Blackbody Radiation

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Rate of Radiation

"The total intensity radiated over all wavelengths increases as the temperature increases." Whence the Stefan-Boltzmann Law

$$\frac{dQ}{dt} = A\sigma T^4,$$

where $\sigma = 5.67 \times 10^{-8} J\cdot s^{-1}\cdot m^{-2}\cdot K^{-4}$.

It should be that

$$\frac{dQ}{dt} \propto u,$$

where u is the *energy density*. Well, let

$$u(\lambda)d\lambda$$

be the radiation energy per unit volume with wavelength in the range λ to $\lambda + d\lambda$. We must choose $u(\lambda)$ such that

$$\frac{dQ}{dt} = \frac{Ac}{4}\int_0^\infty u(\lambda)d\lambda.$$

Density Functions

Wein, in 1896, proposed

$$u(\lambda) \propto \frac{e^{-c_1/\lambda T}}{\lambda^5},$$

but this fails to describe long-wavelengths.

Rayleigh, in 1900, proposed

$$u(\lambda) \propto \frac{T}{\lambda^4}$$

but this produces catastrophe. Consider, for any $\lambda_{max} > 0$,

$$u_{\text{shortwave}} \propto T\int_0^{\lambda_{max}} \frac{d\lambda}{\lambda^4}$$

Planck's Distribution

Planck, in 1900, proposed

$$u(\lambda) = \frac{8\pi hc}{\lambda^5 (e^{hc/\lambda kT} - 1)}.$$

Notice that the integral converges. Let $z = hc/\lambda kT$, then

$$\begin{aligned} u &= \int_0^\infty u(\lambda) d\lambda \\ &= 8\pi \left(\frac{k^4 T^4}{h^3 c^3} \right) \int_0^\infty \frac{z^3}{e^z - 1} \\ &= \frac{8\pi^5}{15} \left(\frac{k^4 T^4}{h^3 c^3} \right). \end{aligned}$$

Bose Gases

> [Consider] the distribution of the energy U among N oscillators of frequency [f]. If U is viewed as divisible without limit, then an infinite number of distributions are possible. We consider however—and this is the essential point of the whole calculation—U as made up of an entirely determined number of finite equal parts, and we make use of the natural constant $h = [6.626 \times 10^{-34} J \cdot s]$. This constant when multiplied by the common frequency of the oscillators gives the element of energy ε. (Max Planck, 1900)

Modeling a Photon Gas

Recall: the allowed energies of the quantum harmonic oscillator are

$$E_n = 0, 2hf, 2hf, \ldots,$$

the partition function is

$$Z = \frac{1}{1 - e^{-hf/kT}},$$

the average energy is

$$\overline{E} = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} = \frac{hf}{e^{hf/kT} - 1},$$

and the average number of hf "units" in oscillator is

$$\overline{n}_{PL} = \frac{1}{e^{hf/kT} - 1}.$$

A "photon" is a ultra-relativistic boson with no chemical potential. So the occupancy is given by $\overline{n}_{PL} = \overline{n}_{BE}$. In a box of volume $V = L^3$, we have

$$\begin{aligned}
\epsilon &= hf && \text{from the photoelectric relation} \\
&= \frac{hc}{\lambda} && \text{where } c \text{ is the speed of light} \\
&= \frac{hc|\vec{n}|}{2L} && \text{where } \vec{n} \in \mathbb{N}^3
\end{aligned}$$

The Planck Distribution becomes a function of ϵ:

$$\overline{n}_{PL}(\epsilon) = \frac{1}{e^{\epsilon/kT} - 1}.$$

> From one point of view, we can analyze the electromagnetic field in a box or cavity in terms of a lot of harmonic oscillators, treating each mode of oscillation according to quantum mechanics as a harmonic oscillator. From a different point of view, we can analyze the same physics in terms of identical Bose particles. And the results of both ways of working *are always in exact agreement.* There is no way to make up your mind whether the electromagnetic field is really to be described as a quantized harmonic oscillator or by giving how many photons are in each condition. The two views turn out to be mathematically identical. (Richard Feynman, 1965)

Characteristics of Photon Gas

Let's consider the black box at a temperature T. The number of photons in thermal equilibrium at temperature T in a volume V is

$$\begin{aligned}
N &= 2\sum_{n_x}\sum_{n_y}\sum_{n_z}\overline{n}_{PL}(\epsilon) \\
&= 2\sum_{n_x,n_y,n_z}\frac{1}{e^{\epsilon/kT}-1} \\
&= 2\sum_{n_x,n_y,n_z}\frac{1}{e^{hc|\vec{n}|/2LkT}-1} \\
&= 2\int_0^{\infty} dn \int_0^{\pi/2} d\theta \int_0^{\pi/2} d\phi n^2 \sin\theta \frac{1}{e^{hcn/2LkT}-1}
\end{aligned}$$

$$= 2\left(\frac{\pi}{2}\right)\int_0^\infty \frac{n^2}{e^{hcn/2LkT}-1}dn \qquad \text{area of an eight of the unit sphere}$$
$$= \pi\left(\frac{2L}{hc}\right)^3\int_0^\infty \frac{\epsilon^2}{e^{\epsilon/kT}-1}d\epsilon \qquad \text{changed variables to } \epsilon = hcn/2L$$
$$= \pi\left(\frac{2LkT}{hc}\right)^3\int_0^\infty \frac{x^2}{e^{x}-1}dx \qquad \text{changed variables to } x = \epsilon/kT$$
$$= 8\pi V\left(\frac{kT}{hc}\right)^3\Gamma(3)\zeta(3)$$
$$= (2.4041)8\pi V\left(\frac{kT}{hc}\right)^3.$$

Consequently, the average energy of these photons is

$$\begin{aligned}
U &= 2\sum_{n_x}\sum_{n_y}\sum_{n_z}\epsilon\overline{n}_{PL}(\epsilon)\\
&= 2\sum_{n_x,n_y,n_z}\frac{hc|\vec{n}|}{2L}\frac{1}{e^{hc|\vec{n}|/2LkT}-1}\\
&= 2\int_0^\infty dn\int_0^{\pi/2}d\theta\int_0^{\pi/2}d\phi n^2\sin\theta\frac{hcn}{2L}\frac{1}{e^{hcn/2LkT}-1}\\
&= \cdots\\
&= 8\pi V\left(\frac{k^4T^4}{h^3c^3}\right)\int_0^\infty\frac{x^3}{e^x-1}dx.
\end{aligned}$$

Bibliography

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