Chapter 4

Elementary Set Theory

"A set is a Many that allows itself to be thought of as a One."
Georg Cantor

So far, we have dealt extremely informally with infinity. One consequence of a loose treatment of this topic is that we are easily led into puzzling paradoxes that resist resolution. We are now going to begin a more careful examination of the infinite. We start with an introduction to one of the core pillars of modern mathematical thought: set theory. Mathematician use set theoretical language in the definitions of virtually every idea in their discipline.

Logicians and set theorists would characterize our development as **naive set theory**. We will provide some suggestions for further reading if you wish to study the more formal approach of **axiomatic set theory**.

Most important ideas in mathematics have evolved over many centuries and through the contributions of many researchers. Their origins are difficult to trace accurately. Set Theory is an exception. It began with a single paper in 1874 by Georg Cantor: "On a Characteristic Property of All Real Algebraic Numbers" ("Über eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen,", *Crelle's Journal für Mathematik* 77 (1874) 258 - 262).

Set theory provides a way to talk about and build with collections of objects, treating the collection as a single entity while keeping access to its individual constituents. Since it is impossible to define all words, we will assume that we know what it means for something to be a set and what it means for something to be an element of a set. The words **set** and **collection** will be used synonymously. Some examples of sets in the real world might be a herd of elephants, a pile of coins, a basket of apples, or a team of soccer players.

Notation: If $\mathbf{A}$ is a set, then $\mathbf{x} \in \mathbf{A}$ means that $\mathbf{x}$ *is an element of* $\mathbf{A}$, or

equivalently, **x** *is a member of* **A**, or **x** *belongs to* **A** or **x** *is in* **A.** We write **x** $\notin$ **A** to indicate that **x** is not a member of **A**.
The set with no elements is called the **empty set** and is represented by the symbol **Ø.**

Notation : One way to describe a set is to list all of its elements, enclosed within curly braces and separated by commas.

Example 1 : The notation **A** = {1,2,3} means that **A** is the set whose elements are the integers 1, 2 and 3.

Notation: Suppose that **P** is a "well defined property" that an object may or may not possess. We use the notation {**x**: **x** has property **P**} to denote the set of all objects with property **P**.

Example 2 : The notation **A** = { **x**: **x** is a positive integer less than 4} also describes the set whose elements are the integers 1,2 and 3.

Definition 1 : Two sets are **equal** if and only if they have precisely the same elements. If **A** and **B** are equal sets, we write **A** = **B**.

To demonstrate that two sets are not equal, you can find an element that belongs to one of the sets, but not the other.

Question 1 : Are the sets {1,2,3} and {1,3,2} equal?

Question 2 : Are the sets {1,2,3} and {1,2,3,3} equal?

Exercise 16 : Let **A** be the set of negative numbers whose squares are negative and let **B** be the set of two-headed men who served as United States President before 2016. Show that these two sets are equal.

Definition 2 : If **A** is a set, the statement that **B** is a **subset** of **A** means that **B** is a set, and that each element of **B** is an element of **A.** If **A** is a set, the notation **B** $\subset$ **A** means that **B** is a subset of **A.**

Definition 3 : If **A** is a set and **B** is a subset of **A**, then **B** is a **proper subset** of **A** if and only if there is an element of **A** which is not an element of **B.**

AN IMPORTANT NOTE: One way to show two sets are equal is to show that each one is a subset of the other one.

Theorem 1. *Suppose **A**, **B** and **C** are any sets. Then the following statements are true:*

(a) $\boldsymbol{A} = \boldsymbol{A}$*,*

(b) if $\boldsymbol{A} = \boldsymbol{B}$*, then* $\boldsymbol{B} = \boldsymbol{A}$*, and*

(c) if $\boldsymbol{A} = \boldsymbol{B}$ *and* $\boldsymbol{B} = \boldsymbol{C}$*, then* $\boldsymbol{A} = \boldsymbol{C}$*.*

Definition 4 : If **X** is a set and **A** is a subset of **X,** then the **relative complement** of **A**, denoted **X - A,** is the set of all elements of **X** which are not elements of **A.** In our notation **X - A** = { **x** $\in$ **X** : **x** $\notin$ **A**, }

Example 3 : If $\mathbf{X} = \{1,2,3,4,5\}$ and $\mathbf{A} = \{2,4\}$, then $\mathbf{X} - \mathbf{A} = \{1,3,5\}$.

Definition 5 : If each of **A** and **B** is a set, the **union** of **A** and **B,** denoted $\mathbf{A} \cup \mathbf{B}$ is the set **C** such that **x** is an element of **C** if and only if either **x** is an element of **A** or of **B.**: $\mathbf{A} \cup \mathbf{B} = \{\mathbf{x} : \mathbf{x} \in \mathbf{A} \text{ or } \mathbf{x} \in \mathbf{B}\}$.

Note that the union of two sets **A** and **B** is the set whose elements are all the elements of the set **A** and all the elements of the set **B** but no other elements. To be a member of the union of two sets, it is necessary and sufficient to belong to at least of the sets.

Example 4 : If $\mathbf{A} = \{1,2,3\}$ and $\mathbf{B} = \{3,4\}$, then $\mathbf{A} \cup \mathbf{B} = \{1,2,3,4\}$.

The union operation, which takes two sets and produces a third set, satisfies several nice properties. In particular, we have

Theorem 2. *Suppose* ***A****,* ***B*** *and* ***C*** *are any sets. Then*

(a) (Idempotency) $\boldsymbol{A} \cup \boldsymbol{A} = \boldsymbol{A}$,

(b) (Commutativity) $\boldsymbol{A} \cup \boldsymbol{B} = \boldsymbol{B} \cup \boldsymbol{A}$, *and*

(c) (Associativity) $\boldsymbol{A} \cup (\boldsymbol{B} \cup \boldsymbol{C}) = (\boldsymbol{A} \cup \boldsymbol{B}) \cup \boldsymbol{C}$.

Definition 6 : If **G** is a collection, each element of which is a set, the **union** of the sets of **G** is the set **X** such that **y** is an element of **X** if and only if there is an element **g** of **G** such that **y** is an element of **g**.

We use the notation

$$\bigcup_{\mathbf{g} \in \mathbf{G}} \mathbf{g}$$

for this union.

Definition 7 : If each of **A** and **B** is a set, then the **intersection**, or **common part** of **A** and **B,** denoted $\mathbf{A} \cap \mathbf{B}$**,** is the set **C** such that **x** is an element of **C** if and only if **x** is an element of **A** **and** **x** is an element of **B.**:
$\mathbf{A} \cap \mathbf{B} = \{\mathbf{x} : \mathbf{x} \in \mathbf{A} \text{ and } \mathbf{x} \in \mathbf{B}\}$.

Example 5 : If $\mathbf{A} = \{1,2,3\}$ and $\mathbf{B} = \{3,4\}$, then $\mathbf{A} \cap \mathbf{B} = \{3\}$.

To be a member of the intersection of two sets, it is necessary and sufficient to belong to both of the sets.The intersection operation, which takes two sets and produces a third set, satisfies several nice properties. In particular, we have

Theorem 3. *Suppose* ***A****,* ***B*** *and* ***C*** *are any sets. Then*

(a) (Idempotency) $\boldsymbol{A} \cap \boldsymbol{A} = \boldsymbol{A}$,

(b) (Commutativity) $\boldsymbol{A} \cap \boldsymbol{B} = \boldsymbol{B} \cap \boldsymbol{A}$, *and*

(c) (Associativity) $\boldsymbol{A} \cap (\boldsymbol{B} \cap \boldsymbol{C}) = (\boldsymbol{A} \cap \boldsymbol{B}) \cap \boldsymbol{C}$.

Definition 8 : If **A** and **B** are sets and have no element in common, then **A** and **B** are **disjoint**. Since the intersection of two disjoint sets is the empty set, we can write $\mathbf{A} \cap \mathbf{B} = \emptyset$.

Definition 9 : If **G** is a nonempty collection of sets, then the **intersection** or **common part** of the sets of **G** is the set **C** such that **x** is an element of **C** if and only if for each set **g** of **G**, **x** is an element of **g**.

We use the notation

$$\bigcap_{\mathbf{g} \in \mathbf{G}} \mathbf{g}$$

for this intersection.

The next theorem makes some assertions about how unions, intersections and complements interact with each other.

Theorem 4. *Let $\boldsymbol{X}$ be a set and $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ be subsets of $\boldsymbol{X}$. Then*

(a) $\boldsymbol{X} - (\boldsymbol{X} - \boldsymbol{A}) = \boldsymbol{A}$,

(b) $\boldsymbol{X} - (\boldsymbol{A} \cup \boldsymbol{B}) = (\boldsymbol{X} - \boldsymbol{A}) \cap (\boldsymbol{X} - \boldsymbol{B})$,

(c) $\boldsymbol{X} - (\boldsymbol{A} \cap \boldsymbol{B}) = (\boldsymbol{X} - \boldsymbol{A}) \cup (\boldsymbol{X} - \boldsymbol{B})$, *and*

(d) $\boldsymbol{A} \cap (\boldsymbol{B} \cup \boldsymbol{C}) = (\boldsymbol{A} \cap \boldsymbol{B}) \cup (\boldsymbol{A} \cap C)$.

Exercise 17 : Suppose **A** , **B** , **C** and **D** are sets. Show that $(\mathbf{A} \cup \mathbf{B}) \cap (\mathbf{C} \cup \mathbf{D}) = \mathbf{A} \cap \mathbf{C}) \cup (\mathbf{B} \cap \mathbf{C}) \cup (\mathbf{A} \cap \mathbf{D}) \cup (\mathbf{B} \cap \mathbf{D})$

Theorem 5. *Suppose* $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ *and* $\boldsymbol{D}$ *are sets. Then*

(a) if $\boldsymbol{A} \subset \boldsymbol{B}$ *and* $\boldsymbol{B} \subset \boldsymbol{C}$, *then* $\boldsymbol{A} \subset \boldsymbol{C}$,

(b) $(\boldsymbol{A} \cap \boldsymbol{B}) \subset \boldsymbol{A} \subset (\boldsymbol{A} \cup \boldsymbol{B})$, *and*

(c) if $\boldsymbol{A} \subset \boldsymbol{B}$ *and* $\boldsymbol{C} \subset \boldsymbol{D}$, *then* $(\boldsymbol{A} \cup \boldsymbol{C}) \subset (\boldsymbol{B} \cup \boldsymbol{D})$ *and* $(\boldsymbol{A} \cap \boldsymbol{C}) \subset (\boldsymbol{B} \cap \boldsymbol{D})$.

Chapter 6

Functions

"A single idea, if it is right, saves us the labor of an infinity of experiences."
Jacques Maritain

Galileo's pairing of positive integers with their squares and Bolzano's pairing of points on one line segment with points on a longer line segment both involve assigning to each member of one set a member of a second set. A core mathematical concept, that of a *function*, captures this notion and plays a prominent role in the modern understanding of most parts of mathematics, especially those sections dealing with infinity. In this section, we provide a careful definition of a function in terms of elementary set theory and investigate some possible properties of functions.

Definition 10 : An **ordered pair** is a combination of two objects, in which the first entry is distinguished from the second. If the first entry is a and the second entry is b, the notation for an ordered pair is (a,b). The pair is "ordered" in that (a,b) differs from (b,a) unless $a=b$. We often use the words **term** or **coordinate** to refer to an entry of an ordered pair. Thus the first term of (a,b) is a and the second term is b. Set theorists formally define the ordered pair (a,b) as the set $\{\{a\},\{a,b\}\}$.

Exercise 27 : Use the formal definition to write out explicitly the ordered pairs $(3,4)$ and $(4,3)$ and show they are different sets.

Exercise 28 : Show that if $a \neq b$, then under the formal definition of ordered pairs, it is true that $(a,b) \neq (b,a)$.

Definition 11 : The statement that **F** is a **function** means that **F** is a collection of ordered pairs, such that no two of these pairs have the same first term.

Example 8 : Let **F** be the set defined by

$$\mathbf{F} = \{(1, Washington), (2, Adams), (43, Obama), (4,3), (6, Adams)\}$$

.

Then **F** is a function which has 5 ordered pairs. Note that the ordered pair (4,3) belongs to **F** but the ordered pair (3,4) does not.

Exercise 29 : Show that the set **G** defined by

$$\mathbf{F} = \{(1, Washington), (2, Adams), (1, Obama), (4,3), (6, Adams)\}$$

also contains 5 distinct ordered pairs, but **G** is not a function.

Definition 12 : Suppose that **F** is a function. The **domain** of **F** is the set **X** such that **x** is an element of **X** if and only if **x** is the first term of some element of **F**. The **range** of **F** is the set **Y** such that **y** is an element of **Y** if and only if **y** is a second term of some element of **F**. If **x** is the first term of an element of **F**, then **F**(**x**), the **value of F at x**, denotes the second term of the ordered pair of **F** whose first term is **x** The function **F** is said to be a function from **X onto Y**. If **Z** is a set such that **Y** is a subset of **Z**, then **F** is a function from **X into Z**. The notation **F:X** $\rightarrow$ **Z** means that **F** is a function from **X** into **Z**. If **A** is a subset of **X,** then **F**(**A**) denotes the set of all elements **F**(**a**) where **a** is an element of **A.** If **W** is a subset of **Z**, then $\mathbf{F}^{-1}$(**W**) denotes $\{$**x**: **x** is in **X** and **(x,w)** is an element of **F** for some **w** in **W**$\}$. The set **F(A)** is called the **image** of **A** under **F** and $\mathbf{F}^{-1}$(**W**) is called the **inverse image** of **W** under **F.**

Exercise 30 : For the function **F** defined in Example 8, show that

(a) the domain is $\{$1,2,3,4,6$\}$,

(b) the range is $\{$Washington, Adams, Obama, 3$\}$

(c) **F**(2) = **F**(6)

(d) $\mathbf{F}^{-1}$($\{$Adams,3$\}$) = $\{$2, 4, 6$\}$.

Example 9 : Let **R** denote the set of all real numbers. Let **F** be $\{(\mathbf{x},\mathbf{x}^2)$: **x** is an element of **R** $\}$. Then **F** is a function. The domain of **F** is **R**, the range of **F** is $\{$**x**: **x** is in **R** and $\mathbf{x} \geq 0\}$, **F**(2) = 4, and we have **F**:**R** $\rightarrow$ **R**.

Definition 13 : Suppose that **F** is a function. The statement that **F** is **one-to-one** means that no two elements of **F** have the same second term. In other words, if **x** and **y** are distinct elements of the domain of **F**, then **F**(**x**) is different from **F**(**y**). Note in the last example, **F** is not one-to-one because **F**(2) = **F**(-2).

Question 3 : Let **f** be a function from a set **X** into a set **Y** and let **A** and **B** be subsets of **X**. Which of the following statements are always true?

(a) $\mathbf{f}(\mathbf{A} \cup \mathbf{B}) = \mathbf{f}(\mathbf{A}) \cup \mathbf{f}(\mathbf{B})$.

(b) $\mathbf{f}(\mathbf{A} \cap \mathbf{B}) = \mathbf{f}(\mathbf{A}) \cap \mathbf{f}(\mathbf{B})$.

(c) **Y - f(A) = f(X - A)** for each subset **A** of **X**.

Note that if **C** and **D** are sets, then **C** - **D** denotes the set of all elements of **C** which are not members of **D**.

Chapter 1

Review

Read the definitions, prove the theorems and do the exercises.

Definitions 1. Suppose A and B are subsets of a set X. We define the following:

1. $A \cup B := \{x \mid x \in A \text{ or } x \in B\}$;
2. $A \cap B := \{x \mid x \in A \text{ and } x \in B\}$;
3. $A \setminus B := \{x \mid x \in A \text{ and } x \notin B\}$.

Theorem 2. Let A be a subset of the set X. Then $X \setminus (X \setminus A) = A$.

Theorem 3. (DeMorgan's Laws) Let A and B be subsets of a set X. Then

1. $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$ and
2. $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$.

Definition 4. Let I be an indexing set. For each $\delta \in I$, let A_δ be a set. We define the following two sets:

1. $\bigcup\limits_{\delta \in I} A_\delta = \{s \mid \text{ there exists } \delta \in I \text{ such that } s \in A_\delta\}$ and
2. $\bigcap\limits_{\delta \in I} A_\delta = \{s \mid s \in A_\delta \text{ for all } \delta \in I\}$.

Theorem 5. (Generalised DeMorgan's Laws) Let $\{A_\delta \mid \delta \in I\}$ be a collection of subsets of a set X. Then

1. $X \setminus (\bigcup\limits_{\delta \in I} A_\delta) = \bigcap\limits_{\delta \in I} (X \setminus A_\delta)$ and

2. $X \setminus (\bigcap_{\delta \in I} A_\delta) = \bigcup_{\delta \in I} (X \setminus A_\delta)$.

Theorem 6. Let A and B be subsets of a set X. Then $A \setminus B = A \cap (X \setminus B)$.

Definitions 7. Let X and Y be sets and $f \subseteq X \times Y$.

1. We say f is a *function* if for each $x \in X$ there is a unique $y \in Y$ such that $(x,y) \in f$. In this case, we write $f : X \to Y$ and $f(x) = y$ for the pairs $(x,y) \in f$.
2. Suppose $f : X \to Y$ and $A \subseteq X$ and $B \subseteq Y$. Then the *image* of A under f is the set
$$f(A) := \{y \in Y \mid y = f(x) \text{ for some } x \in A\}.$$
The *inverse image* of B under f is the set
$$f^{-1}(B) := \{x \mid f(x) \in B\}.$$

Theorem 8. Let $f : X \to Y$ be a function and let B and C be subsets of Y. Then

1. $f^{-1}(B \cup C) = f^{-1}(B) \cup f^{-1}(C)$,
2. $f^{-1}(B \cap C) = f^{-1}(B) \cap f^{-1}(C)$,
3. $f^{-1}(Y \setminus C) = X \setminus f^{-1}(C)$,
4. $f^{-1}(B \setminus C) = f^{-1}(B) \setminus f^{-1}(C)$.

Definition 9. Let $f : X \to Y$ and $g : Y \to Z$ be a functions. Then g *composed with* f denoted $g \circ f$, is the function $g \circ f : X \to Z$ such that for each $x \in X$, $f \circ g(x) = f(g(x)))$.

Theorem 10. Let $f : X \to Y$ and $g : Y \to Z$ be a functions and let B be a subset of Y. Then
$$(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B).)$$

Theorem 11. Let $f : X \to Y$ be a function and let A_1 and A_2 be subsets of X. Then

1. $A_1 \subseteq A_2 \implies f(A_1) \subseteq f(A_2)$,
2. $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$,