MATH 3430-02 WEEK 7-3

Key Words: Constant coefficient differential operators.

This lecture we answer the following question more carefully:

For the equation

$$y'' + ay' + by = (a_0 + a_1 t + \cdots + a_k t^k)e^{\alpha t},$$

why do we guess the form of $y_p(t)$ the way we did in [**7-1**]*?*

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a. Let $D = \dfrac{\mathrm{d}}{\mathrm{d}t}$. In other words, for any differentiable function $f(t)$, we have

$$D(f(t)) = f'(t).$$

b. Analogously,

$$D^n(f(t)) = f^{(n)}(t).$$

The result can be understood as that of applying D successively n times to $f(t)$.

c. Let $D - g(t)$, where $g(t)$ is an arbitrary function, be the differential operator defined by

$$(D - g(t))f(t) = f'(t) - g(t)f(t).$$

d. More generally, we can define an n**-th order linear differential operator** by

$$P = p_n(t)D^n + p_{n-1}(t)D^{n-1} + \cdots + p_1(t)D + p_0(t)$$

by

$$P(f(t)) = p_n(t)f^{(n)}(t) + p_{n-1}f^{(n-1)}(t) + \cdots + p_1(t)f'(t) + p_0(t)f(t),$$

where $p_n(t) \neq 0$.

e. If we have two linear differential operators P and Q, we define their product by

$$(PQ)f(t) = P(Q(f(t))).$$

(By this definition, we cannot take the 'product' PQ naively, even when it is tempting to do so. For example, when $P = D$, $Q = 1$,

$$(PQ)f(t) = D((1)f(t)) = f'(t).$$

In this case, naive calculation $PQ = D(1) = 0$ will lead to an error.)

Q1. What is $D^2(\cos 2t)$?

Q2. What is $[D(D-t)]f(t)$? What is $[(D-t)D]f(t)$? Are they equal?

In general, if P, Q are two linear differential operators, $PQ \neq QP$. However...

Q3. Let λ, μ be constants. Simplify $[(D-\lambda)(D-\mu)]f(t)$ and $[(D-\mu)(D-\lambda)]f(t)$. Are they equal?

Q4. From your simplification in **Q3**, the expression

$$(D-\lambda)(D-\mu) = D^2 - \underline{\qquad\qquad} D + \underline{\qquad\qquad}.$$

Note that λ, μ are just the roots of the characteristic polynomial of the constant coefficient 2nd order differential operator on the right.

Q5. From the discussion above, we can write the equation

$$y'' - 2y' - 8y = e^t$$

in the form

$$[(D - \underline{\quad})(D - \underline{\quad})]y(t) = e^t.$$

Q6. Let r be a constant, simplify

$$(D-r)e^{rt}.$$

Q7. Let r be a constant, simplify

$$(D-r)(t^k e^{rt}), \qquad (k \geq 1).$$

Q8. Now find a particular solution for
$$(D-2)y(t) = te^{2t}.$$

Q9. Find a particular solution for
$$(D-2)^2 y(t) = te^{2t}.$$

Q10. Let r be a constant, simplify
$$(D-r)e^{\alpha t}. \quad (\alpha \neq r)$$

Q11. Let r be a constant, simplify
$$(D-r)(t^k e^{\alpha t}). \quad (k \geq 1, \alpha \neq r).$$

Q12. It follows that, when $r \neq \alpha$,
$$\begin{aligned}(D-r)((b_0 + b_1 t + \cdots + b_k t^k)e^{\alpha t}) = &\underline{\hspace{3cm}}e^{\alpha t}\\ + &\underline{\hspace{3cm}}te^{\alpha t}\\ + &\underline{\hspace{3cm}}t^2 e^{\alpha t}\\ \vdots&\\ + &\underline{\hspace{3cm}}t^k e^{\alpha t}.\end{aligned}$$

(The blanks are all constants.)

If the right-hand-side equals to $(a_0 + a_1 t + \cdots + a_k t^k)e^{\alpha t}$, the constants b_i are determined by the linear system

$$\begin{pmatrix} \\ \\ \\ \\ \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_k \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_k \end{pmatrix},$$

which always has a unique solution. This justifies our 'naive guess', when $r \neq \alpha$, for solving the equation

$$(D - r)y(t) = (a_0 + a_1 t + \cdots + a_k t^k)e^{\alpha t}.$$

Doing this twice handles equations such as

$$[(D - r_1)(D - r_1)]y(t) = (a_0 + a_1 t + \cdots + a_k t^k)e^{\alpha t},$$

where $r_1, r_2 \neq \alpha$. (Generalizations into n-th ($n > 2$) order equations is immediate, but we'll be contented with $n = 1, 2$ for now.)

When you encounter, say

$$(D - 3)(D - 2)y(t) = t^2 e^{2t}.$$

You know that a particular solution arises when (by seeing $(D - 2)y(t)$ as a whole)

$$(D - 2)y(t) = (At^2 + Bt + C)e^{2t},$$

for some constants A, B and C. From the above (**Q7**),

$$y(t) = \left(\frac{1}{3}At^3 + \frac{1}{2}Bt^2 + Ct\right)e^{2t}$$

is a particular solution.

In some sense, we are able to solve a constant coefficient ODE using only linear algebra.