Theorem

This chapter is a big 'ol collection of odds and ends that are useful for the prelim exam. The first part will just be talking about some theorems that are really important in Algebraic Topology and will be used later in the proofs of some important theorems. The second part, however, contains the more important material relating to transverse submanifolds.

- Sard and Whitney -

Theorem 6.10 (Sard's Theorem). Suppose M and N are smooth manifolds with or without boundaryand $F : M \to N$ is a smooth map. Then the set of critical values of F has measure zero in N.

Corollary 6.11. Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is a smooth map. If $\dim(M) < \dim(N)$, then $F(M)$ has measure zero in N.

Theorem 6.12 (Whitney Embedding Theorem). Every smooth n-manifold with or without boundaryadmits a proper smooth embedding into $\mathbb{R}^{2n+1}$.

Note: Just because you can, doesn't mean that you should.

Theorem 6.21 (Whitney Approximation Theorem). Let M be a smooth manifold with or without boundary. If $F : M \to \mathbb{R}^k$ is continuous, then given any positive continuous function $\delta : M \to \mathbb{R}$, there exists a smooth function $\widetilde{F}$ such that $|F(x) - \widetilde{F}(x)| < \delta(x)$ for all $x \in M$. Furthermore, if F is smooth on a closed subset, then $\widetilde{F}$ can be chosen to agree with F on that set.

Definition. If M is a m-dimensional embedded submanifold of $\mathbb{R}^n$ then the ***normal bundle of M*** is defined to be

$$NM = \{(x, v) \in T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n \ : \ x \in M, v \in T_x\mathbb{R}^n \ s.t. \ \ v \cdot w = 0 \ \ \forall w \in T_xM\}$$

Theorem 6.23. If $M \subseteq \mathbb{R}^n$ is an embedded m-dimensional submanifold, then NM is an embedded n-dimensional submanifold of $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$.

Definition. Thinking of NM as a submanifold of $\mathbb{R}^n \times \mathbb{R}^n$, we define $E : NM \to \mathbb{R}^n$ by

$$E(x, v) = x + v.$$

Definition. If $M \subseteq \mathbb{R}^n$ is an embedded m-dimensional submanifold, a ***tubular neighborhood of M*** is a neighborhood U of M in $\mathbb{R}^n$ that is the diffeomorphic image under E of an open subset $V \subseteq NM$ of the form

$$V = \{(x, v) \in NM \ : \ |v| < \delta(x)\},$$

for some positive continuous function $\delta : M \to \mathbb{R}$.

Theorem 6.24 (Tubular Neighborhood Theorem). Every embedded submanifold of $\mathbb{R}^n$ has a tubular neighborhood.

- Transversality -

Definition. Suppose M is a smooth manifold. Two embedded submanifolds $S, \S' \subseteq M$ are said to ***intersect transversely*** if for each $p \in S \cap S'$, the tangent spaces T_pS and T_pS' together span T_pM.

Note: T_pS and T_pS' are allowed to intersect non-trivially.

Definition. I f$F : N \to M$ is a smooth map and $S \subseteq M$ is an embedded submanifold, we say that F is ***transverse to S*** if for every $x \in F^{-1}(S)$, the spaces $T_{F(x)}S$ and $dF_x(T_xN)$ together span $T_{F(x)}M$.

Theorem 6.30 (More General Level Set Theorem). Suppose M and N are smooth manifolds and $S \subseteq M$ is an embedded submanifold.

(a) If $F : N \to M$ is a smooth map that is transverse to S, then $F^{-1}(S)$ is an embedded submanifold of N whose codimension is equal to the codimension of S in M.

(b) If $S' \subseteq M$ is an embedded submanifold that intersects S transversely, then $S \cap S'$ is an embedded submanifold of M whose codimension is equal to the sum of the codimensions of S and S'.

7. Lie Groups

In this chapter we introduce Lie groups, which are smooth manifolds that are also groups in which multiplication and inversion are smooth maps. Besides providing many examples of interesting manifolds themselves, they are essential tools in the study of more general manifolds, primarily because of the role they play as groups of symmetries of other manifolds. Our aim in this chapter is to introduce Lie groups and some of the tools for working with them, and to describe an some examples.

Definition. A ***Lie group*** is a smooth manifold G (without boundary) that is also a group in the algebraic sense, with the property that the multiplication map $m : G \times G \to G$ and inversion map $i : G \to G$, given by

$$m(g, h) = gh, \qquad i(g) = g^{-1},$$

are both smooth. A Lie group is, in particular, a ***topological group***.

Proposition 7.1. If G is a smooth manifold with a group structure such that the map $G \times G \to G$ given by $(g, h) \mapsto gh^{-1}$ is smooth, then G is a Lie group.

Definition. If G is a Lie group, any element $g \in G$ defines maps $L_g, R_g : G \to G$, called ***left translation*** and ***right translation***, respectively, by

$$L_g(h) = gh, \qquad R_g(h) = hg.$$

Example 7.2 (Lie Groups). Each of the following manifolds is a Lie group with the indicated group operation.

(a) The ***general linear group*** $GL_n(\mathbb{R})$ is the set of invertible $n \times n$ matrices with real entries. It is a group under matrix multiplication, and it is an open submanifold of the vector space $M_n(\mathbb{R})$. Multiplication is smooth because the matrix entries of a product matrix AB are polynomials in the entries of A and B. Inversion is smooth by Cramer's rule.

(b) Suppose G is an arbitrary Lie group and $H \subseteq G$ is an ***open subgroup***. The group operations are restrictions of those of G so they are smooth, and so H is a Lie group.

(c) $\mathbb{R}^n$ under addition is a Lie group under addition.

(d) $\mathbb{R}^*$, the multiplicative group of $\mathbb{R}$, is a Lie group.

Definition. If G and H are Lie groups, a ***Lie group homomorphism from G to H*** is a smooth map $F : G \to H$ that is also a group homomorphism. It is called a ***Lie group isomorphism*** if it is also a diffeomorphism.

Theorem 7.5. Every Lie group homomorphism has constant rank.

Definition. Suppose G is a Lie group. A ***Lie subgroup of G*** is a subgroup of G endowed with a topology and a smooth structure making it into a Lie group and an *immersed* submanifold of G.

Proposition 7.11. Let G be a Lie group, and suppose $H \subseteq G$ is a subgroup that is also an embedded submanifold. Then H is a Lie subgroup.

Lemma 7.12. Suppose G is a Lie group and $H \subseteq G$ is an open subgroup. Then H is an embedded Lie subgroup. In addition, H is closed, so it is a union of connected components of G.

Proposition 7.14. Suppose G is a Lie group, and $W \subseteq G$ is any neighborhood of the identity.

(a) S generates an open subgroup of G

(b) If W is connected, it generates a connected open subgroup of G.

(c) If G is connected, W generates G.

Definition. If G is a Lie group, the connected component of G containing the identity is called the ***identity component of* G**.

Proposition 7.15. Let G be a Lie group and let $G_)$ be its identity component. Then G_0 is a normal subgroup of G, and is the only connected open subgroup. Every connected component of G is diffeomorphic to G_0.

Proposition 7.16. Let $F : G \to H$ be a Lie group homomorphism. The kernel of G is a properly embedded Lie subgroup of G, whose codimension is equal to the rank of F.

Proposition 7.17. If $F : G \to H$ is an injective Lie group homomorphism, the image of F has a unique smooth manifold structure such that $F(G)$ is a Lie subgroup of H and $F : G \to F(G)$ is a Lie group isomorphism.

8. Vector Fields

This next chapter is mainly concerned with a familiar object from Calc. III: vector fields. Unlike in Calc. III where we viewed vector field as a continuous map form an open subset $U \subseteq \mathbb{R}^n$ to $\mathbb{R}^n$, for a general smooth manifold, we will view a vector field as a particular type of continuous map from M to its tangent bundle.

- Vector Fields on Manifolds -

Definition. If M is a smooth manifold with or without boundary, a ***vector field on M*** is a section of the map $\pi : TM \to M$. More concretely, a vector field is a continuous map $X : M \to TM$, usually written $p \mapsto X_p$, with the property that

$$\pi \circ X = \mathrm{Id}_M,$$

or equivalently, $X_p \in T_pM$ for each $p \in M$.

Definition. X is called a ***smooth vector field*** if it's smooth as a map from M to TM. Otherwise, it is called a ***rough vector field***.

Definition. Let X be a rough vector field on M, and let $(U, (x^i))$ be a chart on M. Given a vector field X, there exist functions $X^1, \dots, X^n : U \to \mathbb{R}$ such that

$$X_p = X^i(p) \left.\frac{\partial}{\partial x^i}\right|_p .$$

These are called the ***component functions of X***.

Proposition 8.1 (Smoothness Criterion for Vector Fields). Let M be a smooth manifold with or without boundary, and let $X : M \to TM$ be a rough vector field. If $(U, (x^i))$ is any smooth coordinate chart on M, then the restriction of X to U is smooth if and only if its component functions with respect to this chart are smooth.

Example 8.2 (Coordinate Vector Fields). If $(U, (x^i))$ is any smooth chart on M, the assignment

$$p \mapsto \left.\frac{\partial}{\partial x^i}\right|_p$$

determines a vector field on U, called the ***ith coordinate vector field*** and denoted by $\frac{\partial}{\partial x^i}$. It is smooth because its component functions are constant.

Example 8.4 (The Angle Coordinate Vector Field on the Circle). Let θ be any angle coordinate on a proper open subset $U \subseteq \mathbb{S}^1$, and let $d/d\theta$ denote the corresponding coordinate vector field. Because any other angle coordinate $\widetilde{\theta}$ on $V \subseteq \mathbb{S}^1$ is related to θ on $U \cap V$ by $\widetilde{\theta} = \theta + 2\pi n$. Then we have that

$$\frac{d}{d\widetilde{\theta}} = \frac{d}{d\theta}$$

so there is a globally defined coordinate vector field even though there is no globally defined θ.

Lemma 8.6 (Extension Lemma for Vector Fields). Let M be a smooth manifold with or without boundary, and let $A \subseteq M$ be a closed subset. Suppose X is a smooth vector field along A. Given any open subset U containing A, there exists a smooth global vector field $\widetilde{X}$ on M such that $\widetilde{X}|_A$ and $\text{supp}(\widetilde{X}) \subseteq U$.

Note: We will use the symbol $\mathfrak{X}(M)$ to denote the set of all smooth vector fields on M. We sill note that $\mathfrak{X}(M)$ is a vector space under pointwise addition and scalar multiplication.

Proposition 8.8. Let M be a smooth manifold with or without boundary

(a) If X and Y are smooth vector fields on M and $f, g \in C^\infty(M)$, then $fX + gY$ is a smooth vector field.

(b) $\mathfrak{X}(M)$ is a module over the ring $C^\infty(M)$.

Definition. Let M be a smooth manifold. A k-tuple $(X_1, \dots X_k)$ of vector fields on a subset $A \subseteq M$ is ***linearly independent*** if for all $p \in A$, the vectors $(X_1(p), \dots, X_k(p))$ are linearly independent. And $(X_1, \dots X_k)$ is said to ***span the tangent bundle*** if $(X_1(p), \dots, X_k(p))$ spans T_pM for all $p \in A$

Definition. A ***local frame field*** for M is an ordered n-tuple of vector fields $(E_1, \dots, E_n)$ on an open subset $U \subseteq M$ such that $(E_1(p), \dots, E_n(p))$ form a basis for T_pM for all $p \in M$. A local frame field is called a ***global frame field*** if $U = M$ and a smooth frame field if $E_1, \dots, E_n$ are all smooth.

Example 8.10 (Local and Global Frames).

(a) The standard coordinate vector fields form a smooth global frame for $\mathbb{R}^n$.

(b) If $(U, (x^i))$ is any smooth coordinate chart for a smooth manifold M (possibly with boundary), then the coordinate vector fields form a smooth local frame $(\partial/\partial x^i)$ on U, called a ***coordinate frame***. Every point of M is in the domain of such a frame.

(c) The vector field $d/d\theta$ described in Example 8.4 constitutes a smooth global frame for the circle.

Proposition 8.11 (Completion of Local Frames). Let M be a smooth n-manifold with or without boundary.

(a) If $(X_1, \dots, X_k)$ is a linearly independent k-tuple of smooth vector fields on an open subset $U \subseteq M$, with $1 \leq k < n$, then for each $p \in U$ there exist smooth vector fields $X_{k+1}, \dots, X_n$ in a neighborhood V of p such that $(X_1, \dots, X_n)$ is a smooth local frame for M on $U \cap V$.

(b) If $(v_1, \dots, v_k)$ is a linearly independent k-tuple of vectors in T_pM for some $p \in M$, with $1 \leq k \leq n$, then there exists a smooth local frame (X_i) on a neighborhood of p such that $X_i(p) = v_i$ for $i = 1, \dots, k$.

(c) If $(X_1, \dots, X_n)$ is a linearly independent n-tuple of smooth vector fields along a closed subset $A \subseteq M$, then there exists a smooth local frame $(\widetilde{X}_1, \dots \widetilde{X}_n)$ on some neighborhood of A such that $\widetilde{X}_i|_A = X_i$ for $i = 1, \dots, n$.

Lemma 8.13 (Gram-Schmidt Algorithm for Frames). Suppose (X_j) is a smooth local frame for $T\mathbb{R}^n$ over an open subset $U \subseteq \mathbb{R}^n$. Then there is a smooth orthonormal frame (E_j) over U such that $\text{span}(E_1(p), \dots, E_j(p)) = \text{span}(X_1(p), \dots, X_j(p))$ for each $j = 1, \dots, n$ and each $p \in U$.

Definition. A smooth manifold is called parallelizable if it admits a smooth global frame field.

Note: A vector field $X \in \mathfrak{X}(M)$ defines a map

$$X : C^\infty(M) \to C^\infty(M)$$

as you might expect $Xf \in C^\infty(M)$ is defined by

$$Xf(p) = X_p(f).$$

Proposition 8.14. Let M be a smooth manifold with or without boundary, and let $X : M \to TM$ be a rough vector field. The following are equivalent:

(a) X is smooth.

(b) For every $f \in C^\infty(M)$, the function Xf is smooth on M.

(c) For every ope subset $U \subseteq M$ and every $f \in C^\infty(M)$, the function Xf is smooth on U.

Definition. A map $XC^\infty(M) \to C^\infty(M)$ is called a ***derivation*** if it is linear over $\mathbb{R}$, and if

$$X(fg) = f\,Xg + g\,Xf$$

for all $f, g \in C^\infty(M)$.

Note: All $X \in \mathfrak{X}$ are derivations.

Proposition 8.15. Let M be a smooth manifold with or without boundary. A map $D : C^\infty(M) \to C^\infty(M)$ is a derivation if and only if it is of the form $Df = Xf$ for some smooth vector field $X \in \mathfrak{X}(M)$.

A note on Vector Fields and Smooth Manifolds:

Let $F : M \to N$ be a smooth pan and let $X \in \mathfrak{X}(M)$. For each $p \in M$ the differential of F gives a vector

$$dF_p(X_p) \in T_{F(p)}N.$$

However, this does NOT generally produce a vector filed on N:

- If F is not surjective, then parts of the produce vector field will be undefined.
- If F is not injective, then dF may assign multiple vectors to some points in N.

This leads us to the following definition:

Definition. Let $F : M \to N$ be smooth $X \in \mathfrak{X}(M)$. For each $p \in M$, dF gives a vector $dF_p(X_p) \in T_{F(p)}N$. If there exists a $Y \in \mathfrak{X}(N)$ such that

$$dF_p(X_p) = Y_{F(p)}$$

we say that X and Y are ***F-related***.

Proposition 8.16. Suppose $F : M \to N$ is a smooth map between manifolds with or without boundary, $X \in \mathfrak{X}(M)$, and $Y \in \mathfrak{X}(N)$. Then X and Y are F-related if and only if for every smooth real-valued function f defined on an open subset of N,

$$X(f \circ F) = (Yf) \circ F.)$$

Proposition 8.19. Suppose M and N are smooth manifold with or without boundary, and $F : M \to N$ is a diffeomorphism. For every $X \in \mathfrak{X}(M)$, there is a unique smooth vector field on N that is F-related to X.

Definition. In the situation of the preceding proposition, we denote the unique vector field that is F-related to X by F_*X, and call it the ***pushforward of X by F***.

- Lie Bracket and Lie Algebra -

The Lie Bracket is a way of introducing a product structure on the space of smooth vector fields $\mathfrak{X}(M)$.

Definition. Given $X, Y \in \mathfrak{X}(M)$, the ***Lie Bracket*** of X and Y is the operation $[X, Y] : C^\infty(M) \to C^\infty(M)$ defined by

$$[X, Y](f) = X(Yf) - Y(Xf).$$

Lemma 8.25. The Lie bracket of any pair of smooth vector fields is a smooth vector field.

Proposition 8.26 (Coordinate Formula for the Lie Bracket). Let X, Y be smooth vector fields on a smooth manifold M with or without boundary, and let $X = X^i \frac{\partial}{\partial x^i}$ and $Y = Y^j \frac{\partial}{\partial x^j}$ be the coordinate expressions for S and Y in terms of some smooth local coordinates (x^i) for M. Then $[X, Y]$ has the following coordinate expression:

$$[X, Y] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j},$$

or more concisely,

$$[X, Y] = (XY^j - YX^j) \frac{\partial}{\partial x^j}.$$

Corollary. The condition that coordinate vector fields $\left\{ \frac{\partial}{\partial x^i} \right\}$ associated to any local coordinate chart satisfy

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right] = 0 \quad \forall i, j$$

is equivalent to the statement that mixed partials commute.

Proposition 8.28 (Properties of the Lie Bracket). The Lie bracket satisfies the following identities for all $X, Y, Z \in \mathfrak{X}(M)$:

(a) BINLEARITY: For $a, b \in \mathbb{R}$,

$$[aX + bY, Z] = a[X, Z] + b[Y, Z],$$
$$[Z, aX + bY] = a[Z, X] + b[Z, Y].$$

(b) ANTISYMMETRY:

$$[X, Y] = [Y, X]$$

(c) JACOBI IDENTITY:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

(d) For $f, g \in C^\infty(M)$,

$$[fX, gY] = fg[X,Y] + (fXg)Y - (gYf)X.$$

Proposition 8.30 (Naturality of the Lie Bracket). Let $F : M \to N$ be a smooth map between manifolds with or without boundary, and let $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \mathfrak{X}(N)$ be vector fields such that X_i is F-related to Y_i for $i = 1, 2$. Then $[X_1, X_2]$ if F-related to $[Y_1, Y_2]$.

Corollary 8.31 (Pushforwards of Lie Brackets). Suppose $F : M \to N$ is a diffeomorphism and $X_1, X_2 \in \mathfrak{X}(M)$. Then $F_*[X_1, X_2] = [F_*X_1, F_*X_2]$.

Corollary 8.32 (Brackets of Vector Fields Tangent to Submanifolds). Let M be a smooth manifold and let S be an immersed submanifold with or without boundaryin M. If Y_1 and Y_2 are smooth vector fields on M that are tangent to S, then $[Y_1, Y_2]$ is also tangent to S.

Definition. let G be a Lie group. A vector field X on G is said to be ***left-invariant*** if it is invariant under all left translations, in the sense that it is L_g-related to itself for every $g \in G$. More explicitly, this means

$$d(L_g)_{g'}(X_{g'}) = X_{gg'}, \quad \text{for all } g, g' \in G.$$

Proposition 8.33. Let G be a Lie group, and suppose X and Y are smooth left-invariant vector fields on G. Then $[X, Y]$ is also left-invariant.

Definition. A ***Lie algebra*** (over $\mathbb{R}$) is a real vector space $\mathfrak{g}$ endowed with a map called the ***bracket*** from $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, usually denoted by $(X, Y) \mapsto [X, Y]$, that satisfies the following properties for all $X, Y, Z \in \mathfrak{g}$:

(i) BINLEARITY: For $a, b \in \mathbb{R}$,

$$[aX + bY, Z] = a[X, Z] + b[Y, Z],$$
$$[Z, aX + bY] = a[Z, X] + b[Z, Y].$$

(ii) ANTISYMMETRY:

$$[X, Y] = [Y, X]$$

(iii) JACOBI IDENTITY:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

Definition. if $\mathfrak{g}$ is a Lie algebra, a linear subspace $\mathfrak{h} \subseteq g$ is called a ***Lie subalgebra of $\mathfrak{g}$*** if it is closed under brackets.

Definition. If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras, a linear map $A : \mathfrak{g} \to \mathfrak{h}$ is called a ***Lie algebra homomorphism*** if it preserves brackets. An invertible Lie algebra homomorphism is called a ***Lie algebra isomorphism***.

Example 8.36 (Lie Algebras).

(a) The space $\mathfrak{X}(M)$ of all smooth vector fields on a smooth manifold M is a Lie algebra under the Lie bracket.

(b) If G is a Lie group, the set of all smooth left-invariant vector field on G is a Lie subalgebra of $\mathfrak{X}(G)$ and is therefore a Lie algebra.

(c) The vector space $M_n(|R)$ of $n \times n$ matrices becomes an n^2-dimensional Le algebra under the ***commutator bracket***:

$$[A, B] = AB - BA.$$

Bilinearity and antisymmetry are obvious from the definition, and the Jacobi identity follows from a straightforward calculation. When we are regarding $M_n(\mathbb{R})$ as a Lie algebra with this bracket, we denote it by $\mathfrak{gl}_n(\mathbb{R})$

Definition. Any vector space becomes a Lie algebra if we define all brackets to be zero. Such a Lie algebra is said to be ***abelian***.

Definition. The Lie algebra of all smooth left-invariant vector on a Lie group G is called the ***Lie algebra of G***, and is denoted by $\text{Lie}(G)$.

Theorem 8.37. Let G e a Lie group. The evaluation map $\varepsilon : \text{Lie}(G) \to T_eG$, given by $\varepsilon(X) = X_e$, is a vector space isomorphism. Thus, $\text{Lie}(G)$ is finite-dimensional, with dimension equal to $\dim(G)$.

Corollary 8.38. Every left-invariant rough vector field on a Lie group is smooth.

Definition. If G is a Lie group, a local or global frame consisting of left-invariant vector fields is called a ***left-invariant frame***.

Theorem 8.39. Every Lie group admits a left-invariant smooth global frame, and therefore every Lie group is parallelizable.

Proposition 8.41 (Lie Algebra of the General Linear Group). The composition of the natural maps

$$\text{Lie}(GL_n(\mathbb{R})) \to T_{I_n}GL_n(\mathbb{R}) \to \mathfrak{gl}_n(\mathbb{R})$$

gives a Lie algebra isomorphism between $\text{Lie}(GL_n(\mathbb{R}))$ and the matrix algebra $\mathfrak{gl}_n(\mathbb{R})$.

Proposition 8.42. If V is any finite-dimensional real vector space, the composition of canonical isomorphisms given by

$$\text{Lie}(GL(V)) \to T_{\text{Id}}GL(V) \to \mathfrak{gl}(V)$$

yields a Lie algebra isomorphism between $\text{Lie}(GL(V))$ and $\mathfrak{gl}(V)$.

Theorem 8.44 (Induced Lie Algebra Homomorphism). Let G and H be Lie groups, and let $\mathfrak{g}$ and $\mathfrak{h}$ be their Lie algebras. Suppose $F : G \to H$ is a Lie group homomorphism. For every $X \in \mathfrak{g}$, there is a unique vector field in $\mathfrak{h}$ that is F-related to X. With this vector field denoted by F_*X, the map $F_* : \mathfrak{g} \to \mathfrak{h}$ so defined is a Lie algebra homomorphism.

Definition. The map $F_* : \mathfrak{g} \to \mathfrak{h}$ is called the ***induced Lie algebra homomorphism***.

Proposition 8.45 (Properties of Induced Homomorphisms).

(a) The homomorphism $(\text{Id}_G)_* : \text{Lie}(G) \to \text{Lie}(G)$ induced by the identity map of G is the identity of $\text{Lie}(G)$.

(b) If $F_1 : G \to H$ and $F_2 : H \to K$ are Lie group homomorphisms, then

$$(F_2 \circ F_2)_* = (F_2)_* \circ (F_1)_* : \text{Lie}(G) \to \text{Lie}(K).$$

(c) Isomorphic Lie groups have isomorphic Lie algebras.

Theorem 8.46 (the Lie Algebra of a Lie Subgroup). Suppose $H \subseteq G$ is a Lie subgroup, and $\iota : H \hookrightarrow G$ is the inclusion map. There is a Lie subalgebra $\mathfrak{h} \subseteq \mathrm{Lie}(G)$ that is canonically isomorphic to $\mathrm{Lie}(H)$, characterized by either of the following descriptions:

$$\begin{aligned}\mathfrak{h} &= \iota_*(\mathrm{Lie}(H)) \\ &= \{X \in \mathrm{Lie}(H) \ : \ X_e \in T_e H\}.\end{aligned}$$

Example 8.47 (The Lie Algebra of $O(n)$). The orthogonal group $O(n)$ is a Lie subgroup of $GL_n(\mathbb{R})$ that turns out to be equal to the level set $\Phi^{-1}(I_n)$, where $\Phi : GL_n(\mathbb{R}) \to M_n(\mathbb{R}) : A \mapsto A^T A$. We have that $T_{I_n}O(n)$ is the same as the kernel of $d\Phi_{I_n}$, which with a little computation, we have that $d\Phi_{I_n}(B) = B^T + B$, so

$$\begin{aligned}T_{I_n}O(n) &= \{B \in \mathfrak{gl}_n(\mathbb{R}) \ : \ B^T + B = 0\} \\ &= \{\text{skew-symmetric } n \times n \text{ matrices}\}.\end{aligned}$$

We denote this subspace of $\mathfrak{gl}_n(\mathbb{R})$ by $\mathfrak{o}(n)$. Theorem 8.46 them implies that $\mathfrak{o}(n)$ is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R})$ that is canonically isomorphic to $\mathrm{Lie}(O(n))$.

9. Integral Curves and Flows

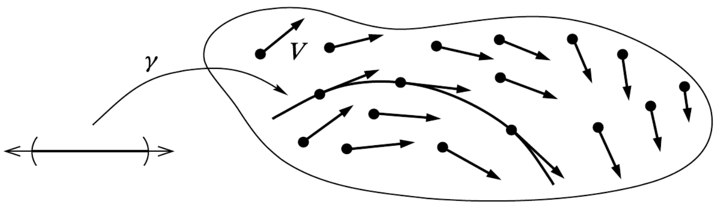
The primary objects associated with smooth vector fields are *integral curves*, which are smooth curves whose velocity at each point is equal to the value of the vector field there. The collection of all integral curves of a given vector field on a manifold determines a family of diffeomorphism of (open subsets of) the manifold called a *flow*. In applied mathematics, these objects are often used to approximate solutions to differential equations. The methodology for doing this will become more clear as we discuss the fundamental theorem on flows.

- Integral Curves and Flows -

Definition. Suppose M is a smooth manifold with or without boundary. If V is a vector field on M, an ***integral curve of V*** is a differentiable curve $\gamma : J \to M$ whose velocity at each point is equal to the value of V at that point:

$$\gamma'(t) = V_{\gamma(t)} \quad \text{for all } t \in J.$$

If $0 \in J$, the point $\gamma(0)$ is called the ***starting point of*** $\boldsymbol{\gamma}$.



Proposition 9.2. Let V be a smooth vector field on a smooth manifold M. For each point $p \in M$, there exists $\varepsilon > 0$ and a smooth curve $\gamma(-\varepsilon, \varepsilon) \to M$ that is an integral curve of V starting at p.

Lemma 9.3 (Rescaling Lemma). Let V be a smooth vector field on a smooth manifold M, let $J \subseteq \mathbb{R}$ be an interval, and let $\gamma : J \to M$ be an integral curve of V. For any $a \in \mathbb{R}$, the curve $\widetilde{\gamma} : \widetilde{J} \to M$ defined by $\widetilde{\gamma}(t) = \gamma(at)$ is an integral curve of the vector field aV, where $\widetilde{J} = \{t \ : \ at \in J\}$.

Lemma 9.4 (Translation Lemma). Let V, M, J and γ be as in the preceding lemma. For any $b \in \mathbb{R}$, the curve $\widehat{\gamma} : \widehat{J} \to M$ defined by $\widehat{\gamma}(t) = \gamma(t+b)$ is also an integral curve of V, where $\widehat{J} = \{t \ : \ t+b \in J\}$.

Proposition 9.5 (Naturality of Integral Curves). Suppose M and N are smooth manifold and $F : M \to N$ is a smooth map. Then $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are F-related if and only if F tales integral curve so of X to integral curves of Y, meaning that for each integral curve of X, $F \circ \gamma$ is an integral curve of Y.

Definition. We define a ***global flow*** on M (also called a ***one-parameter group action***) to be a continuous left $\mathbb{R}$-action on M; that is, a continuous map $\theta : \mathbb{R} \times M \to M$ satisfying the following properties for all $s, t \in \mathbb{R}$ and $p \in M$:

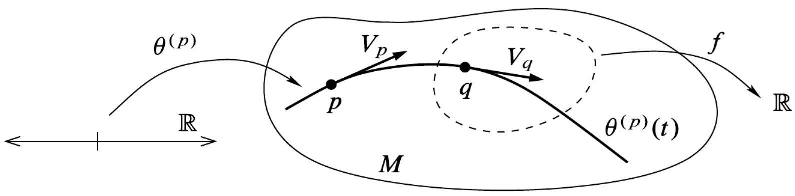
$$\theta(t, \theta(s, p)) = \theta(t + x, p), \qquad \theta(0, p) = p.$$

Definition. Given a global flow θ on M, we define two collections of maps as follows:

- For each $t \in \mathbb{R}$, define a continuous map $\theta_t : M \to M$ by
$$\theta_t(p) = \theta(t, p).$$
- For each $p \in M$, define a curve $\theta^{(p)} : \mathbb{R} \to M$ by
$$\theta^{(p)}(t) = \theta(t, p).$$

Definition. If $\theta : \mathbb{R} \times M \to M$ is a smooth global flow, for each $p \in M$ we define a tangent vector $V_p \in T_pM$ by
$$V_p = \theta^{(p)\prime}(0).$$
The assignment $p \mapsto V_p$ is a (rough) vector field om M, which is called the ***infinitesimal generator of $\boldsymbol{\theta}$***.

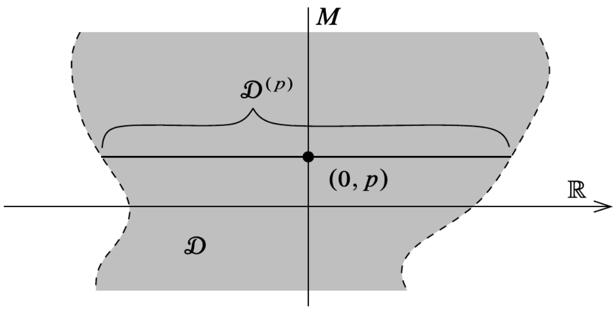
Proposition 9.7. Let $\theta : \mathbb{R} \times M \to M$ be a smooth global flow on a smooth manifold M. The infinitesimal generator V of θ is a smooth vector field on M, and each curve $\theta^{(p)}$ is an integral curve of V.



Example 9.9 (Smooth Vector Field that does not Generate a Smooth Global Flow). Let $M = \mathbb{R}^2 \backslash \{0\}$ with standard coordinates (x, y), and let V be the vector field $\frac{\partial}{\partial x}$ on M. The unique integral curve of V starting at $(-1, 0) \in M$ is $\gamma(t) = (t - 1, 0)$. However, in this case, γ cannot be extended continuously past $t = 1$. This is intuitively evident because of the "hole" in M at the origin.

Definition. If M is a manifold, a ***flow domain*** for M is an open subset $\mathcal{D} \subseteq \mathbb{R} \times M$ with the property that for each $p \in M$, the set $D^{(p)} = \{t \in \mathbb{R} \ : \ (t, p) \in \mathcal{D}\}$ is an open interval containing 0.

Definition. A ***flow*** (or ***local flow***) on M is a continuous map $\theta : \mathcal{D} \to M$, where $\mathcal{D} \subseteq \mathbb{R} \times M$ is a flow domain, that satisfies the following group laws: for all $p \in M$
$$\theta(0, p) = p,$$
and for all $s \in \mathcal{D}^{(p)}$ and $t \in \mathcal{D}^{(\theta(s,p))}$ such that $s + t \in \mathcal{D}^{(p)}$,
$$\theta(t, \theta(s, p)) = \theta(t + x, p).$$



Definition. If θ is a flow, we define $\theta_t(p) = \theta^{(p)} = \theta(t,p)$ whenever $(t,p) \in \mathcal{D}$. For each $t \in \mathbb{R}$, we also define

$$M_t = \{p \in M \ : \ (t,p) \in \mathcal{D}\},$$

so that

$$p \in M_t \Leftrightarrow t \in \mathcal{D}^{(p)} \Leftrightarrow (t,p) \in \mathcal{D}.$$

If θ is smooth, the ***infinitesimal generator of θ*** is defined by $V_p = \theta^{(p)\prime}(0)$.

Proposition 9.11. If $\theta : \mathcal{D} \to M$ is a smooth flow, then the infinitesimal generator V of θ is a smooth vector field, and each cure $\theta^{(p)}$ is an integral curve of V.

Definition. A ***maximal integral curve*** is one that cannot be extended to an integral curve on any larger open interval, and a ***maximal flow*** is a flow that admits no extension to a flow on a larger flow domain.

Theorem 9.12 (Fundamental Theorem on Flows). Let V be a smooth vector field on a smooth manifold M. There is a unique smooth maximal flow $\theta : \mathcal{D} \to M$ whose infinitesimal generator is V. This flow has the following properties:

(a) For each $p \in M$, the curve $\theta^{(p)} : \mathcal{D}^{(p)} \to M$ is the unique maximal integral curve of V starting at p.

(b) If $s \in \mathcal{D}^{(p)}$, then $\mathcal{D}^{(\theta(s,p))}$ is the interval $\mathcal{D}^{(p)} - s = \{t - s \ : \ t \in \mathcal{D}^{(p)}\}$.

(c) For each $t \in \mathbb{R}$, the set M_t is open in M, and $\theta_t : M_t \to M_{-t}$ is a diffeomorphism with inverse θ_{-t}.

Definition. We say that a smooth vector field is ***complete*** if it generates a global flow, or equivalently if each of its maximal integral curves is defined for all $t \in \mathbb{R}$.

Lemma 9.15 (Uniform Time Lemma). Let V be a smooth vector field on a smooth manifold M, and let θ be its flow. Suppose there is a positive number ε such that for every $p \in M$, the domain $\theta^{(p)}$ contains $(-\varepsilon, \varepsilon)$. Then V is complete.

Theorem 9.16. Every compactly supported smooth vector field on a smooth manifold is complete.

Corollary 9.17. On a compact smooth manifold, every smooth vector field is complete

Theorem 9.18. Every left-invariant vector field on a Lie group is complete.

Theorem 9.19 (Flowout Theorem). Suppose M is a smooth manifold, $S \subseteq M$ is an embedded k-dimensional submanifold, and $V \in \mathfrak{X}(M)$ is a smooth vector field that is nowhere tangent to S. Let $\theta\mathcal{D} \to M$ be the flow of V, let $\mathcal{O} = (\mathbb{R} \times S) \cap \mathcal{D}$ and let $\Phi = \theta|_{\mathcal{O}}$.

(a) $\Phi : \mathcal{O} \to M$ is an immersion.

(b) $\frac{\partial}{\partial t} \in \mathfrak{X}(\mathcal{O})$ is Φ-related to V.

(c) There exists a smooth positive function $\delta : S \to \mathbb{R}$ such that the restriction of Φ to $\mathcal{O}_\delta$ is injective, where $\mathcal{O}_\delta \subseteq \mathcal{O}$ is the flow domain

$$\mathcal{O}_\delta = \{(t.p) \in \mathcal{O} \ : \ |t| < \delta(p)\}.$$

Thus, $\Phi(\mathcal{O}_\delta)$ is an immersed submanifold of M containing S, and V is tangent to this submanifold.

(d) If S has codimension 1, then $\Phi|_{\mathcal{O}_\delta}$ is a diffeomorphism onto an open submanifold of M.

Definition. The submanifold $\Phi(\mathcal{O}_\delta) \subseteq M$ in the previous theorem is called a ***flowout from S along V***.

Definition. If V is a vector field on M, a point $p \in M$ is said to be a ***singular point of V*** if $V_p = 0$, and a ***regular point*** otherwise.

Proposition 9.21. Let V be a smooth vector filed on a smooth manifold M, and let $\theta : \mathcal{D} \to M$ be the flow generated by V. If $p \in M$ is a singular point of V, then $\mathcal{D}^{(p)} = \mathbb{R}$ and $\theta^{(p)}$ is the constant curve $\theta^{(p)}(t) \equiv p$. If p is a regular point, then $\theta^{(p)} : \mathcal{D}^{(p)} \to M$ is a smooth immersion.

Theorem 9.22 (Canonical Form Near a Regular Point). Let V be a smooth vector field on a smooth manifold M, and let $p \in M$ be a regular point of V. There exist smooth coordinates (s^i) on some neighborhood of p in which B has the coordinate representation $\frac{\partial}{\partial s^1}$. If $S \subseteq M$ is any embedded hypersurface (submanifold with codimension 1) with $p \in S$ and $V_P \notin T_pS$, then the coordinates can also be chosen so that s^1 is a local defining function for S.

Example 9.23. Let $W = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ on $\mathbb{R}^2$. The flow of W is given by

$$\theta_t(x, y) = (x\cos(t) - y\sin(t), x\sin(t) + y\cos(t).$$

The point $(1, 0) \in \mathbb{R}^2$ is a regular point of W because $W_{(1,0)} = \left.\frac{\partial}{\partial y}\right|_{(1,0)} \neq 0$. Because W has nonzero y-coordinate there, we can take S to be the x-axis, parameterized by $X(s) = (s, 0)$. We define $\Psi : \mathbb{R}^2 \to R^2$ by

$$\Psi(t, s) = \theta_t(s, 0) = (s\cos(t), s\sin(t)),$$

and then solve locally for (t, s) in terms of (x, y) to obtain the following coordinate map in a neighborhood of $(1, 0)$:

$$(t, s) = \Psi^{-1}(x, y) = \left(\tan^{-1}\left(\frac{y}{x}\right), \sqrt{x^2 + y^2}\right).$$

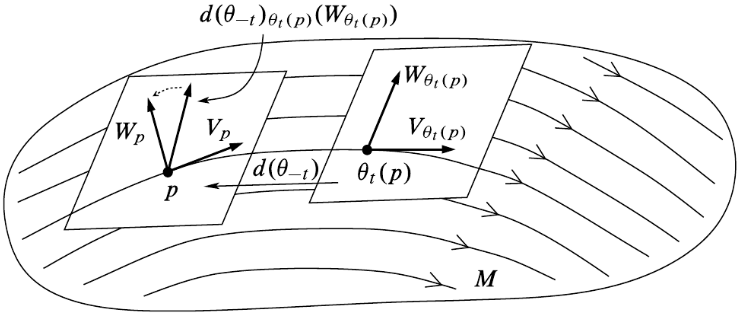
It is easy to check that $W = \frac{\partial}{\partial t}$ in these coordinates.

- Lie Derivatives -

Definition. Let M be a smooth manifold, V a smooth vector field on M, and θ the flow of V. For any smooth vector field W on M, define a rough vector field on M, denoted by $\mathcal{L}_V W$ and called the ***Lie derivative of W with respect to V***, by

$$\begin{aligned}(\mathcal{L}_V W)_p &= \left.\frac{d}{dt}\right|_{t=0} d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}) \\ &= \lim_{t\to 0} \frac{d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}) - W_p}{t},\end{aligned}$$

provided the derivative exists.



Lemma 9.36. Suppose M is a smooth manifold with or without boundary, and $V, W \in \mathfrak{X}(M)$. If $\partial M \neq \varnothing$, assume in addition that V is tangent to ∂M. Then $(\mathcal{L}_p W)_p$ exists for every $p \in M$, and $\mathcal{L}_V W$ is a smooth vector field.

Theorem 9.38. If M is a smooth manifold and $V, W \in \mathfrak{X}(M)$ then $\mathcal{L}_V W = [V, W]$.

Corollary 9.39. Suppose M is a smooth manifold with or without boundary, and $V, W, X \in \mathfrak{X}(M)$.

(a) $\mathcal{L}_V W = -\mathcal{L}_W V$.

(b) $\mathcal{L}_V [W, X] = [\mathcal{L}_V W, X] + [W, \mathcal{L}_V X]$.

(c) $\mathcal{L}_{[V,W]} X = \mathcal{L}_V \mathcal{L}_W X - \mathcal{L}_W \mathcal{L}_V X$.

(d) If $g \in C^\infty(M)$, then $\mathcal{L}_V(gW) = (Vg)W + g\mathcal{L}_V W$.

(e) If $F : M \to N$ is a diffeomorphism, then $F_*(\mathcal{L}_V X) = \mathcal{L}_{F_* V} F_* X$.

Proposition 9.41. Suppose M is a smooth manifold with or without boundaryand $< W \in \mathfrak{X}(M)$. If $\partial M \neq \varnothing$, assume also that V is tangent to ∂M. Let θ be the flow of V. For any (t_0, p) in the domain of θ,

$$\left.\frac{d}{dt}\right|_{t=t_0} d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}) = d(\theta_{-t_0})\big((\mathcal{L}_V W)_{\theta_{t_0}(p)}\big)$$

Definition. Let M be a smooth manifold and $V, W \in \mathfrak{X}(M)$. We say that ***V and W commute*** if $VWf = WVf$ for every smooth function f, or equivalently if $[V, W] \equiv 0$.

Definition. If θ is a smooth flow, a vector field W is said to be ***invariant under θ*** it W is θ_t-related to itself for each t, i.e., if

$$(d\theta_t)_p(W_p) = W_{\theta_t(p)}$$

for all (t, p) in the domain of θ.

Theorem 9.42. For smooth vector fields V and W on a smooth manifold M, the following are equivalent:

(a) V and W commute.

(b) W is invariant under the flow of V.

(c) V is invariant under the flow of W.

Corollary 9.43. Every smooth vector field is invariant under its own flow.

Definition. if θ, ψ are flows on M, we say that ***θ and ψ commute*** if for all $p \in M$ whenever $J, K \subseteq \mathbb{R}$ are open intervals containing 0 such that one of the expression $\theta_t \circ \psi_s(p)$ or $\psi_s \circ \theta_t(p)$ is defined for all $(s,t) \in J \times K$, both are defined and they are equal.

Theorem 9.44. Smooth vector fields commute if and only if their flows commute.

Definition. Suppose M is a smooth n-manifold. A smooth local frame (E_i) for M is called a ***commuting frame*** if $[E_i, E_j] = 0$ for all i and j.

Theorem 9.46 (Canonical Form for Commuting Vector Fields). Let M be a smooth n-manifold, and let $(V_1, \ldots, V_k)$ be a linearly independent k-tuple of smooth commuting vector fields on an open subset $W \subseteq M$. For each $p \in W$, there exists a smooth coordinate chart $(U, (s^i))$ centered at p such that $V_i = \frac{\partial}{\partial s^i}$ for $i = 1, \ldots, k$. If $S \subseteq W$ is an embedded codimension-k submanifold and p is a point of S such that T_pS is complementary to the span of $V_1|_p, \ldots, V_k|_p)$, then the coordinates can also be chosen such that $S \cap U$ is the slice define by $s^1 = \cdots = s^k = 0$.

Example 9.47. Consider the following two vector fields on $\mathbb{R}^2$:

$$V = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \qquad W = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

A computation shows that $[V, W] = 0$. And Example 9.23 showed us that the flow of V is given by

$$\theta_t(x,y) = (x\cos(t) - y\sin(t), x\sin(t) + y\cos(t),$$

and an easy verification show that the flow of W is

$$\eta_t(x,y) = (e^t x, e^t y).$$

At $p = (1,0)$, V_p and $W + p$ are linearly independent. Because $k = n = 2$ in this case, we can take the subset S to be the single point $\{(1,0)\}$, and define $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\Phi(s,t) = \eta_t \circ \theta_s(1,0) = (e^t\cos(s), e^t\sin(s)).$$

In this case, we can solve for $(s,t) = \Phi^{-1}(x,y)$ explicitly in a neighborhood of $(1,0)$ to obtain the coordinate map

$$(s,t) = \left(\tan^{-1}\left(\frac{y}{x}\right), \log\left(\sqrt{x^2+y^2}\right)\right).$$

11. The Cotangent Bundle

In this chapter, we will primarily be talking about the cotangent space at a point p. This can be thought of as either the space of linear functionals on the tangent space, or as the dual space to T_pM. The union of all these dual spaces will be called the cotangent bundle. We will discover that, in many ways, the cotangent spaces of manifolds behave much more nicely when acted on by smooth maps than the regular tangent spaces, and this will eventually lead us to a really nice formula for Stokes's Theorem in Chapter 16.

Definition. Let V be a finite dimensional vector space over $\mathbb{R}$. Then a ***covector*** is a linear functional on V.

Definition. The space of all covectors of a vector space V is called the ***dual space*** of V and is denoted V^*.

Proposition 11.1. Let V be a finite dimensional vector space. Given any basis $E_1, \ldots, E_n)$ for V, let $\varepsilon^1, \ldots, \varepsilon^n \in V^*$ be the covectors defined by

$$\varepsilon^i(E_j) = \delta^i_j,$$

where δ^i_j is he Kronecker delta. Then $(\varepsilon^1, \ldots, \varepsilon^n)$ is a basis for V^*, called he ***dual basis to (E_j)***. Therefore, $\dim(V^*) = \dim(V)$.

Definition. Let $A : V \to W$ be a linear map between two vector spaces. The ***dual map*** or ***transpose*** $A^* : W^* \to V^*$ is defined for all $\omega \in W^*$ and $v \in V$ by

$$(A^*\omega)(v) = \omega(Av).$$

Proposition 11.4. The dual map satisfies the following properties:

(a) $(A \circ B)^* = B^* \circ A^*$.

(b) $(\mathrm{Id}_V)^* : V^* \to V^*$ is the identity map of V^*.

Definition. Let M be a smooth manifold with or without boundary. For each $p \in M$ we define the ***cotangent space at p***, denoted by T_p^*M to be the dual space to T_pM:

$$T_p^*M = (T_pM)^*.$$

Elements of T_p^*M are called ***tangent covectors at p***, or just ***covectors at p***.

Definition. For any smooth manifold M with or without boundary, the disjoint union

$$T^*M = \bigsqcup_{p \in M} T_p^*M$$

is called the ***cotangent bundle of*** M.

Definition. Given a local coordinate chart (x^i) on $U \subseteq M$ the sections $dx^i : U \to T^*M$ defined by

$$dx^i(p) = dx^i|_p \in T^*M$$

are the ***coordinate covector fields*** on M. The give rise to natural coordinates for T^*U defined by

$$\pi^{-1}(U) = T^*U \subseteq T^*M.$$

Definition. A (local or global) section of T^*M is called a ***covector field*** of a ***(differential) 1-form***.

Definition. In any smooth local coordinates on an open subset $U \subseteq M$, a (rough) covector filed ω cam be written in terms of the coordinate covector fields (λ^i) as $\omega = \omega_i \lambda^i$ for n functions $\omega_i : U \to \mathbb{R}$ called the ***component functions of*** $\boldsymbol{\omega}$. They are characterized by

$$\omega_i(p) = \omega_p \left(\left. \frac{\partial}{\partial x^i} \right|_p \right).$$

Note: If ω is a (rough) covector field and X, then we can form a function $\omega(X) : M \to \mathbb{R}$ by

$$\omega(X)(p) = \omega_p(X_p), \qquad p \in M.$$

Proposition 11.5 (Smoothness Criteria for Covector Fields). Let M be a smooth manifold with or without boundary, and let $\omega : M \to T^*M$ be a rough covector field. The following are equivalent:

(a) ω is smooth.

(b) In every smooth coordinate chart, the component functions of ω are smooth.

(c) Each point of M is contained in some coordinate chart in which ω has smooth component functions.

(d) For every smooth vector field $X \in \mathfrak{X}(M)$, the function $\omega(X)$ is smooth on M

(e) For every open subset $U \subseteq M$ and every smooth vector field X on U, the function $\omega(X) : U \to \mathbb{R}$ is smooth on U.

Definition. let M be a smooth manifold with or without boundary, and let $U \subseteq M$ be an open subset. A ***local coframe for M over U*** is an ordered n-tuple of covector fields $(\varepsilon^1, \dots, \varepsilon^n)$ defined n U such that $(\varepsilon^i|_p)$ forms a basis for T_p^*M at each point $p \in U$. If $U = M$, it is called a ***global coframe***.

Note: We will denote the set of smooth covector fields on M by $\mathfrak{X}^*(M)$.

Definition. Le M be a smooth manifold and $f \in C^\infty(M)$. The ***differential of f*** is the covector field df on M defined by

$$df_p(v) = vf \quad \text{for } v \in T_pM.$$

Proposition 11.20 (Properties of the Differential). Let M be a smooth manifold with or without boundary, and let $f, g \in C^\infty(M)$.

(a) If a and B are constants, then $d(af_bg) = a\,df + b\,dg$.

(b) $d(fg) = f\,dg + g\,df$.

(c) $d(f/g) = (g\,df - f\,dg)/g^2$ on the set where $g \neq 0$.

(d) If $J \subseteq \mathbb{R}$ is an interval containing the image of f, and $h : J \to \mathbb{R}$ is a smooth function, then $d(h \circ f) = (h' \circ f)df$.

(e) If f is constant, then $df = 0$.

Proposition 11.22. If f is a smooth real-valued function on a smooth manifold M with or without boundary, then $df = 0$ if and only if f is constant on each component of M.

Proposition 11.23 (Derivative of a Function Along a Curve). Suppose M is a smooth manifold with or without boundary, $\gamma : J \to M$ is a smooth curve, and $f : M \to \mathbb{R}$ is a smooth function. Then the derivative of the real-valued function $f \circ \gamma : J \to \mathbb{R}$ is given by

$$(f \circ \gamma)'(t) = df_{\gamma(t)}(\gamma'(t)).$$

Definition. Let $F > M \to N$ be a smooth map between smooth manifolds with or without boundary, and let $p \in M$ be arbitrary. The differential $dF_p : T_pM \to T_{F(p)}N$ yields a dual linear map

$$dF_p^* : T_p^*N \to T_p^*M,$$

called the ***(pointwise) pullback by F at p***, or the ***cotangent map of F***.

Definition. Given a smooth map $F : M \to N$ and a covector field ω on N, define a rough covector field $F^*\omega$ on M, called the ***pullback of ω by F***, by

$$(F^*\omega)_p = dF_p^*(\omega_{F(p)}).$$

It acts on a vector $v \in T_pM$ by

$$(F^*\omega)_p(v) = \omega_{F(p)}(dF_p(v)).$$

Proposition 11.25. Let $F : M \to N$ be a smooth map between smooth manifolds with or without boundary. Suppose u is a continuous real-valued function on N, and ω is a covector field on N. Then

$$F^*(u\omega) = (u \circ F)F^*\omega.$$

If in addition u is smooth, then

$$F^*du = d(u \circ F).$$

Definition. If $S \subseteq M$ is an immersed submanifold and $\iota : S \hookrightarrow M$ is the inclusion map, and $\omega \in \mathfrak{X}^*(M)$, then $\iota^*\omega \in \mathfrak{X}^*(S)$. And if $v \in T_pS \subseteq T_pM$ then $\iota^*\omega(v) = \omega(d\iota_*(v))\omega(v)$, so $\iota^*\omega$ is just the restriction of ω to T_pS.

Definition. Suppose $[a, b] \subseteq \mathbb{R}$ is a compact interval, and ω is a smooth covector on $[a, b]$. If we let t denote the standard coordinate on $\mathbb{R}$, then ω can be written $\omega_t = f(t)dt$ for some smooth function $f : [a, b] \to \mathbb{R}$. We define the ***integral of ω over* [a,b]** to be

$$\int_{[a,b]} \omega = \int_a^b f(t)\, dt.$$

Definition. Let M be a manifold with or without boundary, we define a ***curve segment in M*** to be a continuous curve $\gamma[a, b] \to M$ whose domain is a compact interval, and a ***smooth curve segment*** to be a smooth continuous curve $\gamma[a, b] \to M$ when $[a, b]$ is considered as a manifold with boundary.

Definition. If $\gamma : [a, b] \to M$ is a smooth curve segment and ω is a smooth covector field on M, we define the ***line interval of ω over γ*** to be the real number

$$\int_\gamma \omega = \int_{[a,b]} \gamma^*\omega.$$

Example 11.36 (Common Example). Let $M = \mathbb{R}^2 \backslash \{0\}$, let ω be the covector field on M given by

$$\omega = \frac{x\,dy - y\,dx}{x^2 + y^2},$$

and let $\gamma : [0, 2\pi] \to M$ be the curve segment defined by $\gamma(t) = (\cos(t), \sin(t))$. Since $\gamma^*\omega$ can be computed by substituting $x = \cos(t)$ and $y = \sin(t)$ everywhere in the formula for ω, we find that

$$\int_\gamma \omega = \int_{[0,2\pi]} \frac{\cos(t)(\cos(t)\,dt) - \sin(t)(-\sin(t)\,dt)}{\sin(t)^2 + \cos(t)^2} = \int_0^{2\pi} dt = 2\pi.$$

Proposition 11.38. If $\gamma : [a, b] \to M$ is a piecewise smooth curve segment, the line integral of ω over γ also be expressed as the ordinary integral

$$\int_\gamma \omega = \int_a^b \omega_{\gamma(t)}(\gamma'(t))\,dt.$$

Theorem 11.39 (Fundamental Theorem for Line Integrals). Let M be a smooth manifold with or without boundary. Suppose f is a smooth real-valued function on M and $\gamma : [a, b] \to M$ is a piecewise smooth curve segment in M. Then

$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a)).$$

Definition. A smooth covector field ω on a smooth manifold M with or without boundaryis said to be ***exact*** on M if there is a function $f \in C^\infty(M)$ such that $\omega = df$. In this case, the function f is called a ***potential for ω***.

Definition. We say that a smooth covector field ω is ***conservative*** if the line integral of ω over every piecewise smooth closed curve segment is zero.

Proposition 11.40. A smooth covector field ω is conservative if and only if its line integrals are path-independent.

Theorem 11.42. Let M be a smooth manifold with or without boundary. A smooth covector field on M is conservative if and only if it is exact.

Corollary. If $\omega = df$ is exact then

$$\omega = a_i(x)dx^i$$

where

$$a_i(x) = \frac{\partial f}{\partial x^i}(x).$$

Then for all i, j we have

$$\frac{\partial a_i}{\partial x^j} = \frac{\partial a_j}{\partial x^i} = \frac{\partial^2 f}{\partial x^i \partial x^j}.$$

Definition. A covector field $\omega = a_i dx^i$ is called ***closed*** if

$$\frac{\partial a_i}{\partial x^j} = \frac{\partial a_j}{\partial x^i}.$$

Proposition 11.44. Every exact covector field is closed.

Proposition 11.45. Let ω be a smooth covector field on a smooth manifold M with or without boundary. The following are equivalent:

(a) ω is closed.

(b) ω satisfies

$$\frac{\partial \omega_j}{\partial x^i} = \frac{\partial \omega_i}{\partial x^j}$$

for all i, j in some smooth chart around every point.

(c) For any open subset $U \subseteq M$ and smooth vector fields $X, Y \in \mathfrak{X}(U)$,

$$(X(\omega(Y)) - Y(\omega(X)) = \omega([X, Y]).$$

Theorem 11.49 (Poincarѐ Lemma for Covector Fields). If M is simply connected then every closed covector filed on M is exact.

Corollary 11.50 (Local Exactness of Closed Covector Fields). Let ω be a closed covector field on a smooth manifold M with or without boundary. Then every point of M has a neighborhood on which ω is exact.

Note: Pullbacks preserve closedness and exactness.

14. Differential Forms

Differential forms are an essential and beautiful part of Differential Geometry. They are, in fact, the objects that we will use in order to thoroughly define the concept of integration on an arbitrary smooth manifold. And, as we will find in Chapter 16, their very existence is what leads to a very elegant and concise way of expressing Stokes's Theorem.

Definition. Let V, W be finite dimensional vector fields, and let $\alpha \in V^*$ and $\beta \in W^*$. The ***tensor product*** $\alpha \otimes \beta$ is a bilinear product

$$\alpha \otimes \beta : V \times W \to \mathbb{R} : (v, w) \mapsto \alpha(v)\beta(w).$$

Note: To define $v \otimes w$ for $v \in V$ and $w \in W$, use the canonical identification with the double dual.

Definition. Let $\alpha, \beta \in V^*$. The ***wedge product*** of α and β is defined by

$$\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha.$$

Definition. The space of ***alternating k-vectors*** or ***alternating k-covectors*** of V^* is

$$\Lambda^k V^* = \text{span}\{\alpha^1 \wedge \cdots \wedge \alpha^k \ : \ \alpha^i \in V^*\}$$

Note: For $0 \leq k \leq n = \dim(V)$, $\dim(\Lambda^k V^*) = \binom{n}{k}$

Proposition 14.11 (Properties of the Wedge Product). Suppose $\omega, \omega', \eta, \eta'$, and ξ are multicovectors on a finite-dimensional vector space V.

(a) Binlearity: For $a, a' \in \mathbb{R}$,

$$[(a\omega + a'\omega') \wedge \eta = a(\omega \wedge \eta) + a'(\omega' \wedge \eta),$$
$$\eta \wedge (a\omega + a'\omega') = a(\eta \wedge \omega) + a'(\eta \wedge \omega').$$

(b) Associativity:

$$\omega \wedge (\eta \wedge \xi) = (\omega \wedge \eta) \wedge \xi.$$

(c) Anticommutativity: For $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^\ell(V^*)$

$$\omega \wedge \eta = (-1)^{k\ell} \eta \wedge \omega.$$

(d) If (ε^i) is any basis for V^* and $I = (i_1, \ldots, i_k)$ is any multi-index, then

$$\varepsilon^{i_1} \wedge \varepsilon^{i_k} = \varepsilon^I.$$

(e) For any covectors $\omega^1, \ldots, \omega^k$ and vectors $v_1, \ldots, v_k$,

$$\omega^1 \wedge \cdots \wedge (v_1, \ldots, v_k) = \det(\omega^j(v_i)).$$

Definition. A k-covector η is ***decomposable*** if it can be expressed as

$$\eta = \omega^1 \wedge \cdots \wedge \omega^k$$

where ω^i is a covector.

Definition. for any n-dimensional vector space V, the ***exterior algebra*** of V is the vector space

$$\Lambda(V^*) = \bigotimes_{k=0}^{n} \Lambda^k V$$

Definition. Let V be a finite-dimensional vector space. For each $v \in V$, we define a linear map $i_v : \Lambda^k(V^*) \to \Lambda^{k-1}(V^*)$, called ***interior multiplication by v***, as follows:

$$i_v\omega(w_1, \ldots, w_{k-1}) = \omega(v, w_1, \ldots, w_{k-1}).$$

In other words, $i_v\omega$ is obtained from ω by inserting v into the first slot. This can also be denoted by $v \lrcorner \omega$.

Lemma 14.13. Let V be a finite-dimensional vector space and $v \in V$.

(a) $i_v \circ i_v = 0$.

(b) If $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^\ell(V^*)$,

$$i_v(\omega \wedge \eta) = (i_v\omega) \wedge \eta + (-1)^k \omega \wedge (i_v\eta)$$

Definition. The ***bundle of alternating k-tensors*** on a smooth manifold M is

$$\Lambda^k(T^*M) = \bigsqcup_{p \in M} \Lambda(T_p^*M).$$

Definition. A ***differential k-form*** on M is a continuous section of $\Lambda^k(T^*M)$. The vector space of smooth k-forms is denoted

$$\Omega^k(M) = \Gamma(\Lambda(T^*M)),$$

and the integer k is called the ***degree*** of the form.

Definition. WE define the vector space $\Omega^*(M)$ to be

$$\Omega^*(M) = \bigoplus_{k=0}^{n} \Omega^k(M),$$

and this is an associative, anticommutative graded algebra.

Lemma 14.16. Suppose $F : M \to N$ is smooth.

(a) $F^* : \Omega^k(N) \to \Omega^k(M)$ is linear over R.

(b) $F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta)$.

(c) In any smooth chart

$$F^*\left(\sum_I{}' \omega_i \, dy^{i_1} \wedge \cdots \wedge dy^{i_k}\right) = \sum_I{}' (\omega_i \circ F)\, d(y^{i_1} \circ F) \wedge \cdots \wedge d(y^{i_k} \circ F)$$

Proposition 14.20 (Pullback Formula for Top-Degree Forms). Let $F : M \to N$ be a smooth map between n-manifolds with or without boundary. If (x^i) and (y^j) are smooth coordinates on open subsets $U \subseteq M$ and $V \subseteq N$, respectively, and u is a continuous real-valued function on V, then the following holds on $U \cap F^{-1}(V)$:

$$F^*(u\, dy^1 \wedge \cdots \wedge dy^n) = (u \circ F)(\det(DF))\, dx^1 \wedge \cdots \wedge dx^n,$$

where DF represents the Jacobian matrix of F in these coordinates.

Definition. Let $\omega = \sum_{|I|=k} f_I \, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ be a k-form on $\mathbb{R}^n$. The ***exterior derivative*** of ω is the $(k+1)$-form

$$d\omega = \sum_{|I|=k} df_I \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Proposition 14.23 (Properties of the Exterior Derivative on $\mathbb{R}^n$).

(a) d is linear over $\mathbb{R}$.

(b) If ω is a smooth k-form and η is a smooth 1-form on an open subset $U \subseteq \mathbb{R}^n$ or $\mathbb{H}^n$, then

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

(c) $d \circ d \equiv 0$

(d) d commutes with pullbacks.

Theorem 14.24 (Existence and Uniqueness of Exterior Differentiation). Suppose M is a smooth manifold with or without boundary. There are unique operators $d : \Omega^k(M) \to \Omega^{k+1}(M)$ for all k, called ***exterior differentiation***, satisfying the following four properties:

(i) d is linear over $\mathbb{R}$.

(ii) IF $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$, then

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge \eta.$$

(iii) $d \circ d \equiv 0$.

(iv) for $f \in \Omega^0(M) = C^\infty(M)$, df is the differential of f, given by $df(X) = Xf$.

Proposition 14.26 (Naturality of the Exterior Derivative). If $F : M \to N$ is a smooth map, then for each k the pullback map $F^* : \Omega^k(N) \to \Omega^k(M)$ commutes with d: for all $\omega \in \Omega^k(N)$,

$$F^*(d\omega) = d(F^*\omega).$$

Now, the particularly astute will notice that the exterior derivative seems to act very similarly to the vector field operations that we encountered in Calculus III. This is no happy accident! In fact, when we define the three operations

$$\begin{aligned} \flat &: \mathfrak{X}(\mathbb{R}^3) \to \Omega^1(\mathbb{R}^3) : X^i e^i \mapsto X_i e^i \\ \beta &: \mathfrak{X}(\mathbb{R}^3) \to \Omega^1(\mathbb{R}^3) : X \mapsto X \lrcorner (dx \wedge dy \wedge dz) \\ * &: C^\infty(\mathbb{R}^3) \to \Omega^3(\mathbb{R}^3) : f \mapsto f\, dx \wedge dy \wedge dz, \end{aligned}$$

then we can construct the following pretty commutative diagram:

$$\begin{tikzcd} C^\infty(\mathbb{R}^3) \arrow[r, "\mathrm{grad}"] \arrow[d, "\mathrm{Id}"] & \mathfrak{X}(\mathbb{R}^3) \arrow[r, "\mathrm{div}"] \arrow[d, "\flat"] & \mathfrak{X}(\mathbb{R}^3) \arrow[r, "\mathrm{curl}"] \arrow[d, "\beta"] & C^\infty(\mathbb{R}^3) \arrow[d, "*"] \\ \Omega^0(\mathbb{R}^3) \arrow[r, "\mathrm{d}"'] & \Omega^1(\mathbb{R}^3) \arrow[r, "\mathrm{d}"'] & \Omega^2(\mathbb{R}^3) \arrow[r, "\mathrm{d}"'] & \Omega^3(\mathbb{R}^3) \end{tikzcd}$$

This is really nice to keep in mind whenever you are teaching Calc. III.

Proposition 14.29 (Exterior Derivative of a 1-Form). For any smooth 1-form ω and smooth vector fields X and Y,

$$d\omega(X, Y) = (X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]).$$

Proposition 14.30. Let M be a smooth n-manifold with or without boundary, let (E_i) be a smooth local frame for M, and let (ε^i) be the dual coframe. For each i, let b^i_{jk} denote the component functions of the exterior derivative of ε^i in this frame , and for each j, k, let c^i_{jk} be the component functions of the Lie bracket $[E_j, E_k]$:

$$d\varepsilon^i = \sum_{j<k} b^i_{jk} \varepsilon^j \wedge \varepsilon^k; \qquad [E_j, E_k] = c^i_{jk} E_i.$$

Then $b^i_{jk} = -c^i_{jk}$.

Definition. A k-form on $\mathbb{R}^n$ is called ***closed*** if $d\omega = 0$ and ***exact*** if $\omega = d\eta$ for some $(k-1)$-form η on $\mathbb{R}^n$.

Proposition 14.33. Suppose M is a smooth manifold $V \in \mathfrak{X}(M)$, and $\omega\eta \in \Omega^*(M)$. Then

$$\mathcal{L}_V(\omega \wedge \eta) = \mathcal{L}_V \omega \wedge \eta + \omega \wedge (\mathcal{L}_V \eta).$$

Theorem 14.35 (Cartan's Magic Formula). On a smooth manifold M, for any smooth vector field V and any smooth differential form ω,

$$\mathcal{L}_V \omega = V \lrcorner (d\omega) + d(V \lrcorner \omega).$$

Corollary 14.36. If V is a smooth vector filed and ω is a smooth differential form, then

$$\mathcal{L}_V(d\omega) = d(\mathcal{L}_V \omega).$$

15. Orientations

This section is mainly here to provide structure and terminology for what is to come in Chapter 16.

Definition. Let V be a real vector space of dimension $n \geq 1$. We say that two ordered bases (E_i) and $\widetilde{E}_j)$ for V are ***consistently oriented*** if the transition matrix (B_i^j), defined by

$$E_i = B_i^j \widetilde{E}_j,$$

has positive determinant.

Note: The motivation for this definition is that we want things with the same orientation to not flip the sign on the volume form.

Definition. If $\dim(V) = N \geq 1$, we define an ***orientation for V*** as an equivalence class of ordered bases. If $(E_1, \ldots, E_n)$ is any ordered basis for V, we denote the orientation that it determines by $[E_1, \ldots, E_n]$. A vector space together with a choice of orientation is called an ***oriented vector space***. If V is oriented, then any ordered basis $(E_1, \ldots, E_n)$ that is in the given orientation is said to be ***positively oriented***.

Definition. Let M be a smooth manifold with or without boundary. We define a ***pointwise orientation*** on M to be a choice of orientation of each tangent space.

Definition. Let M be a smooth n-manifold with or without boundary, endowed with a pointwise orientation. If (E_i) is a local frame for TM, we say that (E_i) is ***(positively) oriented*** if $(E_1|_p, \ldots E_n|_p)$ is a positively oriented basis for T_pM at each point.

Proposition 15.5 (The Orientation Determined by an n-form). Let M be a smooth n-manifold with or without boundary. Any nonvanishing N-form ω on M determines a unique orientation of M for which ω is positively oriented at each point. Conversely, if M is given an orientation, then there is a smooth nonvanishing n-form on M that is positively oriented at each point.

Definition. Any nonvanishing n-form on M is called an ***orientation form***.

Definition. A smooth coordinate chart on an oriented smooth manifold with or without boundaryis said to be ***(positively) oriented*** if the coordinate frame $\left(\frac{\partial}{\partial x^i}\right)$ is positively oriented. A smooth atlas $\{(U_\alpha, \varphi_\alpha)\}$ is said to be ***consistently oriented*** if for each α, β the transition map $\varphi_\beta \circ \varphi_\alpha^{-1}$ has positive Jacobian determinant everywhere on $\varphi_\alpha(U_\alpha \cap U_\beta)$.

Proposition 15.6 (The Orientation Determined by a Coordinate Atlas). Let M be a smooth positive-dimensional manifold with or without boundary. Given any consistently oriented smooth atlas for M, there is a unique orientation for M with the property that each chart in the given atlas is positively oriented. Conversely, if M is oriented and either $\partial M = \varnothing$ or $\dim(M) > 1$, then the collection of all oriented smooth charts is a consistently oriented atlas for M.

Definition. Let M and N be oriented smooth manifold with or without boundary, and suppose $F : M \to N$ is a local diffeomorphism. If M and N are positive-dimensional, we say that F is ***orientation-preserving*** if for each $p \in M$, the isomorphism dF_p takes oriented basis of T_pM to oriented basis of $T_{F(p)}N$.

Proposition 15.15 (The Pullback Orientation). Suppose M and N are smooth manifolds with or without boundary. If $F : M \to N$ is a local diffeomorphism and N is oriented, then M has a unique orientation, called the ***pullback orientation induced by F***, such that F is orientation-preserving.

Proposition 15.17. Every parallelizable smooth manifold is orientable.

Proposition 15.21. Suppose M is an oriented smooth n-manifold with or without boundary, S is an immersed hypersurface with or without boundaryin M, and N is a vector field along S that is nowhere tangent to S. Then S has a unique orientation such that for each $p \in S$, $(E_1, \ldots, ER_{n-1})$ is an oriented basis for T_pS if and only if $(N_p, E_1, \ldots, E_{n-1})$ is an oriented basis for T_pM. If ω is an orientation form for M, then $\iota_S^*(N \lrcorner \omega)$ is an orientation form for S with respect to this orientation, where $\iota_S : S \hookrightarrow M$ is inclusion.

Proposition 15.24 (The Induced Orientation on a Boundary). Let M be an oriented smooth n-manifold with boundary, $n \geq 1$. Then ∂M is orientable, and all outward-pointing vector fields along ∂M determine the same orientation on ∂M.

16. Integration on Manifolds

Really all we want to talk about here is Stokes's Theorem.

Proposition 16.1. Suppose D and E are open domains of integration in $\mathbb{R}^n$ or $\mathbb{H}^n$, and $G : \overline{D} \to \overline{E}$ is a smooth map that restricts to an orientation-preserving or orientation-reversing diffeomorphism from D to E. If ω is an n-form on $\overline{E}$, then

$$\int_D G^*\omega = \begin{cases} \int_E \omega & \text{if } G \text{ is orientation-preserving,} \\ -\int_E \omega & \text{if } G \text{ is orientation-reversing.} \end{cases}$$

Lemma 16.2. Suppose U is an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$, and K is a compact subset of U. Then there is an open domain of integration D such that $K \subseteq D \subseteq \overline{D} \subseteq U$.

Proposition 16.3. Suppose U, V are open subsets of $\mathbb{R}^n$ or $\mathbb{H}^n$, and $G : U \to V$ is an orientation preserving or orientation reversing diffeomorphism. If ω is a compactly supported n-form on V, then

$$\int_V \omega = \pm \int_U G^*\omega,$$

with the positive sing if G is orientation-preserving, and the negative sign otherwise.

Definition. Suppose that ω is an n-form on M that is compactly supported in the domain of a single smooth chart (U, φ) that is either positively or negatively oriented. We define the ***integral of ω over M*** to be

$$\int_M \omega = \pm \int_{\varphi(U)} (\varphi^{-1})^*\omega.$$

Proposition 16.4. With ω as above, $\int_M \omega$ does not depend on the choice of smooth chart whose domain contains $\operatorname{supp}(\omega)$.

Proposition 16.5. The definition of $\int_M \omega$ above does not depend on the choice of chart or partition of unity.

Proposition 16.6 (Properties of Integrals of Forms). Suppose M and N are non-empty oriented smooth n-manifolds with or without boundary, and ω, η are compactly supported n-forms on M.

(a) Linearity: If $a, b \in \mathbb{R}$, then

$$\int_M a\omega + b\eta = a\int_m \omega + b\int_M \eta.$$

(b) Orientation Reversal: If $-M$ denotes M with the opposite orientation, then

$$\int_{-M} \omega = -\int_M \omega.$$

(c) Positivity: If ω is a positively oriented orientation form, then $\int_M \omega > 0$.

(d) Diffeomorphism Invariance: If $F : N \to M$ is an orientation-preserving or orientation reversing diffeomorphism

$$\int_M \omega = \begin{cases} \int_N F^*\omega & \text{if } F \text{ is orientation-preserving,} \\ -\int_N F^*\omega & \text{if } F \text{ is orientation-reversing.} \end{cases}$$

Theorem 16.11 (Stokes's Theorem). Let M be an oriented smooth n-manifold with boundary, and let ω be a compactly supported smooth $(n-1)$-form on M. Then

$$\int_m d\omega = \int_{\partial M} \omega.$$

Corollary 16.13 (Integrals of Exact Forms). If M is a compact oriented smooth manifold without boundary, then the integral of every exact form over M is zero:

$$\int_M d\omega = 0 \quad \text{if } \partial M = 0.$$

Corollary 16.14 (Integrals of Closed Forms over Boundaries). Suppose M is a compact oriented smooth manifold with boundary. If ω is a closed form on M, then the integral of ω over ∂M is zero:

$$\int_{\partial M} \omega = 0 \quad \text{if } d\omega = 0 \text{ on } M.$$

Corollary 16.15. Suppose M is a smooth manifold with or without boundary, $S \subseteq M$ is an oriented compact smooth k-dimensional submanifold (without boundary), and ω is a closed k-form on M. If $\int_S \omega \neq 0$, then both of the following are true:

(i) ω is not exact on M.

(ii) S is not the boundary of an oriented compact smooth submanifold with boundary in M.