MATH 3430-02 WEEK 3-1

Key Words: Euler's Approximation Method.

Review.
Q1. Suppose that $y_1(t)$ and $y_2(t)$ are two distinct solutions of the same 1-st order ODE

$$y' = y^2 + e^{-t}.$$

Is it possible for the graphs of y_1 and y_2 to intersect at some point? Explain.

Now we begin to learn some *numerical methods* for 'solving' 1-st order ODEs. Here 'solving' means getting approximations of $y(t)$ (usually for a particular t) that are accurate enough.

The simplest numerical method is called the method of Euler. Consider the initial value problem:

$$y' = f(t, y), \qquad y(t_0) = y_0.$$

Suppose that we want to approximate $y(T)$ for some $T > t_0$. An idea is subdividing the interval $[t_0, T]$ into n equal pieces:

$$t_0 < t_1 < t_2 < \cdots < t_n = T$$

with $\Delta t = t_{i+1} - t_i = \dfrac{T - t_0}{n}$.

Q2. What is the derivative of y at t_0?

Q3. Pretending that $y'(t)$ does not change on $[t_0, t_1]$, what is an approximate value of $y(t_1)$?

Q4. What, then, is $y'(t_1)$?

Q5. Pretending that $y'(t)$ does not change on $[t_1, t_2]$, what is an approximate value of $y(t_2)$?

Q6. Suppose that you've find approximation for $y(t_k)$, write down a recursive formula to compute $y(t_{k+1})$:

$$y(t_{k+1}) = \underline{\hspace{6cm}}.$$

Q7. For the following initial value problem and for $n = 2, 5, 10$, find approximations for $y(1)$. Compare your results with the true value.

$$y' = y, \qquad y(0) = 1.$$

Q8. A pseudo-code for a computer program 'Euler' may look like this:

(with **initial time** t_0*,* **ultimate time** T*,* **number of subintervals** n*.)*

float $t0$, T, $y0$;
int n; i;
array t, y;

$t[0] \leftarrow t0$;
$y[0] \leftarrow y0$;
stepSize$\leftarrow$ ____________;

for (_______;________; _____) {

$y[i+1] \leftarrow$ __________________________;

$t[i+1] \leftarrow$ __________________________;

}

return ____________;

Q9. Now, on your computer, program using Euler's method to approximate $y(5)$ for $n = 5, 10, 50, 200$ and for $y(t)$ satisfying the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - x + \frac{1}{2}y, \qquad y(0) = 1.$$

Find the true solution $y(t)$, then compare your results with the true value of $y(1)$.
(Below is a sample graph in the case when $n = 10$. The smooth curve is the graph of the true solution; the 'curve' given by line segments is the Euler approximation. The arrows represent the slope field *of the ODE above. The size of each arrow indicates the magnitude of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at that point. This graph is generated using* Mathematica®.*)*

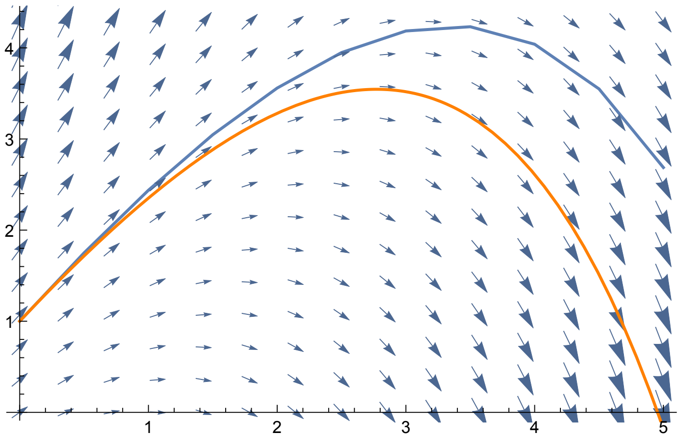

FIGURE 1. $n = 10$ (or $\Delta t = 0.5$).

Q10.

(1) If the graph of a solution $y(t)$ of a first order ODE is **concaving up** (i.e., $y'' > 0$), then a good enough Euler approximation of this solution will be an ________________-estimate.

(2)If the graph of a solution $y(t)$ of a first order ODE is **concaving down** (i.e., $y'' < 0$), then a good enough Euler approximation of this solution will be an ________________-estimate.

Sketch some pictures to illustrate your answers. (You may plot a solution curve as well as some nearby slope fields of the ODE. Then you'll probably see why we refer to 'a good enough' Euler approximation.)