# Chapter 6

## Functions

"A single idea, if it is right, saves us the labor of an infinity of experiences."
Jacques Maritain

Galileo's pairing of positive integers with their squares and Bolzano's pairing of points on one line segment with points on a longer line segment both involve assigning to each member of one set a member of a second set. A core mathematical concept, that of a *function*, captures this notion and plays a prominent role in the modern understanding of most parts of mathematics, especially those sections dealing with infinity. In this section, we provide a careful definition of a function in terms of elementary set theory and investigate some possible properties of functions.

**Definition 10 :** An **ordered pair** is a combination of two objects, in which the first entry is distinguished from the second. If the first entry is $a$ and the second entry is $b$, the notation for an ordered pair is $(a,b)$. The pair is "ordered" in that $(a,b)$ differs from $(b,a)$ unless $a=b$. We often use the words **term** or **coordinate** to refer to an entry of an ordered pair. Thus the first term of $(a,b)$ is $a$ and the second term is $b$. Set theorists formally define the ordered pair $(a,b)$ as the set $\{\{a\},\{a,b\}\}$.

**Exercise 27 :** Use the formal definition to write out explicitly the ordered pairs $(3,4)$ and $(4,3)$ and show they are different sets.

**Exercise 28 :** Show that if $a \neq b$, then under the formal definition of ordered pairs, it is true that $(a,b) \neq (b,a)$.

**Definition 11 :** The statement that **F** is a **function** means that **F** is a collection of ordered pairs, such that no two of these pairs have the same first term.

**Example 8 :** Let **F** be the set defined by

$$\mathbf{F} = \{(1, Washington), (2, Adams), (43, Obama), (4,3), (6, Adams)\}$$

.

Then **F** is a function which has 5 ordered pairs. Note that the ordered pair (4,3) belongs to **F** but the ordered pair (3,4) does not.

**Exercise 29 :** Show that the set **G** defined by

$$\mathbf{F} = \{(1, Washington), (2, Adams), (1, Obama), (4,3), (6, Adams)\}$$

also contains 5 distinct ordered pairs, but **G** is not a function.

**Definition 12 :** Suppose that **F** is a function. The **domain** of **F** is the set **X** such that **x** is an element of **X** if and only if **x** is the first term of some element of **F**. The **range** of **F** is the set **Y** such that **y** is an element of **Y** if and only if **y** is a second term of some element of **F**. If **x** is the first term of an element of **F**, then **F**(**x**), the **value of F at x**, denotes the second term of the ordered pair of **F** whose first term is **x** The function **F** is said to be a function from **X onto Y**. If **Z** is a set such that **Y** is a subset of **Z**, then **F** is a function from **X into Z**. The notation **F:X** $\rightarrow$ **Z** means that **F** is a function from **X** into **Z**. If **A** is a subset of **X,** then **F**(**A**) denotes the set of all elements **F**(**a**) where **a** is an element of **A.** If **W** is a subset of **Z**, then $\mathbf{F}^{-1}$(**W**) denotes $\{$**x**: **x** is in **X** and **(x,w)** is an element of **F** for some **w** in **W**$\}$ . The set **F(A)** is called the **image** of **A** under **F** and $\mathbf{F}^{-1}$(**W**) is called the **inverse image** of **W** under **F.**

**Exercise 30 :** For the function **F** defined in Example 8, show that

(a) the domain is $\{1,2,3,4,6\}$,

(b) the range is {Washington, Adams, Obama, 3}

(c) **F**(2) = **F**(6)

(d) $\mathbf{F}^{-1}$({Adams,3} ) = $\{2, 4, 6\}$.

**Example 9 :** Let **R** denote the set of all real numbers. Let **F** be $\{(\mathbf{x},\mathbf{x}^2)$: **x** is an element of **R** $\}$. Then **F** is a function. The domain of **F** is **R**, the range of **F** is $\{$**x**: **x** is in **R** and $\mathbf{x} \geq 0\}$, **F**(2) = 4, and we have **F**:**R** $\rightarrow$ **R**.

**Definition 13 :** Suppose that **F** is a function. The statement that **F** is **one-to-one** means that no two elements of **F** have the same second term. In other words, if **x** and **y** are distinct elements of the domain of **F**, then **F**(**x** ) is different from **F**(**y**). Note in the last example, **F** is not one-to-one because **F**(2) = **F**(-2).

**Question 3 :** Let **f** be a function from a set **X** into a set **Y** and let **A** and **B** be subsets of **X**. Which of the following statements are always true?

(a) $\mathbf{f}(\mathbf{A} \cup \mathbf{B}) = \mathbf{f}(\mathbf{A}) \cup \mathbf{f}(\mathbf{B})$.

(b) $\mathbf{f}(\mathbf{A} \cap \mathbf{B}) = \mathbf{f}(\mathbf{A}) \cap \mathbf{f}(\mathbf{B})$.

(c) **Y - f(A) = f(X - A)** for each subset **A** of **X**.

Note that if **C** and **D** are sets, then **C** - **D** denotes the set of all elements of **C** which are not members of **D**.