# Logic Reminder

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January 16, 2017

## Contents



## 1 Introduction

Mathematical language is very rigid and depends on simple logic. This document is designed as a quick reminder of the conventions and basic concepts.

## 2 Truth Values

Suppose that $P$ is a statement. For example, $P =$ "the dog is brown". Alternatively, we can have statements that depend on an input, like $P(x) =$ "$x$ is brown". Here, $P(x)$ is a property of the input $x$. Then the truth value of $P$ is *true* or *false* depending on whether the statement is correct or wrong. For example, if $P =$ "the Sun never rises", then most likely, the truth value of $P$ is false. If the statement $P(x)$ depends on $x$, the truth value also depends on $x$. For example, $P(\text{my dog})$ could be true while $P(\text{the Sun})$ is false.

## 3 "Or" and "And"

Now take $P$ and $Q$ any statements. We can make various sentences out of $P$ and $Q$. For example,

$$P \text{ and } Q$$

This sentence as a whole has a truth value, and this truth value depends on that of $P$ and $Q$. In fact, the sentence is true only if

- $P$ is true and $Q$ is true

In any other case (for example, $P$ is false and $Q$ is true), the sentence itself is false.

Another example is

$$P \text{ or } Q$$

When is this sentence true? In math, "or" is not exclusive. This means, that there are three cases when this sentence can be true.

- If $P$ is true and $Q$ is true.
- If $P$ is true and $Q$ is false.
- If $P$ is false and $Q$ is true.

I.e., you only need one of the two to be true, but they can also both be true.

## 4 Negation

Now, negating sentences, (determining there opposite), is extremely important. For example, (not $P$) is true exactly when $P$ is false and (not $P$) is false exactly when $P$ is true. But sometimes, we have more elaborate statements to negate.

For example, when is "$P$ or $Q$" a false sentence? From what we saw above, there is only one way this can be false, i.e.,

- if $P$ is false and $Q$ is false

That is

$$\text{not } (P \text{ or } Q) = (\text{not } P) \text{ and } (\text{not } Q)$$

Similarly, when is "$P$ and $Q$" a false sentence? Well, when one of $P$ and $Q$ is false, so

$$\text{not } (P \text{ and } Q) = (\text{not } P) \text{ or } (\text{not } Q).$$

## 5 Conditionals

Often, we make sentences such as

$$\text{if } P, \text{ then } Q$$

This is the same as saying "$P$ implies $Q$".

When is such a statement true? Well, the requirement is that, $Q$ be true whenever $P$ be true. So, the sentence is true when

- $P$ is true and $Q$ is true
- $P$ is false and $Q$ is true
- $P$ is false and $Q$ is false

In the last two cases, since $P$ is false, we ask for nothing! For example,

$$\text{if 2 is odd, I am 10 years old}$$

That's just always true. Whether I'm 10 or not, since 2 is just not odd!

So, when is "if $P$, then $Q$" false then? Well, there's only one way for this to be false. That is, if

- $P$ is true and $Q$ is false

That is,

$$\text{not (if } P\text{, then } Q) = P \text{ and not } Q$$

**Exercise 5.1** (Contrapositive)**.** Check that

$$\text{if } P\text{, then } Q = \text{if (not } Q\text{), then (not } P)$$

This is called the *contrapositive*.

Another popular phrasing is

$$P \text{ if and only if } Q$$

This sentence is true exactly when $P$ and $Q$ have the same truth value. That is, when

- $P$ is true and $Q$ is true
- $P$ is false and $Q$ is false

Therefore, we have

$$\text{not } (P \text{ if and only if } Q) = (P \text{ and (not } Q)) \text{ or } ((\text{not } P) \text{ and } Q)$$

**Exercise 5.2.** Check that

$$P \text{ if and only if } Q = (\text{if } P \text{ then } Q) \text{ and } (\text{if } Q \text{ then } P)$$

and that

$$P \text{ if and only if } Q = (\text{not } P) \text{ if and only if } (\text{not } Q)$$

## 6 Quantifiers

Finally, we turn to sentence $P(x)$ that depend on the meaning of $x$. If we plug in a specific $x$, like $x$=“my dog”, then $P(x)$ becomes a precise statement and we can go to the previous sections to discuss its truth value and how it behaves within sentences. However, we often makes statements like:

$$\text{for all } x,\ P(x)$$

or

$$\text{there exists } x \text{ such that } P(x)$$

When are such statements false? Well, “for all $x$, $P(x)$” is false if there is at least one $x$ such that $P(x)$ is false. In other words

$$\text{not (for all } x,\ P(x)) = \text{there exists } x \text{ such that (not } P(x))$$

Similarly, when is “there exists $x$ such that $P(x)$” false? Well, exactly when all $x$ make $P(x)$ false. That is,

$$\text{not (there exists } x \text{ such that } P(x)) = \text{for all } x,\ \text{(not } P(x))$$