MATH 3430-02 WEEK 1-3

Key Words: Two Applications: Population Models; Orthogonal Trajectories

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A certain species has population $p(t)$ at time t. When the population is small relative to the capacity of the environment, one may hypothesize: *The rate of change of p at time t is proportional to the population at time t.* In a formula, this becomes the a first order ODE:

$$(*) \qquad \boxed{\phantom{\frac{dp}{dt}}} = rp(t).$$

where r is a positive constant.

Q1. Solve Equation $(*)$.

We see that solutions of $(*)$ represent *exponential growth.*

The previous population model may have poor prediction when t is large, since, in reality, population cannot be too large. Here is another hypothesis, which takes into account the environmental capacity T, a positive constant:

$$(**) \qquad \frac{dp}{dt} = \boxed{}$$

As before, the right-hand-side of this equation represents the rate of change of $p(t)$.

Note that, when p is very small, p^2 is negligible when compared to p, so the equation is similar to $p' = rp$, which models the exponential growth. However, when p becomes larger, p^2 will play a significant role by pulling the population down. This matches reality well.

Q2. At which p value is $p(t)$ increasing with time; at which p value is $p(t)$ decreasing; at which p value is $p(t)$ remaining the same? Indicate your answer in a horizontal p-axis.

The diagram above suggests that $p(t)$ would tend to T unless the original population is zero. There are two other ways to view this. One by the so-called *slope field plot*; another by finding explicit solutions.

Q3. For simplicity, take $r = T = 1$ and sketch a slope field (i.e., a field of line dashes whose slope equals to p' at that point) for the equation $p' = p(1 - p)$. Then sketch some integral curves of the slope field that you have drawn.

Q4. Again, assume that $r = T = 1$. Solve the ODE $p' = p(1 - p)$ explicitly. Describe the behavior of solutions as $t \to \infty$.

Now we turn to another application of 1-st order ODEs. This is an application to geometry.

In the plane $\mathbb{R}^2$, we sometimes have a family $\mathcal{F}$ of curves (i.e., curves that are parametrized by a constant). As an analogy, think of the contour map. A question is, can we find curves that are everywhere orthogonal to the curves in the family $\mathcal{F}$. Continuing the analogy, we are looking for paths of quickest descent on a map.

Q5. Without calculating, find curves that are everywhere orthogonal to the family of concentric circles:

$$x^2 + y^2 = r,$$

where r is a positive parameter.

Q6. What are the curves that are everywhere orthogonal to the family of hyperbolas:

$$y^2 - x^2 = c?$$

In this case, we have three steps to go: For any hyperbola in the family,

1. Find the slope of the hyperbola at any point;
2. Find the slope of the orthogonal direction;
3. Establish an ODE for orthogonal curves.

For step **1**, we differentiate the equation $y^2 - x^2 = c$, obtaining:

$$\boxed{\qquad\qquad\qquad} = 0,$$

which is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \boxed{\qquad} \qquad \text{(unless } y = 0\text{)}.$$

For step **2**, we find the *negative reciprocal* of the slope found in the previous part, obtaining

$$\boxed{\qquad}$$

For step **3**, use the slope found in step **2** to establish a new 1-st order equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \boxed{\qquad}$$

Q7. Solve the previous ODE.