Zeros of Polynomial Functions

Given a polynomial function $p(x)$, if $p(a) = 0$, then a is a zero or root of $p(x)$.

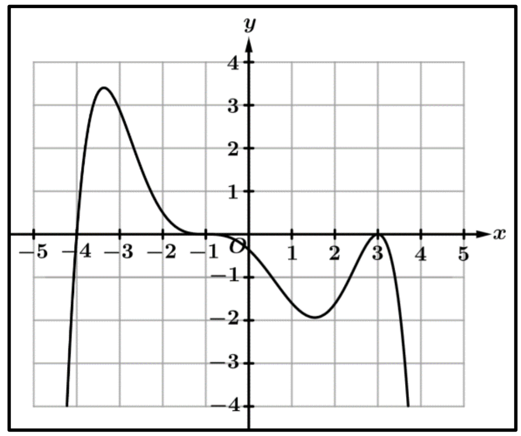
If a is a real number, then if $x = a$ is a zero of p, then $(x - a)$ is a linear factor of p.

Repeated Zeros (Multiplicity)

If a linear factor $(x - a)$ is repeated n times, the corresponding zero of the polynomial has a multiplicity n.

Typically, we know that the graph of a polynomial passes through the zeros on the graph. However, when a zero has a multiplicity greater than 1, the graph will behave differently near the zero.

The function $y = -.01(x + 4)(x + 1)^3(x - 3)^2$ is graphed to the right.
Notice the behavior around the zeros of the function.



Multiplicity

The multiplicity of a zero is the degree of its factor.

We can include the multiplicity when we list the zeros:

$$x = -4, x = -1 \text{ (mult. 3)}, x = 3 \text{ (mult. 2)}$$

At $x = 3$, the multiplicity is 2. The graph of the polynomial is tangent to the x axis (the graph bounces off the x axis).
The graph of a polyomial will always be tangent to the x axis at any zero with an **even** multiplicity.

Example 1: For each of the following polynomials, determine the degree of the polynomial, find all real zeros, and state the multiplicity for each zero.

a) $f(x) = -2x^3(x + 1)(x - 4)^2$

degree: $3 + 1 + 2 = 6$

zeros: $x = 0$ (mult. 3), $x = -1$

$x = 4$ (mult. 2)

b) $g(x) = 3(x^2 - 4)(x - 2)^4$

$= 3(x + 2)(x - 2)(x - 2)^4$

$= 3(x + 2)(x - 2)^5$

degree: $1 + 5 = 6$

zeros: $x = -2, x = 2$ (mult. 5)

c) $y = (x^3 - x^2 - 6x)(x^2 - 7x + 12)$

$= x(x^2 - x - 6)(x - 3)(x - 4)$

$= x(x + 2)(x - 3)(x - 3)(x - 4)$

$= x(x + 2)(x - 3)^2(x - 4)$

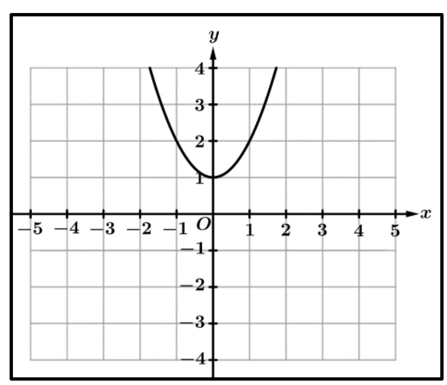
degree: $1 + 1 + 2 + 1 = 5$

zeros: $x = 0, x = -2, x, 3$ (mult. 2),

$x = 4$

Complex Roots

Some polynomials have roots that contain an imaginary number. This means you will not see then on the graph.



The graph of $f(x) = x^2 + 1$ is shown to the right.
To find the zeros of $f(x)$, we set $x^2 + 1 = 0$.

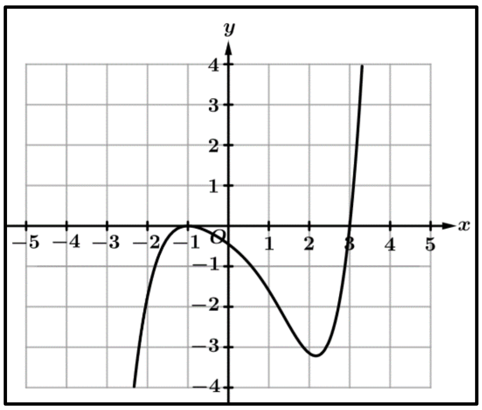
$x^2 = -1 \rightarrow \quad x = \pm\sqrt{-1} = \pm i$

Key Understanding: All imaginary roots come in __pairs__. If $a + bi$ is a root of $f(x)$, then so is __$a - bi$__. These are called **conjugate pairs**.

Example 2: Determine the conjugate of the following complex numbers.

a. $4i$, $-4i$ b. $-i$, i c. $2 - 3i$, $2 + 3i$ d. $-4 + 2i$, $-4 - 2i$

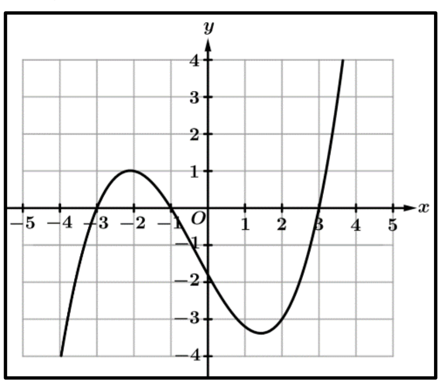
Fundamental Theorem of Algebra
A polynomial of degree n has exactly n complex zeros when counting multiplicities.



Example 3: The graph of the polynomial function $f(x)$ is shown in the figure above. It is known that $x = i\sqrt{3}$ is a zero of f. If f has degree n, what is the least possible value of n?

Zeros: $x = -1$ (Mult. 2); $x = 3$; $x = i\sqrt{3}$; $x = -i\sqrt{3}$ $n \geq 5$ since there are at least 5 zeros.

Polynomial Inequalities



Consider the function $f(x)$ above.

Reminder
When we write "$f(x)$", we are referring to the __y__-value on the graph of $f(x)$.
$f(x) > 0$ means the graph of $f(x)$ is __above__ the __x__-axis
$f(x) < 0$ means the graph of $f(x)$ is __below__ the __x__-axis

Example 4:

a) Where does $f(x) = 0$?
$x = -3, -1, 3$

b) Where is $f(x) > 0$?
$-3 < x < -1$ and $x > 3$

c) Where is $f(x) \leq 0$?
$x \leq -3$ and $-1 \leq x \leq 3$

Solving Nonlinear Inequalities (Polynomials)

1. Solve $f(x) = 0$.
2. Create a **sign chart** with the solutions from Step 1.
3. **Test values** in each interval to see if the values in the interval are $\underline{\ \ +\ \ }$ or $\underline{\ \ -\ \ }$.
4. **Interpret** the sign chart to answer the given inequality from the problem.

NOTE: Be sure to write your answer in **interval notation** and think about the **endpoints**!

Example 5: Solve $(x-3)(x+1)(x+4) > 0$

$$< - \overbrace{-}^{-} - \underbrace{+}_{-4} - \overbrace{-}^{+} - \underbrace{+}_{-1} - \overbrace{-}^{-} - \underbrace{+}_{3} - \overbrace{-}^{+} - >$$

$(-4,-1)$ and $(3,\infty)$ $\quad -4 < x < -1$ and $x > 3$

Example 6: Solve $(x+2)^2(x-5) \leq 0$

$$< - \overbrace{-}^{-} - \underbrace{+}_{-2} - \overbrace{-}^{-} - \underbrace{+}_{5} - \overbrace{-}^{+} - >$$

$(-\infty, 5]$ $\quad x \leq 5$

Determining the Degree a Polynomial Given a Table of Values

If given a table of values with equal width input intervals, we can determine the degree of a polynomial by examining successive differences in the output values. The number of successive differences needed for the differences to be constant is equal to the degree n of the polynomial.

Example 7: Determine the degree of the polynomials represented in the tables below.

a)

x	$f(x)$
1	-2
3	-3
5	-1
7	4
9	12

> -1, > 2, > 5, > 8

> 3, > 3, > 3

degree 2 because the second differences are constant.

b)

x	$g(x)$
0	-2
3	0
6	10
9	27
12	50

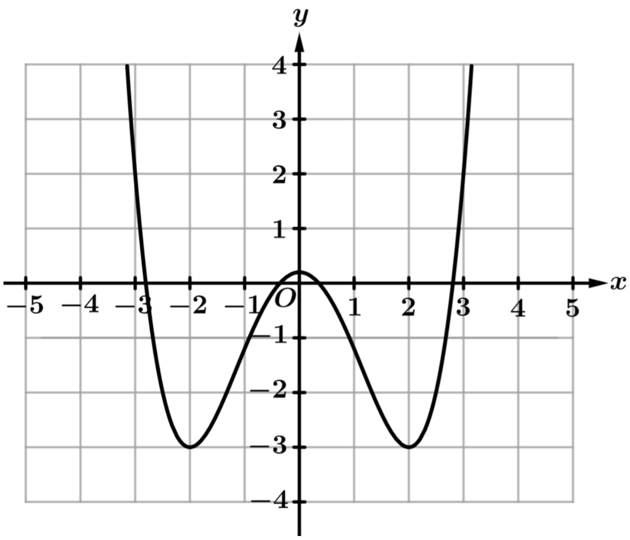
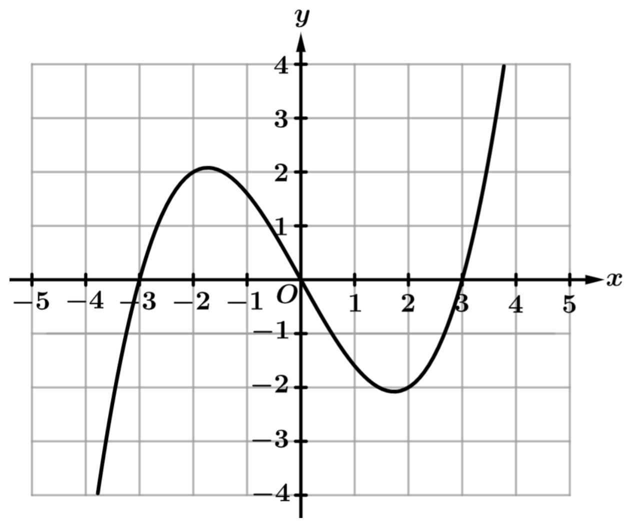
> 2, > 10, > 17, > 23

> 8, > 7, > 6

> -1, > -1

degree 3 because the third differences are constant.

Even and Odd Functions

Even Functions	**Odd Functions**
An even function is symmetric over the y axis. $f(-x) = f(x)$	An odd function is symmetric about the origin. $g(-x) = -g(x)$
	
$f(x) = x^4 - 8x^2 + 1$	$g(x) = x^3 - 9x$

Example 8: Determine if the following polynomials are even, odd, or neither.

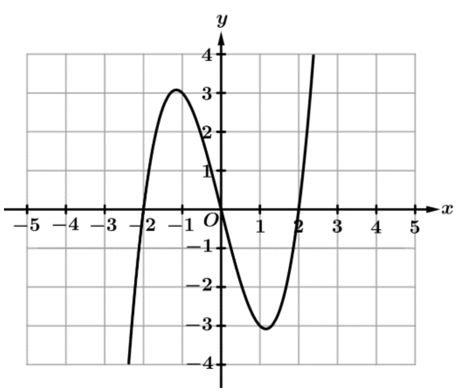
a) $h(x) = 2x^4 - x^2 + 5$

$h(-1) = 2(-1)^4 - (-1)^2 + 5 = 2 - 1 + 5 = 6$
$h(1) = 2(1)^4 - (1)^2 + 5 = 2 - 1 + 5 = 6$
even function

b) $k(x) = x^3 + 3x - 1$

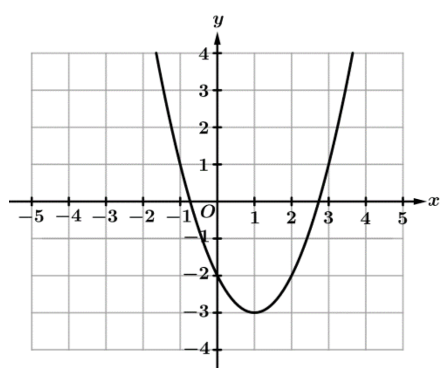
$k(-1) = (-1)^3 + 3(-1) - 1 = -1 - 3 - 1 = -5$
$k(1) = (1)^3 + 3(1) - 1 = 1 + 3 - 1 = 3$
Neither

c)



$f(-1) = 3 \qquad f(1) = -3$
odd function

d)



$f(-1) = 1 \qquad f(1) = -3$
Neither