MATH 3430-02 WEEK 2-1

Key Words: Exact Equations; Integrating Factors

Q1. Write down 3 examples of 1st order ODEs that we know how to integrate by hand. Specify the method needed to integrate them.

Q2. Consider the following ODE. Does it belong to one of the types that you already know how to solve?

$$(*) \qquad (2x + 3ye^{3x}) + (3y^2 + e^{3x})\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Q3. Continuing from **Q2**, let

$$M(x,y) = 2x + 3ye^{3x}, \qquad N(x,y) = 3y^2 + e^{3x}, \qquad F(x,y) = x^2 + y^3 + ye^{3x}.$$

(1) How do M_y (i.e. $\partial M/\partial y$) and N_x relate?
(2) How are $M(x,y)$ and $N(x,y)$ related to $F(x,y)$?
(3) How does $\dfrac{\mathrm{d}F}{\mathrm{d}x}(x, y(x))$ compare with the left-hand-side of the equation $(*)$?
(4) Now find the implicit general solution of $(*)$.

The observation in **Q3** is quite general. We summarize this in a definition, a theorem and a proposition.

Definition. *An 1-st order ODE of the form*

$$(**) \qquad M(x,y) + N(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

is said to be **exact** *if there exists a function $F(x,y)$ such that*

$$F_x = M, \qquad F_y = N.$$

Theorem. *A 1-st order ODE of the form $(**)$ is exact if and only if*

$$M_y = N_x.$$

Proposition. *An exact equation of the form $(**)$ has general implicit solutions*

$$F(x,y) = C,$$

where $F(x,y)$ is as in the definition; C is a constant.

In summary, the **Theorem** tells us how to verify exactness; the **Proposition** tells us that the key to finding a solution is to look for a function $F(x,y)$ satisfying $F_x = M$ and $F_y = N$. The main calculation in solving an exact equation lies in finding such an $F(x,y)$.

Q4. Find the general implicit solution of the first order ODE:

$$(3x + y - \cos x) + (x + e^y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Q5. Solve the following initial value problem:

$$(y\cos x + 2xe^y) + (\sin x + x^2e^y - 1)\frac{\mathrm{d}y}{\mathrm{d}x} = 0, \qquad y(0) = 1.$$

What if a given 1-st order ODE of the form (**) is not exact, but can be made exact after being multiplied by an integrating factor $\mu(x,y)$:

$$(\dagger) \qquad \mu(x,y)M(x,y) + \mu(x,y)N(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0?$$

Q6. (1) Write down the equation that μ must satisfy in order for $(\dagger)$ to be exact.
(2) What if μ depends on x only?
(3) What if μ depends on y only?

Q7. Suppose that the equation

$$\frac{y^2}{2} + 2ye^x + (y + e^x)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

has an integrating factor of the form $\mu(x)$. Find such an integrating factor.

Q8. Suppose that the equation

$$(3xy + y^2) + (x^2 + xy)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

has an integrating factor of the form $\mu(x)$. Solve this equation.