JOURNAL OF INQUIRY-BASED LEARNING IN MATHEMATICS
No. **43**, (Jan. 2017)

# Introduction to Point-Set Topology

Shay Fuchs

University of Toronto Mississauga

# Contents

# Preface

Topology is a relatively new branch of mathematics. A simple way to describe topology is as a 'rubber sheet geometry'. Topologists study those properties of shapes that remain the same when the shapes are stretched or bended. Over the past century, topology has become a well-defined mathematical discipline, providing a framework and a language that serves almost all branches of mathematics. Topological notions arise naturally in many problems in mathematics and science.

These notes cover the most basic ideas of point-set topology. After a quick review of sets and functions, and the Euclidean space $\mathbb{R}^n$, topological spaces are introduced, along with related notions, such as open sets, bases, closed sets, interior and closure, and more. Subspaces and product spaces are discussed, and several standard and non-standard examples are given. The last few chapters cover continuity, quotient spaces, connectedness and compactness.

The notes are designed to be used with an Inquiry-Based-Learning approach, where students work independently on proving statements and solving exercises, and then present them to the class. The class works as a group to clarify and validate the arguments before moving to the next task. The instructor rarely lectures, and serves as a moderator, guiding the discussion by asking questions and providing comments when necessary.

Through the IBL approach, I believe this course achieves much more than just teaching point-set topology. During the course, many skills are developed: Communicating mathematical ideas, working independently, paying attention to detail, crafting convincing arguments, presenting work to others, and more. These skills, once acquired, serve the learner in other mathematics courses, and beyond. Many IBL practitioners describe the learning experience in such courses as a 'transforming experience', with which I tend to agree.

Shay Fuchs

# To the Student

These notes serve as the primary (and mostly the only) written source for this course - a one-semester course in Point-Set Topology. They contain Definitions, Exercises, and various statements (Propositions, Claims, Theorems) without proof. Your job, as a student, is to read the definitions and attempt to prove the statements and solve the exercises presented here, on your own. Then, you will present some of your work to the whole class. Your presentation will be followed by a class discussion, in which we will try to clarify and validate your arguments.

This is, of course, not an easy task, and might be overwhelming at times, in particular in the first few weeks of classes. However, you are not alone. We, the course staff, are here to help, guide, and often cheer you up throughout this journey. You will see that, as time goes by, you become better in proving theorems, communicating your ideas, and generating your own original solutions. In other words, you will become better in **doing mathematics**, and in **thinking like a mathematician**.

Students who have taken this course in the past found this experience to be extremely rewarding and valuable, and have grown tremendously throughout the course. You will acquire skills that will serve you in the future, both in mathematical and non-mathematical situations. Being able to think, analyze, and come up with your own ideas, as well as communicating them to other people, are core skills in our society, and are essential to being successful and having a fulfilling career.

This method of teaching is called Inquiry Based Learning (IBL). It is a student-centered approach, meaning that it puts you, the student, at the center of attention. There will be almost no lectures in the course, and my role, as the instructor, is to moderate discussions and have a conversation with you, as we go through the course material.

Before we begin, here are a few important remarks.

- **Making mistakes is an important and integral part of the learning process.**

  You are a student, and you are learning new material. It is nothing but natural that mistakes will happen on a regular basis. There is nothing wrong with that. In fact, there is quite a lot one can learn from mistakes. Remember, your presentations are not part of a competition or an oral exam. It is all about seeing your work and then discussing and learning from it.

Yes - there will be a mark assigned to each of your presentations, but as long as you work regularly, and make a genuine effort to attempt the assigned problems, your mark will not suffer from the mistakes you have made.

- **Give yourself enough time to work on a problem.**

  Many of the problems in these notes are not straightforward, and you might need to spend one, two, three or five hours, before realizing what needs to be done. The time you spend on a problem is <u>not</u> a waste of time (even when your first few attempts are unsuccessful). Without noticing, you are actually learning and deepening your understanding as you are attempting a new problem. If you feel completely stuck, come and see the course staff, and we will help you and give you some guidelines and hints. Real problems are not easily solvable - they require time and effort.

- **Refrain from using other resources and from searching solutions online.**

  There is nothing more damaging to your learning process, than obtaining a polished written full solution to a problem you were supposed to work on. By doing that, you are missing the joy of taking part in the creative act (i.e., struggling, being frustrated, developing deep understanding, and having the rewarding feeling when you finally get it). You will gain very little by obtaining a solution to a problem, and if you end up presenting the solution to the class, it will be highly noticeable that it is not your own work (not to mention the dishonest aspect of this act).

  If you are frustrated, stuck, or clueless about a certain proof or an exercise, come to see me, or the teaching assistant, or send me an e-mail. I promise that after you come and talk to me, you will have some idea of what to try next.

And now, after this long introduction, let us get started. I truly believe you will enjoy this course, and have a transformative experience. Good Luck!

# Chapter 1

## Set Theory and Functions

In topology, we study structures called Topological Spaces, which will be defined precisely in Chapter 3. The basic tools we use to study these structures are sets and functions, and so we review them here, and introduce a few new related notions.

**Definition 1.** *The **intersection** of two sets $A$ and $B$ is $A \cap B = \{x\colon x \in A \text{ and } x \in B\}$, and their **union** is $A \cup B = \{x\colon x \in A \text{ or } x \in B\}$. The **difference** between $A$ and $B$ is $A \setminus B = \{x\colon x \in A \text{ and } x \notin B\}$, (also denoted as $A - B$). Finally, if a set $A$ is a subset of a universal set $U$, we define its **complement** as $A^c = \{x\colon x \in U \ \text{ and } \ x \notin A\}$.*

**Definition 2.** *Given a collection of sets $\{A_\alpha\}_{\alpha \in J}$ (where $J$ is some index set), we can define their **intersection** and **union** similarly:*

$$\bigcup_{\alpha \in J} A_\alpha = \{x\colon x \in A_\alpha \text{ for some } \alpha \in J\},$$
$$\bigcap_{\alpha \in J} A_\alpha = \{x\colon \text{ For any } \alpha \in J,\ x \in A_\alpha\}.$$

**Exercise 3.** *For each $n \in \mathbb{N}$, define $A_n = (-n, n) \subseteq \mathbb{R}$.*
*Find $\displaystyle\bigcup_{n \in \mathbb{N}} A_n$ and $\displaystyle\bigcap_{n \in \mathbb{N}} A_n$.*

*(<u>Note</u>: $\displaystyle\bigcup_{n \in \mathbb{N}} A_n$ and $\displaystyle\bigcap_{n \in \mathbb{N}} A_n$ can also be written as $\displaystyle\bigcup_{n=1}^{\infty} A_n$ and $\displaystyle\bigcap_{n=1}^{\infty} A_n$, respectively.*
*Also, in this course, $\mathbb{N} = \{1, 2, 3, \dots\}$, which means that $0$ is NOT a natural number.)*

**Exercise 4.** *For each $n \in \mathbb{N}$, define $A_n = \left[-\frac{2}{n}, \frac{2}{n}\right] \subseteq \mathbb{R}$.*
*Find $\displaystyle\bigcup_{n \in \mathbb{N}} A_n$ and $\displaystyle\bigcap_{n \in \mathbb{N}} A_n$.*

**Exercise 5.** *Let $P = (1, \infty) \subseteq \mathbb{R}$. Find $\displaystyle\bigcup_{r \in P} \left(\frac{1}{r}, 3r\right)$ and $\displaystyle\bigcap_{r \in P} \left(\frac{1}{r}, 3r\right)$.*
*(<u>Note</u>: $\left(\frac{1}{r}, 3r\right)$ denotes an open interval.)*

**Exercise 6.** *Let $P = (2,\infty) \subseteq \mathbb{R}$, and define $B_\alpha = \left(-\frac{1}{\alpha} - 1, \alpha\right) \subseteq \mathbb{R}$ for each $\alpha \in P$. Find $\bigcup_{\alpha \in P} B_\alpha$ and $\bigcap_{\alpha \in P} B_\alpha$.*

**Proposition 7.** *Let $X$ be a set, and $\{A_\alpha\}_{\alpha \in J}$ a collection of sets. Then:*

$$X \setminus \left(\bigcap_{\alpha \in J} A_\alpha\right) = \bigcup_{\alpha \in J} (X \setminus A_\alpha),$$
$$X \setminus \left(\bigcup_{\alpha \in J} A_\alpha\right) = \bigcap_{\alpha \in J} (X \setminus A_\alpha).$$

**Exercise 8.** *Derive similar formulas for complements of unions and intersections.*

**Definition 9.** *Let $f\colon A \to B$ be a function.*

*(a) The **image** of a set $C \subseteq A$ is the set $f(C) = \{f(x)\colon x \in C\} \subseteq B$. The set $f(A)$ is called the **image of** $f$.*

*(b) The **pre-image** of a set $D \subseteq B$ is the set $f^{-1}(D) = \{x \in A\colon f(x) \in D\} \subseteq A$.*

**Exercise 10.** *Draw diagrams for the definition of the image and the pre-image.*

**Exercise 11.** *Consider the function $f\colon \mathbb{R} \to \mathbb{R} \quad , \quad f(x) = 2 - x^2$.*

*(a) Find $f([1,3])$, $f((-1,1))$ and $f([-1,1])$.*

*(b) Find $f([0,\infty))$ and $f(\mathbb{R})$.*

*(c) Find $f^{-1}([-3,-2])$ and $f^{-1}(\{0,1,2,3\})$.*

*(d) Find $f^{-1}(\mathbb{N})$ and $f^{-1}((-\infty,0])$.*

*(e) What is the image of $f$ ?*

**Definition 12.** *Let $f\colon A \to B$ be a function. Then $f$ is **injective** (or **an injection**, or **one-to-one**) if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ (or, equivalently, if $f(x_1) = f(x_2)$ implies $x_1 = x_2$ for all $x_1, x_2 \in A$). The function $f$ is **surjective** (or **a surjection**, or **onto**) if $f(A) = B$.*

**Exercise 13.** *Let $f\colon A \to B$ be a function, and $C \subseteq A$ a set.*

*(a) Prove that $C \subseteq f^{-1}(f(C))$.*

*(b) Find an example in which $C \neq f^{-1}(f(C))$.*

*(c) Prove that if $f$ is injective, then $C = f^{-1}(f(C))$.*

**Exercise 14.** *Let $f\colon A \to B$ be a function, and $D \subseteq B$ a set.*

*(a)* *Prove that* $f(f^{-1}(D)) \subseteq D$.

*(b)* *Find an example in which* $f(f^{-1}(D)) \neq D$.

*(c)* *Prove that if* $f$ *is surjective, then* $f(f^{-1}(D)) = D$.

**Proposition 15.** *Let* $f\colon A \to B$ *be a function,* $C_1, C_2 \subseteq A$ *and* $D_1, D_2 \subseteq B$. *Then:*

*(a)* $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$.

*(b)* $f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$. *Equality holds if* $f$ *is injective.*

*(c)* $f^{-1}(D_1 \cup D_2) = f^{-1}(D_1) \cup f^{-1}(D_2)$.

*(d)* $f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2)$.

**Proposition 16.** *Let* $f\colon A \to B$ *be a function,* $\{C_\alpha\}_{\alpha \in J}$ *a collection of subsets of A, and* $\{D_\beta\}_{\beta \in I}$ *a collection of subsets of B. Then:*

*(a)* $f\left(\bigcup_{\alpha \in J} C_\alpha\right) = \bigcup_{\alpha \in J} f(C_\alpha)$.

*(b)* $f\left(\bigcap_{\alpha \in J} C_\alpha\right) \subseteq \bigcap_{\alpha \in J} f(C_\alpha)$. *Equality holds if* $f$ *is injective.*

*(c)* $f^{-1}\left(\bigcup_{\beta \in I} D_\beta\right) = \bigcup_{\beta \in I} f^{-1}(D_\beta)$.

*(d)* $f^{-1}\left(\bigcap_{\beta \in I} D_\beta\right) = \bigcap_{\beta \in I} f^{-1}(D_\beta)$.

**Exercise 17.** *Let* $g\colon [-2,2] \to \mathbb{R}$, $g(x) = \sqrt{4-x^2}$, *and let* $A_\alpha = [\alpha - 2, \alpha + 1]$ *(for* $\alpha \in \mathbb{R}$*). Find* $g\left(\bigcap_{0 \le \alpha \le 1} A_\alpha\right)$ *and* $\bigcap_{0 \le \alpha \le 1} g(A_\alpha)$.

**Proposition 18.** *Let* $f\colon A \to B$ *and* $g\colon B \to C$ *be two functions, and* $E \subseteq C$ *a set. Then* $(g \circ f)^{-1}(E) = f^{-1}(g^{-1}(E))$.

# Chapter 2

## A Quick Look at $\mathbb{R}^n$

You have already encountered the Euclidean spaces $\mathbb{R}$ (the real number line), $\mathbb{R}^2$ (the plane) and $\mathbb{R}^3$ (the three-dimensional space). More generally, we can talk about the Euclidean space $\mathbb{R}^n$ (for $n \in \mathbb{N}$). We now take a quick look at these spaces, and their topological (or geometrical) properties. Many abstract notions in topology can be motivated through a concrete observation or phenomena in $\mathbb{R}^n$ .

**Definition 19.** *Let $\underline{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. The number $||\underline{x}|| = \sqrt{x_1^2 + \cdots + x_n^2}$ is called the **norm** (or the **length**) of $\underline{x}$. If $\underline{x} = (x_1, \ldots, x_n)$ and $\underline{y} = (y_1, \ldots, y_n)$ are two points in $\mathbb{R}^n$, then **the (Euclidean) distance** between $\underline{x}$ and $\underline{y}$ is $d(x, y) = ||\underline{x} - \underline{y}|| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$.*

**Definition 20.** *Given $\underline{x} \in \mathbb{R}^n$ and $r > 0$, the **open ball at $\underline{x}$ with radius** $r$ is the set $B_r(\underline{x}) = \{\underline{y} \in \mathbb{R}^n : d(\underline{x}, \underline{y}) < r\}$. A set $U \subseteq \mathbb{R}^n$ is **open** (in $\mathbb{R}^n$) if each $\underline{x} \in U$ is the center of some open ball in $\mathbb{R}^n$ contained in $U$.*

*Note: In $\mathbb{R}^2$, we often use the term* disks *rather than* balls.

**Exercise 21.** *(a) Show that the open interval $(-1, 1)$ is open in $\mathbb{R}$.*

*(b) Show that the closed interval $[-1, 1]$ is **not** open in $\mathbb{R}$.*

*(c) Show that $\mathbb{R} \setminus \mathbb{N}$ is open in $\mathbb{R}$.*

*(d) Is the set of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ open in $\mathbb{R}$ ?*

**Exercise 22.** *(a) Show that the open unit disk $D = \{(x, y) : x^2 + y^2 < 1\}$ is an open set in $\mathbb{R}^2$.*

*(b) Show that $\mathbb{R}^2$ is open in $\mathbb{R}^2$.*

*(c) Is $\phi$ (the empty set) open in $\mathbb{R}^2$ ? Explain.*

**Proposition 23.** *(a) A point $\{\underline{x}\} \subseteq \mathbb{R}^n$ is **not** an open set in $\mathbb{R}^n$.*

*(b) Any open ball $B_r(\underline{x})$ in $\mathbb{R}^n$ is an open set in $\mathbb{R}^n$ (accompany your proof with a diagram).*

**Theorem 24.** *(a) $\mathbb{R}^n$ and $\phi$ are open sets in $\mathbb{R}^n$.*

*(b) If $\{U_\alpha\}_{\alpha\in J}$ is a collection of open sets in $\mathbb{R}^n$, then their union $\bigcup_{\alpha\in J} U_\alpha$ is also open in $\mathbb{R}^n$.*

*(c) If $V_1,\ldots,V_k$ are open sets in $\mathbb{R}^n$, then their intersection*

$$\bigcap_{i=1}^{k} V_i = V_1\cap\ldots\cap V_k$$

*is also open in $\mathbb{R}^n$.*

**Exercise 25.** *Find an example of an infinite collection of open sets in $\mathbb{R}^2$, whose intersection is not open.*

**Theorem 26.** *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a function. Then $f$ is continuous iff the pre-image of any open set in $\mathbb{R}^m$ is open in $\mathbb{R}^n$ (i.e., if $f^{-1}(U)$ is open in $\mathbb{R}^n$ whenever $U$ is open in $\mathbb{R}^m$).*

**<u>Note</u>:** The precise definition of continuity is needed for the proof. Do not attempt to prove this theorem if you haven't seen $\varepsilon-\delta$ definitions of limits and continuity.

**Exercise 27.** *Draw a couple of diagrams to illustrate Theorem 26 .*

**Exercise 28.** *(a) Use Theorem 26 to prove that $f\colon \mathbb{R}\to\mathbb{R}$ , $f(x) = 3x-1$ is a continuous function.*

*(b) Use Theorem 26 to prove that $g\colon \mathbb{R}\to\mathbb{R}$,*

$$g(x) = \begin{cases} 3x & \textit{for } x \geq 1 \\ 2x & \textit{for } x < 1 \end{cases}$$

*is **not** a continuous function.*

**Conclusion 29.** *Knowing the collection of open sets in $\mathbb{R}^n$ and $\mathbb{R}^m$ is enough in order to determine whether a function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is continuous.*

# Chapter 3

## Topological Spaces

In this Chapter we finally define what is a topological space. Before proceeding, take a quick look again at Theorem 24. This theorem outlines a few basic properties of open sets in $\mathbb{R}^n$. To prove that theorem, we had to use the definition of open sets, which depended on our ability to measure distance between points in $\mathbb{R}^n$. In order to generalize the notion of open sets to other spaces, in which distances are not necessarily defined, we incorporate Theorem 24 into our definition of a topology, as follows:

**Definition 30.** *Let $X$ be a set. A **topology** on $X$ is a collection $\mathscr{T}$ of subsets of $X$, having the following three properties:*

*(a) $\phi \in \mathscr{T}$ and $X \in \mathscr{T}$.*

*(b) If $\{U_\alpha\}_{\alpha \in J}$ is a collection of sets in $\mathscr{T}$, then $\bigcup_{\alpha \in J} U_\alpha \in \mathscr{T}$ (closure under arbitrary unions).*

*(c) If $V_1, V_2, \ldots, V_k \in \mathscr{T}$, then $V_1 \cap V_2 \cap \cdots \cap V_k = \bigcap_{i=1}^{k} V_i \in \mathscr{T}$ (closure under finite intersections).*

*A set $X$ endowed with a topology $\mathscr{T}$ is called a **topological space**. Sets in $\mathscr{T}$ are called **open sets**.*

**Exercise 31.** *Let $X = \{x, y, z\}$.*

*(a) Is $\mathscr{T} = \{\{x,y\}, \{x\}, \{x,z\}, \phi, \{x,y,z\}\}$ a topology on $X$?*

*(b) Is $\mathscr{T} = \{\{x\}, \{y\}, \{z\}, \phi, \{x,y,z\}\}$ a topology on $X$?*

*Add diagrams to your solutions.*

**Definition 32.** *Let $X$ be a set.*

- *$\mathscr{T} = \{\phi, X\}$ is called the **indiscrete topology** on $X$.*

- $\mathscr{T} = P(X) = \{$*all subsets of* $X\}$ *is called the* ***discrete*** *topology on* $X$.

**Claim 33.** *The collection* $\mathscr{T} = \{U : U$ *is an open set in* $\mathbb{R}^n\}$ *is a topology on* $\mathbb{R}^n$ *(called* ***the standard topology*** *on* $\mathbb{R}^n$*).*

**Exercise 34.** *In each case, prove that the collection* $\mathscr{T}$ *defines a topology on* $X$ *.*

*(a)* $X = \mathbb{R}^2$ ; $\mathscr{T} = \{$*All open disks centered at the origin*$\} \cup \{\phi\} \cup \{X\}$ .

*(b)* $X$ *is any infinite set* ; $\mathscr{T} = \{U \subseteq X \ : U^c$ *is finite*$\} \cup \{\phi\}$ .
*(Note: $U^c$ is the complement of $U$ in $X$.)*

The collection of open sets in a topological space can be very large, and difficult to work with. In certain cases, we may identify a sub-collection of open sets which, in some sense, generates all the open sets in our space. The following definition of a basis formalizes this idea.
(As a comparison, think of what you have learned about bases in linear algebra. A basis for a vector space is a subset of vectors that "generates" the whole space...)

**Definition 35.** *Let* $(X, \mathscr{T})$ *be a topological space. A collection* $\mathscr{B} \subseteq \mathscr{T}$ *of open sets is called a* ***basis*** *for the topology* $\mathscr{T}$ *if every (nonempty) open set is a union of sets in* $\mathscr{B}$.

**Exercise 36.** *Verify the following two statements:*

- *The collection of open discs* $\mathscr{B} = \{B_r(\underline{x}) : \underline{x} \in \mathbb{R}^2\ ,\ r > 0\}$ *is a basis for the standard topology on* $\mathbb{R}^2$.
- *The collection* $\mathscr{B} = \{\{x\} : x \in \mathbb{R}\}$ *is a basis for the discrete topology on* $\mathbb{R}$.

**Question:** Given a set $X$, and a collection $\mathscr{B}$ of subsets of $X$. Under what conditions can $\mathscr{B}$ serve as a basis for some topology on $X$? The next Theorem gives an answer to this question.

**Theorem 37.** *Let $X$ be a set, and $\mathscr{B}$ a collection of subsets of $X$, satisfying the following two conditions:*

*(a) For each $x \in X$, there is a $B \in \mathscr{B}$ containing $x$ (i.e., the collection $\mathscr{B}$ 'covers' $X$).*

*(b) If $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \cap B_2$, then there is a set $B_3 \in \mathscr{B}$ for which $x \in B_3 \subseteq B_1 \cap B_2$.*

*Then the collection*

$$\mathscr{T} = \{U \subseteq X : U \text{ is a union of sets in } \mathscr{B}\} \cup \{\phi\}$$

*is a topology on $X$, and $\mathscr{B}$ is a basis for this topology.*
*(Remark: $\mathscr{T}$ is called the topology* ***generated*** *by the basis $\mathscr{B}$.)*

**Exercise 38.** *For each* $n \in \mathbb{Z}$ *, define the set* $B(n)$ *as follows:*

$$B(n) = \begin{cases} \{n\} & \text{if } n \text{ is odd,} \\ \{n-1\,, n\,, n+1\} & \text{if } n \text{ is even.} \end{cases}$$

*Prove that the collection* $\mathscr{B} = \{B(n)\ : n \in \mathbb{Z}\}$ *is a basis for a topology on* $\mathbb{Z}$ *(called* ***the digital line topology****). Draw a diagram showing* $\mathbb{Z}$ *and the basic open sets in this topology.*

**Exercise 39.** *Prove that the collection of all open rectangles in* $\mathbb{R}^2$ *with sides parallel to the axes is a basis for a topology on* $\mathbb{R}^2$*. (What is this topology?)*

**Exercise 40.** *Let* $\mathscr{B} = \{(a,b] \subseteq \mathbb{R} \colon a < b\}$.

*(a) Show that* $\mathscr{B}$ *is a basis for a topology* $\mathscr{T}$ *on* $\mathbb{R}$ *(called* ***the upper-limit topology*** *on* $\mathbb{R}$*).*

*(b) Show that any open interval* $(a,b) \subseteq \mathbb{R}$ *is an open set in the upper-limit topology.*

**Exercise 41.** *Explain why the collection of all closed intervals,*

$$\{\,[a\,,b]\ :\ a\,,b \in \mathbb{R} \quad \text{and} \quad a < b\,\}\ ,$$

***cannot*** *serve as the basis for a topology on the real line.*

**Exercise 42.** *Show that the collection* $\mathscr{C} = \{(a\,,b]\ :\ a < b \text{ and } a,b \in \mathbb{Q}\}$ *is a basis for a topology on the real line, which is* ***different*** *from the upper-limit topology (defined in Exercise 40).*

# Chapter 4

## The Subspace Topology

If $A$ is a subspace of a topological space $X$, is it possible to turn $A$ into a topological space in a natural way? In other words, is there a way to construct a topology for $A$, from the given topology of the larger space $X$ ?

These questions often arise in mathematics when new structures are introduced, and motivate the definition of sub-structures (such as vector subspaces, subgroups, subfields, etc.). The following claim shows how the topology, of a topological space $X$, induces a topology on each of its subsets.

**Claim 43.** *If $(X, \mathscr{T})$ is a topological space, and $A \subseteq X$ is a subset, then the collection $\mathscr{T}' = \{U \cap A \colon U \in \mathscr{T}\}$ is a topology on $A$ (called the **subspace topology** on $A$).*

**Exercise 44.** *Consider the topological space $X = \mathbb{R}$ (with the standard topology), and the subspace $A = [0,1]$. For each set, decide whether it is open in $\mathbb{R}$, in $A$, in both, or in none of the spaces. Justify your answers briefly.*

$$\left[0, \tfrac{1}{3}\right) \quad , \quad (0.6,1] \quad , \quad (0.1,0.4) \quad , \quad \{1\} \quad , \quad [0.7,1] \quad , \quad [0,0.1) \cup (0.9,1] \quad .$$

**Exercise 45.** *Let $A = \{0\} \cup [1,2]$ with the subspace topology induced from $\mathbb{R}$ (with the standard topology).*

*(a) Show that $\{0\}$ is open in $A$, while $\{2\}$ is not open in $A$.*

*(b) Show that $[1,2]$ is open in $A$.*

**Exercise 46.** *Let $E = \{0, \pm 2, \pm 4, \pm 6, \dots\}$ be the set of even numbers. Show that the subspace topology on $E$ (induced from $\mathbb{R}$) is the discrete topology on $E$.*

**Exercise 47.** *Consider the following subspace of $\mathbb{R}$: $Y = \left\{0, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\right\}$.*

*Is the set $\{0\}$ open in $Y$ ? Is $\{1\}$ open in $Y$ ?*

**Exercise 48.** *Let $X = \mathbb{R}^2$ with the standard topology, and consider the subspace*

$$A = \{(x,y)\colon x \geq 0 \text{ and } y \geq 0\}\ .$$

*Show that $U = \{(r\cos\theta, r\sin\theta)\colon 0 \leq r < 1 \text{ and } 0 \leq \theta \leq \frac{\pi}{2}\}$ is open in $A$, but not in $X$.*

**Claim 49.** *Let $Y$ be an open set in a topological space $X$. Prove that if $A$ is open in $Y$ (with respect to the subspace topology), then it is also open in $X$.*

**Proposition 50.** *If $X$ is a topological space with a basis $\mathscr{B}$, and $A \subseteq X$ is a subset, then $\mathscr{B}' = \{B \cap A\colon B \in \mathscr{B}\}$ is a basis for the subspace topology on $A$.*

# Chapter 5

## The Product Topology on $X \times Y$

In this Chapter we discuss yet another way of constructing a new topology from given topologies. If $X$ and $Y$ are two topological spaces, can we define a topology on $X \times Y$ in a natural way? Here is a review of the definition of a Cartesian product.

**Definition 51.** *The* ***(Cartesian) product*** *of two sets A and B, is the set*

$$A \times B = \{(a,b) : a \in A \text{ and } b \in B\} .$$

If $X$ and $Y$ are topological spaces, it is tempting to try and define a topology on $X \times Y$ by taking the open sets to be sets of the form $U \times V$, where $U$ is open in $X$ and $V$ is open in $Y$. However, this approach does not work, as the collection of such sets will not satisfy Definition 30. Another problem that arises is that many open sets in $\mathbb{R}^2$ are not products of open sets in $\mathbb{R}$ (see Part (c) of the following exercise).

**Exercise 52.** *(a) Express the set $Y = \{(x,y) \in \mathbb{R}^2 : \ x^2 \text{ is an integer}\}$ as a product of two sets in $\mathbb{R}$.*

*(b) Express the set $Z = \{(x,y) \in \mathbb{R}^2 : |x| \leq 3 \text{ and } 4y^2 = 1\}$ as a product of two sets in $\mathbb{R}$.*

*(c) Find an* ***open set*** *in $\mathbb{R}^2$ which is not the product of two open sets in $\mathbb{R}$.*

*Even though products of open sets in $X$ and $Y$ may not give a topology on $X \times Y$, they do satisfy the requirements of Theorem 37 for being a basis. This is the content of the following claim. Once the claim is proved, we can define the product topology as the topology generated by this basis.*

**Claim 53.** *Let $X$ and $Y$ be two topological spaces. Then*

$$\mathscr{B} = \{U \times V : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$$

*is a basis for a topology on $X \times Y$ (called the* ***product topology*** *on $X \times Y$).*

**Theorem 54.** *Let $X$ and $Y$ be two topological spaces. If $\mathscr{B}$ is a basis for the topology on $X$, and $\mathscr{C}$ is a basis for the topology on $Y$, then $\mathscr{D} = \{B \times C \colon B \in \mathscr{B} \text{ and } C \in \mathscr{C}\}$ is a basis for the product topology on $X \times Y$.*

**Exercise 55.** *Prove that the product topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is equal to the standard topology on $\mathbb{R}^2$ (defined in Claim 33).*

**Exercise 56.** *Let $X$ and $Y$ be two topological spaces, and $\pi \colon X \times Y \to Y$ the function given by $\pi(x,y) = y$ ($\pi$ is called the **projection** onto the second factor). Prove that if $O$ is open in $X \times Y$, then $\pi(O)$ is open in $Y$ (we say that $\pi$ is an **open map**).*

# Chapter 6

## Closed Sets

We now define closed sets in a topological space as complements of open sets. Informally, we often think of closed sets as sets "that contain their boundary". If you have taken a course in real analysis, you might have seen a definition that formalizes this idea (in the context of metric spaces).
**Recall:** Unless mentioned otherwise, we assume that $\mathbb{R}^n$ is equipped with the standard topology.

**Definition 57.** *A set $A$ in a topological space $X$ is* ***closed*** *if its complement, $A^c = X \setminus A$, is open.*

**Exercise 58.** *(a) Show that a closed interval $[a,b]$ (with $a < b$) is a closed set in $\mathbb{R}$.*

*(b) If $x$ is a real number, show that $\{x\}$ is a closed set in $\mathbb{R}$.*

*(c) Prove that the interval $[-1,1)$ is not an open nor a closed set in $\mathbb{R}$.*

**Exercise 59.** *Let $X = (-1,1) \cup [3,4]$, with the subspace topology (induced from $\mathbb{R}$).*
*Prove that $A = (-1,1)$ is both an open and a closed set in $X$ (we call such sets* ***clopen*** *sets).*

**Exercise 60.** *Prove that $A = \{(x,y) : x \leq 0 \text{ and } y \leq 0\}$ is a closed set in $\mathbb{R}^2$.*

**Exercise 61.** *Consider the real line with the upper-limit topology (defined in Exercise 40).*
*Prove that in this topology, an interval of the form $(a,b]$ is a closed set.*

**Exercise 62.** *If $X$ is any set, equipped with the discrete topology (see Definition 32), what are the closed sets in $X$ ?*

**Theorem 63.** *Let $X$ be a topological space. Then:*

*(a) $\phi$ and $X$ are closed sets in $X$.*

*(b) If $\{C_\alpha\}_{\alpha \in J}$ is a collection of closed sets in $X$, then $\bigcap_{\alpha \in J} C_\alpha$ is also a closed set in $X$.*

*(c) If $C_1, C_2, \ldots, C_k$ are closed sets in $X$, then $\bigcup_{i=1}^{k} C_i = C_1 \cup \cdots \cup C_k$ is also a closed set in $X$.*

**Exercise 64.** *If $\{C_\alpha\}_{\alpha \in J}$ is a collection of closed sets in $\mathbb{R}$, must $\bigcup_{\alpha \in J} C_\alpha$ be also a closed set in $\mathbb{R}$? Give a proof or a counterexample.*

**Proposition 65.** *Let $Y$ be a subspace of a topological space $X$. Then a set $A \subseteq Y$ is closed in $Y$ iff $A = C \cap Y$ for some set $C$ closed in $X$.*

**Exercise 66.** *Prove that if $Y$ is a closed subspace of $X$, and $A \subseteq Y$ is closed in $Y$, then $A$ is also closed in $X$.*

**Proposition 67.** *Let $X, Y$ be two topological spaces. If $A$ is a closed set in $X$, and $B$ is a closed set in $Y$, then $A \times B$ is a closed set in $X \times Y$.*

# Chapter 7

## Closure and Interior

**Definition 68.** *Let $A$ be a set in a topological space $X$.*

(a) *The **interior** of $A$, denoted $int(A)$, is the union of all open sets (in $X$) contained in $A$.*

(b) *The **closure** of $A$, denoted $\overline{A}$, is the intersection of all closed sets (in $X$) containing $A$.*

We often think of the closure of $A$ as the smallest closed set (in $X$) containing $A$. Similarly, the interior of $A$ can be thought of as the largest open set contained in $A$.

**Exercise 69.** *In $\mathbb{R}$, what are the interior and the closure of a half-open/half-closed interval $(a,b]$ (with $a<b$)? Explain.*

**Exercise 70.** *In $\mathbb{R}^2$, what are the interior and the closure of $A=(-1,1)\times[-1,1]$? Explain.*

**Claim 71.** *Let $X$ be a topological space, and $A\subseteq X$ a subset. Then:*

(a) *$int(A)$ is open in $X$, and $\overline{A}$ is closed in $X$.*

(b) *$int(A)\subseteq A\subseteq\overline{A}$*

(c) *If $A$ is open, then $int(A)=A$, and if $A$ is closed, then $\overline{A}=A$.*

Working with the definition of the closure of a set can be difficult, as one needs to consider the collection of all closed set containing a given set $A$, and compute their intersection (did you feel this difficulty when working on Exercise 70 ?). The following theorem provides an alternate (and useful) tool for computing closures of sets in a topological space.

**Theorem 72.** *Let $X$ be a topological space, and $A\subseteq X$ a subset. Then*

(a) *$x\in\overline{A}$ iff for every open set $U\subseteq X$ containing $x$, we have $U\cap A\neq\phi$.*

(b) *If $\mathscr{B}$ is a basis for the topology on $X$, then $x \in \overline{A}$ iff for every $B \in \mathscr{B}$ containing $x$, we have $B \cap A \neq \phi$.*

**Exercise 73.** *In $\mathbb{R}$, what are the interior and the closure of $\mathbb{Q}$ (the rational numbers)? Explain.*

**Exercise 74.** *Let $D = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\right\}$. Find, with proof, the closure of $D$ in $\mathbb{R}$.*

**Exercise 75.** *Let $A = (-1,0)$. What is the closure of $A$ in $(-1,1]$ ? Explain.*

**Exercise 76.** *Consider the real line $\mathbb{R}$ (with the standard topology), and the subset $A = \left\{(-1)^{n+1} - \dfrac{1}{n} \;:\; n \in \mathbb{N}\right\}$. Find, with proof, the closure $\overline{A}$.*

**Exercise 77.** *Let $X$ be a topological space. For each statement, decide whether it is TRUE or FALSE. Justify your arguments.*

(a) *For any $A,B \subseteq X$, we have $\overline{A \cup B} = \overline{A} \cup \overline{B}$.*

(b) *For any $A,B \subseteq X$, we have $\overline{A \cap B} = \overline{A} \cap \overline{B}$.*

**Exercise 78.** *Let $\{A_\alpha\}_{\alpha \in J}$ be a collection of subsets of a topological space $X$.*
*Prove that $\displaystyle\bigcup_{\alpha \in J} \overline{A_\alpha} \subseteq \overline{\bigcup_{\alpha \in J} A_\alpha}$, and give an example where equality fails.*

**Exercise 79.** *Let $X,Y$ be two topological spaces, $A \subseteq X$ and $B \subseteq Y$.*
*Prove that in $X \times Y$, we have $\overline{A \times B} = \overline{A} \times \overline{B}$.*

**Exercise 80.** (a) *Show that if $A$ and $B$ are sets in a topological space, and $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$ .*

(b) *Show that if $A$ is a set in a topological space $X$, then $int(A^c) = (\overline{A})^c$ .*

**Exercise 81.** *Let $\mathbb{R}_f$ be the set of real numbers, together with the topology*

$$\mathscr{T} = \{U \subseteq \mathbb{R} : \; U^c \text{ is finite}\} \cup \{\phi\} \; .$$

*Prove that if $U$ is a nonempty open set in $\mathbb{R}_f$, then $\overline{U}$ is equal to the whole space.*

# Chapter 8

## Continuous Functions

In calculus, continuity was defined using limits (remember? A function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous at $x = a$ if $\lim_{x\to a} f(x) = f(a)$ ). However, the notion of a limit does not generalize easily to topological spaces. There is, however, an alternative. Theorem 26 gives us an equivalent characterization of continuity, that uses open sets (and avoids limits). This observation motivates the following definition.

**Definition 82.** *Let $X$ and $Y$ be two topological spaces. A function $f\colon X \to Y$ is **continuous** if for each open set $V$ in $Y$, $f^{-1}(V)$ is an open set in $X$.*

**Claim 83.** *Let $X$ and $Y$ be two topological spaces, $\mathscr{B}$ a basis for the topology on $Y$, and $f\colon X \to Y$ a function. Then $f$ is continuous if and only if $f^{-1}(B)$ is an open set in $X$ for every $B \in \mathscr{B}$.*

**Exercise 84.** *Use Claim 83 to prove that $f\colon \mathbb{R} \to \mathbb{R}$ , $f(x) = -3x+2$ is a continuous function.*

**Exercise 85.** *In this exercise $\mathbb{R}$ denotes the set of real numbers with the **standard** topology, while $\mathbb{R}_u$ denotes the set of real numbers with the **upper-limit** topology (defined in Exercise 40).*

*(a) Is the function $f\colon \mathbb{R}_u \to \mathbb{R}$ , $f(x) = x$ continuous? Explain.*

*(b) Is the function $f\colon \mathbb{R} \to \mathbb{R}_u$ , $f(x) = x$ continuous? Explain.*

*(c) Is the function $f\colon \mathbb{R}_u \to \mathbb{R}$ , $f(x) = \begin{cases} 1 & \text{for } x > 0 \\ -1 & \text{for } x \leq 0 \end{cases}$ continuous? Explain.*

**Exercise 86.** *Let $X = \mathbb{R}^2$ (as a set) , together with the topology*

$$\mathscr{T} = \{\text{All open disks centered at the origin}\} \cup \{\phi\} \cup \{X\}$$

*(you DO NOT need to prove that this is a topology).*

(a) *Is the function* $f\colon X \to X$, $f(x,y) = (2x,3y)$ *continuous? Explain.*

(b) *Is the function* $g\colon X \to X$, $g(x,y) = (-x,-y)$ *continuous? Explain.*

**Proposition 87.** *Let* $X,Y$ *be two topological spaces. Prove that the map* $\pi_1\colon X \times Y \to X$, $\pi_1(x,y) = x$ *is continuous. (Similarly,* $\pi_2\colon X \times Y \to Y$, $\pi_2(x,y) = y$ *is continuous.)*

**Proposition 88.** *Let* $X$ *and* $Y$ *be two topological spaces, and* $f\colon X \to Y$ *a function. Then the following are equivalent:*

(a) $f$ *is a continuous function.*

(b) *For every closed set* $C$ *in* $Y$, $f^{-1}(C)$ *is a closed set in* $X$.

(c) *For any* $p \in X$ *, the following condition holds:*
*For each open set* $V$ *in* $Y$ *containing* $f(p)$*, there is an open set* $U$ *in* $X$ *containing* $p$*, such that* $f(U) \subseteq V$*. (This condition is called* ***continuity at*** **p***.)*

**Exercise 89.** *Consider the function* $g\colon \mathbb{R} \to \mathbb{R}$,

$$g(x) = \begin{cases} x-3 & \textit{for } x \in \mathbb{Q}, \\ 3-x & \textit{for } x \notin \mathbb{Q}. \end{cases}$$

(a) *Prove that* $g$ *is* ***not*** *a continuous function.*

(b) *Prove that* $g$ *is continuous at* $x = 3$.

**Theorem 90.** *Let* $X,Y$ *and* $Z$ *be three topological spaces.*

(a) *If* $f\colon X \to Y$ *and* $g\colon Y \to Z$ *are continuous functions, then* $g \circ f\colon X \to Z$ *is also continuous.*

(b) *If* $h\colon Z \to X \times Y$, $h(z) = (h_1(z), h_2(z))$ *, then* $h$ *is continuous if and only if* $h_1\colon Z \to X$ *and* $h_2\colon Z \to Y$ *are continuous.*

**Definition 91.** *Let* $X$ *and* $Y$ *be two topological spaces. A function* $f\colon X \to Y$ *is called a* ***homeomorphism*** *if* $f$ *is bijective, continuous, and its inverse* $f^{-1}\colon Y \to X$ *is also continuous.*

The notion of a homeomorphism is a very important one. If two topological spaces $X$ and $Y$ are homeomorphic (i.e., there is a homeomorphism between them), then, from a topological point of view, $X$ and $Y$ are essentially the same space (possibly described in different ways). One of the fundamental problems in topology is to determine whether two spaces are homeomorphic or not.

**Exercise 92.** *Show that the function* $f\colon (-1,1) \to \mathbb{R}$, $f(x) = \dfrac{2x}{1-x^2}$ *is a homeomorphism, and find its inverse. (Note: You can use results from calculus.)*

**Exercise 93.**
*Let $S^1 = \{(x,y) \colon x^2 + y^2 = 1\}$ be the unit circle (with the subspace topology induced from $\mathbb{R}^2$).*
*Prove that $G \colon [0,1) \to S^1$ , $G(t) = (\cos(2\pi t), \sin(2\pi t))$ is **not** a homeomorphism.*

**Exercise 94.** *Denote by $\mathbb{R}$ the real line with the **standard** topology, and by $\mathbb{R}_d$ the real line with the **discrete** topology. Is the function $f \colon \mathbb{R}_d \to \mathbb{R}$, $f(x) = x$ a homeomorphism?*

# Chapter 9

## The Quotient Topology

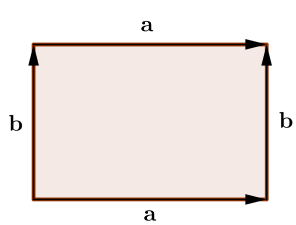
Many geometrical objects can be constructed through "gluing" or "cut-and-paste" techniques. We often use diagrams to describe a space that is formed by gluing parts of it to another. When two points or edges are labeled with the same letter, it means that they need to be glued to each other (and arrows on edges must align when glued).

**Exercise 95.** *Describe and/or draw the object obtained from each of the 'gluing' diagrams.*
*(Note: The goal here is to get you to think informally and develop intuition for the "gluing" procedure. In this exercise, you are not required to provide rigorous proofs.)*

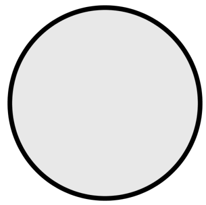
*(a)*

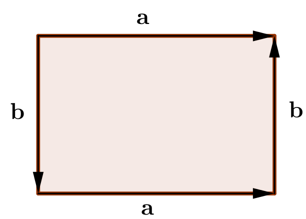

*(glue the two endpoints.)*

*(b)*




*(c)*

*(d)*  *(collapse the boundary into a single point.)*

*(e)* 


To formalize this procedure, we use the notion of equivalence relations (a notion that you have probably encountered before). Let us start by recalling its definition.

**Definition 96.** *Given a set $A$, an **equivalence relation** on $A$ is a subset $R \subseteq A \times A$ such that:*

*(a) For any $a \in A$, $(a,a) \in R$ (reflexivity).*

*(b) For any $a,b \in A$, if $(a,b) \in R$ then $(b,a) \in R$ (symmetry).*

*(c) For any $a,b,c \in A$, if $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$ (transitivity).*

*If $(x,y) \in R$ we often write $x \sim y$ or $x \approx y$.*

**Exercise 97.** *(a) Is the order relation '$>$' an equivalence relation on $\mathbb{R}$ ?*

*(b) Is the relation '$a \sim b$ if and only if $a-b$ is divisible by 3' an equivalence relation on $\mathbb{Z}$ ?*

*(c) Is the relation '$a \sim b$ if and only if $\frac{a}{b} \in \mathbb{Q}$' an equivalence relation on $\mathbb{R} \setminus \{0\}$ ?*

**Definition 98.** *If $\sim$ is an equivalence relation on $A$, then the **equivalence class** of $x \in A$ is the set of all the elements in $A$ which are equivalent to $x$: $\{y \in A \ : \ y \sim x\}$. We denote the equivalence class of $x$ by $[x]$. The set of all equivalence classes will be denoted as $A/\sim$ (and called **the quotient space**). The map $p\colon A \to A/\sim$ , $p(x) = [x]$ is called the **quotient map**.*

**Exercise 99.** *Denote by $\mathbb{R}_+$ the set of all **positive** real numbers, and let*

$$R = \{(x,y) \in \mathbb{R}^2_+ \ : \ \ln x \cdot \ln y > 0 \ \ or \ \ x = y = 1\} \ .$$

*(a) Prove that $R$ is an equivalence relation on $\mathbb{R}_+$ .*

*(b) Describe the equivalence classes of this relation.*

**Theorem 100.** *Let $R$ be an equivalence relation on a set $A$, and $x, y \in A$. Then either $[x] = [y]$ or $[x] \cap [y] = \phi$.*

**Conclusion 101.** *If $R$ is an equivalence relation on $A$, then the equivalence classes induce a partition on $A$.*

We are now ready to present the definition of quotient topology. Given a topological $X$, equipped with an equivalence relation, we form the "glued" space $Y$ by identifying points which are equivalent to each other. The quotient space $X/\sim$ will be thought of as the new space (obtained from the gluing procedure). However, if we want $X/\sim$ to be a topological space (and not just a set), we must also define a topology on it. This is done in the following claim.

**Claim 102.** *Let $X$ be a topological space, $\sim$ an equivalence relation on $X$, and $p\colon X \to X/\sim$ the quotient map. Then*

$$\mathscr{T} = \{U \subseteq X/\sim \; : \; p^{-1}(U) \text{ is open in } X\}$$

*is a topology on $X/\sim$ (called **the quotient topology**).*

**Exercise 103.** *What are the open sets in $\mathbb{R}_+/\sim$, where $\sim$ in the equivalence relation defined in Exercise 99 ?*

**Exercise 104.** *Let $X$ be a topological space and $\sim$ an equivalence relation on $X$. For each statement, decide whether it is TRUE or FALSE. Explain.*

*(a) The quotient map $p\colon X \to X/\sim$ , $p(x) = [x]$ must be surjective (onto).*

*(b) The quotient map $p\colon X \to X/\sim$ , $p(x) = [x]$ must be continuous.*

**Claim 105.** *Let $X$ and $Z$ be two topological spaces, $\sim$ an equivalence relation on $X$, and $f\colon X/\sim \to Z$ a function. Then $f$ is continuous if and only if $f \circ p$ is continuous.*

**Exercise 106.** *Define an equivalence relation on the plane $X = \mathbb{R}^2$ (with the standard topology) as follows:*

$$(x_0, y_0) \sim (x_1, y_1) \qquad \text{if and only if} \qquad x_0^2 - y_0 = x_1^2 - y_1 \; .$$

*The quotient space $X/\sim$ is homeomorphic to a familiar space. What is it? (*Hint*: Set $g(x, y) = x^2 - y$ .)*

**Exercise 107.** *Let $\sim$ be the equivalence relation on $\mathbb{R}$ given by: '$a \sim b$ if and only if $a - b \in \mathbb{Z}$'.*
*Prove that the function $f\colon \mathbb{R}/\sim \to S^1$ , $f([x]) = (\cos(2\pi x), \sin(2\pi x))$ is a homeomorphism.*
*(<u>Note</u>: $S^1 = \{(x, y) \in \mathbb{R}^2 \; : \; x^2 + y^2 = 1\}$ is the unit circle, with the subspace topology induced from $\mathbb{R}^2$.)*

# Chapter 10

## Connected Spaces

Informally speaking, a topological space is connected, if it is "made out of one piece". The following definition makes this notion precise.

**Definition 108.** *A topological space $X$ is **connected** if it <u>cannot</u> be expressed as a union of two non-empty and disjoint open sets (i.e., if there are no two open sets $U,V$ for which $U \neq \phi$, $V \neq \phi$, $U \cap V = \phi$ and $U \cup V = X$).*

*<u>Remark</u>: A pair $(U,V)$ of open sets in $X$, satisfying $U \neq \phi$, $V \neq \phi$, $U \cap V = \phi$ and $U \cup V = X$ is called **a separation** of $X$.*

Look carefully at the definition: It defines the meaning of **being connected**, by specifying what it means **not to be connected**.

**Exercise 109.**

*(a) Prove that the topological space $X = \{-1\} \cup (0,1) \cup [2,3]$ (as a subspace of $\mathbb{R}$) is not a connected space.*

*(b) Prove that the set of irrational numbers (as a subspace of $\mathbb{R}$) is not a connected space.*

*(c) Prove that the topological space $Y = \{(x,0) : x \in \mathbb{R}\} \cup \{(x,e^x) : x \in \mathbb{R}\}$ (as a subspace of $\mathbb{R}^2$) is not a connected space.*

**Claim 110.** *The real line $\mathbb{R}$, with the standard topology, is a connected space.*

(**<u>Note</u>:** We will not go over the proof of this claim. A formal proof will use some of the axioms for the real numbers, such as the Least Upper Bound axiom.)

**Exercise 111.** *Show that every topological space $X$, that has the indiscrete topology, is connected (see Definition 32).*

**Claim 112.** *A topological space $X$ is connected if and only if the only clopen sets (i.e., sets which are open and closed) are $X$ and $\phi$.*

**Theorem 113.** *Let $X$ and $Y$ be two topological spaces, and $f\colon X \to Y$ a continuous surjection. If $X$ is a connected space, then $Y$ is also connected.*

**Conclusion 114.** *Let $X$ and $Y$ be two topological spaces, and assume that they are homeomorphic (i.e., that there exists a homeomorphism from $X$ to $Y$). Then $X$ is connected if and only if $Y$ is connected.*

**Exercise 115.**

*(a) Prove that any open interval $(a,b) \subseteq \mathbb{R}$ is a connected space (use Conclusion 114).*

*(b) Prove that any closed interval $[a,b] \subseteq \mathbb{R}$ is a connected space (use part (a) and Definition 108).*

**Exercise 116.** *Prove that $S^1$ (the unit circle) is a connected space. (Hint: Think of $S^1$ as a quotient space.)*

**Proposition 117.** *Let $X$ be a topological space, and $(U,V)$ a separation of $X$.*
*If $A \subseteq X$ is connected, then either $A \subseteq U$ or $A \subseteq V$.*

**Theorem 118.** *Let $X$ be a topological space, and $\{A_\alpha\}_{\alpha\in J}$ a collection of subspaces. If each $A_\alpha$ is connected, and $\bigcap_{\alpha\in J} A_\alpha \neq \phi$, then $Y = \bigcup_{\alpha\in J} A_\alpha$ is connected.*

(Hint: Use Proposition 117 .)

**Theorem 119.** *If $X$ and $Y$ are two connected spaces, then $X \times Y$ is also connected.*

(Hint: Fix a point $(a,b) \in X \times Y$ and prove that each of the spaces $T_x = (\{x\} \times Y) \cup (X \times \{b\})$ is a connected space. Use Theorem 118. Then show that $X \times Y = \bigcup_{x\in X} T_x$ is also connected by using Theorem 118 again. A picture might be helpful.)

**Conclusion 120.** *The topological spaces $\mathbb{R}^2$, $\mathbb{R}^3$, $[0,1]\times[0,1]$ and the torus $S^1 \times S^1$ are all connected spaces.*

**Theorem 121.** *If $f\colon [a,b] \to \mathbb{R}$ is a continuous function, and $L$ is between $f(a)$ and $f(b)$, then $f(c) = L$ for some $c \in [a,b]$.*

(Hint: Assume by contradiction that no such $c$ exists, and consider the sets $U = f^{-1}((-\infty, L))$ and $V = f^{-1}((L,\infty))$. Do you recognize this Theorem ? )

We now present another related notion - path-connectedness. A claim below states that path-connectedness implies connectedness.

**Definition 122.** *A topological space $X$ is said to be **path-connected**, if for every $p,q \in X$, there is a continuous function $f\colon [a,b] \to X$ with $f(a) = p$ and $f(b) = q$ (such an $f$ is called a **path** from $p$ to $q$).*

**Exercise 123.**

*(a) Show that the unit square $[0,1]^2 \subseteq \mathbb{R}^2$ is path-connected.*

*(b) Show that $\mathbb{R}^2 \setminus \{(x,x)\colon x > 0\}$ is path-connected.*

**Theorem 124.** *Let $X$ and $Y$ be two topological spaces, and $f\colon X \to Y$ a continuous surjection. If $X$ is a path-connected space, then $Y$ is also path-connected.*

**Claim 125.** *Every path-connected space is also a connected space.*

(Hint: Proof by contradiction.)

**Exercise 126.**

*(a) Is the space $A = \{(x,y) \ : \ y^2 > (x^2+1)^2 \ \}$ (as a subspace of $\mathbb{R}^2$ ) connected? Prove your arguments.*

*(b) Prove that*

$$Y = \{(a,b) \in \mathbb{R}^2 \ : \ a \in \mathbb{Q} \quad or \quad b \in \mathbb{Q}\} \quad ,$$

*as a subspace of $\mathbb{R}^2$ (with the standard topology) is path-connected (and hence connected).*

**Remark:** There are spaces which are connected but not path-connected. One example is $S = \{\left(x, \sin\left(\frac{1}{x}\right)\right) \ : \ x > 0\} \cup (\{0\} \times [-1,1])$. This space (as a subspace of $\mathbb{R}^2$) is connected, but not path-connected. We will not prove this formally, but try to convince yourself using a picture...

# Chapter 11

## Compact Spaces

Compactness is a central notion in Topology, Analysis, Geometry and other areas of mathematics. Yet, it can take some time to wrap your head around it and appreciate its importance. At first, the definition of compactness may look strange, which makes it difficult to see what compactness is all about.

If you have a finite (nonempty) set of numbers, it always has a largest and smallest elements. This statement is not valid, in general, when the set of numbers is infinite. However, if the set of numbers is the image of a continuous function $f\colon [a,b] \to \mathbb{R}$, then it does have a largest and smallest elements (this is the Extreme Value Theorem you have seen in first year calculus). This hints that a closed interval $[a,b]$, though it is an infinite set of numbers, has some "finiteness" properties. The fact that a closed interval "does not go to infinity" (and contains its endpoints) makes it behave, in some sense, like a finite set.

Compact sets in $\mathbb{R}^n$ can be defined as closed and bounded sets. However, this does not generalize to arbitrary topological spaces, where boundedness does not make sense. Only in the 19th century mathematicians came up with a characterization of closed and bounded sets (in $\mathbb{R}^n$) in terms of open sets only. This led to the following general definition of a compact topological space.

**Definition 127.** *A topological space $X$ is said to be* ***compact*** *if for every collection of open sets $\{U_\alpha\}_{\alpha\in J}$ with $X = \bigcup_{\alpha\in J} U_\alpha$, there is a finite sub-collection $\{U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_k}\}$ satisfying $X = U_{\alpha_1} \cup U_{\alpha_2} \cup \cdots \cup U_{\alpha_k}$. (One says that $X$ is compact if every open cover for $X$ has a finite subcover.)*

**Exercise 128.**

*(a) Show that $\mathbb{R}$ (with the standard topology) is not a compact space.*

*(b) Show that $[-2,2)$ (as a subspace of $\mathbb{R}$) is not a compact space.*

**Exercise 129.** *Show that $A = \left\{0, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\right\}$ is a compact space.*

**Theorem 130.** *The interval $[0,1]$ (as a subspace of $\mathbb{R}$) is a compact space.*

(We will **not** prove this theorem, as the proof uses some of the axioms for the real number system.)

**Theorem 131.** *Let $X,Y$ be two topological spaces, and $f\colon X \to Y$ a continuous surjection. If $X$ is a compact space, then so is $Y$.*

**Conclusion 132.** *If $X$ and $Y$ are two homeomorphic topological spaces, then $X$ is compact if and only if $Y$ is compact.*

**Exercise 133.** *Show that any closed interval $[a,b]$ (as a subspace of $\mathbb{R}$) is a compact space.*

**Exercise 134.** *Use Theorem 131 to show that the unit circle $S^1$ (as a subspace of $\mathbb{R}^2$) is a compact space.*

**Theorem 135.** *If $X$ is a compact space, and $C \subseteq X$ is a closed set, then $C$ is compact.*

**Claim 136.** *Let $C$ be a compact subset of $\mathbb{R}^n$ (i.e., $C$ is a compact space in the subspace topology), and let $x_0 \notin C$. Then there are open sets $U,V \subseteq \mathbb{R}^n$ with $x_0 \in U$, $C \subseteq V$ and $U \cap V = \phi$.*

**Theorem 137.** *If $C \subseteq \mathbb{R}^n$ is a compact subspace of $\mathbb{R}^n$, then $C$ is a closed set.*

**Note:** We would like to prove now that the product of two compact spaces is also compact. Surprisingly, the proof is not trivial (Try it! Where does the difficulty arise?). The following proposition is an important tool that will be used in the proof of this fact.

**Proposition 138.** *Let $X$ and $Y$ be two topological spaces with $Y$ a compact space, and let $x_0$ be a point in $X$. If $N$ is an open set in $X \times Y$, containing $\{x_0\} \times Y$, then there is an open set $W \subseteq X$ containing $x_0$, such that $W \times Y \subseteq N$.*

*(Hint: Draw a picture! As $N$ is open in $X \times Y$, it is a union of basic sets in the product topology ... )*

**Theorem 139.** *If $X$ and $Y$ are two compact spaces, then so is $X \times Y$.*

(Hint: Given a cover for $X \times Y$, show that for each $x \in X$ one can create a "tube" around $\{x\} \times Y$, covered by finitely many open sets from that cover. Use Proposition 138 .)

**Conclusion 140.** *The square $[0,1]^2$, the cube $[0,1]^3$, the torus $S^1 \times S^1$, and the Klein Bottle are all compact spaces.*

**Definition 141.** *A set $A \subseteq \mathbb{R}^n$ is* ***bounded*** *if it is contained in some open ball in $\mathbb{R}^n$ .*

**Theorem 142.** *In $\mathbb{R}^n$ with the standard topology, a subset A is compact if and only if it is closed and bounded.*

(Hint: Part of this Theorem was already proved. For the rest, use Theorems 135 and 139 . )

**Exercise 143.**

*(a) Is $\mathbb{Z}$ compact (as a subspace of $\mathbb{R}$) ?*

*(b) Let $A = \left\{(-1)^{n+1} - \frac{1}{n} \ : \ n = 1,2,3\dots\right\}$. Is A compact (as a subspace of $\mathbb{R}$) ? Is $\overline{A}$ compact? Explain.*

*(c) Is $[0,1]^2 \cap \mathbb{Q}^2$ (as a subspace of $\mathbb{R}^2$) compact?*

**Exercise 144.** *Explain why there is **no** continuous function $f\colon [-1,1] \to (0,\infty)$ which is onto.*

**Theorem 145.** *If $X \neq \phi$ is a compact space, and $f\colon X \to \mathbb{R}$ a continuous function, then f has an absolute maximum and an absolute minimum.*
*(Do you recognize this Theorem?)*

**Exercise 146.** *Explain why at any time, there is always a warmest and a coldest spot on the surface of the earth (assume that temperature varies continuously).*