Journal of Inquiry-Based Learning in Mathematics
No. **38**, (July 2015)

Discovering Infinity

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Contents

Acknowledgments

I am indebted to the Academy for Inquiry-Based Learning of the Educational Advancement Foundation whose grant support made possible the creation of these notes. Special thanks go to the editor David Clark and the anonymous referee for their many helpful suggestions on improving an earlier draft.

I am also indebted to several generations of Middlebury College students who have worked through several different versions of these notes and exhibited patience, perseverance and an eager desire to become active learners.

To the Instructor

These notes grew out of a First Year Seminar for liberal arts students, *Discovering Infinity* which I have offered several times at Middlebury College. Classes in the First Year Seminar program are small with enrollment usually capped at 15 or 16 students. The subject matter is meant to be interdisciplinary and there is a strong focus on writing. Here is the course description for my most recent offering of *Discovering Infinity*:

> *Infinity* has intrigued poets, artists, philosophers, musicians, religious thinkers, physicists, astronomers, and mathematicians throughout the ages. Beginning with puzzles and paradoxes that show the need for careful definition and rigorous thinking, students will examine the idea of infinity within mathematics, discovering their own theorems and proofs about the infinite. Our central focus is the evolution of the mathematician's approach to infinity, for it is here that the concept has its deepest roots and where our greatest understanding lies. In the final portion of the course, we will consider representation of the infinite in literature and the arts.

The principal texts for the course were Amir Aczel's *The Mystery of the Aleph: Mathematics, the Kabbalah, and the Search for Infinity*, Eli Maor's *To Infinity and Beyond: A Cultural History of the Infinite*, and an early version of these notes. Students also read essays, short stories, and plays by such writers as Jorge Luis Borges, M. C. Escher, Tom Stoppard, William Hazlett Upson, Philip Dick, and Brendan Kneale. They listened to music by Per Nørgård ("Achilles and the Tortoise") and Epoch of Unlight ("The Continuum Hypothesis," "Cardinality" and "Ad Infinitum") and looked at paintings and sculpture by Escher and Max Bill. Thus they were introduced to a broad range of ways of looking at the infinite.

As the course description promises, however, the central focus is the mathematician's approach to infinity. The intellectual hero of the story, as in these notes, is Georg Cantor. A primary goal of the course, as it is in these notes, is for students to discover their solutions, proofs or counterexamples to the stated exercises, theorems, and questions. As students gain more understanding of the material, they are encouraged to make conjectures of their own and attempt to prove them.

We follow a modified Moore method. At the beginning of each class period, students turn in a slip of paper with their names and the numbers of the theorems, problems and questions they have been able to solve without reference to the mathematical literature. I call on them to present their work to the rest of the class which acts as a sounding board, asking questions if they can't follow an argument, pointing out flaws in the reasoning, or suggesting ways to repair the mistakes. I encourage a cooperative rather than competitive spirit.

I caution my students that many, perhaps most, of them do not as yet really know how to construct a complete and correct mathematical proof. Their previous mathematics courses probably did not stress this aspect. Learning this process will be a valuable part of this course for them. They will fairly quickly discover that some of the theorems are easy to prove, while others may be quite difficult. They should not necessarily try to prove all of the theorems, but rather be willing to devote a good deal of time to those which intrigue them. It is possible that some of the theorems are false as stated. There may be even be a few that no one in the world has yet proved or disproved. There is a certain amount of redundancy in these notes. You may discover that one theorem follows from another previously proved one with only a sentence or two for a supporting argument. Don't even be surprised if you see that we have, in essence, presented the same theorem twice in lightly (or more heavily) disguised clothing.

My assumptions are that students want to be active learners, not passive recipients of lecture or textbook material. It is not sufficient for them to exhibit on a test that they have understood what the author has written or the professor has presented. They must want a piece of the action. They want to *do* mathematics as much as possible.

In a course that makes extensive use of some of these notes, I would advise trying to create a *research community* as much as it possible at this beginning undergraduate level. When mathematicians do research, they develop conjectures about what might be true. After coming up with a tentative proof or an outline of an attack on an question, they give a preliminary chalkboard talk to their colleagues where they invite criticism and skepticism. They then revise their proofs in response to what they've heard. Eventually, if they are lucky, they construct a proof which is accepted by their peers. Then they write it up in a careful manner and submit it to a journal for review. Referees examine the paper and provide feedback to the editor and the author. Another round of revision may occur before the paper is formally published.

It is possible to incorporate a version of almost all of these aspects of research level mathematics in a course based on these notes. The main difference is that, at least initially, the conjectured theorems and exercises are coming from the instructor or the author. Once a student has presented a proof on the board that the class is willing to accept as valid, the student does a formal write up and submits it to me as editor of our local *Journal of Infinity Studies*. I assign two or three students to read over the submission and to prepare referee reports. At the end of the term, we publish the journal. I have included as an appendix some guidelines I give to the reviewers.

In any class, there will be some students who have relatively few appearance at the blackboard and papers in the journal. To make certain that I have a sufficient amount of written work to use in assessing performance in addition to the papers, referee reports, and class discussion, I also require each student to submit formal work on a periodic basis, usually once every two weeks. Here are my instructions to them:

> Write up clear, careful and complete proofs you have discovered on your own of two theorems stated in the notes. (You may also include proofs of conjectures that have been added to our notes). These proofs will be evaluated on the basis of rigor, correctness and clarity. Greatest weight will be given to proofs of results which have yet to be presented in class. You may also present proofs of theorems correctly presented in class if your arguments are fundamentally different in approach. You may still earn a satisfactory evaluation on your submission if both proofs are essentially similar to ones which have been presented in class if your argument is a very clear and rigorously correct one with full reasons given for all claims.
>
> Normally, I will grade proofs of theorems written up on a 0 - 5 scale. I will attempt to assign points in a manner consistent with the descriptions of expectations in Table 1 below.

These notes contain more than enough problems to occupy students for many weeks. I have supplemented the definitions, examples, exercises, questions, and theorems with some of the fascinating historical material and paradoxes that have played important roles in the evolution of our understanding of the mathematics of infinity. There are suggestions for extensive projects or term papers as well as a guide to further reading.

What does the instructor do? The best, short piece of advice I have found comes from Philip Nanzetta and George Strecker who promise that teaching from this style of notes will be a “memorable experience.” They write

> “You forget just how difficult it is to quietly sit and watch someone present a proof different from the one you have in mind, or one that is too sketchy or is burdensomely detailed. But this pain is ultimately worth it. The reward of actually seeing a stu-

Table 1: Grading Guidelines for Written Work

Score	Characteristics
5	Logically correct and complete. A solid mathematical argument with full reasons given for all substantial claims. Very clearly written.
4	Logically correct and complete, but argument may not be well-organized. Fairly clearly written, but might be difficult to follow at one or two places.
3	Essentially logically correct and mostly complete, but may have minor subtle errors. Writing needs improvement.
2	A sound overall strategy to prove the theorem, but the argument contains important conceptual errors, logical mistakes, or incorrect use of definitions. Writing needs substantial improvement.
1	Fundamental flaws in the purported proof that can not be patched without major revision of the argument. Poorly written.
0	No Submission

> dent who didn't even know what a proof was at the beginning present a beautiful, polished proof after some months of work justifies the pain on your part and the effort on his. Patience is called for, and criticism, and trick questions, and traps. Blind alleys must be followed to the end. But most of the time you must sit and be quiet."

I would add three comments:

First, you are free, of course, to choose whatever level of intervention in the classroom you feel is right for you and your class. Be aware that most students will look first at you to signal confirmation that the presentation they have just heard is correct or an indication that there are problems. It is far easier to move your students to be independent learners if you wean them away as quickly as possible from reliance on you. You should make it very clear and explicit in advance just what level of intervention the students can expect from you.

Second, each student is different from every other student in background, intelligence, motivation, learning style, and psychological make up. The more you know about each student as an individual, the more effective you can be in "teaching" a Moore method course. Some students, for example, may require very gentle coaxing and considerable support outside of the formal class setting to succeed; others may prosper best with a sterner and more challenging demeanor.

Third, consider R. L. Moore's immortal words: "That student is taught best who is told the least."

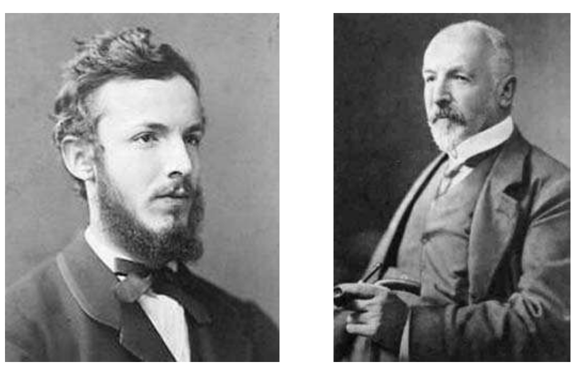
Chapter 1

Introduction

“It is in the contemplation of the infinite that man attains his greatest good.”
Giordano Bruno

Infinity has puzzled and fascinated poets, artists, philosophers, musicians, religious thinkers, physicists, astronomers, and mathematicians throughout the ages. Gazing into the night sky eons ago, humans asked if the heavens stretched out forever. If so, does the universe have a center and if not, what is on the other side?

For millenia awe, fear, ignorance and false starts marked our understanding of the infinite. Then one man – who died less than a century ago – caused a intellectual revolution. Working at a third rate university, battling increasingly frequent bouts of mental depression and plagued by colleagues actively plotting to ruin his reputation and destroy his career, Georg Cantor (1845 – 1918) opened an entirely new way to think about infinity that shook the foundations of mathematics. Before we look in detail at what Cantor accomplished and how he did it, let’s take a brief look at infinity in the history of ideas.



Georg Ferdinand Ludwig Philipp Cantor
Born: March 3, 1845 in St Petersburg, Russia
Died: January 6, 1918 in Halle, Germany

Perhaps the earliest written reference to infinity in Western thought comes from a fragment by the sixth century BCE Greek philosopher Anaximander of Miletus (612-545 BCE). Anaximander used the word $\alpha\pi\varepsilon\iota\rho\o\nu$, “apeiron,“ meaning “unlimited“, “infinite“ or “indefinite“ from the Greek *a* (without) and *peiras* (end or limit). He asserted that everything which exists originates from the apeiron and eventually returns there. The apeiron is the eternal originative substance without limits in time, space, quantity or quality.

Paradoxes and puzzles quickly arose in Greek philosophy that showed the need for careful definition and rigorous thinking. The most famous of these are due to Zeno of Elea, who lived from about 490 BCE to about 430, and were recorded in the writings of Aristotle. We’ll briefly sketch two of these paradoxes, that of *Achilles and the Tortoise* and the *Dichotomy Paradox*.

The *Paradox of Achilles and the Tortoise* concerns a footrace between the swift athlete Achilles and the very slowly moving tortoise. Suppose, for example, that Achilles runs 10 times faster than the tortoise. To give the race some semblance of fairness, we grant the tortoise a head start, say 1000 yards so Achilles begins at position 0 and the tortoise at position 1000. Zeno argues that Achilles will never catch up to the tortoise. Achilles must first reach the tortoise’s initial position at marker 1000. While Achilles is racing towards this spot, the tortoise is crawling ahead and will reach position 1100. Achilles must then run to spot 1100, an additional 100 yards, but this gives the tortoise time to move forward 10 yards. It will then take Achilles some further time to run that distance, by which time the tortoise will have advanced farther; and then more time still to reach this point, while the tortoise moves ahead. Thus, whenever Achilles reaches somewhere the tortoise has been, he still has farther to go. Therefore, because there are an infinite number of points Achilles must reach where the tortoise has already been, he can never overtake his slower moving opponent. Although Zeno’s argument appears flawless, its conclusion contradicts our common experience that faster runners always overtake slower ones if the track is long enough.

The *Dichotomy Paradox* is similar, in some sense, to the story of Achilles and the Tortoise, but with a more startling conclusion: *motion is impossible*. Aristotle states the paradox succinctly: “That which is in locomotion must arrive at the half-way stage before it arrives at the goal.” Suppose you wish to move from your current position to the exit door. You must first move to the point A1 halfway between you and the door. But to get to A1, you need to step to the point A2 halfway between you and A1. This, of course, necessitates that you step to the spot A3, halfway between you and A2. Repeating this argument, we see that there are infinitely many steps you need to reach, but we each can only do finitely many things. Motion necessarily means taking infinitely many steps, something humans can not do; thus, motion is not possible.

Zeno's paradoxes have been, and continue to be, sources of debate and discussion. The British philosopher Bertrand Russell (1872 - 1970) described them as "immeasurably subtle and profound".

Exercise 1 : Most people disagree with the conclusions of these paradoxes; they believe that motion is possible and that a faster runner will overcome the head start of a slower runner and eventually pass him (although some concur with Zeno that motion is an illusion). If the conclusions of the paradoxes are false, where do you find flaws in the reasoning?

Exercise 2 : A candy company is currently offering a promotion. Each chocolate bar contains a coupon and if you collect 10 coupons, you can redeem them for a free bar. If the selling price for the chocolate bar is $1.00, what is the actual cost to you for the chocolate? Note that for $1.00, you get the chocolate and a coupon. If you happen to have, for example, $100, then you can initially purchase 100 bars. You can then take the 100 coupons and redeem them for 10 bars. Extracting the coupons from these 10 bars, you can redeem them for an additional bar. You have spent $100 and have 111 bars and one coupon. You will have spent less than $1 per bar, but how much less?

There are also very interesting and important ideas concerning infinity that arose in Eastern thought. Jainism, for example, is a philosophy and religion founded in India around the Sixth Century BCE. The Jaina cosmology posits time as eternal and without form. The world has always existed. Space pervades everything and is without form. All objects exist in space which is divided into the space of the universe and the space of the non-universe. There is a central region of the universe in which all living beings, including men, animals, gods and devils, live. At least five types of infinity occur in Jainist thought: infinite in one direction, infinite in two directions, infinite in area, infinite everywhere and perpetually infinite, although these are not rigorously defined.

Chapter 2

Paradoxes of the Infinite

"Mankind will never be able to refute completely the Eleatic philosopher Zeno because the infinite is inexhaustible."
Gustav Ivanovich Naan

The history of humanity's wrestling with the concept of the infinite is a long, complex and fascinating one. We touch here in this section on a mere handful of highlights and individuals (Aristotle, Archimedes, Galileo, Bolzano) as they point the way towards Cantor's monumental work.

Faced with conundrums like Zeno's Paradoxes, the great Greek philosopher Aristotle (384 – 322 BCE) made a careful distinction between **Potential Infinity** and **Actual Infinity**. Actual infinity is the idea that numbers or other types of mathematical objects, can form an actual, existing completed totality. An Actual Infinity would be something which is completed, definite and consist of infinitely many elements. Aristotle viewed this as a paradoxical idea; the Actual Infinite does not exist in nature and should be excluded from theory and logical reasoning. A Potential Infinity, on the other hand, made good sense for Aristotle. A potentially infinite collection is a finite collection which can always be extended. Take a finite set of positive integers, for example. We can always add 1 to the largest integer in the collection to extend it to a bigger finite collection. There is no barrier to continuing this process. As Aristotle wrote "For generally the infinite has this mode of existence: one thing is always being taken after another, and each thing that is taken is always finite, but always different." In a similar vein, Euclid's formulation of the postulates for plane geometry has an axiom that essentially states that a line segment may be extended in either direction as far as one likes; the segment remains finite in length but always has the potential to be made longer.

Aristotle's insistence that one should avoid the Actual Infinite and stick only with the Potential Infinite largely dominated mathematics for a long period. Restricting themselves to arguments based on Potential Infinity, the ancient Greek mathematicians gave rigorous proofs of many important discoveries in geometry, including measurements of lengths of curves and the areas they enclosed. Archimedes (287 - 212 BCE), the most eminent mathematician in the century after Aristotle, perfected proof techniques using Potential Infinity. Historians had long thought that the Greeks, particularly Archimedes, totally avoided Actual Infinity. Very recent discoveries, however, have altered our understanding of the past. Within the last decade, work on a centuries old parchment manuscript, called the *Palimpsest*, revealed the remains of a scraped-off text: a previously unknown work of Archimedes. In a scant dozen lines of Greek, scholars find a proof by Archimedes where he made explicit and systematic use of Actual Infinity. Archimedes argued that the number of lines inside a rectangle is equal to the number of triangles inside a prism. He even writes of the number (*plethos*) of such objects, making him the first person ever to consider an infinite numerosity.

Galileo's Paradox

In *Two New Sciences*, his last scientific work, Galileo Galilei (1564 - 1642) constructs a dialogue between Salviati and Simplicio about "the difficulties which arise when we attempt, with our finite minds, to discuss the infinite." Salviati observes that some positive integers, such as 1, 4, 9 and 25 are the squares of positive integers, while others are not squares. Since the set of positive integers is made up both squares and non-squares, Simplicio acknowledges "most certainly" that there are more integers than squares. Not only are the larger proportion of integers nonsquares, but Salviati points out "the proportionate number of squares diminishes as we pass to larger numbers, Thus up to 100 we have 10 squares, that is, the squares constitute 1/10 part of all the numbers; up to 10000, we find only 1/100 part to be squares; and up to a million only 1/1000 part." For such reasons, and also because a basic principle that the "whole must be greater than any of its parts," it seems obvious that there are far more positive integers than there are perfect squares.

Yet Galileo then has Salviati argue that each square has a unique positive square root: "If I should ask further how many squares there are one might reply truly that there are as many as the corresponding number of roots, since every square has its own root and every root its own square, while no square has more than one root and no root more than one square." But each positive integer is the square root of its square, so there are as many roots as there are positive integers. Hence there are just as many squares as there are positive integers.

We can illustrate the pairing of positive integers with their squares by a table

integer :	1	2	3	4	5	…	n	…
$\Updownarrow$	$\Updownarrow$	$\Updownarrow$	$\Updownarrow$	$\Updownarrow$	$\Updownarrow$	$\Updownarrow$	$\Updownarrow$	$\Updownarrow$
square :	1	4	9	16	25	…	n^2	…

In this 1634 volume, Galileo has presented a paradox: on the one hand, there are clearly many more positive integers than squares, but on the other hand there are exactly the same number.

Faced with this apparent paradox, Galileo can only conclude

> "So far as I see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all the numbers, nor the latter greater than the former; and finally the attributes 'equal,' 'greater,' and 'less,' are not applicable to infinite, but only to finite, quantities."

Exercise 3 : Instead of matching up each positive integer n with its square, suppose you match it with n^{100}, the one-hundredth power of n. Thus 1 gets paired with $1^{100} = 1$, 2 is paired with 2^{100}, 3 with 3^{100}, and so on. Note that $2^{100} = 1267650600228229401496703205376$ has 30 digits. How many digits are in 3^{100}? Show that despite the large, increasing gaps between consecutive 100th powers, we can still pair up all the positive integers with all the hundredth powers.

Exercise 4 : The millionth power of 1 is 1 while the millionth power of 2 has more than 300,000 digits. To write out fully the millionth power of 3 would require a book of 100 pages to accommodate all of its 477,121 digits. Thus the gaps between successive millionth powers are astronomical; the millionth powers occur "increasingly rarely" among the positive numbers. Is it possible to pair up all of the positive integers with all of the millionth powers? What about the billionth powers?

Bolzano's Paradox

While Galileo's Paradoxes arises in the context of simple arithmetic, the paradox of Bolzano concerns elementary geometry. Bernhard Placidus Johann Nepomuk Bolzano (1781 –1848) was a mathematician, logician, philosopher, theologian, and Catholic priest from Bohemia. His major writings on infinity, *Paradoxien des Unendlichen (Paradoxes of the Infinite)* was only published three years after his death. It remained largely unknown for another half-century.

Imagine a triangle ΔAPB with horizontal side $\overline{AB}$. Pick a point C on the

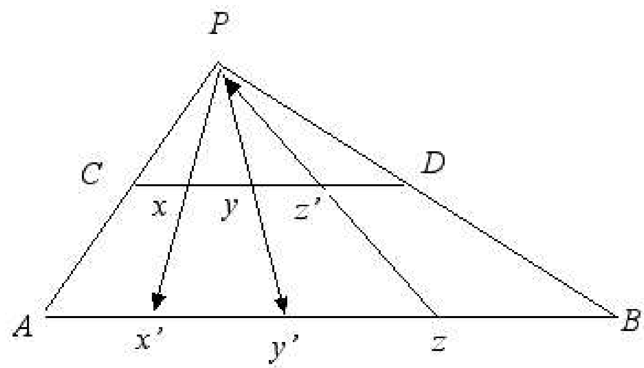


Figure 2.1: A pictorial representation of Bolzano's Paradox. The arrow-tipped lines show how to pair a point on $\overline{AB}$ with a mate on $\overline{CD}$

interior of the side $\overline{AP}$ and select D on the side $\overline{PB}$ so that the segment $\overline{CD}$ is parallel to $\overline{AB}$. Since $\overline{CD}$ is shorter than $\overline{AB}$, it should contain fewer points than $\overline{AB}$. In fact, $\overline{CD}$ is congruent to a short segment of $\overline{AB}$ so again the "whole being larger than any of its parts," implies that there are more points in $\overline{AB}$ than $\overline{CD}$.

On the other hand, Bolzano observes, we can pair up all the points in $\overline{CD}$ with the points of $\overline{AB}$ so each point of $\overline{CD}$ is paired with a unique point of $\overline{AB}$ and each point of $\overline{AB}$ is mated with a point of $\overline{CD}$. To accomplish this pairing, pick any point x on $\overline{CD}$ and draw the line segment from P to x. Extend the segment $\overline{Px}$ until it intersects $\overline{AB}$; call the point of intersection x'. If we begin with two different points x and y on $\overline{CD}$, this projection from P onto $\overline{AB}$ produces different points x' and y'. Note that every point on $\overline{CD}$ is projected onto a point on $\overline{AB}$. Conversely, if we begin with any point on $\overline{AB}$ and draw the line connecting that point with P, this line will intersect the segment $\overline{CD}$ in a unique point. See Figure 2.1.

Exercise 5 : Suppose we choose a point D on the interior of the segment $\overline{PB}$ so that $\overline{CD}$ is not parallel to $\overline{AB}$. Is it possible to pair up all the points on $\overline{CD}$ and $\overline{AB}$?

Exercise 6 : Let C_1 and C_2 be circles of radii 1 and 2, respectively, but with the same center. Show that you can pair up all the points inside the circle C_1 with the points inside the circle C_2 even though the larger circle bounds an area 4 times as big as the area bounded by the smaller circle.

Galileo, Bolzano and other thinkers across the ages were stymied about paradoxes which showed that the whole could be equal to one of its proper parts where infinite sets were concerned. They generally cautioned that we could not rigorously conclude anything about infinity.

It was the genius of Cantor to realize that the apparent *problem* with some infinite sets – they can paired with a proper part – is, in fact, their essential property. If we take that property as the defining characteristic of infinite sets, then we can discover and prove a number of amazing results about infinity. You will be examining Cantor's approach in much detail later in these notes. Let's take a few minutes to examine a most peculiar overnight accommodation for weary travelers, the Hilbert Hotel.

Chapter 3

The Hilbert Hotel

"Tristram Shandy, as we know, employed two years in chronicling the first two days of his life, and lamented that, at this rate, material would accumulate faster than he could deal with it, so that, as years went by, he would be farther and farther from the end of his history. Now I maintain that, if he had lived for ever, and had not wearied of his task, then, even if his life had continued as eventfully as it began, no part of his biography would have remained unwritten. For consider: the hundredth day will be described in the hundredth year, the thousandth in the thousandth year, and so on. Whatever day we may choose as so far on that he cannot hope to reach it, that day will be described in the corresponding year. Thus any day that may be mentioned will be written up sooner or later, and therefore no part of the biography will remain permanently unwritten. This paradoxical but perfectly true proposition depends upon the fact that the number of days in all time is no greater than the number of years."

Bertrand Russell

The eminent German mathematician David Hilbert (1862 - 1943) recognized the importance of Cantor's achievement and the central role it would play in the future development of mathematics. "No one," Hilbert proclaimed, "shall expel us from the Paradise that Cantor has created."

To illustrate the strange and paradoxical nature of infinite sets, Hilbert told the story of a fictional grand hotel, now universally known as the Hilbert Hotel. The Hilbert Hotel is very large. There is a room for every natural number; that is, Room n exists for every positive integer n . Each room is a single and can hold one guest.

Exercise 7 : Suppose every room in the Hilbert Hotel is currently occupied by a guest. A tourist stops at the front desk and requests a room. The clerk reports that the hotel is currently occupied ("All the rooms are filled"), but the tourist need not worry. "Our motto," says the clerk, "is that there's always room for you at the Hilbert Hotel." The clerk announces on the public address system (there's a speaker in every room) that all guests should move from the room they are currently in to the room next door which has one higher room number. Discuss why this scheme frees up a room for our new tourist while all the other guests still have a room.

Exercise 8 : Could the clerk have accomplished the same goal by having each person move to room next door which had one *lower* room number?

Exercise 9 : What should you do if you are the clerk, the hotel is fully occupied, and you want to accommodate two guests who show up asking for separate rooms?

Exercise 10 : Solve the desk clerk's problem if ten travelers show up, requesting ten rooms.

Exercise 11 : The night is still young for our poor clerk. Imagine that at 6 pm, the Hilbert Bus Company's airport shuttle #1 arrives, carrying a passenger for every natural number. [We will the terms *natural number* and *positive integer* as synonyms. The term *counting number* is also sometimes used.] Show how the clerk can find a room for each of these infinitely many passengers even though the hotel was fully occupied when they arrived.

Exercise 12 : What should the clerk do if shuttles #1 and #2 arrive simultaneously, each carrying a passenger for every natural number? Show that if the clerk is clever, all of these visitors can be assigned a room.

Exercise 13 : What should the clerk do if shuttles #1, #2 and #3 arrive simultaneously, each carrying a passenger for every natural number?

Exercise 14 : *The Almost Ultimate Nightmare:* The Hilbert Bus Company has a bus for each natural number and each bus contains a passenger for each natural number; that is shuttle #1 has passengers numbered 1,2,3,. . . as does shuttle #2, shuttle #3, etc. There's obviously no one person who can be on more than one bus so we have an infinite collection of shuttles each of which has a different infinite set of potential lodgers. Show that a very clever clerk can find a room for all these passengers.

Exercise 15 : *The Ultimate Nightmare?* Shuttle #1 arrives and it carries one passenger for every real number between 0 and 1. Can the clerk offer a room to all these tourists?

Chapter 4

Elementary Set Theory

"A set is a Many that allows itself to be thought of as a One."
Georg Cantor

So far, we have dealt extremely informally with infinity. One consequence of a loose treatment of this topic is that we are easily led into puzzling paradoxes that resist resolution. We are now going to begin a more careful examination of the infinite. We start with an introduction to one of the core pillars of modern mathematical thought: set theory. Mathematician use set theoretical language in the definitions of virtually every idea in their discipline.

Logicians and set theorists would characterize our development as **naive set theory**. We will provide some suggestions for further reading if you wish to study the more formal approach of **axiomatic set theory**.

Most important ideas in mathematics have evolved over many centuries and through the contributions of many researchers. Their origins are difficult to trace accurately. Set Theory is an exception. It began with a single paper in 1874 by Georg Cantor: "On a Characteristic Property of All Real Algebraic Numbers" ("Über eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen,", *Crelle's Journal für Mathematik* 77 (1874) 258 - 262).

Set theory provides a way to talk about and build with collections of objects, treating the collection as a single entity while keeping access to its individual constituents. Since it is impossible to define all words, we will assume that we know what it means for something to be a set and what it means for something to be an element of a set. The words **set** and **collection** will be used synonymously. Some examples of sets in the real world might be a herd of elephants, a pile of coins, a basket of apples, or a team of soccer players.

Notation: If $\mathbf{A}$ is a set, then $\mathbf{x} \in \mathbf{A}$ means that $\mathbf{x}$ *is an element of* $\mathbf{A}$, or

equivalently, **x** *is a member of* **A**, or **x** *belongs to* **A** or **x** *is in* **A.** We write **x** $\notin$ **A** to indicate that **x** is not a member of **A**.
The set with no elements is called the **empty set** and is represented by the symbol **Ø.**

Notation : One way to describe a set is to list all of its elements, enclosed within curly braces and separated by commas.

Example 1 : The notation **A** = {1,2,3} means that **A** is the set whose elements are the integers 1, 2 and 3.

Notation: Suppose that **P** is a "well defined property" that an object may or may not possess. We use the notation {**x**: **x** has property **P**} to denote the set of all objects with property **P**.

Example 2 : The notation **A** = { **x**: **x** is a positive integer less than 4} also describes the set whose elements are the integers 1,2 and 3.

Definition 1 : Two sets are **equal** if and only if they have precisely the same elements. If **A** and **B** are equal sets, we write **A** = **B**.

To demonstrate that two sets are not equal, you can find an element that belongs to one of the sets, but not the other.

Question 1 : Are the sets {1,2,3} and {1,3,2} equal?

Question 2 : Are the sets {1,2,3} and {1,2,3,3} equal?

Exercise 16 : Let **A** be the set of negative numbers whose squares are negative and let **B** be the set of two-headed men who served as United States President before 2016. Show that these two sets are equal.

Definition 2 : If **A** is a set, the statement that **B** is a **subset** of **A** means that **B** is a set, and that each element of **B** is an element of **A.** If **A** is a set, the notation **B** $\subset$ **A** means that **B** is a subset of **A.**

Definition 3 : If **A** is a set and **B** is a subset of **A**, then **B** is a **proper subset** of **A** if and only if there is an element of **A** which is not an element of **B.**

AN IMPORTANT NOTE: One way to show two sets are equal is to show that each one is a subset of the other one.

Theorem 1. *Suppose* ***A****,* ***B*** *and* ***C*** *are any sets. Then the following statements are true:*

(a) ***A*** *=* ***A****,*

(b) if ***A*** *=* ***B****, then* ***B*** *=* ***A****, and*

(c) if ***A*** *=* ***B*** *and* ***B*** *=* ***C****, then* ***A*** *=* ***C****.*

Definition 4 : If **X** is a set and **A** is a subset of **X,** then the **relative complement** of **A**, denoted **X - A,** is the set of all elements of **X** which are not elements of **A.** In our notation **X - A** = { **x** $\in$ **X** : **x** $\notin$ **A**,}

Example 3 : If $\mathbf{X} = \{1,2,3,4,5\}$ and $\mathbf{A} = \{2,4\}$, then $\mathbf{X} - \mathbf{A} = \{1,3,5\}$.

Definition 5 : If each of **A** and **B** is a set, the **union** of **A** and **B,** denoted $\mathbf{A} \cup \mathbf{B}$ is the set **C** such that **x** is an element of **C** if and only if either **x** is an element of **A** or of **B.**: $\mathbf{A} \cup \mathbf{B} = \{\mathbf{x} : \mathbf{x} \in \mathbf{A} \text{ or } \mathbf{x} \in \mathbf{B}\}$.

Note that the union of two sets **A** and **B** is the set whose elements are all the elements of the set **A** and all the elements of the set **B** but no other elements. To be a member of the union of two sets, it is necessary and sufficient to belong to at least of the sets.

Example 4 : If $\mathbf{A} = \{1,2,3\}$ and $\mathbf{B} = \{3,4\}$, then $\mathbf{A} \cup \mathbf{B} = \{1,2,3,4\}$.

The union operation, which takes two sets and produces a third set, satisfies several nice properties. In particular, we have

Theorem 2. *Suppose* $\boldsymbol{A}$, $\boldsymbol{B}$ *and* $\boldsymbol{C}$ *are any sets. Then*

(a) (Idempotency) $\boldsymbol{A} \cup \boldsymbol{A} = \boldsymbol{A}$,

(b) (Commutativity) $\boldsymbol{A} \cup \boldsymbol{B} = \boldsymbol{B} \cup \boldsymbol{A}$, *and*

(c) (Associativity) $\boldsymbol{A} \cup (\boldsymbol{B} \cup \boldsymbol{C}) = (\boldsymbol{A} \cup \boldsymbol{B}) \cup \boldsymbol{C}$.

Definition 6 : If **G** is a collection, each element of which is a set, the **union** of the sets of **G** is the set **X** such that **y** is an element of **X** if and only if there is an element **g** of **G** such that **y** is an element of **g**.

We use the notation

$$\bigcup_{\mathbf{g} \in \mathbf{G}} \mathbf{g}$$

for this union.

Definition 7 : If each of **A** and **B** is a set, then the **intersection**, or **common part** of **A** and **B,** denoted $\mathbf{A} \cap \mathbf{B}$, is the set **C** such that **x** is an element of **C** if and only if **x** is an element of **A** **and** **x** is an element of **B.**:
$\mathbf{A} \cap \mathbf{B} = \{\mathbf{x} : \mathbf{x} \in \mathbf{A} \text{ and } \mathbf{x} \in \mathbf{B}\}$.

Example 5 : If $\mathbf{A} = \{1,2,3\}$ and $\mathbf{B} = \{3,4\}$, then $\mathbf{A} \cap \mathbf{B} = \{3\}$.

To be a member of the intersection of two sets, it is necessary and sufficient to belong to both of the sets.The intersection operation, which takes two sets and produces a third set, satisfies several nice properties. In particular, we have

Theorem 3. *Suppose* $\boldsymbol{A}$, $\boldsymbol{B}$ *and* $\boldsymbol{C}$ *are any sets. Then*

(a) (Idempotency) $\boldsymbol{A} \cap \boldsymbol{A} = \boldsymbol{A}$,

(b) (Commutativity) $\boldsymbol{A} \cap \boldsymbol{B} = \boldsymbol{B} \cap \boldsymbol{A}$, *and*

(c) (Associativity) $\boldsymbol{A} \cap (\boldsymbol{B} \cap \boldsymbol{C}) = (\boldsymbol{A} \cap \boldsymbol{B}) \cap \boldsymbol{C}$.

Definition 8 : If **A** and **B** are sets and have no element in common, then **A** and **B** are **disjoint**. Since the intersection of two disjoint sets is the empty set, we can write $\mathbf{A} \cap \mathbf{B} = \mathbf{Ø}$.

Definition 9 : If **G** is a nonempty collection of sets, then the **intersection** or **common part** of the sets of **G** is the set **C** such that **x** is an element of **C** if and only if for each set **g** of **G**, **x** is an element of **g**.

We use the notation

$$\bigcap_{\mathbf{g} \in \mathbf{G}} \mathbf{g}$$

for this intersection.

The next theorem makes some assertions about how unions, intersections and complements interact with each other.

Theorem 4. *Let* $\boldsymbol{X}$ *be a set and* $\boldsymbol{A}$, $\boldsymbol{B}$, *and* $\boldsymbol{C}$ *be subsets of* $\boldsymbol{X}$. *Then*

(a) $\boldsymbol{X} - (\boldsymbol{X} - \boldsymbol{A}) = \boldsymbol{A}$,

(b) $\boldsymbol{X} - (\boldsymbol{A} \cup \boldsymbol{B}) = (\boldsymbol{X} - \boldsymbol{A}) \cap (\boldsymbol{X} - \boldsymbol{B})$,

(c) $\boldsymbol{X} - (\boldsymbol{A} \cap \boldsymbol{B}) = (\boldsymbol{X} - \boldsymbol{A}) \cup (\boldsymbol{X} - \boldsymbol{B})$, *and*

(d) $\boldsymbol{A} \cap (\boldsymbol{B} \cup \boldsymbol{C}) = (\boldsymbol{A} \cap \boldsymbol{B}) \cup (\boldsymbol{A} \cap C)$.

Exercise 17 : Suppose **A** , **B** , **C** and **D** are sets. Show that $(\mathbf{A} \cup \mathbf{B}) \cap (\mathbf{C} \cup \mathbf{D}) = \mathbf{A} \cap \mathbf{C}) \cup (\mathbf{B} \cap \mathbf{C}) \cup (\mathbf{A} \cap \mathbf{D}) \cup (\mathbf{B} \cap \mathbf{D})$

Theorem 5. *Suppose* $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ *and* $\boldsymbol{D}$ *are sets. Then*

(a) if $\boldsymbol{A} \subset \boldsymbol{B}$ *and* $\boldsymbol{B} \subset \boldsymbol{C}$, *then* $\boldsymbol{A} \subset \boldsymbol{C}$,

(b) $(\boldsymbol{A} \cap \boldsymbol{B}) \subset \boldsymbol{A} \subset (\boldsymbol{A} \cup \boldsymbol{B})$, *and*

(c) if $\boldsymbol{A} \subset \boldsymbol{B}$ *and* $\boldsymbol{C} \subset \boldsymbol{D}$, *then* $(\boldsymbol{A} \cup \boldsymbol{C}) \subset (\boldsymbol{B} \cup \boldsymbol{D})$ *and* $(\boldsymbol{A} \cap \boldsymbol{C}) \subset (\boldsymbol{B} \cap \boldsymbol{D})$.

Chapter 5

Well-Ordering and Mathematical Induction

"God made the integers. All else is the work of man."
Leopold Kronecker

Mathematical Induction is a powerful proof technique based on an intuitively appealing assumption about the usual counting numbers: **The Well-Ordering Axiom:** *Every non-empty set of positive integers has a smallest element*; that is, if **A** is a set all of whose elements are positive integers and **A** is not the empty set, then there exists an element s in **A** such that $s \leq x$ for all x in **A**.

The Well-Ordering Axiom often comes into play when we want to establish the truth of some proposition that speaks about a property of all the positive integers or the set of positive integers that exceeds some particular value. We will illustrate the process with several examples.

Example 6 : We'll prove that for each positive integer n, the sum of the integers from 1 to n is equal to $\frac{n(n+1)}{2}$. It's easy to check the validity of this proposition for the first few values of n:

n	*sum*	$\frac{n(n+1)}{2}$
1	$1 = 1$	$\frac{1(2)}{2} = 1$
2	$1+2 = 3$	$\frac{2(3)}{2} = 3$
3	$1+2+3 = 6$	$\frac{3(4)}{2} = 6$

Strictly speaking, we do not need to check these cases individually for the proof, but working through the details may help you better understand the proposition. For the proof itself, we let **A** be the set of all positive integers n for which the proposition is **FALSE**. If **A** is the empty set, then the proposition is true for all positive integers and we are done. Suppose, then, that **A** is nonempty. By our axiom, **A** has a smallest element, s; that is, the proposition is false for $n = s$ but the proposition is true for all positive integers less than s.

Since the proposition is false when $n = s$, we have

$$1+2+3+...+s \neq \frac{s(s+1)}{2}$$

Observe that $s \neq 1$ since the proposition is true when $n = 1$, which we saw by direct computation. Hence s is a positive integer greater than or equal to 2, Thus, $s-1$ is also a positive integer. But $s-1$ is smaller than s so $s-1$ is not an element of **A** and hence the proposition is true for $n = s-1$; that is,

$$1+2+3+...+(s-1) = \frac{(s-1)s}{2}$$

but now

$$\begin{aligned} &1+2+3...+s \\ &\quad = [1+2+3+...+(s-1)]+s \\ &\quad = \frac{(s-1)s}{2}+s \\ &\quad = s[\frac{s-1}{2}+1] = s[\frac{s-1}{2}+\frac{2}{2}] \\ &\quad = s[\frac{s-1+2}{2}] = s[\frac{s+1}{2}] = \frac{s(s+1)}{2} \text{ that is,} \end{aligned}$$

$$1+2+3+...+s = \frac{s(s+1)}{2}$$

We have thus arrived at a logical contradiction. We have shown both that 1 + 2 + 3 + ... +s is equal to and is not equal to $\frac{s(s+1)}{2}$. The contradiction must follow from a false assumption, namely the assumption that our set **A** is nonempty. Thus the set of values for which our proposition is false is the empty set. Our proposition is therefore true for all positive integers.

Example 7 : Here is a second example. The proposition is *Every positive integer is the product of a nonnegative power of 2 and an odd integer*. Let's check a few positive integers:

$$1 = 2^0 \times 1, 2 = 2^1 \times 1, 3 = 2^0 \times 3, 4 = 2^2 \times 1, 20 = 2^2 \times 5, 100 = 2^2 \times 25$$

As in the previous example, we'll show that the set of positive integers for which the proposition is false is the empty set. We'll do this by assuming it is nonempty and arrive at a contradiction. Thus, let **A** be the set of positive integers n which can **not** be written as the product of an odd integer and a nonnegative power of 2.

Assume **A** is nonempty and let s be the smallest element in **A**. Now $s \neq 1$ since $1 = 2^0 \times 1$. Furthermore, s can not be an odd integer because every odd integer equals itself multiplied by 2^0. Thus s is an even positive integer that equals or exceeds 2. Since s is even, it is divisible by 2 so $\frac{s}{2}$ is a positive integer which is less than s. Thus the proposition is true for $\frac{s}{2}$ and we can write

$$\frac{s}{2} = 2^m \times N$$

where m is a nonnegative integer and N is an odd integer. But this equality yields

$$s = 2 \times \frac{s}{2} = 2 \times 2^m \times N = 2^{m+1} \times N$$

which is the product of a positive power of 2 and an odd integer. Thus the proposition is also true for s. We have the contradiction that the proposition is both true and false for s. This contradiction forces us to conclude that our initial assumption (**A** is nonempty) is false.

Here are some exercises to give you practice with this important proof technique:

Exercise 18 : Prove that the sum of the first n positive odd integers is n^2.

Exercise 19 : Prove that the sum of the squares of the first n positive integers is $\frac{n(n+1)(2n+1)}{6}$

Exercise 20 : For every positive integer n, show that the integer $n^3 + 5n$ is an integer multiple of 6.

Exercise 21 : For every positive integer n, prove that there exists an n-digit number divisible by 5^n all of whose digits are odd.

Exercise 22 : Prove that the sum of the reciprocals of the squares of the first n positive integers is always less than 2.

Exercise 23 : What, if anything, is wrong with the following argument? *Proposition: If n is any positive integer, then every set with exactly n people contains only people of the same gender.*

*Proof: Let **A** be the set of positive integers for which the proposition is false and assume **A** is nonempty. Let s be the smallest member of **A**. Now s can not be 1, since every set of 1 person has only members of one gender. Thus s is at least 2 so the proposition if true for* $s-1$. *Let **P** be any set of s people. Label the people in this set 1,2,3,..,s. Then the subset **Q** of people 1,2,3,..,* $s-1$ *and the subset **R** of people 2,3,...,s are each sets of* $s-1$ *individuals. Since the proposition is true for* $s-1$, *everyone on **Q** has the same gender and everyone in **R** also has the same gender. Thus individuals 1,2,..,* $s-1$

have the same gender and persons 2,3,...,s have the same gender. But the person with label 2 belongs to both ***Q*** *and* ***R*** *so everyone in* ***P*** *has the same gender, which gives us a contradiction.*

Exercise 24 : Formalize and prove the following, commonly used statement of the *Principle of Mathematical Induction:* If

(a) whenever a statement about a positive integer n is true for $n = k$, it is also true for its successor, $n = k+1$, and

(b) the statement is true for $n = 1$

then the statement will be true for every positive integer n.

Exercise 25 : Show that there is a **unique** way to write a positive integer as a product of a nonnegative power of 2 and an odd integer; that is, if $s = 2^m \times N$ and $s = 2^{m'} \times N'$ where m and m' are nonnegative integers and N and N' are odd integers, then $m = m'$ and $N = N'$.

Exercise 26 : Write the following argument by Martin Gardner as a formal proof and discuss its validity:

> "The question arises: Are there any uninteresting numbers? We can prove that there are none by the following simple steps. If there are dull numbers, then we can divide all numbers into two sets – interesting and dull. In the set of dull numbers there will be only one number that is the smallest. Since it is the smallest uninteresting number it becomes, *ipso facto*, an interesting number. We must therefore remove it from the dull set and place it in the other. But now there will be another smallest uninteresting number. Repeating this process will make any dull number interesting."

Chapter 6

Functions

"A single idea, if it is right, saves us the labor of an infinity of experiences."
Jacques Maritain

Galileo's pairing of positive integers with their squares and Bolzano's pairing of points on one line segment with points on a longer line segment both involve assigning to each member of one set a member of a second set. A core mathematical concept, that of a *function*, captures this notion and plays a prominent role in the modern understanding of most parts of mathematics, especially those sections dealing with infinity. In this section, we provide a careful definition of a function in terms of elementary set theory and investigate some possible properties of functions.

Definition 10 : An **ordered pair** is a combination of two objects, in which the first entry is distinguished from the second. If the first entry is a and the second entry is b, the notation for an ordered pair is (a,b). The pair is "ordered" in that (a,b) differs from (b,a) unless $a=b$. We often use the words **term** or **coordinate** to refer to an entry of an ordered pair. Thus the first term of (a,b) is a and the second term is b. Set theorists formally define the ordered pair (a,b) as the set $\{\{a\},\{a,b\}\}$.

Exercise 27 : Use the formal definition to write out explicitly the ordered pairs $(3,4)$ and $(4,3)$ and show they are different sets.

Exercise 28 : Show that if $a \neq b$, then under the formal definition of ordered pairs, it is true that $(a,b) \neq (b,a)$.

Definition 11 : The statement that **F** is a **function** means that **F** is a collection of ordered pairs, such that no two of these pairs have the same first term.

Example 8 : Let **F** be the set defined by

$$\mathbf{F} = \{(1, Washington), (2, Adams), (43, Obama), (4,3), (6, Adams)\}$$

.

Then **F** is a function which has 5 ordered pairs. Note that the ordered pair (4,3) belongs to **F** but the ordered pair (3,4) does not.

Exercise 29 : Show that the set **G** defined by

$$\mathbf{F} = \{(1, Washington), (2, Adams), (1, Obama), (4,3), (6, Adams)\}$$

also contains 5 distinct ordered pairs, but **G** is not a function.

Definition 12 : Suppose that **F** is a function. The **domain** of **F** is the set **X** such that **x** is an element of **X** if and only if **x** is the first term of some element of **F**. The **range** of **F** is the set **Y** such that **y** is an element of **Y** if and only if **y** is a second term of some element of **F**. If **x** is the first term of an element of **F**, then **F**(**x**), the **value of F at x**, denotes the second term of the ordered pair of **F** whose first term is **x** The function **F** is said to be a function from **X onto Y**. If **Z** is a set such that **Y** is a subset of **Z**, then **F** is a function from **X into Z**. The notation **F:X** $\to$ **Z** means that **F** is a function from **X** into **Z**. If **A** is a subset of **X,** then **F**(**A**) denotes the set of all elements **F**(**a**) where **a** is an element of **A.** If **W** is a subset of **Z**, then $\mathbf{F}^{-1}$(**W**) denotes $\{$**x**: **x** is in **X** and **(x,w)** is an element of **F** for some **w** in **W**$\}$. The set **F(A)** is called the **image** of **A** under **F** and $\mathbf{F}^{-1}$(**W**) is called the **inverse image** of **W** under **F.**

Exercise 30 : For the function **F** defined in Example 8, show that

(a) the domain is $\{$1,2,3,4,6$\}$,

(b) the range is $\{$Washington, Adams, Obama, 3$\}$

(c) **F**(2) = **F**(6)

(d) $\mathbf{F}^{-1}$($\{$Adams,3$\}$) = $\{$2, 4, 6$\}$.

Example 9 : Let **R** denote the set of all real numbers. Let **F** be $\{$(**x**,$\mathbf{x}^2$): **x** is an element of **R** $\}$. Then **F** is a function. The domain of **F** is **R**, the range of **F** is $\{$**x**: **x** is in **R** and **x** $\geq 0\}$, **F**(2) = 4, and we have **F**:**R** $\to$ **R**.

Definition 13 : Suppose that **F** is a function. The statement that **F** is **one-to-one** means that no two elements of **F** have the same second term. In other words, if **x** and **y** are distinct elements of the domain of **F**, then **F**(**x**) is different from **F**(**y**). Note in the last example, **F** is not one-to-one because **F**(2) = **F**(-2).

Question 3 : Let **f** be a function from a set **X** into a set **Y** and let **A** and **B** be subsets of **X**. Which of the following statements are always true?

(a) $\mathbf{f}(\mathbf{A} \cup \mathbf{B}) = \mathbf{f}(\mathbf{A}) \cup \mathbf{f}(\mathbf{B})$.

(b) $\mathbf{f}(\mathbf{A} \cap \mathbf{B}) = \mathbf{f}(\mathbf{A}) \cap \mathbf{f}(\mathbf{B})$.

(c) **Y - f(A)** = **f(X - A)** for each subset **A** of **X**.

Note that if **C** and **D** are sets, then **C** - **D** denotes the set of all elements of **C** which are not members of **D**.

(d) If **f** is a one-to-one function and **g** is a subset of **f**, then **g** is a one-to-one function.

(e) Statement (a) if the union of two sets is replaced by the union of an arbitrary collection of subsets of **X**?

(f) Statement (b) if the intersection of two sets is replaced by the intersection of an arbitrary collection of subsets of **X**?

(g) If **W** is any subset of **Y,** then $\mathbf{f(f^{-1}(W)) = W}$.

(h) $\mathbf{f^{-1}(f(A)) = A}$.

(i) If **f** is a function from **X onto Y** and **W** is a subset of **Y,** then

$\mathbf{X - f^{-1}(Y\text{-}W) = f^{-1}(W)}$.

Theorem 6. *If **F** is a one-to-one function from **X** onto **Y** and* $G = \{(\boldsymbol{b},\boldsymbol{a}) : (\boldsymbol{a},\boldsymbol{b}) \in \boldsymbol{F}\}$*, then **G** is a one-to-one function from **Y** onto **X**.*

Definition 14 : Suppose **F** is a function from **X** into **Y** and **G** is a function from **Y** into **Z**. The **composition and F and G** is the set of ordered pairs $(\mathbf{x},\mathbf{z})$ such that $\mathbf{x} \in \mathbf{X}$, $\mathbf{z} \in \mathbf{Z}$ and there exists some $\mathbf{y} \in \mathbf{Y}$ such that $(\mathbf{x},\mathbf{y}) \in \mathbf{F}$ and $(\mathbf{y},\mathbf{z}) \in \mathbf{G}$. We denote the composition of **F** followed by **G** as $\mathbf{G} \circ \mathbf{F}$.

Example 10 : Let **X** be the set of integers, **Y** the set of non-negative integers and **Z** the set of positive integers greater than or equal to 3. If $\mathbf{F} = \{(\mathbf{x},\mathbf{x}^2) : \mathbf{x} \in \mathbf{X}\}$ and $\mathbf{G} = \{(\mathbf{y},\mathbf{y}+8) : \mathbf{y} \in \mathbf{Y}\}$, then (-7, 57) is an element in the composition of **F** and **G** since $(-7,49) \in \mathbf{F}$ and $(49,57) \in \mathbf{G}$.

Exercise 31 : If $(\mathbf{x},\mathbf{z})$ is an element in the composition of functions **F** and **G**, then $\mathbf{z} = \mathbf{G}(\mathbf{F}(\mathbf{x}))$.

Theorem 7. *If **F** is a function from **X** into **Y** and **G** is a function from **Y** into **Z**, then the composition of **F** and **G** is a function from **X** into **Z**.*

Theorem 8. *If **F** is a a one-to-one function from **X** onto **Y** and **G** is a one-to-one function from **Y** onto **Z**, then the composition of **F** and **G** is a one-to-one function from **X** onto **Z**.*

Chapter 7

Set Equivalence

"I realise that in this undertaking I place myself in a certain opposition to views widely held concerning the mathematical infinite and to opinions frequently defended on the nature of numbers."
Georg Cantor

We come now to a concept that is essential to our understanding of infinite sets, that of **set equivalence**. You will see that Galileo and Bolzano were really dealing with set equivalences in their paradoxes.

The idea is that two sets are equivalent if it is possible to pair off members of the first set with members of the second, with no leftover members on either side. Imagine two very large sets that might exist in the real world: the seats in a university football stadium and the people in a big crowd who want to attend a game in the stadium. One way to determine which of the two sets is larger would be to count the number of seats, count the number of fans and compare the two numbers. An alternative method is simply to let all the people into the stadium and ask them to sit down when they find a seat. If all the seats are occupied and some people are still standing, then we know the size of the crowd was bigger than the capacity of the stadium. If everyone is seated and there are some empty seats, then there are fewer people than seats. If every seat is filled and no one standing, then the two sets have the same size. With this method, we can determine the relative sizes of the two sets without knowing exactly how many elements there are in each. It was Cantor's brilliant idea to realize that we can apply the same method to decide if two infinite sets have the same size.

Definition 15 : Suppose that **X** and **Y** are sets. Then the statement that **X is equivalent to Y**, denoted $\mathbf{X} \sim \mathbf{Y}$, means that there is a one-to-one function from **X** onto **Y**.

Example 11 : Let $\mathbf{F} = \{ (n, n+1)$: n is a non-negative integer$\}$. Then **F** is a one-to-one function from the set of non-negative integers onto the set of positive integers.

Set equivalence is such an important idea in mathematics that it should not come as a surprise that other terms have been used for this concept. Some mathematicians call two equivalent sets **equinumerous** or **equipotent** or may write that the two sets **have the same size** or are of **equal cardinality**. While we use the notation $\mathbf{X} \sim \mathbf{Y}$, you may encounter the notations $\|\mathbf{X}\| = \|\mathbf{Y}\|$ or $\overline{\overline{\mathbf{X}}} = \overline{\overline{\mathbf{Y}}}$.

Notation: Henceforth, we will let **J** denote the set of positive integers, **R** denote the set of real numbers, and **Q** denote the set of rational numbers.

Example 12 : Let **F** be $\{\, (x, \frac{1}{1+x}) :$ x is a positive real number$\}$. Then **F** is a one-to-one function from the positive real numbers onto the open interval (0,1).

Example 13 : Let **F** be $\{(x,\ 2^x) :$ x is a real number$\}$. Then **F** is a one-to-one function from the real numbers onto the set of positive real numbers.

Example 14 : Let **F** be $\{\, (x, x)$: x is in **R** and $-1 \le x \le 1\,\} \cup \{\, (x, 2-(1/x))$: x is a real number greater than $1\} \cup \{\, x, -2-(1/x)$: x is a real number less than $-1\}$ Then **F** is a one-to-one function from **R** onto $\{x$: x is in **R** and $-2 < x < 2\}$.

Example 15 : Let **G** be $\{\, (x, 2 - x)$: x is in **R** and $\boldsymbol{x} \le \mathbf{1}\} \cup \{\, (x, 1/x)$: x is in **R** and $\boldsymbol{x} > \mathbf{1}\}$. Then **G** is a one-to-one function from **R** onto $\{x$: x is in **R** and $x > 0\}$.

Exercise 32 : Show that if **J** is the set of positive integers and **B** is the set of even positive integers, then **J** $\sim$ **B**.

Exercise 33 : *Galileo's Paradox:* Show that if **J** is the set of positive integers and **B** is the set of positive squared integers, then **J** $\sim$ **B**.

Exercise 34 : *Bolzano's Paradox:* Let **A** be the closed interval $[-1,1]$ of real numbers and let **B** be the closed interval $[-3,3]$ of real numbers. Show that **A** $\sim$ **B**. Thus two intervals of different lengths can be equivalent.

Exercise 35 : Show that if **J** is the set of positive integers and **Z** is the set of integers, then **J** $\sim$ **Z**.

Exercise 36 : Show that the set of all real numbers is equivalent to the open interval; (0,1) of real numbers.

Exercise 37 : Let **C** = $\{1/n$: n is a positive integer$\}$ and let **D = C** $\cup$ $\{\mathbf{0}\}$**.** Show that **C** $\sim$ **D**.

Exercise 38 : Let **A** be the closed unit interval; that is, **A** = $\{x : x$ is a real number and $0 \le x \le 1\}$. Let **B** be the open unit interval; that is,
B = $\{x : x$ is a real number and $0 < x < 1\,\}$. Is **A** $\sim$ **B**? Is **B** $\sim$ **A** ?

Exercise 39 : Let **A** be the set of points on or inside the circle with center at the origin and radius 1 in the plane. Let **B** be the set of points on or inside the circle with center at the origin and radius 2 in the plane. Show that **A** $\sim$ **B**.

Exercise 40 : Let $\mathbf{A}$ be the set of points less than 1 unit from the origin in the plane and let $\mathbf{B}$ be the set of all points in the plane. Show that $\mathbf{A} \sim \mathbf{B}$.

Theorem 9. *If $\boldsymbol{X}$ is a set, then $\boldsymbol{X} \sim \boldsymbol{X}$.*

Theorem 10. *Suppose $\boldsymbol{X}$ and $\boldsymbol{Y}$ are sets. If $\boldsymbol{X} \sim \boldsymbol{Y}$, then $\boldsymbol{Y} \sim \boldsymbol{X}$.*

Theorem 11. *Suppose that $\boldsymbol{X}$, $\boldsymbol{Y}$, and $\boldsymbol{Z}$ are sets. If $\boldsymbol{X} \sim \boldsymbol{Y}$ and $\boldsymbol{Y} \sim \boldsymbol{Z}$, then $\boldsymbol{X} \sim \boldsymbol{Z}$.*

Chapter 8

Infinite Sets

"The fear of infinity is a form of myopia that destroys the possibility of seeing the actual infinite, even though it in its highest form has created and sustains us, and in its secondary transfinite forms occurs all around us and even inhabits our minds."
Georg Cantor

We now have the necessary machinery in place to present the definition of an infinite set.

Definition 16 : Suppose that **A** is a set. Then **A** is **infinite** if and only if **A** is equivalent to some proper subset of itself. A set **A** is **finite** if and only if it is not infinite.

Observe that under this definition, we know a positive attribute of an infinite set: it must be equivalent to at least one of its subsets. An infinite set may be equivalent to many different subsets.

Note also that we have **defined a finite set** as a collection which is not infinite. Thus we can not assume to be true those properties of sets you may have called finite in the past. You must prove those properties are true. In particular, we can not make statements such as "A finite set has precisely *n* elements for some nonnegative integer *n*" until we can prove these assertions.

Our first task is to show that sets we've always thought were infinite actually turn out to be infinite under this definition.

Exercise 41 : The set **J** of positive integers is infinite.

Exercise 42 : The set **R** of real numbers is infinite.

Definition 17 : The **unit interval I** is the set of nonnegative real numbers which do not exceed 1; that is, **I** = $\{\, x \in \mathbf{R} \colon 0 \le x \le 1 \}$. The **unit square S** is the set of points in the plane both of whose coordinates are nonnegative and do not exceed 1; that is, **S** = $\{\ (x,y) \colon\ 0 \le x \le 1, 0 \le y \le 1\ \}$. The **unit cube C** is the set of points in three-dimensional space all of whose coordinates are nonnegative and do not exceed 1; that is, **C** = $\{\ (x,y,z) : 0 \leq x \leq 1,$ $0\ \leq\ y\ \leq 1,\ 0 \leq z \leq 1\}$.

Exercise 43 : The unit interval **I** is infinite.

Exercise 44 : The unit square **S** is infinite.

Exercise 45 : The unit cube **C** is infinite.

Now let's establish a pair of results about infinite sets that should be "intuitively obvious" but certainly require proof. The first asserts that a set equivalent to an infinite set should be infinite. The second states that any set which has an infinite subset must itself be infinite.

Theorem 12. *Suppose **A** and **B** are sets. If **A** $\sim$ **B** and **A** is infinite, then **B** is infinite.*

Theorem 13. *Suppose that **A** and **B** are sets. If **A** is a subset of **B** and if **A** is infinite, then **B** is infinite.*

Theorem 14. *If **A** is an infinite set and p is an element not in **A**, then the set $\boldsymbol{A} \cup \{p\}$ is infinite.*

Theorem 15. *The union of two infinite sets is an infinite set.*

Exercise 46 : Give an example of two infinite sets whose intersection is an infinite set.

Exercise 47 : Give an example of two infinite sets **X** and **A** so that the relative complement **X – A** is infinite.

Theorem 16. *If **A** is an infinite set and p is an element of **A**, then the set **A** - $\{p\}$ is infinite.*

Theorem 17. *Let **A** be an infinite set. Suppose p and q are distinct elements of **A**, and **B** = $\{p,q\}$. Then **A – B** is infinite.*

Use Mathematical Induction to establish the following result:

Theorem 18. *Let **A** be an infinite set. Suppose $p_1, p_2, p_3, \ldots, p_n$ are distinct elements of **A** for some positive integer m. If **B** = $\{p_1, p_2, p_3, \ldots, p_n\}$. Then **A – B** is infinite.*

Theorem 19. *If **A** is an infinite set, then there are distinct subsets **B** and **C** such that **A** $\sim$ **B** and **A** $\sim$ **C**.*

Exercise 48 : In the preceding theorem, can **B** and **C** always be chosen so they are disjoint?

The next several theorems and exercises explore more subtle properties of set equivalence.

Definition 18 : If **f**: **A** $\rightarrow$ **A** is a function from a set **A** into itself and n is a positive integer, then the **nth iterate** $\mathbf{f}^{(n)}$ is defined recursively by the rules

$$\mathbf{f}^{(1)} = \mathbf{f}, \mathbf{f}^{(2)} = \mathbf{f} \circ \mathbf{f}, \mathbf{f}^{(3)} = \mathbf{f} \circ \mathbf{f}^{(2)}, \ldots, \mathbf{f}^{(n+1)} = \mathbf{f} \circ \mathbf{f}^{(n)}.$$

Exercise 49 : Suppose **f** is a one–to–one function from a set **X** into itself; that is, **f**: $\mathbf{X} \to \mathbf{X}$ and **Y** is a set such that $\mathbf{f}(\mathbf{X}) \subset \mathbf{Y} \subset \mathbf{X}$. Let **Z** = **Y** - **f**(**X**) and

$$\mathbf{S} = \mathbf{Z} \cup \mathbf{f}(\mathbf{Z}) \cup \mathbf{f}(\mathbf{f}(\mathbf{Z})) \cup \mathbf{f}(\mathbf{f}(\mathbf{f}(\mathbf{Z}))) \cup \ldots = \mathbf{Z} \cup \bigcup_{n \in J} \mathbf{f}^{(n)}(\mathbf{Z})$$

Define a new function **g** by the rule

$$\mathbf{g}(x) = \begin{cases} x & \text{if } x \in \mathbf{S} \\ \mathbf{f}(x) & \text{if } x \in \mathbf{X} - \mathbf{S} \end{cases}$$

Explain why each of the following claims is true:

1. $\mathbf{X} = \mathbf{S} \cup (\mathbf{X} - \mathbf{S})$.
2. $\mathbf{g}(\mathbf{X}) = \mathbf{g}(\mathbf{S}) \cup \mathbf{g}(\mathbf{X} - \mathbf{S}) = \mathbf{S} \cup \mathbf{f}(\mathbf{X} - \mathbf{S})$.
3. $\mathbf{f}(\mathbf{S}) = \mathbf{f}(\mathbf{Z}) \cup \mathbf{f}^{(2)}(\mathbf{Z}) \cup \mathbf{f}^{(3)}(\mathbf{Z}) \cup \ldots$.
4. $\mathbf{S} = \mathbf{Z} \cup \mathbf{f}(\mathbf{S})$.
5. $\mathbf{g}(\mathbf{X}) = \mathbf{S} \cup \mathbf{f}(\mathbf{X} - \mathbf{S}) = \mathbf{Z} \cup \mathbf{f}(\mathbf{S}) \cup \mathbf{f}(\mathbf{X} - \mathbf{S}) = \mathbf{Z} \cup \mathbf{f}(\mathbf{X})$.
6. $\mathbf{Z} \cup \mathbf{f}(\mathbf{X}) = [\, \mathbf{Y} - \mathbf{f}(\mathbf{X}) \,] \cup \mathbf{f}(\mathbf{X}) = \mathbf{Y}$.
7. $\mathbf{g}(\mathbf{X}) = \mathbf{Y}$ so **g** maps **X** onto **Y**.
8. **g** is one-to-one on each of the sets **S** and **X** - **S**.
9. $\mathbf{g}(\mathbf{S}) = \mathbf{S}$.
10. $\mathbf{g}(\mathbf{X} - \mathbf{S}) = \mathbf{f}(\mathbf{X} - \mathbf{S}) = \mathbf{f}(\mathbf{X}) - \mathbf{f}(\mathbf{S})$.
11. **f**(**X**) and **Z** are disjoint.
12. $\mathbf{f}(\mathbf{X}) = \mathbf{f}(\mathbf{X}) - \mathbf{Z}$.
13. $\mathbf{f}(\mathbf{X}) - \mathbf{f}(\mathbf{S}) = \mathbf{f}(\mathbf{X}) - [\, \mathbf{Z} \cup \mathbf{f}(\mathbf{S})] = \mathbf{f}(\mathbf{X}) - \mathbf{S}$.
14. $\mathbf{S} \cap [\mathbf{f}(\mathbf{X}) - \mathbf{f}(\mathbf{S})\,] = \varnothing$.
15. **g** is a one-to-one **f**unction **f**rom **X** onto **Y**.

Theorem 20. *Suppose that **A** is a set, **B** is a subset of **A**, **C** is a subset of **B** and **A** $\sim$ **C**. Then **A** $\sim$ **B**.*

Theorem 21. *Suppose that **A** is a set, **M** is a set, **B** is a subset of **A**, **N** is a subset of **M**, **A** $\sim$ **N** and **M** $\sim$ **B**. Then **A** $\sim$ **M**.*

Definition 19 : If **A** and **B** are sets , then we say **B weakly dominates A**, notation $\mathbf{B} \succeq \mathbf{A}$ or $\mathbf{A} \preceq \mathbf{B}$, if there is a one-to-one function from **A** into **B**. The set **B strongly dominates A**, notation $\mathbf{B} \succ \mathbf{A}$ or $\mathbf{A} \prec \mathbf{B}$ if there is a one-to-one function from **A** into **B** but there is no one-to-one function from **B** into **A**.

Theorem 22. *If **B** strongly dominates **A**, then **B** weakly dominates **A**.*

Exercise 50 : Let **A** be the set of real numbers and let **B** be the set of points in the plane. Show that **B** weakly dominates **A**.

Exercise 51 : Find an example of two sets where one of them weakly dominates the other, but does not strongly dominate it.

Theorem 23. *If **B** weakly dominates **A** and **A** is infinite, then **B** is infinite.*

Theorem 24. *If **C** strongly dominates **B** and If **B** strongly dominates **A**, then If **C** strongly dominates **A**; that is, if $\mathbf{C} \succ \mathbf{B}$ and $\mathbf{B} \succ \mathbf{A}$, then $\mathbf{C} \succ \mathbf{A}$.*

Theorem 25. *If **A** weakly dominates **B** and **B** weakly dominates **A**, then **A** and **B** are equivalent.*

The previous theorem has been called the Cantor-Schröder-Bernstein Theorem. Cantor gave the first statement of this theorem in 1897. His proof, published in 1895, relied on a controversial postulate about sets called the **Axiom of Choice**. In 1896 – 97, Schröder (1841 – 1902) published two proofs, both eventually shown to be in error. Felix Bernstein (1878 – 1956), then a 19–year-old in Cantor's seminar, discovered a different proof in 1897. Richard Dedekind (1831 – 1916) had actually found a proof not using the Axiom of Choice in 1887, but did not publish it. The proofs presented by Schröder and Bernstein also avoid this axiom. Because of problems with Schröder's arguments, his name is often omitted and the theorem nowadays is simply called the Cantor – Bernstein Theorem. Some historians suggest the most appropriate name should be the Cantor – Dedekind – Bernstein Theorem.

We can express the statement of the theorem notationally as

$$\text{if } \mathbf{A} \preceq \mathbf{B} \text{ and } \mathbf{B} \preceq \mathbf{A}, \text{ then } \mathbf{A} \sim \mathbf{B}.$$

For nonnegative numbers, the symbols $\leq, \geq, <, >$ indicate an *ordering by magnitude*: for example, $a < b$ means a is *less than* or *smaller in size than b*; we call say b is *greater than a*. Similarly, we read the statement $a \leq b$ as *a is less than or equal to b* or *b is at least as large as a*. You are familiar with the elementary properties of these relations among numbers. For example, if $a \leq b$ and $b \leq c$, then $a \leq c$ and if $a \leq b$ and $b \leq a$, then $a = b$.

The symbols we introduced in this chapter $\preceq, \succeq, \prec, \succ$ are meant to convey similar relations between the magnitude or *size* of sets. Thus, for example, the statement $\mathbf{A} \preceq \mathbf{B}$ can be thought of as asserting that the set **B** is *as* *least as large in size* as the set **A**. Similarly, you can think of the assertion that $\mathbf{A} \prec \mathbf{B}$ as conveying the idea that **B** has a larger size than **A**. We use the symbol $\sim$ for equivalence of sets to stand for equality of size of sets.

The last several theorems of this chapter show some justification for using these symbols: some relations between sizes of numbers and sizes of sets are exactly parallel. The *transitive* property of numbers ($a \leq b$ and $b \leq c$ implies $a \leq c$) also holds for sets ($\mathbf{A} \preceq \mathbf{B}$ and $\mathbf{B} \preceq \mathbf{C}$ implies $\mathbf{A} \preceq \mathbf{C}$). Of course, each of the results about the magnitude of sets requires proof. Using similar shaped symbols does not by itself guarantee that similar theorems will be true.

These theorems only become very interesting if there are infinite sets of different sizes. If it were the case that whenever **A** and **B** are infinite sets, it was true that **A** and **B** are equivalent, then we could never have one infinite set strongly dominating another infinite set.

When Cantor began investigating the properties of infinite sets, he did not know if they were all the same size or not. In the next several chapters, we will begin to examine this question and discover the interesting answer.

Chapter 9

Countable Sets

“Mathematics are the result of mysterious powers which no one understands, and which the unconscious recognition of beauty must play an important part. Out of an infinity of designs a mathematician chooses one pattern for beauty’s sake and pulls it down to earth.”
Marston Morse

Mathematicians often use the phrase the **Natural Numbers** to refer to the set **J** of positive integers. Many of the infinite sets we meet in our early mathematical education are sets which, at least qualitatively, “look like” collections of natural numbers: all of the integers, the powers of ten, and the multiples of three are examples. Speaking more carefully, we would say that each such set is equivalent to some infinite subset of the positive integers. In this section, you will prove some theorems about sets which have this property.

Definition 20 : Suppose that **A** is a set. Then **A** is **countable** if and only if there is a subset **H** of **J** such that $\mathbf{A} \sim \mathbf{H}$.

Theorem 26. *Every subset of the positive integers **J** is countable.*

Theorem 27. *If **A** is a countable set and p is an element not in **A**, then $\mathbf{A} \cup \{p\}$ is a countable set.*

Theorem 28. *If **A** is a countable set and $p \in \mathbf{A}$, then $\mathbf{A} - \{p\}$ is countable.*

Theorem 29. *A set **A** is an infinite countable set if and only if $\mathbf{A} \sim \mathbf{J}$.*

Theorem 30. *If **A** and **B** are infinite countable sets, then $\mathbf{A} \sim \mathbf{B}$.*

Exercise 52 : Partition the positive integers into two disjoint infinite subsets.

Exercise 53 : Partition the positive integers **J** into three infinite subsets such that each pair of subsets is disjoint; that is, write **J** as $\mathbf{J} = \mathbf{A} \cup \mathbf{B} \cup \mathbf{C}$ where **A**, **B**, and **C** are each infinite and $\mathbf{A} \cap \mathbf{B} = \emptyset$, $\mathbf{A} \cap \mathbf{C} = \emptyset$, and $\mathbf{B} \cap \mathbf{C} = \emptyset$.

Theorem 31. *If $\boldsymbol{A}$ and $\boldsymbol{B}$ are disjoint sets, each of which is countable, then the union $\boldsymbol{A} \cup \boldsymbol{B}$ is a countable set.*

Theorem 32. *If $\boldsymbol{A}$ is a countable set and $\boldsymbol{B}$ is a countable set, then the union $\boldsymbol{A} \cup \boldsymbol{B}$ is a countable set.*

Exercise 54 : For each positive integer k, let $\mathbf{A}_k$ be the set of positive odd multiples of 2^k Show that

$$\bigcup_{k \in \mathbf{J}} \mathbf{A}_k$$

is the set of all positive even integers and that for any two distinct positive integers i and j, $\mathbf{A}_i$ and $\mathbf{A}_j$ are disjoint.

In proving the following theorem, you might want to consider first the case where every pair of distinct sets in the collection is disjoint.

Theorem 33. *If $\boldsymbol{G}$ is a countable collection of sets, each of which is countable, then the union of the sets of $\boldsymbol{G}$ is countable.*

Theorem 34. *If $\boldsymbol{A}$ is an infinite subset of the set of positive integers $\boldsymbol{J}$, then $\boldsymbol{A} \sim \boldsymbol{J}$.*

Theorem 35. *Every infinite set contains an infinite countable subset.*

Exercise 55 : Explain why the previous theorem implies that the set of positive integers $\mathbf{J}$ is taken as the model of a "smallest" infinite set.

Is there an infinite set larger in size than the positive integers? One property of this set is its *discreteness*. There is a gap of length one between successive positive integers: there is no integer between n and $n+1$. A likely candidate for a larger size infinite set would be a set which is *dense* in the sense that every interval no matter how small contains a member of the set. Such a set would seemingly contain a myriad of points "tightly packed" together. The set of ordinary fractions displays such a property. Could do fractions be an infinite set whose size exceeds the size of the integers?

Definition 21 : A number is **rational** if can be written as quotient $\frac{a}{b}$ of two integers a and b where $b \neq 0$.

Theorem 36. *The set $\boldsymbol{Q}$ of rational numbers is an infinite set.*

Theorem 37. *The sum, difference and product of two rational numbers is a rational number.*

Theorem 38. *If r and s are rational numbers with $s \neq 0$, then the quotient $\frac{r}{s}$ is a rational number.*

Theorem 39. *If r and s are rational numbers with $r < s$, then the average $\frac{r+s}{2}$ is rational number with $r < \frac{r+s}{2} < s$.*

Theorem 40. *The unit interval $[0,1]$ contains an infinite set of rational numbers.*

Theorem 41. *For each positive integer n, the interval [0, $\frac{1}{n}$] contains an infinite set of rational numbers.*

Theorem 42. *For each rational number a and each positive integer n, the interval $[a, a+\frac{1}{n}]$ contains an infinite set of rational numbers.*

Exercise 56 : Show that the preceding theorem implies that even the unit interval $[0,1]$ contains an infinite set of subintervals, each of which contains an infinite set of rational numbers. Moreover, each of these subintervals itself contains an infinite collection of subintervals each of which contains an infinite set of rationals.

Despite the seeming preponderance of rational numbers guaranteed by this last exercise, Cantor was able to prove his first remarkable theorem about infinite sets:

Theorem 43. *The rational numbers* **Q** *form a countable set.*

Consider using Theorem 33 to prove the countability of the rationals.

The positive integers and the rational numbers, although they look qualitatively different because one is discrete and the other is dense, both have the same size as infinite sets. In looking for an infinite set of possibly larger size, Cantor then examined the algebraic numbers.

Definition 22 : An **algebraic number** is a real number which is the solution of a polynomial equation each of whose coefficients is an integer.

Example 16 : The rational number $\frac{2}{3}$ is algebraic since it is the solution of the equation $3x-2=0$.

Theorem 44. *Every rational number is algebraic.*

Example 17 : $\sqrt{2}$ is algebraic since it a solution of the equation $x^2-2=0$.

Exercise 57 : Show that $\sqrt{2}+\sqrt{3}$ is a solution of the polynomial equation $x^4-10x^2+1=0$ and hence $\sqrt{2}+\sqrt{3}$ is algebraic.

Exercise 58 : Show that $\sqrt{2}\times\sqrt{3}$ is algebraic.

Exercise 59 : Find a polynomial equation with integer coefficients for which $\sqrt{3}+\sqrt{7}$ is algebraic.

Definition 23 : The **degree** of a polynomial is the highest exponent for a term with a non-zero coefficient. Thus the degree of $P(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_2x^2+a_1x+a_0$ is n if a_n is nonzero.

Example 18 : The degree of $P(x)=3x^7+4x^5-2x^4-11x^3+x^2-5x+21$ is 7.

Definition 24 : The **height** of a polynomial is degree of the polynomial plus the sum of the absolute values of its coefficients.

Example 19 : The height of the polynomial $x^4-2x^3+7x^2-3x-1$ is $4+|1|+|-2|+|7|+|-3|+|-1|=4+1+2+7+3+1=18$. Note that since the polynomial has degree 4, it has at most 4 roots; that is, there are at most 4 solutions to the equation $x^4 - 2x^3 + 7x^2 - 3x - 1 = 0$.

Exercise 60 : How many polynomials with integer coefficients have height 3? List them all explicitly.

Theorem 45. *For each positive integer n the set of polynomials of height n with integer coefficients is finite.*

Theorem 46. *The set of algebraic numbers is infinite.*

Here the second of Cantor's surprising results:

Theorem 47. *The set of algebraic numbers is countable.*

So far, all of the infinite sets we have seen are of the same size. The positive integers, the rationals, and the algebraic numbers are each countable sets. In the next chapter, we will see how to build an infinite set which is genuinely of larger size.

Chapter 10

The Power Set of a Set

"Startled, Ralph realized that the boys were falling still and silent, feeling the beginnings of awe at the power set. . ."
William Golding, *Lord of the Flies*

We've seen that if you start with two sets, then you can build a new set by taking their union or their intersection. There is also a way to create a new, interesting set from a single set **A** by forming the collection of all of the subsets of **A**, including the empty set and **A** itself. We call this set the **Power Set of A**.

Definition 25 : The **Power Set** of a set **A**, denoted $\mathscr{P}(\mathbf{A})$ or $\mathbf{2^A}$ is the collection of all subsets of **A.**

Example 20 : If **A** = $\{$ Earth, Venus, Mercury$\}$, then $\mathbf{2^A}$ = $\{$ **Ø**, $\{$Earth$\}$, $\{$Venus$\}$, $\{$Mercury$\}$, $\{$Earth, Venus$\}$, $\{$Earth, Mercury$\}$, $\{$Venus, Mercury$\}$, $\{$Earth, Venus, Mercury$\}\}$.

Exercise 61 : List all the elements of $\mathbf{2^A}$ for each of the following sets **A**:

(a) **A** = $\{1\}$

(b) **A** = $\{1,2\}$

(c) **A**= **Ø**

Exercise 62 : Let **A** = $\{a,b,c\}$ and **B** = $\mathbf{2^A}$. Suppose **f** is the function from **A** to **B** defined by **f** = $\{(a,\{a\}),(b,\{a,c\}),(c,\mathbf{Ø})\}$. Let **S** be the set of all elements x in **A** such that $x \notin$ **f(x)**. Show there is no element x in **A** such that $\mathbf{f}(x) = \mathbf{S}$.

Theorem 48. *If **f** is any function from any set **A** into $\mathbf{2^A}$ and **S** is the set of all elements x in **A** such that $x \notin$ **f(x)**, then there is no element x in **A** such that $\mathbf{f}(x) = \mathbf{S}$.*

Exercise 63 : Show if for any set **A**, there is a subset **B** of $\mathbf{2^A}$ such that $\mathbf{A} \sim \mathbf{B}$.

Theorem 49. *There is no set $\mathbf{A}$ such that $\mathbf{A} \sim \mathbf{2^A}$.*

Theorem 50. *If $\mathbf{A}$ is an infinite set, then the set $\mathbf{2^A}$ is infinite.*

One of the consequences of the last two theorems is that if **A** is an infinite set, $\mathbf{B} = \mathbf{2^A}$ and $\mathbf{C} = \mathbf{2^B}$, then these three sets are infinite and no two of them are equivalent. (Why are **A** and **C** not equivalent?) Thus, there are at least three different “sizes“ of infinity. In fact, there are a great many different sizes of infinite sets.

Theorem 51. *The power set of a set strictly dominates the set.*

Although Cantor discovered many fascinating and unexpected facts about infinite sets, the preceding result is often referred to as *Cantor’s Theorem*. His proof appeared in an 1891 paper “ Über eine elementare Frage der Mannigfaltigkeitslehre.” Cantor’s Theorem leads to what is considered a truly remarkable and exciting fact about the universe of infinite sets: there are infinitely many different sizes of infinity! More precisely,

Theorem 52. *There exists an infinite collection of infinite sets, no two of which are equivalent.*

Exercise 64 : Show that the set of positive integers **J** is not the power set of any set.

Exercise 65 : Discuss the truth of the following proposition: If **A** and **B** are infinite sets which are not countable, then $\mathbf{A} \sim \mathbf{B}$.

Chapter 11

Some Results About Finite Sets

"I am incapable of conceiving infinity, and yet I do not accept finity."
Simone de Beauvoir

We argued earlier than our intuition may fail us when we consider infinite sets since we typically have little or no experience dealing with infinite sets in "real life." On the other hand, we encounter finite sets all the time. We've developed a strong intuitive sense of what's true about finite sets that has not failed us in previous mathematical settings. One approach to infinite sets would simply be to define an infinite set as any set which is not finite. Of course, we would also have to provide a careful, rigorous definition of **finite set** if we use this approach.

To make any progress in understanding infinity in the mathematical world and resolve the various paradoxes created over the centuries, we desperately needed a careful definition of "infinite set." Cantor supplied such a definition. Cantor's definition, a set is infinite if it is equivalent to a proper subset, provides, moreover, a property with which we can work to establish propositions and theorems about infinity. We know something positive about an infinite set at the outset: there is at least one proper subset equivalent to the original set. This fact provides a foundation upon which we may hope to build a rich structure.

If we adopt Cantor's definition, then the most natural way to define a Finite Set is that it's a collection which is not infinite. That's a perfectly logical approach and it's the one we took. The major consequence of using this definition is that we must now *prove* all those properties we've always believed were true (and obvious!) about finite sets. We'll set about doing this in this chapter.

Definition 26 : A set **A** is **finite** if it is not infinite; that is, there is no one-to-one function from **A** onto any of its proper subsets.

Exercise 66 : The empty set is finite.

Exercise 67 : The set **A** = $\{1\}$ is finite.

Exercise 68 : The set **A** = $\{1, 2\}$ is finite.

Theorem 53. *If **A** is a finite subset of a set **X** and **p** is an element of **X** but does not belong to **A**, then the set $\boldsymbol{A} \cup \{\boldsymbol{p}\}$ is finite.*

Theorem 54. *Suppose n is a positive integer and $\boldsymbol{Z_n} = \{\ \boldsymbol{z}$**:** $\boldsymbol{z}$ is an element of **J** and $\boldsymbol{z} \leq \boldsymbol{n}\}$ Then $\boldsymbol{Z_n}$ is finite.*

Theorem 55. *If $m \neq n$, then $\boldsymbol{Z_m}$ is not equivalent to $\boldsymbol{Z_n}$*

Theorem 56. *The power set of $\boldsymbol{Z_n}$ is equivalent to $\boldsymbol{Z_{2^n}}$.*

Theorem 57. *Suppose that **X** is a non-empty finite set. Then there is a positive integer n such that **X** is equivalent to $\boldsymbol{Z_n}$.*

Theorem 58. *If **B** is a finite subset of an infinite set **A**, then the set **A** - **B** is infinite.*

Theorem 59. *If **B** is a finite subset of an infinite countable set **A**, then the set **A** - **B** is an infinite countable set.*

Theorem 60. *If **B** is a finite subset of an infinite set **A**, then **A** - **B** is equivalent to **A**.*

Strictly speaking, the next several exercises and theorems are not about finite sets. They do, however, represent natural generalizations of the preceding trio so we include them here.

Exercise 69 : Find an example where **B** is an infinite countable subset of an infinite set **A**, but the set **A** - **B** is not infinite.

Exercise 70 : Find an example where **B** is an infinite countable subset of an infinite set **A**, but the set **A** - **B** is infinite, but not countable.

Theorem 61. *If **B** is an infinite countable subset of an infinite set **A** where **A** is not equivalent to **B**, then the set **A** - **B** is infinite.*

Theorem 62. *If **B** is an infinite countable subset of an infinite countable set **A**, then the **A** - **B** is a countable set.*

Theorem 63. *If **B** is an infinite countable subset of an infinite set **A** where **A** is not equivalent to **B**, then **A** - **B** is equivalent to **A**.*

Chapter 12

The Rational Numbers and The Real Numbers

“Take the collection of all positive whole numbers n and denote it by (n); further, imagine the collection of all positive real numbers x and denote it by (x); the question is simply whether (n) and (x) can be corresponded so that each individual of one collection corresponds to one and only one individual of the other.”
Georg Cantor, November 29, 1873 letter to Dedekind

Georg Cantor did not leap into the world of infinite sets casually. His investigations actually grew out of wrestling with questions about particular subsets of the real numbers. We can think of the real numbers as those numbers that have decimal representations. Alternatively, and what turns out to be equivalently, we can view the real numbers as the points on a line. Thus the real numbers correspond to a simple geometric object - a line - which is one of the most important objects in all of mathematics. Calculus, for example, is simply the study of certain properties of functions from one set of real numbers to another set of real numbers.

The idea of representing real numbers as decimals traces back at least as far as Simon Stevin (1548 –1620), a Flemish mathematician, physicist and military engineer. The link between points on the line and decimal representations should be a familiar one to you. To find the decimal representation of a point in the closed interval $I = [0,1]$, for example, begin by partitioning the interval into 10 equally long subintervals. The left hand end points of these subintervals correspond to the numbers 0, $\frac{1}{10}, \frac{2}{10}, \ldots, \frac{9}{10}$. with decimal representations 0, .1, .2, $\cdots$, .9. For any other point P, locate the subinterval which contains it. If P is in the leftmost subinterval, then its first decimal digit will be 0. If P lies in the second subinterval, then its first decimal digit will be 1, and so on. To obtain the second decimal digit, divide the subinterval containing P into 10 equal sub-subintrevals and locate which of these holds P. If P is in the kth sub-subinterval, then its second decimal will be $k-1$. Proceed in a similar manner to generate each of the decimal places.

The discussion above is admittedly an informal one. A more rigorous approach is definitely needed, especially for developing the major results of calculus. The appendix *Suggested Projects* introduces some ways to do this. For our purposes, we need only assume the simplest properties of the reals; *e.g.*, multiplying a real number by 10 corresponds to shifting the decimal point one place to the right.

In this section, you will provide the details and proofs for a deeper study of the real numbers. Recall the definition that a number r is **rational** if it can be written as the quotient of two integers $r = \frac{m}{n}$ where $n \neq 0$. We also have, by previous results, that the sum, difference and product of two rational numbers r and s is rational. The quotient of r divided by s is rational if $s \neq 0$; if $s = 0$, the quotient is undefined.

Exercise 71 : Decimal representations are often called *base 10* representations. Describe how you might obtain a *binary* or *base 2* representation by starting with the unit interval $[0,1]$ and bisecting it into two halves, then bisecting each of the halves, bisecting each of these sub-subintervals, and so on. Assign a binary digit of 0 if the point is in the left half of a subinterval and 1 if it is in the right half. Then each real number would have a representation as a string of 0's and 1's.

In Exercise 33, you showed that the set of real numbers and the set of numbers in the open interval (0,1) are equivalent. Thus, we need only to work with the interval (0,1) when we are investigating the size of the set of reals. We shall assume that every real number x between 0 and 1 has a decimal representation

$$x = .a_1a_2a_3a_4....a_n...$$

where the dots ... indicate that we have a decimal digit a_n for each positive integer n where each decimal digit a_n is one of the integers
$\{0,1,2,3,4,5,6,7,8,9\}$.

Some real numbers have two different decimal representations.

Example 21 : The real number $\frac{1}{2}$ can be represented as .500000.. where every digit after the first one is a 0 or it can be represented as .499999... where every digit after the first one is a 9.

Exercise 72 : Explain why .500000... and .499999... both represent $\frac{1}{2}$.

Exercise 73 : Find two different decimal representations for the number $\frac{3}{4}$.

Exercise 74 : Explain why no real number between 0 and 1 can have three distinct decimal representations.

Exercise 75 : Let **A** be the set of real numbers between 0 and 1 which have two different decimal representations. Show that **A** is infinite. Show that **A** is countable.

Exercise 76 : Is there a smallest positive real number? If so, what is its decimal representation?

Exercise 77 : Is there a real number "immediately to the right" of 1? If so, what is its decimal representation? Is there a real number "immediately to the left of 1"?

Theorem 64. *If p and q are distinct real numbers with $p < q$,, then their average $r = \frac{p+q}{2}$ satisfies $p < r < q$ and thus is distinct from both p and q.*

Exercise 78 : If p is a point on the real line, is there a point q "next to" p?

We call a decimal representation a **repeating decimal** or a **recurring decimal** if at some stage it comes **periodic**; that is, there appears at some decimal place a fixed block of digit that repeats itself endlessly. Perhaps the simplest and most familiar example is the decimal representation for one-third: $\frac{1}{3} = 0.33333...$ where the one-digit long block 3 repeats indefinitely from the first decimal place onward. We will draw a horizontal bar (called a *viniculum*) above the block of repeating digits; e.g., $\frac{1}{3} = .\overline{3}$

Example 22 : The decimal representation of the rational number $\frac{5857}{19980}$ is

$$\frac{5857}{19980} = .29314314314.. = .29\overline{314}.$$

Here the block 314 repeats indefinitely beginning in the third decimal place.

We say that a decimal representation with a repeating final 0 **terminates** before these zeros and we usually do not write the zeros. Thus instead of writing .6579000... or $.6579\overline{0}$ one simply writes .6579. The decimal is also called a **terminating decimal**.

Exercise 79 : Show that terminating decimals represent rational numbers of the form $\frac{m}{n}$ where n is the product of a power of 2 and a power of 5.

The standard procedure of long division, where we divide m by n, will generate a decimal representation of the rational number $\frac{m}{n}$. At each stage in the division process, we generate a new decimal digit and we have a remainder which is one of the integers 0,1,2,..., $n-1$. If the remainder at any stage is 0, then we have a terminating decimal and the division process ceases. Otherwise, we continue. Since there are only n distinct possible remainders, by the time we have generated the first $n+1$ decimal digits, we must have encountered some remainder k at least twice, But then the decimal digits that come after the second time k is a remainder will be exactly the same as the decimal digits that came right after the first time we had k as remainder; that is, we will have a repeating decimal.

Exercise 80 : Find the decimal representations of $\frac{8}{11}$ and $\frac{3}{7}$.

If we start with a rational number expressed as a fraction, then the standard long division algorithm shows that the decimal representation will either be terminating or repeating. On the other hand, if we begin with a repeating or terminating decimal, then it must represent a rational number. We'll discuss now how to convert a repeating decimal into a fraction of integers.

Example 23 : We begin with a specific example: $.237237237237.... = \overline{237}$ Let's name our number $a = 0.237237237..$ and multiply it by a power of 10 until we get two different numbers that agree after the decimal point. Then subtract the smaller number from the larger so that the repeating decimal parts cancel each other in the subtraction:

$$\begin{aligned} a &= 0.237237237237\ldots \\ 10a &= 2.37237237237\ldots \\ 100a &= 23.7237237237\ldots \\ 1000a &= 237.237237237237\ldots \\[1em] 1000a &= 237.237237237237\ldots \\ a &= \quad 0.237237237237\ldots \\ 999a &= 237 \end{aligned}$$

$a = \frac{237}{999}$, which in lowest terms is $\frac{79}{333}$.

Example 24 : Here is a slightly more complicated example where our decimal representation begins with one block of digits before it switches over to a different block that repeats: $a = .146\overline{32} = .14632323232...$ To convert this decimal representation into a quotient of two integers, we want to find two different multiples of a by a power of 10 which agree completely after the decimal point. In this case

$$\begin{aligned} a &= .14632323232.. \\ 10^3 a &= 146.323232..... \\ 10^5 a &= 14632.323232.. \end{aligned}$$

so that

$$\begin{aligned} 10^5 a &= 14632.323232.. \\ 10^3 a &= \quad 146.323232..... \\ (10^5 - 10^3)a = 99000a &= 14486 \end{aligned}$$

Thus $a = .146\overline{32} = .14632323232..$ is the decimal representation of the rational number $\frac{14486}{99000} = \frac{7243}{49500}$.

Exercise 81 : Each of the following repeated decimals represents a rational number in the form $\frac{m}{m}$. Find m and n.

(a) $0.\overline{315}$

(b) $0.388\overline{4290}$

(c) $0.4425\overline{59}$

Exercise 82 : Outline a procedure that will convert any repeating decimal into a quotient of two integers. Assume there is an initial block of k digits after the decimal point followed by a repeating block of n digits.

Theorem 65. *A real number is rational if and only if it has a repeating decimal representation.*

It is easy to specify a decimal representation which is not repeating. After the decimal point, write down 1 followed by one 0. Then write 1 followed by two 0's, then 1 followed by three 0's, 1 followed by four 0's, and so on. This forms the number whose initial string of decimal digits looks like

$$z = .1010010001000010000010000000....$$

Exercise 83 : Show that the nth decimal digit of the number z is 1 exactly when n is of the form $n = \frac{k(k+1)}{2}$ for some positive integer k;otherwise the nth digit is 0.

Exercise 84 : Show that the decimal representation for the number z we have specified is not a repeating decimal.

One consequence of this description of z is a constructive proof that not all real numbers are rational: we have explicitly described one such number! It can be shown that many commonly occurring real numbers, such as $\sqrt{2}$ and π are not rational.

Example 25 : We'll present an outline of a proof that $\sqrt{2}$ is not rational. You should fill in the details. Assume to the contrary that $\sqrt{2}$ is rational and write it as fraction in lowest terms:

$$\sqrt{2} = \frac{p}{q}$$

where p and q have no common factors; in particular, at least one of them must be odd. We shall arrive at the contradiction that, in fact, both p and q must be even. Squaring both sides of the equation yields

$$2 = \frac{p^2}{q^2}$$

or

$$p^2 = 2q^2$$

Now $2q^2$ is divisible by 2 so p^2 is also divisible by 2. But if p^2 is divisible by 2, p must also be divisible by 2; otherwise p would be odd and the square of an odd integer is also odd. Since p is divisible by 2, there is an integer k so that $p = 2k$. Thus

$$2q^2 = p^2 = (2k)^2 = 4k^2$$

which implies that

$$q^2 = 2k^2$$

but this forces q to be an even number as well. Our assumption that $\sqrt{2}$ is rational led to a contradiction, so $\sqrt{2}$ is not a rational number.

Exercise 85 : Prove that $\sqrt{3}$ is not rational.

Definition 27 : A real number is **irrational** if it is not rational.

Theorem 66. *If n is a positive integer which is not a perfect square, then $\sqrt{n}$ is irrational.*

Theorem 67. *If a is a nonzero rational number and b is an irrational number, then $a+b$ and ab are irrational.*

Exercise 86 : Provide examples of irrational numbers a and b such that $a+b$ and ab are rational.

Theorem 68. *The set of rational numbers is an infinite, countable set.*

Theorem 69. *The set of real numbers in the open interval (0,1) is infinite.*

Exercise 87 : For each subset **A** of the positive integers **J**, we can define a real number a = .$a_1a_2a_3$... in $[0,1)$ by the rule a_n = 1 if n is an element of **A** and 0 otherwise. Let **f** be the set of ordered pairs $\{(\mathbf{A},a)\}$ defined by this rule.

(a) Show that **f** is a function.

(b) Determine **f(J)**.

(c) Determine **f(Ø)**.

(d) Determine **f(E)** if **E** is the set of even positive integers.

(e) Find the subset **A** of positive integers so that **f(A)** is the decimal representation of the fraction $\frac{1}{90}$.

(f) Show that **f** is a one-to-one function from the power set of **J** onto the interval $[0,\frac{1}{9}]$.

Theorem 70. *The power set of the positive integers is equivalent to the set of numbers in the open interval (0,1).*

Exercise 88 : Cantor's Diagonal Argument. Suppose **f** is any function from the set of positive integers into the open interval (0,1) of real numbers. Define a number b = .$b_1b_2b_3$... in (0,1) by the rule

$$b_n = \begin{cases} 5 & \text{if } n\text{th digit of } f(n) = 3, \\ 3 & \text{if } n\text{th digit of } f(n) \neq 3 \end{cases}$$

Note that $b \neq f(1)$ since the first digit of b is different from the first digit of $f(1)$ and $b \neq f(2)$ since the second digit of b is different from the first digit of $f(2)$. Show that b is not in the range of **f**.

Theorem 71. *No function from the positive integers to the real numbers in the open interval (0,1) can be an onto function.*

Theorem 72. *The set of positive integers and the set of real numbers in the open interval (0,1) are not equivalent.*

Theorem 73. *If $\boldsymbol{Z}$ is the set of integers and $\boldsymbol{R}$ is the set of reals, then $\boldsymbol{Z} \prec \boldsymbol{R}$.*

Cantor's proof that the integers and the real numbers constituted different kinds of infinite sets – that one infinite set of numbers essential to many areas of mathematics could be larger than another infinite set occurring throughout the discipline – was a major breakthrough in the history of mathematics.

Georg Ferdinand Ludwig Philipp Cantor was born in St. Petersburg, Russia on March 3, 1845 but moved to Germany at age 11 with his family. His father was a successful merchant and stockbroker originally from Denmark and his Russian mother was quite musical; Cantor himself was an excellent violinist. He studied engineering at the Polytechnic of Zurich, then mathematics at the universities of Berlin and Göttingen, completing his doctoral dissertation in 1867. He joined the faculty of the University of Halle where he spent the rest of his academic career. Cantor suffered from a number of disabling bouts of depression brought on most likely by a bipolar disorder, exacerbated by hostility to his ideas and an inability to settle some of the conjectures he made, Cantor made contributions to number theory and the representation of functions as trigonometric series, but his major achievement was his work on set theory and infinity.

Some mathematicians quickly realized the significance of Cantor's discoveries and the new world of infinity they opened up. As we have seen, Hilbert described it as a "paradise" and "the finest product of mathematical genius and one of the supreme achievements of purely intellectual human activity." In 1904, the British Royal Society awarded Cantor its Sylvester Medal, the highest honor it can confer for mathematical work. But there were harsh critics as well. The eminent French mathematician Henri Poincaré expressed disapproval,claiming that future generations would consider Cantor's set theory as "a disease from which one has recovered." Decades after Cantor's death, the philosopher Ludwig Wittgenstein lamented that mathematics is "ridden through and through with the pernicious idioms of set theory," which he dismissed as "utter nonsense" that is "laughable" and "wrong."

The fiercest attacks in Cantor's lifetime came from his former teacher and mentor Leopold Kronecker(1823 - 1891). Kronecker staunchly believed that the only numbers worthy of consideration were the integers. For Kronecker, irrational numbers were fictions and had no place in mathematics. Kronecker vilified Cantor, calling him a "charlatan" a "renegade" and a "corrupter of youth." He did all he could to suppress Cantor's ideas, which he labeled "humbug." Among other tactics, Kronecker delayed or suppressed completely Cantor's and his followers' publications, launched both written and verbal personal attacks against him, belittled his ideas in front of his students and blocked Cantor's ambition of gaining a position at the prestigious University of Berlin. Cantor died on January 6, 1918 in the Halle sanatorium where he had spent the final year of his life.

In the end, Cantor's idea prevailed. The results he found that seemed astonishing and counterintuitive to his contemporaries are part of every mathematician's working knowledge today. Here is another result that certainly was surprising in the late 19th Century:

Theorem 74. *The power set of the set of positive integers* $\boldsymbol{J}$ *is equivalent to* $\boldsymbol{R}$*, the set of real numbers.*

The rational numbers and the algebraic numbers form countable sets. The real numbers form an infinite set of a size which is larger than these sets. What can we say about the real numbers which are not algebraic?

Definition 28 : A real number is *transcendental* if it is not algebraic.

Theorem 75. *The set of transcendental numbers* $\boldsymbol{T}$ *is infinite.*

Theorem 76. *The set of transcendental numbers* $\boldsymbol{T}$ *is equivalent to the set of real numbers.*

From this last theorem, we see that in an informal sense, "most" real numbers are transcendentals. A simple way to show that a number is algebraic is to display a polynomial equation with integer coefficients for which it is a solution. It is not as easy to demonstrate that a particular number is transcendental. It is known, but difficult to prove, that both π and e (the base of the natural logarithm) are transcendental. These results were not established until the late 19th Century. Even today, it is not known whether $\pi + e$ and πe are algebraic or transcendental.

We'll close this section with some results on how the rationals, irrationals and transcendentals are distributed among the real numbers.

Theorem 77. *Every interval of positive length contains a rational number.*

Theorem 78. *Every interval of positive length contains an irrational number.*

Theorem 79. *Every interval of positive length contains a transcendental number.*

Theorem 80. *Every interval of positive length contains an infinite subset of rational numbers.*

Theorem 81. *Every interval of positive length contains an infinite subset of irrational numbers.*

Theorem 82. *Every interval of positive length contains an infinite subset of transcendental numbers.*

Chapter 13

Infinity and Dimension

"I am so in favor of the actual infinite that instead of admitting that Nature abhors it, as is commonly said, I hold that Nature makes frequent use of it everywhere, in order to show more effectively the perfections of its Author. Thus I believe that there is no part of matter which is not - I do not say divisible - but actually divisible; and consequently the least particle ought to be considered as a world full of an infinity of different creatures."
Georg Cantor

Exercise 89 : Let **A** be the unit interval and **B** be the unit square where we always choose the repeating decimal when we have a choice of representing a number as a repeating or a terminating decimal; e.g., we represent $\frac{3}{4}$ as .749999.. = .74$\overline{9}$ rather than .75. Define the "shuffling" sets **f** and **g** by the following

$$\mathbf{f} = \{(.x_1x_2x_3x_4x_5\ldots, (.x_1x_3x_5x_7, .x_2x_4x_6x_8\ldots) : (.x_1x_2x_3x_4x_5\ldots) \in \mathbf{A}\}$$

$$\mathbf{g} = \{((.x_1x_2x_3x_4\ldots, .y_1y_2y_3y_4\ldots), .x_1y_1x_2y_2x_3y_3..) : (.x_1x_2x_3x_4\ldots, .y_1y_2y_3y_4\ldots) \in \mathbf{B}\}$$

(a) Show that $\mathbf{f}(\frac{47}{99}) = \mathbf{f}(.47474747..) = (.444\ldots, .777) = (\frac{4}{9}, \frac{7}{9})$.

(b) Show that $\mathbf{f}(\frac{2861}{9999} = (\frac{26}{99}, \frac{81}{99})$.

(c) Compute $\mathbf{g}(.\overline{369}, .\overline{24})$

(d) Compute, if possible $\mathbf{f}(\frac{1}{2})$. Does it make a difference if we represent $\frac{1}{2}$ as .5000... or as .49999...?

(e) Is **f** a function?

(f) If we agree only to represent real numbers as repeating decimals (*e.g.* we use .4999... instead of .5000..), is **f** a function?

(g) Is **f** a one-to-one function?

(h) Does **f** map **A** onto **B**?

(i) Is **g** a function?

(j) Is **g** a one-to-one function?

(k) Does **g** map **B** onto **A**?

Having shown that the real numbers are not countable, Cantor had discovered that two of the most important sets in mathematics, the reals and the rationals, represent two different "sizes" of infinity. Cantor then wondered if there were other familiar mathematical sets that represented even more different sizes of infinity. A natural place to look was geometric objects of higher dimension. What could Cantor discover about the nature of the infinity that the unit square and the unit cube, for example, possessed?

In a January 5, 1874 letter to Dedekind, Cantor expressed his initial belief that the square and the line were of different orders of infinity:

> "Can a surface (say a square that includes the boundary) be uniquely referred to a line (say a straight line segment that includes the end points) so that for every point on the surface there is a corresponding point of the line and, conversely, for every point of the line there is a corresponding point of the surface? I think that answering this question would be no easy job, despite the fact that the answer seems so clearly to be 'no' that proof appears almost unnecessary. "

Cantor alternated between believing (a) that the unit interval and the unit square represented different levels of infinity and (b) there were more points in the square than the interval. Finally, in 1877, he discovered a rigorous proof that two sets are equivalent. Indeed, Cantor proved that sets of different dimension still contained the same number of points. Cantor wrote to his friend Richard Dedekind: "Je le vois, mais je ne le crois pas!" ("I see it, but I don't believe it!")

Theorem 83. *If **I** is the unit interval and **S** is the unit square, then* $\boldsymbol{I} \sim \boldsymbol{S}$.

Theorem 84. *If **I** is the unit interval and **C** is the unit cube then* $\boldsymbol{I} \sim \boldsymbol{C}$.

Definition 29 : We can easily generalize the definition of the unit interval (dimension 1), the unit square (dimension 2) and the unit cube (dimension 3) to higher dimensions. The *k*th dimensional unit box $\mathbf{B^k}$ is the set

$$\{(x_1, x_2, x_3, \ldots x_k) : 0 \le x_j \le 1, 1 \le j \le k\}.$$

Theorem 85. *For any two positive integers m and n,* $\boldsymbol{B^m} \sim \boldsymbol{B^n}$.

Chapter 14

Some Perplexing Questions

"For nothing worthy proving can be proven, Nor yet disproven"
Alfred, Lord Tennyson, *The Ancient Sage*

We conclude this initial venture into set theory and infinity with a brief look at two major problems. The first, shown in the guise of Russell's Paradox, points out the need for a more formalized approach to this entire subject. The second, illustrated by the Continuum Hypothesis, challenges what we know and what is knowable about the real numbers.

Our development of Cantor's approach to infinite sets rests on the notions of elementary set theory that we presented. Mathematicians call our treatment of sets *Naive Set Theory*. We used ordinary natural language to describe the basic properties of sets and operations on sets to produce new sets. We did not subject words such as *and, or, not* or *some* to a rigorous definition, but rather relied on an assumption that we share a common intuitive understanding of these terms. The naive approach helps develop a facility for working with sets and can provide a foundation for verifying the correctness of Cantor's discoveries about the infinite natures of concrete sets such as the rationals and the reals.

Unfortunately, *Naive Set Theory* is fraught with difficulties that can lead to disturbing paradoxes. In particular, the naive approach does not stipulate carefully enough what assemblages of things can suitably be called sets. Bertrand Russell showed how allowing for the "set of all sets" can lead to logical difficulties.

Russell's Paradox

Definition 30 : A set is **ordinary** if it does not contain itself as an element. A set is **extraordinary** if it does contain itself as a member. By our definitions, every set is either ordinary or it is extraordinary.

Example 26 : The set $\mathbf{A} = \{1, 2, 3\}$ is an ordinary set. Each element of **A** is a positive number. The set $\{1, 2, 3\}$ is not an element of **A** although it is a subset of **A**.

Example 27 : The set of positive integers is an ordinary set.

Example 28 : The set **A** of all abstract ideas is an extraordinary set as **A** itself is an abstract idea.

Example 29 : The set **C** of all things which are not dogs is an extraordinary set since **C** is not a dog.

Russell's Paradox concerns the set $\mathfrak{R}$ of all ordinary sets. Every element of $\mathfrak{R}$ is an ordinary set and every ordinary set is a member of $\mathfrak{R}$. Russell asks the question "Is $\mathfrak{R}$ an ordinary set or is it an extraordinary set?"

Theorem 86. *Russell's Paradox: Let $\mathfrak{R}$ be the set of all ordinary sets.*

(a) If $\mathfrak{R}$ is ordinary, then it is extraordinary.

(b) If $\mathfrak{R}$ is extraordinary, then it is ordinary.

Russell's Paradox demonstrated that an intuitive or "naive" approach to set theory was not an adequate basis for mathematics. One of Hilbert's students, Ernst Zermelo (1871 - 1953) presented the first rigorous approach to the subject through the creation of an **axiomatic set theory** in 1905. In the early 1920's, Abraham Fraenkel formulated an improved collection of axioms which strengthened Zermelo's approach. Zermelo-Fraenkel Set Theory with the Axiom of Choice (ZFC) is currently the standard form of axiomatic set theory and is the most commonly used foundation of modern mathematics. The **Axiom of Choice** states that if we are given any collection of nonempty sets, then it is possible to build a new set which contains precisely one element from each member of the collection.

The Continuum Hypothesis

Cantor was the first person to understand that there were at least two different levels of infinity. In particular, as you have seen, the integers and the real numbers are infinite sets which are not equivalent to one another. Since the integers are contained within the real numbers, the latter set has a "larger size" than the smaller. In his search for other infinities, Cantor looked at common geometric figures of higher dimension. But these sets - the solid square and the solid cube - proved to be equivalent to the real line. The situation did not change by looking at higher dimensional analogues. Cantor did find that he could construct towers of infinite sets of different sizes by starting with any infinite set, then taking its power set, followed by the power set of the power set, followed by the power set of that set, etc.

Another place Cantor examined with some care, but without a final resolution, was within the real numbers themselves. Could there lurk within the real numbers, Cantor asked, a subset representing a third type of infinity, a size somehow intermediate between the integers and the reals?

Question 4 : Is there an infinite set of real numbers which is neither countable nor equivalent to set of all real numbers?

Exercise 90 : Examine several sets of real numbers each of which contains the positive integers and determine if each of them is equivalent to the reals, equivalent to the rationals, or equivalent to neither.

Cantor could not find a third level of infinity within the reals. In 1877, he formulated a conjecture that such sets do not exist. Another term for the real numbers was **the continuum** so Cantor's conjecture - which he was never able to prove or disprove - became known as **The Continuum Hypothesis**.

The Continuum Hypothesis: If $\mathbf{Z}$ is the set of integers, $\mathbf{R}$ is the set of reals and $\mathbf{A}$ is any set such that $\mathbf{Z} \subset \mathbf{A} \subset \mathbf{R}$, then either $\mathbf{A} \sim \mathbf{Z}$ or $\mathbf{A} \sim \mathbf{R}$.

The Continuum Hypothesis essentially asserts "There is no set whose size is larger than the size of the positive integers but smaller than the size of the real numbers" or, more succinctly, "There is no infinity between the integers and the reals."

Exercise 91 : Show that The Continuum Hypothesis can be formulated as: There is no set $\mathbf{B}$ such that $\mathbf{Z} \prec \mathbf{B} \prec \mathbf{R}$.

Cantor subsequently generalized this conjecture to a much broader claim. Having shown that the power set of the integers is equivalent to the real numbers; that is, $\mathbf{2^Z} \sim \mathbf{R}$, Cantor formulated what is now known as **The Generalized Continuum Hypothesis**: there is no level of infinity between the size of an infinite set and the size of its power set.

The Generalized Continuum Hypothesis If $\mathbf{A}$ is any infinite set, then there is no set $\mathbf{B}$ such that $\mathbf{A} \prec \mathbf{B} \prec 2^{\mathbf{A}}$.

We should note that power sets of large sets do occur in some natural settings.

Exercise 92 : Let $\mathbf{F}$ be the set of all functions from the real numbers $\mathbf{R}$ to the real numbers. Show that $\mathbf{F} \sim 2^{\mathbf{R}}$.

Cantor believed that the Continuum Hypothesis was true and was confident that the theory of sets he had created would provide the tools to substantiate its validity. But a proof eluded him despite years of trying. At one point in 1884, Cantor thought he had found an extraordinarily simple proof, only to discover a flaw in his argument two months later. Weeks later, he was convinced he had shown the conjecture false but he then saw his new claim – that there were infinitely many different level of infinity between the integers and the reals – was invalid. This back and forth pattern –he had a proof of the hypothesis, he had a proof it was false – persisted. As Amir Aczel describes it

> "...Cantor continued his touch-and-go relationships with the continuum hypothesis. After weeks of intensive work he would suddenly be convinced that he had found a proof of the theorem. Then he would find a fatal flaw in his derivations, and a few weeks later he would suddenly be sure that he had found a proof of the opposite result. Through this ordeal, and aggravated by Kronecker's continuing attacks, Cantor slowly went mad."

David Hilbert presented the major speech at the International Congress of Mathematicians held in Paris in 1900. In his address, titled "Mathematical Problems," he listed establishing the truth or falsehood of the Continuum Hypothesis as the first of the twenty-three challenges mathematicians should address in the future. Some 40 years later, Kurt Gödel, (1906 – 1978) showed that Cantor's guess can never be disproved from the other axioms of mathematics. In 1963, Paul Cohen (1934 – 2007) showed that the Continuum Hypothesis was unprovable. More formally, their work showed that if ZFC was a consistent set of axioms, then adding the truth of the Continuum Hypothesis as an additional axiom still yielded a consistent system but that adding the falsity of the Continuum Hypothesis instead also resulted in a consistent system. The Continuum Hypothesis is *independent* of the standard Zermelo–Fraenkel axioms of set theory.

It's no wonder to us then why Cantor was unable to establish the truth or falsity of the Continuum Hypothesis. We know now that the axioms of set theory Cantor was using were not up to the task. Cantor came to believe very strongly, in almost a religious sense, that the Continuum Hypothesis must be true. Many of his nervous breakdowns which led to confinement in an asylum came after he put prolonged effort on this problem. He died in a mental hospital in Halle, decades before Gödel and Cohen did their work.

For many mathematicians, Gödel's and Cohen's results raised fundamental philosophical questions about the nature of truth. There are different schools of philosophy within the mathematical community. *Formalists* believe that mathematical statements don't have an intrinsic truth or falsity – about the only thing that can asserted about a statement is whether it can be proved in a given axiom system. To a formalist, it makes little sense to ask whether the Continuum Hypothesis is true or false: if the Continuum Hypothesis can't be decided within the standard axiom system of mathematics, then the hypothesis must be inherently vague. In contrast, *Platonists* affirm mathematical objects such as sets have an existence in an ideal universe; an axiomatic system is simply a tool to determine which statements about those objects are true. To Platonists, the Continuum Hypothesis feels like a concrete statement that should be true or false. If the standard axioms can't settle the Continuum Hypothesis, it's not that the hypothesis is meaningless. The problem is that the axiom system needs improvement.

The question of how to modify an axiom system such as ZFC to make it strong enough to settle the Continuum Hypothesis while preserving the "self-evident" nature of the axioms remains an open research question in set theory. Recent results by Hugh Woodin indicate that there may be a choice of axioms which could tell us more about the smallest uncountably infinite sets and that these axioms imply that the continuum hypothesis is false. "If there's a simple solution to the continuum hypothesis, it must be that it is false," Woodin has written. If it is false, then there are indeed infinite sets bigger than the counting numbers and smaller than the real numbers.

At this point in history, although we might be close, we still do not know the final resolution of Cantor's question about the nature of the real numbers represented by the Continuum Hypothesis. Perhaps you may contribute to our eventual understanding.

"This – the number of points on a line – is still the big unsolved riddle of mathematics. The enigma of infinity still haunts us..."
Reviel Netz

Appendix A: Suggested Projects

"In mathematics the art of proposing a question must be held of higher value than solving it."
Georg Cantor

Our investigations so far, challenging and possibly frustrating for you at times, are now seen as the very beginning of the rich and deep modern theory of sets. In this section, we outline a few suggested topics for you to investigate,

1. As we have seen, Cantor's approach, which **defines** an infinite set as one which is equivalent to a proper subset has the advantage of giving us a property of infinite sets we can use to prove a number of properties of such sets. Note, however, that Cantor defines **finite set** in a negative way: a set is finite if it is not infinite. We are left with the task of proving all of the properties of finite sets we've always assumed true are in fact true; in particular, a nonempty finite set is equivalent to $\mathbf{Z_n}$ for some nonnegative integer *n*. An alternative approach to infinite sets would be to define a finite set to be one equivalent to some $\mathbf{Z_n}$ and then **define** an infinite set to be one which is not finite.

   Work through the theorems about infinite sets and see how many of these you can prove with this definition. In particular, can you prove if **A** is a nonempty set which is not equivalent to any $\mathbf{Z_n}$ then **A** must be equivalent to one of its proper subsets?

2. A third approach to finite and infinite sets is to try to have the best of both worlds. *Define* an infinite set as one which is equivalent to a proper subset and *define* a finite set to be one which is empty or equivalent to some $\mathbf{Z_n}$. Now you know something positive about each type of set. What remains to be shown? One theorem you would have to prove is that every set is either finite or infinite and that no set can be both.

3. *Peano Axioms* The positive integers (also called the **Natural Numbers**) are fundamental to all of arithmetic. We have assumed, without

explicit statements, that the addition and multiplication operations of these numbers are well-defined and satisfy commutative, associative and distributive laws. For example, if a , b and c are any positive integers, then $a+b$ and $a \times b$ are positive integers with the properties that $a+b=b+a$, $a \times b = b \times$, $a+(b+c)=(a+b)+c$, $a \times (b \times c) = (a \times b) \times c$, $a \times (b+c) = a \times b + a \times c$, etc.
Several mathematicians in the late 19th Century provided rigorous axiomatic treatments of the positive integers. Giuseppe Peano (1858 - 1932) provided the most widely adopted set of axioms. Peano's Axioms assume the existence of a set **J** with the following properties:

- There exists an element 1 in **J**.
- For every element $n \in$ **J**, there is a unique **successor** element **(**S)(n).
- 1 is not the successor of any element in **J**.
- Every pair of distinct elements has distinct successors: If $m \neq n$, then **(**$S)(n) \neq$ **(**$S)(n)$.
- If **A** is any subset of **J** which contains 1 and which also contains the successor of any element in **A**, then **A = J**.

Investigate how addition and multiplication can be defined in terms of the successor idea so that the commutative, associative and distributive laws hold. See the references by Edmund Landau and Inder Rana for more information.

4. *Defining the Rationals.* Investigate a more rigorous definition and development of the rational numbers based on defining the rationals as the set of ordered pairs (r,s) of integers where the second term is nonzero. Two rational numbers (a,b) and (r,s) are defined to be equal if and only if the integers as and br are equal. We define addition of rationals by $(a,b)+(r,s) = (as+br,bs)$ and multiplication by $(a,b) \times (r,s) = (ar,bs)$. Since the basic arithmetic operations for rationals are defined in terms of corresponding operations within the integers, their properties will follow from the corresponding ones for the integers. Show that addition and multiplication of rationals are commutative and association, that the distributive laws hold, and discuss how division of rationals should be carried out.

5. *Defining the Reals* The least rigorous part – and the part conceptually most advanced to present carefully – of these notes is our lack of a careful definition of a **real number**. There are several ways of using the rational numbers to define the reals. One approach uses sets of rational numbers called **Dedekind cuts**. A nonempty proper subset **A** of **Q** is a Dedekind cut if (1) **A** has no maximum; that is, for each $r \in$ **A**, there is an $s \in$ **A** with $s > r$ and (2) if $r \in$ **A** then **A** contains

every rational number less than r. The real numbers are then defined to be the set of all Dedekind cuts. See Section 8.4 and Chapter 1 of Steve Abbott's *Understanding Analysis* for details.

6. Cantor himself provided a different approach to defining the real numbers that you may wish to examine. By a **rational sequence**, we simply mean a function $\mathbf{f}$ from the positive integers $\mathbf{J}$ into the rational numbers $\mathbf{Q}$. A rational sequence is called a **Cauchy rational sequence** if for every positive rational number ε, there is a positive integer N such that
$$|\mathbf{f}(n) - \mathbf{f}(m)| < \varepsilon \text{ whenever } m, n > N.$$
Cantor's idea was to essentially define the real numbers to be the set of all Cauchy rational sequences.

7. *Transcendental Numbers.* The transcendental numbers are a fascinating part of the real number system. They are linked to a geometric problem of antiquity: can you construct, using only a straightedge and a pair of compasses, a square with the same area as a unit circle? Transcendental numbers also arise when examining which irrational numbers can be approximated by rapidly converging sequences of rational numbers. Almost all we know about transcendental numbers was discovered in the last century and a half. The books by Ivan Nivens provide a good place to begin studying them.

8. *Cardinal Numbers*. While we have used the idea of set equivalence to capture the idea that two sets "have the same size," we have carefully avoided talking about "the size of a set." While we have discussed what it means for one set to be "of larger size" than another set, we have not actually defined the "size of a set." Investigate how mathematicians, beginning with Cantor, have developed a careful definition of the **cardinality** or **size** that made it possible to extend the definition of addition and multiplication of the integers to addition and multiplication of cardinal numbers. Investigate how the size of a set is defined and what are properties of cardinal arithmetic. The *Further Reading* section below gives some references; Halmos' book *Naive Set Theory* is a good starting point.

9. *Ordinal Numbers*. Just as Cantor created the cardinal numbers to extend the measure of the size of a finite set to measure the sizes of infinite sets, he also created in 1897 another extension of the natural numbers called **ordinal numbers**. We can use a positive integer to describe a finite set's size, but we can also it to describe the *position* of an element in a sequence: the first object, the second object and so on. The ordinal numbers play an important role in discussing ordered sets.

We can define addition and multiplication of ordinals, but these operations, unlike cardinal arithmetic, are not commutative. Thus more surprising results abound. Again, starting with Halmos is suggested.

10. We have mentioned the importance of **Axiomatic Set Theory**, the **Continuum Hypothesis** and the **Axiom of Choice**. Further investigation of any of these topics would prove enlightening. See the suggestions in the section below on **Further Reading**.

Appendix B: Further Reading

"It is hard to be finite upon an infinite subject, and all subjects are infinite."
Herman Melville

We'll conclude with some suggested readings that will take you deeper into many mathematical aspects of the infinite.

Interesting accounts of infinity for the general reader include:

Barrow, John, *The Infinite Book: A Short Guide to the Boundless, Timeless and Endless,* Pantheon, 2005.

Maor, Eli: *To Infinity and Beyond: A Cultural History of the Infinite*, Princeton University Press, 1991.

Moore, A. W.: *The Infinite*, Routledge, 1991.

Vilenkin, Naum: *In Search of Infinity*, translated by Abe Shenitzer, Boston: Birkhäuser.

Rucker, Rudy: *Infinity and the Mind: The Science and Philosophy of the Infinite*, Princeton University Press, 2004.

To learn more about Zeno's paradoxes, examine

Salmon, Wesley: *Zeno's Paradoxes*, Hackett, 2001.

Mazur, Joseph: *Zeno's Paradox: Unraveling the Ancient Mystery Behind the Science of Space and Time*, Plume, 2008.

Faris, J. A. : *The Paradoxes of Zeno*, Avebury, 1996.

You can follow the developing story of the Archimedes palimpsest at the website http://www.archimedespalimpsest.org. A recent study of Archimedes' work is Hirshfeld, Alan: *Eureka Man: The Life and Legacy of Archimedes*, Walker, 2009.

Stillman Drake was the leading Galileo scholar of the Twentieth Century. Drake's translation of Galileo's *Two New Sciences* was published by the University of Wisconsin Press in 1974. You may wish to look at two other books by Drake: *Galileo At Work*. University of Chicago Press, 1978 and *Galileo: Pioneer Scientist*, University of Toronto Press, 1990.

Bolzano's *Paradoxes of the Infinite* is included in *The Mathematical Works of Bernard Bolzano*, edited by Steve Russo: Oxford University Press, 2005.

The classic introduction to elementary set theory is the text by Paul Halmos, *Naive Set Theory*, Springer, 1998. Halmos provides an introduction to cardinal and ordinal numbers as well. In *An Outline of Set Theory* (Springer, 1986), James Henle provides a treatment of axiomatic set theory within a problem oriented approach. Keith Devlin's *The Joy of Sets: Fundamentals of Contemporary Set Theory* (Springer, 1993) gives another lively account of Zermelo-Fraenkel set theory.

The definitive biography of Cantor is Joseph Dauben's *Georg Cantor: His Mathematics and Philosophy of the Infinite*, Princeton University Press, 1990. Amir Aczel offers a shorter study of Cantor's life and work, centered on his struggle with the Continuum Hypothesis: *The Mystery of the Aleph: Mathematics, the Kabbalah, and the Search for Infinity*, Washington Square Press, 2001.

Two of Cantor's most important papers are available in English translation: (1) "On a Property of the Set of Real Algebraic Numbers," in volume 2, pp. 839-843 and (2) "On an Elementary Question in the Theory of Manifolds," in volume 2, pp. 923-940 of *From Kant to Hilbert. A Source Book in the Foundations of Mathematics* edited by William Ewald, Oxford: Clarendon Press, 1999. Philip E. B. Jourdain offers an English translation and long introduction to some of Cantor's work of the 1890 's in *Contributions to the Founding of the Theory of Transfinite Numbers*, Nabu Press, 2010. This is a reprint of a book originally published in 1913.

For a more complete history of the Cantor–Bernstein Theorem and extended discussions of more than 30 proofs, see Arie Hinkis, *Proofs of the Cantor–Bernstein Theorem: A Mathematical Excursion*, Birkhäuser, 2013.

For careful construction and axiomatization of the integers, rational and real numbers, consult these treatments:

Abbott, Stephen, *Understanding Analysis*, Springer, 2001,

Landau, Edmund, *Foundations of Analysis*, AMS Chelsea, 2001,

Rana, Inder, *From Numbers to Analysis*, World Scientific, 1998,

Rosenlicht, Maxwell, *Introduction to Analysis*, Dover, 1985, and

Rudin, Walter, *Principles of Mathematical Analysis*, McGraw-Hill, 1976.

You can find out more about the Continuum Hypothesis in these advanced level treatments:

Gödel, Kurt, *The Consistency of the Continuum Hypothesis*, Ishi Press, 2008,

Cohen, Paul, *Set Theory and the Continuum Hypothesis*, Dover, 2008, and

Woodin, W. H. "The Continuum Hypothesis, volume 48 of *Notices of the American Mathematical Society* : Part I (June/July, pp. 567 – 576) and Part II (August, pp. 681 – 690), Part II. These papers are available at at http://www.ams.org/notices/200106/fea-woodin.pdf and t http://www.ams.org/notices/200107/fea-woodin.pdf.

Major book-length works on the Axiom of Choice include:

Howard, Paul and Rubin, Jean, *Consequences of the Axiom of Choice*, American Mathematical Society, 1998,

Jech, Thomas, *The Axiom of Choice*, North-Holland, 1973,

Moore, Gregory, *Zermelo's Axiom of Choice: Its Origins, Development, and Influence*, Springer-Verlag, 1982, and

Rubin, Herman and Jean, *Equivalents of the Axiom of Choice*, North-Holland, 1985.

John Stillwell's *Roads To Infinity: The Mathematics of Truth and Proof*, A K Peters, 2010 focuses on the interaction between set theory and logic in a lively account accessible to readers who have worked carefully through these notes.

To learn more about irrational and transcendental numbers, I would recommend you begin with Ivan Niven's *Numbers: Rational and Irrational*, Mathematical Association of America, 1961 and then move on to his more advanced treatment *Irrational Numbers*, Mathematical Association of America, 2005.

Appendix C: *The Journal of Infinity Studies*

The *Journal of Infinity Studies* endeavors to publish full, clear and complete proofs of all Theorems, answers to all Questions, solutions to all Exercises, and resolutions of all Conjectures presented in *Discovering Infinity*.

Referee Reports

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(http://atom.iop.org/atom/usermgmt.nsf/refereeservices)

You may be asked to referee for publication a result discovered by one of your peers and written up for review. In most cases, the author has presented the result successfully in class. The formal written exposition of the result may, however, contain logical errors that were not discovered during the verbal presentation. The exposition may also be difficult to follow and/or it may contain typographical, grammatical and other errors.

As a referee your work is invaluable. We thank you for the time and effort you spend on reviewing the papers of your peers for publication in our *Journal of Infinity Studies.*

General Procedure

Papers submitted for publication are generally sent to two independent referees who are asked to report on the scientific quality and originality of the work and its presentation. The referees will normally be other students who have claimed that they have also developed correct proofs. The referees will complete a written report and return it to the journal editor. The editor will, in turn, send the referee reports to the author. You should bear this in mind when preparing your report. The identity of referees is strictly confidential and we ask that you do not transmit your report directly to the author. We are committed to publishing only high-quality material in our journal. If there is sufficient agreement between the referees,

(a) the paper may be accepted,

(b) the referees' reports may be sent to the author for amendments or revision of the article, or

(c) if the paper contains too many errors for the referees to comment fully on the scientific content, the author will be asked to make major revisions and then resubmit the article.

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When authors make revisions to their article in response to the referees' comments, they are asked to submit a list of changes and any replies for transmission to the referees. The revised version is usually returned to at least one of the original referees who is then asked whether the revisions have been carried out satisfactorily.

Instructions for Referees

When a manuscript is sent to you for review, you will be asked to confirm that you are able to report, and whether you are able to do so by the given deadline or would like an extension. It is important that you let us know as soon as possible whether or not you will be able to review the article, as we will not usually select an alternative referee until we have heard from you, and this can cause delays in publication.

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Your report form should be divided into sections which deal with accuracy, scientific quality, scientific content and interpretation.

Please indicate your assessment of the article. We ask also that you supply comments suitable for transmission to the authors. It would be of great help if you can address the following key points when you assess the article and write your report.

1. **Technical**

   - Mathematical merit: is the work logically rigorous, accurate and correct?
   - Clarity: are ideas expressed clearly and concisely? Are the concepts understandable? Is the discussion written in a way that is easy to read and understand?
   - Completeness: Are clear reasons given for all assertions? Are the appropriate previously proved Theorems cited? Are all major relevant definitions included? Are any new concepts clearly and completely defined?
   - Notation: Does the author's use of mathematical notation help or hinder understanding of the exposition? If nonstandard notation is used, does the author explain what it means?
   - English: Indicate where corrections to grammar, syntax, and punctuation are needed. It is especially helpful if you correct the English where the scientific meaning is unclear.

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   - Title: is it adequate and appropriate for the content of the article? Does the title indicate which Theorem, Question, Exercise or Conjecture is the subject of the paper?
   - Statement of Result: does the paper clearly state the content of Theorem, Question Problems or Conjecture?
   - Diagrams and figures: are they clear and essential?
   - Text and mathematics: are they brief but still clear? If you recommend shortening, please suggest what should be omitted.
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