MATH 3430-02 WEEK 7-1

Key Words: 'Guessing' method for finding $y_p(t)$ (equation having constant coefficients).

Setting: Second order constant coefficient ODE, inhomogeneous, need to find a particular solution.

A natural question is: *Since the ODEs considered have constant coefficients, we know how to find all homogeneous solutions. As a result, the method of variation of parameters would yield a particular solution. Why introduce another method?*

The answer is simple: The formulae for variation of parameters involve integrals, and integrals are not always easy to compute. On the other hand, sometimes, guessing the form of a solution, then determining the constants therein, can be easier.

Q1. Consider the ODE

$$y'' - 2y' - 3y = te^{\alpha t},$$

where α is a parameter. A 'natural' guess of a particular solution is of the form $y(t) = v(t)e^{\alpha t}$. What do you think $v(t)$ should look like?

Q2. Now substitute $y(t) = v(t)e^{\alpha t}$ into the equation, we obtain the following equation in $v(t)$.

Q3. In cases of $\alpha = -1, 0, 1, 3$, find appropriate $v(t)$'s. What do you observe?

Q4. Consider the ODE

$$y'' - 4y' + 4y = t^2 e^{\alpha t},$$

In each case of $\alpha = 0, 1, 2$, find $v(t)$ such that $v(t)e^{\alpha t}$ is a particular solution. What do you observe?

Fact. For an equation of the form

$$y'' + ay' + by = (a_0 + a_1 t + \cdots + a_k t^k)e^{\alpha t}.$$

If α is a root (with multiplicity r) of $p(\lambda) = \lambda^2 + a\lambda + b$, then the ODE has a particular solution of the form

$$y_p(t) = t^r(c_0 + c_1 t + \cdots + c_k t^k)e^{\alpha t}.$$

Q5. What is the form of a particular solution of the following ODE?

$$y'' + 2y' - 8y = (1 + 2t)e^{2t}.$$

Then determine the constants in your guess.

Q6. To find a particular solution of

$$y'' - 2y' + 3y = e^t \cos \sqrt{2}t,$$

first note that this equation itself is the real part of

$$(y + ih)'' - 2(y + ih)' + 3(y + ih) = e^{(1+\sqrt{2}i)t}.$$

In other words, if $z(t)$ is a complex solution of

$$z'' - 2z' + 3z = e^{(1+\sqrt{2}i)t},$$

then $\mathrm{Re}(z)$ is a solution of the original ODE.

Now for the equation in $z(t)$, the characteristic polynomial is

$$p(\lambda) = \underline{\hspace{5cm}}.$$

Is $\alpha = 1 + \sqrt{2}i$ a root of $p(\lambda)$? If so, what is the multiplicity?

Now make a guess of the form of z_p:

$$z_p(t) = \underline{\hspace{4cm}}.$$

Determine $z_p(t)$, based on your guess, by substituting it in the equation for $z(t)$:

Now find

$$y_p(t) = \mathrm{Re}(z_p(t)) = \underline{\hspace{5cm}}.$$