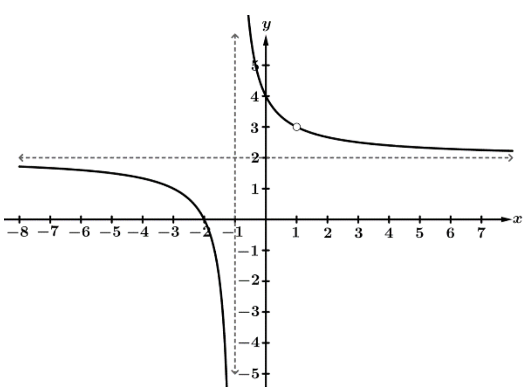
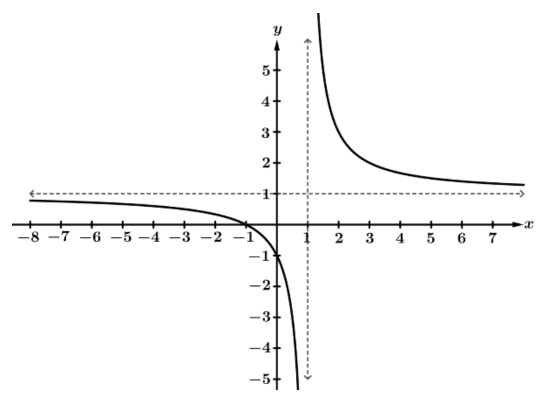
Recall: A rational function is the quotient of two polynomials.

Rational Function: $y = \dfrac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are both polynomials.

Since we are dividing by a polynomial, rational functions have restrictions on their domain. We know that we cannot divide by 0, so we must consider any x values where $g(x) = 0$, and restrict them from the domain. These x values will be the location of either a vertical asymptote or a hole in the graph.

Vertical asymptotes and holes both occur when the denominator of a rational function equals 0. So how can we distinguish between the two when working with a rational equation?

<table>
<tr><th colspan="2">Vertical Asymptotes and Holes</th></tr>
<tr><td colspan="2">A hole occurs when the factor in the denominator cancels out with factors in the numerator.
A vertical asymptote occurs when a factor in the denominator cannot cancel out with factors in the numerator.</td></tr>
<tr><th>Holes</th><th>Vertical Asymptotes</th></tr>
<tr><td>Equation: $f(x) = \dfrac{(x-1)(x+2)}{(x-1)}$ $g(x) = \dfrac{(x-1)^3}{2(x-1)^2}$ The functions f and g both have a hole at $x = 1$.</td><td>Equation: $f(x) = \dfrac{(x-3)(x+2)}{(x-1)}$ $g(x) = \dfrac{(x-1)(x+2)}{(x-1)^2}$ The functions f and g both have a vert. asymptote at $x = 1$</td></tr>
<tr><td>Graph: </td><td>Graph: </td></tr>
<tr><td>The graph above has a hole at $x = 1$</td><td>The graph above has a vertical asymptote at $x = 1$</td></tr>
<tr><td>$\lim\limits_{x\to1^-} f(x) = 3$ $\lim\limits_{x\to1^+} f(x) = 3$</td><td>$\lim\limits_{x\to1^-} f(x) = -\infty$ $\lim\limits_{x\to1^+} f(x) = +\infty$</td></tr>
</table>

Example 1: For each function below, determine the x values of any holes or vertical asymptotes.

a) $f(x) = \dfrac{(x-2)(x+3)}{(x+3)(x-5)}$

b) $y = \dfrac{(x+1)(x-2)^2}{(x-2)(x+1)^2}$

c) $g(x) = \dfrac{1}{x^3+4x}$

Example 2: Write a left and a right limit statement as x approaches 2 for each of the following functions.

a) $f(x) = \dfrac{(x-1)(x+3)}{(x-2)}$

Left:

Right:

b) $g(x) = \dfrac{(x-2)(x+4)}{(x-2)(x-3)}$

Left:

Right:

c) $h(x) = \dfrac{(x-4)(x-2)}{(x-2)^2(x-1)}$

Left:

Right:

Example 3: Write an equation of a rational function with the following limit properties.

a) $\lim\limits_{x\to 3^-} f(x) = 5$ $\quad \lim\limits_{x\to 3^+} f(x) = 5$

$\lim\limits_{x\to 1^-} f(x) = -\infty$ $\quad \lim\limits_{x\to 1^+} f(x) = +\infty$

b) $\lim\limits_{x\to -2^-} f(x) = 4$ $\quad \lim\limits_{x\to -2^+} f(x) = 4$

$\lim\limits_{x\to -1^-} f(x) = +\infty$ $\quad \lim\limits_{x\to -1^+} f(x) = +\infty$

Example 4: Sketch a picture of a rational function that has the following properties.

a) $f(x)$ has a hole at $x = 1$

As x approaches 4 from the left, $f(x)$ increases without bound.

As x approaches 4 from the right, $f(x)$ decreases without bound.

b) $g(x)$ has holes at $x = -2$ and $x = 3$

As x approaches -1 from the left, $g(x)$ decreases without bound.

As x approaches -1 from the right, $g(x)$ decreases without bound.