Because logarithmic functions and exponential functions are inverse functions, the characteristics of their graphs will have inverse relationships.

### Key Characteristics of Logarithmic Functions

A logarithmic function has the general form

$$f(x) = a\log_b x, \quad b > 0$$

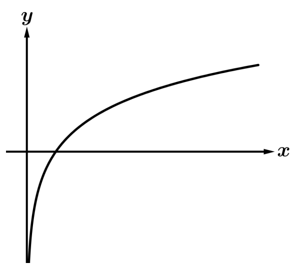
where $a$ and $b$ are constants with $a \neq 0$ and $b \neq 1$.

**Domain**: $(0, \infty)$

**Range**: $(-\infty, \infty)$

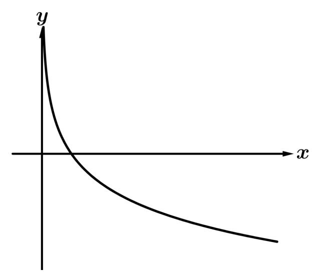
**Logarithmic Functions**

$a > 0$ and $b > 1$




**Logarithmic Functions**

$a < 0$ and $b > 1$




**Increasing vs. Decreasing**

Logarithmic functions are **always increasing** or **always decreasing**! They will never switch from one to the other, so they have **no relative (local) extrema** (unless on a closed interval).

**Concave Up vs. Concave Down**

Logarithmic functions are **always concave up** or **always concave down**! They will never switch concavity, so they have **no points of inflection**.

**End Behavior**

For logarithmic functions in general form, as the input values $(x)$ increase without bound, the output values $(y)$ will increase/decrease without bound.

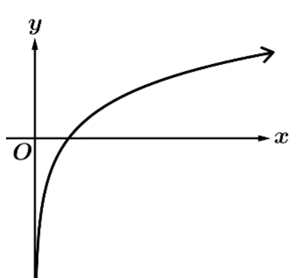
Since logarithmic functions have a restricted domain, they are vertically asymptotic to $x = 0$. As a result, the left end behavior will occur as $x \to 0^+$.

**End Behavior Limit Statements**

$\lim\limits_{x\to 0^+} a\log_b x = \pm\infty$ and $\lim\limits_{x\to +\infty} a\log_b x = \pm\infty$

**Example 1**: Write limit statements for the end behavior of the following logarithmic functions.

a)

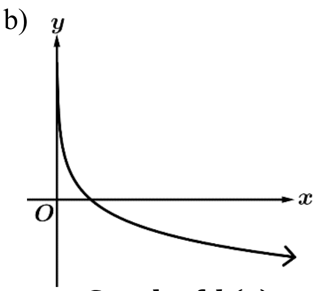



**Graph of $f(x)$**

Left: $\lim\limits_{x\to 0^+} f(x) = -\infty$

Right: $\lim\limits_{x\to\infty} f(x) = \infty$

b)



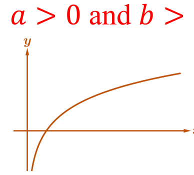

**Graph of $h(x)$**

Left: $\lim\limits_{x\to 0^+} h(x) = \infty$

Right: $\lim\limits_{x\to\infty} f(x) = -\infty$

c) $g(x) = 2\log_3 x$

$a > 0$ and $b > 1$



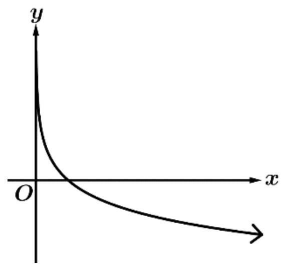

Left: $\lim\limits_{x\to 0^+} g(x) = -\infty$

Right: $\lim\limits_{x\to\infty} f(x) = \infty$


Solutions by Ted Gott tedg20776@gmail.com

**Example 2**: For each of the following, determine if the logarithmic function is increasing/decreasing and concave up/down.

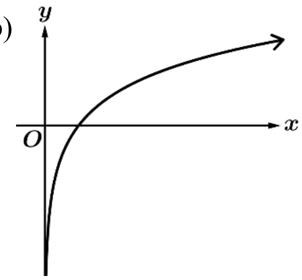
a)




Concave Up or Concave Down

Increasing or Decreasing

b)

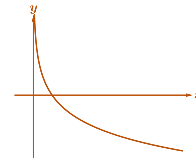



Concave Up or Concave Down

Increasing or Decreasing

c) $h(x) = -4\log_6 x$

$a < 0$ and $b > 1$



Concave Up or Concave Down

Increasing or Decreasing

**Example 3:** Selected values of the several logarithmic functions are shown in the tables below. For each table, find the value of the constant $k$.

| $x$ | $f(x)$ |
|---|---|
| 1 | 1 |
| 2 | 2 |
| $k$ | 3 |
| 8 | 4 |
| 16 | 5 |

| $x$ | $g(x)$ |
|---|---|
| $k$ | 0 |
| 6 | 5 |
| 18 | 10 |
| 54 | 15 |
| 162 | 20 |

| $x$ | $h(x)$ |
|---|---|
| 4 | 10 |
| 5 | 0 |
| 7 | $-10$ |
| $k$ | $-20$ |
| 19 | $-30$ |

| $x$ | $l(x)$ |
|---|---|
| $e^{-2}$ | 7 |
| $e$ | 14 |
| $k$ | 21 |
| $e^7$ | 28 |
| $e^{10}$ | 35 |

$x$ values are powers of 2
$1 = 2^0$ $2 = 2^1$
$k = 2^2 = 4$
$f(x) = \log_2 x + 1$

$x$ values are powers of 3 doubled
$6 = 3^1 \cdot 2$ $18 = 3^2 \cdot 2$
$k = 3^0 \cdot 2 = 2$
$g(x) = 5\log_3\left(\frac{x}{2}\right)$

$x - 3$ are powers of 2
$4 - 3 = 1 = 2^0$
$5 - 3 = 2 = 2^1$
$7 - 3 = 4 = 2^2$
$k - 3 = 8 = 2^3 \Rightarrow k = 11$
$19 - 3 = 16 = 2^4$
$h(x) = -10(\log_2(x-3) - 1)$

The exponents form a linear pattern $p = -2 + 3n$ for $n = 0,1,2,3,4.$
$k = e^{(-2+3\cdot 2)} = e^4$
$l(x) = \frac{7}{3}(5 + \ln(x))$

**Example 4**: Find the domain and range of the following logarithmic functions.

a) $f(x) = 2\log_3 x$

**Domain**: $(0, \infty)$

**Range**: $(-\infty, \infty)$

b) $g(x) = -5\log_2(x-3)$

**Domain**: $(3, \infty)$

**Range**: $(-\infty, \infty)$

c) $h(x) = 8\log(2x+3)$

**Domain**: $\left(-\frac{3}{2}, \infty\right)$

**Range**: $(-\infty, \infty)$