MATH 3430-02 WEEK 4-3

Key Words: Constant Coefficient Homogeneous Linear 2nd order ODEs; Method of Characteristic Polynomial; Complex Numbers.

A linear second order ODE takes the form

$$y'' + p(t)y' + q(t)y = g(t).$$

A **constant coefficient homogeneous** linear second order ODE takes the form:

$$y'' + ay' + by = 0.$$

Q1. For the following ODE

$$y''(t) + 5y'(t) + 6y(t) = 0,$$

suppose that one is looking for solutions of the form $y = e^{rt}$. (There is a little guessing here.)

(1) *Which value(s) should r take?*

(2) From these values of r, which solutions of the ODE have you obtained?

(3) Have you found all solutions of this ODE? If so, what's the general expression?

(4) What if, in addition, we impose the initial condition

$$y(0) = 1, \quad y'(0) = 3?$$

Q2. Now follow a similar procedure *(guess e^{rt}→find r→general solution→determine constants)* to solve the following initial value problem:

$$y''(t) + 2y'(t) - 8y(t) = 0, \quad y(0) = 2, \quad y'(0) = 0.$$

Associated to a constant coefficient homogeneous second order ODE

$$y'' + ay' + by = 0$$

is the so-called **characteristic polynomial**:

$$p(r) := r^2 + ar + b.$$

This is a factor you would obtain when you make the substitution $y(t) = e^{rt}$ for the left-hand-side of the ODE. From the previous two examples, we have observed the following:

> *If, in the current case (const. coeff., homogeneous, 2nd order), the characteristic polynomial $p(r)$ has* two distinct real roots r_1, r_2*. Then the general solution of the homogeneous equation is*
>
> $$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

Q3. Write down the characteristic polynomial and the general solutions of the ODE

$$y'' - y' - 20y = 0.$$

However, we may encounter cases that are not covered by the statement above. There are two other possibilities:

I. *The characteristic polynomial has no real roots, but two complex roots that are complex conjugates of each other.*

II. *The characteristic polynomial is of the form* $p(r) = (r - r_0)^2$*, in which case we say that* $p(r)$ *has a repeated root* r_0*.*

We'll try to deal with **Case I** today, and leave **Case II** to next time.

Q4. What is a homogeneous constant coefficient second order ODE with the characteristic polynomial

$$p(r) = r^2 + 4?$$

What are the roots of $p(r)$?

After **Q4**, you may ask, can we claim the two solutions: e^{2it} and e^{-2it}? But what do these expressions mean? Exponential functions whose powers are complex numbers? In fact, we need to make sense of them.

Q5. Write down the Taylor series of e^x, expanded at $x = 0$:

$$e^x = \underline{\hspace{8cm}}.$$

Now, in your expression above, replace x by it, and simplify:

$$e^{it} = \underline{\hspace{8cm}}$$

$$= \underline{\hspace{8cm}}.$$

The terms in e^{it} with even powers form a series

$$\underline{\hspace{8cm}},$$

which is the power series of the function ____________.

The terms in e^{it} with odd powers form a series

$$\underline{\hspace{8cm}},$$

which is the power series of the function ____________.

From this argument, we obtain the important formula:

$$e^{it} = \underline{\hspace{3cm}} + i \cdot \underline{\hspace{3cm}}.$$

A particular case is when you set $t = \pi$ and obtain

$$e^{i\pi} = \underline{\hspace{3cm}}.$$

(Some people regard this as the most beautiful equation in mathematics, which purely contains -1, π, e and i, each member having, in history, extended the human knowledge of numbers.)

From the above, we obtain, letting $z = a + ib$,

$$\boxed{e^z = e^{(a+ib)} = e^a e^{ib} = e^a(\cos b + i \sin b).}$$

(Here, you may ask, how can you split e^{a+ib} as $e^a e^{ib}$? This is a property of e^z where z can be complex numbers. For details, refer to Analytic Function Theory*.)*

One more property: Letting t be a real variable, r a complex number.

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{rt} = re^{rt}.$$

Q6. Back to **Q4** with the ODE $y'' + 4y = 0$. If we write a 'solution' as

$$y(t) = e^{2it} = \underline{\hspace{3cm}},$$

you'll notice this is not a real-valued function.

How to find real-valued solutions? The answer is simple: *Just take, respectively, the real and imaginary parts of a complex solution.* This is because, if both $z(t)$ and $\bar{z}(t)$ satisfy the homogeneous linear ODE $L[y] = 0$, then their linear combination

$$\mathrm{Re}(z) = \frac{1}{2}(z + \bar{z}), \quad \mathrm{Im}(z) = \frac{1}{2i}(z + \bar{z})$$

would also satisfy $L[y] = 0$.

What is the general solution of $y'' + 4y = 0$?

Q7. Find the general solution of the ODE

$$y'' - 2y' + 10y = 0.$$