NOTES ON DERIVED ALGEBRAIC GEOMETRY

COLTON GRIFFIN

When studying any kind of cohomology theory, one defines it as a certain family of quotients related to a cochain complex. For example, given a smooth manifold M, we may consider the de Rham complex $(C^{\bullet}_{\mathrm{dR}}(M), d)$ of differential forms on M with the exterior derivative d. The de Rham cohomology $H^{\bullet}_{\mathrm{dR}}(M)$ is given by the space of closed forms (that is, differential forms ω such that $d\omega = 0$) modulo exactness (that is, we identify $\omega_1 \sim \omega_2$ if $\omega_1 - \omega_2 = d\tau$ for some differential form τ). This has the structure of a graded $\mathbb{R}$-algebra.

If one is only concerned with constructing invariants of geometric objects (with respect to some kind of equivalence), then it suffices to consider cohomology in many applications. Indeed, cochain complexes associated to some geometric object X are generally not unique, and many different cochain complexes can provide the same cohomology theory. For example, the de Rham theorem asserts that de Rham cohomology is isomorphic to the singular cohomology $H^{\bullet}_{\mathrm{sing}}(M; \mathbb{R})$ with coefficients in $\mathbb{R}$.

In algebraic geometry, working at the level of cohomology is sometimes not enough. The classical Bézout formula says that if $X, Y \subseteq \mathbb{CP}^2$ are smooth algebraic curves of degrees n and m that intersect transversely (no tangent intersections), then the number of intersection points is nm. Cohomologically speaking, we can assign to a curve X its fundamental class $[X] \in H^2(\mathbb{CP}^2; \mathbb{Z})$. If X and Y intersect transversely, then we have $[X] \smile [Y] = [X \cap Y]$ (here $\smile$ is the cup product), where $[X \cap Y] \in H^4(\mathbb{CP}^2; \mathbb{Z}) \cong \mathbb{Z}$ just counts the intersection number. Here the intersection is given by the fibered product $X \cap Y = X \times_{\mathbb{CP}^2} Y$. This formula does not work if the intersection is not transverse, but there are generalizations of this formula that do correctly count the intersection number. The correct formula is given by Serré's intersection formula, which uses Tor homology theory to correctly calculate the intersections. The downside to this formula is that it does not have such a nice interpretation in terms of fibered products. From a derived algebro-geometric perspective, the fix here is to use the derived fibered product, which (in the right cochain complex) gives us the correct generalization of Bézout's formula. Bézout's formula lies in the realm of intersection theory, which in some sense motivated the development of derived algebraic geometry as its own subject of study.

The language of ∞-categories is the correct context to develop derived geometry since relations are not required to hold up to a strict notion of isomorphism or equivalence. With ∞-categories, we allow for an infinite chain of relations holding up to other relations, which hold up to other relations, which... and so on. This is a good setting to study objects like cochain complexes, for example, and the derived category of cochain complexes associated to an abelian category (which derived geometry takes its name from) can be shown to display the structure of an ∞-category (see Section 2.4).

The goal of these notes is to introduce the basic ideas of derived algebraic geometry from the perspective of ∞-categories. More specifically, we will treat derived algebraic geometry as a theory of geometric objects whose local geometry is that of derived (or simplicial) commutative rings. Lurie has developed this theory as well as a higher

Date: August 3, 2025.

algebraic version of algebraic geometry called spectral algebraic geometry [Lur3], but we will not cover this.

We make no claim to originality; after all, we are not presenting any novel results. This writing is based heavily on the work of Adeel Khan [Kha1], [Kha3], [Kha4], [Kha5]. We recommend reading his work for proofs of some of the statements we omit. Of course, we also recommend Lurie's work for his treatment of ∞-categories and derived geometry [Lur1], [Lur2] [Lur3].

We assume the reader is familiar with basic category theory, but otherwise nothing else. Familiarity with algebraic geometry would be helpful to understand the motivation of these notes, but it is not strictly necessary. We will often omit proofs as derived algebraic geometry is a very dense subject, but we will try to leave references to where the proofs can be found.

Acknowledgments. We thank Mona Merling for advising us on this writing for the minor portion of the oral exam.

Notation. Here is a list of notational choices.

- Categories and ∞-categories will be referred to in bold font, such as **Set** or $\mathbf{Grpd}_\infty$. An arbitrary category or ∞-category will be denoted as $\mathcal{C}$.
- If $\mathcal{C}$ is a category, then we refer to its object class as $\mathcal{C}$ itself and not $\mathrm{Ob}\,\mathcal{C}$.
- All rings are commutative, associative, and unital unless stated otherwise.
- Whenever we write diagrams in a category, solid arrows (such as $\longrightarrow$) refer to given data in category, and dashed arrows (such as $\dashrightarrow$) refer to morphisms that are stated to exist as part of a theorem. Usually, we write $\exists$ above it as well. Informally, such a diagram means that the dashed arrow can be "filled in" with an actual arrow such that the resulting diagram commutes.
- Hom, Mor, and Maps all mean the same thing (referring to a class of morphisms in a category or ∞-category), but we use them in different contexts. Generally, we use Hom when referring to morphisms of algebraic objects, Mor for morphisms in an abstract category, and Maps for morphisms of geometric objects.

Convention. Here are some conventions on names of categories (by category we mean 1-category).

(1) Given two categories $\mathcal{C}$ and $\mathcal{D}$ we denote the category of functors $F\colon \mathcal{C} \to \mathcal{D}$ to be $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. All functors are assumed to be covariant.
(2) We write the opposite category of $\mathcal{C}$ as $\mathcal{C}^{\mathrm{op}}$, which is the category formed by flipping the direction of all morphisms in $\mathcal{C}$. A presheaf is just a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$, that is a contravariant functor.
(3) Throughout the notes, we will refer to $(n, 1)$-categories as n-categories and $(\infty, 1)$-categories as ∞-categories. We will occasionally break this convention, and we will try to be clear when we do so.
(4) We denote the category of sets as **Set**, whose objects are sets and morphisms are set maps.
(5) We denote the category of topological spaces as **Top**, whose objects are topological spaces and morphisms are continuous maps.
(6) We denote the category of commutative rings as **CRing**, whose objects are commutative rings and morphisms are ring homomorphisms. Rings are assumed to be associative and unital.
(7) We denote the category of R-modules for $R \in \mathbf{CRing}$ as $\mathbf{Mod}_R$, whose objects are R-modules and morphisms are R-module homomorphisms.
(8) We denote **Grpd** as the 2-category of groupoids, whose objects are groupoids (small categories whose morphisms are all invertible), 1-morphisms are functors

between groupoids, and 2-morphisms are (invertible) natural transformations of functors.

1. Background in ∞-categories

We begin by giving a basic introduction to ∞-categories and sheaves. We start with a very heavy warning.

Remark 1.0.1. ∞-categories are similar to categories in the sense that one has a class of objects together with some way of relating them via morphisms. The perspective we apply for ∞-categories more or less requires the object class to be small (that is, a set), but many commonplace categories are not small, such as **Set**. From a foundations perspective, it is desirable to avoid large classes since they are outside the purview of set theory.

While it seems nice to allow oneself to work with sets of arbitrary size, in practice most sets one encounters have relatively small cardinality (maybe unless you are a logician). It is reasonable, then, to require the *universe axiom*, which asserts that every set (that you care about, at least) is contained in some universal set U, sometimes called a *Grothendieck universe*. This bounds the size of **Set**, making it easier to work with. This introduces new issues, such as showing that certain constructions in category theory are in some sense independent of the choice of the universe.

We will not discuss universes any further in these notes, and we will largely ignore size issues, asking that the reader trust these can be resolved. For those desiring a better treatment of size issues with ∞-categories we recommend [Lur4, §4.7].

1.1. Simplicial sets.

Let Δ be the *simplex category* of finite ordinals $[n] = \{0, \ldots, n\}$ for $n \geq 0$ with (non-strictly) order-preserving maps. We may think of $[n]$ itself as a category whose objects are $0, \ldots, n$ and the morphisms are $i \to j$ whenever $i \leq j$.

Definition 1.1.1. The category of *simplicial objects* with values in a category $\mathcal{C}$ is the presheaf category $\mathbf{sSet}(\mathcal{C}) \overset{\text{def}}{=} \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$. The category of simplicial sets is the category $\mathbf{sSet} \overset{\text{def}}{=} \mathbf{sSet}(\mathbf{Set})$.

Remark 1.1.2. Equivalently, a simplicial set X consists of n-simplices of arbitrary dimensions $n \geq 0$ together with maps $X(\alpha)\colon X_m \to X_n$ for every order-preserving map $\alpha\colon [n] \to [m]$. These maps are compositions of two kinds of maps:

- The face maps $d_i^n\colon X_n \to X_{n-1}$ for $0 \leq i \leq n$, induced by $d^i\colon [n-1] \to [n]$. This is injective and misses i.
- The degeneracy maps $s_i^n\colon X_n \to X_{n+1}$ for $0 \leq i \leq n$, induced by $s^i\colon [n+1] \to [n]$ that is surjective and hits i twice.

One may check that X is uniquely determined by the sets X_n and the face and degeneracy maps, which need to satisfy the following *simplicial identities* (induced by the corresponding identities for d^i, s^i):

$$\begin{aligned}
d_i d_j &= d_{j-1} d_i \\
d_i s_j &= s_{j-1} d_i \\
d_j s_j &= 1 = d_{j+1} s_j \\
d_i s_j &= s_j d_{i-1} \\
s_i s_j &= s_{j+1} s_i
\end{aligned}$$

Example 1.1.3. The *standard n-simplex* Δ^n is the image of $[n]$ under the Yoneda embedding $\Delta \hookrightarrow \mathbf{sSet}$. The k-simplices are given by

$$\Delta^n_k = \mathrm{Hom}_\Delta([k],[n]).$$

Also note that the Yoneda lemma implies that for any simplicial set X, we have a bijection

$$\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, X) \cong X_n. \tag{1.1.1}$$

Definition 1.1.4. Here are some conventions:

(1) Given $0 \leq i \leq n$, we let $\partial_i\Delta^n \subset \Delta^n$ denote the sub-simplicial set generated by the ith face $d_i(\mathrm{id}_{[n]}) = d^i : [n-1] \to [n]$.
(2) We let $\partial\Delta^n \subset \Delta^n$ denote the sub-simplicial set generated by all its faces $\partial_i\Delta^n$. We set $\partial\Delta^0 = \varnothing$, the constant simplicial set with value the empty set. It is the initial object in $\mathbf{sSet}$.
(3) The *ith horn* $\Lambda^n_i \subseteq \Delta^n$ is generated by all faces except the ith one. It is called *inner* if $0 < i < n$ and *outer* otherwise.

Definition 1.1.5. An ∞*-category* is a simplicial set $\mathcal{C}$ such that every inner horn admits a filler. That is, for all $0 < i < n$ we have

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "\forall"] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

An ∞-category $\mathcal{C}$ is said to be an ∞*-groupoid* if the above condition holds without the word “inner.” A functor of ∞-categories or ∞-groupoids is a morphism of simplicial sets. We denote the corresponding ∞-categories as $\mathbf{Cat}_\infty$ and $\mathbf{Grpd}_\infty$.

Definition 1.1.6. Let X be a simplicial set, for example an ∞-category.

(1) The set X_0 is the set of *objects* of X. We write $x \in X$ instead of $x \in X_0$.
(2) The set X_1 is the set of *morphisms* in X. Given a morphism $f \in X_1$ we write $f : x_0 \to x_1$ for $x_0 = d_1(f)$ and $x_1 = d_0(f)$. These are called the *source* and *target* of f respectively.
(3) For an object $x \in X$, we define its *identity morphism* $\mathrm{id}_x \stackrel{\mathrm{def}}{=} s_0(x) \in X_1$.
(4) Given a 2-simplex $\sigma \in X_2$ with $d_2(\sigma) = f$, $d_0(\sigma) = g$, $d_1(\sigma) = h$, we write it graphically as follows:

$$\begin{tikzcd} & c_1 \arrow[dr, "g"] & \\ c_0 \arrow[ur, "f"] \arrow[rr, "h"'] & & c_2 \end{tikzcd}$$

In this situation, we say that h is a *composite* of f and g. We write $h \simeq g \circ f$.

Definition 1.1.7. Let X be a simplicial set.

(1) We say that two morphisms $f, g : x \to y$ are *homotopic* if there exists a 2-simplex $\sigma : \Delta^2 \to X$ of the form

$$\begin{tikzcd} & y \arrow[dr, "\mathrm{id}"] & \\ x \arrow[ur, "f"] \arrow[rr, "g"'] & & y \end{tikzcd}$$

This defines an equivalence relation $f \sim g$ on the set of morphisms $x \to y$.

(2) The *homotopy category* $\mathrm{h}(X)$ is the category whose set of objects is X_0 and, for $x, y \in X_0$, the set $\mathrm{Hom}_{\mathrm{h}(X)}(x, y)$ is the set of equivalence classes of morphisms $x \to y$. We can compose such equivalence classes and check that this gives rise to a well-defined category $\mathrm{h}(X)$.

Definition 1.1.8. Let X be an ∞-category. A morphism $f\colon x \to y$ is an *isomorphism* if there is a morphism $g\colon y \to x$ such that $f \circ g \simeq \mathrm{id}_y$ and $g \circ f \simeq \mathrm{id}_x$. One can show that a morphism is an isomorphism if and only if it induces an isomorphism in $\mathrm{h}(X)$.

Definition 1.1.9. Let $\mathcal{C}$ be a (small) category. We define its *nerve* $N(\mathcal{C})$ to be the simplicial set given as follows. For every $n \geq 0$, we set

$$N(\mathcal{C})_n = \mathrm{Fun}([n], \mathcal{C}).$$

For every $\alpha\colon [m] \to [n]$ in Δ, the induced map $\alpha^*\colon N(\mathcal{C})_n \to N(\mathcal{C})_m$ is given by composition: $([n] \to \mathcal{C}) \mapsto ([m] \xrightarrow{\alpha} [n] \to \mathcal{C})$.

More explicitly, the n-simplices are paths of length n in $\mathcal{C}$:

$$c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} c_n$$

The face map d_i takes this to

$$c_0 \xrightarrow{f_1} \dots \xrightarrow{f_{i-1}} c_{i-1} \xrightarrow{f_{i+1} \circ f_i} c_{i+1} \xrightarrow{f_{i+2}} \dots \xrightarrow{f_n} c_n$$

for $0 < i < n$, and it discards f_0 (resp. f_n) if $i = 0$ (resp. $i = n$). The degeneracy map s_i takes it to

$$c_0 \xrightarrow{f_1} \dots \xrightarrow{f_i} c_i \xrightarrow{\mathrm{id}_{c_i}} c_i \xrightarrow{f_{i+1}} \dots \xrightarrow{f_n} c_n$$

Lemma 1.1.10. *Every inner horn $\Lambda^n_i \to N(\mathcal{C})$ admits a unique filler $\Delta^n \to N(\mathcal{C})$. In particular, $N(\mathcal{C})$ is an ∞-category.*

Proposition 1.1.11. *The assignment $\mathcal{C} \mapsto N(\mathcal{C})$ determines a fully faithful functor $N\colon \mathbf{Cat} \to \mathbf{sSet}$. It admits a left adjoint given by $\mathrm{h}\colon \mathbf{sSet} \to \mathbf{Cat}$, the homotopy category functor. Moreover, for every category $\mathcal{C}$ we have $\mathcal{C} \simeq \mathrm{h}\, N(\mathcal{C})$.*

If $\mathcal{C}$ is a category, then $N(\mathcal{C})$ is a Kan complex if and only if $\mathcal{C}$ is a groupoid. Moreover, we have the following:

Lemma 1.1.12. *An ∞-category $\mathcal{C}$ is an ∞-groupoid if and only if every morphism is an isomorphism, or equivalently if $\mathrm{h}(\mathcal{C})$ is a groupoid.*

Definition 1.1.13. Given a topological space X, its *fundamental groupoid* $\Pi_\infty(X)$ is the simplicial set whose n-simplices are continuous maps $\Delta^n \to X$ from the standard topological n-simplex.

Theorem 1.1.14 (Milnor)**.** *Given a topological space X, the fundamental groupoid $\Pi_\infty(X)$ is an ∞-groupoid, and the assignment $X \mapsto \Pi_\infty(X)$ determines an equivalence from the homotopy category of topological spaces to the homotopy category of ∞-groupoids. That is, we have a homotopy equivalence $X \simeq Y$ if and only if $\Pi_\infty(X) \simeq \Pi_\infty(Y)$ as ∞-groupoids.*

There is a natural sense in which spaces form an ∞-category, denoted $\mathbf{Spc}$. This theorem suggests that we should really identify $\mathbf{Spc} \simeq \mathbf{Grpd}_\infty$.

Definition/Proposition 1.1.15. Let K be a simplicial set and $\mathcal{C}$ an ∞-category. We define a simplicial set $\mathrm{Fun}(K, \mathcal{C})$ with

$$\mathrm{Fun}(K, \mathcal{C})_n = \mathrm{Hom}_{\mathbf{sSet}}(K \times \Delta^n, \mathcal{C}).$$

This defines an ∞-category, using induced face and degeneracy maps. We call this the ∞-category of functors from K to $\mathcal{C}$. A natural transformation of functors $F \to G$ is a morphism in $\mathrm{Fun}(K, \mathcal{C})$.

Definition 1.1.16. Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories.

(1) We say F is an *equivalence* if there exists a functor $G \colon \mathcal{D} \to \mathcal{D}$ such that $F \circ G \simeq \mathrm{id}$ and $G \circ F \simeq \mathrm{id}$.
(2) By naturality of F, we induce maps of hom-sets
$$\mathrm{Hom}_{\mathcal{C}}(c_1, c_2) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(c_1), F(c_2)).$$
We say F is *fully faithful* if this map is a homotopy equivalence.
(3) We say F is *essentially surjective* if for every object $d \in \mathcal{D}$ there is $c \in \mathcal{C}$ such that $d \cong F(c)$.

Lemma 1.1.17. *A functor of ∞-categories is an equivalence if and only if it is essentially surjective and fully faithful.*

1.2. **Limits and colimits.** Here we discuss limits and colimits of ∞-categories. For more details, see [Lur4, 02VY].

Let $\mathcal{I}$ be a simplicial set and $\mathcal{C}$ an ∞-category. Given $X \in \mathcal{C}$, we denote $\underline{X} \colon \mathcal{I} \to \mathcal{C}$ as the constant functor to X, that is $\underline{X}(i) = X$ for all $i \in \mathcal{I}$. Note that this gives rise to a functor $\mathcal{C} \to \mathrm{Fun}(\mathcal{I}, \mathcal{C})$ given by mapping $f \colon X \to Y$ to a natural transformation $\underline{f} \colon \underline{X} \to \underline{Y}$.

Definition 1.2.1. Let $\mathcal{I}$ be a simplicial set and $\mathcal{C}$ an ∞-category. Let $F \colon \mathcal{I} \to \mathcal{C}$ be a map of simplicial sets. We say that $X \in \mathcal{C}$ is a *limit* of F if there exists a natural transformation $\alpha \colon \underline{X} \to F$ with the following universal property:

For every $Y \in \mathcal{C}$ the induced functor of ∞-groupoids
$$\mathrm{Hom}_{\mathcal{C}}(Y, X) \longrightarrow \mathrm{Hom}_{\mathrm{Fun}(\mathcal{I},\mathcal{C})}(\underline{Y}, F),$$
$$(Y \to X) \longmapsto (\underline{Y} \to \underline{X} \xrightarrow{\alpha} F)$$
is an equivalence. We write the limit of F as
$$X \simeq \varprojlim_{i \in \mathcal{I}} F(i) = \lim_{i \in \mathcal{I}} F(i).$$

We say that X is a *colimit* of F if it is a limit of $F^{\mathrm{op}} \colon \mathcal{I}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$, or equivalently if there is a natural transformation $\beta \colon F \to \underline{X}$ such that for all $Y \in \mathcal{C}$ the induced functor
$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathrm{Fun}(\mathcal{I},\mathcal{C})}(F, \underline{Y}),$$
$$(X \to Y) \longmapsto (F \xrightarrow{\beta} \underline{X} \to \underline{Y})$$
is an equivalence. We write the colimit of F as
$$X \simeq \varinjlim_{i \in \mathcal{I}} F(i) = \operatorname*{colim}_{i \in \mathcal{I}} F(i).$$

Definition 1.2.2. Given a simplicial diagram $X_\bullet : \Delta^{\mathrm{op}} \to \mathcal{C}$, we refer to its colimit as its *geometric realization* (assuming it exists), denoted as
$$|X_\bullet| \overset{\mathrm{def}}{=} \varinjlim_{[n] \in \Delta} X_n \in \mathcal{C}.$$

Remark 1.2.3. We note that (co)limits for ∞-categories are not like usual (co)limits, they correspond to homotopy (co)limits. For example, if X is a topological space, then the limit in $\mathbf{Grpd}_\infty$ of the diagram

$$\mathrm{pt} \xrightarrow{x} \Pi_\infty(X) \xleftarrow{x} \mathrm{pt}$$

is the loop space $\Omega_x(X)$, since a commutative square of the form

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & \mathrm{pt} \arrow[d, "x"] \\ \mathrm{pt} \arrow[r, "x"'] & \Pi_\infty(X) \end{tikzcd}$$

encodes a self-homotopy of the constant map $x : Y \to \Pi_\infty(X)$, that is a loop in X based at x. However, the limit of this diagram in **Top** is just a point. The homotopy limit, however, is the loop space $\Omega_x(X)$ based at x.

1.3. **Sites and sheaves.** Now we discuss sheaves, which are in some sense the primary object of study in algebraic geometry.

Definition 1.3.1. Given a category $\mathcal{C}$ and an object $x \in \mathcal{C}$, the *slice category* $\mathcal{C}_{/x}$ is the category whose objects are arrows $c \to x$ in $\mathcal{C}$ and morphisms are $c \to d$ whenever $(c \to d \to x) = (c \to x)$. We can also define the slice category $\mathcal{C}_{x/}$ consisting of arrows of the form $x \to c$.

Example 1.3.2. The category of R-algebras is $\mathbf{CAlg}_R \overset{\text{def}}{=} \mathbf{CRing}_{R/}$.

Definition 1.3.3. Let $\mathcal{C}$ be a category. A *Grothendieck topology* for $\mathcal{C}$ consists of a set τ of families of maps $\{\phi_i \colon U_i \to U\}_{i \in I}$ such that the following hold:

(1) For any isomorphism $V \xrightarrow{\phi} U$ we have $\{V \xrightarrow{\phi} U\} \in \tau$.
(2) If $\{U_i \to U\}_{i \in I} \in \tau$ and $\{V_{ij} \to U_i\}_{j \in J_i} \in \tau$ for all i, then $\{V_{ij} \to U\}_{i,j} \in \tau$.
(3) If $\{U_i \to U\}_i \in \tau$ and $V \to U$ is a morphism, then $U_i \times_U V$ exists for all i and $\{U_i \times_U V \to V\}_i \in \tau$.

A *site* is a category $\mathcal{C}$ together with a Grothendieck topology τ.

Definition 1.3.4. Let $\mathcal{C}$ be a locally small category. A *sieve on* $c \in \mathcal{C}$ is a subfunctor $S \subset \mathrm{Hom}(-, c)$. A Grothendieck topology on $\mathcal{C}$ may be equivalently defined as an assignment of each object $c \in \mathcal{C}$ to a collection of sieves on c, called *covering sieves*, such that:

(1) If $g : d \to c$ is a morphism and S is a covering sieve for c, then the pullback g^*S (defined as the collection of morphisms of the form $h\colon a \to d$ such that $g \circ h \in S$) is also a covering sieve.
(2) If F is a sieve on c such that $\bigcup_d \{g \mid d \to c \mid g^*S \text{ covers } d\}$ covers c, then F is a covering sieve for c.
(3) The *maximal sieve* $\mathrm{id} : \mathrm{Hom}(-, c) \to \mathrm{Hom}(-, c)$ is a covering sieve for c.

Remark 1.3.5. An ∞-categorical notion of Grothendieck topologies and sites can be given using the language of covering sieves (see [Lur1, Section 6.2.2]). This requires developing an ∞-categorical version of the slice category $\mathcal{C}_{/x}$, but this is not too difficult to do. However, it is known that there is a bijective correspondence between the Grothendieck (∞-)topologies on an ∞-category $\mathcal{C}$ and the Grothendieck topologies on $\mathrm{h}(\mathcal{C})$ (see [Lur1, Remark 6.2.2.3]), so we omit the definition. We refer to these ∞-categorical versions of sites as ∞-*sites*.

Example 1.3.6. Let X be a topological space. Then the category $\mathrm{Opens}(X)$ of opens $U \subseteq X$ (with morphisms given by $U \to V$ whenever $U \subseteq V$) admits a Grothendieck topology given by families $\{U_i \to U\}_{i \in I}$ such that $\bigcup_{i \in I} U_i = U$. When working with sheaves over a topological space, one often assumes this to be the underlying Grothendieck topology.

Example 1.3.7. Let X be a scheme (see Section 3 if you are not familiar with algebraic geometry already). The following is a nonexhaustive list of different Grothendieck topologies for schemes:

(1) The *small Zariski site* X_{Zar} is the full subcategory of $\mathbf{Sch}_{/X}$ consisting of open immersions $U \to X$, and the Grothendieck topology is given by families of open immersions $\{U_i \to X\}$ for which $\bigcup_i U_i = X$.
(2) The *big Zariski site* $(\mathbf{Sch}_{/X})_{\mathrm{Zar}}$ of a scheme X is the category $\mathbf{Sch}_{/X}$ together with the Grothendieck topology τ given by declaring $\{U_i \to U\}_i \in \tau$ if and only if the morphisms $U_i \to U$ are all open immersions and $\bigcup_i U_i = U$.
(3) The *small étale site* $X_{\text{ét}}$ and the *big étale site* $(\mathbf{Sch}_{/X})_{\text{ét}}$ of a scheme X is defined the same way as above, except we replace "open immersion" with "étale." When $X = \mathrm{Spec}(\mathbb{Z})$, we call this the *étale topology* on $\mathbf{Sch} = \mathbf{Sch}_{/\mathbb{Z}}$.
(4) The *small smooth site* X_{sm} and the *big smooth site* $(\mathbf{Sch}_{/X})_{\mathrm{sm}}$ of a scheme X is defined the same way as above except we replace "open immersion" with "smooth and locally of finite presentation." When $X = \mathrm{Spec}(\mathbb{Z})$, we call this the *smooth topology* on $\mathbf{Sch} = \mathbf{Sch}_{/\mathbb{Z}}$.
(5) The *big fppf site* $(\mathbf{Sch}_{/X})_{\mathrm{fppf}}$ is the category $\mathbf{Sch}_{/X}$ equipped with a Grothendieck topology given by defining a covering family $\{U_i \to U\}_i$ such that $\bigcup_i U_i \to U$ is surjective and $U_i \to U$ is flat and locally of finite presentation for all i.
(6) The *big fpqc site* $(\mathbf{Sch}_{/X})_{\mathrm{fpqc}}$ is the category $\mathbf{Sch}_{/X}$ equipped with a Grothendieck topology given by defining a covering family $\{U_i \to U\}_i$ such that $\coprod_i U_i \to U$ is faithfully flat and every quasi-compact open subset of U is the image of a quasi-compact open subset of $\coprod_i U_i$.

Definition 1.3.8. Let $(\mathcal{C}, \tau)$ be an ∞-site and $\mathcal{V}$ an ∞-category (if $\mathcal{C}$ is a 1-category, then we think of it as an ∞-category via the nerve construction). A *presheaf on $\mathcal{C}$ with values in $\mathcal{V}$* is a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$. A presheaf F is a *sheaf* if it satisfies the following conditions:

(1) F respects finite coproducts in $\mathcal{C}$, sending them to finite products in $\mathcal{V}$.
(2) For every morphism $f : U \to X$ in τ, let $U_\bullet$ denote the Čech nerve of f, that is the simplicial object

$$\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} U \times_X U \times_X U \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} U \times_X U \rightrightarrows U$$

in which the nth object is the n-fold fiber product of U with itself over X. Then the canonical map

$$F(X) \to \varprojlim_{[n] \in \Delta} F(U_n)$$

is invertible. Equivalently, $F(X)$ is the limit of $F(U_\bullet)$ via the diagram

$$\cdots \substack{\longleftarrow\\\longleftarrow\\\longleftarrow\\\longleftarrow} F(U \times_X U \times_X U) \substack{\longleftarrow\\\longleftarrow\\\longleftarrow} F(U \times_X U) \leftleftarrows F(U) \longleftarrow F(X)$$

We denote $\mathrm{Shv}(\mathcal{C}; \mathcal{V}) \subset \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{V})$ as the full subcategory of sheaves. We may also dualize this notion by flipping all of the arrows, giving the notion of a *precosheaf* and *cosheaf*.

Proposition 1.3.9. *Let $V^\bullet \colon \Delta \to \mathcal{V}$ be a cosimplicial diagram in an ∞-category $\mathcal{V}$. If $\mathcal{V}$ is equivalent to a 1-category, then the limit of $V^\bullet$ is identified with the equalizer of $V^0 \rightrightarrows V^1$:*

$$\varprojlim_{[n]\in\Delta} V^n \simeq \varprojlim(V^0 \rightrightarrows V^1).$$

Similarly, if $\mathcal{V}$ is equivalent to a 2-category, then the limit is identified with the 2-limit:

$$\varprojlim_{[n]\in\Delta} V^n \simeq \varprojlim(V^0 \rightrightarrows V^1 \substack{\rightarrow\\\rightarrow\\\rightarrow} V^2).$$

Remark 1.3.10. One may interpret ∞-categories as limits of n-categories in the following sense. We say that an ∞-category $\mathcal{V}$ is an *n-category* if all mapping groupoids $\mathrm{Maps}_{\mathcal{V}}(V, V')$ are $(n-1)$-truncated (that is they have trivial higher homotopy groups $\pi_i \mathrm{Maps}_{\mathcal{V}}(V, V') = 0$ for $i \geq n$).

In this case the limit of $V^\bullet$ is isomorphic to the limit of the restriction $V|_{\Delta_{\leq n}}$ to the full subcategory of Δ spanned by the objects $[0], [1], \ldots, [n]$. This follows from a variant of Quillen's Theorem A because the inclusion $\Delta_{\leq n} \hookrightarrow \Delta$ is an *n-final*, which is to say the category $\Delta_{\leq n} \times_\Delta \Delta_{/[m]}$ has n-connected nerve for every $[m] \in \Delta$.

Furthermore, note that a limit over $\Delta_{\leq 1}$ is isomorphic to the limit over the subcategory where the morphism $[1] \to [0]$ is discarded, i.e. it is the equalizer of the two parallel arrows $V^0 \twoheadrightarrow V^1$.

Construction 1.3.11. Let $\mathcal{C}$ be a 2-category (not necessarily a (2,1)-category). The *Duskin nerve* (or just *nerve*) of $\mathcal{C}$ is a simplicial set $N^D(\mathcal{C})$ in which the n-simplices are given by:

- Objects $C_i \in \mathcal{C}$ for $0 \leq i \leq n$;
- Morphisms $f_{ij} \colon C_i \to C_j$ for $1 \leq i < j \leq n$;
- 2-morphisms $\mu_{ijk} \colon f_{jk} \circ f_{ij} \Rightarrow f_{ik}$ for $0 \leq i < j < k \leq n$.

This data is required to satisfy some compatibility conditions involving μ_{ijl}, μ_{ikl}, and μ_{jkl} for $1 \leq i < j < k < l \leq n$.

The difference between $N^D(\mathcal{C})$ and the nerve of $\mathcal{C}$ as a 1-category is, more or less, that the diagrams

$$\begin{tikzcd} C_i \arrow[r, "f_{ij}"] \arrow[rr, bend right, "f_{ik}"'] & C_j \arrow[r, "f_{jk}"] & C_k \end{tikzcd}$$

Only commute up to the natural transformation μ_{ijk} (which is not necessarily invertible).

Theorem 1.3.12 (Duskin)**.** *Let $\mathcal{C}$ be a 2-category. The following are equivalent:*

(1) $\mathcal{C}$ is a (2,1)-category; i.e. the 2-morphisms of $\mathcal{C}$ are all invertible.
(2) $N^D(\mathcal{C})$ is a weak Kan complex (that is, an ∞-category).

Convention 1.3.13. In the language of the above remark, we will regard n-categories as ∞-categories satisfying the properties defined above. For example, a 1-category $\mathcal{C}$ may be regarded as an ∞-category via the nerve construction. Similarly, a (2,1)-category $\mathcal{D}$ may be regarded as an ∞-category via the Duskin nerve construction.

One important instance of a sheaf is when the site $\mathcal{C}$ is generated by a basis subcategory $\mathcal{C}_0$, as we define here:

Definition 1.3.14. Let $\mathcal{C}$ be a site with Grothendieck topology generated by a class of morphisms S. Let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full subcategory that is closed under fibered products and finite coproducts, and regard it with the Grothendieck topology generated by $S \cap \mathcal{C}_0$ (the subclass of morphisms in S whose source and target belong to $\mathcal{C}_0$). We say that $\mathcal{C}_0$

is a *basis* for $\mathcal{C}$ if for every object $X \in \mathcal{C}$ there exists a collection of morphisms $(Y_i \to X)_i$ such that:

(1) $Y_i \in \mathcal{C}_0$ for all i.
(2) The coproduct $\coprod_i Y_i$ exists in $\mathcal{C}$.
(3) $\coprod_i Y_i \to X$ belongs to S.

Theorem 1.3.15. *Let $\mathcal{V}$ be an ∞-category admitting limits. In the above situation, the functor*

$$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{V}) \to \mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathcal{V})$$

restricts to an equivalence

$$\mathrm{Shv}(\mathcal{C}; \mathcal{V}) \to \mathrm{Shv}(\mathcal{C}_0; \mathcal{V}).$$

Here is a more precise version of the above statements.

Definition 1.3.16. Let $i\colon \mathcal{C}_0 \hookrightarrow \mathcal{C}$ be a fully faithful functor of categories. Let $F_0\colon \mathcal{C}_0^{\mathrm{op}} \to \mathcal{V}$ be a presheaf with values in an ∞-category $\mathcal{V}$ admitting limits. The *right Kan extension* $F \overset{\mathrm{def}}{=} i_*(F_0)$ of F_0 is the unique limit-preserving functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$ that restricts to F_0. Explicitly, it is given by

$$F(X) \simeq \varprojlim_{(Y,f)} F(Y),$$

where the limit is taken over the category of pairs (Y, f) such that $Y \in \mathcal{C}_0$ and $f\colon i(Y) \to X$ is a morphism in $\mathcal{C}$ (where morphisms $(Y', f') \to (Y, f)$ are morphisms $Y' \to Y$ that are compatible with f and f').

Theorem 1.3.17. *Let $\mathcal{V}$ be an ∞-category admitting limits, and let $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$ be a presheaf. Then F is a sheaf if and only if the following conditions hold:*

(1) $F_0 \overset{\mathrm{def}}{=} F|_{\mathcal{C}_0}$ is a sheaf on $\mathcal{C}_0$.
(2) F is the right Kan extension of F_0 along $\mathcal{C}_0 \to \mathcal{C}$.

Proof. See [Aok, Section A.8] for a proof. $\square$

2. Pointed and Stable ∞-categories

In this section we cover the basic notions of a nice class of ∞-categories, namely stable ∞-categories. The writings of this section are largely based on [Lur2, §1], which we recommend viewing for more details on proofs of some statements.

2.1. Pointed ∞-categories.

Definition 2.1.1. An ∞-category $\mathcal{C}$ is *pointed* if it has an object $0 \in \mathcal{C}$ that is initial and final:

$$\mathrm{Mor}_{\mathcal{C}}(0, X) \simeq * \simeq \mathrm{Mor}_{\mathcal{C}}(X, 0), \qquad \forall X \in \mathcal{C}.$$

The object 0 is called a *zero object*.

Given a zero object $0 \in \mathcal{C}$, for any two objects $x, y \in \mathcal{C}$ we always have a zero morphism $0\colon x \to y$ given by the composition of the morphisms $x \to 0$ and $0 \to y$.

Lemma 2.1.2. *Let $\mathcal{C}$ be an ∞-category.*

(1) The space of zero objects in $\mathcal{C}$ is either empty or contractible.
(2) $\mathcal{C}$ is pointed if and only if there is an initial object $\varnothing$, a final object $*$, and a morphism $* \to \varnothing$. If either of these conditions is satisfied, then $\varnothing \simeq *$ is the zero object.

Definition 2.1.3. Let $\mathcal{C}$ be a pointed ∞-category. A *triangle* in $\mathcal{C}$ is a diagram[1] $\Delta^1 \times \Delta^1 \to \mathcal{C}$ of the form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ 0 \arrow[r] & Z \end{tikzcd}$$

A triangle is said to be a

- *fiber sequence* if it is a pullback.
- *cofiber sequence* if it is a pushout.

Sometimes a (co)fiber sequence is referred to as (co)exact.

Let $g : X \to Y$ be a morphism. A *kernel/fiber of* g is a fiber sequence

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d, "g"] \\ 0 \arrow[r] & Y \end{tikzcd}$$

A *cokernel/cofiber of* g is a cofiber sequence

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & W \end{tikzcd}$$

We write $W = \mathrm{fib}(g)$ and $Z = \mathrm{cofib}(g)$.

2.2. **Reminder on Spectra.** Let $X = (X, x_0)$ be a pointed topological space. Recall that the reduced suspension is defined to be

$$\Sigma X = (X \times I)/(X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times I).$$

Also recall that ΩX is the set of pointed loops at x_0 (with compact open topology)

Definition 2.2.1. A *(classical) spectrum* E is a collection of pointed topological spaces $(E_n)_{n\geq 0}$ together with structure maps $\Sigma E_n \to E_{n+1}$. A morphism of spectra $E \to E'$ is a collection of maps $f_n : E_n \to E'_n$ commuting with the structure maps.

We define the homotopy groups of a spectrum E as

$$\pi_n(E) = \mathrm{colim}_k\, \pi_{n+k}(E_k).$$

A spectrum E is a Ω-*spectrum* if the adjoints of the structure maps are weak equivalences.

Example 2.2.2. Here are two common examples of spectra.

(1) Given X we have the *suspension spectrum* $\{\Sigma^n X\}_n$ with identity morphisms. The *sphere spectrum* is when $X = S^0$, and its homotopy groups are the *stable homotopy groups of spheres*.
(2) Let G be abelian group. Recall the equivalence of Eilenberg-MacLane spaces $K(G, n) \simeq \Omega K(G, n+1)$. The spectrum $\{K(G, n)\}_n$ with bonding maps adjoint to this equivalence defines the *Eilenberg-MacLane spectrum associated to* G. This represents singular cohomology.

[1]Note that the diagram is, amusingly, not quite a triangle.

Loops and suspensions of spaces are given respectively by the (homotopy) pullback and pushout squares

$$\begin{array}{ccc} \Omega X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & X \end{array} \qquad \begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma X \end{array}$$

Definition 2.2.3. We define the stable homotopy category **SH** to be localization of spectra by weak equivalence (by weak equivalence of a pair $E \to F$ we mean that this map induces isomorphic cohomology theories).

2.3. **Spectra and stable ∞-categories.** Let $\mathcal{C}$ be a pointed ∞-category, and let M^{Σ} (respectively M^{Ω}) be the full subcategory of squares in $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ of the form

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0' & \longrightarrow & X' \end{array}$$

where 0 and $0'$ are zero objects of $\mathcal{C}$ and the above is a pushout square (resp. pullback). Assume that fibers (cofibers) exist. Then by [Lur1, Proposition 4.3.2.15], evaluation at X (resp. X') gives us a trivial Kan fibration $M^{\Sigma} \to \mathcal{C}$ (resp. $M^{\Omega} \to \mathcal{C}$). Let $s\colon \mathcal{C} \to M^{\Sigma}$ (resp. $s\colon \mathcal{C} \to M^{\Omega}$) be a section of this fibration, and let $e\colon M^{\Sigma} \to \mathcal{C}$ (resp. $e\colon M^{\Omega} \to \mathcal{C}$) be evaluation at X' (resp. X). Then let $\Sigma = e \circ s\colon \mathcal{C} \to \mathcal{C}$ (resp. $\Omega = e' \circ s'\colon \mathcal{C} \to \mathcal{C}$) defines the *suspension* (*loop space* functor).

Example 2.3.1. When $\mathcal{C} = \mathbf{Spc}_*$ is the ∞-category of pointed spaces, then Σ and Ω give the usual (reduced) suspension and loop space functors.

Definition/Proposition 2.3.2. A pointed ∞-category $\mathcal{C}$ is *stable* if it satisfies any of the following equivalent conditions:

1. $\mathcal{C}$ admits finite limits and the loop space functor $\Omega\colon \mathcal{C} \to \mathcal{C}$ is an equivalence.
2. $\mathcal{C}$ admits finite colimits and the suspension functor $\Sigma\colon \mathcal{C} \to \mathcal{C}$ is an equivalence.
3. $\mathcal{C}$ admits finite limits and colimits, and any commutative square is a pullback if and only if it is a pushout.
4. For every morphism $g : X \to Y$ fibers and cofibers exist, and a triangle in $\mathcal{C}$ is a fiber sequence if and only if it is a cofiber sequence.

Lemma 2.3.3. *Let $\mathcal{C}$ be pointed ∞-category with all the desired fibers/cofibers. Then Σ is left adjoint to Ω, and when $\mathcal{C}$ is stable they are equivalent.*

If $c \in \mathcal{C}$ is a final object, we can define the ∞-category of *pointed objects* $\mathcal{C}_*$ to be the full subcategory of $\mathcal{C}^{\Delta^1}$ consisting of morphisms $c \to d$.

Definition 2.3.4. Let $\mathcal{C}$ be an ∞-category with finite limits. We define its *stabilization* $\mathrm{sp}(\mathcal{C})$ to be the limit of the sequence

$$\mathcal{C}_* \xleftarrow{\Omega} \mathcal{C}_* \xleftarrow{\Omega} \dots$$

If $\mathcal{C} = \mathbf{Spc}$ (∞-category of spaces) we write $\mathbf{Sp} = \mathrm{sp}(\mathbf{Spc})$ for the ∞-*category of spectra*.

Proposition 2.3.5. *Let $\mathcal{C}$ be a ∞-category with finite limits. Then $\mathrm{sp}(\mathcal{C})$ is stable.*

We write $X \to X[n] = \Sigma^n X$ to be the nth power suspension and $X[-n] = \Omega^n X$ to be the nth power loop space.

2.4. **Triangulated categories.** To give some motivation as to why one would want to consider ∞-categories for algebraic geometry, we give a brief overview of [Lur2, §1.3] and the proof that a stable ∞-category is the "correct" generalization of a triangulated category. Triangulated and derived categories were first studied by Grothendieck and Verdier in the context of abelian categories.

Definition/Proposition 2.4.1. Here are two equivalent definitions of an additive category.

(1) A category $\mathcal{C}$ is *preadditive* if $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ has the structure of an abelian group, and composition of morphisms is bilinear. It can be shown that every finitary product in $\mathcal{C}$ is a finitary coproduct and vice versa. A preadditive category $\mathcal{C}$ is *additive* if it admits all finitary coproducts.
(2) A category $\mathcal{C}$ is *semiadditive* if it has a zero object and all binary biproducts. One can show that $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ has the structure of an abelian monoid for all objects X, Y. Then, a semiadditive category $\mathcal{C}$ is additive if all morphisms have additive inverses.

Furthermore, we say $\mathcal{C}$ is *pre-abelian* if it is additive and every morphism admits a kernel and a cokernel object.

Proposition 2.4.2. *Let $\mathcal{C}$ be a pre-abelian category. Then for every morphism $f : X \to Y$ there is a canonical decomposition*

$$X \twoheadrightarrow \mathrm{coker}(\ker(f) \to X) \to \ker(Y \to \mathrm{coker}(f)) \hookrightarrow Y.$$

Definition 2.4.3. A category $\mathcal{C}$ is *abelian* if it is pre-abelian and for every morphism $f\colon X \to Y$ the canonical map $\mathrm{coker}(\ker(f) \to X) \to \ker(Y \to \mathrm{coker}(f))$ is an isomorphism. A functor of abelian categories $F\colon \mathcal{A} \to \mathcal{B}$ is *additive* if for all $A, A' \in \mathcal{A}$ the induced map $\mathrm{Hom}_{\mathcal{A}}(A, A') \to \mathrm{Hom}_{\mathcal{A}}(F(A), F(A'))$ is a homomorphism of abelian groups.

Here we review some basic objects in the study of abelian categories.

Definition 2.4.4. Let $\mathcal{A}$ be an abelian category.

- Given a sequence of the form
$$A \xrightarrow{f} B \xrightarrow{g} C,$$
we say the sequence is *exact at* B if $g \circ f = 0$ and the induced map $\mathrm{im}(f) \hookrightarrow \ker(g)$ is an isomorphism.
- A *short exact sequence* is a sequence of the form
$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0,$$
which is exact at A, B, and C.
- A *(long) exact sequence* is a sequence of the form
$$\ldots \longrightarrow A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \longrightarrow \ldots,$$
which is exact at every term in the sequence.

Definition 2.4.5. Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor of abelian categories. We denote an arbitrary short exact sequence in $\mathcal{A}$ as

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.$$

The functor F induces a (not necessarily exact) sequence in $\mathcal{B}$

$$0 \longrightarrow F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \longrightarrow 0.$$

- We say F is *exact* if for every short exact sequence in $\mathcal{A}$ the induced sequence in $\mathcal{B}$ is exact.
- We say F is *left exact* if for every short exact sequence in $\mathcal{A}$ the induced sequence in $\mathcal{B}$
$$0 \longrightarrow F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$$
is exact.
- We say F is *right exact* if for every short exact sequence in $\mathcal{A}$ the induced sequence in $\mathcal{B}$
$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \longrightarrow 0$$
is exact.
- We say F is *half exact* if for every short exact sequence in $\mathcal{A}$ the induced sequence in $\mathcal{B}$
$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$$
is exact.

Lemma 2.4.6. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor of abelian categories.*

- *The functor F is exact if and only if for every exact sequence*
$$A \longrightarrow B \longrightarrow C$$
the induced sequence
$$F(A) \longrightarrow F(B) \longrightarrow F(C)$$
is exact.
- *The functor F is left exact if and only if for every exact sequence*
$$0 \longrightarrow A \longrightarrow B \longrightarrow C$$
the induced sequence
$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C)$$
is exact.
- *The functor F is left exact if and only if for every exact sequence*
$$A \longrightarrow B \longrightarrow C \longrightarrow 0$$
the induced sequence
$$F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0$$
is exact.

Definition/Proposition 2.4.7. Let $\mathcal{A}$ be a category.

- An object P is *projective* if for every epimorphism $g\colon P \to Y$ and morphism $f\colon X \twoheadrightarrow Y$ there is a morphism $\overline{g}\colon P \to X$ such that the following diagram commutes:
$$\begin{tikzcd} P \arrow[d, dashed, "\exists \overline{g}"'] \arrow[dr, "g"] & \\ X \arrow[r, two heads, "f"'] & Y \end{tikzcd}$$
If $\mathcal{A}$ is an abelian category, then P is projective if and only if the hom functor
$$\mathrm{Hom}(P, -)\colon \mathcal{A} \to \mathbf{Ab}$$
is exact.

- An object I is *injective* if for every monomorphism $f\colon X \hookrightarrow Y$ and morphism $g\colon X \to I$ there is a unique morphism $\overline{g}\colon Y \to I$ such that the following diagram commutes:

$$\begin{tikzcd} I & \\ X \arrow[u, "g"] \arrow[r, hook, "f"'] & Y \arrow[ul, dashed, "\exists \overline{g}"'] \end{tikzcd}$$

If $\mathcal{A}$ is an abelian category, then I is injective if and only if the hom functor

$$\operatorname{Hom}(-, I)\colon \mathcal{A} \to \mathbf{Ab}$$

is exact.

Definition 2.4.8. Let $\mathcal{A}$ be an additive category.

- A *chain complex* $(A_\bullet, d^A_\bullet)$ is a sequence of objects $A_\bullet = (A_i)_{i\in\mathbb{Z}}$ in $\mathcal{A}$ together with a sequence of morphisms $d^A_\bullet = (d^A_i\colon A_i \to A_{i-1})_{n\in\mathbb{Z}}$ such that $d_{i-1} \circ d_i = 0$ for all $i \in \mathbb{Z}$. We write this in a sequence as

$$\cdots \longleftarrow A_{i-1} \xleftarrow{d^A_i} A_i \xleftarrow{d^A_{i+1}} A_{i+1} \longleftarrow \cdots$$

A *morphism/chain map* of chain complexes $f_\bullet\colon (A_\bullet, d^A_\bullet) \to (B_\bullet, d^B_\bullet)$ is a sequence of morphisms $(f_i\colon A_i \to B_i)_{i\in\mathbb{Z}}$ such that the following diagram commutes:

$$\begin{tikzcd} \cdots & A_{i-1} \arrow[l] \arrow[d, "f_{i-1}"] & A_i \arrow[l, "d^A_i"'] \arrow[d, "f_i"] & A_{i+1} \arrow[l, "d^A_{i+1}"'] \arrow[d, "f_{i+1}"] & \cdots \arrow[l] \\ \cdots & B_{i-1} \arrow[l] & B_i \arrow[l, "d^B_i"'] & B_{i+1} \arrow[l, "d^B_{i+1}"'] & \cdots \arrow[l] \end{tikzcd}$$

Given a chain complex $A_\bullet$, we define its *ith homology object* to be

$$H_i(A_\bullet) \stackrel{\text{def}}{=} \ker d_i / \operatorname{im} d_{i+1}.$$

- A *cochain complex* $(A^\bullet, d^\bullet_A)$ is a sequence of objects $A^\bullet = (A^i)_{i\in\mathbb{Z}}$ in $\mathcal{A}$ together with a sequence of morphisms $d^\bullet_A = (d^i_A\colon A^i \to A^{i+1})_{i\in\mathbb{Z}}$ such that $d^{i+1} \circ d^i = 0$ for all $i \in \mathbb{Z}$. We write this in a sequence as

$$\cdots \longrightarrow A^{i-1} \xrightarrow{d^{i-1}_A} A^i \xrightarrow{d^i_A} A^{i+1} \longrightarrow \cdots$$

A *morphism/cochain map* of cochain complexes $f^\bullet\colon (A^\bullet, d^\bullet_A) \to (B^\bullet, d^\bullet_B)$ is a collection of morphisms $f^i : A^i \to B^i$ for all i such that the following diagram commutes:

$$\begin{tikzcd} \cdots \arrow[r] & A^{i-1} \arrow[r, "d^{i-1}_A"] \arrow[d, "f^{i-1}"] & A^i \arrow[r, "d^i_A"] \arrow[d, "f^i"] & A^{i+1} \arrow[r] \arrow[d, "f^{i+1}"] & \cdots \\ \cdots \arrow[r] & B^{i-1} \arrow[r, "d^{i-1}_B"] & B^i \arrow[r, "d^i_B"] & B^{i+1} \arrow[r] & \cdots \end{tikzcd}$$

Given a cochain complex $A^\bullet$, we define its *ith cohomology object* to be

$$H^i(A^\bullet) \stackrel{\text{def}}{=} \ker d^i / \operatorname{im} d^{i-1}.$$

Chain complexes and cochain complexes are equivalent up to change of notation, but we work with both.

Definition/Proposition 2.4.9. Let $f_\bullet, g_\bullet \colon A_\bullet \to B_\bullet$ be two morphisms of chain complexes. We say $f_\bullet$ and $g_\bullet$ are *(chain) homotopic* (denoted $f_\bullet \sim g_\bullet$) if there is a sequence of morphisms $(h_i \colon A_i \to B_{i+1})_{i\in\mathbb{Z}}$ such that for all $i \in \mathbb{Z}$

$$f_i - g_i = h_{i-1} \circ d_i^A + d_{i+1}^B \circ h_i.$$

If this is the case, then $f_\bullet$ and $g_\bullet$ induce the same morphism $H_i(A_\bullet) \to H_i(B_\bullet)$ on homology.

Let $f^\bullet, g^\bullet \colon A^\bullet \to B^\bullet$ be two morphisms of chain complexes. We say $f^\bullet$ and $g^\bullet$ are *(cochain) homotopic* (denoted $f \sim g$) if there is a sequence of morphisms $(h^i \colon A^i \to B^{i-1})_{i\in\mathbb{Z}}$ such that for all $i \in \mathbb{Z}$

$$f^i - g^i = h^{i-1} \circ d_A^i + d_B^{i+1} \circ h^i.$$

If this is the case, then $f^\bullet$ and $g^\bullet$ induce the same morphism $H^i(A^\bullet) \to H^i(B^\bullet)$ on homology.

Definition 2.4.10. A category $\mathcal{C}$ is *triangulated* if it is additive and has a morphism $T : X \mapsto X[1]$ and a collection of *distinguished/exact triangles*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

satisfying several properties:

(TR1) (a) Every morphism $X \xrightarrow{f} Y$ extends to a distinguished triangle (DT).
(b) The collection of DTs is stable under isomorphism.
(c) Given an object $X \in \mathcal{C}$, the diagram $X \xrightarrow{\mathrm{id}_X} X \longrightarrow 0 \longrightarrow X[1]$ is a DT.

(TR2) A diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is a DT if and only if the rotated diagram

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$

is a DT.

(TR3) We have a diagram of DTs

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "g"] \arrow[d] & Z \arrow[r, "h"] \arrow[d, dashed, "\exists"] & X[1] \arrow[d] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & X'[1]
\end{tikzcd}$$

(TR4) Given three DTs

$$X \xrightarrow{f} Y \xrightarrow{u} Y/X \xrightarrow{d} X[1]$$

$$Y \xrightarrow{g} Z \xrightarrow{v} Z/Y \xrightarrow{d'} Y[1]$$

$$X \xrightarrow{g\circ f} Z \xrightarrow{w} Z/X \xrightarrow{d''} X[1]$$

There exists a fourth DT:

$$\begin{tikzcd}
X \arrow[rr, "g\circ f"] \arrow[dr, "f"'] & & Z \arrow[rr, "v"] \arrow[dr, "w"] & & Z/Y \arrow[rr, dashed, "\theta"] \arrow[dr, "d'"] & & Y/X[1] \\
& Y \arrow[ur, "g"'] \arrow[dr, "u"] & & Z/X \arrow[ur, dashed, "\psi"'] \arrow[dr, "d''"] & & Y[1] \arrow[ur, "u[1]"'] & \\
& & Y/X \arrow[ur, dashed, "\phi"'] \arrow[rr, "d"] & & X[1] \arrow[ur, "f[1]"'] & &
\end{tikzcd}$$

Remark 2.4.11. Triangulated categories in general are notoriously difficult to work with. Usually they don't admit all colimits, and the axioms are rather complicated to work with. This structure arose from the study of derived categories for abelian categories. They are nice because they allow you to speak about homotopy colimits of chain complexes (which we will define shortly), but they don't allow you to work with any higher structure, making them somewhat limited. Nevertheless, they have many applications in algebraic geometry, particularly for D-modules.

Theorem 2.4.12. *Let $\mathcal{C}$ be a stable ∞-category. Then the homotopy category* $\mathrm{h}(\mathcal{C})$ *has the structure of a triangulated category.*

Sketch of proof. Our shift functor is Σ, which is an equivalence. We define a diagram of the form $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ to be a DT if there is a diagram

$$\begin{tikzcd}
X \arrow[r, "\widetilde{f}"] \arrow[d] & Y \arrow[r] \arrow[d, "\widetilde{g}"] & 0 \arrow[d] \\
0' \arrow[r] & Z \arrow[r, "\widetilde{h}"] & W
\end{tikzcd}$$

such that:

(1) $0, 0'$ are zero objects.
(2) Both squares are pushout diagrams.
(3) The morphisms $\widetilde{f}$ and $\widetilde{g}$ are f and g.
(4) The map $h : Z \to X[1]$ is the composition of (the homotopy class of) $\widetilde{h}$ with the equivalence $W \simeq X[1]$.

One may verify that $\mathrm{h}(\mathcal{C})$ is triangulated by checking all of the axioms.

Definition 2.4.13. Let $\mathcal{D}$ be a triangulated category. A t-structure on $\mathcal{D}$ is a pair of full subcategories $\mathcal{D}_{\leq 0}$, $\mathcal{D}_{\geq 0}$ (stable under isomorphisms) such that:

(1) For $X \in \mathcal{D}_{\geq 0}$, $Y \in \mathcal{D}_{\leq 0}$ we have $\mathrm{Hom}_{\mathcal{D}}(X, Y[-1]) = 0$.
(2) We have inclusions $\mathcal{D}_{\geq 0}[1] \subseteq \mathcal{D}_{\geq 0}$, $\mathcal{D}_{\leq 0}[-1] \subseteq \mathcal{D}_{\leq 0}$.
(3) For all $X \in \mathcal{D}$ there is a fiber sequence $X' \to X \to X''$ such that $X' \in \mathcal{D}_{\geq 0}$ and $X'' \in \mathcal{D}_{\leq 0}[-1]$.

A t-structure on a stable ∞-category $\mathcal{C}$ is a t-structure on $\mathrm{h}\mathcal{C}$, and we denote $\mathcal{C}_{\leq 0}$, $\mathcal{C}_{\geq 0}$ as the full subcategories of $\mathcal{C}$ given by objects belonging to $(\mathrm{h}\mathcal{C})_{\leq 0}$, $(\mathrm{h}\mathcal{C})_{\geq 0}$. We define the *heart* to be $\mathcal{C}^{\heartsuit} = \mathcal{C}_{\leq 0} \cap \mathcal{C}_{\geq 0}$.

If $\mathcal{D}$ is the derived category associated to an abelian category, then we let

$$\mathcal{D}_{\leq 0} \simeq \{X \mid \forall i > 0,\, H^i(X) = 0\}$$
$$\mathcal{D}_{\geq 0} \simeq \{X \mid \forall i < 0,\, H^i(X) = 0\}$$
$$\mathcal{D}^{\heartsuit} \simeq \{X \mid \forall i \neq 0,\, H^i(X) = 0\}.$$

Definition/Proposition 2.4.14. Let $\mathcal{A}, \mathcal{B}$ be abelian categories.

- An *injective resolution* of an object A in $\mathcal{A}$ is a cochain complex $I^\bullet$ defined for $i \geq 0$ together with a morphism $\varepsilon\colon A \to I^0$ such that I^i is injective for all $i \geq 0$ and the sequence
$$0 \longrightarrow A \xrightarrow{\ \varepsilon\ } I^0 \longrightarrow I^1 \longrightarrow \dots$$
is exact. A projective resolution may be defined similarly by replacing "injective" with "projective" and reversing all arrows.
- We say $\mathcal{A}$ *has enough injectives* if for every object A there is an injective object I with a monomorphism $A \hookrightarrow I$, and $\mathcal{A}$ *has enough projectives* if for every object A there is a projective object P with an epimorphism $P \twoheadrightarrow A$. If $\mathcal{A}$ has enough injectives/projectives, then every object has an injective/projective resolution that is unique up to chain homotopy.
- Let $\mathcal{A}$ be an abelian category with enough injectives and let $F\colon \mathcal{A} \to \mathcal{B}$ be a covariant left exact functor. The *right derived functors* R^iF for $i \geq 0$ are defined as $R^iF(A) = h^i(F(I^\bullet))$, where $I^\bullet$ is a chosen injective resolution of A. Replace "injective" with "projective" to get left derived functors L^iF of a right exact functor F.
- The left/right derived functors are additive functors $\mathcal{A} \to \mathcal{B}$, and they are independent of the choice of projective/injective resolution up to natural isomorphism. Assuming the left/right derived functors are well-defined, F is naturally isomorphic to L^0F and R^0F.

Let $\mathcal{A}$ be an abelian category with enough projectives (respectively injectives); then we can construct a stable ∞-category $D^-(\mathcal{A})$ (resp. $D^+(\mathcal{A})$) whose objects are chain (resp. cochain) complexes in $\mathcal{A}$, and its homotopy category is the usual derived category of $\mathcal{A}$, which is a triangulated category.

Definition 2.4.15. Let k be a ring. A *dg category* $\mathcal{C}$ over k consists of the following data:

(1) A class of objects.
(2) For every pair $X, Y \in \mathcal{C}$ a chain complex $\operatorname{Hom}(X,Y)_\bullet$ of k-modules.
(3) For every triple $X, Y, Z \in \mathcal{C}$ a composition map
$$\operatorname{Hom}(Y,Z)_\bullet \otimes_k \operatorname{Hom}(X,Y)_\bullet \longrightarrow \operatorname{Hom}(X,Z)_\bullet,$$
which splits into a collection of k-bilinear maps
$$\circ\colon \operatorname{Hom}(Y,Z)_p \times \operatorname{Hom}(X,Y)_q \longrightarrow \operatorname{Hom}(X,Z)_{p+q}$$
satisfying associativity and the graded Leibniz rule $d(g \circ f) = dg \circ f + (-1)^p g \circ df$.

Assume $k = \mathbb{Z}$ unless stated otherwise.

Definition 2.4.16 (Dg nerve)**.** Let $\mathcal{C}$ be a dg category. We associate to $\mathcal{C}$ the dg nerve $\mathrm{N}_{\mathrm{dg}}(\mathcal{C})$. For $n \geq 0$ define $\mathrm{N}_{\mathrm{dg}}(\mathcal{C})_n$ as the set of pairs $(\{X_i\}_{0 \leq i \leq n}, \{f_I\})$, where:

(1) $X_i \in \mathcal{C}$ for all i.
(2) For every $I = \{i_- < i_m < i_{m-1} < \ldots < i_1 < i_+\} \subseteq [n]$ with $m \geq 0$, f_I is an element of $\operatorname{Hom}(X_{i_-}, X_{i_+})_m$ satisfying
$$df_I = \sum_{1 \leq j \leq m} (-1)^j \left(f_{I \setminus \{i_j\}} - f_{\{i_j < \ldots < i_1 < i_+\}} \circ f_{\{i_- < i_m < \ldots < i_j\}} \right).$$
If $\alpha\colon [m] \to [n]$ is nondecreasing, then this induces a map $\mathrm{N}_{\mathrm{dg}}(\mathcal{C})_n \to \mathrm{N}_{\mathrm{dg}}(\mathcal{C})_m$ given by
$$(\{X_i\}_{0 \leq i \leq n}, \{f_I\}) \mapsto (\{X_{\sigma(j)}\}_{0 \leq j \leq m}, \{g_J\}),$$

where
$$g_J = \begin{cases} f_{\alpha(J)}, & \text{if } \alpha|_J \text{ is injective} \\ \mathrm{id}_{X_i}, & \text{if } J = \{j, j'\}, \quad \alpha(j) = \alpha(j') = i \\ 0, & \text{else.} \end{cases}$$

- A 0-simplex of $\mathrm{N_{dg}}(\mathcal{C})$ is an object of $\mathcal{C}$.
- A 1-simplex is a 0-degree closed morphism $X \xrightarrow{f} Y$, that is $df = 0$.
- A 2-simplex is a triple of 0-degree closed morphisms

$$X \xrightarrow{f} Y$$
$$Y \xrightarrow{g} Z$$
$$X \xrightarrow{h} Z$$

with an element $z \in \mathrm{Hom}(X, Z)_1$ such that $dz = (g \circ f) - h$.

Theorem 2.4.17. *The dg nerve* $\mathrm{N_{dg}}(\mathcal{C})$ *is an* ∞*-category.*

Proof. Given $\Lambda^n_i \to \mathrm{N_{dg}}(\mathcal{C})$, we want to extend to $\Delta^n \to \mathrm{N_{dg}}(\mathcal{C})$. We let

$$f_{[n]} = 0$$
$$f_{[n]-\{i\}} = \sum_{0<p<n} (-1)^{p-i} f_{\{p,\dots,n\}} \circ f_{\{0,\dots,p\}} - \sum_{0<p<n, p\neq i} (-1)^{p-i} f_{[n]-\{p\}}.$$

One must verify that this gives us the desired lift. □

Remark 2.4.18. There is a notion of localization for ∞-categories, and $\mathrm{N_{dg}}(\mathcal{C})$ can be thought of as localizing by "quasi-isomorphisms."

We want to put a dg category structure on $\mathcal{A}$:

Definition 2.4.19. Let $\mathcal{A}$ be an additive category, and let $M_\bullet$, $N_\bullet$ be chain complexes on $\mathcal{A}$. For all p we let

$$\mathrm{Hom}(M_\bullet, N_\bullet)_p = \prod_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(M_n, N_{n+p}).$$

The differential is given by $(df)(x) = d(f(x)) - (-1)^p f(dx)$. Composition also gives us a tensor product satisfying graded Leibniz, giving $\mathrm{Ch}(\mathcal{A})$ the structure of a dg category.

We write $\mathrm{Ch}^-(\mathcal{A})$ (respectively $\mathrm{Ch}^+(\mathcal{A})$) for the full subcategory of chain complexes $M_\bullet$ such that $M_n \simeq 0$ for $n \ll 0$ (resp. $n \gg 0$).

Definition 2.4.20. If $\mathcal{A}$ is an abelian category with enough injective (respectively projective) objects, we let $\mathrm{D}^+(\mathcal{A}) = \mathrm{N_{dg}}(\mathrm{Ch}^+(\mathcal{A}_{\mathrm{inj}}))$ (resp. $\mathrm{D}^-(\mathcal{A}) = \mathrm{N_{dg}}(\mathrm{Ch}^-(\mathcal{A}_{\mathrm{proj}}))$), where $\mathcal{A}_{\mathrm{inj}}$ (resp. $\mathcal{A}_{\mathrm{proj}}$) is the full subcategory of injective (resp. projective) objects.

Proposition 2.4.21. *Let* $\mathcal{A}$ *be an additive category. Then* $\mathrm{N_{dg}}(\mathrm{Ch}(\mathcal{A}))$ *is a stable* ∞*-category.*

3. Algebraic geometry

In this section we cover the basic setup of algebraic geometry. Most proofs will be sketched or omitted, referring to other sources. We recommend [SP] as a reference and [GW] as a book series in algebraic geometry.

3.1. **Background in commutative algebra.** Algebraic geometry is more or less a geometric treatment of commutative algebra. As such, we introduce some basic notions in commutative algebra.

Construction 3.1.1. Let $R \to S$ be a ring morphism. We define the module of *Kähler differentials* $\Omega_{S/R}$ to be the S-module such that we have the following S-module isomorphism for every S-module M:

$$\mathrm{Hom}_S(\Omega_{S/R}, M) \xrightarrow{\cong} \mathrm{Der}_R(S_R, M_R),$$

where the right hand side is the set of derivations from S to M as R-modules (morphisms satisfying the Leibniz rule $d(fg) = f\,dg + g\,df$). The S-module $\Omega_{S/R}$ can also be constructed as a quotient of the free S-module with generators $(ds)_{s\in S}$ by the relations

$$dr = 0, \qquad d(s+t) = ds + dt, \qquad d(st) = s\,dt + t\,ds,$$

where $r \in R$ and $s, t \in S$.

The last equivalent construction of $\Omega_{S/R}$ is given as follows: define I to be the kernel of the multiplication map $S \otimes_R S \to S$. Then we define $\Omega_{S/R} = I/I^2$. The universal derivation d is given by $ds = 1 \otimes s - s \otimes 1$. This specific construction of Kähler differentials is especially important in algebraic geometry.

Definition 3.1.2. Let $\phi \colon A \to B$ be a ring morphism. With this, we can view B as an A-algebra.

(1) We say ϕ is *flat* if B is a flat A-module, which is to say if $M \to M' \to M''$ is an exact sequence of A-modules then $M \otimes_A B \to M' \otimes_A B \to M'' \otimes_A B$ is an exact sequence of B-modules. If the converse holds as well, we say ϕ is *faithfully flat.*
(2) We say ϕ is *of finite type* if B is a finitely generated A-module (i.e. there is a surjection $A[x_1, \ldots, x_n] \to B$ for some n).
(3) We say ϕ is *of finite presentation* if B is a finitely presented A-module (i.e. $B \cong R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ for some n, m and polynomials f_i).
(4) We say ϕ is *formally étale* (resp. *formally smooth*, *formally unramified*) if for every A-algebra R and ideal $I \subset R$ such that $I^2 = 0$, the induced map $\mathrm{Hom}_A(B, R) \to \mathrm{Hom}_A(B, R/I)$ is bijective (resp. surjective, injective).
(5) We say ϕ is *unramified* (respectively *G-unramified*) if it is formally unramified and of finite type (resp. of finite presentaion).
(6) We say ϕ is *étale* (respectively *smooth*) if it is formally étale (resp. formally smooth) and of finite presentation.

Note that a finitely presented ring morphism $\phi \colon A \to B$ is of finite type since finitely presented A-modules are finitely generated.

Lemma 3.1.3. *A ring morphism $\phi \colon A \to B$ is formally unramified if and only if $\Omega_{B/A} = 0$.*

3.2. **Classical schemes.** While this paper is primarily about derived algebraic geometry, we roughly outline the classical theory here. We begin by defining ringed spaces and schemes. Note that for this section, we refer to sheaves exclusively as sheaves on $\mathrm{Opens}(X)$ for a space X.

Definition 3.2.1. We fix the following definitions:

(1) A *ringed space* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$.
(2) A *locally ringed space* is a ringed space such that for all $x \in X$ the stalk $\mathcal{O}_{X,x}$ (also denoted as $\mathcal{O}_x$) is a local ring.

(3) We denote $\mathfrak{m}_{X,x}$, or just $\mathfrak{m}_x$, as the unique maximal ideal of $\mathcal{O}_{X,x}$.
(4) We denote the *residue field* of $\mathcal{O}_{X,x}$ as $\kappa_{X,x} \overset{\text{def}}{=} \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$.
(5) A morphism of ringed spaces $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of a continuous map $f : X \to Y$ and a map $f^\#; \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings on Y.
(6) A *local ring map* $f : A \to B$ between local rings is a ring map such that it maps the unique maximal ideal of A into that of B: $f(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$.
(7) A morphism of locally ringed spaces $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces such that for all $x \in X$ the induced ring map $f_x^\# : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local ring map.
(8) A morphism of (locally) ringed spaces is an *isomorphism* if it has a two-sided inverse, or equivalently if f is a homeomorphism and $f^\#$ is an isomorphism of sheaves.

Many geometric objects are ringed spaces. For example, any differentiable manifold M is a locally ringed space, where the sheaf of rings assigns $U \subseteq M$ to the ring $\mathcal{O}_M(U)$ of differentiable functions on U. Here a manifold is modeled locally by Euclidean space, but in algebraic geometry we replace Euclidean space with something else.

Construction 3.2.2. We construct affine schemes explicitly. Let R be a ring, and consider its set of prime ideals $\mathrm{Spec}(R)$. A topology on $\mathrm{Spec}(R)$ is given by the sets $V(T) = \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p} \supset T\}$ for $T \subset R$. A basis $\mathcal{B}$ is given by the *standard open sets* $D(f) = \mathrm{Spec}(R) \setminus V(\{f\})$ for $f \in R$. This basis may be arranged into a full subcategory $\mathcal{B}(\mathrm{Spec}(R)) \hookrightarrow \mathrm{Opens}(\mathrm{Spec}(R))$ whose objects are the standard open sets $D(f)$ for $f \in R$ and morphisms are inclusions (that is $D(f) \hookrightarrow D(g)$ whenever $D(f) \subset D(g)$).

Given an R-module M, we construct a presheaf $\widetilde{M} : \mathrm{Basis}(\mathrm{Spec}(R)) \to \mathbf{Mod}_R$ on the basis of standard opens of $\mathrm{Spec}(R)$ with values in $\mathbf{Mod}_R$ given via the localization $\widetilde{M}(D(f)) = M_f$. One may check that the stalk at a point $\mathfrak{p} \in \mathrm{Spec}(R)$ is given by

$$\widetilde{M}_{\mathfrak{p}} = \operatorname*{colim}_{f \in R \setminus \mathfrak{p}} M_f = M_{\mathfrak{p}}.$$

Furthermore, using some results in commutative algebra (see [SP, 01HR]) we can see that $\widetilde{M}$ is a sheaf on the basis of standard opens. Furthermore, using sheaf properties (see [SP, 009H]) we can check that there is a unique sheaf on $\mathrm{Spec}(R)$ with values in $\mathbf{Mod}_R$ that agrees with $\widetilde{M}$ on the basis of standard opens. We denote this sheaf by $\widetilde{M}$ as well. In particular, the sheaf $\widetilde{R} = \mathcal{O}_{\mathrm{Spec}(R)}$ is a sheaf of rings whose stalks are local rings, giving $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ the structure of a locally ringed space.

We summarize some of the observations above in the following lemma:

Lemma 3.2.3. *Let R be a ring and M an R-module (or R-algebra). In the above construction, we have the following:*

(1) For every $f \in R$, we have $\Gamma(D(f), \widetilde{M}) = M_f$.
(2) Let f be a unit. Then $D(f) = \mathrm{Spec}(R)$, and $\Gamma(\mathrm{Spec}(R), \widetilde{M}) = M$.
(3) Suppose $D(g) \subset D(f)$. Then the induced map on sections is the canonical R_f-module map $M_f \to M_g$.
(4) The stalk at $\mathfrak{p} \in \mathrm{Spec}(R)$ is $\widetilde{M}_{\mathfrak{p}} = M_{\mathfrak{p}}$.

If M is an R-algebra, then the above holds with all relations at the level of R-algebras.

Proof. See [SP, 01HR]. □

Definition 3.2.4. Let $(X, \mathcal{O}_X)$ be a locally ringed space.

(1) We say that $(X, \mathcal{O}_X)$ is an *affine scheme* if it is isomorphic to $(\operatorname{Spec}(R), \mathcal{O}_{\operatorname{Spec}(R)})$ for some ring R.
(2) We say that $(X, \mathcal{O}_X)$ is a *scheme* if every $x \in X$ has an open neighbourhood $U \ni x$ such that $(U, \mathcal{O}_X|_U)$ is isomorphic to an affine scheme. A morphism of schemes is a morphism of locally ringed spaces. The category of schemes will be denoted as **Sch**, and the full subcategory of affine schemes will be denoted as **AffSch**.

The first example of an affine scheme is the *affine space* $\mathbb{A}^n_R = \operatorname{Spec} R[x_1, \ldots, x_n]$. Usually R is a field in practice.

Construction 3.2.5 (Proj construction)**.** Let $S = \bigoplus_{i \geq 0} S_i$ be an $\mathbb{N}$-graded ring, where we denote an element $r \in S$ as $r = \sum_{i \geq 0} r_i$. We let $S_+ = \bigoplus_{i>0} S_i$. A *homogeneous ideal* of S is an ideal $I \subset S$ such that $r \in I$ implies $r_i \in I$ for all $i \geq 0$. We define the set $\operatorname{Proj}(S)$ to be

$$\operatorname{Proj}(S) = \{\mathfrak{p} \subset S \text{ homogeneous prime ideal} \mid S_+ \not\subset \mathfrak{p}\}.$$

We define a Zariski topology on $\operatorname{Proj}(S)$ by declaring the closed subsets to be of the form

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Proj}(S) \mid \mathfrak{a} \subseteq \mathfrak{p}\},$$

where $\mathfrak{a}$ is a homogeneous ideal of S. Equivalently, we let $D(\mathfrak{a}) = \operatorname{Proj}(S) \setminus V(\mathfrak{a})$, which provides us all of the open subsets. For $f \in S_+$ a homogeneous element of positive degree, we denote $D(f) \stackrel{\text{def}}{=} D(Sf)$, where Sf is the ideal generated by f. A basis for this Zariski topology is given by the collection of all such $D(f)$. By carrying out a similar argument to Construction 3.2.2, one may construct a sheaf of rings for $\operatorname{Proj}(S)$ that gives it the structure of a scheme (which follows from showing that $D(f)$ is an affine scheme, isomorphic to $\operatorname{Spec}(S_{(f)})$). See [SP, 01M3] for more details.

This construction provides a functor from the category of $\mathbb{N}$-graded rings to schemes, one that provides us with many examples of schemes. For example, this construction provides us with the *projective space* $\mathbb{P}^n_A \stackrel{\text{def}}{=} \operatorname{Proj}(A[x_1, \ldots, x_n])$, a space in which the local functions are homogeneous polynomials with coefficients in A.

Definition 3.2.6. Let S be a scheme. An *S-scheme* is a scheme X together with a morphism $X \to S$. A *morphism* of S-schemes $X \to Y$ is a morphism of schemes such that $X \to S$ factors through Y, i.e. $(X \to Y \to S) = (X \to S)$. Equivalently, the category of S-schemes is the slice category $\mathbf{Sch}_{/S}$.

We have the following property of affine schemes:

Lemma 3.2.7. *Let X be a locally ringed space and Y an affine scheme. The map*

$$\operatorname{Mor}(X, Y) \to \operatorname{Hom}(\Gamma(Y, \mathcal{O}_Y), \Gamma(X, \mathcal{O}_X))$$

that sends f to $f^\#$ is a bijection.

Proof. See [SP, 01HX]. □

Note that $\mathbb{Z}$ is initial in **CRing**, that is every ring has a unique morphism $\mathbb{Z} \to R$ by mapping $1 \mapsto 1_R$. The above lemma with this observation gives us the following:

Corollary 3.2.8. *The following hold:*

(1) The affine scheme $\operatorname{Spec}(\mathbb{Z})$ *is final in the category of locally ringed spaces, and in particular it is final in* **Sch** *and* **AffSch**.

(2) *The category* **AffSch** *is equivalent to* $\mathbf{CRing}^{\mathrm{op}}$. *The two functors inducing the equivalence are* $\mathrm{Spec}\colon \mathbf{CRing}^{\mathrm{op}} \to \mathbf{AffSch}$ *and the global sections functor* $\Gamma(-,\mathcal{O}_{-})\colon \mathbf{AffSch} \to \mathbf{CRing}^{\mathrm{op}}$.

Another way of phrasing the first statement in the corollary is that **Sch** is equivalent to $\mathbf{Sch}_{/\,\mathrm{Spec}(\mathbb{Z})}$. The second statement is incredibly important as we will see later (really, commit it to memory!), but also not very surprising in some ways. After all, the definition of an affine scheme $\mathrm{Spec}(R)$ requires the input of a ring $R \in \mathbf{CRing}$, and a ring morphism $R \to S$ induces a morphism of topological spaces $\mathrm{Spec}(R) \to \mathrm{Spec}(S)$.

Lemma 3.2.9. *Let $X \to S$ and $Y \to S$ be S-schemes. Then the fibered product $X \times_S Y$ exists. If $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$, and $S = \mathrm{Spec}(R)$ are all affine, then A, B are R-algebras and $X \times_S Y = \mathrm{Spec}(A \otimes_R B)$.*

Proof. See [SP, 01JL]. □

Since $\mathrm{Spec}(\mathbb{Z})$ is final in **Sch**, the fibered product $X \times_{\mathrm{Spec}(\mathbb{Z})} Y$ is also the product in **Sch**. The coproduct of schemes is given by the disjoint union of topological spaces. At the level of affine schemes, we have

$$\mathrm{Spec}(R_1) \sqcup \ldots \sqcup \mathrm{Spec}(R_n) = \mathrm{Spec}\left(R_1 \times \ldots \times R_n\right).$$

This is an example of the equivalence $\mathbf{AffSch} \simeq \mathbf{CRing}^{\mathrm{op}}$. Products of infinitely many rings are very hard to work with, and this identity does not always hold if we don't look at finite coproducts.

Remark 3.2.10. One other common way to construct new schemes out of old ones is to glue schemes together. One must define what it means to give an appropriate set of *gluing data* for locally ringed spaces, but it can be shown that this gluing data gives rise to new locally ringed spaces. If all the locally ringed spaces are schemes, then so is the result of the gluing data.

One of the first examples of a non-affine scheme is affine space over a field k with a doubled origin, where we glue two copies of $\mathbb{A}^n_k$ along their respective copies of $(\mathbb{A}^n_k)^\times = \mathbb{A}^n_k \setminus \{P\}$, where P corresponds to the maximal ideal $(x_1, \ldots, x_n)$. We denote this as $\mathbb{A}^n_k \sqcup_{(\mathbb{A}^n_k)^\times} \mathbb{A}^n_k$. It can be shown that this is affine if and only if $n = 1$, with $\mathbb{A}^1_k \sqcup_{(\mathbb{A}^1_k)^\times} \mathbb{A}^1_k \cong \mathrm{Spec}\, k[x, x^{-1}]$.

There are many different adjectives used to describe schemes and their morphisms. Here are a few of these terms, but most of them will not be used further.

Remark 3.2.11. In the convention of algebraic geometry, we say a topological space X is *quasi-compact* if every open cover $\{U_i\}_{i \in I}$ has a finite subset that is also an open cover of X. We say X is *compact* if it is quasi-compact and Hausdorff, that is two distinct points in X admit disjoint open neighborhoods. Being Hausdorff is equivalent to the diagonal $\Delta_X \subset X \times X$ being closed in $X \times X$ with respect to the product topology.

Definition 3.2.12. Let X be a scheme.

(1) We say X is *connected/irreducible* if it is connected/irreducible as a topological space.
(2) We say X is *reduced* if for every open set U, the ring $\mathcal{O}_X(U)$ has no nilpotent elements (equivalently if the stalks $\mathcal{O}_x$ have no nilpotent elements for all x).
(3) We say X is *integral* if for every open set $U \subseteq X$, the ring $\mathcal{O}_X(U)$ is an integral domain. This is equivalent to being reduced and irreducible.

(4) We say X is *locally Noetherian* if it admits a covering by open affine subsets $\{\operatorname{Spec} A_i\}_{i\in I}$, where A_i is a Noetherian ring for all i. We say X is *Noetherian* if it is locally Noetherian and *quasi-compact*, or equivalently if the index set I in the affine open cover $\{\operatorname{Spec} A_i\}_{i\in I}$ can be finite.
(5) We say X is *Gorenstein/Cohen-Macaulay* if it is locally Noetherian and the stalks $\mathcal{O}_x$ are all Gorenstein/Cohen-Macaulay local rings (look up what this means).

Definition 3.2.13. Let $f\colon X\to Y$ be a morphism of schemes. The *diagonal morphism* $\Delta_X\colon X\to X\times_Y X$ associated to f is the unique morphism obtained by the universal property of the fibered product:

$$\begin{tikzcd} & X \arrow[dl, "\mathrm{id}_X"'] \arrow[d, dashed, "\exists!\,\Delta_X"] \arrow[dr, "\mathrm{id}_X"] & \\ X & X\times_Y X \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\mathrm{pr}_2"] & X \end{tikzcd}$$

Definition 3.2.14. Let $f : X\to Y$ be a morphism of schemes.

(1) We say f is an *open immersion* if the underlying map of topological spaces is a homeomorphism of X with an open subset of $U\subset X$, and the induced sheaf morphism $\mathcal{O}_Y\to f_*\mathcal{O}_X$ induces an isomorphism $\mathcal{O}_Y|_U\cong (j_*\mathcal{O}_X)|_U$ (of sheaves on U).
(2) We say f is a *closed immersion* if the underlying map of topological spaces is a homeomorphism of X with a closed subset $C\subset X$, and the sheaf homomorphism $f^\#\colon\mathcal{O}_Y\to f_*\mathcal{O}_X$ is surjective.
(3) We say f is *quasi-compact* if for every quasi-compact open subset $V\subseteq Y$ the preimage $f^{-1}(V)\subseteq X$ is quasi-compact.
(4) We say f is *quasi-separated* if the diagonal morphism $\Delta_X\colon X\to X\times_Y X$ is quasi-compact.
(5) We say f is *separated* if the diagonal morphism $\Delta_X\colon X\to X\times_Y X$ is a closed immersion.
(6) Suppose there exists a covering $\{V_i\}_{i\in I}$ of Y by open affine subsets $V_i\cong\operatorname{Spec} A_i$, and for each $i\in I$ a covering $\{U_{ij}\}_{J_i}$ of $f^{-1}(V_i)$ by open affine subschemes $U_{ij}\cong\operatorname{Spec} B_{ij}$ of X.
   - We say f is *locally of finite type/presentation* if for all $i\in I$ and $j\in J_i$ the ring morphism $A_i\to B_{ij}$ is locally of finite type/presentation.
   - We say f is *(formally) étale/smooth/unramified* if for all $i\in I$ and $j\in J_i$ the ring morphism $A_i\to B_{ij}$ is (formally) étale/smooth/unramified.
(7) We say f is *of finite type* if it is locally of finite type and quasi-compact.
(8) We say f is *of finite presentation* if it is locally of finite presentation, quasi-compact, and quasi-separated.
(9) We say f is *flat* if the induced map on stalks $f_x\colon\mathcal{O}_{Y,f(x)}\to\mathcal{O}_{X,x}$ is a flat map of rings for all $x\in X$. We say f is *faithfully flat* if it is flat and surjective.

3.3. **The functor of points and stacks.** This explicit construction of schemes we have given is convenient, but it does not tell us how to interpret such objects categorically. This is given by the functor of points perspective:

Definition 3.3.1. Given a scheme X, we define its *generalized functor of points*

$$\begin{aligned} h^X\colon\ \mathbf{Sch}^{\mathrm{op}} &\longrightarrow \mathbf{Set}\\ T &\longmapsto \operatorname{Mor}_{\mathbf{Sch}}(T,X). \end{aligned}$$

The *functor of points* associated to X is $h^X|_{\mathbf{AffSch}}$, that is the same functor restricted to $\mathbf{AffSch}^{\mathrm{op}}\simeq\mathbf{CRing}$.

One may check that the functor of points is a sheaf. By Yoneda's lemma, we can see that if h^X is naturally isomorphic to h^Y, then $X \cong Y$. Via the construction of schemes, a functor $f\colon T \to X$ is determined by how it maps each member of an affine open cover $\{U_i\}_i$ of T, so it suffices to consider when T is affine.

Proposition 3.3.2. *If $h^X|_{\mathbf{AffSch}}$ is naturally isomorphic to $h^Y|_{\mathbf{AffSch}}$, then $X \cong Y$. That is, a scheme is determined by its functor of points.*

Thus, an affine scheme $\mathrm{Spec}(R)$ is equivalently given by the functor

$$\begin{aligned} h_R\colon \mathbf{CRing} &\longrightarrow \mathbf{Set}, \\ S &\longmapsto \mathrm{Hom}(R, S). \end{aligned}$$

With this, we may identify a scheme X with its generalized functor of points h^X.

A great deal of interesting geometry arises from attempting to replace **Set** with some other category, such as **Grpd**. After all, the forgetful functor $\mathbf{Grpd} \to \mathbf{Set}$ comes with a left adjoint $\mathbf{Set} \hookrightarrow \mathbf{Grpd}$ by declaring a set S to be a groupoid in which all morphisms are identity maps. This motivates the definition of a stack:

Definition 3.3.3. A *prestack* is a presheaf $\mathcal{X}\colon \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Grpd}$. Given a Grothendieck topology τ on **Sch**, a *stack* is a prestack that is a sheaf with respect to τ. We denote the 2-category of stacks (with respect to a given topology τ) as $\mathbf{Stk} \overset{\mathrm{def}}{=} \mathrm{Shv}(\mathbf{Sch}; \mathbf{Grpd})$ (see Convention 1.3.13 for why this is an ∞-category).

Usually the Grothendieck topology τ is the étale topology or something stronger (such as the fpqc topology). It can be shown that the generalized functor of points $h^X\colon \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set} \hookrightarrow \mathbf{Grpd}$ defines a stack with respect to the fpqc topology.

Treating **Grpd** as a 2-category, a stack is a functor $\mathcal{X}\colon \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Grpd}$ such that for every finite collection of morphisms $(U_i \to U)_i \in \tau$, the diagram

$$\mathcal{X}(U) \longrightarrow \prod_i \mathcal{X}(U_i) \rightrightarrows \prod_{i,j} \mathcal{X}(U_i \times_U U_j) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \prod_{i,j,k} \mathcal{X}(U_i \times_U U_j \times_U U_k)$$

is a limit diagram in **Grpd**.

3.4. **Quasi-coherent sheaves.** Here we introduce quasi-coherent sheaves from a categorical perspective. One many think of these as a kind of representation theory for schemes. The categorical perspective allows us to characterize quasi-coherent sheaves as a right Kan extension, which is useful for generalizations of schemes such as stacks or even derived stacks.

Definition 3.4.1. Let $X = (X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module is a sheaf $\mathcal{F}$ of abelian groups on X together with two morphisms (addition and scalar multiplication) of sheaves

$$\mathcal{F} \times \mathcal{F} \to \mathcal{F}, \qquad \mathcal{O}_X \times \mathcal{F} \to \mathcal{F},$$

such that when we restrict to an open subset $U \subseteq X$, the induced morphisms

$$\mathcal{F}(U) \times \mathcal{F}(U) \to \mathcal{F}(U), \qquad \mathcal{O}_X(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$$

endow $\mathcal{F}(U)$ with the structure of a $\mathcal{O}_X(U)$-module. A morphism of $\mathcal{O}_X$-modules $\mathcal{F}_1 \to \mathcal{F}_2$ is a morphism of sheaves such that the induced map $\mathcal{F}_1(U) \to \mathcal{F}_2(U)$ is a morphism of $\mathcal{O}_X(U)$-modules.

Definition 3.4.2. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is *quasi-coherent* if for every inclusion of opens $U \subseteq V \subseteq X$ the natural map

$$\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V) \to \mathcal{F}(V)$$

is an isomorphism. A morphism of quasi-coherent sheaves is a morphism of $\mathcal{O}_X$-modules. We denote the category of quasi-coherent modules as $\mathrm{QCoh}(X)$.

Given a morphism of ringed spaces $f : X \to Y$, we have an adjoint pair of functors

$$f^* \colon \mathrm{QCoh}(X) \rightleftarrows \mathrm{QCoh}(Y) : f_*,$$

where f^* is the inverse image functor and f_* is the direct image functor. For a pair of morphisms $f : X \to Y$ and $g : Y \to Z$ there is an invertible natural transformation

$$(g \circ f)^* \to f^* \circ g^*.$$

This natural transformation is rarely the identity, but it does determine a commutative diagram in the 2-category **Grpd**. The assignment $X \mapsto \mathrm{QCoh}(X)$ then determines a functor

$$\mathrm{QCoh}\colon \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Cat}$$

into the 2-category of categories.

Definition 3.4.3. Let $\pi\colon \mathcal{E} \to \mathcal{C}$ be a functor of categories. Let $f : C \to D$ be a morphism in $\mathcal{C}$ and $\widetilde{\mathcal{D}} \in \mathcal{E}$ a lift of D (so that $\pi(\widetilde{D}) = D$). Let $\widetilde{f}\colon \widetilde{C} \to \widetilde{D}$ be a lift of f, i.e. $\pi(\widetilde{C}) = C$ and $\pi(\widetilde{f}) = f$. We say $\widetilde{f}$ is π-cartesian if for all $E \in \mathcal{E}$ the canonical map

$$\mathrm{Mor}_{\mathcal{E}}(E, \widetilde{C}) \to \mathrm{Mor}_{\mathcal{E}}(E, \widetilde{D}) \times_{\mathrm{Mor}_{\mathcal{C}}(\pi(E),D)} \mathrm{Mor}_{\mathcal{C}}(\pi(E), \pi(\widetilde{C}))$$

is bijective. Informally speaking, $\widetilde{f}$ is terminal among all lifts of f with target $\widetilde{D}$.

Definition 3.4.4. Let $\pi\colon \mathcal{E} \to \mathcal{C}$ be a functor of categories. We say that π is a *cartesian fibration* if for every $E \in \mathcal{E}$, $C \in \mathcal{C}$, and morphism $f : C \to \pi(E)$ there is a lift $\widetilde{C} \in \mathcal{E}$ of C and a π-cartesian morphism $\widetilde{f}\colon \widetilde{C} \to E$ lifting f.

Example 3.4.5. Let $\mathrm{QCoh}_{\mathbf{Sch}}$ denote the category of pairs $(X, \mathcal{F})$ such that $X \in \mathbf{Sch}$ and $\mathcal{F} \in \mathrm{QCoh}(X)$. A morphism $(X', \mathcal{F}') \to (X, \mathcal{F})$ is a morphism $f : X' \to X$ together with a morphism $\phi\colon f^*\mathcal{F} \to \mathcal{F}'$ in $\mathrm{QCoh}(X')$. Given morphisms $(f, \phi)\colon (X', \mathcal{F}') \to (X, \mathcal{F})$ and $(g, \psi)\colon (X'', \mathcal{F}'') \to (X', \mathcal{F}')$, the composition is defined by

$$f \circ g\colon X'' \to X' \to X$$

and

$$g^*f^*\mathcal{F} \xrightarrow{g^*\phi} g^*\mathcal{F}' \xrightarrow{\psi} \mathcal{F}''.$$

Then the projection $(X, \mathcal{F}) \mapsto X$ determines a functor

$$\mathrm{QCoh}_{\mathbf{Sch}} \to \mathbf{Sch},$$

which is a cartesian fibration.

Construction 3.4.6. Let $\mathcal{C}$ be a fixed category. We denote by $\mathrm{Cart}(\mathcal{C})$ the 2-category whose objects are cartesian fibrations $\pi\colon \mathcal{E} \to \mathcal{C}$, whose 1-morphisms $(\pi' : \mathcal{E}' \to \mathcal{C}) \to (\pi\colon \mathcal{E} \to \mathcal{C})$ are morphisms $f : \mathcal{E}' \to \mathcal{E}$ that are compatible with π and π', and whose 2-morphisms are invertible natural transformations $\theta\colon f \to g$ for $f, g\colon \pi' \to \pi$ such that for all $E' \in \mathcal{E}'$, $\pi : \mathcal{E} \to \mathcal{C}$ sends

$$\theta_{E'}\colon f(E') \to g(E')$$

to the identity of $\pi(f(E')) = \pi(g(E'))$.

Theorem 3.4.7 (Grothendieck construction)**.** *For every category $\mathcal{C}$, there is an equivalence of ∞-categories*

$$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}) \to \mathrm{Cart}(\mathcal{C}).$$

There are ∞-categorical versions of this theorem in which we replace **Cat** with the ∞-category $\mathbf{Cat}_\infty$, not that they will be relevant for us.

Definition 3.4.8. Given $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$, the corresponding cartesian fibration is called the *unstraightening* of F. Given a cartesian fibration $\pi\colon \mathcal{E} \to \mathcal{C}$, the *straightening* of π is the (essentially unique) presheaf of categories whose unstraightening is π.

Definition 3.4.9. We let $\mathrm{QCoh}\colon \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Cat}$ denote the unstraightening of the cartesian fibration $\mathrm{QCoh}_{\mathbf{Sch}} \to \mathbf{Sch}$. We let $\mathrm{QCoh}^{\simeq}\colon \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Grpd}$ denote the presheaf of groupoids given by sending

$$X \mapsto \mathrm{QCoh}(X)^{\simeq},$$

where $(-)^{\simeq}$ indicates that we discard all non-invertible morphisms.

3.5. **Descent for quasi-coherent sheaves.** The goal of this section is to show that the functor $\mathrm{QCoh}\colon \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Cat}$ satisfies étale descent, or more generally descent for the fpqc topology on **Sch**.

Definition 3.5.1. Given a cosimplicial diagram $X^\bullet\colon \Delta \to \mathcal{C}$ in an ∞-category, we refer to its limit as the *totalization* of $X^\bullet$, and write

$$\mathrm{Tot}(X^\bullet) \overset{\mathrm{def}}{=} \varprojlim_{\Delta} X^\bullet.$$

Definition 3.5.2. Let Δ_+ denote the category whose objects are the finite sets $[n] = \{0, \ldots, n\}$ for all $n \geq -1$, where $[-1] = \varnothing$, and whose morphisms are order-preserving maps. Given an ∞-category $\mathcal{C}$, an *augmented cosimplicial diagram* in $\mathcal{C}$ is a functor $X^\bullet\colon \Delta_+ \to \mathcal{C}$. We will depict $X^\bullet$ by the diagram

$$X^{-1} \to X^0 \rightrightarrows X^1 \substack{\to\\\to\\\to} X^2 \substack{\to\\\to\\\to\\\to} \cdots.$$

We say this is a *limit diagram* if the induced morphism of cosimplicial diagrams

$$X^{-1}_{\mathrm{cst}} \to X^\bullet|_\Delta$$

exhibits X^{-1} as the limit (totalization) of $X^\bullet|_\Delta$.

Definition 3.5.3. Let $\Delta_{-\infty}$ denote the category whose objects are the finite sets $[n]$ for all $n \geq -1$ and whose morphisms $[m] \to [n]$ are given by order-preserving maps $[m] \cup \{-\infty\} \to [n] \cup \{-\infty\}$ that preserve $-\infty$. A *splitting* of a cosimplicial diagram $X^\bullet\colon \Delta \to \mathcal{C}$ is an extension to a functor $X^\bullet\colon \Delta_{-\infty} \to \mathcal{C}$. A simplicial diagram $X^\bullet$ is *split* if it admits a splitting. In this case, $X^\bullet|_{\Delta_+}$ is a limit diagram, i.e. we have $\mathrm{Tot}(X^\bullet|_\Delta) \simeq X^{-1}$.

Theorem 3.5.4 (Descent criterion)**.** *Let $\mathcal{C}^\bullet\colon \Delta_+ \to \mathbf{Cat}_\infty$ be an augmented cosimplicial diagram of ∞-categories, which we depict as*

$$\mathcal{C}^{-1} \xrightarrow{F} \mathcal{C}^0 \rightrightarrows \mathcal{C}^1 \substack{\to\\\to\\\to} \mathcal{C}^2 \substack{\to\\\to\\\to\\\to} \cdots. \tag{3.5.1}$$

Suppose the following conditions hold:

(1) The functor $F\colon \mathcal{C}^{-1} \to \mathcal{C}^0$ is conservative.

(2) For every morphism $\alpha\colon [m] \to [n]$ in Δ_+, let $\beta\colon [m+1] \to [n+1]$ denote the unique morphism that commutes with $\delta^0\colon [m] \to [m+1]$ and $\delta^0\colon [n] \to [n+1]$, and consider the commutative square

$$\begin{tikzcd} \mathcal{C}^m \arrow[r, "d^0"] \arrow[d, "\alpha"] & \mathcal{C}^{m+1} \arrow[d, "\beta"] \\ \mathcal{C}^n \arrow[r, "d^0"] & \mathcal{C}^{n+1} \end{tikzcd}$$

Then the horizontal arrows admit right adjoints $d^{0,R}$ that also commute with the vertical arrows; more precisely, the natural transformation

$$\alpha \circ d^{0,R} \xrightarrow{\text{unit}} d^{0,R} \circ d^0 \circ \alpha \circ d^{0,R} \simeq d^{0,R} \circ \beta \circ d^0 \circ d^{0,R} \xrightarrow{\text{counit}} d^{0,R} \circ \beta$$

is invertible.

(3) The functor F preserves totalizations of F-split simplicial diagrams in $\mathcal{C}^{-1}$. That is, for every cosimplicial diagram $X^\bullet$ in $\mathcal{C}^{-1}$ whose image $F(X^\bullet)$ is split, the canonical map $F(\mathrm{Tot}(X^\bullet)) \to \mathrm{Tot}(F(X^\bullet))$ is invertible.

Then the induced functor

$$C^{-1} \to \mathrm{Tot}(\mathcal{C}^\bullet|_\Delta)$$

is an equivalence of ∞-categories. That is, Eq. (3.5.1) is a limit diagram.

Proof. This result is a corollary of an ∞-categorical version of the monadicity theorem of Barr-Beck. See [Lur3, Corollary 4.7.5.3]. □

Now consider the functor

$$\mathbf{CRing} \to \mathbf{Cat}, \qquad R \mapsto \mathbf{Mod}_R,$$

sending a commutative ring R to the category of R-modules, and a ring homomorphism $\phi : R \to S$ to the extension of scalars functor

$$\phi^* \stackrel{\text{def}}{=} (-) \otimes_R S \colon \mathbf{Mod}_R \to \mathbf{Mod}_S\,.$$

Under the equivalence $\mathbf{AffSch}^{\mathrm{op}} \simeq \mathbf{CRing}$ this functor is equivalently the restriction of $\mathrm{QCoh}\colon \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Cat}$ to affine schemes.

Theorem 3.5.5. *The functor $R \mapsto \mathbf{Mod}_R$ satisfies descent for the flat topology, i.e. the Grothendieck topology on* **CRing** *generated by faithfully flat ring homomorphisms.*

Lemma 3.5.6. *Let $f : X \to Y$ be a morphism of schemes. If f is faithfully flat and quasi-compact, then the inverse image functor $\mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ is conservative.*

Proof. Let $(V_j \to X)_j$ be a family of morphisms, where $\{V_j\}_j$ is a open covering of Y by affine opens. For each j, let $V'_j \stackrel{\text{def}}{=} V_j \times_Y X = f^{-1}(V_j) \subseteq X$. Since f is quasi-compact, V'_j is quasi-compact. Then there exists a finite family of morphisms $(U_{i,j} \hookrightarrow V'_j)_i$ such that $U_{i,j} \subseteq V'_j$ are affine opens covering V'_j. We have the commutative square

$$\begin{array}{ccc} \coprod_i U_{i,j} & \xrightarrow{f_j} & V_j \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where $f_j\colon \coprod_i U_{i,j} \to V'_j \to V_j$ is a faithfully flat morphism of affine schemes for all j. This is because faithfully flat morphisms are stable under composition and base change (in this case the base change is just the pullback of V_j).

Now let $\phi\colon \mathcal{F} \to \mathcal{G}$ be a morphism of quasi-coherent sheaves on Y, whose inverse image $f^*(\phi)\colon f^*(\mathcal{F}) \to f^*(\mathcal{G})$ is invertible. We aim to show that ϕ is also invertible. It will suffice to show that each restriction $\phi|_{V_j}$ is invertible. Since f_j is a faithfully flat morphism of affine schemes, we know f_j^* is conservative (since it corresponds to extension of scalars of a module along a faithfully flat morphism of rings). Moreover, it then suffices to show that ϕ is invertible after inverse image along $\coprod_i U_{i,j} \to V_j \subseteq Y$. The latter morphism factors through f, hence is invertible by assumption. □

With this Lemma, we obtain the following:

Theorem 3.5.7. *The presheaf of categories* $\mathrm{QCoh}\colon \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Cat}$ *satisfies descent for the* fpqc *topology on* $\mathbf{Sch}$.

Corollary 3.5.8. *The presheaf of groupoids* $\mathrm{QCoh}^{\simeq}\colon \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Grpd}$ *satisfies descent. In particular, it is a stack.*

Proof. The functor $\mathbf{Cat} \to \mathbf{Grpd}$ sending $\mathcal{C} \mapsto \mathcal{C}^{\simeq}$ preserves limits since it is the right adjoint to the inclusion $\mathbf{Grpd} \hookrightarrow \mathbf{Cat}$. Thus the result follows. □

3.6. **Quasi-coherent sheaves on stacks.** Since the full subcategory $\mathbf{AffSch} \subseteq \mathbf{Sch}$ is a basis with respect to the Zariski topology, we can use Theorem 1.3.17 to get the following:

Corollary 3.6.1. *The presheaf of categories* $\mathrm{QCoh}^{\mathrm{op}}\colon \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Cat}$ *is the right Kan extension of its restriction to* $\mathbf{AffSch}$.

Under the equivalence $\mathbf{AffSch}^{\mathrm{op}} \simeq \mathbf{CRing}$, $\mathrm{QCoh}|_{\mathbf{AffSch}}$ is identified with the functor $\mathbf{CRing} \to \mathbf{Cat}$ sending $R \mapsto \mathbf{Mod}_R$. Thus we have the following characterization of QCoh.

Corollary 3.6.2. *For every scheme X, there is a canonical equivalence*

$$\mathrm{QCoh}(X) \simeq \varprojlim_{(R,x)} \mathbf{Mod}_R,$$

where the limit is taken of the category of pairs (R, x) consisting of a ring $R \in \mathbf{CRing}$ and $x \in X(\mathrm{Spec}(R)) = \mathrm{Mor}_{\mathbf{Sch}}(R, X)$ is an R-point, where morphisms $(R, x) \to (R', x')$ consist of a ring morphism $R \to R'$ such that the induced functor $X(R) \to X(R')$ sends x to x'.

This motivates a simple definition of quasi-coherent sheaves for a stack.

Definition 3.6.3. We define $\mathrm{QCoh}\colon \mathbf{Stk}^{\mathrm{op}} \to \mathbf{Cat}$ to be the right Kan extension of the presheaf $\mathrm{Spec}(R) \mapsto \mathbf{Mod}_R$ along the inclusion $\mathbf{AffSch} \hookrightarrow \mathbf{Stk}$. That is, we set

$$\mathrm{QCoh}(\mathcal{X}) \stackrel{\mathrm{def}}{=} \varprojlim_{(R,x)} \mathbf{Mod}_R,$$

where the limit is over the category of pairs (R, x) consisting of a ring $R \in \mathbf{CRing}$ and $x \in \mathcal{X}(R)$.

Remark 3.6.4. By definition, a quasi-coherent sheaf $\mathcal{F}$ on a stack $\mathcal{X}$ consists of the following data:

(1) For every $R \in \mathbf{CRing}$ and every R-point $x \in \mathcal{X}(R)$, an R-module $\mathcal{F}(x)$.
(2) For every ring homomorphism $R \to R'$, R-point $x \in \mathcal{X}(R)$, and R'-point $x' \in \mathcal{X}(R')$ such that $\mathcal{X}(R) \to \mathcal{X}(R')$ sends $x \mapsto x'$, an R'-module isomorphism

$$\alpha_{x,x}\colon \mathcal{F}(x) \otimes_R R' \simeq \mathcal{F}(x').$$

This data is subject to the following cocycle condition: for every pair of ring morphisms $R \to R'$ and $R' \to R''$, R-point $x \in \mathcal{X}(R)$, R'-point $x \in \mathcal{X}(R')$, and R''-point $x \in \mathcal{X}(R'')$ such that $\mathcal{X}(R) \to \mathcal{X}(R')$ sends $x \mapsto x'$ and $\mathcal{X}(R') \to \mathcal{X}(R'')$ sends $x' \mapsto x''$, there is a commutative diagram

$$\begin{tikzcd}
(\mathcal{F}(x) \otimes_R R') \otimes_{R'} R'' \arrow[r, "\alpha_{x,x'}"] \arrow[d, equal] & \mathcal{F}(x') \otimes_{R'} R'' \arrow[r, "\alpha_{x',x''}"] & \mathcal{F}(x'') \arrow[d, equal] \\
\mathcal{F}(x) \otimes_R R'' \arrow[rr, "\alpha_{x,x''}"] & & \mathcal{F}(x'')
\end{tikzcd}$$

4. Algebraic and derived categories

Now that we have briefly introduced schemes and stacks, we build up to the subject of derived algebraic geometry. In short, the idea is to make an ∞-categorical version of a stack. It seems reasonable to replace **Grpd** with $\mathbf{Grpd}_\infty$, but what should we replace schemes with? This depends on your choice of replacement of affine schemes. One approach is to replace with an appropriate notion of spectrum for $\mathbb{E}_\infty$-rings[2] (if you know what those are), in which case we arrive at *spectral algebraic geometry.* Here we work instead with *animated* (or *simplicial*) *rings.* For a comparison of the two generalizations see [Lur3, §25]. Note that if we look at their analogs of R-algebras, then the two theories are more or less equivalent when R is a $\mathbb{Q}$-algebra.

We begin with a discussion of algebraic categories and their animation.

4.1. Algebraic categories.

We say a category $\mathcal{C}$ is *locally small* if $\operatorname{Hom}_{\mathcal{C}}(X, Y)$ is a set for all $X, Y \in \mathcal{C}$.

Definition 4.1.1. A category $\mathcal{C}$ is *algebraic* if there exists a locally small full subcategory $\mathcal{F}_{\mathcal{C}} \subseteq \mathcal{C}$ closed under finite coproducts, and an equivalence

$$\operatorname{Fun}_{\Pi}\left(\mathcal{F}_{\mathcal{C}}^{\mathrm{op}}, \mathbf{Set}\right) \xrightarrow{\simeq} \mathcal{C}.$$

The restriction along the Yoneda embedding $\mathcal{F}_{\mathcal{C}} \hookrightarrow \operatorname{Fun}_{\Pi}\left(\mathcal{F}_{\mathcal{C}}^{\mathrm{op}}, \mathbf{Set}\right)$ gives the inclusion $\mathcal{F}_{\mathcal{C}} \hookrightarrow \mathcal{C}$. The subscript Π indicates the full subcategory of finite product-preserving functors.

A reflexive pair is a commutative diagram of the form

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B \arrow[l, "s"'] \end{tikzcd}$$

A *reflexive coequalizer* is a colimit of this diagram, or equivalently a coequalizer for the diagram ignoring the morphism s.

Definition 4.1.2. Let $\mathcal{C}$ be a category.

(1) An object $X \in \mathcal{C}$ is *compact* if $\operatorname{Hom}_{\mathcal{C}}(X, -)\colon \mathcal{C} \to \mathbf{Set}$ preserves filtered colimits.
(2) A compact object $X \in \mathcal{C}$ is *projective* if $\operatorname{Hom}_{\mathcal{C}}(X, -)\colon \mathcal{C} \to \mathbf{Set}$ preserves reflexive coequalizers.

Given $\mathcal{C}$, we can explicitly say what $\mathcal{F}_{\mathcal{C}}$ is up to equivalence.

Proposition 4.1.3. *A category $\mathcal{C}$ is algebraic if and only if the equivalence in Definition 4.1.1 holds with $\mathcal{F}_{\mathcal{C}}$ being the full subcategory of compact projective objects.*

Proof. Let $\mathcal{C} = \operatorname{Fun}_{\Pi}(\mathcal{F}_{\mathcal{C}}^{\mathrm{op}}, \mathbf{Set})$ for some $\mathcal{F}_{\mathcal{C}}$ satisfying the conditions in the definition. Then the objects in $\mathcal{F}_{\mathcal{C}}$ may be identified with the contravariant hom-functors $X = h^A = \operatorname{Hom}_{\mathcal{F}_{\mathcal{C}}}(-, A)$ for $A \in \operatorname{Ob} \mathcal{F}_{\mathcal{C}}$. Then for a functor $F \in \mathcal{C}$ we have by the Yoneda lemma

$$\operatorname{Hom}_{\mathcal{C}}(h^A, F) \cong F(A).$$

Now consider a filtered colimit $\varinjlim_{i \in I} \operatorname{Hom}_{\mathcal{C}}(h^A, F_i)$. Then it suffices to justify the middle step in the following:

$$\varinjlim_{i \in I} \operatorname{Hom}_{\mathcal{C}}(h^A, F_i) \cong \varinjlim_{i \in I} F_i(A) \cong \left(\varinjlim_{i \in I} F_i\right)(A) \cong \operatorname{Hom}_{\mathcal{C}}\left(h^A, \varinjlim_{i \in I} F_i\right).$$

[2] Here we write "$\mathbb{E}_\infty$-ring" to mean $\mathbb{E}_\infty$-ring spectra. By "spectrum" above we mean to replace the object $\operatorname{Spec}(A)$ with an appropriate étale version $\operatorname{Spét}(A)$ for when A is an $\mathbb{E}_\infty$-ring. See [Lur3, Definitions 1.2.3.3 and 1.4.2.5] for more details.

The middle step follows from using the fact that $\varinjlim_{i\in I} F_i(-)$ is indeed an object in $\mathcal{C}$ since filtered colimits respect finite products in **Set**. Checking the universal property for colimits shows the middle step. This shows h^A is compact. We can similarly check that h^A is projective by Yoneda and using the fact that reflexive coequalizers also respect finite products in **Set**. This shows that every object in $\mathcal{F}_{\mathcal{C}}$ is compact projective. □

When referring to an algebraic category $\mathcal{C}$, assume $\mathcal{F}_{\mathcal{C}}$ is the category of compact projective objects. The proof also implies that $\mathcal{C}$ is the free completion of $\mathcal{F}_{\mathcal{C}}$ by filtered colimits and reflexive coequalizers.

Definition 4.1.4. Let $\mathcal{C}$ be a category admitting colimits and $\mathcal{D}$ a category admitting filtered colimits and reflexive coequalizers. We define $\mathrm{Fun}'(\mathcal{C},\mathcal{D})$ be the full subcategory of functors that preserve filtered colimits and reflexive coequalizers.

Proposition 4.1.5. *Let $\mathcal{C}$ be an algebraic category. For every category $\mathcal{D}$ admitting filtered colimits and reflexive coequalizers, the canonical functor*

$$\mathrm{Fun}'(\mathcal{C},\mathcal{D}) \to \mathrm{Fun}(\mathcal{F}_{\mathcal{C}},\mathcal{D})$$

is an equivalence.

Proposition 4.1.6. *Let $\mathcal{C}$ be an additive algebraic category. Then the forgetful functor*

$$\mathrm{Fun}_{\Pi}(\mathcal{F}_{\mathcal{C}}^{\mathrm{op}},\mathbf{Ab}) \to \mathrm{Fun}_{\Pi}(\mathcal{F}_{\mathcal{C}}^{\mathrm{op}},\mathbf{Set}) \simeq \mathcal{C}$$

is an equivalence.

Proof. This is just because every finite product-preserving functor $X\colon \mathcal{F}_{\mathcal{C}}^{\mathrm{op}} \to \mathbf{Set}$ automatically takes values in abelian groups. □

If a category $\mathcal{F}$ is generated under finite coproducts by an element $1 \in \mathcal{F}$ so that every object is $1^{\oplus n}$ for some $n \geq 0$, then we may think of $\mathrm{Fun}_{\Pi}(\mathcal{F}^{\mathrm{op}},\mathbf{Set})$ as the category of sets X equipped with some operations $X^{\times m} \to X^{\times n}$, which are encoded by elements of $\mathrm{Hom}_{\mathcal{F}}(1^{\oplus n}, 1^{\oplus m})$.

Here some examples of algebraic categories.

Example 4.1.7. The category of sets is algebraic: we have

$$\mathbf{Set} \simeq \mathrm{Fun}_{\Pi}(\mathbf{Fin}^{\mathrm{op}},\mathbf{Set}),$$

where **Fin** is the full subcategory of finite sets. Given a set X, we define a product-preserving functor $\mathbf{Fin}^{\mathrm{op}} \to \mathbf{Set}$ sending $Y \mapsto \mathrm{Hom}(Y,X)$, which gives us a functor $\mathbf{Set} \to \mathbf{Fin}_{\Pi}(\mathbf{Fin}^{\mathrm{op}},\mathbf{Set})$. Given a product-preserving functor $F\colon \mathbf{Fin}^{\mathrm{op}} \to \mathbf{Set}$, we get the following data:

(1) We have sets $F_0 = F(\varnothing)$, $F_1 = F(\{1\})$, $F_2(\{1,2\})$, ...
(2) For every map of sets $\{1,2,\dots,n\} \to \{1,2,\dots,m\}$ and induced map $F_m \to F_n$.
(3) Canonical isomorphisms $F_0 \simeq \mathrm{pt} \overset{\mathrm{def}}{=} \{*\}$ and $F_n \simeq (F_1)^{\times n}$ for $n \geq 1$.

The assignment $F \mapsto F_1$ defines a functor from the right hand side to **Set**, and it is straightforward to see these functors define an equivalence of categories. Moreover, the compact projective objects in **Set** are exactly the finite sets.

Example 4.1.8. For a ring R, the category $\mathbf{Mod}_R$ of R-modules is algebraic:

$$\mathbf{Mod}_R \simeq \mathrm{Fun}_{\Pi}(\mathcal{F}_{\mathbf{Mod}_R}^{\mathrm{op}},\mathbf{Set}),$$

where $\mathcal{F}_{\mathbf{Mod}_R}$ is the category of finitely generated free R-modules (that is, $R^{\oplus n}$ for $n \geq 0$). The compact projective objects are precisely objects belonging to $\mathcal{F}_{\mathbf{Mod}_R}$.

Note that $\mathcal{F}_{\mathbf{Mod}_R}$ is equivalent to the category whose objects are natural numbers, morphisms are matrices of elements of R

$$\mathrm{Hom}(n, m) = \mathrm{Hom}_{\mathbf{Mod}_R}(R^{\oplus n}, R^{\oplus m}) = \mathrm{Mat}_{m\times n}(R),$$

and the composition rule is just matrix multiplication.

An object of $\mathrm{Fun}_\Pi(\mathcal{F}^{\mathrm{op}}_{\mathbf{Mod}_R}, \mathbf{Set})$ amounts to the data of sets $F_0, F_1, \ldots$ with isomorphisms $F_n \simeq (F_1)^{\times n}$ for all n, and for every matrix $\phi \in \mathrm{Mat}_{m\times n}(R)$, a map $F_\phi \colon F_m \to F_n$. This data is subject to identities imposed by functoriality.

From this data, we let $M \overset{\mathrm{def}}{=} F_1 \in \mathbf{Set}$, which is equipped with various operations encoded by matrices. For example, every $(n \times 1)$-matrix $\phi = (a_i)_{i=1}^n$ encodes the operation $F_\phi \colon M^{\times n} \to M$ given by the linear combination

$$(x_i)_{i=1}^n \mapsto \sum_i a_i x_i.$$

In particular:

(1) There is a (commutative) addition map $M \times M \to M$ given by $(x_1, x_2) \mapsto x_1 + x_2$, encoded by the matrix $(1, 1)$.
(2) There is a zero element $0 \in M$, encoded as the map $0\colon \mathrm{pt} \to M$ coming from the empty matrix.
(3) There is an additive inverse map $M \to M$, $x_1 \mapsto -x_1$ encoded by the (1×1)-matrix (-1).

Associativity and distributivity of addition and multiplication follows from the associativity of morphisms.

With the equivalence given above, we can also see that the category $\mathbf{CAlg}_R$ of commutative R-algebras is algebraic:

$$\mathbf{CAlg}_R \simeq \mathrm{Fun}_\Pi(\mathcal{F}^{\mathrm{op}}_{\mathbf{CAlg}_R}, \mathbf{Set}),$$

where $\mathcal{F}_{\mathbf{CAlg}_R}$ is the category of finitely generated polynomial R-algebras. These are the compact projective objects in $\mathbf{CAlg}_R$.

Example 4.1.9 (Clausen-Scholze)**.** A *profinite set/space* X is a topological space that is homeomorphic to a (small) limit of a diagram of finite discrete spaces. This is equivalent to being Hausdorff, quasi-compact, and totally disconnected (see [SP, 08ZW]). These form a full subcategory of $\mathbf{Top}$, which we denote as $\mathbf{Prof}$. We can assign a Grothendieck topology on $\mathbf{Prof}$ by declaring a family of morphisms $\{U_i \to U\}_{i\in I}$ to be a covering if and only if there is a finite subset $J \subset I$ such that $\coprod_{i\in J} U_i \to U$ is a surjective morphism in $\mathbf{Top}$.

A *condensed set* is a functor $X\colon \mathbf{Prof}^{\mathrm{op}} \to \mathbf{Set}$ such that:

(1) X carries finite disjoint unions to finite products (here $X(\varnothing)$ is a singleton).
(2) For every surjection $T \to S$ in $\mathbf{Prof}$, the morphism $X(S) \to X(T)$ is an equalizer for the pair of maps $X(T) \rightrightarrows X(T \times_S T)$.

equivalently, a condensed set is a sheaf on the site $\mathbf{Prof}$ with values in $\mathbf{Set}$ with respect to the Grothendieck topology defined above. $\mathbf{Cond}$ is algebraic with $\mathcal{F}_{\mathbf{Cond}}$ given by extremally disconnected sets (meaning the closure of any open set is open).

4.2. **Animation and its meaning.** Now that we have described algebraic categories, we define the animation of an algebraic category $\mathcal{C}$. We recall that $\mathcal{C} \simeq \mathrm{Fun}_\Pi(\mathcal{F}^{\mathrm{op}}_{\mathcal{C}}, \mathbf{Set})$; the idea here is to replace $\mathbf{Set}$ with a suitable ∞-category. Depending on the choice, we arrive at different ∞-categories.

Definition 4.2.1. Let $\mathcal{C}^=$ be an ∞-category admitting colimits and $\mathcal{D}$ be a category admitting filtered colimits and geometric realizations. We define $\mathrm{Fun}'(\mathcal{C}, \mathcal{D})$ be the full ∞-subcategory of functors that preserve filtered colimits and geometric realizations.

Definition 4.2.2. Let $\mathcal{C}$ be an algebraic category. Given an ∞-category $\mathcal{C}^+$ admitting colimits, we say that a fully faithful functor $\mathcal{F}_{\mathcal{C}} \hookrightarrow \mathcal{C}^+$ exhibits $\mathcal{C}^+$ as an *animation* of $\mathcal{C}$ if it has the following universal property: For every ∞-category $\mathcal{D}$ admitting filtered colimits and geometric realizations, restriction induces an equivalence

$$\mathrm{Fun}'(\mathcal{C}^+, \mathcal{D}) \to \mathrm{Fun}(\mathcal{F}_{\mathcal{C}}, \mathcal{D}),$$

Roughly, $\mathcal{C}^+$ is freely generated by $\mathcal{F}_{\mathcal{C}}$ under filtered colimits and geometric realizations.

We denote the animation of an algebraic category $\mathcal{C}$ as $\mathrm{Anim}(\mathcal{C})$. An animated object of $\mathcal{C}$ is an object of $\mathrm{Anim}(\mathcal{C})$.

The animation of an algebraic category $\mathcal{C}$ is only defined up to equivalence of ∞-categories, so determining an appropriate presentation of the animation will be a common challenge.

Proposition 4.2.3. *Let $\mathcal{C}$ be an algebraic category.*

(1) The functors $\mathbf{Fin} \hookrightarrow \mathbf{Set} \hookrightarrow \mathbf{Grpd}_\infty$ *exhibit* $\mathbf{Grpd}_\infty$ *as an animation of* $\mathbf{Set}$.

(2) The Yoneda embedding

$$\mathcal{F}_{\mathcal{C}} \hookrightarrow \mathrm{Fun}_{\Pi}(\mathcal{F}_{\mathcal{C}}^{\mathrm{op}}, \mathbf{Grpd}_\infty)$$

exhibits the target as an animation of $\mathcal{C}$.

Proof. See [Lur1, Proposition 5.5.8.15]. □

Notation 4.2.4. We define the ∞-category of *anima* to be $\mathbf{Ani} = \mathrm{Anim}(\mathbf{Set})$. We refer to an element of $\mathbf{Ani}$ as an animum. The above proposition states that this is equivalent to $\mathbf{Grpd}_\infty$, but it puts an emphasis on the maps that define an animum/space/∞-groupoid. We can think of an animum/space/∞-groupoid as an object with a collection of maps satisfying certain rules. These rules can be interpreted algebraically, homotopically, or simplicially depending on your preference.

Example 4.2.5. Here are some examples of animations of categories:

(1) The ∞-category $\mathbf{Grpd}_\infty$ is the animation of the category of sets [Lur1, Prop. 5.5.8.15]. This is because a functor $F \in \mathrm{Fun}_{\Pi}(\mathbf{Fin}^{\mathrm{op}}, \mathbf{Grpd}_\infty)$ is completely determined by the ∞-groupoid $F(\{*\})$ because of how it maps products.

(2) We denote the animation $\mathrm{Anim}(\mathbf{CAlg}_R)$ by $\mathbf{ACAlg}_R$, the ∞-category of *animated (commutative) R-algebras.* When $R = \mathbb{Z}$ we write $\mathbf{ACRing} = \mathbf{ACAlg}_{\mathbb{Z}}$, and we refer to its objects as *animated (commutative) rings.*

Remark 4.2.6. Recall that an R-algebra may be represented by a morphism of rings $R \to S$. Similarly, the ∞-category $\mathbf{ACAlg}_R$ may be equivalently thought of as the ∞-slice category $\mathbf{ACRing}_{R/}$.

Definition 4.2.7. A spectrum X is connective if the nth component X_n is n-connective for all $n \geq 0$ (that is $\pi_i(X_n)$ for all $i < n$). The category $\mathbf{Sp}_{\geq 0}$ is the limit of the following tower in the ∞-category of ∞-categories:

$$(\mathbf{Spc}_*)_{\geq 0} \xleftarrow{\Omega} (\mathbf{Spc}_*)_{\geq 1} \xleftarrow{\Omega} \dots,$$

where $(\mathbf{Sp}_*)_{\geq n}$ is the full subcategory of n-connective pointed spaces.

Informally, a spectrum X is an *infinite delooping* of X_0, or that it gives X_0 the structure of an *infinite loop space.*

Definition 4.2.8. An animated object $X \in \mathrm{Anim}(\mathcal{C})$ is *discrete* if the corresponding functor $X\colon \mathcal{F}_{\mathcal{C}}^{\mathrm{op}} \to \mathbf{Grpd}_\infty$ takes values in sets. The inclusion $\mathbf{Set} \hookrightarrow \mathbf{Grpd}_\infty$, regarding sets as discrete ∞-groupoids, induces a fully faithful functor $\mathcal{C} \hookrightarrow \mathrm{Anim}(\mathcal{C})$ that identifies $\mathcal{C}$ with the discrete objects of $\mathrm{Anim}(\mathcal{C})$. There is a left adjoint $\pi_0\colon \mathrm{Anim}(\mathcal{C}) \to \mathcal{C}$, sending

$$(\mathcal{F}^{\mathrm{op}} \to \mathbf{Grpd}_\infty) \mapsto (\mathcal{F}^{\mathrm{op}} \to \mathbf{Grpd}_\infty \xrightarrow{\pi_0} \mathbf{Set}),$$

where $\pi_0\colon \mathbf{Grpd}_\infty \to \mathbf{Set}$ sends an ∞-groupoid to its set of isomorphism classes of objects (or connected components).

Example 4.2.9. For a ring R, we denote the animation $\mathrm{Anim}(\mathbf{Mod}_R)$ by $\mathrm{D}(R)_{\geq 0}$ (or $\mathrm{D}(R)^{\leq 0}$), that is an *animated R-module* M is a finite product-preserving functor of the form $M\colon \mathcal{F}_{\mathbf{Mod}_R}^{\mathrm{op}} \to \mathbf{Grpd}_\infty$. This is the ∞-category version of a right-bounded derived category, or the ∞-categorical localization at quasi-isomorphisms of chain complexes with $H_i = H^{-i} = 0$ for $i < 0$.

4.3. **Derived categories.**

Construction 4.3.1. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of algebraic categories. Its restriction to $\mathcal{F}_{\mathcal{C}}$ gives rise by universal property of animation to an essentially unique functor $\mathbf{L}F : \mathrm{Anim}(\mathcal{C}) \to \mathrm{Anim}(\mathcal{D})$ such that:

(1) $\mathbf{L}F$ preserves filtered colimits and geometric realizations.
(2) There is a commutative diagram

$$\begin{tikzcd} \mathcal{F}_{\mathcal{C}} \arrow[r, "F|_{\mathcal{F}_{\mathcal{C}}}"] \arrow[d, hook] & \mathcal{D} \arrow[r, hook] & \mathrm{Anim}(\mathcal{D}) \\ \mathrm{Anim}(\mathcal{C}) \arrow[urr, "\mathbf{L}F"'] & & \end{tikzcd}$$

Moreover, $\mathbf{L}F$ preserves finite coproducts (and hence all colimits) if and only if F preserves finite coproducts. We call $\mathbf{L}F$ the *left derived functor* or *animation* of F.

Remark 4.3.2. We have canonical isomorphisms

$$\pi_0 \mathbf{L}F(X) \simeq F(\pi_0(X)),$$

functorial in $X \in \mathrm{Anim}(\mathcal{C})$, where π_0 is left adjoint to the fully faithful functor $\mathcal{C} \hookrightarrow \mathrm{Anim}(\mathcal{C})$.

Proposition 4.3.3. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors of algebraic categories that preserve filtered colimits and reflexive coequalizers. Then there is a canonical equivalence*

$$\mathbf{L}G \circ \mathbf{L}F \simeq \mathbf{L}(G \circ F) \tag{4.3.1}$$

under either of the following conditions:

(1) F sends $\mathcal{F}_{\mathcal{C}}$ to $\mathcal{F}_{\mathcal{D}}$ (more generally, we can ask that for $X \in \mathcal{F}_{\mathcal{C}}$ the object $F(X)$ is a filtered colimit of objects in $\mathcal{F}_{\mathcal{D}}$).
(2) $\mathbf{L}G\colon \mathrm{Anim}(\mathcal{D}) \to \mathrm{Anim}(\mathcal{E})$ preserves discrete objects (more generally, it is enough that for every $X \in \mathcal{F}_{\mathcal{C}}$, $\mathbf{L}G(F(X)) \in \mathrm{Anim}(\mathcal{E})$ is discrete).

Proof. Check the universal property on both sides in either case. □

Definition 4.3.4. The *derived ∞-category* of $\mathcal{C}$ is the stable ∞-category

$$\mathrm{D}(\mathcal{C}) = \mathrm{Fun}_\Pi(\mathcal{F}_{\mathcal{C}}^{\mathrm{op}}, \mathbf{Sp}). \tag{4.3.2}$$

We let $\mathrm{D}(\mathcal{C})_{\geq 0}$ be the essential image of the fully faithful embedding $\mathrm{Anim}(\mathcal{C}) \hookrightarrow \mathrm{D}(\mathcal{C})$. We say an object in $\mathrm{D}(\mathcal{C})$ is *connective* if it lies in $\mathrm{D}(\mathcal{C})_{\geq 0}$.

Proposition 4.3.5. *Let $\mathcal{C}$, $\mathcal{D}$ be additive algebraic categories and $F\colon \mathrm{Anim}(\mathcal{C}) \to \mathrm{Anim}(\mathcal{D})$ a functor.*

(1) If F commutes with Ω, then it extends uniquely to a functor

$$F\colon \mathrm{D}(\mathcal{C}) \to \mathrm{D}(\mathcal{D})$$

such that $\Omega^{\infty-n}\circ \simeq F\circ\Omega^{\infty-n}$ for all $n \geq 0$. Informally, we map $(X_i)_{i\geq 0}$ to $(F(X_i))_{i\geq 0}$.

(2) If F preserves colimits and zero objects, then it extends uniquely to a colimit-preserving functor

$$F\colon \mathrm{D}(\mathcal{C}) \to \mathrm{D}(\mathcal{D})$$

such that $F\circ\Sigma^{\infty-n} \simeq \Sigma^{\infty-n}\circ F$ for all $n\geq 0$. Here $\Sigma^{\infty-n}\colon \mathrm{Anim}(-)\to \mathrm{D}(-)$ is left adjoint to $\Omega^{\infty-n}$.

4.4. **Animated algebra.**

Example 4.4.1. Let $f : R \to S$ be a ring morphism. We recall three canonical functors between $\mathbf{Mod}_R$ and $\mathbf{Mod}_S$:

(1) The *extension of scalars* functor $f_! \overset{\text{def}}{=} (-)\otimes_R S\colon \mathbf{Mod}_R \to \mathbf{Mod}_S$.
(2) The *coextension of scalars* functor $f_* \overset{\text{def}}{=} \mathbf{Mod}_R(S,-)\colon \mathbf{Mod}_R \to \mathbf{Mod}_S$.
(3) The *restriction of scalars* functor $f^* \overset{\text{def}}{=} (-)_R\colon \mathbf{Mod}_S \to \mathbf{Mod}_R$, where we identify an action of $r\in R$ on an S-module N via $r\cdot_R N \overset{\text{def}}{=} f(r)\cdot_S N$.

Here we have adjunctions

$$f_! : \mathbf{Mod}_R \leftrightarrows \mathbf{Mod}_S : f^*, \qquad f^* : \mathbf{Mod}_R \leftrightarrows \mathbf{Mod}_S : f_*.$$

Since $f_!$ and f^* are both left adjoints, they must preserve colimits (such as filtered colimits and reflexive coequalizers). These functors all give rise to animated functors between $\mathrm{D}(R)_{\geq 0}$ and $\mathrm{D}(S)_{\geq 0}$. We denote the animation of extension of scalars as

$$\mathbf{L}f_! \overset{\text{def}}{=} (-)\otimes^{\mathbf{L}}_R S\colon \mathrm{D}(R)_{\geq 0} \to \mathrm{D}(S)_{\geq 0}.$$

We aim to show that $\mathbf{L}f_!$ is left adjoint to $\mathbf{L}f^*$. For this, it suffices to show that $\mathbf{L}(f_!\circ f^*) \simeq \mathbf{L}f_!\circ\mathbf{L}f^*$ and $\mathbf{L}(f^*\circ f_!) \simeq \mathbf{L}f^*\circ\mathbf{L}f_!$ since we have unit and counit transformations $\mathrm{id}_{\mathbf{Mod}_R} \to f^*\circ f_!$ and $f_!\circ f^* \to \mathrm{id}_{\mathbf{Mod}_S}$ defining the adjunction.

(1) Note that $f_!(R^n) = R^n\otimes_R S \cong S^n$ as S-modules, so $f_!$ maps $\mathcal{F}_{\mathbf{Mod}_R}$ to $\mathcal{F}_{\mathbf{Mod}_S}$. This implies that $\mathbf{L}f^*\circ\mathbf{L}f_! \simeq \mathbf{L}(f^*\circ f_!)$ by letting $M = R^n$ by condition (1) from Proposition 4.3.3.
(2) Let M be an R-module. Lazard's theorem says that M is flat if and only if it can be written as a filtered colimit of finitely generated free modules. Then for M flat the functor $(-)\otimes^{\mathbf{L}}_R S$ commutes with this filtered colimit, and $(-)\otimes^{\mathbf{L}}_R S$ agrees with $(-)\otimes_R S$ on finitely generated free modules. Therefore, we have

$$M\otimes^{\mathbf{L}}_R S \simeq M\otimes_R S.$$

This implies that $(-)\otimes^{\mathbf{L}}_R S$ preserves discrete objects, so $\mathbf{L}f_!\circ\mathbf{L}f^* \simeq \mathbf{L}(f_!\circ f^*)$ by condition (2) from Proposition 4.3.3.

Construction 4.4.2. Given a ring R, Kähler differentials define a functor

$$\Omega_{-/R}\colon \mathbf{CAlg}_R \to \mathbf{CAlgMod}_R, \qquad A \mapsto (A, \Omega_{A/R}). \tag{4.4.1}$$

where $\mathbf{CAlgMod}_R$ is the category of pairs (A, M) such that A is an R-algebra, M is an A-module, and morphisms $(A, M) \to (A', M')$ are given by R-algebra morphisms $A \to A'$ together with A'-module morphisms $M\otimes_R A' \to M'$.

It is straightforward to check that $\mathbf{CAlgMod}_R$ is an algebraic category, so we can animate to get a functor

$$\Omega^{\mathrm{anim}}_{-/R} \colon \mathbf{ACAlg}_R \longrightarrow \mathrm{Anim}(\mathbf{CAlgMod}_R)$$

such that $\Omega^{\mathrm{anim}}_{-/R}$ is a section of the canonical projection $\mathrm{Anim}(\mathbf{CAlgMod}_R) \to \mathbf{ACAlg}_R$ (which is in turn the animation of the usual projection $\pi \colon \mathbf{CAlgMod}_R \to \mathbf{CAlg}_R$ given by $(A, M) \mapsto A$. The image of $A \in \mathbf{ACAlg}_R$ under this map may be regarded as a pair $(A, \mathbf{L}\Omega_{A/R})$, where $\mathbf{L}\Omega_{A/R} \in \mathrm{D}(A)_{\geq 0} = \mathrm{Anim}(\mathbf{Mod}_A)$.

Definition 4.4.3. In the above construction, the (relative) *cotangent complex* of $A \in \mathbf{ACAlg}_R$ is $\mathbf{L}_{A/R} \overset{\mathrm{def}}{=} \mathbf{L}\Omega_{A/R} \in \mathrm{D}(A)_{\geq 0}$.

By construction, we have $\pi_0 \mathbf{L}_{A/R} \simeq \Omega_{\pi_0(A)/\pi_0(R)}$.

Proposition 4.4.4. *The following hold:*

(1) Let $A \to B$ be a morphism in $\mathrm{Anim}(\mathbf{CAlg}_R)$. Then there is an exact triangle

$$\mathbf{L}_{A/R} \otimes^{\mathbf{L}}_A B \to \mathbf{L}_{B/R} \to \mathbf{L}_{B/A}$$

in $\mathrm{D}(B)$.

(2) For every cocartesian square of the form

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array}$$

in $\mathbf{ACAlg}_R$ (exhibiting B' as the derived tensor product $A' \otimes^{\mathbf{L}}_A B$, there is a canonical isomorphism

$$\mathbf{L}_{B/A} \otimes^{\mathbf{L}}_B B' \simeq \mathbf{L}_{B'/A'}.$$

Definition 4.4.5. Let $R \in \mathbf{CRing}$, $A \in \mathbf{ACAlg}_R$, and $M \in \mathrm{D}(R)_{\geq 0}$. The canonical morphism $A \oplus M \to A$, given informally by $(a, m) \mapsto a$, induces a map

$$\mathrm{Hom}_{\mathbf{ACAlg}_R}(A, A \oplus M) = \mathrm{Hom}_{\mathbf{ACAlg}_R}(A, A).$$

The space $\mathrm{Der}_R(A, M)$ or R-linear derivations of A with values in M is the homotopy fiber of the map at the point id_A. Concretely, a point of $\mathrm{Der}_R(A, M)$ is a dashed arrow making the below diagram commute:

$$\begin{tikzcd} R \arrow[r] \arrow[d] & A \oplus M \\ A \arrow[r, equal] \arrow[ur, dashed] & A \arrow[u] \end{tikzcd}$$

The trivial derivation is the map $A \mapsto A \oplus 0 \simeq A$.

Theorem 4.4.6. *Let $A \in \mathbf{ACAlg}_R$. Then the cotangent complex $\mathbf{L}_{A/R}$ corepresents the functor $\mathrm{D}(A)_{\geq 0} \to \mathbf{Grpd}_\infty$, $M \mapsto \mathrm{Der}_R(A, M)$. That is, there are canonical isomorphisms*

$$\mathrm{Hom}_{\mathrm{D}(A)_{\geq 0}}(\mathbf{L}_{A/R}, M) \simeq \mathrm{Der}_R(A, M).$$

Proof. If $A = R[T_1, \ldots, T_m]$ is a polynomial algebra, then $\mathbf{L}_{A/R} \simeq \Omega_{A/R}$, which is free on m generators. In general, we write A as a geometric realization of a simplicial diagram of filtered systems of polynomial algebras. $\square$

Definition/Proposition 4.4.7. Let $A \to B$ be a morphism of animated commutative rings. We say that it is *flat* if it satisfies the following equivalent conditions:

(1) The functor $(-)\otimes^{\mathbf{L}}_A B\colon \mathrm{D}(A)_{\geq 0}\to \mathrm{D}(B)_{\geq 0}$ is left-exact, i.e. it preserves discrete objects.
(2) The induced ring morphism $\pi_0(A)\to\pi_0(B)$ is flat, and the canonical morphisms

$$\pi_i(A)\otimes_{\pi_0(A)}\pi_0(B)\to\pi_i(B)$$

are bijective for all $i\geq 0$.

We say that $A\to B$ is *faithfully flat* if it is flat and $\pi_0(A)\to\pi_0(B)$ is faithfully flat.

5. Derived algebraic geometry

This section is a basic overview of algebraic geometry from the functor of points perspective.

5.1. **Derived stacks.**

Definition 5.1.1. Let R be a commutative ring.

(1) A *derived stack* is a functor $X\colon \mathbf{ACRing}\to\mathbf{Grpd}_\infty$ satisfying étale descent (i.e. is a sheaf with respect to the étale topology). We write the ∞-category of derived stacks as $\mathbf{DStk}\overset{\text{def}}{=}\mathrm{Shv}(\mathbf{ACRing}^{\mathrm{op}},\mathbf{Grpd}_\infty)$.
(2) A *derived stack over* R is a functor $X\colon \mathbf{ACAlg}_R\to\mathbf{Grpd}_\infty$ satisfying étale descent. We denote the ∞-category as $\mathbf{DStk}_R\overset{\text{def}}{=}\mathrm{Shv}(\mathbf{ACAlg}_R^{\mathrm{op}},\mathbf{Grpd}_\infty)$.
(3) A *derived affine scheme* is the derived stack $\mathrm{Spec}(A)\colon \mathbf{ACRing}\to\mathbf{Grpd}_\infty$ given by $B\mapsto\mathrm{Hom}(A,B)$, which is corepresented by an animated ring $A\in\mathbf{ACRing}$.

Remark 5.1.2. Let $X\to\mathrm{Spec}(A)$ be a morphism of derived stacks with affine target. Then X may be regarded as a functor $X\colon\mathbf{ACAlg}_A\to\mathbf{Grpd}_\infty$, where $\mathbf{ACAlg}_R=\mathbf{ACRing}_{A/}$ is the ∞-category of animated A-algebras, via a canonical equivalence

$$\mathrm{Shv}(\mathbf{ACRing}^{\mathrm{op}},\mathbf{Grpd}_\infty)_{/\,\mathrm{Spec}(A)}\simeq\mathrm{Shv}(\mathbf{ACAlg}_A^{\mathrm{op}};\mathbf{Grpd}_\infty).$$

Definition 5.1.3. Let $j:U\to X$ be a morphism of derived stacks.

(1) If X and U are affine, we say j is an open immersion if it is étale (i.e. $\mathcal{O}_X\to\mathcal{O}_U$ is an étale morphism of derived commutative rings) and $U_{\mathrm{cl}}\to X_{\mathrm{cl}}$ is an open immersion of classical affines.
(2) If X is affine, we say j is an open immersion if it is a monomorphism (the diagonal $U\to U\times_X U$ is an isomorphism) and there exists a collection of affines $(U_\alpha)_\alpha$ and a surjection $\coprod_\alpha U_\alpha\to U$ such that each composite $U_\alpha\to X$ is an open immersion of affines.
(3) In general, we say j is an open immersion if for every affine S and a morphism $S\to X$, the base change map $U\times_X S\to S$ is an open immersion to an affine.

Definition 5.1.4. Let X be a derived stack.

(1) We say that X is *0-Artin* or a *derived algebraic space* if there exists an étale surjection $U\twoheadrightarrow X$, where U is a disjoint union of affine derived schemes.
(2) We say that X is a *derived algebraic stack* if there exists a smooth surjection $U\twoheadrightarrow X$, where U is a derived algebraic space.
(3) We say a morphism of derived stacks $f\colon X\to Y$ is *representable by derived algebraic stacks*, or *1-representable*, if for every affine derived scheme V and every morphism $v\colon V\to Y$, the derived fiber $X\times^{\mathbf{R}}_Y V$ is a derived algebraic space.

(4) We say a morphism of derived stacks $f\colon X \to Y$ is *representable by derived algebraic stacks*, or *representable*, if for every affine derived scheme V and every morphism $v\colon V \to Y$, the derived fiber $X \times^{\mathbf{R}}_Y V$ is a derived algebraic stack.

Definition 5.1.5. Given a derived stack X, the restriction of the functor X along $\mathbf{CRing} \hookrightarrow \mathbf{ACRing}$ is called the *classical truncation* of X, denoted X_{cl}. For example, $\mathrm{Spec}(A)_{\mathrm{cl}} \simeq \mathrm{Spec}(\pi_0(A))$ for any $A \in \mathbf{ACAlg}_R$. In one sense, X is a *higher stack.* If the ∞-groupoid $X(R)$ is 1-truncated for every $R \in \mathbf{CRing}$, then $X_{\mathrm{cl}}\colon \mathbf{CRing} \to \mathbf{Grpd}$ is a stack. For example, this will be the case if X is a derived algebraic space.

If the ∞-groupoid $X(R)$ is 1-truncated for every ring R, then $X_{\mathrm{cl}}\colon \mathbf{CRing} \to \mathbf{Grpd}$ is a stack. This will be the case if X is a derived algebraic stack.

Definition 5.1.6. We fix the following definitions:

(1) We say X is a *derived scheme* if there exists a collection $(U_\alpha \hookrightarrow X)_\alpha$ of open immersions, where U_α are affine schemes, and a surjection $\coprod_\alpha U_\alpha \twoheadrightarrow X$.
(2) A morphism $f : X \to Y$ of derived stacks is *schematic* if for every affine V and every morphism $V \to Y$, the fiber $X \times^{\mathbf{R}}_Y V$ is a derived scheme.
(3) A schematic morphism $f : X \to Y$ of derived stacks is *smooth* (resp. étale) if for every affine V and every morphism $V \to Y$, there exists a collection of open immersions $(U_\alpha \hookrightarrow X \times_Y V)_\alpha$, where each U_α is affine and each composite $U_\alpha \to X \times_Y V \to V$ is a smooth (resp. étale) morphism of affines.

Remark 5.1.7. A derived scheme X is 0-truncated, in the sense that the functor $X\colon \mathbf{ACAlg}_R \to \mathbf{Grpd}_\infty$ takes values in sets (discrete ∞-groupoids).

5.2. Quasi-coherent sheaves.

Definition 5.2.1. Given a derived stack X, we define the ∞-*category of quasi-coherent sheaves* $\mathrm{D}_{\mathrm{qc}}(X)$ to be the limit

$$\mathrm{D}_{\mathrm{qc}}(X) \overset{\mathrm{def}}{=} \varprojlim_{(A,x)} \mathrm{D}(A) \tag{5.2.1}$$

over the category of pairs (A, x), where $A \in \mathbf{ACAlg}_R$ and $x \in X(A)$. In other words, $\mathrm{D}_{\mathrm{qc}}\colon \mathbf{DStk}^{\mathrm{op}} \to \mathbf{Cat}_\infty$ is the right Kan extension of the presheaf $\mathrm{Spec}(A) \mapsto \mathrm{D}(A)$ along the inclusion $\mathbf{AffSch} \hookrightarrow \mathbf{DStk}$. We refer to objects of $\mathrm{D}_{\mathrm{qc}}(X)$ as *quasi-coherent complexes* on X. Note that $\mathrm{D}_{\mathrm{qc}}(X)$ is stable since this property is preserved under limits.

Remark 5.2.2. The presheaf $\mathrm{D}_{\mathrm{qc}}\colon \mathbf{DStk}^{\mathrm{op}} \to \mathbf{Cat}_\infty$ is the unique limit-preserving functor extending $\mathrm{D}_{\mathrm{qc}}\colon \mathbf{AffSch}^{\mathrm{op}} \to \mathbf{Cat}_\infty$, $\mathrm{Spec}(A) \mapsto \mathrm{D}_{\mathrm{qc}}(\mathrm{Spec}(A)) \simeq \mathrm{D}(A)$ (as a right Kan extension). In particular, it sends colimits of derived stacks to limits of ∞-categories.

Definition 5.2.3. Let R be a ring.

(1) We say that a derived R-module $M \in \mathrm{D}(R)$ is *perfect* if it is contained in the full subcategory of $\mathrm{D}(R)$ generated by R under finite colimits, finite limits, and direct summands. This forms a full subcategory of $\mathrm{D}(R)$, which we denote as $\mathrm{Perf}(R)$.
(2) Let X be a stack. We say that a quasi-coherent complex $\mathcal{F} \in \mathrm{D}_{\mathrm{qc}}(X)$ is *perfect* if for every (R, x) the derived R-module $\mathbf{R}\Gamma(\mathrm{Spec}(R), x^*\mathcal{F})$ is perfect.

We write $\mathrm{D}_{\mathrm{perf}}(X) \subseteq \mathrm{D}_{\mathrm{qc}}(X)$ for the full subcategory spanned by perfect subcomplexes, so that

$$\mathrm{D}_{\mathrm{perf}}(X) \simeq \varprojlim_{(R,x)} \mathrm{D}_{\mathrm{perf}}(R),$$

where $\mathrm{D}_{\mathrm{perf}}(X)$ is the ∞-category of perfect derived R-modules.

Theorem 5.2.4. *The functor*

$$\mathbf{ACAlg}_R \to \mathbf{Cat}_\infty, \qquad A \mapsto \mathrm{D}(A)$$

is a sheaf for the flat topology. The same holds for $A \mapsto \mathrm{D}(A)_{\geq 0}$ and $A \mapsto \mathrm{Perf}(A)$.

Example 5.2.5. If X is affine, then we have the tautological equivalence given by right derived global sections:

$$\mathbf{R}\Gamma(X,-)\colon\ \mathrm{D}_{\mathrm{qc}}(X) \to \mathrm{D}(R).$$

Example 5.2.6. Let G be a group scheme over R acting on a derived stack X. The *quotient stack* $[X/G]$ is the colimit of the action groupoid

$$X_\bullet = [\ldots \substack{\to\\\to\\\to} G \times X \rightrightarrows X].$$

This gives us a limit diagram of ∞-categories

$$\mathrm{QCoh}([X/G]) \to \mathrm{QCoh}(X) \rightrightarrows \mathrm{QCoh}(G \times X) \substack{\to\\\to\\\to} \mathrm{QCoh}(G \times G \times X) \substack{\to\\\to\\\to\\\to} \ldots$$

In other words, the canonical functor $\mathrm{QCoh}([X/G]) \to \mathrm{Tot}(\mathrm{QCoh}(X_\bullet))$ is an equivalence, where $X_\bullet = [\ldots \substack{\to\\\to\\\to} G \times X \rightrightarrows X]$ is the action groupoid whose colimit is the quotient stack $[X/G]$. We call

$$\mathrm{QCoh}^G(X) \overset{\mathrm{def}}{=} \mathrm{Tot}(\mathrm{QCoh}(X_\bullet))$$

the *G-equivariant derived ∞-category of quasi coherent sheaves on X*, its objects we call *G-equivariant quasi-coherent sheaves* on X, are quasi-coherent sheaves $\mathcal{F}$ on X with a (specified) isomorphism $\mathrm{act}^*\mathcal{F} \simeq \mathrm{pr}^*\mathcal{F}$ on $G \times X$, as well as a homotopy coherent system of isomorphisms on the higher terms $G^{\times n} \times X$. Thus quasi-coherent sheaves on $[X/G]$ are equivalent to quasi-coherent sheaves on X that are G-equivariant in a homotopy coherent sense.

5.3. **Cotangent complexes.**

Definition 5.3.1. Let $X\colon \mathbf{ACAlg}_R \to \mathbf{Grpd}_\infty$ be a derived stack. We say that X *admits a cotangent complex* $\mathbf{L}_{X/R}$ if and only if the following conditions hold:

(1) For every $A \in \mathbf{ACAlg}_R$ and every $x \in X(A)$, denote by $F_x(N)$ the fiber at x of the map $X(A \oplus N) \to X(A)$ for every $N \in \mathrm{D}(A)_{\geq 0}$. Then the functor $F_X(-)$ is corepresented by a derived A-module M_x that is eventually coconnective (i.e. $M_x[n] \in \mathrm{D}(A)_{\geq 0}$ for some n).
(2) For every morphism $A \to B$ in $\mathbf{ACAlg}_R$ and every $N \in \mathrm{N}(B)_{\geq 0}$, the commutative square

$$\begin{tikzcd} X(A \oplus N) \arrow[r] \arrow[d] & X(B \oplus N) \arrow[d] \\ X(A) \arrow[r] & X(B) \end{tikzcd}$$

is cartesian.

Under these conditions, there exists an object $\mathbf{L}_{X/R} \in \mathrm{D}_{\mathrm{qc}}(X)$ such that $x^*\mathbf{L}_{X/R} \simeq M_x$ for every $A \in \mathbf{ACAlg}_R$ and $x \in X(A)$ (modulo the equivalence $\mathrm{D}_{\mathrm{qc}}(\mathrm{Spec}(A)) \simeq \mathrm{D}(A)$). We will often write $\mathbf{L}_X \simeq \mathbf{L}_{X/R}$.

Remark 5.3.2. As in the affine case (animated commutative rings), it is possible to talk about *relative* cotangent complexes $\mathbf{L}_{X/Y}$ for a morphism $X \to Y$ of derived stacks. Here we have $\mathbf{L}_{X/R} \simeq \mathbf{L}_{X/\,\mathrm{Spec}(R)}$, and by the following theorem there is an exact triangle

$$f^*\mathbf{L}_Y \to \mathbf{L}_X \to \mathbf{L}_{X/Y}$$

in the stable ∞-category $\mathrm{D}_{\mathrm{qc}}(X)$.

Theorem 5.3.3. *The following hold:*

(1) *Let S be a derived stack and $f\colon X \to Y$ a morphism over S. If X and Y admit cotangent complexes over S, then f admits a relative cotangent complex such that there is an exact triangle*
$$f^*\mathbf{L}_{Y/S} \to \mathbf{L}_{X/S} \to \mathbf{L}_{X/Y}$$
in $\mathrm{D}_{\mathrm{qc}}(X)$.

(2) *Let $f : X \to Y$ be a morphism of derived stacks. If f admits a relative cotangent complex, then for every morphism $Y' \to Y$ the derived base change $X \times^{\mathbf{R}}_Y Y' \to Y'$ admits a relative cotangent complex, and moreover there is a canonical isomorphism*
$$p^*\mathbf{L}_{X/Y} \simeq \mathbf{L}_{X\times^{\mathbf{R}}_Y Y'/Y'}$$
in $\mathrm{D}_{\mathrm{qc}}(X \times^{\mathbf{R}}_Y Y')$, where $p\colon X \times^{\mathbf{R}}_Y Y' \to X$ is the canonical projection.

Definition 5.3.4. Let $f : X \to Y$ be a morphism of derived stacks. Given an animated ring $A \in \mathbf{ACRing}$, an A-point $x \in X(A)$, and a connective derived A-module $\mathrm{D}(A)_{\geq 0}$, consider the commutative square

$$\begin{array}{ccc} X(A \oplus M) & \longrightarrow & X(A) \\ \downarrow & & \downarrow \\ X(B \oplus M) & \longrightarrow & X(B) \end{array}$$

induced by the morphism of animated rings $A \oplus M \to A$, $(a, m) \mapsto a$. This gives rise to a map of ∞-groupoids
$$X(A \oplus M) \to Y(A \oplus M) \times_{Y(A)} X(A).$$
The ∞-groupoid $\mathrm{Der}_x(X/Y, M)$ of *derivations of f at x with values in M* is the fiber of the above map at the point (y', x), where $y' \in Y(A\oplus M)$ is the image of $y = f(x) \in Y(A)$ along the trivial derivation $A \to A \oplus M$, $a \mapsto (a, 0)$.

Definition 5.3.5. Let $f : x \to Y$ be a morphism of (derived) stacks and $\mathcal{L} \in \mathrm{D}_{\mathrm{qc}}(X)$ an *eventually connective* quasi-coherent complex on X (i.e. $\mathcal{L}[n]$ is connective for some integer n). We say that $\mathcal{L}$ is a *(relative) cotangent complex* for $f\colon X \to Y$ if for every $A \in \mathbf{ACRing}$ and $x : \mathrm{Spec}(A) \to X$, the inverse image $\mathbf{L}x^*\mathcal{L}$ corepresents the functor $\mathrm{Der}_x(X/Y, -)$. That is, there are isomorphisms
$$\mathrm{Maps}_{\mathrm{D}(A)}(\mathbf{L}x^*\mathcal{L}, M) \simeq \mathrm{Der}_x(X/Y, M)$$
functorial in $M \in \mathrm{D}(A)_{\geq 0}$, where by abuse of notation we identify $\mathbf{L}x^*\mathcal{L} \in \mathrm{D}_{\mathrm{qc}}(\mathrm{Spec}(A))$ with the derived A-module $\mathbf{R}\Gamma(\mathrm{Spec}(A), \mathbf{L}x^*\mathcal{L}) \in \mathrm{D}(A)$.

Theorem 5.3.6. *Let $f : X \to Y$ be a 1-representable morphism of derived stacks.*

(1) *There exists a cotangent complex $\mathbf{L}_{X/Y}$ for f.*

(2) *The cotangent complex $\mathbf{L}_{X/Y}$ is (-1)-connective. That is, for every derived scheme U and every smooth morphism $p : U \to X$, the inverse image $p^*\mathbf{L}_{X/Y}$ is (-1)-connective, i.e.*
$$H^{-i}(U, \mathbf{L}_{X/Y}) = \pi_i\mathbf{R}\Gamma(U, \mathbf{L}_{X/Y}) = \pi_i \mathrm{Maps}_{\mathrm{D}(U)}(\mathcal{O}_U, p^*\mathbf{L}_{X/Y}) = 0$$
for all $i < -1$. If f is representable by derived algebraic spaces (or derived Deligne-Mumford stacks), then $\mathbf{L}_{X/Y}$ is connective.

Proposition 5.3.7. *The following hold:*

(1) *Let $f : X \to Y$ be a morphism of derived stacks over a derived stack S. Then there is an exact triangle*
$$\mathbf{L}f^*\mathbf{L}_{Y/S} \to \mathbf{L}_{X/S} \to \mathbf{L}_{X/Y}$$
in $\mathrm{D}_{\mathrm{qc}}(X)$.

(2) *For every cartesian square*
$$\begin{tikzcd} X' \arrow[r] \arrow[d, "p"] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$
in **DStk** *(exhibiting X' as the fiber product $X \times^{\mathbf{R}}_Y Y'$), there is a canonical isomorphism*
$$\mathbf{L}p^*\mathbf{L}_{X/Y} \simeq \mathbf{L}_{X'/Y'}.$$

5.4. **Moduli of sheaves.**

Definition 5.4.1. Let $\mathcal{M}_{\mathrm{Perf}}$ denote the functor
$$\mathcal{M}_{\mathrm{Perf}} \colon \mathbf{ACAlg}_R \to \mathbf{Grpd}_\infty, \qquad A \mapsto \mathrm{Perf}(A)^{\simeq}$$
sending A to the ∞-groupoid of perfect derived A-modules, with the superscript meaning we take the underlying ∞-groupoid by discarding all non-invertible morphisms. By Theorem 5.2.4, this satisfies étale descent and hence defines a derived stack over R.

Construction 5.4.2. By the Yoneda lemma, there is a *universal perfect complex*
$$\mathcal{E}^{\mathrm{univ}} \in \mathrm{Perf}(\mathcal{M}_{\mathrm{Perf}}),$$
which is defined such that for every animated ring A and every morphism $x \colon \mathrm{Spec}(A) \to \mathcal{M}_{\mathrm{Perf}}$ classifying a perfect complex $\mathcal{E} \in \mathrm{D}_{\mathrm{Perf}}(A)$, we have
$$x^*(\mathcal{E}^{\mathrm{univ}}) \simeq \mathcal{E}.$$

Remark 5.4.3. Let $\mathrm{D}_{\mathrm{pscoh}}(A) \subseteq \mathrm{D}_{\mathrm{qc}}(A)$ be the full subcategory of *pseudocoherent* derived A-modules (sometimes called *almost perfect* A-modules) and $\mathrm{D}_{\mathrm{coh}}(A) \subseteq \mathrm{D}_{\mathrm{pscoh}}(A)$ is the full subcategory of *coherent* derived A-modules.

We can also consider the stacks $\mathcal{M}_{\mathrm{D}_{\mathrm{coh}}}$ and $\mathcal{M}_{\mathrm{D}_{\mathrm{pscoh}}}$ sending $A \in \mathbf{ACAlg}_R$ to $\mathrm{D}_{\mathrm{coh}}(A)^{\simeq}$ to $\mathrm{D}_{\mathrm{coh}}(A)^{\simeq}$ $\mathrm{D}_{\mathrm{pscoh}}(A)^{\simeq}$, respectively. There are open immersions of derived stacks
$$\mathcal{M}_{\mathrm{D}_{\mathrm{perf}}} \hookrightarrow \mathcal{M}_{\mathrm{D}_{\mathrm{coh}}} \hookrightarrow \mathcal{M}_{\mathrm{D}_{\mathrm{pscoh}}},$$
but these larger stacks do not admit a cotangent complex (because the perfect complexes are precisely the dualizable objects in $\mathrm{D}_{\mathrm{pscoh}}(X)$).

Definition 5.4.4. Let X be a smooth proper scheme over R.

(1) The *moduli stack of perfect complexes over* X is the derived mapping stack
$$\mathcal{M}_{\mathrm{Perf}(X)} = \underline{\mathrm{Maps}}_{\mathrm{Spec}(R)}(X, \mathcal{M}_{\mathrm{Perf}}).$$
For $A \in \mathbf{ACAlg}_R$, its A-points are morphisms $X_A \overset{\mathrm{def}}{=} X \times \mathrm{Spec}(A) \to \mathcal{M}_{\mathrm{Perf}}$ over $\mathrm{Spec}(A)$, i.e. perfect complexes on X_A.

(2) Given a group scheme G over R, the *moduli stack of G-torsors on X* (a.k.a. *principal G-bundles* on X) is the derived mapping stack
$$\mathcal{M}_{\mathrm{Bun}_G(X)} = \underline{\mathrm{Maps}}(X, BG).$$
For $A \in \mathbf{ACAlg}_R$, its A-points are morphisms $X_A \to BG$ over $\mathrm{Spec}(A)$, i.e. G-torsors on X_A.

(3) The *moduli stack of vector bundles on* X is the substack $\mathcal{M}_{\mathrm{Vect}(X)} \subseteq \mathcal{M}_{\mathrm{Perf}(X)}$ defined as follows: for $A \in \mathbf{ACAlg}_R$, an A-point of $\mathcal{M}_{\mathrm{Perf}(X)}$ belongs to $\mathcal{M}_{\mathrm{Vect}(X)}$ if and only if the corresponding perfect complex $\mathcal{F} \in \mathrm{D}_{\mathrm{perf}}(X_A)$ is of Tor-amplitude $[0,0]$, i.e. it is connective and flat over X_A.

(4) The *moduli stack of coherent sheaves on* X is the substack $\mathcal{M}_{\mathrm{Coh}(X)}$ of the derived mapping stack
$$\mathcal{M}_{\mathrm{D}_{\mathrm{coh}}(X)} = \underline{\mathrm{Maps}}(X, \mathrm{D}_{\mathrm{coh}})$$
defined as follows: for $A \in \mathbf{ACAlg}_R$, an A-point of $\mathcal{M}_{\mathrm{D}_{\mathrm{coh}}(X)}$ belongs to $\mathcal{M}_{\mathrm{Coh}(X)}$ if and only if the corresponding coherent complex $\mathcal{F} \in \mathrm{D}_{\mathrm{coh}}(X_A)$ is connective and flat over $\mathrm{Spec}(A)$.

Remark 5.4.5. Since vector bundles are locally trivial, there is a canonical isomorphism of derived stacks
$$\mathcal{M}_{\mathrm{Vect}(X)} \simeq \coprod_{n \geq 0} \mathcal{M}_{\mathrm{Bun}_{\mathrm{GL}_n}(X)}.$$

Remark 5.4.6. The classical truncation of $\mathcal{M}_{\mathrm{Coh}(X)}$ can be identified with the usual moduli stack of coherent sheaves on X. That is, if A is an ordinary R-algebra, the ∞-groupoid of A-points of $\mathcal{M}_{\mathrm{Coh}(X)}$ is equivalent to the 1-groupoid of coherent sheaves on X_A that are flat over $\mathrm{Spec}(A)$.

5.5. **The cotangent complex of the moduli of perfect complexes.**

Lemma 5.5.1. *Let A be an animated ring and $M \in \mathrm{D}_{\mathrm{perf}}(A)$ a perfect derived A-module. For every $N \in \mathrm{D}_{\mathrm{perf}}(A)_{\geq 0}$ denote by $F_M(N)$ the fiber at M of the map of ∞-groupoids*
$$\mathrm{D}_{\mathrm{perf}}(A \oplus N)^{\simeq} \to \mathrm{D}_{\mathrm{perf}}(A)^{\simeq}$$
given by extending scalars along the canonical homomorphism $A \oplus N \to A$. Then we have canonical isomorphisms
$$F_M(N) \simeq \mathrm{Maps}_{\mathrm{D}(A)}(M \otimes_A^{\mathbf{L}} M^\vee[-1], N),$$
which are natural in N.

Theorem 5.5.2. *The perfect complex*
$$\mathbf{L}_{\mathcal{M}_{\mathrm{Perf}}} = \mathcal{E}^{\mathrm{univ}} \otimes^{\mathbf{L}} \mathcal{E}^{\mathrm{univ},\vee}[-1]$$
is a cotangent complex for the derived stack $\mathcal{M}_{\mathrm{Perf}}$.

Proof. Let $A \in \mathbf{ACRing}$ and $x\colon \mathrm{Spec}(A) \to \mathcal{M}_{\mathrm{Perf}}$ be an A-point classifying a perfect derived A-module $M \in \mathrm{D}_{\mathrm{perf}}(A)$. By Lemma 5.5.1, the animum of derivations $\mathrm{Der}_x(X, M)$ (relative to $\mathrm{Spec}(\mathbb{Z})$) is corepresented by $M \otimes^{\mathbf{L}} M^\vee[-1]$. Moreover, if $A \to B$ is a morphism of animated rings then we have an isomorphism
$$(M \otimes_A^{\mathbf{L}} M^\vee[-1]) \otimes_A^{\mathbf{L}} B \simeq N \otimes_B^{\mathbf{L}} N^\vee[-1]$$
where $N = M \otimes_A^{\mathbf{L}}$ is the perfect derived B-module classified by the composition
$$\mathrm{Spec}(B) \to \mathrm{Spec}(A) \xrightarrow{x} \mathcal{M}_{\mathrm{Perf}}.$$
It follows that the collection $(M \otimes^{\mathbf{L}} M^\vee[-1])$ assembles into a perfect complex on $\mathcal{M}_{\mathrm{Perf}}$ by varying the pair (A, x). This gives us a cotangent complex for $\mathcal{M}_{\mathrm{Perf}}$. By construction, this perfect complex is just $\mathcal{E}^{\mathrm{univ}} \otimes^{\mathbf{L}} \mathcal{E}^{\mathrm{univ},\vee}[-1] \in \mathrm{D}_{\mathrm{perf}}(\mathcal{M}_{\mathrm{Perf}})$. $\square$

6. Moduli Stacks

Definition 6.0.1. Let S be a derived stack and X, Y be derived stacks over S. The *derived mapping stack* $\underline{\mathrm{Maps}}_S(X,Y)$ is the functor

$$\begin{aligned} \underline{\mathrm{Maps}}_S(X,Y)\colon\ \mathbf{ACRing} &\longrightarrow \mathbf{Grpd}_\infty, \\ R &\longmapsto \mathrm{Maps}_S(X \times^{\mathbf{R}}_S \mathrm{Spec}(R), Y). \end{aligned}$$

Sometimes we denote $X_R \overset{\mathrm{def}}{=} \mathrm{Maps}_S(X \times^{\mathbf{R}}_S \mathrm{Spec}(R), Y)$.

Definition 6.0.2. The *evaluation morphism*

$$\mathrm{ev}\colon\ \mathrm{Maps}_S(X,Y) \times^{\mathbf{R}}_S X \to Y$$

is the morphism classified by the identity map $\mathrm{id}_{\mathrm{Maps}_S(X,Y)}$.

Proposition 6.0.3. *Let $f : X \to Y$ be a morphism of derived algebraic stacks. If f is proper, representable, and of finite Tor-amplitude, then $\mathbf{L}f^*\colon \mathrm{D}_{\mathrm{qc}}(X) \to \mathrm{D}_{\mathrm{qc}}(Y)$ admits a left adjoint $f_\#$. For perfect complexes $\mathcal{F} \in \mathrm{D}_{\mathrm{perf}}(X)$, it is given by the formula $f_\#(\mathcal{F}) \overset{\mathrm{def}}{=} \mathbf{R}f_*(\mathcal{F}^\vee)^\vee$.*

Proof. If $\mathcal{F} \in \mathrm{D}_{\mathrm{qc}}(X)$ is perfect, hence dualizable, we may set $f_\#(\mathcal{F})$ as in the statement. It is clear that this defines a left adjoint to $\mathbf{L}f^*\colon \mathrm{D}_{\mathrm{perf}}(Y) \to \mathrm{D}_{\mathrm{perf}}(X)$ as follows. For every $\mathcal{G} \in \mathrm{D}_{\mathrm{perf}}(Y)$, we have

$$\begin{aligned} \mathrm{Maps}(\mathbf{R}f_*(\mathcal{F}^\vee)^\vee, \mathcal{G}) &\simeq \mathrm{Maps}(\mathcal{O}_Y, \mathcal{G} \otimes^{\mathbf{L}} \mathbf{R}f_*(\mathcal{F}^\vee)) \\ &\simeq \mathrm{Maps}(\mathcal{G}^\vee, \mathbf{R}f_*(\mathcal{F}^\vee)) \\ &\simeq \mathrm{Maps}(\mathbf{L}f^*(\mathcal{G})^\vee, \mathcal{F}^\vee) \\ &\simeq \mathrm{Maps}(\mathcal{F}, \mathbf{L}f^*(\mathcal{G})). \end{aligned}$$

Here we used the fact that the functor $A \otimes^{\mathbf{L}} (-)$ is left adjoint to $A^\vee \otimes^{\mathbf{L}} (-)$.

If $\mathcal{F}$ is not perfect, it is equivalent to define $\mathbf{L}f^* f_\#(\mathcal{F})$ for every morphism $v\colon \mathrm{Spec}(R) \to Y$ and $R \in \mathbf{ACRing}$. By left transposition from the derived base change formula $\mathbf{L}f^*\mathbf{R}v_* \simeq \mathbf{R}u_*\mathbf{f}^*_R$, we have

$$\mathbf{L}v^* f_\#(\mathcal{F}) \simeq f_{R,\#}\mathbf{L}u^*(\mathcal{F})$$

whenever $f_\#$ and $f_{R,\#}$ both exist. Therefore, we may replace Y by $\mathrm{Spec}(R)$ and assume Y is affine.

In the case where $Y = \mathrm{Spec}(R)$ is affine, X is a derived algebraic space that is in particular separated over $\mathrm{Spec}(R)$, hence quasi-compact and quasi-separated. For such X, there is a generalization of a theorem of Thomason that asserts that every quasi-coherent complex can be written as a filtered colimit of perfect complexes (see [Lur3, Proposition 9.6.1.1] or [Kha2, Theorem 1.40]). It follows that there is a unique colimit preserving functor $f_\#\colon \mathrm{D}_{\mathrm{qc}}(X) \to \mathrm{D}_{\mathrm{qc}}(Y)$ that restricts to $\mathbf{R}f_*(-^\vee)^\vee$ on perfect complexes.

If $\mathcal{F} \in \mathrm{D}_{\mathrm{qc}}(X)$ is a colimit of a filtered system $(\mathcal{F}_\alpha)_\alpha$ of perfect complexes and $\mathcal{G} \in \mathrm{D}_{\mathrm{qc}}(Y)$, then

$$\begin{aligned}
\mathrm{Maps}(f_\#(\mathcal{F}), \mathcal{G}) &\simeq \mathrm{Maps}(\varinjlim_\alpha f_\#(\mathcal{F}_\alpha), \mathcal{G}) \\
&\simeq \varprojlim_\alpha \mathrm{Maps}(f_\#(\mathcal{F}_\alpha), \mathcal{G}) \\
&\simeq \varprojlim_\alpha \mathrm{Maps}(\mathcal{F}_\alpha, \mathbf{L}f^*(\mathcal{G})) \\
&\simeq \mathrm{Maps}(\varinjlim_\alpha \mathcal{F}_\alpha, \mathbf{L}f^*(\mathcal{G})) \\
&\simeq \mathrm{Maps}(\mathcal{F}, \mathbf{L}f^*(\mathcal{G})).
\end{aligned}$$

Therefore $f_\#$ is left adjoint to $\mathbf{L}f^*$. □

Theorem 6.0.4. *Suppose S is algebraic and X and Y are derived stacks over S. Set $H \overset{\mathrm{def}}{=} \underline{\mathrm{Maps}}_S(X, Y)$ and consider the diagram*

$$H \xleftarrow{\pi} X \times_S^{\mathbf{R}} H \xrightarrow{\mathrm{ev}} Y,$$

where π is the projection. If X is proper of finite Tor-amplitude and representable over S, and Y admits a cotangent complex $\mathbf{L}_{Y/S}$ over S, then the perfect complex $\mathbf{L}_{H/S} \simeq \pi_\# \mathbf{L}\,\mathrm{ev}^*(\mathbf{L}_{Y/S})$ is a relative cotangent complex for H over S.

6.1. The Artin-Lurie representability theorem.

Theorem 6.1.1. *Let k be a commutative ring, which we assume is finite type over a field (or more generally a G-ring), and X a derived stack over k. Then X is algebraic if and only if the following conditions hold:*

1. *X admits a cotangent complex $\mathbf{L}_X$.*
2. *The restriction of X to ordinary k-algebras takes values in 1-groupoids.*
3. Almost of finite presentation: *For any $n \geq 0$ the functor $X\colon \mathbf{ACAlg}_k \to \mathbf{Grpd}_\infty$ preserves filtered colimits when restricted to n-truncated algebras.*
4. Integrability: *For every complete local noetherian k-algebra R, the canonical map $X(R) \to \varprojlim_n X(R/\mathfrak{m}_R^n)$ is invertible, where $\mathfrak{m}_R \subseteq R$ is the unique maximal ideal.*
5. *Nil-completeness: For every $R \in \mathbf{ACAlg}_k$, the canonical map*
$$X(R) \to \varinjlim_n X(\tau_{\leq n}(R))$$
is invertible.
6. *Infinitesimal cohesion: For every cartesian square in $\mathbf{ACAlg}_k$*
$$\begin{tikzcd} A' \arrow[r] \arrow[d] & A \arrow[d, two heads] \\ B' \arrow[r, two heads] & B \end{tikzcd}$$
such that $A \to B$ and $B' \to B$ are surjective on π_0 with nilpotent kernel, X sends the square to a cartesian square.

It is possible to check the conditions of Theorem 6.1.1 for the moduli stacks we have been considering (except $\mathcal{M}_{\mathrm{Perf}(X)}$, which doesn't satisfy condition (ii)). The main point is to check the computation of the cotangent complex.

Theorem 6.1.2. *Let k be a G-ring and X an algebraic space that is proper and of finite Tor-amplitude over k. Then the following derived stacks are algebraic:*

(1) The moduli stack $\mathcal{M}_{\mathrm{Vect}(X)}$ of vector bundles over X.
(2) The moduli stack $\mathcal{M}_{\mathrm{Bun}_G(X)}$ of G-bundles over X for every smooth group scheme G over k.
(3) The moduli stack $\mathcal{M}_{\mathrm{Coh}(X)}$ of coherent sheaves on X, if X is smooth over k.

Moreover, the above are smooth (hence classical) if X is a relative curve, and quasi-smooth if X is a relative surface.

7. Cohomology of Stacks

7.1. **Abelian sheaves.**

Definition 7.1.1. Let k be a base field and $\mathbf{Sch}_k$ the category of k-schemes that are locally of finite type. Given $X \in \mathbf{Sch}_k$ and a commutative ring Λ, we denote by $\mathrm{D}(X;\Lambda)$ the following:

(1) If $k = \mathbb{C}$, the stable ∞-category $\mathrm{Shv}(X(\mathbb{C}), \mathrm{D}(\Lambda))$ of sheaves on the topological space $X(\mathbb{C})$ with values in the derived ∞-category of Λ-modules.
(2) If $\Lambda = \mathbb{Z}/n\mathbb{Z}$ for $n \neq \mathrm{Char}(k)$, the stable ∞-category $\mathrm{Shv}(X_{\text{ét}}, \mathrm{D}(\Lambda))$ of sheaves on the small étale site $X_{\text{ét}}$ with values in the derived ∞-category of Λ-modules.

Theorem 7.1.2. *The presheaf $\mathrm{D}^*\colon \mathbf{Sch}_k^{\mathrm{op}} \to \mathbf{Cat}_\infty$ determined by the assignment*

$$X \mapsto \mathrm{D}(X;\Lambda), \qquad f \mapsto f^*$$

satisfies descent for the étale topology.

Construction 7.1.3. Let **AlgStk** denote the ∞-category of algebraic stacks that are locally of finite type over k.

Theorem 7.1.4 (Six operations)**.** *We have the following operations on the ∞-categories $\mathrm{D}(X)$ for $X \in \mathbf{AlgStk}_k$:*

(1) An adjoint pair of bifunctors

$$\otimes : \mathrm{D}(X) \times \mathrm{D}(X) \to \mathrm{D}(X)$$
$$\underline{\mathrm{Hom}} : \mathrm{D}(X) \times \mathrm{D}(X) \to \mathrm{D}(X)$$

for all $X \in \mathbf{AlgStk}_k$.

(2) For every morphism $f : X \to Y$ in $\mathbf{AlgStk}_k$, an adjoint pair

$$f^*\colon\ \mathrm{D}(Y) \to \mathrm{D}(X), \qquad f_*\colon\ \mathrm{D}(X) \to \mathrm{D}(Y).$$

(3) For every morphism $f : X \to Y$ in $\mathbf{AlgStk}_k$, an adjoint pair

$$f_!\colon\ \mathrm{D}(Y) \to \mathrm{D}(X), \qquad f^!\colon \mathrm{D}(X) \to \mathrm{D}(Y).$$

Moreover, these functors satisfy the following properties:

(1) (Base change formula) *For every cartesian square there is a canonical isomorphism*

$$q^* f_! \simeq g_! p^*.$$

(2) (Projection formula) *For every morphism $f : X \to Y$ in $\mathbf{AlgStk}_k$, there is a canonical isomorphism*

$$f_!(-) \otimes (-) \simeq f_!(- \otimes f^*(-)).$$

(3) (Forgetting supports) *If f has a proper diagonal map, then there is a canonical morphism*

$$f_! \to f_*,$$

which is invertible when f is proper.

(4) (Étale pullback) *If f is étale, then there is a canonical isomorphism $f^! \simeq f^*$.*
(5) (Localization) *If $X \in \mathbf{Stk}_k$ and $i\colon Z \hookrightarrow X$ is a closed immersion with a complementary open immersion $jU \hookrightarrow X$, then there are canonical exact triangles*

$$f_! f^* \to \mathrm{id} \to i_! i^*$$
$$i_! i^* \to \mathrm{id} \to j_! j^*.$$

Proof. This is the main result in [LZ]. □

7.2. **Cohomology.** Given a (derived) algebraic stack X that is locally of finite type over k, let $a_X\colon X \to \mathrm{Spec}(k)$ denote the projection. We define

$$C^\bullet(X;\Lambda) \overset{\text{def}}{=} R\Gamma(f_* f^*\Lambda) \simeq R\Gamma(X;\Lambda_X)$$
$$C^{\mathrm{BM}}_\bullet(X;\Lambda) \overset{\text{def}}{=} R\Gamma(f_* f^!\Lambda) \simeq R\Gamma(X;\omega_X),$$

where $\Lambda_X = f^*\Lambda$ and $\omega_X = f^!\Lambda$ denote the constant and dualizing sheaves respectively. These are the complexes of cochains and Borel-Moore chains on X, respectively. We also write

$$H^\bullet(X;\Lambda) \overset{\text{def}}{=} H^\bullet(C^\bullet(X;\Lambda)) \simeq H^\bullet(X;\Lambda_X)$$
$$H^{\mathrm{BM}}_\bullet(X;\Lambda) \overset{\text{def}}{=} H^{-\bullet}(C^{\mathrm{BM}}_\bullet(X;\Lambda)) \simeq H^{-\bullet}(X;\omega_X),$$

Proposition 7.2.1. *The following hold:*

(1) Proper pushforward: *Let $f : X \to Y$ be a proper morphism in $\mathbf{AlgStk}_k$. Then there is a canonical morphism*

$$f_*\colon C^{\mathrm{BM}}_\bullet(X;\Lambda) \longrightarrow C^{\mathrm{BM}}_\bullet(Y;\Lambda).$$

(2) Étale pullback: *Let $f : X \to Y$ be an étale morphism in $\mathbf{AlgStk}_k$. Then there is a canonical morphism*

$$f^!\colon C^{\mathrm{BM}}_\bullet(Y;\Lambda) \longrightarrow C^{\mathrm{BM}}_\bullet(C;\Lambda).$$

(3) Localization triangle: *Let $X \in \mathbf{Stk}_k$ and $i : Z \hookrightarrow X$ be a closed immersion with complementary open immersion $j : U \hookrightarrow X$. Then there is a canonical exact triangle*

$$C^{\mathrm{BM}}_\bullet(Z;\Lambda) \xrightarrow{i_*} C^{\mathrm{BM}}_\bullet(X;\Lambda) \xrightarrow{j^!} C^{\mathrm{BM}}_\bullet(U;\Lambda) \longrightarrow C^{\mathrm{BM}}_\bullet(Z;\Lambda)[1].$$

We also have the following, which is a consequence of Theorem 7.1.2:

Corollary 7.2.2. *On the ∞-category $\mathbf{AlgStk}_k$, the presheaves*

$$X \mapsto C^\bullet(X;\Lambda), \quad f \mapsto f^*$$
$$X \mapsto C^{\mathrm{BM}}_\bullet(X;\Lambda), \quad f \mapsto f^!$$

satisfy descent for the étale topology.

7.3. **Intersection theory.** Let $f : X \to Y$ be a homotopically smooth morphism of derived Artin stacks. We define the *normal bundle* to be $N_{X/Y} \overset{\text{def}}{=} T_{X/Y}[1]$. It is the moduli of sections of the 1-shifted tangent complex $\mathbf{L}^\vee_{X/Y}[1]$. More precisely, it is the derived Artin stack whose functor of points

$$\mathbf{DSch}_X^{\mathrm{op}} \to \mathbf{Grpd}_\infty$$

is given by the assignment

$$(T \xrightarrow{t} X) \mapsto \mathrm{Maps}_{\mathrm{D_{qc}}(T)}(\mathbf{L}t^*\mathcal{E}, \mathcal{O}_T).$$

If $f\colon X\to Y$ is a regular closed immersion between schemes, then the tangent complex $\mathbf{L}^\vee_{X/Y}\simeq\mathcal{N}^\vee_{X/Y}[-1]$ is the (-1)-shifted normal sheaf, so $N_{X/Y}$ is just the usual normal bundle.

Definition/Theorem 7.3.1. Let $f: X\to Y$ be a homotopically smooth morphism of derived Artin stacks. The *normal deformation* $D_{X/Y}$ is the derived mapping stack

$$D_{X/Y}=\underline{\mathrm{Maps}}_{Y\times\mathbb{A}^1}(Y\times\{0\},X\times\mathbb{A}^1).$$

(1) If X and Y are n-Artin, then $D_{X/Y}$ is $(n+1)$-Artin.
(2) There is a commutative diagram of cartesian squares

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "0"] & X\times\mathbb{A}^1 \arrow[d, "\widehat{f}"] & X\times\mathbb{G}_m \arrow[l] \arrow[d, "f\times\mathrm{id}"] \\
N_{X/Y} \arrow[r] \arrow[d] & D_{X/Y} \arrow[d] & Y\times\mathbb{G}_m \arrow[l] \arrow[d] \\
\{0\} \arrow[r] & \mathbb{A}^1 & \mathbb{G}_m \arrow[l]
\end{tikzcd}$$

Proof. See [Kha1, §1.4] or [HKR]. □

Construction 7.3.2. Let $f: X\to Y$ be a homotopically smooth morphism of derived algebraic stacks locally of finite type over k. There is a canonical map

$$\mathrm{sp}_{X/Y}\colon C^{\mathrm{BM}}_\bullet(Y;\Lambda)\to C^{\mathrm{BM}}_\bullet(N_{X/Y};\Lambda),$$

which is defined as the composite

$$\begin{aligned}C^{\mathrm{BM}}_\bullet(Y;\Lambda)&\hookrightarrow C^{\mathrm{BM}}_\bullet(Y;\Lambda)\oplus C^{\mathrm{BM}}_\bullet(Y;\Lambda)(1)[1]\\ &\simeq C^{\mathrm{BM}}_\bullet(Y\times\mathbb{G}_m;\Lambda)[-1]\xrightarrow{\partial}C^{\mathrm{BM}}_\bullet(N_{X/Y};\Lambda),\end{aligned}$$

where the splitting comes from the unit section of $\mathbb{G}_m$ and ∂ is the boundary map in the localization triangle

$$C^{\mathrm{BM}}_\bullet(N_{X/Y};\Lambda)\longrightarrow C^{\mathrm{BM}}_\bullet(D_{X/Y};\Lambda)\longrightarrow C^{\mathrm{BM}}_\bullet(Y\times\mathbb{G}_m;\Lambda)\xrightarrow{\partial}C^{\mathrm{BM}}_\bullet(N_{X/Y};\Lambda)[1].$$

Notation 7.3.3. Given $d\in\mathbb{Z}$, we write $\langle d\rangle\overset{\mathrm{def}}{=}(d)[2d]$, where (d) denotes the Tate twist (dth tensor power of (1)).

Construction 7.3.4. Let $f: X\to Y$ be a morphism in $\mathbf{AlgStk}_k$. Suppose f is quasi-smooth, that is homotopically 1-smooth, of relative virtual dimension $\mathrm{vd}(f)\overset{\mathrm{def}}{=}\mathrm{rk}(\mathbf{L}_{X/Y})=d$. Then $\mathbf{L}_{X/Y}$ is in Tor-amplitude $[-1,1]$, and we have a generalized homotopy invariance isomorphism

$$C^{\mathrm{BM}}_\bullet(X;\Lambda)\simeq C^{\mathrm{BM}}_\bullet(N_{X/Y};\Lambda)\langle d\rangle$$

since the projection $N_{X/Y}\to X$ is of relative dimension $-d$. The *quasi-smooth pullback*, or *virtual pullback*, is the canonical map

$$f^!\colon C^{\mathrm{BM}}_\bullet(Y;\Lambda)\xrightarrow{\mathrm{sp}_{X/Y}}C^{\mathrm{BM}}_\bullet(N_{X/Y};\Lambda)\simeq C^{\mathrm{BM}}_\bullet(X;\Lambda)\langle -d\rangle. \tag{7.3.1}$$

Definition 7.3.5. Let X be a quasi-smooth derived algebraic stack of relative virtual dimension d over $\mathrm{Spec}(k)$. The projection $a_X\colon X\to\mathrm{Spec}(k)$ gives rise to the pullback

$$a_X^!\colon C^{\mathrm{BM}}_\bullet(\mathrm{Spec}(k))\to C^{\mathrm{BM}}_\bullet(X)\langle -d\rangle$$

and hence to the canonical element

$$[X]\in C^{\mathrm{BM}}_\bullet(X)\langle -d\rangle\quad\to\quad[X]\in H^{\mathrm{BM}}_{2d}(X)(-d),$$

which is called the *virtual fundamental class* of X.

Remark 7.3.6. The element $[X] \in C_\bullet^{\mathrm{BM}}(X)\langle -d\rangle$ corresponds to a canonical morphism

$$\Lambda_X\langle d\rangle \to a_X^!(\Lambda)$$

in $\mathrm{D}(X;\Lambda)$. This gives rise to a natural transformation

$$a_X^*(-)\langle d\rangle \to a_X^*(-) \otimes a_X^!(\Lambda) \longrightarrow a_X^!(-).$$

By adjunction we get a trace map $(a_X)_! a_X^*\langle d\rangle \to \mathrm{id}$. If $f\colon X \to Y$ is a quasi-smooth morphism of relative virtual dimension d, we similarly get a natural transformation

$$\mathrm{tr}_f\colon f_! f^*\langle d\rangle \to \mathrm{id}\,.$$

Theorem 7.3.7 (Poincaré duality)**.** *The following hold:*

1. *If $f : X \to Y$ is smooth, then the natural transformation $f^*(-)\langle d\rangle \to f^!(-)$ is invertible. Equivalently, tr_f is the counit of an adjunction $f_! \dashv f^*\langle d\rangle$.*
2. *For any smooth algebraic stack X in $\mathbf{AlgStk}_k$, cap product with $[X]$ determines a canonical isomorphism*

$$(-) \cap [X]\colon C^\bullet(X) \to C_\bullet^{\mathrm{BM}}(X)\langle -d\rangle.$$

Proof. If $f\colon X \to Y$ is smooth, then the diagonal $\Delta_X\colon X \to X \times_Y X$ is quasi-smooth. This gives a natural transformation tr_{Δ_X} that gives rise to a counit for the adjunction. The second statement follows from the first, see [Kha3]. □

If X is a smooth scheme over k, then the cup product in homology gives rise by Poincaré duality to an intersection product

$$C_\bullet^{\mathrm{BM}}(X)\langle -p\rangle C_\bullet^{\mathrm{BM}}(X)\langle -q\rangle \to C_\bullet^{\mathrm{BM}}(X)\langle -p-q+d\rangle.$$

If Y is quasi-smooth and proper over X of virtual dimension d, then the virtual fundamental class gives us a class in $C_\bullet^{\mathrm{BM}}(X)\langle -d\rangle$ by proper pushforward.

Theorem 7.3.8 (Non-transverse Bézout formula)**.** *Let Y and Z be smooth closed subvarieties of X, of dimension p and q respectively. Then there is a canonical homotopy equivalence*

$$[Y]\cdot[Z] \simeq [Y \times_X^{\mathbf{R}} Z]$$

in $C_\bullet^{\mathrm{BM}}(X)\langle -p-q+d\rangle$.

Proof. A more general version of this statement for derived Artin stacks and its proof can be found in [Kha1, Theorem 3.22]. □

Remark 7.3.9. The classical Bézout formula says that if $X, Y \subseteq \mathbb{CP}^2$ are smooth algebraic curves of degrees n and m that intersect transversely (no tangent intersections), then the number of intersection points is nm. Cohomologically, we can assign to a curve X its fundamental class $[X] \in H^2(\mathbb{CP}^2;\mathbb{Z})$. If X and Y intersect transversely, then $[X] \smile [Y] = [X \cap Y]$, where $[X \cap Y] \in H^4(\mathbb{CP}^2;\mathbb{Z}) \cong \mathbb{Z}$ just counts the intersection number. The above theorem is a vast generalization of this basic fact through the lens of derived geometry. There are other intermediate points that do not need to use derived geometry, such as Serre's intersection formula. The formula requires the full use of the Tor functor and homological algebra.

References

[AKLPR] D. Aranha, A. A. Khan, A. Latyntsev, H. Park, C. Ravi, *Localization theorems for algebraic stacks*. arXiv:2207.01652 (2022).

[Aok] K. Aoki, *Tensor triangular geometry of filtered objects and sheaves*. arXiv:2001.00319 (2020).

[Avr] L. Avramov, *Locally complete intersection homomorphisms and a conjecture of Quillen on the vanishing of cotangent homology*. Ann. Math. **150** (1999), no. 2, 455–487.

[BF] K. Behrend, B. Fantechi, *The intrinsic normal cone*, Invent. Math. **128** (1997), no. 1, 45–88.

[CS] K. Çesnavicius, P. Scholze, *Purity for flat cohomology*. arXiv:1912.10932 (2019).

[Ci] D.-C. Cisinski, *Higher categories and homotopical algebra*. Cambridge Studies in Advances Mathematics **180** (2019).

[Con] B. Conrad, *Arithmetic moduli of generalized elliptic curves*. J. Inst. Math. Jussieu **6** (2007), no. 2, 209–278.

[GR] D. Gaitsgory, N. Rozenblyum, *A study in derived algebraic geometry. Vols. I–II*. Math. Surv. Mono. **221** (2017).

[GZ] P. Gabriel, M. Zisman, *Calculus of fractions and homotopy theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band **35** (1967). Springer–Verlag.

[GW] U. Görtz, T. Wedhorn, *Algebraic Geometry: I and II*. Springer Spektrum Wiesbaden (2020).

[GP] T. Graber, R. Pandharipande, *Localization of virtual classes*. Invent. Math. **135** (1999), no. 2, 487–518.

[Gro] Ph. Gross, *Tensor generators on schemes and stacks*. Algebr. Geom. **4**, no. 4, 501–522 (2017).

[HKR] J. Hekking, A. A. Khan, D. Rydh, *Deformation to the normal cone and blow-ups via derived Weil restrictions*. In preparation.

[HS] A. Hirschowitz, C. Simpson, *Descente pour les n-champs*. math.AG/9807049 (2001).

[KRy] A. A. Khan, D. Rydh, *Virtual Cartier divisors and blow-ups*. arXiv:1802.05702 (2018).

[KRa] A. A. Khan, C. Ravi, *Equivariant generalized cohomology via stacks*. arXiv:2209.07801 (2022).

[Kha1] A. A. Khan, *Virtual fundamental classes for derived stacks I*. arXiv:1909.01332 (2019).

[Kha2] A. A. Khan, *K-theory and G-theory of derived algebraic stacks*. Jpn. J. Math. **17** (2022), 1–61.

[Kha3] A. A. Khan, *Absolute Poincaré duality in étale cohomology*. Forum Math. Sigma **10** (2022), no. 10, e99. arXiv:2111.02875.

[Kha4] A. A. Khan, *Lectures on Algebraic Stacks*. arXiv:2310.12456 (2023).

[Kha5] A. A. Khan, *An Introduction to Derived Algebraic Geometry*. Kiem, YH. (eds) Moduli Spaces, Virtual Invariants and Shifted Symplectic Structures. KIAS Springer Series in Mathematics, vol 4. Springer, Singapore.

[Kon] M. Kontsevich, *Enumeration of rational curves via torus actions*, in: The moduli space of curves (Texel Island, 1994), 335–368, Progr. Math. **129** (1995).

[LMB] G. Laumon, L. Moret-Bailly, *Champs algébriques*. Ergebnisse der Mathematik und ihrer Grenzgebiete **39** (2000).

[LZ] Y. Liu, W. Zheng, *Enhanced six operations and base change theorem for higher Artin stacks*. arXiv:1211.5948 (2012).

[Lur1] J. Lurie, *Higher Topos Theory*. Ann. Math. Stud. **170** (2009).

[Lur2] J. Lurie, *Higher Algebra*, version of 2017-09-18. Available at: `https://www.math.ias.edu/~lurie/papers/HA.pdf`.

[Lur3] J. Lurie, *Spectral algebraic geometry*, version of 2018-02-03. Available at: `https://www.math.ias.edu/~lurie/papers/HA.pdf`.

[Lur4] J. Lurie, *Kerodon*, `https://kerodon.net`.

[Ryd1] D. Rydh, *Noetherian approximation of algebraic spaces and stacks*. J. Algebra **422** (2015), 105–147.

[Ryd2] D. Rydh, *Approximation of sheaves on algebraic stacks*. Int. Math. Res. Not. **2016** (2016), no. 3, 717–737.

[SP] J. de Jong, *The Stacks Project*, `https://stacks.math.columbia.edu`.

[Se] J.-P. Serre, *Algèbre locale. Multiplicités*. Lecture Notes in Mathematics **11** (1989).

[TV] B. Toën, M. Vaquié, *Moduli of objects in dg-categories*. Ann. Sci. Éc. Norm. Supér. **40** (2007), no. 3, 387–444.

[TV2] B. Toën, G. Vezzosi, *Homotopical algebraic geometry. II: Geometric stacks and applications*. Mem. Am. Math. Soc. **902** (2008).

[Tho] R. W. Thomason, *Equivariant resolution, linearization, and Hilbert's fourteenth problem over arbitrary base schemes*. Adv. Math. **65** (1987), 16–34.

[Ver] J.-L. Verdier, *Le théorème de Riemann-Roch pour les intersections complètes*, in: Séminaire de géométrie analytique, Paris, France, 1974–75. Astérisque **36-37** (1976), 189–228.

David Rittenhouse Lab, University of Pennsylvania, 209 South 33rd Street, Philadelphia, PA 19104-6395