Math Review for Stat 110

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1 Sets

> A set is a Many that allows itself to be thought of as a One.
> – Georg Cantor

> Amazon should put their cloud in a cloud, so the cloud will have the redundancy of the cloud.
> – @dowens

A set is a collection of objects. The objects can be anything: numbers, people, cats, courses, even other sets! The language of sets allows us to talk precisely about *events* (see the *Translating Between Probability and Sets* handout). If S is a set, then the notation $x \in S$ indicates that x is an element (a.k.a. member) of the set S (think of the set as a club, with very precisely defined criteria for membership). The set may be finite or infinite. If A is a finite set, we write $|A|$ for the number of elements in A, which is called its *cardinality*.

For example:

1. $\{1, 3, 5, 7, \dots\}$ is the set of all odd numbers;

2. {Worf, Jack, Tobey} is the set of Joe's cats;

3. $[3, 7]$ is the closed interval consisting of all real numbers between 3 and 7;

4. $\{\mathrm{HH}, \mathrm{HT}, \mathrm{TH}, \mathrm{TT}\}$ is the set of all possible outcomes if a coin is flipped twice (where, for example, HT means the first flip lands Heads and the second lands Tails);

5. {Stat 110} is the set of prerequisites for Stat 123.

To describe a set (when it's tedious or impossible to list out its elements), we can give a rule that says whether each possible object is or isn't in the set. For example, $\{(x, y) : x \text{ and } y \text{ are real numbers and } x^2 + y^2 \leq 1\}$ is the disc in the plane of radius 1, centered at the origin.

1.1 The Empty Set

> Bu Fu to Chi Po: “No, no! You have merely painted what is! Anyone can paint what is; the real secret is to paint what isn’t.”
> Chi Po: “But what is there that isn’t?”
> – Oscar Mandel, *Chi Po and the Sorcerer: A Chinese Tale for Philosophers and Children*

> ‘Take some more tea,’ the March Hare said to Alice very earnestly. ‘I’ve had nothing yet,’ Alice replied in an offended tone, ‘so I can’t take more.’ ‘You mean you can’t take less,’ said the Hatter: ‘It’s very easy to take more than nothing.’
> – Lewis Carroll

The smallest set, which is both subtle and important, is the *empty set*, which is the set that has no elements whatsoever. It is denoted by $\emptyset$ or by $\{\}$. Make sure not to confuse $\emptyset$ with $\{\emptyset\}$! The former has no elements, while the latter has one element. If we visualize the empty set as an empty paper bag, then we can visualize $\{\emptyset\}$ as a paper bag inside of a paper bag.

1.2 Subsets

If A and B are sets, then we say A is a subset of B (and write $A \subseteq B$) if every element of A is also an element of B. For example, the set of all integers is a subset of the set of all real numbers. A general strategy for showing that $A \subseteq B$ is to let x be an arbitrary element of A, and then show that x must also be an element of B. For practice, check that $\emptyset$ is a subset of every set! A general strategy for showing that $A = B$ for two sets A and B is to show that each is a subset of the other.

1.3 Unions, Intersections, and Complements

> I won’t use Google+ until I can do arbitrary unions, intersections, and complements of circles.
> – @stat110

The *union* of two sets A and B, written as $A \cup B$, is the set of all objects that are in A or B (or both). The *intersection* of A and B, written as $A \cap B$, is the set of all objects that are in both A and B. We say that A and B are *disjoint* if $A \cap B = \emptyset$. For n sets $A_1, \ldots, A_n$, the union $A_1 \cup A_2 \cdots \cup A_n$ is the set of all objects that are in *at least one* of the A_j's, while the intersection $A_1 \cap A_2 \cdots \cap A_n$ is the set of all objects that are in *all* of the A_j's.

In many applications, all the sets we're working with are subsets of some set S (in probability, this may be the set of all possible outcomes of some experiment). When S is clear from the context, we define the *complement* of a set A to be the set of all objects in S that are *not* in A; this is denoted by A^c.

Unions, intersections, and complements can be visualized easily using Venn diagrams, such as the one below. The union is the entire shared region, while the intersection is the sliver of points that are in both A and B. The complement of A is all points in the rectangle that are outside of A.

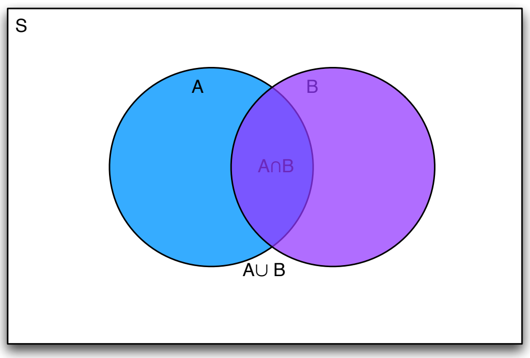

Figure 1: A Venn diagram.

Note that the area of the region $A \cup B$ is the area of A plus the area of B, minus the area of $A \cap B$ (this is a basic form of what is called the *inclusion-exclusion principle*).

De Morgan's Laws give an elegant, useful duality between unions and intersections:

$$(A_1 \cup A_2 \cdots \cup A_n)^c = A_1^c \cap A_2^c \cdots \cap A_n^c$$

$$(A_1 \cap A_2 \cdots \cap A_n)^c = A_1^c \cup A_2^c \cdots \cup A_n^c$$

It is much more important to *understand* De Morgan's laws (why they're true and how to use them) than to *memorize* them! The first says that not being in at least one of the A_j is the same thing as not being in A_1, nor being in A_2. For example, let A_j be the set of all people who like the jth Star Wars prequel (for $j \in \{1, 2, 3\}$). Then $(A_1 \cup A_2 \cup A_3)^c$ is the set of people for whom it is *not* the case that they like at least one of the prequels, but that's the same as $A_1^c \cap A_2^c \cap A_3^c$, the set of people who don't like *The Phantom Menace*, don't like *Attack of the Clones*, and don't like *Revenge of the Sith*.

For practice prove the following facts (writing out your reasoning, not just drawing Venn diagrams)s:

1. $A \cap B$ and $A \cap B^c$ are disjoint, with $(A \cap B) \cup (A \cap B^c) = A$.
2. $A \cap B = A$ if and only if $A \subseteq B$.
3. $A \subseteq B$ if and only if $B^c \subseteq A^c$.
4. $|A \cup B| = |A| + |B| - |A \cap B|$ if A and B are finite sets.

2 Functions

> The concept of function is of the greatest importance, not only in pure mathematics but also in practical applications. Physical laws are nothing but statements concerning the way in which certain quantities depend on others when some of these are permitted to vary.
> – Courant, Robbins, and Stewart, *What is mathematics?*

Let A and B be sets. A *function* from A to B is a (deterministic) rule that, given an element of A as input, provides an element of B as an output. That is, a function from A to B is a machine that takes an x in A and "maps" it to some y in B. Different x's can map to the same y, but each x only maps to one y. Here A is called the *domain* and B is called the *target*. The notation $f : A \to B$ says that f is a function mapping A into B.

Of course, we have many familiar examples, such as the function f given by $f(x) = x^2$, for all real x. It is important to distinguish between f (the function) and $f(x)$ (the value of the function when evaluated at x). That is, f is a rule, while $f(x)$ is a number for each number x. The function g given by $g(x) = e^{-x^2/2}$ is exactly the same as the function g given by $g(t) = e^{-t^2/2}$; what matters is the rule, not the name we use for the input.

A function f from the real line to the real line is *continuous* if $f(x) \to f(a)$ as $x \to a$, for any value of a. It is called *right continuous* if this is true when approaching from the right, i.e., $f(x) \to f(a)$ as $x \to a$ while ranging over values with $x > a$.

In general though, A needn't consist of numbers, and f needn't be given by an explicit formula. For example, let A be the set of all positive-valued, continuous functions on $[0, 1]$, and f be the rule that takes a function in A as input, and gives the area under its curve (from 0 to 1) as output.