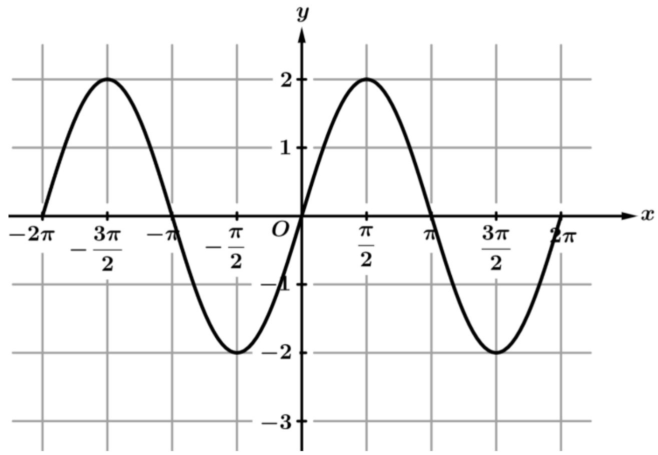
In Unit 1, we solved inequalities of polynomial and rational functions. Similarly, we can solve inequalities involving trigonometric functions.



Graph of *f*

Example 1: The graph of f is defined for $-2\pi \le x \le 2\pi$, as shown. What are all values of x for which $f(x) \le 0$?

$f(x) \le 0$ when $x = -2\pi$ or $-\pi \le x \le 0$ or $\pi \le x \le 2\pi$

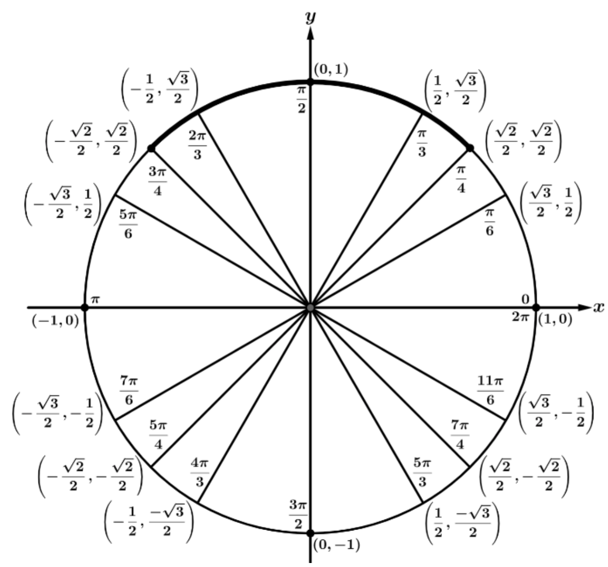
Solving inequalities is easy when we are given a graph—if only all problems were so simple! However, we must also understand how to solve inequalities in analytical form (algebraic form). For simpler inequalities that only involve one trigonometric expression, we can often just use our understanding of the unit circle to solve.

Example 2: What are all values of θ, $0 \le \theta < 2\pi$, for which $\sin\theta \ge \frac{\sqrt{2}}{2}$?

Since sine is the vertical displacement from the horizontal axis, we are looking for all possible angle measures where the y-coordinate is at least $\frac{\sqrt{2}}{2}$. The figure to the right highlights the portion of the unit circle that satisfies this inequality. Using the figure, what is the solution to the inequality

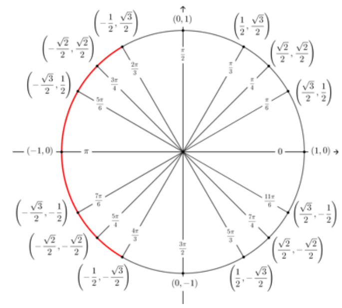
$\sin\theta \ge \frac{\sqrt{2}}{2}$?

$$\frac{\pi}{4} \le x \le \frac{3\pi}{4}$$



Example 3: Let $f(x) = 4 + 2\cos x$ and let $g(x) = 3$.

What are all values of x in the xy-plane, $0 \le x \le 2\pi$, for which $f(x) < g(x)$?



$4 + 2\cos x < 3$

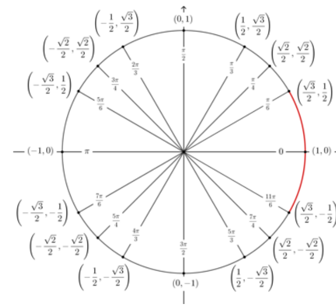
$2\cos x < -1$

$\cos x < -\frac{1}{2}$

$\frac{2\pi}{3} < x < \frac{4\pi}{3}$

Example 4: Let $h(x) = 3\cos x$ and let $k(x) = \sqrt{3} + \cos x$.

What are all values of x in the xy-plane, $0 \le x \le 2\pi$, for which $h(x) > k(x)$?



$3\cos x > \sqrt{3} + \cos x$

$2\cos x > \sqrt{3}$

$\cos x > \frac{\sqrt{3}}{2}$

$0 \le x < \frac{\pi}{6}$ or $\frac{11\pi}{6} < x \le 2\pi$

Using Sign Charts with Inequalities

As we did in Unit 1, we can also use sign charts to help us solve inequalities involving trigonometric functions.

Solving Trigonometric Inequalities

To solve the inequality $f(x) < 0$ or $f(x) > 0$, we can take the following steps.

Step 1: Set $f(x) = 0$ and solve for x. (We may need to factor to solve)

Step 2: Create a sign chart with the solutions marked from Step 1. (Be sure to include any domain restrictions)

Step 3: Test a value in one of the intervals by plugging it into the factored expression to see if the result of positive (+) or negative (-). (It is usually easiest to pick a value on one of the axes if possible)

Step 4: Label the remaining intervals as positive or negative.

Step 5: Interpret the sign chart to answer the inequality from the given question.

Example 5: What are all values of θ, $0 \le \theta \le 2\pi$, for which $2\sin^2\theta + \sin\theta < 0$?

$2\sin^2\theta + \sin\theta = 0$

$\sin\theta\,(2\sin\theta + 1) = 0$

$\sin\theta = 0 \Rightarrow \theta = 0, \pi, 2\pi$

$2\sin\theta + 1 = 0 \Rightarrow \sin\theta = -\frac{1}{2} \Rightarrow \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

$$\underbrace{+}_{0}\ \overbrace{-----}^{\substack{(+)(+)\\ +}}\ \underbrace{+}_{\pi}\ \overbrace{-----}^{\substack{(-)(+)\\ -}}\ \underbrace{+}_{\frac{7\pi}{6}}\ \overbrace{-----}^{\substack{(-)(-)\\ +}}\ \underbrace{+}_{\frac{11\pi}{6}}\ \overbrace{-----}^{\substack{(-)(+)\\ -}}\ \underbrace{+}_{2\pi}$$

$\pi < \theta < \frac{7\pi}{6}$ or $\frac{11\pi}{6} < \theta < 2\pi$

Example 6: What are all values of θ, $0 \le \theta \le 2\pi$, for which $2\cos^2\theta + \cos\theta - 1 \ge 0$?

$2\cos^2\theta + \cos\theta - 1 = 0$

$(2\cos\theta - 1)(\cos\theta + 1) = 0$

$2\cos\theta - 1 = 0 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}, \frac{5\pi}{3}$

$\cos\theta + 1 = 0 \Rightarrow \cos\theta = -1 \Rightarrow \theta = \pi$

$$\underbrace{+}_{0}\ \overbrace{-----}^{\substack{(+)(+)\\ +}}\ \underbrace{+}_{\frac{\pi}{3}}\ \overbrace{-----}^{\substack{(-)(+)\\ -}}\ \underbrace{+}_{\pi}\ \overbrace{-----}^{\substack{(-)(+)\\ -}}\ \underbrace{+}_{\frac{5\pi}{3}}\ \overbrace{-----}^{\substack{(+)(+)\\ +}}\ \underbrace{+}_{2\pi}$$

$0 \le \theta \le \frac{\pi}{3}$ or $\theta = \pi$ or $\frac{5\pi}{3} \le \theta \le 2\pi$