MATH 3430-02 WEEK 10-1

Key Words: The Laplace Transform (s-shifting and t-shifting); Step functions.

We start with two useful properties of the Laplace transform. Officially they are both called 'shifting theorems', but to distinguish, we'll call one s-shifting, the other t-shifting, for obvious reasons that will become clear.

Q1. Suppose that $f(t)$ is of exponential order (i.e., appropriate for $\mathcal{L}$). Find a formula for

$$\mathcal{L}\{e^{ct}f(t)\}.$$

If $F(s)$ is defined on (α, ∞), where is $\mathcal{L}\{e^{ct}f(t)\}$ defined?

Let's call the formula above the 's-shifting property' of $\mathcal{L}$.

Q2. Let $H_c(t)$ $(c \geq 0)$ ('H' referring to a scientist named Oliver Heaviside) be the following function

$$H_c(t) = \begin{cases} 0, & t < c, \\ 1, & t \geq c. \end{cases}$$

Plot the graph of $H_c(t)$ for $c = 1$. $H_c(t)$ is called a **step function**.

Q3. First plot the graph of t^2; then plot the graph of $H_2(t)(t-2)^2$.

Q4. In general, what is the relation between $f(t)$ and $H_c(t)f(t-c)$?

(We call the latter the shifting to right by c of the former. Note that the part of $f(t)$ for $t<0$ is truncated.)

Q5. Express the Laplace transform

$$\mathcal{L}\{H_c(t)f(t-c)\}$$

in terms of $F(s)$.

Let's call this formula the 't-shifting property' of $\mathcal{L}$.

Q6. Find the following Laplace transforms or inverse transforms.

(1) $\mathcal{L}\{e^{3t}\sin t\}$;

(2) $\mathcal{L}\{H_2(t)\cos(t-2)\}$;

(3) $\mathcal{L}\{H_3(t)\}$;

(4) $\mathcal{L}\{H_5(t)(t-3)\}$;

(5) $\mathcal{L}\{e^{2t}H_7(t)t\}$;

(6) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+4s+5}\right\}$;

(7) $\mathcal{L}^{-1}\left\{\dfrac{e^{-3s}}{s-2}\right\}$;

Q7. One can use the step functions to express other discontinuous functions.

(1) Suppose that

$$g(t) = \begin{cases} 0, & t < \pi, \\ 1, & \pi \le t < 2\pi, \\ 0, & t \ge 2\pi. \end{cases}$$

Plot the graph of $g(t)$; then express $g(t)$ as the difference between two step functions.

(2) Let $g(t)$ be the one as above, plot the graph of the function $g(t)\sin(t)$.

(3) Suppose that

$$h(t) = \begin{cases} 0, & t < 1, \\ e^{5t}, & 1 \le t < 2, \\ t, & 2 \le t < 5, \\ e^{-t}, & t \ge 5. \end{cases}$$

Write $h(t)$ in a closed form using the step functions.

Q8. For the $h(t)$ in **Q7**, find $\mathcal{L}\{h(t)\}$.

Next time, we'll see such discontinuous functions appearing as the forcing term (right-hand-side) of an ODE.