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Point-Set Topology

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Contents

Chapter 1

Review

Read the definitions, prove the theorems and do the exercises.

Definitions 1. Suppose A and B are subsets of a set X. We define the following:

1. $A \cup B := \{x \mid x \in A \text{ or } x \in B\}$;
2. $A \cap B := \{x \mid x \in A \text{ and } x \in B\}$;
3. $A \setminus B := \{x \mid x \in A \text{ and } x \notin B\}$.

Theorem 2. Let A be a subset of the set X. Then $X \setminus (X \setminus A) = A$.

Theorem 3. (DeMorgan's Laws) Let A and B be subsets of a set X. Then

1. $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$ and
2. $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$.

Definition 4. Let I be an indexing set. For each $\delta \in I$, let A_δ be a set. We define the following two sets:

1. $\bigcup_{\delta \in I} A_\delta = \{s \mid \text{ there exists } \delta \in I \text{ such that } s \in A_\delta\}$ and
2. $\bigcap_{\delta \in I} A_\delta = \{s \mid s \in A_\delta \text{ for all } \delta \in I\}$.

Theorem 5. (Generalised DeMorgan's Laws) Let $\{A_\delta \mid \delta \in I\}$ be a collection of subsets of a set X. Then

1. $X \setminus (\bigcup_{\delta \in I} A_\delta) = \bigcap_{\delta \in I} (X \setminus A_\delta)$ and

2. $X \setminus \left(\bigcap_{\delta \in I} A_\delta\right) = \bigcup_{\delta \in I} (X \setminus A_\delta)$.

Theorem 6. Let A and B be subsets of a set X. Then $A \setminus B = A \cap (X \setminus B)$.

Definitions 7. Let X and Y be sets and $f \subseteq X \times Y$.

1. We say f is a *function* if for each $x \in X$ there is a unique $y \in Y$ such that $(x,y) \in f$. In this case, we write $f : X \to Y$ and $f(x) = y$ for the pairs $(x,y) \in f$.
2. Suppose $f : X \to Y$ and $A \subseteq X$ and $B \subseteq Y$. Then the *image* of A under f is the set
$$f(A) := \{y \in Y \mid y = f(x) \text{ for some } x \in A\}.$$
The *inverse image* of B under f is the set
$$f^{-1}(B) := \{x \mid f(x) \in B\}.$$

Theorem 8. Let $f : X \to Y$ be a function and let B and C be subsets of Y. Then

1. $f^{-1}(B \cup C) = f^{-1}(B) \cup f^{-1}(C)$,
2. $f^{-1}(B \cap C) = f^{-1}(B) \cap f^{-1}(C)$,
3. $f^{-1}(Y \setminus C) = X \setminus f^{-1}(C)$,
4. $f^{-1}(B \setminus C) = f^{-1}(B) \setminus f^{-1}(C)$.

Definition 9. Let $f : X \to Y$ and $g : Y \to Z$ be a functions. Then *g composed with f* denoted $g \circ f$, is the function $g \circ f : X \to Z$ such that for each $x \in X$, $f \circ g(x) = f(g(x)))$.

Theorem 10. Let $f : X \to Y$ and $g : Y \to Z$ be a functions and let B be a subset of Y. Then
$$(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B).)$$

Theorem 11. Let $f : X \to Y$ be a function and let A_1 and A_2 be subsets of X. Then

1. $A_1 \subseteq A_2 \implies f(A_1) \subseteq f(A_2)$,
2. $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$,

3. $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$ and the reverse inclusion fails, and
4. $f(A_1) \setminus f(A_2) \subseteq f(A_1 \setminus A_2)$ and the reverse inclusion fails.

Theorem 12. Let $f : X \to Y$ be a function. Let A be a subset of X, and let B be a subset of Y. Then

1. $A \subseteq f^{-1}(f(A))$,
2. $f(f^{-1}(B)) \subseteq B$, and
3. $f(X) \setminus f(A) \subseteq f(X \setminus A)$.

Chapter 2

Topological Spaces

Definition 13. A *topological space* (X,τ) is a set X and a family τ of subsets of X satisfying the following conditions:

1. the empty set $\emptyset$ and X are members of τ;
2. if A and B are in τ, then $A \cap B$ is in τ; and
3. if I is an indexing set, and A_δ is in τ for each $\delta \in I$, then $\bigcup_{\delta \in I} A_\delta$ is in τ.

The members of τ are called *open sets* and τ is called a *topology* on X.

Exercise 14. Find all the possible topologies on the set

$$X := \{a, b, c\}.$$

Exercise 15. Let X be any set. Show that the power set of X is a topology on X. We call this topology the *discrete topology* on X.

Exercise 16. Consider the collection

$$\tau := \{U \subseteq \mathbb{R} \mid U = \emptyset \text{ or } \mathbb{R} \setminus U \text{ is finite }\}$$

where $\mathbb{R}$ denotes the set of real numbers. Show τ defines a topology on $\mathbb{R}$. This is called the *finite complement topology*.

Definition 17. Let (X,τ) be a topological space. A subset A of X is called a *closed* set iff $X \setminus A$ is open.

Exercise 18. Let (X,τ) be a topological space. Suppose $A \subseteq X$.

1. Is it true that A must be either open or closed? (Prove it or find a counter example.)
2. Can A be both open and closed? (Justify your answer.)

Theorem 19. Let (X,τ) be a topological space and A and B be subsets of X.

1. If A and B are closed, then $A \cup B$ is closed.
2. If I is an indexing set, and A_δ is a closed subset of X for each $\delta \in I$, then $\bigcap\limits_{\delta \in I} A_\delta$ is closed.

Theorem 20. For any topological space (X,τ), the sets $\emptyset$ and X are closed.

Theorem 21. Let (X,τ) be a topological space and A be subset of X. Then A is open if and only if for each $x \in A$, there is an open set O_x containing x such that $O_x \subseteq A$.

Definition 22. Let (X,τ) be a topological space, and let A be a subset of X. The *interior* of A, denoted $\operatorname{int}(A)$ is the union of all open subsets contained in A.

Theorem 23. The interior satisfies the following:

1. $\operatorname{int}(\emptyset) = \emptyset$;
2. $\operatorname{int}(X) = X$;
3. $\operatorname{int}(A)$ is open;
4. $\operatorname{int}(\operatorname{int}(A)) = \operatorname{int}(A)$;
5. $\operatorname{int}(A \cap B) = \operatorname{int}(A) \cap \operatorname{int}(B)$;
6. $A = \operatorname{int}(A)$ if and only if A is open.

Definition 24. Let (X,τ) be a topological space, x be an element of X and A be a subset of X. Then x is said to be a *boundary point* of A if every open set containing x has nonempty intersection with both A and $X \setminus A$. The set of all boundary points of A is called the *boundary* of A and is denoted $\partial(A)$.

Theorem 25. Let (X,τ) be a topological space. For every subset A of X, $\partial(A) = \partial(X \setminus A)$.

Theorem 26. Let (X,τ) be a topological space. For every subset A of X, the sets $\text{int}(A)$ and $\partial(A)$ are mutually disjoint.

Theorem 27. Let (X,τ) be a topological space. For every subset A of X, the set $\partial(A)$ is closed.

Theorem 28. Let (X,τ) be a topological space and A be a subset of X. Then A is closed if and only if $\partial(A) \subseteq A$.

Definition 29. Let (X,τ) be a topological space and A be a subset of X. The *closure* of A, denoted $\overline{A}$, is the intersection of all closed sets containing A.

Theorem 30. Let (X,τ) be a topological space and A be a subset of X. Then A is closed if and only if $\overline{A} = A$.

Exercise 31. Consider the set of real numbers $\mathbb{R}$.

1. Let O be an open set with respect to the finite complement topology as defined in Exercise 16. Describe $\overline{O}$.
2. Let A be open set with respect to the discrete topology as defined in Exercise 15. Describe $\overline{A}$.

Theorem 32. Let (X,τ) be a topological space and A be a subset of X. An element x is in $\overline{A}$ if and only if every open set containing x intersects A.

Theorem 33. Let (X,τ) be a topological space and A be a subset of X. Then $\overline{A} = \text{int}(A) \cup \partial(A)$.

Theorem 34. Let (X,τ) be a topological space with A and B be subsets of X. The closure operation satisfies the following:

1. $A \subseteq \overline{A}$;
2. $\overline{\overline{A}} = \overline{A}$; and
3. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Definition 35. Let (X,τ) be a topological space, x be an element of X and A be a subset of X. Then x is called a *limit point* (also called a *cluster point* or an *accumulation point*) of A if every open set containing x contains an element of A different from x. We denote the set of all limit points of a set A by A'.

Theorem 36. Let (X,τ) be a topological space and A be a subset of X. Then A is closed if and only if $A' \subseteq A$.

Theorem 37. Let (X,τ) be a topological space and A be a subset of X. Then $\overline{A} = A \cup A'$.

Definition 38. Let τ and σ be topologies on X. We say that τ is *finer* (or larger) than σ if $\sigma \subseteq \tau$. In this case, we also say that σ is *coarser* (or smaller) than τ.

Exercise 39. Give some examples of finer/coarser topologies on a set X. (For example, $X = \mathbb{R}$ or $X = \{a,b,c\}$.) Also find examples of two topologies on a space X that are not comparable in this way.

Chapter 3

Basis

Definition 40. A family $\mathscr{B}$ of subsets of a set X is a *base* or *basis* for a topology on X if the following two conditions are satisfied:

1. for each $x \in X$, there is a $B \in \mathscr{B}$ such that $x \in B$; and
2. if A and B are in $\mathscr{B}$ and $x \in A \cap B$, then there is a C in $\mathscr{B}$ such that $x \in C$ and $C \subseteq A \cap B$.

Theorem 41. Let $\mathscr{B}$ be a basis for a topology on a set X. Let

$$\tau := \{\emptyset\} \cup \{U \mid U \text{ is the union of members of } \mathscr{B}\}.$$

Then τ is a topology on X.

Definition 42. The topology τ defined in Theorem 41 is called the *topology generated by* $\mathscr{B}$.

Exercise 43. Let $\mathscr{B}$ be the collection of *open intervals* in $\mathbb{R}$. That is

$$\mathscr{B} := \{(a,b) \mid a,b \in \mathbb{R} \text{ and } a < b\} \text{ where } (a,b) := \{x \in \mathbb{R} \mid a < x < b\}.$$

1. Show $\mathscr{B}$ is a basis for a topology on $\mathbb{R}$. The topology generated by $\mathscr{B}$ is called the *standard topology on* $\mathbb{R}$.
2. Compare the standard topology on $\mathbb{R}$ with finite complement topology. Is the standard topology coarser, finer or neither?

Exercise 44. Let

$$\mathscr{C} := \{[a,b) \mid a,b \in \mathbb{R} \text{ and } a < b\} \text{ where } [a,b) := \{x \in \mathbb{R} \mid a \leq x < b\}.$$

1. Show $\mathscr{C}$ is a basis for a topology on $\mathbb{R}$. The topology generated by $\mathscr{C}$ is called the *lower limit topology on* $\mathbb{R}$.

2. Show that $[0,1)$ is both open and closed (that is *clopen*) in the lower limit topology.

3. Compare the lower limit topology with the standard topology on $\mathbb{R}$. Is the standard topology coarser, finer or neither?

Exercise 45. Let

$$\mathscr{D} := \{(a,\infty) \mid a \in \mathbb{R}\} \text{ where } (a,\infty) := \{x \in \mathbb{R} \mid a < x\}.$$

1. Show $\mathscr{D}$ is a basis for a topology on $\mathbb{R}$. The topology generated by $\mathscr{D}$ is called the *right open ray topology on* $\mathbb{R}$.

2. Compare the right open ray topology with the standard topology on $\mathbb{R}$. Is the standard topology coarser, finer or neither?

3. Compare the right open ray topology with the lower limit topology on $\mathbb{R}$. Is the lower limit topology coarser, finer or neither?

Theorem 46. A family $\mathscr{B}$ of subsets of X is a basis for a given topology τ on X if and only if the following two conditions are true:

1. for each U in τ and each $x \in U$, there is a $B \in \mathscr{B}$ such that $x \in B$ and $B \subseteq U$, and

2. $\mathscr{B} \subseteq \tau$.

Exercise 47. Find a minimal basis for the discrete topology on a set X.

Chapter 4

Subspace Topology

Theorem 48. Let (X,τ) be a topological space and $Y \subseteq X$. Then

$$\tau_Y := \{Y \cap U \mid U \in \tau\}$$

is a topology on Y.

Definition 49. The topological space (Y,τ_Y) is called the *relative* (or *subspace*) topology on Y. Sets in τ_Y are called *open in* Y or *open relative to* Y. Similar terminology is used for closed sets.

Theorem 50. Let (Y,τ_Y) be a subspace of (X,τ) and $A \subseteq Y$. Then

1. A is closed in Y if and only if $A = Y \cap F$, where F is closed subset in X;
2. an element x in Y is a τ_Y-limit point of A if and only if x is a τ-limit point of A; and
3. the τ_Y-closure of A is the intersection of Y and the τ-closure of A.

Theorem 51. Let (Y,τ_Y) be a subspace of (X,τ) and $A \subseteq Y$. Then

1. if A is closed in Y and Y is closed in X, then A is closed in X;
2. if A is open in Y and Y is open in X, then A is open in X.

Exercise 52. Let $Y = (0,1] \cup \{2\}$ which is a subset of $\mathbb{R}$. Determine whether each of the following subsets of Y is open, closed or neither relative to Y.

1. $(0,.5)$
2. $(0,.5]$

3. $(.6, 1)$
4. $(.6, 1]$
5. $\{1\}$
6. $\{2\}$

Chapter 5

Product Topology

Definition 53. Let (X,τ) and (Y,σ) be topological spaces. The *box topology* on $X \times Y$ is the topology generated by the basis that contains all sets of the form $U \times V$ where U is open in X and V is open in Y.

Theorem 54. If $\mathscr{B}$ is a basis for the topology of X and $\mathscr{D}$ is a basis for the topology on Y, then the collection of sets of the form $B \times D$ where $B \in \mathscr{B}$ and $D \in \mathscr{D}$ is a basis for the box topology on $X \times Y$.

Exercise 55. Describe the box topology on $\mathbb{R}^2$ when $\mathbb{R}$ is equipped with the standard topology.

Definitions 56. Let A be an indexing set and for each $\alpha \in A$ let (X_α, τ_α) be a topological space.

1. The Cartesian product of the family $\{X_\alpha\}_{\alpha\in A}$, denoted
$$\prod_{\alpha\in A} X_\alpha,$$
is the set of all A-tuples $(x_\alpha)_{\alpha\in A}$ such that $x_\alpha \in X_\alpha$, for each $\alpha \in A$.

2. The *box topology* on the Cartesian product is the topology generated by the basis of sets of the form
$$\prod_{\alpha\in A} U_\alpha,$$
where U_α is open in X_α for each $\alpha \in A$.

3. The *product topology* on the Cartesian product is the topology generated by the basis of sets of the form
$$\prod_{\alpha\in A} U_\alpha,$$
where U_α is open in X_α for each $\alpha \in A$ and $U_\alpha = X_\alpha$ except for finitely many values of α.

Exercise 57. Show that the collections described in items (2) and (3) of Definition 56 are indeed bases.

Theorem 58. Let A be an indexing set and for each $\alpha \in A$ let (X_α, τ_α) be a topological space. If A is finite, then the box and product topologies on the Cartesian product of the family $\{X_\alpha\}_{\alpha \in A}$ are the same.

Exercise 59. Let X be the set of all sequences of elements with entries in $\{0,1\}$. Thus,

$$X = \prod_{i \in \mathbb{N}} \{0,1\}.$$

If we equip $\{0,1\}$ with the discrete topology, describe the basis for the product topology on X.

Definition 60. Let X be a topological space. We say a sequence (x_n) in X converges to a point x in X if and only if for every open set U containing x, there exists N such that if $n > N$ then $x_n \in U$.

Exercise 61. Consider the Cartesian product X defined in Exercise 59. Define a sequence (x_n) in X as follows

$$\begin{aligned} x_1 &= (1,0,0,0,\dots) \\ x_2 &= (0,1,0,0,\dots) \\ x_3 &= (0,0,1,0,\dots) \text{ etc.} \end{aligned}$$

(Thus (x_n) is a sequence of sequences!)

1. Does this sequence converge in the product topology?
2. Does this sequence converge in the box topology?

Chapter 6

Continuity and homeomorphisms

Definition 62. Let (X,τ) and (Y,σ) be topological spaces and $f : X \to Y$. We say f is *continuous* if the inverse image of each open set of Y is open in X.

Theorem 63. Let (X,τ) and (Y,σ) be topological spaces and $f : X \to Y$. Then f is continuous if and only if the inverse image of each closed set of Y is closed in X.

Theorem 64. Let (X,τ) and (Y,σ) be topological spaces and $f : X \to Y$ and $\mathscr{B}$ be a basis for Y. Then f is continuous if and only if the inverse image of each element of $\mathscr{B}$ is open.

Theorem 65. Let (X,τ) and (Y,σ) be topological spaces and $f : X \to Y$. The following are equivalent:

1. f is continuous;
2. for every $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$; and
3. for every $B \subseteq Y$, $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$.

Exercise 66. Show that the usual definition for continuity of functions from $\mathbb{R}$ to $\mathbb{R}$ agrees with the topological definition.

Exercise 67. Let (X,τ) and (Y,σ) be topological spaces. Define the map

$$\pi_2 : X \times Y \to Y$$

such that $\pi_2((x,y)) = y$. The map π_2 is called the projection map associated to the second coordinate.

1. Show π_2 is continuous.
2. Show π_2 is an *open map*. That is, show π_2 takes open sets to open sets.

Exercise 68. Let A be an indexing set and for each $\alpha \in A$ let (X_α, τ_α) be a topological space. Fix $\beta \in A$. Define the map

$$\pi_\beta : \prod_{\alpha \in A} X_\alpha \to X_\beta$$

such that $\pi_\beta((x_\alpha)_{\alpha \in A}) = x_\beta$. The map π_β is called the projection map associated to β.

1. Show π_β is continuous with respect to the product topology.
2. Show π_β is an *open map* with respect to the product topology. That is, show π_β takes open sets to open sets.

Theorem 69. Let (X,τ), (Y,σ) and (Z,ν) be topological spaces and $f : X \to Y$ and $g : Y \to Z$ be continuous. Then $g \circ f : X \to Z$ is continuous.

Definition 70. Let (X,τ) and (Y,σ) be topological spaces and $f : X \to Y$. We say f is a *homeomorphism* if the following conditions are satisfied:

1. f is a bijection and
2. both f and f^{-1} are continuous.

Theorem 71. Let (X,τ) and (Y,σ) be topological spaces and $f : X \to Y$ such that f is a bijection. Then the following are equivalent:

1. f is a homeomorphism;
2. if G is a subset of X, then $f(G)$ is open in Y if and only if G is open in X;
3. if F is a subset of Y, then $f^{-1}(F)$ is open in X if and only if F is open in Y; and
4. if E is a subset of X, then $f(\overline{E}) = \overline{f(E)}$.

Chapter 7

Quotient Topology

Definitions 72. Let (X,τ) be a topological space equipped with an equivalence relation $\sim$. For each $x \in X$, the *equivalence class* of x is the set

$$[x] := \{y \in X \mid y \sim x\}.$$

Let $\tilde{X} := \{[x] \mid x \in X\}$. Define the *quotient map* $q : X \to \tilde{X}$ by $q(x) = [x]$. The *quotient topology* on $\tilde{X}$ is the collection:

$$\tau_q = \{U \subseteq \tilde{X} \mid q^{-1}(U) \text{ is open in } X\}.$$

Theorem 73. The collection τ_q is a topology on $\tilde{X}$.

Theorem 74. If τ' is any topology on $\tilde{X}$ so that the quotient map q is continuous, then τ_q is finer than τ'. (Thus τ_q is the 'largest' topology on $\tilde{X}$ in which q is continuous.)

Theorem 75. Let Y be a topological space. Show that $f : \tilde{X} \to Y$ is continuous if and only if $f \circ q$ is continuous.

Exercise 76. Define an equivalence relation on $\mathbb{R}^2$ (with the standard topology) by

$$(x_0,y_0) \sim (x_1,y_1) \text{ if and only if } x_0^2 + y_0^2 = x_1^2 + y_1^2.$$

Show that the quotient topology on $\tilde{X}$ is homeomorphic to $[0,\infty)$ where $[0,\infty)$ is a subspace of the standard topology on $\mathbb{R}$.

Chapter 8

Metric Spaces

Definition 77. A *metric space* (X,d) is a set X together with a function $d : X \times X \to \mathbb{R}$ such that the following conditions are satisfied:

1. $d(x,y) \geq 0$ for all $x,y \in X$;
2. $d(x,y) = 0$ if and only if $x = y$;
3. $d(x,y) = d(y,x)$ for all $x,y \in X$; and
4. $d(x,y) \leq d(x,z) + d(z,y)$ for all $x,y \in X$.

We call d a *metric*.

Definition 78. Let (X,d) be a metric space. For $x \in X$ and $r > 0$, the set

$$B(x,r) := \{y \mid y \in X \text{ and } d(x,y) < r\}$$

is called the *r-ball* (or *r-neighbourhood*).

Theorem 79. Let (X,d) be a metric space. The the collection of all sets of the form $B(x,r)$ such that $x \in X$ and $r \in \mathbb{R}$, is a basis for a topology on X.

Definition 80. The topology generated by the basis of r-balls as in Theorem 79 is called the *metric topology* on X generated by d.

Theorem 81. Consider the function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$d(x,y) = |x-y|.$$

Show that the topology on $\mathbb{R}$ generated by r-balls is the same as the standard topology on $\mathbb{R}$.

Theorem 82. Let (X,d) be a metric space. Show that the metric topology on X is the coarsest topology on X under which the function d is continuous (where $X \times X$ is given the product topology).

Chapter 9

Compactness

Definition 83. Let I be an indexing set and $\Phi = \{A_\delta \mid \delta \in I\}$ be a collection of sets. The collection Φ is called a *cover* of Y if

$$Y \subseteq \bigcup_{\delta \in I} A_\delta.$$

Any subcollection of Φ that also covers Y is called a *subcover*.

Definition 84. Let (X,τ) be a topological space and $Y \subseteq X$. A cover Φ of Y is called an open cover of Y if each member of Φ is an open subset of X.

Definition 85. Let (X,τ) be a topological space and $K \subseteq X$. We say K is *compact* if every open cover of K has a finite subcover.

Theorem 86. Let (X,τ) be a topological space and $Y \subseteq K \subseteq X$. If K is compact and Y is closed, then Y is compact.

Theorem 87. Let (X,τ) and (Y,σ) be topological spaces. Suppose $f : X \to Y$ is a continuous function and $K \subseteq X$ is compact. Then $f(K)$ is compact.

Exercise 88. Consider the topological space (X,τ) where τ is the discrete topology. Describe the compact sets of X.

Exercise 89. Consider the set $\mathbb{R}$ with the standard topology.

1. Is the open interval $(0,1)$ compact in $\mathbb{R}$? Justify your answer.
2. Prove that interval $[0,1]$ is compact in $\mathbb{R}$. (As a reminder, the axioms of the real numbers includes a ‘completeness’ axiom.)

Theorem 90. (The Heine-Borel Theorem.) Consider the set $\mathbb{R}$ with the standard topology. A subset K of $\mathbb{R}$ is compact if and only if K is closed and bounded.

Exercise 91. Define an equivalence relation on $\mathbb{R}$ (with the standard topology) by

$$x \sim y \text{ if and only if } x - y \in \mathbb{Z}.$$

Is the set $\tilde{X}$ compact in the quotient topology?

Chapter 10

Separability

Definitions 92. Suppose (X, τ) is a topological space.

1. The space X is called a *T_0-space* if for each pair of distinct members of X, there is an open set U containing one of the members but not the other.
2. The space X is called a *T_1-space* if for each pair of distinct members x and y of X, there is an open set U containing x but not y.
3. The space X is called a *Hausdorff space* (or a *T_2-space*) if for each pair of distinct members x and y of X, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.
4. The space X is called *regular* if for each closed subset K of X and each point $x \in X$ with $x \notin K$, there exist disjoint open sets U and V such that $K \subseteq U$ and $x \in V$.
5. The space X is called *normal* if for each pair E and F of disjoint closed subsets of X, there exist disjoint open sets U and V such that $E \subseteq U$ and $F \subseteq V$.

Theorem 93. Suppose (X, τ) is a topological space in which singleton sets are closed. That is, $\{x\}$ is closed for every $x \in X$. Then we have the following:

$$X \text{ is normal } \implies X \text{ is regular } \implies X \text{ is Hausdorff} \implies X \text{ is } \mathrm{T}_1 \implies X \text{ is } \mathrm{T}_0.$$

Exercise 94. Let $X = \mathbb{R}$ with the standard topology. Is this a T_0 space? T_1? etc,.

Exercise 95. Let $X = \mathbb{R}$ with the topology generated by open rays. That is, with basis $\{(a, \infty) \mid a \in \mathbb{R}\}$. Is this a T_0 space? T_1? etc,.

Exercise 96. Let $X = \mathbb{R}$ with the finite complement topology. Is this a T_0 space? T_1? etc,.

Theorem 97. A space X is a T_1 space if and only if singleton sets are closed.

Theorem 98. Suppose X is Hausdorff.

1. If $Y \subseteq X$, then Y is Hausdorff in τ_Y.
2. Let collection $\{X_\alpha\}_{\alpha \in A}$ be a collection of Hausdorff spaces. Then

$$\prod_{\alpha \in A} X_\alpha,$$

is Hausdorff with respect to the product topology.

Exercise 99. Let $X = \mathbb{R}$ with the standard topology. Define a relation $\sim$ on X such that

$a \sim b$ if and only if either a and b are both rational or they are both irrational.

1. Show that $\sim$ is an equivalence relation.
2. Describe the quotient topology on $\tilde{X}$.
3. Is $\tilde{X}$ Hausdorff with respect to the quotient topology?

Theorem 100. Suppose X is a Hausdorff space, then

1. finite sets are closed and
2. x is a limit point of a subset A of X if and only if each open set containing x contains infinitely many elements of A.

Theorem 101. A space X is regular if and only if for each $x \in X$ and each open set U containing x, there exists and open set V such that $x \in V$ and $\overline{V} \subseteq U$.

Theorem 102. A space X is normal if and only if for each closed set K and open set U containing K, there exists and open set V such that $K \subseteq V \subseteq \overline{V} \subseteq U$.

Exercise 103. Show that the Hausdorff property is not preserved by continuous functions. That is, find an example for a continuous function $f : X \to Y$ such that X is Hausdorff and Y is not Hausdorff.

Theorem 104. Suppose $f : X \to Y$ is an open bijection and that X is Hausdorff. Then Y is Hausdorff.

Theorem 105. A compact subset of a Hausdorff space is closed.

Theorem 106. An bijective continuous function from a compact space onto a Hausdorff space is a homeomorphism.

Theorem 107. A compact Hausdorff space is regular.

Theorem 108. A compact Hausdorff space is normal.

Chapter 11

Connectedness

Definition 109. A topological space (X,τ) is *connected* if X is not the union of two nonempty disjoint open sets. A subset Y of X is connected if (Y,τ_Y) is connected.

Theorem 110. The space (X,τ) is connected if and only if the only subsets of X which are both open and closed are $\emptyset$ and X.

Exercise 111.

1. Consider $\mathbb{R}$ with the lower limit topology as in Example 44. Is $\mathbb{R}$ connected?
2. Consider $\mathbb{R}$ with the standard topology. Is $\mathbb{R}$ connected?

Theorem 112. Let A and B be connected subsets of a space (X,τ) such that $A \cap B \neq \emptyset$. Then $A \cup B$ is connected.

Theorem 113. Let A be a subset of X. If A is connected and $A \subseteq B \subseteq \overline{A}$, then B is connected.

Theorem 114. The continuous image of a connected space is connected.

Chapter 12

Some important theorems of Point-Set Topology

Exercise 115. Pick one of the items below, look it up, and tell us about it.

1. Urysohn's Lemma and the Tietze Extension Theorem
2. Tychonoff's theorem
3. Compactification theorems
4. Metrization theorems
5. The Baire Category Theorem