Chapter 1

Review

Read the definitions, prove the theorems and do the exercises.

Definitions 1. Suppose A and B are subsets of a set X. We define the following:

1. $A \cup B := \{x \mid x \in A \text{ or } x \in B\}$;
2. $A \cap B := \{x \mid x \in A \text{ and } x \in B\}$;
3. $A \setminus B := \{x \mid x \in A \text{ and } x \notin B\}$.

Theorem 2. Let A be a subset of the set X. Then $X \setminus (X \setminus A) = A$.

Theorem 3. (DeMorgan's Laws) Let A and B be subsets of a set X. Then

1. $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$ and
2. $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$.

Definition 4. Let I be an indexing set. For each $\delta \in I$, let A_δ be a set. We define the following two sets:

1. $\bigcup_{\delta \in I} A_\delta = \{s \mid \text{ there exists } \delta \in I \text{ such that } s \in A_\delta\}$ and
2. $\bigcap_{\delta \in I} A_\delta = \{s \mid s \in A_\delta \text{ for all } \delta \in I\}$.

Theorem 5. (Generalised DeMorgan's Laws) Let $\{A_\delta \mid \delta \in I\}$ be a collection of subsets of a set X. Then

1. $X \setminus (\bigcup_{\delta \in I} A_\delta) = \bigcap_{\delta \in I} (X \setminus A_\delta)$ and

2. $X \setminus (\bigcap_{\delta \in I} A_\delta) = \bigcup_{\delta \in I} (X \setminus A_\delta)$.

Theorem 6. Let A and B be subsets of a set X. Then $A \setminus B = A \cap (X \setminus B)$.

Definitions 7. Let X and Y be sets and $f \subseteq X \times Y$.

1. We say f is a *function* if for each $x \in X$ there is a unique $y \in Y$ such that $(x,y) \in f$. In this case, we write $f : X \to Y$ and $f(x) = y$ for the pairs $(x,y) \in f$.
2. Suppose $f : X \to Y$ and $A \subseteq X$ and $B \subseteq Y$. Then the *image* of A under f is the set

$$f(A) := \{y \in Y \mid y = f(x) \text{ for some } x \in A\}.$$

The *inverse image* of B under f is the set

$$f^{-1}(B) := \{x \mid f(x) \in B\}.$$

Theorem 8. Let $f : X \to Y$ be a function and let B and C be subsets of Y. Then

1. $f^{-1}(B \cup C) = f^{-1}(B) \cup f^{-1}(C)$,
2. $f^{-1}(B \cap C) = f^{-1}(B) \cap f^{-1}(C)$,
3. $f^{-1}(Y \setminus C) = X \setminus f^{-1}(C)$,
4. $f^{-1}(B \setminus C) = f^{-1}(B) \setminus f^{-1}(C)$.

Definition 9. Let $f : X \to Y$ and $g : Y \to Z$ be a functions. Then g *composed with* f denoted $g \circ f$, is the function $g \circ f : X \to Z$ such that for each $x \in X$, $f \circ g(x) = f(g(x)))$.

Theorem 10. Let $f : X \to Y$ and $g : Y \to Z$ be a functions and let B be a subset of Y. Then

$$(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B).)$$

Theorem 11. Let $f : X \to Y$ be a function and let A_1 and A_2 be subsets of X. Then

1. $A_1 \subseteq A_2 \implies f(A_1) \subseteq f(A_2)$,
2. $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$,