MATH 3430-02 WEEK 4-1

Key Words: Higher order vs. More Equations; How to spot a linear equation; Wronskian.

Note: This lecture is not a complete story. We discuss 3 pieces, which may, at the moment, seem quite separate. (1) Encountering ODEs of higher order; (2) The notions of linearity and homogeneity; (3) The Wronskian, for testing linear independence of a list of functions.

If you see the 1-st order ODEs as one's 'training ground', equations of higher order (i.e., the highest derivative in the equation is greater than 1) is more like the reality.

This said, it is not hard for one to have the following observation of a higher order ODE.

Q1. Consider the 3-rd order ODE below:

$$y''' + (y^2 + 1)y'' + \cos y' - e^x = 0.$$

If we introduce the new variables:

$$\begin{aligned} z_0 &:= y, \\ z_1 &:= y', \\ z_2 &:= y'', \end{aligned}$$

These variables would satisfy (expressed completely in terms of z_i's and x):

$$\begin{cases} z_0' = \underline{\hspace{3cm}}, \\ z_1' = \underline{\hspace{3cm}}, \\ z_2' = \underline{\hspace{6cm}}. \end{cases}$$

The 3 first order equations in z_i' is called a 1-st order *system* of ODEs. If a single ODE is of the n-th order in the unknown variables y, we can introduce n new variables $z_0, z_1, ..., z_{n-1}$, representing $y, y', y'', ..., y^{(n-1)}$ and rewrite the original ODE equivalently as a 1-st order system in z_i.

Q2. Express the following 4-th order ODE as a 1-st order system.

$$y^{(4)} + y''' + (y' - y)^2 y'' + y' = 0.$$

Therefore, if we understand a 1-st order system well, we automatically understand a higher order ODE as a special case. In other words, we have now two views available. We'll study second order ODEs (without turning them into 1-st order systems) in Chapter 2; then we study 1-st order systems (using linear algebra) in Chapter 3.

Among n-th order ODEs, there are those that are *linear*, which we understand relatively well.

An n-th order ODE is said to be **linear** *if it can be put in the following form:*

$$y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \cdots + p_1(x)y'(x) + p_0(x)y(x) = g(x),$$

where $p_{n-1}, \cdots, p_1, p_0, g$ are given functions of x.

Q3. Write down, as an example, a second order linear ODE.

In the definition of a linear ODE above, if $g(x) \equiv 0$, we say that the ODE is **homogenous***; otherwise, it is called* **inhomogeneous***.*

Q4. Write down, as an example, a third order linear homogeneous ODE.

I'll explain what's special about being *linear* next time.

Lastly, we introduce the notion of *Wroskian.* First, recall the concept of *linear dependence.*

A list of functions $\{f_1(x), f_2(x), ..., f_n(x)\}$ is said to be **linearly dependent** *if there exist* **constants** $c_1, c_2, ..., c_n$, **not all zero**, *such that*

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) \equiv 0.$$

(The "$\equiv 0$" means being identically zero, i.e., for all x in a prescribed domain.)

It is important, in the process of solving linear ODEs, to know whether a given list of solutions is linearly independent.

Q5. Suppose that $f_1(x), f_2(x)$ satisfy

$$2f_1(x) + 5f_2(x) \equiv 0.$$

This list is, by definition, linearly ________________.

Differentiating the above equation with respect to x on both sides, we obtain

$$2\_\_\_\_\_\_\_\_ + 5\_\_\_\_\_\_\_\_ \equiv 0.$$

Putting the previous two equations together, we have

$$\begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix} \begin{pmatrix} 2 \\ 5 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since the vector $(2, 5) \neq \mathbf{0}$, by linear algebra, we have

$$\det \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix} \equiv \_\_\_\_.$$

Given a list of functions $\{f_1, f_2 \cdots, f_n\}$, define the **Wronskian** of this list to be the function

$$W[f_1, f_2, \cdots, f_n](x) := \det \begin{pmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{pmatrix}.$$

Theorem. *If a list of functions $\{f_1, f_2 \cdots, f_n\}$ is* **linearly dependent**, *then the associated Wronskian satisfies*

$$W[f_1, f_2, \cdots, f_n](x) \equiv \underline{\qquad\qquad}. \qquad \textit{(Always, for any } x!\textit{)}$$

(**Caution.** The converse of this theorem is not true in general. That is, $W \equiv 0$ does not necessarily lead to linear dependence of a list of functions. I'll say more about this later.)

Also useful is the contra-positive of the theorem above:

Theorem. *If, for a list of functions $\{f_1, f_2 \cdots, f_n\}$, one computed that*

$$W[f_1, f_2, \cdots, f_n](x_0) \neq 0$$

at some point x_0, then the list $\{f_1, \cdots, f_n\}$ is __________ __________.

Q6. What is the Wronskian associated to the list $\{\cos x, \sin x\}$? Is the list linearly independent?

Q7. Find the Wronskian associated to the list $\{x \cos x, e^x\}$, evaluated at $x_0 = 0$. Is the list linearly independent?

Q8. Find the Wronskian associated to the list $\{x \cos x, x\}$, evaluated at $x = 0$. Based on this calculation alone, can you say anything about the linear dependence of the given list?