Not Closed Sets and Continuous Functions

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Postulate (Archimedes)**.** For any $x \in \mathbf{R}$, there exists $n \in \mathbf{N}$ such that $x < n$.

Definition 17 (convergent sequence)**.** If $\{x_i\}_{i\in\mathbf{N}} \subseteq \mathbf{R}$, then we say *converges to* x, written as $x_i \to x$, if and only if for every open set U containing x, there is an $N \in \mathbf{N}$ such that $x_i \in U$ for all $i > N$.

Definition 18 (limit point in $\mathbf{R}$)**.** Let A be a subset of $\mathbf{R}$ and x be a point in $\mathbf{R}$. Then x is a *limit point* of A if and only if for every open set U containing x, the intersection $(U \setminus \{x\}) \cap A$ is not empty.

Lemma. *A real number x is a limit point of the set $A \subseteq \mathbf{R}$ if and only if $x = \lim a_i$ for some sequence of real numbers $\{a_i\}_{i\in\mathbf{N}}$ contained in a set A satisfying $a_i \neq x$ for all $i \in \mathbf{N}$.*

Proof. Let A be a subset of the real numbers.

($\Rightarrow$) Suppose a real number x is a limit point of A. Let us construct a sequence: for each $i \in \mathbf{N}$, consider the radius $1/i > 0$; by definition 18 there exists an real number not x (call it a_i) such that

$$a_i \in B(x, 1/i) \cap A.$$

We have defined each term $a_i \neq x$ in the sequence of real numbers $\{a_i\}_{i\in\mathbf{N}} \subseteq A$. To show $\{a_i\}$ converges to x, consider any open set of real numbers U which contains x. By the definition of an open set in $\mathbf{R}$, there exists $\varepsilon > 0$ such that

$$B(x, \varepsilon) \subseteq U.$$

We desire to find an index N after which all terms of the sequence $\{a_i\}$ lie within the open ball $B(x, \varepsilon)$. To do this, choose[1] $N \in \mathbf{N}$ such that

$$N > 1/\varepsilon.$$

Consider the index $j > N$. It is clear

$$j > \frac{1}{\varepsilon} \text{ if and only if } \varepsilon > \frac{1}{j}.$$

[1]We know N exists by Archimedes' postulate.

By construction, the term a_j is contained in $B(x, 1/j)$. Further, the radius $1/j$ is less than ε. Taking both these facts in stride, we know

$$a_j \in B(x, 1/j) \subseteq B(x, \varepsilon).$$

Knowing $B(x, \varepsilon) \subseteq U$, we conclude

$$a_j \in U$$

which is precisely the condition (elaborated in definition 17) to be met for us to say that $\{a_i\}$ converges to x.

($\Leftarrow$) Suppose A contains the sequence $\{a_i\}_{i \in \mathbf{N}}$ where $a_i \neq x$ for all $i \in \mathbf{N}$. Further suppose $\{a_i\}$ converges to x. Let U be an open set containing x. As a consequence of definition 17, we know: there exists $N \in \mathbf{N}$ such that $i > N$ implies $a_i \in U$. Choose the least index[2] j greater than N. Because $a_j \neq x$ is an element of both $\{a_i\}_{i \in \mathbf{N}}$ and U, we know

$$a_j \in (U \setminus \{x\}) \cap \{a_i\}.$$

Because $\{a_i\} \subseteq A$, we know

$$a_j \in (U \setminus \{x\}) \cap A.$$

It follows that

$$(U \setminus \{x\}) \cap A \neq \emptyset$$

which is exactly the condition (elaborated in definition 18) to be must be met for x to be a limit point of A. $\square$

Definition 19 (closed set in $\mathbf{R}$)**.** A set in $\mathbf{R}$ is *closed* if and only if it contains all of its limit points.

Example 40. If $X \subseteq \mathbf{R}$ is finite, then X is closed.

Example 47. Each singleton set $\{1/n\}$ (for all natural numbers n) is closed, but their union (the set of harmonic numbers) is not closed.

Tedious Proof. For all natural numbers n, let $\{1/n\}$ be the singleton set containing the multiplicative inverse of n. Because each set $\{1/n\}$ is finite, each set is closed (see example 40). Let us denote the union

$$\begin{aligned} \bigcup_{n \in \mathbf{N}} \left\{ \frac{1}{n} \right\} &= \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \right\} \\ &:= \mathcal{H}, \end{aligned}$$

where $\mathcal{H}$ represents the set of harmonic numbers.

We will show that $\mathcal{H}$ does not contain its limit point 0. First, let's verify 0 is a limit point of $\mathcal{H}$. To do this, assume U is an open set which contains 0. By definition of an open set in $\mathbf{R}$, there is radius $\varepsilon > 0$ such that the open ball $B(0, \varepsilon)$ is contained in U. We now

[2] The set $\{i \in \mathbf{N} : i > N\}$ is a nonempty subset of the natural numbers, so it has a least element j.

desire a harmonic number which is less than ε. By Archimedes' postulate, we can find a natural number $N > 1/\varepsilon$ such that the harmonic number

$$\frac{1}{N} < \varepsilon.$$

So follows

$$\frac{1}{N} \in B(0,\varepsilon) \subseteq U.$$

Because $\frac{1}{N} \neq 0$, we have

$$\frac{1}{N} \in (U \setminus \{0\}) \cap \mathcal{H},$$

which is exactly the condition which must be met for 0 to be a limit point of $\mathcal{H}$ (see definition 18). As the harmonic numbers are the multiplicative inverses of the natural numbers (each of which have finite magnitude), 0 is not contained in $\mathcal{H}$. We conclude $\mathcal{H}$ is not closed (see definition 19). $\square$

Fancy Proof. For each natural number n, consider the set $\{1/n\}$. Because the set $\{1/n\}$ is finite, it is closed. Now see the union

$$\bigcup_{n\in\mathbf{N}} \left\{\frac{1}{n}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}$$

is identically the sequence $\{1/n\}_{n\in\mathbf{N}}$ which converges[3] to 0. Because there does not exist a natural number n such that $1/n = 0$, we know the union does not contain 0. But, by the lemma, we know 0 is a limit point of the union. So the union is not closed. $\square$

Example 6 (b)**.** The set $\mathbf{Q}$ of rational numbers is neither open not closed in $\mathbf{R}$.

Proof. This is a proof by two contradictions—demonstrating first that $\mathbf{Q}$ cannot be open and second that $\mathbf{Q}$ cannot be closed.

First, suppose that $\mathbf{Q}$ is open. Then there exists an open ball $B(0,\varepsilon)$ containing 0 such that $B(0,\varepsilon) \subseteq \mathbf{Q}$. Because $\mathbf{Q}$ is dense in $\mathbf{R}$, there exists a rational number r such that $0 < r < \varepsilon$. As $0 < r/\sqrt{2} < \varepsilon$, we know the point $r/\sqrt{2}$ is an element of $B(0,\varepsilon)$. With $r \in \mathbf{Q}$, are we to accept that $r/\sqrt{2} \in \mathbf{Q}$? Certainly not. It follows that $\mathbf{Q}$ cannot be open.

Second, suppose that $\mathbf{Q}$ is closed. Then $\mathbf{Q}$ contains all of its limit points. Consider the sequence of rational[4] numbers

$$\{a_i\}_{i\in\mathbf{N}} = \left\{1, 1+\frac{1}{2}, 1+\frac{1}{2+\frac{1}{2}}, 1+\frac{1}{2+\frac{1}{2+\frac{1}{2}}}, \dots\right\}.$$

This proof would be rigorous if we knew the $\{a_i\}$ converge to some real number, call it x. For the sake of example, let's suppose this is a fact. As $a_i \to x$, we have

$$x = 1 + \frac{1}{2+\frac{1}{2+\frac{1}{2+\frac{1}{2+\dots}}}}.$$

[3]For any $\varepsilon > 0$, there exists $N > 1/\varepsilon$ such that $n > N$ implies $|1/n - 0| < \varepsilon$.

[4]Because $\mathbf{Q}$ is an ordered field, i.e., closed under the usual addition and multiplication as well as containing inverses, each term in the sequence is indeed a rational number.

If $\mathbf{Q}$ is closed (and we are supposing it is closed), then (by the lemma) $x \in \mathbf{Q}$. We will see that $x \in \mathbf{Q}$ produces contradiction. Rearranging,

$$x = 1 + \cfrac{1}{1 + 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}}.$$

Since x is defined as the limit of the a_i, we may substitute

$$x = 1 + \frac{1}{1 + x}.$$

We proceed to find

$$\begin{aligned} x - 1 &= \frac{1}{1 + x}, \\ (x - 1)(x + 1) &= 1, \\ x^2 - 2 &= 0, \\ x &= \sqrt{2} \end{aligned}$$

(taking the positive root as x in accordance with the increasing, positive, sequence). Are we to accept that $\sqrt{2} \in \mathbf{Q}$? Certainly not. It follows that $\mathbf{Q}$ cannot be closed. $\square$

Definition 20 (continuous function in $\mathbf{R}$)**.** A function $f \colon D \subseteq \mathbf{R} \to \mathbf{R}$ is *continuous at x* if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $z \in D$ and $|x - z| < \delta$ then $|f(x) - f(z)| < \varepsilon$. We say that f is *continuous* if it is continuous for every point x in its domain D.

Definition 21 (continuous function in $\mathbf{R}$)**.** A function $f \colon \mathbf{R} \to \mathbf{R}$ is *continuous* if and only if[5] for every open set $U \in \mathbf{R}$, $f^{-1}(U)$ is open in $\mathbf{R}$.

Theorem 48. *The two definitions of continuity for real-valued functions on* $\mathbf{R}$ *are equivalent.*

Example 6 (c)**.** Every linear polynomial gives a continuous function (from $\mathbf{R}$ to $\mathbf{R}$).

Distance Proof. Suppose we have a linear polynomial $f \colon \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = kx, \text{ for some real } k.$$

Consider a point $x \in \mathbf{R}$ and a real radius $\varepsilon > 0$. Given any point $z \in \mathbf{R}$ within a distance of ε to x, we will demonstrate the existence of a real radius $\delta > 0$ that bounds the the image $f(z)$ within a distance of δ to the image $f(x)$. To do this, let[6] $\varepsilon = |k|\delta$. Then

$$|x - z| < \delta$$

[5] Metonym: "Given open U, f-inverse U is open too."

[6] If $k = 0$, the image of $\mathbf{R}$ is 0, which is bounded by *any* $\delta > 0$.

implies that

$$\begin{aligned}
|x - z| &< \frac{\varepsilon}{|k|} \\
|k||x - z| &< \varepsilon \\
|kx - kz| &< \varepsilon \\
|f(x) - f(z)| &< \varepsilon
\end{aligned}$$

which is exactly the condition which must be met (see definition 20) for f to be continuous at $x \in \mathbf{R}$. Because we have only required x to be real, we may generalize: f is continuous (at all points in $\mathbf{R}$). $\square$

Open Set Proof. Suppose we have a linear polynomial $f\colon \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = kx, \text{ for some real } k.$$

We desire proof 'for every open subset U of the real numbers, the inverse image $f^{-1}(U)$ is an open subset of the real numbers too.' Let us assume we have an open[7] set U that is a subset of the co-domain ($\mathbf{R}$) of f. Consider a real point[8]

$$x_0 \in f^{-1}(U).$$

Its image is

$$f(x_0) \in U.$$

Because U is open, a real radius $\delta > 0$ exists such that U contains the open ball $B(f(x_0), \delta)$. Choose $\varepsilon = \frac{\delta}{k}$. Suppose x is a real number. The following statements are equivalent:[9]

$$\begin{aligned}
x &\in B(x_0, \varepsilon), \\
x_0 - \varepsilon < x &< x_0 + \varepsilon \\
x_0 - \frac{\delta}{k} < x &< x_0 + \frac{\delta}{k}, \\
kx_0 - \delta < kx &< kx_0 + \delta, \\
kx &\in B(kx_0, \delta), \\
f(x) &\in B(f(x_0), \delta).
\end{aligned}$$

So, whenever x is in $B(x_0, \varepsilon)$, $f(x)$ is in $B(f(x_0), \delta)$. Excellent. Because U contains $B(f(x_0), \delta)$, whenever x is in $B(x_0, \varepsilon)$, we know $f(x)$ is in U. It follows that, whenever x is in $B(x_0, \varepsilon)$, x is in $f^{-1}(U)$. This is exactly the condition which must be met for $f^{-1}(U)$ to be open. Examining definition 21, we conclude that f is continuous on $\mathbf{R}$. $\square$

[7]If $f(x) = 0$ for all $x \in \mathbf{R}$, then an *open* set $U \subseteq$ in the co-domain of f does not exist. In which case f is continuous (if only vacuously.)

[8]Does one always exist? Given $f(x) = kx$, yes, but for general function $f\colon R \to R$, I cannot say.

[9]We are assuming $k \neq 0$.