MATH 3430-02 WEEK 8-3

Key Words: Series method (cont.), Radius of convergence, Euler equations.

Q1. What is a series solution of

$$y'' + \frac{t}{1+t^2}y' + \frac{1}{1+t^2}y = 0?$$

For a series solution, it is important to know when the series converges. There is a simple criterion, which we describe in two steps

i. Suppose that $f(t)$ is an analytic function. If the corresponding complex function $f(z)$ and its derivative are defined on $|z - t_0| < \rho$, then the Taylor series of $f(t)$ expanded at t_0 converges on $|t - t_0| < \rho$. The maximum such ρ is called the **radius of convergence** of the Taylor series. (Note: ρ may be ∞.)

ii. Consider the equation

$$y'' + p(t)y' + q(t)y = 0.$$

If the corresponding analytic functions (with the complex variable z) $p(z)$ and $q(z)$ have respectively radii of convergence ρ_1 and ρ_2, then a series solution centered at t_0 converges on $|t - t_0| < \min\{\rho_1, \rho_2\}$.

Q2. Let $f(t) = \dfrac{1}{1+t^2}$. It follows that $f(z) =$ __________.

$f(z)$ or $f'(z)$ is not defined at $z =$ __________.

It follows (by **i**) that the Taylor expansion at $t = 0$:

$$1 - t^2 + t^4 - t^6 + \cdots$$

converges on the region

$$|t| < \underline{\qquad\qquad}.$$

Q3. From this, you know that the series solution you obtained in **Q1** converges on the region:

$$|t| < \underline{\qquad\qquad}.$$

Q4. Consider the equation

$$(1 + t + t^2)y'' + ty' + y = 0.$$

Suppose that you have found a series solution centered at $t_0 = 0$. What is a radius of convergence of such a series solution?

Q5. Consider the initial value problem

$$y'' + (1 + t^2)y' + e^t y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

Suppose that you have found a series solution centered at $t_0 = 0$. What is a radius of convergence of that series solution?

Q6. Find the first 4 terms in a series solution (centered at $t_0 = 0$) of the equation in **Q5**.

Q7. Consider the following equation:

$$t^2 y'' + ty' + y = 0, \qquad t > 0.$$

Can you obtain a nonzero power series solution (centered at $t = 0$) for this equation?

Q8. Consider equations of the form:

$$t^2y'' + \alpha t y' + \beta y = 0, \qquad t > 0,$$

where α, β are constants.

Instead of looking for power series solutions, we look for solutions of the form $y(t) = t^r$, where r is a complex number.

Substituting $y(t) = t^r$ in the equation, we have:

This implies the following algebraic equation in r:

This leads to three cases, which we'll explain below.