Math 3430 Exam II Solutions

1. a.

$$y''y = (y')^2 \qquad\qquad y(0) = y'(0) = 1 \;\Rightarrow\; y''(0) = 1$$

$$\big\downarrow \; d/dt \qquad\qquad\qquad\qquad\qquad\qquad \Downarrow$$

$$y'''y + \cancel{y''y'} = \overset{1}{\cancel{2}}\,y''y' \qquad\qquad y'''(0) = 1$$

$$\big\downarrow \; d/dt \qquad\qquad\qquad\qquad\qquad\qquad \Downarrow$$

$$y''''y + \cancel{y'''y'} = \cancel{y'''y'} + (y'')^2 \qquad\qquad y^{(4)}(0) = 1.$$

b. $y(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

c. Guess $y(x) = e^x$. check: $\underbrace{(e^x)''e^x}_{e^{2x}} = \underbrace{((e^x)')^2}_{e^{2x}}$.

more, $y(0) = e^0 = 1$, $y'(0) = e^0 = 1$.

2. Set $y(t) = \sum_{n=0}^{\infty} a_n t^n$. $\Rightarrow y'(t) = \sum_{\substack{n=1 \\ (\text{or } 0)}}^{\infty} n a_n t^{n-1}$.

$$\therefore (t^2+1)y' + 3y = \sum_{n=1}^{\infty} n a_n t^{n+1} + \sum_{n=0}^{\infty}(n+1)a_{n+1}t^n + \sum_{n=0}^{\infty} 3a_n t^n$$

$$= \sum_{n=1}^{\infty}(n-1)a_{n-1}t^n + \sum_{n=0}^{\infty}(n+1)a_{n+1}t^n + \sum_{n=0}^{\infty} 3a_n t^n$$

$$\therefore \left.\begin{array}{lll} n=0: & a_1 + 3a_0 = 1 & a_0 = 1 \\ n=1: & 2a_2 + 3a_1 = 2 & \\ n \geq 2: & (n-1)a_{n-1} + (n+1)a_{n+1} + 3a_n = 0 & \end{array}\right\} \text{Recurrence relations}$$

3.(a) $r_1 = -2$, $r_2 = 3$ $\Rightarrow$ $(r+2)(r-3) = r^2 - r - 6$

$$\therefore \boxed{y'' - y' - 6y} = (x^2-1)'' - (x^2-1)' - 6(x^2-1)$$

$$= 2 - 2x - 6x^2 + 6$$

$$\boxed{= -6x^2 - 2x + 8}$$

is the equation.

(b) $\mathcal{L}\{H_3(t)\cos(t-3)\} = e^{-3s}\frac{s}{s^2+1}$; thus $\mathcal{L}\{H_3(t)e^{-t}\cos(t-3)\} = e^{-3(s+1)}\frac{s+1}{(s+1)^2+1}$.

(c) e.g. $y''+2y'+3y = H_2(t)e^{-(t-2)}$.

$y''+t^2y = \cos t$.

4. Applying the Laplace transform yields:

$$\begin{cases} sY_1(s) - Y_2(s) = e^{-s}\frac{1}{s^2} \\ Y_1(s) + sY_2(s) = 0 \end{cases}$$

$$\Rightarrow \quad Y_1(s) = e^{-s}\frac{1}{s(s^2+1)} = e^{-s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)$$

$$Y_2(s) = -e^{-s}\frac{1}{s^2(s^2+1)} = -e^{-s}\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right).$$

Thus, $y_1(t) = \mathcal{L}^{-1}\{Y_1(s)\} = H_1(t) - H_1(t)\cos(t-1)$.

$y_2(t) = -H_1(t)(t-1) + H_1(t)\sin(t-1)$.

Math 3430 Exam II — Supplementary Problems Solutions

1. (1) $e^t y'' - t y' + 3y = 0$ $\quad y(0)=1,\ y'(0)=-1 \Rightarrow y''(0) = -3$

$$\downarrow d/dt \qquad\qquad \Downarrow$$

$$e^t(y''+y''') - y' - t y'' + 3y' = 0 \qquad y'''(0) = 5$$

(2) $y(t) = 1 - t - \frac{3}{2}t^2 + \frac{5}{6}t^3 + \dots$

2. Set $y(t) = \sum_{n=0}^{\infty} a_n t^n$.

$$\therefore \quad y'' + (-2t)y' + y = \sum_{n=0}^{\infty} a_{(n+2)} \cdot (n+2)(n+1)t^n - 2\sum_{n=0}^{\infty} n a_n t^n + \sum_{n=0}^{\infty} a_n t^n$$

$$= \sum_{n=0}^{\infty} \left[(n+1)(n+2)a_{n+2} + (1-2n)a_n \right] t^n$$

$$= 0$$

$$\therefore \quad a_{n+2} = \frac{2n-1}{(n+1)(n+2)} a_n \quad , \qquad a_0 = 1,\ a_1 = 2.$$

3. (1) $y(t) = \sin t * e^t$

$$\downarrow \mathcal{L}$$

$$Y(s) = \frac{1}{s^2+1} \cdot \frac{1}{s-1} \quad \Rightarrow \quad (s^2+1)Y(s) = \frac{1}{s-1}$$

$$\downarrow \mathcal{L}^{-1} \text{ with } y(0) = y'(0) = 0$$

$$\begin{cases} y'' + y = e^t \\ y(0) = y'(0) = 0. \end{cases}$$

(2) Euler Equations

$$t^2 y'' + \alpha t y' + \beta y = 0 \quad (t > 0)$$

$$\downarrow$$

$$r^2 + (\alpha-1)r + \beta = (r+3)^2 = r^2 + 6r + 9$$

$$\therefore \quad \alpha = 7 \ , \ \beta = 9.$$

4. Note that $f(t) = 1 + H_1(t)(t-1)$. So $y''+2y'+4y = f(t)$

$$\begin{aligned}(s^2+2s+4)Y(s) \\ -(s+4)\end{aligned} = \left\{\begin{aligned}&(s^2+2s+4)Y(s) \\ &-y'(0) - 2y(0) \\ &-sy(0)\end{aligned}\right. \xleftarrow{\mathcal{L}} y''+2y'+4y = f(t) \xrightarrow{\mathcal{L}} \frac{1}{s} + e^{-s}\frac{1}{s^2}$$

$$\therefore \quad Y(s) = \underbrace{\frac{s+4}{s^2+2s+4}}_{\displaystyle = \frac{s+1}{(s+1)^2+(\sqrt{3})^2} + \sqrt{3}\frac{\sqrt{3}}{(s+1)^2+(\sqrt{3})^2}} + \underbrace{\frac{1}{s(s^2+2s+4)}}_{\displaystyle = \frac{1}{4}\left(\frac{1}{s} - \frac{s+2}{s^2+2s+4}\right)} + \underbrace{e^{-s}\frac{1}{s^2(s^2+2s+4)}}_{\displaystyle = -\frac{1}{8}e^{-s}\left(\frac{s-2}{s^2} - \frac{s}{s^2+2s+4}\right)}$$

$\downarrow \mathcal{L}^{-1}$

$$\therefore\ y(t) = e^{-t}\left(\cos\sqrt{3}t + \sqrt{3}\sin\sqrt{3}t\right) + \overbrace{\frac{1}{4} - \frac{1}{4}e^{-t}\left(\cos\sqrt{3}t + \frac{1}{\sqrt{3}}\sin\sqrt{3}t\right)}$$

$$- \frac{1}{8}H_1(t)\left(1 - 2(t-1) - e^{-(t-1)}\cos\sqrt{3}(t-1) + \frac{1}{\sqrt{3}}e^{-(t-1)}\sin\sqrt{3}(t-1)\right).$$

5. We can find the equation by taking $\mathcal{L}$ on the expression of $y(t)$

$$y(t) = 3e^{-t}\cos t + H_3(t)e^{-(t-3)}\sin(t-3) - 2\int_0^t e^{\tau}\sin(t-\tau)\,d\tau$$

$\mathcal{L} \downarrow \qquad \downarrow \mathcal{L}$

$$Y(s) = 3\frac{s+1}{(s+1)^2+1} + e^{-3s}\frac{1}{(s+1)^2+1} - 2\frac{1}{s-1}\cdot\frac{1}{s^2+1}.$$

Multiplying both sides by $(s+1)^2+1$ and moving $3(s+1)$ to the left-hand-side, we have:

$$(s^2+2s+2)Y(s) - 3s - 3 = e^{-3s} - 2\underbrace{\frac{s^2+2s+2}{(s-1)(s^2+1)}}$$

$\mathcal{L}^{-1} \swarrow \qquad \mathcal{L}^{-1} \swarrow$

$$\left\{\begin{aligned}&y''+2y'+2y \\ &y(0) = 3,\ y'(0) = -3\end{aligned}\right.$$

$$\begin{aligned}&\delta(t-3) - 2A - 2B\cos t \\ &\qquad - 2C\sin t \\ &= \delta(t-3) - 5 + 3\cos t \\ &\qquad - \sin t\end{aligned}$$

$$= \frac{A}{s-1} + \frac{Bs+C}{s^2+1} \Rightarrow \left\{\begin{aligned}A+B &= 1 \\ -B+C &= 2 \\ A-C &= 2\end{aligned}\right.$$

$$\therefore \quad A = \frac{5}{2}, \quad B = -3/2, \quad C = 1/2$$

$\therefore$ Conclude:

$$\left\{\begin{aligned}&y''+2y'+2y = \delta(t-3) - 5 + 3\cos t - \sin t \\ &y(0) = 3,\quad y'(0) = -3.\end{aligned}\right.$$

Math 3430 Sample Final Solutions

1. (1) False; (the term yy', for example)

   (2) True. Multiplying both sides by $2019x - 15y$ and rearranging terms, we obtain:

   $$(3x - 2019y) - (2019x - 15y)y' = 0,$$

   which is an exact equation.

   (3) False. The equation is linear, but inhomogeneous.

   (4) False. The constant solution are missing from the general expression.

2. Set $M(x,y) = 3x^2y^2 - y\cos(xy) - 2x$
   $N(x,y) = \gamma x^3y - x\cos(xy) - 2y$.
   For exactness, we need: $M_y = N_x$, that is:

   $$6x^2y - \cos(xy) + xy\sin(xy) = 3\gamma x^2y + (-\cos(xy)) + xy\sin(xy).$$

   It follows that $\gamma = 2$.

   To solve the corresponding ODE, we look for a function $F(x,y)$ satisfying

   $$F_x = 3x^2y^2 - y\cos(xy) - 2x$$

   $$\therefore \quad F(x,y) = x^3y^2 - \sin(xy) - x^2 + h(y).$$

   Now $F_y = 2x^3y - x\sin(xy) + h'(y) = N(x,y) = 2x^3y - x\cos(xy) - 2y$.

   Therefore, one could choose

   $$h(y) = -y^2.$$

   It follows that we have solution:

   $$\boxed{x^3y^2 - \sin(xy) - x^2 - y^2 = C}$$

   Using the initial value, we determine $\boxed{C = -1.}$

3. This is a linear 1st order ODE; which can be solved using the method of integrating factors.

   $$\mu = e^{\int p(x)dx} = e^{\int \frac{1}{x}dx} = x.$$

   $$\therefore \quad y(x) = \frac{1}{\mu}\int \mu(x)g(x)\,dx = \frac{1}{x}\int x \cdot \sin x\,dx = \frac{1}{x}\left(-x\cos x + \sin x + C\right).$$

   Using initial value, we determine $C = 0$.

   $$\therefore \quad \boxed{y(x) = -\cos x + \frac{\sin x}{x}}.$$

4. Suppose another solution is of the form $v(x)\, y_1(x)$, where $y_1(x) = x$ is the given solution. We have

$$y_1 v'' + (p y_1 + 2 y_1')\, v' = 0 .$$

letting $z(x) := v'(x)$ and substituting $y_1(x) = x$, we obtain:

$$x z' + \left(-\frac{x^2}{x-1} + 2 \right) z = 0 .$$

$$\therefore \quad z' + \left(-\frac{x}{x-1} + \frac{2}{x} \right) z = 0$$

$$\frac{z'}{z} = \frac{x}{x-1} - \frac{2}{x} \quad \Rightarrow \quad z = e^x \frac{x-1}{x^2} .$$

Now $v = \int z(x)\, dx = \dfrac{e^x}{x}$. Hence $y_2(x) = v \cdot y_1 = e^x$.

General solution : $\boxed{y(x) = C_1 x + C_2 e^x}$

5. $y_0 = 0$. $\quad y_0' = 0(1 + 0^2) = 0$ $\qquad t_0 = 0$

$y_1 = y_0 + \Delta t \cdot y_0' = 0$, $\quad y_1' = 1(1 + y_1^2) = 1$ $\qquad t_1 = 1$

$y_2 = y_1 + \Delta t \cdot y_1' = 1$, $\quad y_2' = 2(1 + y_2^2) = 4$ $\qquad t_2 = 2$

$y_3 = y_2 + \Delta t \cdot y_2' = \boxed{5}$ $\qquad t_3 = 3$

$\therefore \quad y(3) \approx 5 .$

6. Setting $y(x) = \sum_{n=0}^{\infty} a_n x^n$; we have

$$y'' = \sum_{n=0}^{\infty} a_{n+2} (n+1)(n+2) x^n$$

$$(1 - x^2) \cdot y' = \sum_{n=0}^{\infty} a_{n+1} (n+1) x^n - \underbrace{\sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n+1}}_{\displaystyle \parallel \atop \displaystyle \sum_{n=2}^{\infty} a_{n-1} (n-1) x^n}$$

$$x y = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n .$$

Putting together: $y'' + (1-x^2)\, y' + xy$

$$= \sum_{n=0}^{\infty} a_{n+2}(n+1)(n+2)x^n + \sum_{n=0}^{\infty} a_{n+1}(n+1)x^n - \sum_{n=2}^{\infty} a_{n-1}(n-1)x^n + \sum_{n=1}^{\infty} a_{n-1}x^n .$$

$n=0$: $\quad 2a_2 + a_1 = 0$

$n=1$: $\quad 6a_3 + 2a_2 + a_0 = 0$

$n \geq 2$: $\quad (n+1)(n+2)a_{n+2} + (n+1)a_{n+1} + (2-n)a_{n-1} = 0.$

Recurrence relations.

$a_0 = y(0), \quad a_1 = y'(0).$

7. Applying the Laplace transform gives:

$$sY(s) + \mathcal{L}\{e^{-2t} * y(t)\} = e^{-s}.$$

that is, $$sY(s) + \frac{1}{s+2}Y(s) = e^{-s}.$$

$$\therefore Y(s) = \frac{s+2}{(s+1)^2}e^{-s} = e^{-s}\left(\frac{1}{s+1} + \frac{1}{(s+1)^2}\right).$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1} + \frac{1}{(s+1)^2}\right\} = (1+t)e^{-t}.$$

$$\therefore \boxed{y(t) = H_1(t)\, t e^{-(t-1)}.}$$

8. Note that $\det(A-\lambda I) = \det\begin{pmatrix} 2-\lambda & 0 & 3 \\ 0 & 4-\lambda & 1 \\ 0 & 3 & 2-\lambda \end{pmatrix} = (2-\lambda)(\lambda^2-6\lambda+5) = -(\lambda-2)(\lambda-1)(\lambda-5).$

E-vectors: $\lambda=1$, $\lambda=2$, $\lambda=5$

$$A-I = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 3 & 1 \\ 0 & 3 & 1 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} -9 \\ -1 \\ 3 \end{pmatrix}; \quad A-2I = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 2 & 1 \\ 0 & 3 & 0 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad A-5I = \begin{pmatrix} -3 & 0 & 3 \\ 0 & -1 & 1 \\ 0 & 3 & -3 \end{pmatrix} \Rightarrow v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

$$\therefore \boxed{y(t) = c_1 e^{t}\begin{pmatrix} -9 \\ -1 \\ 3 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_3 e^{5t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.}$$

9. $J = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 5 & 3 & 1 \end{pmatrix}. \quad \Rightarrow Q^{-1} = \frac{1}{16}\begin{pmatrix} 1 & -4 & 3 \\ -5 & 4 & 1 \\ 10 & 8 & -2 \end{pmatrix}.$

$$\therefore \boxed{A} = QJQ^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 5 & 3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix}\frac{1}{16}\begin{pmatrix} 1 & -4 & 3 \\ -5 & 4 & 1 \\ 10 & 8 & -2 \end{pmatrix}$$

$$\boxed{= \frac{1}{16}\begin{pmatrix} -23 & -36 & 11 \\ -50 & -8 & 10 \\ -75 & -20 & 47 \end{pmatrix}.}$$