MATH 3430-02 WEEK 1-2

Key Words: Integrating Factors; Initial Value Problems; Asymptotic Behavior of Solutions; Separable Equations (aka. Separation of Variables)

Recall the *Method of Integrating Factors*: To solve the 1-st order linear ODE

$$y' + a(t)y = b(t)$$

we multiply both sides by a function $\mu(t)$, which is to be determined later, obtaining

$$\mu y' + a(t)\mu y = b(t)\mu.$$

The left-hand-side looks like $(\mu y)' = \mu y' + \mu' y$, but not yet: it only happens when

$$a(t)\mu = \mu'.$$

Solving this yields one such μ (integration constant do not appear since we need only one μ)

$$\mu(t) = e^{\int a(t)dt}.$$

Consequently, for this $\mu(t)$, the original equation is equivalent to

$$(\mu y)' = \mu(t)b(t).$$

Direct integration yields

$$\mu y = \int \mu(t)b(t)\mathrm{d}t \quad \Rightarrow \quad \boxed{y = \frac{1}{\mu(t)} \int \mu(t)b(t)\mathrm{d}t, \text{ where } \mu(t) = e^{\int a(t)dt}.}$$

Q1. The factor $1/\mu$ appears in the expression of y. Could this produce an invalid solution as μ may be equal to zero?

Q2. Now solve the following ODE *without* directly applying the formula above:

$$y' + \frac{y}{x} = \sin x.$$

We have seen that solutions of the 1-st order linear ODEs would involve a constant C, produced by the indefinite integral $\int \mu(t)b(t)\mathrm{d}t$. Such a constant can be determined if we know one extra piece of information: the *initial value* of the ODE.

An initial value of a 1-st order ODE contains two pieces of information: an initial time t_0, and the value of y at t_0, say $y_0 = y(t_0)$. To understand this better, think of the motion of free fall, which is governed by the equation $\frac{\mathrm{d}}{\mathrm{d}t}V(t) = -g$, where V stands for the velocity. We cannot determine $V(t)$ unless we know what V is at a particular time.

A differential equaion together with a specification of initial values is called an *initial value problem* (IVP).

Q3. Solve the following initial value problem (you can use the result from **Q2**):

$$y' + \frac{y}{x} = \sin x, \qquad y(\pi) = 1.$$

(Note: It is convenient to solve an initial value problem in two steps: first find the general solution involving an integration constant C, then determine C by substituting the initial values in the solution.)

Sometimes, we are interested in how solutions of an ODE behave when $t \to \pm\infty$, the so-called *asymptotic behavior* of solutions. This can often directly obtained by applying limits to solutions.

Q4. Consider the initial value problem

$$y' - y = 1 + 2\cos t, \qquad y(0) = y_0.$$

What is a value of y_0 such that the solution $y(t)$ remains finite when $t \to \infty$?

Now we move to another class of 1-st order ODEs that we can solve by hand.

Q5. Below, what is a common feature that equations on the List (I) share that distinguishes them from those on List (II)?

(I)	(II)
$\frac{\mathrm{d}y}{\mathrm{d}t} = (1+t^2)e^y$	$\frac{\mathrm{d}y}{\mathrm{d}t} = 2y + t$
$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1+t}{y}$	$\frac{\mathrm{d}y}{\mathrm{d}t} = y(e^y + t)$
$\frac{\mathrm{d}y}{\mathrm{d}t} = y(t + \cos t)$	$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{y+t}{y^2+t^2}$
$\frac{\mathrm{d}y}{\mathrm{d}t} = (1+t+t^3)\sec y$	$\frac{\mathrm{d}y}{\mathrm{d}t} = \cos(t+y)$

A 1-st order ODE is said to be *separable* if it can be put in the following form:

$$p(y)\frac{\mathrm{d}y}{\mathrm{d}t} = q(t).$$

(Note: Broadly speaking, the property of being separable is only possible for 1-st order ODEs.) A method to solve a separable equation is by separating the variables:

$$p(y)\mathrm{d}y = q(t)\mathrm{d}t,$$

then putting integral sign on both sides:

$$\int p(y)\mathrm{d}y = \int q(t)\mathrm{d}t.$$

The result will involve an integration constant C. Usually, you won't be able to explicitly express y as an expression of t, but instead achieving an *implicit solution*, as the following question will show.

Q6. Solve the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1+t^2}{2+\cos y}, \qquad y(0) = 1.$$