MATH 3430-02 WEEK 2-2

Key Words: Picard Iteration; Existence and Uniqueness Theorem; Estimate of Existence Intervals

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Review.
Q1. Examine the following 1-st order ODE. What is a convenient method for solving it?

$$y^2 + y - x\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

This equation also admits an integrating factor of the form $\mu(y)$. Find this integrating factor, then solve a corresponding exact equation.

The new material today might seem theoretical, so it is helpful to know how it connects with the rest of the topic of 1-st order ODEs.

I. We came from a place knowing that it is hard to solve many 1-st order ODEs. Our techniques are limited to particular cases (e.g., linear, separable, exact, etc.). A natural 'next question' would be: *Do we know that solutions exist even though we couldn't explicitly write them down?*

II. Now suppose that we know that for certain t-values, a solution to an initial value problem exists and that it is the only one. *Is there any method to approximate $y(t)$ as closely as we like, for any t?*

For differential equations in general, methods similar to those in **I** can be understood as classical; methods related to **II** is 'well-fledged' (if not 'full-fledged') today, called the *numerical methods.* Restricting to 1-st order ODEs, our current situation is in between **I** and **II**, that is, admitting difficulties in **I** and giving justification to what one might do in **II**.

Consider a 1-st order initial value problem in its most general form:

$$\text{(IVP)} \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y(t)), \qquad y(t_0) = y_0.$$

Integrating on both sides using $\int_{t_0}^{t}$ (for this you need to change the t-variable in the integrand into s) gives:

$$\boxed{}$$

Rearranging terms, we have

$$\text{(INT)} \qquad \boxed{y(t) = }$$

This is a so-called *integral equation* in $y(t)$.

Q2. Verify that a differentiable function $y(t)$ satisfies (IVP) if and only if it satisfies (INT).

It was an remarkable idea of Picard that, though (INT) is an *equation* (since $y(t)$ is involved in both sides of the equal sign), we can still modify (INT) into a *formula*:

$$\text{(ITR)} \qquad y_{N+1}(t) := y_0 + \int_{t_0}^{t} f(s, y_N(s))\mathrm{d}s.$$

This formula generates a new function $y_{N+1}(t)$ from a function $y_N(t)$.

Of course, the formula (ITR) and the equation (INT) are not the same thing. However, the following question is asked:

> *Can one start from an imprecise solution $y_0(t)$, then apply (ITR) successively to obtain $y_1(t)$, $y_2(t)$, ..., $y_N(t)$, $y_{N+1}(t)$,... and getting closer and closer to the actual solution $y(t)$ of (INT)?*

The answer to this question is yes under certain mild conditions on the function f. This idea of determining solutions is called **Picard iteration**.

Q3. Apply the Picard iteration to the initial value problem:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y, \qquad y(0) = 1.$$

What do you notice?

Q4. Apply the Picard iteration to the initial value problem:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2t(y+1), \qquad y(0) = 0.$$

What do you notice?

In general, it is hard to apply Picard iteration for many steps. The main value of Picard iteration lies in proving the following **Existence and Uniqueness Theorem**.

Suppose that, on a rectangular region $\mathcal{R}$ defined by

$$t_0 \le t \le t_0 + a, \qquad |y - y_0| \le b,$$

both f and $\partial f/\partial y$ are continuous.

Let

$$M := \max_{(t,y)\in\mathcal{R}} |f(t,y)|, \qquad \alpha = \min\left\{a, \frac{b}{M}\right\}.$$

There exists a unique solution of the initial value problem

$$y' = f(t,y), \qquad y(t_0) = y_0$$

on the interval $t_0 \le t \le t_0 + \alpha$. Moreover, on this interval, $|y(t) - y_0| \le b$.

We will skip the proof and see some applications of this theorem.

Q5. Apply the theorem above to the IVP

$$y' = t^2 + e^{-y^2}, \qquad y(0) = 0$$

and the rectangular region

$$0 \le t \le \frac{1}{2}, \qquad |y| \le 1.$$

Find M, α and determine an interval on which a solution exists and is unique.

Q6. Apply the theorem to the IVP

$$y' = y^2 + \cos(t^2), \qquad y(0) = 0$$

and the rectangular region

$$0 \leq t \leq 1, \qquad |y| \leq 1.$$

Find M, α and determine an interval on which a solution exists and is unique.
What is a bound of $y(t)$ on this interval?

Q7. Does the existence and uniqueness theorem apply to the following initial value problem?

$$y' = y^{1/3}, \qquad y(0) = 0.$$

What is an obvious solution of this IVP?
Find another solution of this IVP using separation of variables.