MATH 3430-02 WEEK 9-2

Key Words: The Laplace transform (cont.)

Using calculus, you can find:

$$
\begin{aligned}
\mathcal{L}\{1\} &= \frac{1}{s}, \quad (s>0)\\
\mathcal{L}\{t\} &= \frac{1}{s^2}, \quad (s>0)\\
\mathcal{L}\{t^n\} &= \frac{n}{s}\mathcal{L}\{t^{n-1}\}, \quad (s>0)\\
\mathcal{L}\{e^{at}\} &= \frac{1}{s-a}, \quad (s>a)\\
\mathcal{L}\{\cos(at)\} &= \frac{s}{s^2+a^2}, \quad (s>0)\\
\mathcal{L}\{\sin(at)\} &= \frac{a}{s^2+a^2}, \quad (s>0).
\end{aligned}
$$

Moreover, we have the following properties (so far)

$$
\begin{aligned}
\mathcal{L}\{f'(t)\} &= -f(0) + sF(s),\\
\mathcal{L}\{f''(t)\} &= -f'(0) - sf(0) + s^2F(s),\\
\mathcal{L}\{tf(t)\} &= -\frac{\mathrm{d}}{\mathrm{d}s}F(s).
\end{aligned}
$$

In addition, note that $\mathcal{L}$ is linear, namely,

$$\mathcal{L}\{f(t)+g(t)\} = F(s) + G(s), \qquad \mathcal{L}\{cf(t)\} = cF(s).$$

Q1. Fill in the following blanks:

$$
\begin{aligned}
\mathcal{L}\Big\{ \qquad\qquad\qquad\qquad \Big\} &= \frac{1}{s-2},\\
\mathcal{L}\Big\{ \qquad\qquad\qquad\qquad \Big\} &= \frac{2s}{1+s^2},\\
\mathcal{L}\Big\{ \qquad\qquad\qquad\qquad \Big\} &= \frac{1+2s}{1+s^2}.
\end{aligned}
$$

Given $f(t)$, finding $\mathcal{L}\{f(t)\} = F(s)$ is called the Laplace transform of $f(t)$.

Conversely, given $F(s)$, finding a function $f(t)$ such that $\mathcal{L}\{f(t)\} = F(s)$ is called the **inverse Laplace transform**. The inverse Laplace transform is unique up to a certain notion of equivalence. But we'll not pursue this point, since our practice will not touch the subtlety there.

A reason why the Laplace transform is useful is that it turns a differential equation into an algebraic equation. To see this, consider the following example.

Q2. For the initial value problem

$$y'' - y' - 2y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

What are the Laplace transforms of the both sides of this equation (using the initial values)?

Thus you have obtained an algebraic equation in $Y(s)$. Now solve $Y(s)$ in terms of s.

Finally, find the function $f(t)$ that satisfies

$$\mathcal{L}\{y(t)\} = Y(s).$$

This $y(t)$ is the solution of the original ODE.

Q3. Solve the initial value problem

$$y'' + y = \sin t, \qquad y(0) = 2, \quad y'(0) = 1$$

using the method of Laplace transform.

Note that these examples are equations with constant coefficients. (If the coefficients are not constants, then applying the Laplace transform may not yield an algebraic equation in $Y(s)$.) Thus, you may ask, *What is an advantage of the Laplace method?* Next time, we'll counter cases when the right-hand-side of an equation is discontinuous. This is a true 'playground' for Laplace transforms.