MATH 3430-02 WEEK 11-1

Key Words: Convolution; Linear Algebra review.

It is natural to ask whether the Laplace transform satisfies the property:

$$\mathcal{L}\{f(t)g(t)\} \stackrel{?}{=} \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}.$$

What is your answer? You may give an example to illustrate your point.

What is another instance of *'the transformation of a product does not equal to the product of transformations'*?

The main point of this lecture is to point out that there is a way to relate

$$\mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$$

to a single Laplace transform. For this purpose we introduce a new operation, which take two functions as inputs and outputs one, called **convolution**.

Suppose that $f(t), g(t)$ are zero when $t < 0$. The convolution of f, g is defined to be

$$(f * g)(t) := \int_0^t f(t-\tau)g(\tau)\mathrm{d}\tau.$$

The following properties are not hard to verify:

1) $f * g = g * f$. (commutativity)
2) $f * (c_1 g + c_2 h) = c_1 f * g + c_2 f * h$, ($c_1, c_2$ constants). (bilinearity)
3) $f * (g * h) = (f * g) * h$. (associativity)

Q1. What is $\delta(t - t_0) * f(t)$, where $t_0 > 0$?

Q2. Simplify the expression $\mathcal{L}\{f(t) * g(t)\}$.

We therefore have the important formula:

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}.$$

This formula gives us a new perspective of some of the formulae that we know. For example, for $c > 0$,

$$\mathcal{L}\{\delta(t-c) * f(t)\} = \mathcal{L}\{\delta(t-c)\}F(s) = e^{-sc}F(s).$$

This agrees with $\mathcal{L}\{H_c(t)f(t-c)\}$.

Q3. Express $\mathcal{L}^{-1}\left\{\dfrac{2}{s(s^2+s+2)}\right\}$ as a convolution. (You don't have to evaluate the integral.)

Q4. Solve the integro-differential equation:

$$y(t) = 2t - 3\int_0^t y(t-\tau)\sin\tau \mathrm{d}\tau.$$

Recall: Given a square matrix A, if $\mathbf{v} \neq \mathbf{0}$ satisfies

$$A\mathbf{v} = \lambda \mathbf{v}$$

for some λ, then $\mathbf{v}$ is called an **eigenvector** of A, λ the associated **eigenvalue**.
For a specific eigenvalue λ, all eigenvectors associated to it, together with $\mathbf{0}$, form a vector space, called the **eigenspace** of A associated to λ. The notation is E_λ.
An $n \times n$ matrix A is said to be **diagonalizable** if there exists an invertible matrix P and a diagonal matrix D such that

$$A = PDP^{-1};$$

this holds if and only if A has n linearly independent eigenvectors; another equivalent characterization is

$$n = \sum_{\lambda\text{: eigenvalue of } A} \dim(E_\lambda).$$

In practice, finding eigenvalues and eigenvalues follows the steps:
1) Compute $\det(A - \lambda I)$. This is a polynomial; find all roots of this polynomial. Those roots are the eigenvalues of A.
2) For each eigenvalue found in the previous step, solve the equation

$$(A - \lambda I)\mathbf{v} = \mathbf{0}.$$

Q5. What are the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 12 \\ 3 & 1 \end{pmatrix}?$$

Is A diagonalizable?

Q6. What are the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}?$$

Is A diagonalizable?