MATH 3430-02 WEEK 1-1

Key Words: Ordinary Differential Equations (ODEs); Order; 1-st Order Linear Equations; the Method of Integrating Factors.

An *ordinary differential equation* (ODE) is a relation established on a single-variable function $y(t)$ and its derivatives.

Q1. Which of the following is an ODE?

$$\textbf{A.}\ \frac{dy}{dt} = t + y^2; \qquad \textbf{B.}\ \frac{\partial y}{\partial s} = (s^2 + t^2)\frac{\partial y}{\partial t}; \qquad \textbf{C.}\ \frac{d^2y}{dx^2} = x^2 e^y.$$

The *order* of an ODE in $y(t)$ is the highest derivative of y appearing in the equation.

Q2. Determine the order of each of the following ODEs?

$$\textbf{A.}\ \frac{dy}{dt} = t + y^2; \qquad \textbf{B.}\ (1 + y^2)y'' + te^y = 1; \qquad \textbf{C.}\ t^5 y^{(4)} + (y'' + 1)^2 = 1.$$

Ans.

One can classify ODEs by their order. We'll start with the simplest case: 1-st order ODEs, which take the general form:

$$\frac{dy}{dt} = f(t, y),$$

where we assume f to be a continuous function in both t and y.

By a *solution* we mean a differentiable function $y(t)$ that satisfies the ODE.

Q3. What are all the solutions of the 1-st order ODE $y' = t^2 + 1$?

Q4. What are all the solutions of the 1-st order ODE $y' = (t^2 + 1)y$?

In general, there is no explicit way to find solutions for a 1-st order ODE. However, in a particular case, we have the technique to find all solutions: when the ODE is linear.

A 1-st order ODE is said to be *linear* if it can be put in the form:

$$\frac{dy}{dt} + a(t)y = b(t).$$

(To correspond to the notation $y' = f(t, y)$, this is precisely when $f(t, y) = -a(t)y + b(t)$, a linear function in y.)

Q4. Which of the following 1-st order ODEs are linear?

$$\textbf{A.}\frac{dy}{dt} = (e^t + y)y; \qquad \textbf{B.}\frac{dy}{dt} = (t^2 + 1)y + 3; \qquad \textbf{C.}(t^2 + 1)\frac{dy}{dt} - 2ty = 1.$$

The well-known method to solve a 1-st order linear ODE is called *the Method of Integrating Factors.* The idea is starting from the ODE

$$y' + a(t)y = b(t).$$

Multiplying both sides of this equation by $\mu(t)$ (not specified yet), we'll obtain

$$\mu(t)y' + a(t)\mu(t)y = \mu(t)b(t).$$

Key Question: Can we choose $\mu(t)$ such that the left hand side of the previous equation is equal to $(\mu(t)y(t))'$? If so, find an expression of $\mu(t)$; then find a general expression of a solution $y(t)$.

As a consequence, we obtain the formula:

Q5. Find all solutions of the 1-st order linear ODE $y' + 2ty = t$.