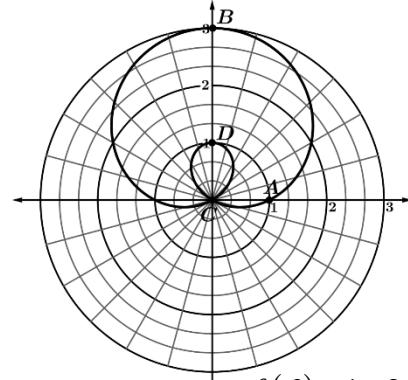


In this section, we will learn ways that we can describe characteristics of the graph of a polar function.

Given a polar function $r = f(\theta)$, we know that r represents the “signed radius” of the function. We use the phrase “signed radius” because r can be a positive or negative value.

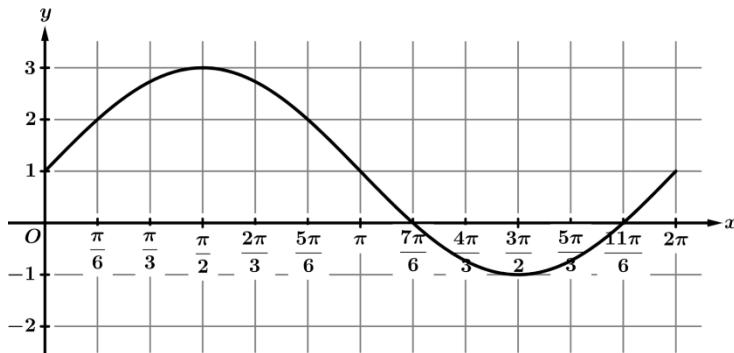
As we trace the graph of a polar function, we are interested in whether the graph of $r = f(\theta)$ is getting closer to the origin or further from the origin over a given interval.

Changes in the Distance from $r = f(\theta)$ to the Origin	
$r = f(\theta)$ is <u>positive</u> and <u>increasing</u>	The distance between $r = f(\theta)$ and the origin is <u>increasing</u>
$r = f(\theta)$ is <u>negative</u> and <u>decreasing</u>	
$r = f(\theta)$ is <u>positive</u> and <u>decreasing</u>	The distance between $r = f(\theta)$ and the origin is <u>decreasing</u>
$r = f(\theta)$ is <u>negative</u> and <u>increasing</u>	



$r = 1 + 2 \sin \theta$

Example 1: The graph of $f(x) = 1 + 2 \sin x$ is shown below for $0 \leq x \leq 2\pi$. Use the graph below to complete the given table with the appropriate intervals.



Description of $f(x)$	Interval(s)
f is <u>positive</u> and <u>increasing</u>	$0 \leq x < \frac{\pi}{2}$ $\frac{11\pi}{6} < x < 2\pi$
f is <u>positive</u> and <u>decreasing</u>	$\frac{\pi}{2} < x < \frac{7\pi}{6}$
f is <u>negative</u> and <u>increasing</u>	$\frac{3\pi}{2} < x < \frac{11\pi}{6}$
f is <u>negative</u> and <u>decreasing</u>	$\frac{7\pi}{6} < x < \frac{3\pi}{2}$

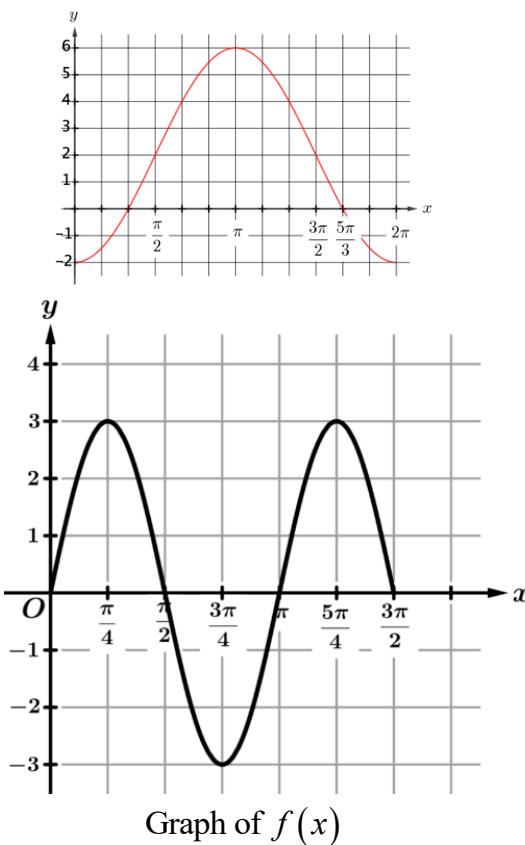
In the polar coordinate system, the graph of $f(x) = 1 + 2 \sin x$ above becomes $f(\theta) = 1 + 2 \sin \theta$, as shown in the table above. The labeled points A, B, C, and D correspond to the intervals found in **Example 1**.

Points on $f(\theta)$	From A to B	From B to C	From C to D	From D to C	From C to A
Interval	$0 < \theta < \frac{\pi}{2}$	$\frac{\pi}{2} < \theta < \frac{7\pi}{6}$	$\frac{7\pi}{6} < \theta < \frac{3\pi}{2}$	$\frac{3\pi}{2} < \theta < \frac{11\pi}{6}$	$\frac{11\pi}{6} < \theta < 2\pi$
$r = f(\theta)$ is	<u>positive</u> and <u>increasing</u>	<u>positive</u> and <u>decreasing</u>	<u>negative</u> and <u>decreasing</u>	<u>negative</u> and <u>increasing</u>	<u>positive</u> and <u>increasing</u>
Distance between $f(\theta)$ and the origin is	increasing	decreasing	increasing	decreasing	increasing

AP Exam Tip: It is often helpful to sketch the graph of a given function in rectangular coordinates when attempting to describe the behavior of a polar function.

Example 2: Consider the graph of the polar function $r = f(\theta)$, where $f(\theta) = 2 - 4 \cos \theta$, in the polar coordinate system for $0 \leq \theta \leq 2\pi$. Which of the following statements is true about the distance between the point with polar coordinates $(f(\theta), \theta)$ and the origin?

- (A) The distance is increasing for $\pi < \theta < \frac{5\pi}{3}$, because $f(\theta)$ is positive and increasing on the interval.
- (B) The distance is increasing for $\frac{5\pi}{3} < \theta < 2\pi$, because $f(\theta)$ is negative and increasing on the interval.
- (C) The distance is decreasing for $\pi < \theta < \frac{5\pi}{3}$, because $f(\theta)$ is positive and decreasing on the interval.
- (D) The distance is decreasing for $\frac{5\pi}{3} < \theta < 2\pi$, because $f(\theta)$ is negative and decreasing on the interval.



Example 3: The graph of $f(x) = 3 \sin(2x)$, where $0 \leq x \leq \frac{3\pi}{2}$ is shown above in the rectangular coordinate system.

If the polar function $r = f(\theta)$, where $f(\theta) = 3 \sin(2\theta)$, is graphed in the polar coordinate system for $0 \leq \theta \leq \frac{3\pi}{2}$,

on which of the following intervals is the distance between the point with polar coordinates $(f(\theta), \theta)$ and the origin decreasing?

- (A) $0 < \theta < \frac{\pi}{4}$ (B) $\frac{\pi}{2} < \theta < \frac{3\pi}{4}$ (C) $\frac{3\pi}{4} < \theta < \pi$ (D) $\pi < \theta < \frac{5\pi}{4}$

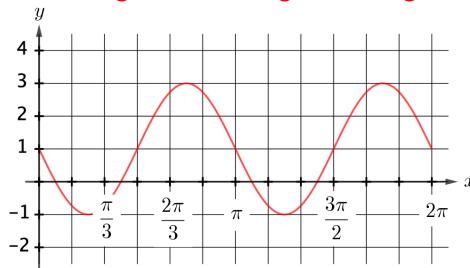
$f(\theta)$ is negative and increasing so distance between the point and the origin is decreasing.

Relative Extrema and Polar Functions

Another characteristic that arises when we study polar functions are relative extrema (minima and maxima). For polar functions, if $r = f(\theta)$ changes from increasing to decreasing (or from decreasing to increasing), then the function has a relative extremum on the interval corresponding to a point relatively closest to or farthest from the origin.

Example 4: Consider the graph of the polar function $r = f(\theta)$, where $f(\theta) = 1 - 2\sin(2\theta)$, in the polar coordinate system for $0 \leq \theta \leq \pi$. Which of the following statements is true about the graph of $r = f(\theta)$?

- (A) The graph of $r = f(\theta)$ has a relative minimum on the interval $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$, because $r = f(\theta)$ changes from negative to positive. *r does change from negative to positive but that does not indicate a relative minimum.*
- (B) The graph of $r = f(\theta)$ has a relative minimum on the interval $\frac{2\pi}{3} < \theta < \pi$, because $r = f(\theta)$ changes from decreasing to increasing. *r changes from increasing to decreasing.*
- (C) The graph of $r = f(\theta)$ has a relative maximum on the interval $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$, because $r = f(\theta)$ changes from positive to negative. *r changes from negative to positive.*
- (D) The graph of $r = f(\theta)$ has a relative maximum on the interval $\frac{2\pi}{3} < \theta < \pi$, because $r = f(\theta)$ changes from increasing to decreasing. *r changes from increasing to decreasing indicating a relative maximum on the interval.*



Average Rate of Change

In previous units, we learned about the average rate of change of a function in rectangular coordinates. In the polar coordinate system, we will find the average rate of change of r with respect to θ over a given interval of θ .

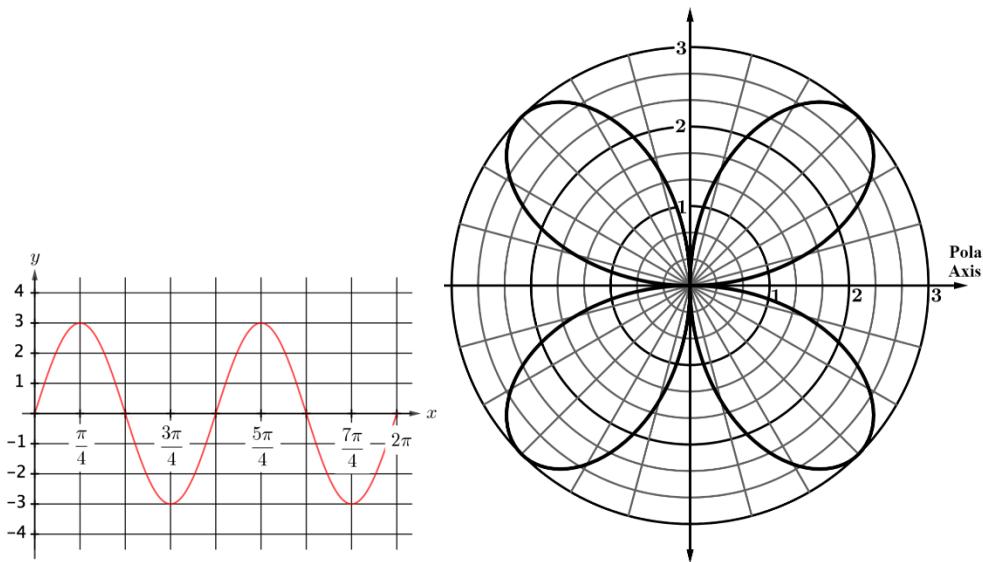
Average Rate of Change of a Polar Function

For the polar function $r = f(\theta)$, the average rate of change of $r = f(\theta)$ over the interval $a \leq \theta \leq b$ is given by the expression $\frac{f(b) - f(a)}{b - a}$.

Geometrically, the average rate of change indicates the rate at which the radius is changing per radian.

Example 5: Consider the graph of the polar function $r = f(\theta)$, where $f(\theta) = 3 - 3\cos\theta$, in the polar coordinate system. What is the average rate of change of $r = f(\theta)$ over the interval $\frac{\pi}{2} \leq \theta \leq \pi$?

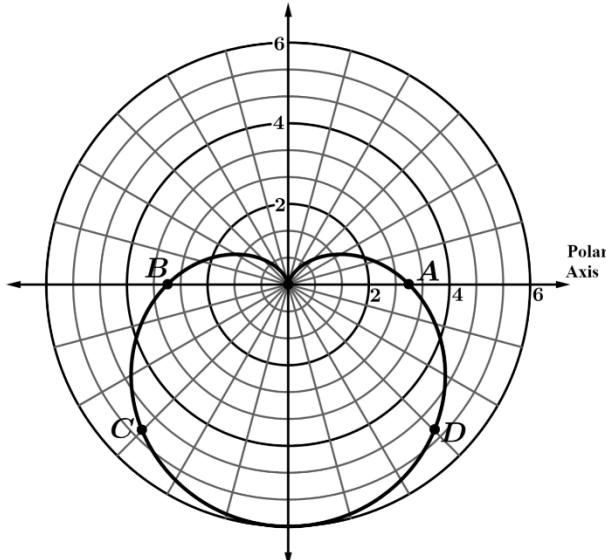
$$\text{Average rate of change} = \frac{f(\pi) - f\left(\frac{\pi}{2}\right)}{\pi - \frac{\pi}{2}} = \frac{[3 - 3(-1)] - [3 - 3(0)]}{\frac{\pi}{2}} = \frac{6 - 3}{\frac{\pi}{2}} = \frac{3}{\frac{\pi}{2}} = \frac{6}{\pi}$$



Example 6: The figure shows the graph of the polar function $r = f(\theta)$, where $f(\theta) = 3\sin(2\theta)$ for $0 \leq \theta \leq 2\pi$, in the polar coordinate system. On which of the following intervals is the average rate of change of $f(\theta)$ equal to zero?

- (A) $\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$ (B) $\frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{4}$ (C) $\frac{3\pi}{4} \leq \theta \leq \frac{7\pi}{4}$ (D) $\frac{\pi}{4} \leq \theta \leq \frac{7\pi}{4}$

The average rate of change is zero because the r value is -3 at both values of θ .



Example 7: In the polar coordinate system, the graph of a polar function $r = f(\theta)$ is shown with a domain of all real values of θ for $0 \leq \theta \leq 2\pi$. On this interval of θ , the graph has no holes, passes through each point exactly one time, and as θ increases, the graph passes through the labeled points A, B, C, and D, in that order. On which of the following intervals is the average rate of change of r with respect to θ least?

- (A) From A to B average rate of change is zero
 (B) From B to C average rate of change is positive because r increased.
 (C) From C to D average rate of change is zero
 (D) From D to A average rate of change is negative because r decreased.

Estimating Values of $r = f(\theta)$ Using the Average Rate of Change

For a given interval, we can use the average rate of change of $r = f(\theta)$ over the interval to estimate other values of $r = f(\theta)$ inside the given interval.

Recall: The point-slope form of a line is given by $y - y_1 = m(x - x_1)$, where (x_1, y_1) is a known point on the line and m is the slope of the line.

We can utilize this concept and create a linear function that will help us approximate a given polar function. We will use the average rate of change of r with respect to θ as our slope, and we can use a point at either end of the given interval as our given point. This leads us to the following

$$f(\theta) \approx f(\theta_1) + \frac{f(b) - f(a)}{b - a}(\theta - \theta_1),$$

where $(f(\theta_1), \theta_1)$ is a known point on the graph of $r = f(\theta)$, and the average rate of change of $r = f(\theta)$ with respect to θ over the interval $a \leq \theta \leq b$ is given by $\frac{f(b) - f(a)}{b - a}$.

θ	$\frac{\pi}{6}$	$\frac{7\pi}{6}$
$f(\theta)$	2	-1

Example 8: The table above gives values of the polar function $r = f(\theta)$ at selected values of θ . Use the average rate of change of $r = f(\theta)$ over the interval $\frac{\pi}{6} \leq \theta \leq \frac{7\pi}{6}$ to approximate $f\left(\frac{5\pi}{6}\right)$.

$$f\left(\frac{5\pi}{6}\right) \approx f\left(\frac{\pi}{6}\right) + \frac{f\left(\frac{7\pi}{6}\right) - f\left(\frac{\pi}{6}\right)}{\frac{7\pi}{6} - \frac{\pi}{6}}\left(\frac{5\pi}{6} - \frac{\pi}{6}\right) = 2 + \frac{-1 - 2}{\pi}\left(\frac{5\pi}{6} - \frac{\pi}{6}\right) = 2 - \frac{3}{\pi}\left(\frac{4\pi}{6}\right) = 2 - \frac{12}{6} = 2 - 2 = 0$$