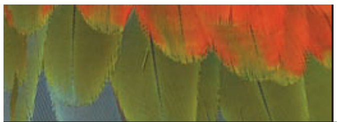
27. *Counting: Hiring* The qualified applicant pool for six management trainee positions consists of seven women and five men.
(a) How many different groups of applicants can be selected for the positions?
(b) How many different groups of trainees would consist entirely of women?
(c) *Probability extension:* If the applicants are equally qualified and the trainee positions are selected by drawing the names at random so that all groups of six are equally likely, what is the probability that the trainee class will consist entirely of women?

4



Chapter Review

SUMMARY

In this chapter we explored basic features of probability.

- The probability of an event A is a number between 0 and 1, inclusive. The more likely the event, the closer the probability of the event is to 1.
- There are three main ways to determine the probability of an event: the method of relative frequency, the method of equally likely outcomes, and intuition.
- The law of large numbers indicates that as the number of trials of a statistical experiment or observation increases, the relative frequency of a designated event becomes closer to the theoretical probability of that event.
- Events are mutually exclusive if they cannot occur together. Events are independent if the occurrence of one event does not change the probability of the occurrence of the other.
- Conditional probability is the probability that one event will occur, given that another event has occurred.
- The complement rule gives the probability that an event will not occur. The addition rule gives the probability that at least one of two specified events will occur. The multiplication rule gives the probability that two events will occur together.
- To determine the probability of equally likely events, we need to know how many outcomes are possible. Devices such as tree diagrams and counting rules, such as the multiplication rule of counting, the permutations rule, and the combinations rule, help us determine the total number of outcomes of a statistical experiment or observation.

In most of the statistical applications of later chapters, we will use the addition rule for mutually exclusive events and the multiplication rule for independent events.

IMPORTANT WORDS & SYMBOLS

Section 4.1
Probability of an event A, $P(A)$
Relative frequency
Law of large numbers
Equally likely outcomes
Statistical experiment
Simple event
Sample space
Complement of event A

Section 4.2
Independent events
Dependent events
~~$A \mid B$~~

$P(A|B)$!

Conditional probability
Multiplication rules of probability (for independent and dependent events)
A and B
Mutually exclusive events
Addition rules (for mutually exclusive and general events)
A or B

Section 4.3
Multiplication rule of counting
Tree diagram
Permutations rule
Combinations rule

(e) Suppose you were assigned to write an article for the student newspaper and you were given a quota (by the editor) of interviewing at least three extroverted professors. How many professors selected at random would you need to interview to be at least 90% sure of filling the quota?

(See Problem 22 of Section 5.3.)

COMMENT Both extroverted and introverted professors can be excellent teachers.

SECTION 5.1

Introduction to Random Variables and Probability Distributions

FOCUS POINTS

- Distinguish between discrete and continuous random variables.
- Graph discrete probability distributions.
- Compute μ and σ for a discrete probability distribution.
- Compute μ and σ for a linear function of a random variable x.
- Compute μ and σ for a linear combination of two independent random variables.

Random Variables

For our purposes, we say that a *statistical experiment* or *observation* is any process by which measurements are obtained. For instance, you might count the number of eggs in a robin's nest or measure daily rainfall in inches. It is common practice to use the letter x to represent the quantitative result of an experiment or observation. As such, we call x a variable.

A quantitative variable x is a **random variable** if the value that x takes on in a given experiment or observation is a chance or random outcome.

A **discrete random variable** can take on only a finite number of values or a countable number of values.

A **continuous random variable** can take on any of the countless number of values in a line interval.

The distinction between discrete and continuous random variables is important because of the different mathematical techniques associated with the two kinds of random variables. all cases

on exam

In most of the cases we will consider a *discrete random variable* will be the result of a count. The number of students in a statistics class is a discrete random variable. Values such as 15, 25, 50, and 250 are all possible. However, 25.5 students is not a possible value for the number of students.

not on exam

Most of the *continuous random variables* we will see will occur as the result of a measurement on a continuous scale. For example, the air pressure in an automobile tire represents a continuous random variable. The air pressure could, in theory, take on any value from 0 lb/in^2 (psi) to the bursting pressure of the tire. Values such as 20.126 psi, 20.12678 psi, and so forth are possible.

GUIDED EXERCISE 1

Discrete or continuous random variables

Which of the following random variables are discrete and which are continuous?

(a) *Measure* the time it takes a student selected at random to register for the fall term. → Time can take on any value, so this is a continuous random variable.

Continued

GUIDED EXERCISE 1 *continued*

(b) *Count* the number of bad checks drawn on Upright Bank on a day selected at random.

The number of bad checks can be only a whole number such as 0, 1, 2, 3, etc. This is a discrete variable.

(c) *Measure* the amount of gasoline needed to drive your car 200 miles.

We are measuring volume, which can assume any value, so this is a continuous random variable.

(d) Pick a random sample of 50 registered voters in a district and find the number who voted in the last county election.

This is a count, so the variable is discrete.

Probability Distribution of a Discrete Random Variable

A random variable has a probability distribution whether it is discrete or continuous.

Probability distribution

A **probability distribution** is an assignment of probabilities to each distinct value of a discrete random variable or to each interval of values of a continuous random variable.

Features of the probability distribution of a discrete random variable

1. The probability distribution has a probability assigned to *each* distinct value of the random variable.
2. The sum of all the assigned probabilities must be 1.

EXAMPLE 1 Discrete probability distribution

Dr. Mendoza developed a test to measure boredom tolerance. He administered it to a group of 20,000 adults between the ages of 25 and 35. The possible scores were 0, 1, 2, 3, 4, 5, and 6, with 6 indicating the highest tolerance for boredom. The test results for this group are shown in Table 5-1.

(a) If a subject is chosen at random from this group, the probability that he or she will have a score of 3 is 6000/20,000, or 0.30. In a similar way, we can use relative frequencies to compute the probabilities for the other scores (Table 5-2). These probability assignments make up the probability distribution. Notice that the scores are mutually exclusive: No one subject has two scores. The sum of the probabilities of all the scores is 1.

TABLE 5-1 **Boredom Tolerance Test Scores for 20,000 Subjects**

Score	Number of Subjects
0	1400
1	2600
2	3600
3	6000
4	4400
5	1600
6	400

TABLE 5-2 **Probability Distribution of Scores on Boredom Tolerance Test**

Score x	Probability $P(x)$
0	0.07
1	0.13
2	0.18
3	0.30
4	0.22
5	0.08
6	0.02
	$\Sigma P(x) = 1$

Mean and standard deviation of a discrete probability distribution

A probability distribution can be thought of as a relative-frequency distribution based on a very large n. As such, it has a mean and standard deviation. If we are referring to the probability distribution of a *population,* then we use the Greek letters μ for the mean and σ for the standard deviation. When we see the Greek letters used, we know the information given is from the *entire population* rather than just a sample. If we have a sample probability distribution, we use $\overline{x}$ (x bar) and s, respectively, for the mean and standard deviation.

The **mean** and the **standard deviation of a discrete population probability distribution** are found by using these formulas:

$\mu = \Sigma x P(x)$; μ is called the **expected value** of x

$\sigma = \sqrt{\Sigma (x - \mu)^2 P(x)}$; σ is called the standard deviation of x

where x is the value of a random variable,
$P(x)$ is the probability of that variable, and
the sum Σ is taken for all the values of the random variable.

Note: μ is the *population mean* and σ is the underlying *population standard deviation* because the sum Σ is taken over *all* values of the random variable (i.e., the entire sample space).

Linear Functions of a Random Variable

Let a and b be any constants, and let x be a random variable. Then the new random variable $L = a + bx$ is called a *linear function of x*. Using some more advanced mathematics, the following can be proved.

Let x be a random variable with mean μ and standard deviation σ. Then the **linear function $L = a + bx$** has mean, variance, and standard deviation as follows:

$$\mu_L = a + b\mu$$
$$\sigma_L^2 = b^2\sigma^2$$
$$\sigma_L = \sqrt{b^2\sigma^2} = |b|\sigma$$

Linear Combinations of Independent Random Variables

Independent random variables

Suppose we have two random variables x_1 and x_2. These variables are *independent* if any event involving x_1 by itself is *independent* of any event involving x_2 by itself. Sometimes, we want to combine independent random variables and examine the mean and standard deviation of the resulting combination.

Let x_1 and x_2 be independent random variables, and let a and b be any constants. Then the new random variable $W = ax_1 + bx_2$ is called a *linear combination of x_1 and x_2*. Using some more advanced mathematics, the following can be proved.

Let x_1 and x_2 be independent random variables with respective means μ_1 and μ_2, and variances σ_1^2 and σ_2^2. For the **linear combination $W = ax_1 + bx_2$**, the mean, variance, and standard deviation are as follows:

$$\mu_W = a\mu_1 + b\mu_2$$
$$\sigma_W^2 = a^2\sigma_1^2 + b^2\sigma_2^2$$
$$\sigma_W = \sqrt{a^2\sigma_1^2 + b^2\sigma_2^2}$$

Note: The formula for the mean of a linear combination of random variables is valid regardless of whether the variables are independent. However, *the formulas for the variance and standard deviation are valid* only if x_1 and x_2 are *independent* random variables. In later work (Chapter 7 on), we will use independent random samples to ensure that the resulting variables (usually means, proportions, etc.) are statistically independent.

EXAMPLE 3 LINEAR COMBINATIONS OF INDEPENDENT RANDOM VARIABLES

Let x_1 and x_2 be independent random variables with respective means $\mu_1 = 75$ and $\mu_2 = 50$, and standard deviations $\sigma_1 = 16$ and $\sigma_2 = 9$.

(a) Let $L = 3 + 2x_1$. Compute the mean, variance, and standard deviation of L.

SOLUTION: L is a linear function of the random variable x_1. Using the formulas with $a = 3$ and $b = 2$, we have

$$\mu_L = 3 + 2\mu_1 = 3 + 2(75) = 153$$
$$\sigma_L^2 = 2^2\sigma_1^2 = 4(16)^2 = 1024$$
$$\sigma_L = |2|\sigma_1 = 2(16) = 32$$

Notice that the variance and standard deviation of the linear function are influenced only by the coefficient of x_1 in the linear function.

ball lands in the gold slot, the contestant wins \$50,000. No other slot pays. What is the probability that the quiz show will have to pay the fortune to three contestants out of 100?

In this problem, the contestant and the quiz show sponsors are concerned about only two outcomes from the wheel of fortune: The ball lands on the gold, or the ball does not land on the gold. This problem is typical of an entire class of problems that are characterized by the feature that there are exactly two possible outcomes (for each trial) of interest. These problems are called *binomial experiments,* or *Bernoulli experiments,* after the Swiss mathematician Jacob Bernoulli, who studied them extensively in the late 1600s.

Features of a binomial experiment

Features of a binomial experiment

1. There are a *fixed number of trials.* We denote this number by the letter n.
2. The n trials are *independent* and repeated under identical conditions.
3. Each trial has only *two outcomes:* success, denoted by S, and failure, denoted by F.
4. For each individual trial, the *probability of success is the same.* We denote the probability of success by p and that of failure by q. Since each trial results in either success or failure, $p + q = 1$ and $q = 1 - p$.
5. The central problem of a binomial experiment is to find the *probability of r successes out of n trials.*

EXAMPLE 4 BINOMIAL EXPERIMENT

Let's see how the wheel of fortune problem meets the criteria of a binomial experiment. We'll take the criteria one at a time.

SOLUTION:

1. Each of the 100 contestants has a trial at the wheel, so there are $n = 100$ trials in this problem.
2. Assuming that the wheel is fair, the *trials are independent,* since the result of one spin of the wheel has no effect on the results of other spins.
3. We are interested in only two outcomes on each spin of the wheel: The ball either lands on the gold, or it does not. Let's call landing on the gold *success* (S) and not landing on the gold *failure* (F). In general, the assignment of the terms *success* and *failure* to outcomes does not imply good or bad results. These terms are assigned simply for the user's convenience.
4. On each trial the probability p of success (landing on the gold) is 1/36, since there are 36 slots and only one of them is gold. Consequently, the probability of failure is

$$q = 1 - p = 1 - \frac{1}{36} = \frac{35}{36}$$

   on each trial.
5. We want to know the probability of 3 successes out of 100 trials, so $r = 3$ in this example. It turns out that the probability the quiz show will have to pay the fortune to 3 contestants out of 100 is about 0.23. Later in this section we'll see how this probability was computed.

Anytime we make selections from a population *without replacement, we do not have independent trials.* However, replacement is often not practical. If the

We have done quite a bit of work to determine your chances of $r = 0, 1, 2$, or 3 successes on three multiple-choice questions if you are just guessing. Now we see that there is only a small chance (about 0.016) that you will get them all correct.

Table 5-9 can be used as a model for computing the probability of r successes out of only *three* trials. How can we compute the probability of 7 successes out of 10 trials? We can develop a table for $n = 10$, but this would be a tremendous task because there are 1024 possible combinations of successes and failures on 10 trials. Fortunately, mathematicians have given us a direct formula to compute the probability of r successes for any number of trials.

General formula for binomial probability distribution

Formula for the binomial probability distribution

$$P(r) = \frac{n!}{r!(n-r)!} p^r q^{n-r} = C_{n,r} p^r q^{n-r}$$

where n = number of trials

p = probability of success on each trial

$q = 1 - p$ = probability of failure on each trial

r = random variable representing the number of successes out of n trials ($0 \le r \le n$)

! = factorial notation. Recall from Section 4.3 that the factorial symbol $n!$ designates the product of all the integers between 1 and n. For instance, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. Special cases are $1! = 1$ and $0! = 1$.

Table for $C_{n,r}$

$C_{n,r} = \dfrac{n!}{r!(n-r)!}$ is the binomial coefficient. Table 2 of Appendix II gives values of $C_{n,r}$ for select n and r. Many calculators have a key designated nCr that gives the value of $C_{n,r}$ directly.

Note: The binomial coefficient $C_{n,r}$ represents the number of combinations of n distinct objects (n = number of trials in this case) taken r at a time (r = number of successes). For more information about $C_{n,r}$, see Section 4.3.

Let's look more carefully at the formula for $P(r)$. There are two main parts. The expression $p^r q^{n-r}$ is the probability of getting one outcome with r successes and $n - r$ failures. The binomial coefficient $C_{n,r}$ counts the number of outcomes that have r successes and $n - r$ failures. For instance, in the case of $n = 3$ trials, we saw in Table 5-8 that the probability of getting an outcome with one success and two failures was pq^2. This is the value of $p^r q^{n-r}$ when $r = 1$ and $n = 3$. We also observed that there were three outcomes with one success and two failures, so $C_{3,1}$ is 3.

Now let's take a look at an application of the binomial distribution formula in Example 5.

EXAMPLE 5 Compute $P(r)$ using the binomial distribution formula

Privacy is a concern for many users of the Internet. One survey showed that 59% of Internet users are somewhat concerned about the confidentiality of their e-mail. Based on this information, what is the probability that for a random sample of 10 Internet users, 6 are concerned about the privacy of their e-mail?

SOLUTION:

(a) This is a binomial experiment with 10 trials. If we assign success to an Internet user being concerned about the privacy of e-mail, the probability of success is 59%. We are interested in the probability of 6 successes. We have

$$n = 10 \qquad p = 0.59 \qquad q = 0.41 \qquad r = 6$$

By the formula,

$$P(6) = C_{10,6}(0.59)^6(0.41)^{10-6}$$

$$= 210(0.59)^6(0.41)^4 \quad \text{Use Table 2 of Appendix II or a calculator.}$$

$$\approx 210(0.0422)(0.0283) \quad \text{Use a calculator.}$$

$$\approx 0.25$$

There is a 25% chance that *exactly* 6 of the 10 Internet users are concerned about the privacy of e-mail.

(b) Many calculators have a built-in combinations function. On the TI-84Plus and TI-83Plus calculators, press the **MATH** key and select **PRB**. The combinations function is designated nCr. Figure 5-2 displays the process for computing $P(6)$ directly on these calculators.

FIGURE 5-2

TI-84Plus/TI-83Plus Display

```
10 nCr 6*.59^6*.
41^(10-6)
      .2503034245
```

Using a Binomial Distribution Table

In many cases we will be interested in the probability of a range of successes. In such cases, we need to use the addition rule for mutually exclusive events. For instance, for $n = 6$ and $p = 0.50$,

$$P(4 \text{ or fewer successes}) = P(r \leq 4)$$

$$= P(r = 4 \text{ or } 3 \text{ or } 2 \text{ or } 1 \text{ or } 0)$$

$$= P(4) + P(3) + P(2) + P(1) + P(0)$$

It would be a bit of a chore to use the binomial distribution formula to compute all the required probabilities. Table 3 of Appendix II gives values of $P(r)$ for selected p values and values of n through 20. To use the table, find the appropriate section for n, and then use the entries in the columns headed by the p values and the rows headed by the r values.

Table 5-10 is an excerpt from Table 3 of Appendix II showing the section for $n = 6$. Notice that all possible r values between 0 and 6 are given as row headers. The value $p = 0.50$ is one of the column headers. For $n = 6$ and $p = 0.50$, you can find the value of $P(4)$ by looking at the entry in the row headed by 4 and the column headed by 0.50. Notice that $P(4) = 0.234$.

TABLE 5-10 Excerpt from Table 3 of Appendix II for $n = 6$

						p								
n	*r*	.01	.05	.10	…	.30	…	.50	…	.70	…	.85	.90	.95
⋮														
6	0	.941	.735	.531	…	.118	…	.016	…	.001	…	.000	.000	.000
	1	.057	.232	.354	…	.303	…	.094	…	.010	…	.000	.000	.000
	2	.001	.031	.098	…	.324	…	.234	…	.060	…	.006	.001	.000
	3	.000	.002	.015	…	.185	…	.312	…	.185	…	.042	.015	.002
	4	.000	.000	.001	…	.060	…	.234	…	.324		.176	.098	.031
	5	.000	.000	.000	…	.010	…	.094	…	.303	…	.399	.354	.232
	6	.000	.000	.000	…	.001	…	.016	…	.118	…	.377	.531	.735

TI-84Plus/TI-83Plus Press the **DISTR** key and scroll to **binompdf**(n, p, r). Enter the number of trials n, the probability of success on a single trial p, and the number of successes r. This gives $P(r)$. For the cumulative probability that there are r or fewer successes, use **binomcdf**(n, p, r).

$P(r) = 4$

$P(r \leq 4)$

```
binompdf(6,.3,4)
              .059535
binomcdf(6,.3,4)
              .989065
```

Excel Menu Choice: **Paste Function** f_x ➤ **Statistical** ➤ **Binomdist.** In the dialogue box, fill in the values r, n, and p. For $P(r)$, use false; for P(at least r successes), use true.

Minitab First, enter the r values 0, 1, 2, . . . , n in a column. Then use menu choice **Calc** ➤ **Probability Distribution** ➤ **Binomial.** In the dialogue box, select Probability for $P(r)$ or Cumulative for P(at least r successes). Enter the number of trials n, the probability of success p, and the column containing the r values. A sample printout is shown in Problem 19 at the end of this section.

Common expressions and corresponding inequalities

examinable

Many times we are asked to compute the probability of a range of successes. For instance, in a binomial experiment with n trials, we may be asked to compute the probability of four or more successes. Table 5-11 shows how common English expressions such as "four or more successes" translate to inequalities involving r.

TABLE 5-11 Common English Expressions and Corresponding Inequalities (consider a binomial experiment with n trials and r successes)

Expression	Inequality
Four or more successes At least four successes No fewer than four successes Not less than four successes	$r \geq 4$ That is, $r = 4, 5, 6, \ldots, n$
Four or fewer successes At most four successes No more than four successes The number of successes does not exceed four	$r \leq 4$ That is, $r = 0, 1, 2, 3,$ or 4
More than four successes The number of successes exceeds four	$r > 4$ That is, $r = 5, 6, 7, \ldots, n$
Fewer than four successes The number of successes is not as large as four	$r < 4$ That is, $r = 0, 1, 2, 3$

not examined

Sampling Without Replacement: Use of the Hypergeometric Probability Distribution

If the population is relatively small and we draw samples without replacement, the assumption of independent trials is not valid and we should not use the binomial distribution.