MATH 3430-02 WEEK 5-1

Key Words: Repeated roots; Reduction of Order (constant coefficient case)

Review.
Q1. Find the solution of the following initial value problem.

$$y'' - 2y' + 5y = 0, \qquad y(0) = 1, \qquad y'(0) = -1.$$

Q2. *(relating to the 2nd order linear Existence and Uniqueness theorem)*
If, in a homogeneous second order linear ODE

$$y'' + p(t)y' + q(t)y = 0,$$

$p(t), q(t)$ are both continuous on an interval containing $t = 0$, is it possiible for $y(t) = t^2$ to be a solution of this ODE?

Q3. Check that $y(t) = t^2$ is a solution of the following linear homogeneous ODE:

$$2t^2y'' - ty' - 2y = 0.$$

Compare this example with your conclusion in **Q2**. Which condition/observation allows $y(t) = t^2$ to be a solution here but not in **Q2**?

This lecture we finish the topic: *constant coefficient homogeneous linear second order ODEs.* There is only one case we haven't dealt with: when the characteristic polynomial has repeated roots.

Q4. Find the characteristic polynomial of the ODE

$$y'' - 2ry' + r^2 y = 0,$$

where r is a real number. Which solution(s) of this ODE do you know so far?

Idea: Can we modify the one known solution to obtain more solutions? Here, by 'modify', we mean multiplying the known solution by a factor $u(t)$. This method is known as **reduction of order**, for a reason you'll see below. It's application is not limited to constant coefficient ODEs, as we'll see next time.

(1) Substitute

$$y_2(t) = u(t)e^{rt}$$

into the original ODE and simplify. Which differential equation does $u(t)$ satisfy?

(2) Note that, in the differential equation $u(t)$ satisfies, every term involving u has at least one derivative. This suggests us to make the substitution:

$$v(t) := u'(t).$$

Now write down the corresponding ODE satisfied by $v(t)$. *(Note that we are now dealing with an ODE of a lower order. This is what 'reduction of order' refers to.)*

(3) Now find a solution $v(t)$, then find a non-constant solution $u(t)$. *(Why wouldn't we like $u(t)$ to end up being a constant? What would $y_2(t)$ be if $u(t)$ is a constant?)*

(4) What is $y_2(t)$ based on your choice of $u(t)$ in (3)?

(5) What is the general solution of the original ODE?

Q5. Solve the initial value problem

$$y'' - 4y' + 4y = 0, \qquad y(0) = 1, \quad y'(0) = 3.$$