MATH 3430-02 WEEK 13-1

Key Words: 1-st order linear systems (cont).

This note will contain two topics. One, an example of $\mathbf{y}' = A\mathbf{y}$ where A is a constant, real diagonalizable matrix; two, the solving method in the case when A is complex diagonalizable.

Example 1. Consider $\mathbf{y}' = A\mathbf{y}$, where

$$A = \begin{pmatrix} 2 & 3 & 6 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{pmatrix}.$$

To see whether A is diagonalizable, we compute the characteristic polynomial

$$\det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & 3 & 6 \\ 0 & 1-\lambda & -2 \\ 0 & 1 & 4-\lambda \end{pmatrix} = (2-\lambda)(\lambda^2 - 5\lambda + 6) = (\lambda - 2)^2(\lambda - 3).$$

Thus, $\lambda = 2$ is an eigenvalue with multiplicity 2, $\lambda = 3$ an eigenvalue with multiplicity 1.
It turns out

$$A - 2I = \begin{pmatrix} 0 & 3 & 6 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \end{pmatrix}.$$

This matrix has null vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}.$$

It follows that the eigenspace E_2 has a basis $\{\mathbf{v}_1, \mathbf{v}_2\}$.
Furthermore,

$$A - 3I = \begin{pmatrix} -1 & 3 & 5 \\ 0 & -2 & -2 \\ 0 & 1 & 1 \end{pmatrix}.$$

This matrix has null vector

$$\mathbf{v}_3 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix},$$

which is an eigenvector associated to $\lambda = 3$.
Thus, A is (real) diagonalizable.
By a theorem from previous note, we have the general solution of $\mathbf{y}' = A\mathbf{y}$ being

$$\mathbf{y}(t) = C_1 e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} + C_3 e^{3t} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

Now we turn to the case when A is diagonalizable, but only in the complex sense.

In the real diagonalizable case, we see that, if $\lambda, \mathbf{v}$ are a eigenvalue-eigenvector pair, then $\mathbf{y}(t) = e^{\lambda t}\mathbf{v}$ is a solution.

The reason is straightforward:

$$\mathbf{y}' = \lambda e^{\lambda t}\mathbf{v} = \lambda \mathbf{y},$$

and

$$A\mathbf{y} = e^{\lambda t} A\mathbf{v} = e^{\lambda t}\lambda \mathbf{v} = \lambda \mathbf{y};$$

so

$$\mathbf{y}' = A\mathbf{y}.$$

In fact, the same algebra holds when $\lambda, \mathbf{v}$ are complex. In that case, let $\lambda = a+bi$, $\mathbf{v} = \mathbf{u}+i\mathbf{w}$, where $a, b, \mathbf{u}, \mathbf{w}$ are real. Consider

$$\mathbf{z}(t) = e^{\lambda t}\mathbf{v}.$$

It is a similar matter to verify that $\mathbf{z}(t)$ satisfies $\mathbf{z}' = A\mathbf{z}$.
Since A is a real matrix, by assumption, to find real solutions of $\mathbf{y}' = A\mathbf{y}$, it suffices to take the real and imaginary parts of $\mathbf{z}(t)$.

Example 2. Consider the system $\mathbf{y}' = A\mathbf{y}$ where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The eigenvalues of A are $\lambda_1 = i$ and $\lambda_2 = -i$; the corresponding eigenvectors (basis) being

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

Note that the two pairs of eigenvalues and eigenvectors relate by complex conjugation. As a result, up to linear combination, $e^{\lambda_1 t}\mathbf{v}_1$ and $e^{\lambda_2 t}\mathbf{v}_2$ share the same real and imaginary parts.
It suffices to look at

$$e^{\lambda_1 t}\mathbf{v}_1 = e^{it}\begin{pmatrix} 1 \\ i \end{pmatrix}.$$

This is simply

$$(\cos t + i\sin t)\begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + i\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

It follows that the general solution of $\mathbf{y}' = A\mathbf{y}$ is

$$\mathbf{y} = C_1\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + C_2\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

This is no surprise, since the original system corresponds to the second order ODEs $y_1''+y_1 = 0$, $y_2 = y_1'$.

Now let's consider a more generic example.

Example 3. Suppose that $A = PDP^{-1}$ with

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 5+2i & 0 \\ 0 & 0 & 5-2i \end{pmatrix}, \qquad P = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3+i & 3-i \\ 0 & 1+2i & 1-2i \end{pmatrix}.$$

We want to find the basis solutions of $\mathbf{y}' = A\mathbf{y}$.
Because of the real eigenvalue-eigenvector pair, one solution is easy to find:

$$\mathbf{y}(t) = e^{3t}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$

Now note that

$$e^{(5+2i)t}\begin{pmatrix}1\\3+i\\1+2i\end{pmatrix}$$

can be rewritten as

$$e^{5t}(\cos 2t + i\sin 2t)\left(\begin{pmatrix}1\\3\\1\end{pmatrix}+i\begin{pmatrix}0\\1\\2\end{pmatrix}\right).$$

Taking real and imaginary parts gives rise to two more (basis) solutions.

$$\mathbf{y}(t)=e^{5t}\begin{pmatrix}\cos 2t\\3\cos 2t-\sin 2t\\\cos 2t-2\sin 2t\end{pmatrix},\qquad \mathbf{y}(t)=e^{5t}\begin{pmatrix}\sin 2t\\\cos 2t+3\sin 2t\\2\cos 2t+\sin 2t\end{pmatrix}.$$

Putting together, the general solutions are

$$\mathbf{y}(t)=C_1e^{3t}\begin{pmatrix}2\\1\\0\end{pmatrix}+C_2e^{5t}\begin{pmatrix}\cos 2t\\3\cos 2t-\sin 2t\\\cos 2t-2\sin 2t\end{pmatrix}+C_3e^{5t}\begin{pmatrix}\sin 2t\\\cos 2t+3\sin 2t\\2\cos 2t+\sin 2t\end{pmatrix}.$$