CORRECTION TO WORKSHEET 3-2 Q12

Dear all,

As you know, there was an error in **Q12** of today's lecture, which caused me a little trouble. In fact, I believe that such a pause was a good one, because the equation is so natural and simple that you'd hope to see how the estimates in the 'E&U Theorem' and the 'Euler's Error' might work for it! Unfortunately, these estimates do not work well, at least not in an obvious way, as I will explain below. This tells us that our methods have limitations, thus one must come up with ways to use them properly or improve these theorems.

To recapture, the IVP in question is

$$y'(t) = y(t), \qquad y(0) = 1.$$

The true solution is $y(t) = e^t$, which extends to all $t \in (-\infty, \infty)$. In particular, $y(1) = e$. Immediately, we can ask the following two questions:

> **I.** *What is the largest interval of existence that the Existence and Uniqueness Theorem can guarantee us?*
>
> **II.** *Is it possible to obtain an estimate of the value of e, pretending that we don't know that $e = 2.71828...$, using Euler's method and the estimation of Euler's error?*

I'll try to approach these two questions using our recently-learned methods.

I

1. Question **I** is essentially a question of choosing an appropriate rectangular region

$$\mathcal{R} = [0, a] \times [1 - b, 1 + b]$$

so that we can obtain as large an α as possible. *(This is where I messed up in my notes; I thought that the initial y_0 was 0, so that we can obtain $\alpha = 1$ using $a = b = 1$!)*

Note that, in our notation,

$$M = \max_{\mathcal{R}} |y| = 1 + b, \qquad \alpha = \min\left\{a, \frac{b}{M}\right\} = \min\left\{a, \frac{b}{1+b}\right\}.$$

So the true constraint that prohibits α from being as big as we like is not a but b, since

$$\frac{b}{b+1} < 1.$$

However, we can choose a sequence $b_k \to \infty$ and obtain

$$\alpha_k := \frac{b_k}{1 + b_k} \to 1, \quad (k \to \infty).$$

In this way, we know that there is a unique solution $y(k)$, defined on the interval $[0, \alpha_k]$, to the initial value problem. Therefore, we have a solution $y(t)$ that is defined, continuous for all $t \in [0, 1)$ (1 is not included). *Is it possible to extend the definition of $y(t)$ to $t = 1$?* In class, I claimed a 'yes'. This is indeed true, however, not obvious from this direct approach.

One way to justify this, without invoking the true solution e^t, is the following.

2. Let $z(t) := y^{1/2}(t)$. Using the equation of $y(t)$, we find that $z(t)$ satisfies the IVP

$$z'(t) = \frac{1}{2}z(t), \qquad z(0) = 1.$$

Using the same argument as that in Paragraph **1**, for $z(t)$ and some new a, b, we have

$$M = \max_{\mathcal{R}} |z/2| = (1+b)/2, \qquad \alpha = \min\left\{a, \frac{b}{M}\right\} = \min\left\{a, \frac{2b}{1+b}\right\}.$$

By choosing $a = b = 1$, we have $\alpha = 1$, and $|z(1) - 1| \le b = 1$ *(by the last part of the Existence and Uniqueness Theorem).* It follows that $z(1)$ is determined, satisfying $|z(1)| < 2$; hence, $y(1)$ is determined, satisfying $|y(1)| = |z(1)|^2 < 4$.

3. The previous paragraph shows that the solution $y(t)$ exists and is unique at least on $[0, 1]$.

4. Analogously, if you set $w(t) = y^{1/k}(t)$, you'd have

$$w'(t) = \frac{1}{k}w(t), \qquad w(0) = 1.$$

Moreover, for this equation in $w(t)$ and some new a, b, we compute

$$M = \max_{\mathcal{R}} |w/k| = (1+b)/k, \quad \alpha = \min\{a, kb/(1+b)\}.$$

By choosing $a = k$ and $b = k - 1$, we obtain $\alpha = k - 1$. This means $w(t)$ exists and is unique on the interval $[0, k-1]$. Therefore, $y(t)$ exists and is unique on $[0, k-1]$. By letting $k \to \infty$, we see that the solution of

$$y' = y, \qquad y(0) = 1$$

can be extended for all $t \ge 0$.

5. From another perspective, *what if we only care about estimating $w(1)$ using Euler's Method?* In this case, we want to choose a not-too-large b, such that $\alpha = 1$. In other words, we choose

$$b = \frac{1}{k-1}.$$

It follows that we have the estimate: $|w(1) - 1| \le b = \dfrac{1}{k-1}$. In other words,

$$w(1) \le 1 + \frac{1}{k-1} = \frac{k}{k-1}.$$

6. As a consequence of Paragraph **5**,

$$y(1) = w(1)^k \le \left(\frac{k}{k-1}\right)^k = \left(1 + \frac{1}{k-1}\right)^k.$$

If you remember the definition $e := \lim_{n\to\infty} \left(1 + \dfrac{1}{k}\right)^k$, you'll see what the previous inequality tells you: *Since $y(1)$ does not depend on k and that the right-hand-side has the limit e, we obtain $y(1) \le e$.* See how 'tight' this estimate is! (since $y(1) = e$.)

II

7. Now we address the second question: *Can we find a good approximation of the value e using Euler's Method?*

In fact, we cannot direcly apply the Existence and Uniqueness Theorem to the equation $y' = y$ to estimate the Euler error at $t = 1$. This is because, for the equation

$$y' = y, \qquad y(0) = 1,$$

the α we can have is at best < 1 and not attaining 1.

8. What if we consider the equation of $z(t)$ instead? That is,

$$z'(t) = \frac{1}{2} z(t), \qquad z(0) = 1.$$

By choosing $a = b = 1$ as in Paragraph **2**, we have

$$\alpha = 1, \quad L = \max_{\mathcal{R}} |f_z| = \frac{1}{2}, \quad D = \max_{\mathcal{R}} |f_t + f_z f| = \frac{1}{4}(1 + b) = \frac{1}{2}.$$

Therefore, if we use Euler to approximate $z(1)$, we would have an error

$$E_n \leq \frac{Dh}{2L}(e^{\alpha L} - 1) = \frac{h}{2}(\sqrt{e} - 1) < \frac{h}{2}(\sqrt{4} - 1) = \frac{h}{2}.$$

(Here, to avoid circular arguments, since we have to pretend not knowing the value of e, we only assume that e < 4.)

9. Therefore, to have an error as small as 0.001, we need an $h \leq 0.002$, which, in our case ($t_0 = 0$, $T = 1$), means

$$n = 500.$$

Let's use Euler's method with $n = 500$ to approximate $z(1) = \sqrt{e}$. *(This should still be possible to compute by hand, though tedious.)* The answer I obtained (using a machine) is:

$$z_{500} \approx 1.64830942,$$

so the actual error is

$$|\sqrt{e} - z_{500}| \approx 0.00041185 < 0.001.$$

10. Furthermore, we can estimate:

$$|e - z_{500}^2| \leq |\sqrt{e} - z_{500}||\sqrt{e} + z_{500}| < 0.001 \times 4 = 0.004.$$

The actual value of z_{500}^2 is

$$z_{500}^2 \approx 2.716923954.$$

So we find that e is between 2.716923954 and 2.720923954.

The true value of e is $2.71828182846\cdots$.

Is using Euler's method computationally efficient in terms finding, say, the first N digits of e accurately? (Of course, there are other methods that compute quickly the first N digits of e.) I shall not pursue further. This is a question for you. (In fact, using the Taylor series of e^t evaluated at $t = 1$ is probably already pretty efficient: $e = 1 + 1 + \frac{1}{2} + \cdots + \frac{1}{k!} + \cdots$. However, this involves divisions, whereas Euler's method only involves multiplication and addition.)

I have to stop here. I didn't intend to write this much, but I hope you'll be interested.

Yours,
Yuhao