1. Consider the complex equation

$$z'' - 3z' - 4z = 2e^{it}.$$

if a particular $z_p(t)$ is found, then $y_p(t) = \operatorname{Im}(z_p(t))$ is a solution of the original real equation.

Now guess $\quad z(t) = T e^{it} \quad$ (T : complex number)

This leads to, by substituting in the (complex) equation,

$$-Te^{it} - 3iTe^{it} - 4Te^{it} = 2e^{it}$$

$$\Leftrightarrow \quad -(5+3i)T\, e^{it} = 2e^{it}$$

$$\Leftrightarrow \quad T = -\frac{2}{5+3i} = -\frac{2(5-3i)}{34}$$

$$= -\frac{1}{17}(5-3i)$$

Therefore $\quad z_p(t) = -\frac{1}{17}(5-3i)\,e^{it}$

$$= -\frac{1}{17}(5-3i)(\cos t + i\sin t).$$

$$\underline{\operatorname{Im}(z_p(t))} = \frac{1}{17}\left(3\cos t - 5\sin t\right).$$

$$\|$$

$y_p(t).$

2. By the characteristic polynomial

$$p(r) = r^2 - r - 2 \qquad , \qquad \text{roots:} \quad r = 2, -1;$$
$$= (r-2)(r+1)$$

the homogeneous solution (space) has basis:

$$\underset{y_1(t)}{\overset{\shortparallel}{e^{2t}}} , \quad \underset{y_2(t)}{\overset{\shortparallel}{e^{-t}}} .$$

We compute $\quad W(y_1, y_2)(t) = \det \begin{pmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{pmatrix} = -3e^{t}$.

By the formula of variation of parameters, a particular solution is:

$$y_p(t) = u_1 y_1 + u_2 y_2,$$

where

$$u_1 = -\int g(t) y_2(t)/W \, dt = -\int \frac{2e^{-t} \cdot e^{-t}}{-3e^{t}} \, dt$$
$$= -\frac{2}{9} e^{-3t}$$

$$u_2 = \int g(t) y_1(t)/W \, dt = \int e^{2t} (2e^{-t}) / (-3e^{t}) \, dt$$
$$= -\frac{2}{3} t .$$

Putting together,

$$y_p(t) = -\frac{2}{9} e^{-t} - \frac{2}{3} t e^{-t} .$$

(or simply $-\frac{2}{3} t e^{-t}$, since $-\frac{2}{9} e^{-t}$ is a homogeneous solution.)

3. Setting $y(t) = \sum_{n=0}^{\infty} a_n t^n$; we have

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} t^n + 2 \sum_{n=0}^{\infty} a_n t^n = t .$$

This implies.

$$\left.\begin{aligned} n=0 &: \quad a_1 + 2a_0 = 0 \\ n=1 &: \quad 2a_2 + 2a_1 = 1 \\ n>1 &: \quad (n+1) a_{n+1} + 2a_n = 0. \end{aligned}\right\} \text{Recurrence relation}$$

Now $a_0 = y(0) = 1$.

$$\therefore \quad a_1 = -2$$

$$a_2 = \frac{5}{2}$$

$$a_3 = -\frac{2}{3} a_2 = -\frac{5}{3}$$

$$a_4 = -\frac{2}{4} a_3 = \frac{5}{6}$$

etc.

4. Setting $y(t) = \sum_{n=0}^{\infty} a_n t^n$, we have

$$\underset{(2)}{\sum_{n=0}^{\infty}} n(n-1) a_n t^{n-2} + \underset{(1)}{\sum_{n=0}^{\infty}} n a_n t^{n-1} - \sum_{n=0}^{\infty} a_n t^{n+1} = 0$$

$$\Updownarrow$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n - \sum_{n=1}^{\infty} a_{n-1} t^n = 0$$

$\therefore$

$n=0$: $\quad 2a_2 + a_1 = 0$

$n \geq 1$: $\quad (n+2)(n+1) a_{n+2} + (n+1) a_{n+1} - a_{n-1} = 0$

$$\left(a_{n+2} = -\frac{1}{n+2} a_{n+1} + \frac{1}{(n+2)(n+1)} a_{n-1} \right).$$

Using initial values,

$$a_0 = 1$$

$$a_1 = 2$$

$$a_2 = -\frac{1}{2} a_1 = -1$$

$$a_3 = -\frac{1}{3} a_2 + \frac{1}{6} a_0 = \frac{1}{2}$$

$$a_4 = -\frac{1}{4} a_3 + \frac{1}{12} a_1 = \frac{1}{24}$$

$$y_p(t) \approx 1 + 2t - t^2 + \frac{1}{2} t^3 + \frac{1}{24} t^4 + \dots$$

5. Set $y(t) = \sum_{n=0}^{\infty} a_n t^n$.

$$(1-t^2)\, y'' - y$$

$$= y'' - t^2 y'' - y$$

$$= \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} - \sum_{\substack{n=2\\(0)}}^{\infty} n(n-1)\, a_n t^n - \sum_{n=0}^{\infty} a_n t^n$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2} t^n - \sum_{n=0}^{\infty} n(n-1)\, a_n t^n - \sum_{n=0}^{\infty} a_n t^n$$

$$= 0.$$
(need)

$$\therefore \quad a_{n+2} = \frac{n^2 - n + 1}{(n+2)(n+1)}\, a_n.$$

(Recurrence relation)

$$y_p(t) = a_0 + a_1 t + \frac{1}{2} a_0 t^2 + \frac{1}{6} a_1 t^3 + \cdots$$

$$= a_0 \left(1 + \frac{1}{2} t^2 + \cdots \right)$$

$$+ a_1 \left(t + \frac{1}{6} t^3 + \cdots \right)$$

6. The homogeneous equation is an Euler equation:

$$t^2y'' + 4ty' + 2y = 0.$$

Associated polynomial is: $r(r-1) + 4r + 2$

$$= r^2 + 3r + 2$$

$$= (r+1)(r+2)$$

$\therefore$ roots: $r = -1, -2$

Basis Homogeneous Solutions: $y_1(t) = t^{-1}$, $y_2(t) = t^{-2}$.

note: $g(t) = \dfrac{e^t}{t^2}$.

$$y_p(t) = u_1 y_1 + u_2 y_2$$

where

$$W[y_1, y_2] = \det\begin{pmatrix} 1/t & 1/t^2 \\ -1/t^2 & -2/t^3 \end{pmatrix} = -1/t^4.$$

$$u_1 = -\int \frac{g(t)y_2}{W}\,dt = -\int \frac{e^t/t^4}{-1/t^4}\,dt = e^t$$

$$u_2 = \int \frac{g(t)y_1}{W}\,dt = \int \frac{e^t/t^3}{-1/t^4}\,dt = \int -te^t\,dt = -te^t + e^t.$$

$$\therefore y_p(t) = e^t t^{-1} + (1-t)e^t \cdot t^{-2} = \frac{e^t}{t^2}.$$

7. 1) $W = u_1 u_2' - u_2 u_1'$

2) $$u_i'' = \frac{e^t u_i'}{1+e^t} - \frac{u_i}{1+e^t} \qquad (i=1,2).$$

3) Using 1) and 2),

$$\frac{dW}{dt} = u_1 u_2'' - u_2 u_1''$$

$$= u_1\left(\frac{e^t u_2'}{1+e^t} - \frac{u_2}{1+e^t}\right) - u_2\left(\frac{e^t u_1'}{1+e^t} - \frac{u_1}{1+e^t}\right)$$

$$= \frac{e^t}{1+e^t}\left(u_1 u_2' - u_2 u_1'\right)$$

$$= \frac{e^t}{1+e^t} W.$$

4) The equation in W is separable:

$$\frac{dW}{W} = \frac{e^t}{1+e^t}\, dt.$$

$$\therefore \quad \ln|W| = \ln(1+e^t) + C.$$

$$\therefore \quad W = A\cdot(1+e^t).$$

(A is a nonzero constant.).

since $W \neq 0$.

9. $y'' + 2y' + 4y = 1 + H_1(t)(t-1)$ $\quad y(0) = 1, \quad y'(0) = 2$

$$\mathcal{L}\downarrow \qquad\qquad \downarrow\mathcal{L}$$

$$-y'(0) - s y(0) + s^2 Y(s) + 2(-y(0) + sY(s)) + 4Y(s) \qquad \frac{1}{s} + e^{-s}\frac{1}{s^2}$$

$$\|$$

$$-2 - s + s^2 Y(s) + 2(-1 + sY(s)) + 4Y(s)$$

$$\|$$

$$(-4-s) + (s^2 + 2s + 4)Y(s)$$

$$\therefore \quad Y(s) = \frac{1}{s^2+2s+4}\left(s+4 + \frac{s+e^{-s}}{s^2}\right)$$

note:

$$\frac{s+4}{s^2+2s+4} = \frac{(s+1)}{(s+1)^2+3} + \sqrt{3}\,\frac{\sqrt{3}}{(s+1)^2+3}.$$

$$\frac{1}{s(s^2+2s+4)} = \frac{1}{4}\left(\frac{1}{s} - \frac{s+2}{s^2+2s+4}\right).$$

$$= \frac{1}{4}\left(\frac{1}{s} - \frac{s+1}{(s+1)^2+3} - \frac{1}{\sqrt{3}}\frac{\sqrt{3}}{(s+1)^2+3}\right)$$

$$e^{-s}\frac{1}{s^2(s^2+2s+4)} = -\frac{e^{-s}}{8}\left(\frac{1}{s} - \frac{2}{s^2} - \frac{s}{s^2+2s+4}\right)$$

$$= -\frac{e^{-s}}{8}\left(\frac{1}{s} - \frac{2}{s^2} - \frac{s+1}{(s+1)^2+3} + \frac{1}{\sqrt{3}}\frac{\sqrt{3}}{(s+1)^2+3}\right)$$

$$\mathcal{L}^{-1}$$

$\therefore \; y(t)$

$$= e^{-t}\left(\cos\sqrt{3}t + \sqrt{3}\sin\sqrt{3}t\right)$$

$$+\frac{1}{4}\left(1 - e^{-t}\left(\cos\sqrt{3}t + \frac{1}{\sqrt{3}}\sin\sqrt{3}t\right)\right)$$

$$-\frac{1}{8}H_1(t)\left(1 - 2(t-1) - e^{-(t-1)}\cos\sqrt{3}(t-1) + \frac{1}{\sqrt{3}}e^{-(t-1)}\sin\sqrt{3}(t-1)\right).$$

10.

$$y'' + y = \sin t - 2\delta(t-4) \qquad y(0) = y'(0) = 0.$$

$$\downarrow \mathcal{L} \qquad \downarrow \mathcal{L}$$

$$(s^2+1)Y(s) = \frac{1}{s^2+1} - 2e^{-4s}$$

$$\therefore \quad Y(s) = \frac{1}{(s^2+1)^2} - 2e^{-4s}\frac{1}{s^2+1}.$$

$$\downarrow \mathcal{L}^{-1}$$

$$\underbrace{(\sin t) * (\sin t)} - 2H_4(t)\sin(t-4).$$

$$\| $$

$$\int_0^t \sin(t-\tau)\sin\tau \, d\tau$$

$$\| $$

$$\int_0^t \frac{1}{2}\Big(\cos(t-2\tau) - \cos(t)\Big)\, d\tau$$

$$\| $$

$$-\frac{1}{4}\sin(t-2\tau)\Big|_{\tau=0}^{t} - \frac{1}{2}t\cos t$$

$$\| $$

$$\frac{1}{2}\sin t - \frac{1}{2}t\cos t.$$

$$\therefore \quad y(t) = \frac{1}{2}(\sin t - t\cos t) - 2H_4(t)\sin(t-4).$$