A **rational function** is simply the quotient (fraction) of two ____polynomials______________.

Rational Function: $y = \dfrac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are both polynomials and $g(x) \neq 0$

The following are all examples of rational functions:

$$y = \frac{2}{x+3} \qquad y = \frac{x^2 - 3x + 1}{3x + 4} \qquad y = \frac{2x^2 + 4x - 6}{x^3 - 7x + 11}$$

End Behavior for Rational Functions

The end behavior of a rational function is determined by the leading terms of the numerator and denominator:

$$f(x) = \frac{ax^n}{bx^d}$$

Case I: The leading terms have the same degree ($n = d$)
Result: $f(x)$ has a horizontal asymptote: $y = \dfrac{a}{b}$

Case II: The denominator dominates the numerator ($n < d$)
Result: $f(x)$ has a horizontal asymptote: $y = 0$

Case III: The numerator dominates the denominator ($n > d$)
Result: $f(x)$ has the end behavior of the polynomial $y = \dfrac{a}{b}x^{n-d}$

Note: If the degree of the numerator is exactly 1 more than the degree of the denominator, then $f(x)$ has a slant (oblique) asymptote.

Example 1: Determine if the following rational functions have a horizontal asymptote, slant asymptote, or neither. If the function has a horizontal asymptote, write the equation of the asymptote.

a) $f(x) = \dfrac{3x^2 + 4x - 7}{5x^2 - 3}$

same degree, $n = d$
horizontal asymptote $y = \frac{3}{5}$

b) $y = \dfrac{2x - 5}{x^2 + 3x + 2}$

denominator dominates $n < d$
horizontal asymptote $y = 0$

c) $g(x) = \dfrac{2x^2 - 4}{5x + 9}$

numerator dominates, $n = d + 1$
slant asymptote

d) $y = \dfrac{4x + 5}{8x - 1}$

same degree, $n = d$
horizontal asymptote $y = \frac{4}{8} = \frac{1}{2}$

e) $k(x) = \dfrac{3}{x^2 + 3x - 7}$

denominator dominates $n < d$
horizontal asymptote $y = 0$

f) $p(x) = -\dfrac{4}{2x + 1}$

denominator dominates $n < d$
horizontal asymptote $y = 0$

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Example 2: Write limit statements to describe the end behavior of the following rational functions

a) $f(x) = \dfrac{2x^3 + 4x - 1}{6x^3 - x^2 + 4}$

Left: $\lim_{x \to -\infty} f(x) = \frac{2}{6} = \frac{1}{3}$

Right: $\lim_{x \to \infty} f(x) = \frac{2}{6} = \frac{1}{3}$

b) $g(x) = \dfrac{5x^2 - 8x + 9}{2x^3 + x - 1}$

Left: $\lim_{x \to -\infty} f(x) = 0$

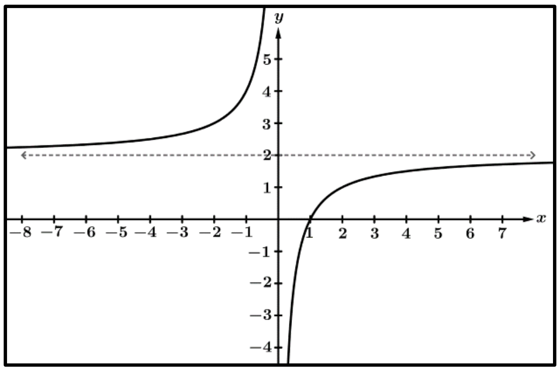
Right: $\lim_{x \to \infty} f(x) = 0$

c) $h(x) = \dfrac{-3x^4 - x^2 + x}{x^3 + 4x + 4}$

Left: $\lim_{x \to -\infty} f(x) = \infty$

Right: $\lim_{x \to \infty} f(x) = -\infty$

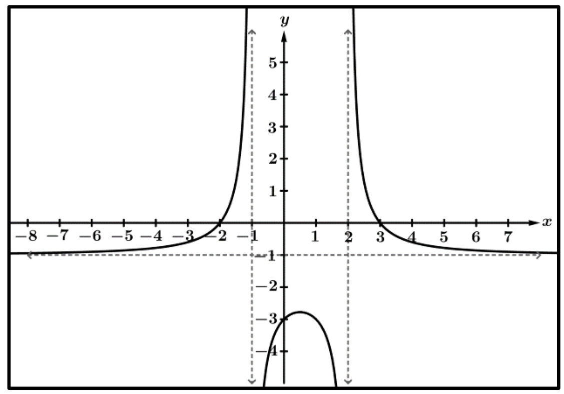
d)



Left: $\lim_{x \to -\infty} f(x) = 2$

Right: $\lim_{x \to \infty} f(x) = 2$

e)



Left: $\lim_{x \to -\infty} f(x) = -1$

Right: $\lim_{x \to \infty} f(x) = -1$

Slant Asymptotes

If the degree of the numerator is exactly 1 greater than the degree of the denominator, a rational function will have a slant asymptote that is parallel to the ratio of leading terms.

$$f(x) = \frac{ax^n + \cdots + c_1}{bx^d + \cdots + c_2}$$ where ax^n and bx^d are the leading terms and $n = d + 1$,

$f(x)$ has a slant asymptote parallel to the line $y = \dfrac{a}{b}x$

Example 3: Which of the following rational functions has a slant asymptote parallel to the line $y = \dfrac{1}{2}x$?

I. $f(x) = \dfrac{x^2 + 3}{2x^2 + x + 6}$ II. $g(x) = \dfrac{x^2 + 4x + 1}{2x^3 + x^2 + 2}$ III. $h(x) = \dfrac{x^2 + 3x + 5}{2x + 4}$ IV. $k(x) = \dfrac{x^4 + x^3 + 5}{2x^2 + x - 1}$

A) I only B) II only C) III only D) I and II only E) III and IV only

$n = d + 1 \quad y = \frac{1}{2}x$