In this Unit, we have worked with exponential functions and logarithmic functions, while utilizing important properties of each when simplifying expressions and finding equivalent representations. Now, we will use those same properties to solve equations and inequalities involving exponential and logarithmic functions.

Below are the important properties we have learned related to exponents and logarithms. We will need to be proficient with these properties so that we can use them to solve equations and inequalities involving exponential and logarithmic functions.

Properties of Exponents		
Product Property	**Power Property**	**Negative Exponent Property**
$b^m b^n = b^{m+n}$	$\left(b^m\right)^n = b^{mn}$	$b^{-n} = \dfrac{1}{b^n}$
Properties of Logarithms		
Product Property	**Quotient Property**	**Power Property**
$\log_b(xy) = \log_b x + \log_b y$	$\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$	$\log_b x^n = n\log_b x$
Converting Between Exponential and Logarithmic Forms		
Exponential Form: $a = b^c \Leftrightarrow$ Logarithmic Form: $\log_b a = c$		

Equations With Only 1 Log or Exponential Function

If an equation or inequality involves only a single logarithm or only a single exponential function, we can solve by converting between exponential and logarithmic forms.

Solving Exponential and Log Equations Involving Only 1 Logarithm or Exponent

1. Isolate the exponential or log expression on one side of the equation.
2. Write the equation to the alternate form (Exponential ↔ Log)
3. Solving for the variable (if necessary).
4. Simplify and/or round answer as directed.

Example 1: Solve the following equations. Leave your answers in exact form.

a) $4^x - 2 = 7$

$4^x = 9$

$x = \log_4 9$

b) $(2)^{x+3} + 5 = 26$

$(2)^{x+3} = 21$

$x + 3 = \log_2 21$

$x = \log_2 21 - 3$

c) $-7 = 3 - 4e^x$

$-10 = -4e^x$

$\dfrac{-10}{-4} = e^x \quad x = \ln\dfrac{5}{2}$

Example 2: Solve the following equations. Leave your answers in exact form.

a) $2\log_3(x+3)+4=10$

$2\log_3(x+3)=6$
$\log_3(x+3)=3$
$x+3=27$
$x=24$

b) $2+\log_5(2x-1)=4$

$\log_5(2x-1)=2$
$2x-1=5^2=25$
$2x=26$
$x=13$

c) $\dfrac{\ln(2-x)}{5}=1$

$\ln(2-x)=5$
$2-x=e^5$
$x=2-e^5$

Equations With Multiple Logarithms

When working with more than one logarithm, we can use our log properties to simplify expressions before solving the equation or inequality. For equations and inequalities involving multiple logarithms, we will generally simplify to one of these forms:

Form 1: $\log_a(x)=b$

Form 2: $\log_a(x)=\log_a(y)$

If we simplify to **Form 1**, then we can finish solving by converting to exponential form as we did with **Example 2**.

If we simplify to **Form 2**, then we can finish solving by setting $x=y$.

Important Note #1: In either case, we need to check for extraneous solutions! Using log properties can sometimes lead to solutions that are not in the domain of the original equation.

Important Note #2: $\log x+\log y=\log z$ does **NOT** imply that $x+y=z$. We must use log properties to simplify $\log x+\log y=\log z$ … we **cannot** just cancel out the logs!

Example 3: Solve the following equations. Check for extraneous solutions when necessary.

a) $\log(3)+\log(x+4)=\log(5x-2)$ $\quad x>\dfrac{2}{5}$

$\log\big(3(x+4)\big)=\log(5x-2)$
$3(x+4)=5x-2$
$3x+12=5x-2$
$14=2x$
$x=7$

b) $\ln(x+1)-\ln(3x-5)=\ln(7)$ $\quad x>\dfrac{5}{3}$

$\ln\left(\dfrac{x+1}{3x-5}\right)=\ln(7)$
$\dfrac{x+1}{3x-5}=7$ $\qquad x+1=7(3x-5)$
$x+1=21x-35$ $\qquad 36=20x$
$x=\dfrac{36}{20}=\dfrac{9}{5}$

c) $\log_3(x+2)+\log_3(x-3)=\log_3(14)$ $\quad x>3$

$\log_3[(x+2)(x-3)]=\log_3(14)$
$(x+2)(x-3)=14$
$x^2-x-6=14$
$x^2-x-20=0$
$(x-5)(x+4)=0$ $\qquad x=5$ $\qquad x=-4<3$
$x=5$ only

d) $\ln(x)-\ln(3)=2$ $\quad x>0$

$\ln\left(\dfrac{x}{3}\right)=2$
$\dfrac{x}{3}=e^2$
$x=3e^2$

Example 4: Solve the following equations. Check for extraneous solutions when necessary.

a) $\log_4(x) - 3\log_4 2 = \log_4(x+7)$

$x > 0 \qquad x + 7 > 0 \Rightarrow x > -7 \qquad \text{D}: x > 0$

$\log_4(x) - \log_4 2^3 = \log_4(x+7)$

$\log_4\left(\frac{x}{8}\right) = \log_4(x+7)$

$\frac{x}{8} = x + 7 \qquad x = 8x + 56$

$-7x = 56 \qquad x = -\frac{56}{7} < 0 \quad \text{No solution}$

b) $\log(x) - 2 = \log(x-7)$

$x > 0 \qquad x - 7 > 0 \Rightarrow x > 7 \quad \text{D}: x > 7$

$\log(x) - \log(x-7) = 2 \to \log\left(\frac{x}{x-7}\right) = 2$

$\frac{x}{x-7} = 10^2 = 100 \to x = 100x - 700$

$99x = 700 \to x = \frac{700}{99}$

Example 5: Let $f(x) = \ln(x+15) - \ln(x)$ and let $g(x) = \ln(x+3)$. Find all values of x where the graphs of f and g intersect.

$\ln(x+15) - \ln(x) = \ln(x+3) \qquad \text{D}: x > 0$

$x + 15 > 0 \Rightarrow x > -15 \qquad x > 0 \qquad x + 3 > 0 \Rightarrow x > -3$

$\ln\left(\frac{x+15}{x}\right) = \ln(x+3) \to \quad \frac{x+15}{x} = x + 3$

$x + 15 = x^2 + 3x$

$0 = x^2 + 2x - 15$

$(x+5)(x-3) = 0 \to x = -5, 3$

$x = 3$ is the only solution in the domain.

Example 6: Let $h(x) = \log_3(2x-3) + \log_3(x+4)$ and let $k(x) = \log_3(8x+2)$. In the xy-plane, what are all x-coordinates of the points of intersection of the graphs of h and k?

$\log_3(2x-3) + \log_3(x+4) = \log_3(8x+2) \qquad \text{D}: x > \frac{3}{2}$

$2x - 3 > 0 \Rightarrow x > \frac{3}{2} \qquad x + 4 > 0 \Rightarrow x > -4$

$8x + 2 > 0 \Rightarrow x > -\frac{2}{8}$

$\log_3[(2x-3)(x+4)] = \log_3(8x+2)$

$(2x-3)(x+4) = 8x + 2$

$2x^2 + 5x - 12 = 8x + 2$

$2x^2 - 3x - 14 = 0$

$(2x-7)(x+2) = 0$

$\boxed{x = \frac{7}{2}} \qquad x = -2 < \frac{3}{2}$

Equations with Multiple Exponential Functions

When an equation has more than one exponential function, we can often solve the equation by finding common bases using properties of exponents.

Example 7: Solve the following equations.

a) $2^{5x-1} = 2^{2x+7}$

$5x - 1 = 2x + 7$

$3x = 8$

$x = \frac{8}{3}$

b) $8^{3x-2} = 2^{x+5}$

$(2^3)^{3x-2} = 2^{x+5}$

$3(3x-2) = x + 5$

$9x - 6 = x + 5 \qquad 8x = 11 \qquad x = \frac{11}{8}$

c) $9^{2x} = 27^{x+2}$

$(3^2)^{2x} = (3^3)^{x+2}$

$4x = 3x + 6$

$x = 6$

d) $\left(\frac{1}{2}\right)^{5x+7} = 4^x$

$(2^{-1})^{5x+7} = (2^2)^x$

$-5x - 7 = 2x \qquad -7x = 7 \qquad x = -1$