MATH 3430-02 WEEK 3-2

Key Words: How accurate is 'Euler'? (Background in the proof: *Lagrange Remainder Theorem, Mean Value Theorem, An Estimate of a sequence* E_k *satisfying* $E_{k+1} = AE_k + B$ *and* $E_0 = 0$, $A, B > 0$.)

Note: Q3-Q11 below are for the derivation of the estimate (∗∗) and for defining the the notations therein. We'll only get to details of these if we have time. If we don't, you are encouraged to try to work through it yourself, and afterwards comparing your answers with pp. 100-102 of the textbook.

Review.

Recall the idea of Euler's method: Knowing the position of a point (t, y)*, we can find the derivative of a solution* $y(t)$ *at that point using the ODE itself* $y' = f(t, y)$*. In other words, we can find the derivative* **at a point** *without taking any derivative. However, to find* $y(t')$ *at a nearby* t'*, Euler's method assumes that* $y'(t)$ *does not change between* t *and* t'*. This is where approximation errors arise.*

Q1. For the 1-st order initial value problem

$$y' = x^2 + y^2, \qquad y(1) = 0,$$

what is the Euler-approximation of $y(3)$ with $n = 2$ steps?

Q2. Is there a $\Delta t > 0$ such that applying Euler's method to solving the initial value problem

$$y' = \cos t + 1, \qquad y(0) = \pi$$

yields an *accurate* solution?

In the two questions above, we don't know the true solution of the IVP in **Q1**; furthermore, we know whether the Euler-approximation is accurate in **Q2** only because we can determine the true solution. Needless to say, one only uses Euler's method in practice because a true solution is difficult to find. But, in this case,

How do you know how accurate an Euler-approximation is?

This is what we'll try to answer this time.

Two prerequisites:

I. Given a 1-st order initial value problem, the Existence and Uniqueness Theorem tells us that, if there is a rectangular region $[t_0, t_0 + a] \times [y_0 - b, y_0 + b]$ on which both f and $\partial f/\partial t$ are continuous, then we can find an interval $[t_0, t_0 + \alpha]$ on which solution exists and is unique. Here α is _________, where _____ is ________________.

It is reasonable to apply Euler's approximation to find $y(T)$ only for those T that belong to the interval $[t_0, t_0+\alpha]$. The reason is obvious: *How can we claim that an approximation is good enough, if we don't even know that a true solution exists and is unique at $t = T$?*

II. In our estimate of error below, we will use the following theorem from Calculus:

Lagrange Remainder Theorem. Let $y(t)$ be a differentiable function. Its d-th Taylor Polynomial, centered at t_0, is, by definition,

$$P_{t_0}^{(d)}(t) := y(t_0) + y'(t_0)(t - t_0) + \frac{y''(t_0)}{2!}(t - t_0)^2 + \cdots + \frac{y^{(d)}(t_0)}{k!}(t - t_0)^d.$$

(Note: $P_{t_0}^{(d)}(t)$ and $y(t)$ has the same derivative at $t = t_0$ up to the d-th order.)

Now, fixing any $t_1 > t_0$, there exists a number $\xi \in [t_0, t_1]$ such that

$$y(t_1) - P_{t_0}^{(d)}(t_1) = \frac{y^{(d+1)}(\xi)}{(d+1)!}(t_1 - t_0)^{d+1}.$$

(Note: The Lagrange Remainder Theorem is a direct consequence of **Cauchy's Mean Value Theorem***, which is in term a consequence of the* **Fermat's Theorem***.)*

Q3. Let $y(t) = 1 + t^2$, $t_0 = 1$, what is $P_{t_0}^{(1)}(t)$? _________. What is $P_{t_0}^{(1)}(2)$?_________ What is $y(2) - P_{t_0}^{(1)}(2)$?_________ Now simplify the expression $\dfrac{y^{(d+1)}(\xi)}{(d+1)!}(t_1 - t_0)^{d+1}$ for $d = 1$, $t_0 = 1$ and $t_1 = 2$:_________. Compare your results with the *Lagrange Remainder Theorem.*

Now we apply the Lagrange Remainder Theorem to analyze the Euler Error. First we fix some notations.

- The initial value problem is

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0,$$

- If possible, let α be the size of the existence interval computed using the 'E&U Theorem'.
- Let $\mathcal{R}$ be the rectangular region $[t_0, t_0 + a] \times [y_0 - b, y_0 + b]$.
- Let $T \in (t_0, t_0 + \alpha]$.
- The step number is n, with

$$t_0 < t_1 < \cdots < t_n = T.$$

- The step-size is a constant

$$h := t_{k+1} - t_k = \frac{T - t_0}{n}.$$

- Let $y(t_k)$ be the **true value** of the solution at t_k.
- The Euler approximation of $y(t_k)$ is y_k, iterated by the formula

$$y_{k+1} = y_k + f(t_k, y_k) \cdot h.$$

- The error at t_k is $E_k := |y(t_k) - y_k|$. Our goal is to estimate E_n (i.e., error at T).

Q4. Use the ODE $y' = f(t, y)$ to find an expression for

$$y''(t) = \underline{\hspace{3cm}}.$$

Q5. By definition, the 1-st Taylor Polynomial of $y(t)$, centered at t_k is

$$P_{t_k}^{(1)}(t) = \underline{\hspace{4cm}}$$

$$\textit{(using } y' = f(t,y)\textit{)} = \underline{\hspace{4cm}}.$$

Therefore,

$$P_{t_k}^{(1)}(t_{k+1}) = \underline{\hspace{1.5cm}} + \left(\underline{\hspace{2.5cm}}\right) h.$$

By the Lagrange Remainder Theorem, there exists a $\xi_k \in [t_k, t_{k+1}]$ such that

$$\boxed{y(t_{k+1})} = P_{t_k}^{(1)}(t_{k+1}) + \underline{\hspace{3.5cm}}$$

$$\textit{(Expanding } P_{t_k}^{(1)}(t_{k+1})\textit{)} = \underline{\hspace{6.5cm}}.$$

Q6. By the iterative formula, we have

$$\boxed{y_{k+1}} = y_k + \underline{\hspace{3cm}}.$$

Q7. By **Q5** and **Q6**, we have

$$\boxed{y(t_{k+1}) - y_{k+1}} = \boxed{y(t_k) - y_k} + h\left(\underline{\hspace{3.5cm}}\right) + \frac{h^2}{2}\left(\underline{\hspace{3cm}}\right).$$

Q8. By the mean value theorem (or the Lagrange Remainder Theorem applied to the y-variable and $d = 0$), the expression in the first parenthesis above equals to

$$f(t_k, y(t_k)) - f(t_k, y_k) = \left(\frac{\partial f}{\partial y}(t_k, \eta_k)\right)(y(t_k) - y_k),$$

for some $\eta_k \in [t_k, t_{k+1}]$.

Q9. From **Q7**, using the general inequality of absolute values $|a + b + c| \leq |a| + |b| + |c|$ and the meaning of the notations E_k, we obtain

$$E_{k+1} \leq E_k + h \cdot \underline{\hspace{1cm}} \cdot \left|\underline{\hspace{1.5cm}}\right| + \frac{h^2}{2}\left|\underline{\hspace{2.5cm}}\right|.$$

You should recognize that the two expressions inside absolute value above as expressions of f and its derivatives, whose expressions we know completely, and whose maximum-possible values we should be able to derive. Therefore, let L, D be such that

$$(*) \quad \boxed{L \geq \max_{(t,y)\in\mathcal{R}} \left|\frac{\partial f}{\partial y}\right|; \qquad D \geq \max_{(t,y)\in\mathcal{R}} \left|\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial y}\right|.}$$

Q10. We can derive from **Q9** and these new notations L, D the inequality:

$$E_{k+1} \leq (1 + \underline{\hspace{1cm}})E_k + \frac{h^2}{2}\underline{\hspace{1cm}}.$$

Now we need to apply a

Lemma. *If* $F_{k+1} \le AF_k + B$, *where* $A, B > 0$, *and* $F_0 = 0$, *then*

$$F_k \le \frac{B}{A-1}(A^k - 1).$$

Q11. Following from the lemma and **Q10**, we have

$$\begin{aligned} E_k &\le \underline{\hspace{4cm}} \\ &= \frac{Dh}{2L}\left((1+hL)^k - 1\right) \\ (\textbf{b.c. } 1 + hL < e^{hl}) \quad &\le \frac{Dh}{2L}(e^{\boxed{}} - 1) \\ (\textbf{b.c. } kh < \alpha) \quad &\le \frac{Dh}{2L}(\underline{\hspace{1.5cm}} - 1). \end{aligned}$$

Since the last expression is independent of k, we have

$$(**) \quad \boxed{E_n \le \frac{Dh}{2L}(e^{\alpha L} - 1).}$$

This is the error estimate of Euler's approximation that we'll use. *In particular, this estimate tells you how small h should be in order to guarantee a small error.*

Q12. (Sorry, this question has an error.) Previously we have applied Euler Approximation to

$$y' = y, \qquad y(0) = 1,$$

and found that, when $T = 1$ and $n = 5$, $y_5 \approx 2.488$. How close is this approximation to the true value?

Now choosing $a = 1$, $b = 1$, we find

$$M = \max_{(t,y)\in\mathcal{R}} |f(t,y)| = \underline{\hspace{3cm}}.$$

Therefore,

$$\alpha = \min\left\{a, \frac{b}{M}\right\} = \underline{\hspace{2cm}}.$$

Moreover,

$$L = \underline{\hspace{3cm}}, \quad D = \underline{\hspace{3cm}}.$$

It follows, by $h = \underline{\hspace{2cm}}$, that

$$E_5 \le \underline{\hspace{3.5cm}}.$$

(Note: Since any solution is concaving up, by $y'' = y' = 1$*, we know that* $y_5 = 2.488$ *is an underestimate. Since* $E_5 \le (0.2)(e-1)$ *and the true solution* $y(1) = e$*, we have*

$$e - 2.488 \le 0.2(e-1),$$

or

$$2.488 < e \le \frac{2.688}{0.8} \approx 3.36.$$

This is quite rough, but supposing that you don't know the approximate value of e, this will give you an idea.)

Q13. In **Q12**, with which h are you guaranteed an error smaller than 0.01?

Q14. Consider the IVP

$$y' = t - y^4, \qquad y(0) = 0.$$

If we let $a = 1,\ b = 1$, we obtain

$M =$ ____________, $\alpha =$ ____________, $L =$ ____________, $D =$ ____________.

Which values of h can yield an Euler approximation of $y(1)$ with error less than 0.001?

Q15. Consider the IVP

$$y' = t^2 + \tan^2 y, \quad y(0) = 0.$$

Let $a = 1/2$, $b = \pi/4$. Find

$M =$ ____________, $\alpha =$ ____________, $L =$ ____________, $D =$ ____________.

Which values of h can yield an Euler approximation of $y(1/2)$ with error less than 0.00001?