MATH 3430-02 WEEK 4-2

Key Words: Linear Operators; What's special about linear; 2nd-order Existence and Uniqueness Theorem (linear case); Reduction of ODE problem to a linear algebra problem.

Ingredients:
(1) A second order linear ODE.

General form: ________________________;

Homogeneous general form: ________________________.

(2) The Wronskian.
Let $y_1(t), y_2(t)$ be differentiable, then

$$W[y_1, y_2](t) = \qquad\qquad\qquad\qquad .$$

Q1. Let L denote the following transformation: it takes a function $y(t)$ as an input and outputs

$$L[y](t) := y''(t) + (t^2 + 1)y'(t) + e^t y(t).$$

By this definition,

$$L[t^2] = \rule{5cm}{0.4pt}.$$

Now let $f(t), g(t)$ be two differentiable functions, c constant. We have

$$L[f_1](t) = \rule{5cm}{0.4pt};$$
$$L[f_2](t) = \rule{5cm}{0.4pt};$$
$$L[f_1 + f_2](t) = \rule{5cm}{0.4pt};$$
$$L[f_1](t) + L[f_2](t) = \rule{5cm}{0.4pt};$$
$$L[cf_1](t) = \rule{5cm}{0.4pt};$$
$$cL[f_1](t) = \rule{5cm}{0.4pt}.$$

From this, you observe

$$L[f_1](t) + L[f_2](t) \;\rule{1cm}{0.4pt} L[f_1 + f_2](t), \qquad L[cf_1](t) \rule{1cm}{0.4pt} cL[f_1](t).$$

In linear algebra, such a correspondence L is called a ____________ ____________.

We call the map L in **Q1** a **linear operator**. In general, a **second order linear operator** takes the form

$$L[y](t) := p_2(t)y'' + p_1(t)y' + p_0(t)y,$$

where $p_2(t) \neq 0$.

A linear operator satisfies the following property: given any $f(t), g(t)$ and constants a, b, we have

$$L[af + bg](t) = aL[f](t) + bL[g](t).$$

Using the language above, a second order linear ODE

$$y'' + p(t)y' + q(t)y = g(t)$$

may be written as

$$L[y](t) = g(t),$$

with $L[y](t) := y'' + p(t)y' + q(t)y$.

The notation L helps us focusing on the algebraic properties of solutions, as we'll see below.

Q2. Let L denote a 2nd order linear operator. Suppose that f, g are both solutions of the homogeneous equation

$$L[y](t) = 0.$$

Then,

$$L[f](t) = \underline{\qquad\qquad}, \qquad L[g](t) = \underline{\qquad\qquad};$$

hence,

$$L[f+g](t) = \underline{\qquad\qquad\qquad\qquad} = \underline{\qquad\qquad},$$

$$L[cf](t) = \underline{\qquad\qquad\qquad} = \underline{\qquad\qquad}.$$

It follows that the set of solutions of $L[y](t) = 0$ forms a ________ ________.

Q3. Suppose that f_1, f_2 are solutions of an inhomogeneous linear equation $L[y](t) = g(t)$; h a solution of the corresponding homogeneous equation $L[y](t) = 0$.

It follows that

$$L[f_1 - f_2](t) = \underline{\qquad\qquad}, \qquad L[f_1 + h](t) = \underline{\qquad\qquad}.$$

In other words, the difference between two general solutions of $L[y](t) = g(t)$ is a homogeneous solution; the sum of a solution of $L[y](t) = g(t)$ with a homogeneous solution is again a solution of $L[y](t) = g(t)$.

Does this remind you of similar results in linear algebra?

Now let's do a little reasoning.

0. Our goal is to solve a second order linear ODE, whose general form is

$$y'' + p(t)y' + q(t)y = g(t),$$

or $L[y](t) = g(t)$, if we specify $L[y](t)$ to be the left-hand-side of the previous equation.

1. To find all solutions, we only need to know two things: (i) a particular solution of $L[y](t) = g(t)$; (ii) all homogeneous solutions (i.e., solutions of $L[y](t) = 0$). This is because (see **Q3**) all solutions of $L[y](t) = g(t)$ are of the form:

$$y(t) = \text{a particular sol.} + \text{a homogenous sol.}$$

2. All **homogeneous solutions** form a vector space V_h, so if we know a basis, we know all of them.

3. To know a basis, we only need to know the dimension of V_h, say d, and a list of d linearly independent homogeneous solutions.

4. Finally, we need to know how to find a particular solution and a list of d linearly independent homogeneous solutions. We'll cover this part in later lectures.

Q4. Let's put this reasoning to practice. (You are not doing this from scratch, since I've found some solutions for you.)

The second order linear ODE:

$$y'' + 4y = e^t.$$

Check that $\cos 2t$ and $\sin 2t$ are homogeneous solutions.

Verify that $\cos 2t$ and $\sin 2t$ are linearly independent.

Suppose we know that the dimension of V_h (space of homogeneous solutions) is 2, a general homogeneous solutions can be written as

$$y_h(t) = \underline{\hspace{4cm}}.$$

Verify that $y_p(t) = \frac{1}{5}e^t$ is a solution of the inhomogeneous equation.

Therefore, a general solutions of $y'' + 4y = e^t$ are

$$y(t) = y_p + y_h = \underline{\hspace{6cm}}.$$

What is d, the dimension of the space of homogeneous solutions?

Ans. For n-th order linear ODEs, d is always n. In particular, for second order linear ODEs, $d = 2$.

This dimension result is the consequence of the following existence & uniqueness theorem.

Consider the second order **homogeneous** *linear initial value problem:*

$$y'' + p(t)y' + q(t)y = 0, \qquad y(t_0) = y_0,\ y'(t_0) = y_0'.$$

If both $p(t), q(t)$ are continuous on some interval (a, b), and $t_0 \in (a, b)$, then there exists **a unique** *solution of the IVP above on the interval (a, b).*

Q5. How is this theorem related to the dimension of V_h? Let's do some more reasoning:

1. Any homogeneous solution $y(t)$ (i.e., satisfying $L[y](t) = 0$) defined on (a, b) has an initial value, say

$$y(t_0) = \alpha, \qquad y'(t_0) = \beta.$$

2. By the existence and uniqueness theorem, there exists a unique homogeneous solution $y_1(t)$ satisfying

$$y_1(t_0) = 1, \qquad y_1'(t_0) = 0;$$

similarly, there exists a unique homogeneous solution $y_2(t)$ satisfying

$$y_2(t_0) = 0, \qquad y_2'(t_0) = 1.$$

3. We compute

$$W[y_1, y_2](t_0) = \underline{\hspace{2cm}}.$$

It follows that y_1, y_2 are ____________ ____________.

4. Consider the function

$$z(t) := \alpha y_1(t) + \beta y_2(t).$$

We have

- $L[z](t) =$ ________________.
- $z(t_0) =$ ________________.
- $z'(t_0) =$ ________________.

Therefore, $z(t)$ is a homogeneous solution with the same initial value as ________. By the uniqueness part of the theorem, we have

$$y(t) \underline{\hspace{1cm}} z(t) = \alpha y_1(t) + \beta y_2(t).$$

5. Based on the previous argument, $\{y_1(t), y_2(t)\}$ is linearly independent and spans the space of homogeneous solutions. Therefore it forms a basis for all homogeneous solutions. It follows that the space of homogeneous solutions has dimension 2.