MATH 3430-02 WEEK 8-1

Key Words: Series solutions I: Successive Differentiation; Series solutions II: Undetermined coefficients (pt. 1).

Assuming that a solution $y(t)$ of the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0$$

to be in a power series (centered at t_0)

$$y(t) = a_0 + a_1(t - t_0) + \cdots + a_k(t - t_0)^k + \cdots ,$$

we may be able to find the a_i, consquently determining a solution (in series form) of the ODE.

For simplicity, we consider $y(t)$ to be a power series centered at $t_0 = 0$:

$$y(t) = a_0 + a_1 t + \cdots + a_k t^k + \cdots$$

Q1. If $y(t)$ defines a differentiable function, then

$$\begin{aligned} y(0) &= \underline{\hspace{2cm}}; \\ y'(0) &= \underline{\hspace{2cm}}; \\ y''(0) &= \underline{\hspace{2cm}}; \\ &\vdots \\ y^{(k)}(0) &= \underline{\hspace{2cm}}; \\ &\vdots \end{aligned}$$

Q2. Consider the initial value problem

$$y'' + y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

The equation implies that

$$\begin{aligned} y(0) &= \underline{\hspace{2cm}}; \\ y'(0) &= \underline{\hspace{2cm}}; \\ y''(0) &= \underline{\hspace{2cm}}; \\ y'''(0) &= \underline{\hspace{2cm}}; \\ y^{(4)}(0) &= \underline{\hspace{2cm}}; \\ &\vdots \\ y^{(4k)}(0) &= \underline{\hspace{2cm}}; \\ y^{(4k+1)}(0) &= \underline{\hspace{2cm}}; \\ y^{(4k+2)}(0) &= \underline{\hspace{2cm}}; \\ y^{(4k+3)}(0) &= \underline{\hspace{2cm}}; \\ &\vdots \end{aligned}$$

It follows that

$$\begin{aligned} y(t) &= a_0 + a_1 t + \cdots + a_k t^k + \cdots \\ &= y(0) + y'(0)t + \underline{\hspace{3cm}} + \cdots + \underline{\hspace{3cm}} + \cdots \\ &= \underline{\hspace{5cm}}. \end{aligned}$$

This is the Taylor series of

$$y(t) = \underline{\hspace{5cm}}.$$

The method above is called *successive differentiation*, for an obvious reason.

Q3. Use the method of successive differentiation to determine a series solution of

$$y'' - ty' + t^2 y = 0, \qquad y(0) = y'(0) = 1$$

up to the t^4 term.

Note that further differentiation would yield rather complicated expressions.

However, there is a different approach that does not seem to need much differentiation.

Q4. Letting

$$y(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_k t^k + \cdots$$

We have

$$-ty'(t) = \underline{\hspace{7cm}}$$

and

$$y''(t) = \underline{\hspace{7cm}}.$$

Therefore,

$$\begin{aligned} 0 = y'' - ty' + t^2 y = {} & \underline{\hspace{4cm}} \\ & + \underline{\hspace{4cm}}\, t \\ & + \underline{\hspace{4cm}}\, t^2 \\ & + \underline{\hspace{4cm}}\, t^3 \\ & + \underline{\hspace{4cm}}\, t^4 \\ & + \underline{\hspace{4cm}}\, t^5 \\ & + \underline{\hspace{4cm}}\, t^6 \\ & \cdots \end{aligned}$$

Now

$$a_0 = \underline{\hspace{2cm}}, \qquad a_1 = \underline{\hspace{2cm}}.$$

It follows that

$$a_2 = \_\_\_\_, \ a_3 = \_\_\_\_, \ a_4 = \_\_\_\_, \ a_5 = \_\_\_\_, \cdots$$

and that

$$y(t) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_.$$

This method is called the method of *undetermined coefficients*.
Next time, we'll introduce a more efficient way to apply this method.

Q5. Use the method of undetermined coefficients to find the first 10 terms in a series solution of

$$y'' - ty = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$