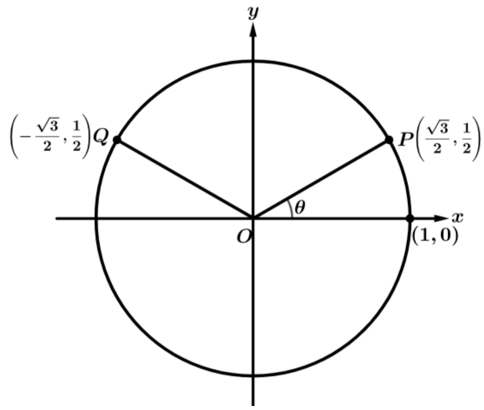
Consider the following two equation statements.

Equation 1: $\sin(x) = \frac{1}{2}$ **Equation 2:** $\sin^{-1}\left(\frac{1}{2}\right) = x$

Are these two equations equivalent?



On the unit circle, we know there are two angles, $0 \le \theta \le 2\pi$, where $\sin(\theta) = \frac{1}{2}$: $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$. However, due to the domain restrictions, $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ only. As a result, when we are solving equations involving trigonometric functions, we must modify our solutions to account for any domain restrictions, or the lack of any domain restrictions.

Finding General Solutions to a Trigonometric Equation

Since trigonometric functions are periodic, we can have infinitely many solutions to an equation if no domain restriction is given. Let's reconsider **Equation 1** from above. We know that $\sin(x) = \frac{1}{2}$ when $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$. However, since sine is periodic with a period of 2π, we also know that $\sin(x) = \frac{1}{2}$ when $x = \frac{\pi}{6} + 2\pi = \frac{13\pi}{6}$ and $x = \frac{5\pi}{6} + 2\pi = \frac{17\pi}{6}$.

In fact, we could keep rotating around the unit circle as many times as we wanted to create infinitely many angles where $\sin(x) = \frac{1}{2}$.

Writing the General Solutions to a Trigonometric Equation		
$\sin(x) = \frac{1}{2}$	$\cos(x) = \frac{1}{2}$	$\tan(x) = 1$
$x = \frac{\pi}{6} + 2\pi k$, where k is an integer.	$x = \frac{\pi}{3} + 2\pi k$, where k is an integer.	$x = \frac{\pi}{4} + \pi k$, where k is an integer.
$x = \frac{5\pi}{6} + 2\pi k$, where k is an integer.	$x = \frac{5\pi}{3} + 2\pi k$, where k is an integer.	Since the period of tangent is π, one equation will capture all solutions.

Example 1: Find all solutions to the equation $\cos(x) = -\frac{\sqrt{2}}{2}$.

$x = \frac{3\pi}{4} + 2\pi k$ and $x = \frac{5\pi}{4} + 2\pi k$

Example 2: Find all solutions to the equation $\sin(x) = -1$.

$x = \frac{3\pi}{2} + 2\pi k$

Example 3: Find all solutions to the equation $\tan(x) = \frac{1}{\sqrt{3}}$.

$x = \frac{\pi}{6} + \pi k$

Solving Trigonometric Equations

1. Isolate the trigonometric function on one side of the equation.
2. Find the corresponding angle measures on the unit circle that satisfy the given equation.
3. Consider any domain restrictions in the problem.
4. Write the solutions and/or the general solutions.

Example 4: What are all values of θ where $6\sin(\theta) = 3\sqrt{2}$?

$\sin(\theta) = \frac{3\sqrt{2}}{6} = \frac{\sqrt{2}}{2}$ $\quad \theta = \frac{\pi}{4} + 2\pi k, \frac{3\pi}{4} + 2\pi k$

Example 5: Let $f(x) = 3 + 2\cos x$ and $g(x) = 2$. In the xy-plane, what are the x-coordinates of the points of intersection of the graph of f and g for $0 \le \theta < 2\pi$?

$3 + 2\cos x = 2$ $\quad 2\cos x = -1$ $\quad \cos x = -\frac{1}{2}$ $\quad x = \frac{2\pi}{3}, \frac{4\pi}{3}$

When we want to square trigonometric functions (or raise them to any exponent), we do not want our notation to be overly complicated or ambiguous.

If we wanted to square the variable x, the notation is very straightforward, and we simply write x^2.

But trig functions can create a bit of an issue if we are not careful. If we wanted to square the expression $\sin x$, we must be careful to clearly communicate our intentions.

If we write $\sin x^2$, then we are only squaring the variable x and not the entire expression $\sin x$, as intended.

So, we must include parentheses around the expression and then square the outside of the parentheses: $(\sin x)^2$.

However, we also have notation that we can use that is both clear and simple.

$$\boxed{\text{We can write } (\sin x)^n \text{ as } \sin^n x. \text{ For example } (\sin x)^2 = \sin^2 x.}$$

Example 6: The function g is given by $g(x) = 3 - 4\sin^2 x$. What are the zeros of g on the interval $0 \le \theta < 2\pi$?

$3 - 4\sin^2 x = 0 \quad -4\sin^2 x = -3 \quad \sin^2 x = \frac{3}{4} \quad \sin x = \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2} \quad x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

Example 7: What are all values of θ, for $0 \le \theta < 2\pi$, where $\cos^2\theta = -\cos\theta$?

$\cos^2\theta + \cos\theta = 0$

$\cos\theta(\cos\theta + 1) = 0$

$\cos\theta = 0 \qquad \cos\theta + 1 = 0$

$\cos\theta = 0 \qquad \theta = \frac{\pi}{2}, \frac{3\pi}{2}$

$\cos\theta = -1 \qquad \theta = \pi$

Example 8: Let $f(x) = 2\sin^2 x$ and $g(x) = \sqrt{3}\sin x$. In the xy-plane, what are the x-coordinates of the points of intersection of the graph of f and g for $0 \le \theta < 2\pi$?

$2\sin^2 x = \sqrt{3}\sin x$

$2\sin^2 x - \sqrt{3}\sin x = 0$

$\sin x\left(2\sin x - \sqrt{3}\right) = 0$

$\sin x = 0 \qquad 2\sin x - \sqrt{3} = 0$

$\sin x = 0 \quad x = 0, \pi$

$\sin x = \frac{\sqrt{3}}{2} \quad x = \frac{\pi}{3}, \frac{2\pi}{3}$