MATH 3430-02 WEEK 12-1

Key Words: 1-st order linear systems.

A **1-st order linear system** of ODEs takes the form

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \\ \vdots \\ y_n'(t) \end{pmatrix} = \begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix} + \begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{pmatrix},$$

where $a_{ij}(t)$, $g_i(t)$ are given continuous functions defined on a domain in $\mathbb{R}$.
It is convenient to use vector and matrix notations to rewrite the above as

$$\mathbf{y}'(t) = A(t)\mathbf{y} + \mathbf{g}(t).$$

An **initial value problem** in this setting is the system above together with an initial condition

$$\mathbf{y}(0) = \mathbf{y}_0 \in \mathbb{R}^n.$$

We focus on the case when A is a constant matrix and $\mathbf{g}(t) = 0$, namely, the '**constant-coefficient homogeneous**' case.
An example is

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \mathbf{x}.$$

This system tells us that

$$x_1' = x_2, \qquad x_2' = -2x_1 - 3x_2.$$

This is equivalent to

$$x_1'' + 3x_1' + 2x_1 = 0.$$

Seeing this, we realize that the original system is a way to rewrite a 2nd-order constant coefficient equation.

In fact, every higher order linear system corresponds to a 1-st order linear system with more variables. For example, in

$$\begin{cases} y'' + 2y' - 3z = 0, \\ z'' - z' + 5y = 0, \end{cases}$$

we can introduce the variables $x_1, ..., x_4$ such that

$$\begin{cases} x_1 = y, \\ x_2 = y', \\ x_3 = z, \\ x_4 = z'. \end{cases}$$

It follows that the system in y, z can be turned into the 1-st order linear system:

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

A linear system that is easiest to solve is one with A being diagonal. Such a system is called 'decoupled', because, despite being a 'system', the equations involved really don't affect each other. For example, consider

$$\mathbf{x}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix} \mathbf{x}.$$

This system is no different than the following three equations:

$$x_1' = x_1,$$
$$x_2' = 2x_2,$$
$$x_3' = -3x_3.$$

Thus, the general solutions of the system is

$$\mathbf{x} = \begin{pmatrix} C_1 e^t \\ C_2 e^{2t} \\ C_3 e^{-3t} \end{pmatrix},$$

where C_1, C_2, C_3 are arbitrary constants.

Of course, in practice, it is rare to have A being diagonal. It is more general, but not more difficult, a situation if we assume A to be **diagonalizable**. This means that we can find an invertible matrix P and a diagonal matrix D such that

$$A = PDP^{-1}.$$

By construction, the diagonal entries of D are eigenvalues of A, the columns of P being the corresponding (basis) eigenvectors. [Note that not every A is diagonalizable!]
Under this assumption, the system

$$\mathbf{x}' = A\mathbf{x}$$

is equivalent to

$$\mathbf{x}' = PDP^{-1}\mathbf{x},$$

or, by noting that P is a constant matrix,

$$(P^{-1}\mathbf{x})' = DP^{-1}\mathbf{x}.$$

It is natural to introduce

$$\mathbf{y} = P^{-1}\mathbf{x},$$

so that the system becomes

$$\mathbf{y}' = D\mathbf{y},$$

a case we have already dealt with.

As an example, consider the system

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \mathbf{x}.$$

The matrix A has the characteristic polynomial

$$p(\lambda) = \det \begin{pmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{pmatrix} = \lambda^2 + 3\lambda + 2.$$

(We remarked in class that this polynomial is the same as the characteristic polynomial of the corresponding 2nd order equation.)
So the eigenvalues of A are

$$\lambda = -1, -2.$$

By solving the systems $A - \lambda I = \mathbf{0}$, we find that $\lambda = -1$ has an eigenvector

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix};$$

$\lambda = -2$ has an eigenvector

$$\mathbf{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Therefore, we can choose

$$D = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \qquad P = \begin{pmatrix} -1 & -1 \\ 1 & -2 \end{pmatrix}.$$

Using notations from the theoretical development above, we have

$$\mathbf{y} = \begin{pmatrix} C_1 e^{-t} \\ C_2 e^{-2t} \end{pmatrix}.$$

And

$$\begin{aligned} \mathbf{x} = P\mathbf{y} &= \begin{pmatrix} -1 & -1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} C_1 e^{-t} \\ C_2 e^{-2t} \end{pmatrix} \\ &= C_1 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} -1 \\ 2 \end{pmatrix}. \end{aligned}$$

Note that the expression of $\mathbf{x}$ is in the form

$$C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2,$$

which is a linear combination of expressions such as

$$e^{\lambda t}\mathbf{v},$$

where λ is an eigenvalue of A, $\mathbf{v}$ a corresponding (basis) eigenvector.
Indeed, this observation is no a coincidence, as the following theorem will explain.

Theorem. *Suppose that A is diagonalizable with (real) eigenvalues*

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$$

with corresponding eigenvectors, which form a basis of $\mathbb{R}^n$:

$$\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n.$$

The general solutions of the system

$$\mathbf{x}' = A\mathbf{x}$$

are

$$\mathbf{x}(t) = \sum_{k=1}^{n} C_k e^{\lambda_k t}\mathbf{v}_k.$$

Proof of the Theorem. The diagonalizability condition gives

$$A\mathbf{v}_k = \lambda_k \mathbf{v}_k.$$

In other words, letting

$$D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}, \qquad P = (\mathbf{v}_1 \ \mathbf{v}_2 \cdots \ \mathbf{v}_n),$$

we have

$$A = PDP^{-1}.$$

The solution to

$$\mathbf{y}' = D\mathbf{y}$$

are given by

$$y_k = C_k e^{\lambda_k t} \quad (k = 1, ..., n).$$

Thus,

$$\mathbf{x} = P\mathbf{y} = (\mathbf{v}_1 \ \mathbf{v}_2 \cdots \mathbf{v}_n) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \sum_{k=1}^{n} y_k \mathbf{v}_k = \sum_{k=1}^{n} C_k e^{\lambda_k t} \mathbf{v}_k.$$

This completes the proof.