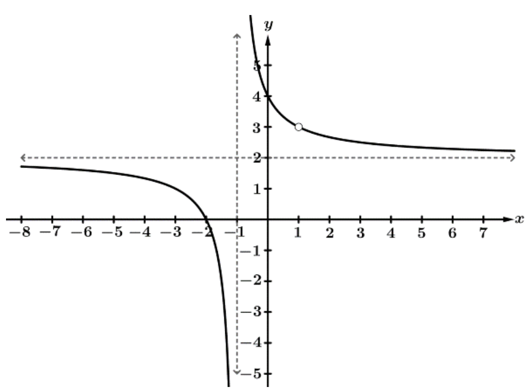
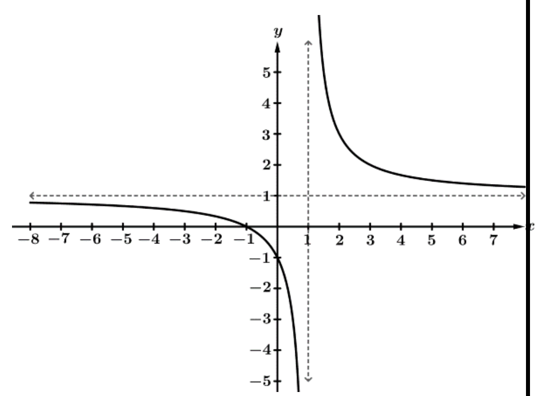
Notes: (Topics 1.9 — 1.10) Rational Functions — Vertical Asymptotes and Holes Solutions

Recall: A rational function is the quotient of two polynomials.

Rational Function: $y = \dfrac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are both polynomials.

Since we are dividing by a polynomial, rational functions have restrictions on their domain. We know that we cannot divide by 0, so we must consider any x values where $g(x) = 0$, and restrict them from the domain. These x values will be the location of either a vertical asymptote or a hole in the graph.

Vertical asymptotes and holes both occur when the denominator of a rational function equals 0. So how can we distinguish between the two when working with a rational equation?

<table>
<tr><th colspan="2">Vertical Asymptotes and Holes</th></tr>
<tr><td colspan="2">A hole occurs when the factor in the denominator cancels out with factors in the numerator.
A vertical asymptote occurs when a factor in the denominator cannot cancel out with factors in the numerator.</td></tr>
<tr><th>Holes</th><th>Vertical Asymptotes</th></tr>
<tr><td>Equation: $f(x) = \dfrac{(x-1)(x+2)}{(x-1)}$ $g(x) = \dfrac{(x-1)^3}{2(x-1)^2}$ The functions f and g both have a hole at $x = 1$.</td><td>Equation: $f(x) = \dfrac{(x-3)(x+2)}{(x-1)}$ $g(x) = \dfrac{(x-1)(x+2)}{(x-1)^2}$ The functions f and g both have a vert. asymptote at $x = 1$</td></tr>
<tr><td>Graph: </td><td>Graph: </td></tr>
<tr><td>The graph above has a hole at $x = 1$</td><td>The graph above has a vertical asymptote at $x = 1$</td></tr>
<tr><td>$\lim_{x\to 1^-} f(x) = 3$ $\lim_{x\to 1^+} f(x) = 3$</td><td>$\lim_{x\to 1^-} f(x) = -\infty$ $\lim_{x\to 1^+} f(x) = +\infty$</td></tr>
</table>

Example 1: For each function below, determine the x values of any holes or vertical asymptotes.

a) $f(x) = \dfrac{(x-2)(x+3)}{(x+3)(x-5)}$

Hole at $x = -3$ because the factor $(x+3)$ cancels out of the denominator with a factor in the numerator.

Vertical asymptote at $x = 5$ because the factor $(x-5)$ does not cancel out of the denominator with a factor in the numerator.

b) $y = \dfrac{(x+1)(x-2)^2}{(x-2)(x+1)^2}$

Hole at $x = 2$ because the factor $(x-2)$ cancels out of the denominator with a factor in the numerator.

Vertical asymptote at $x = -1$ because one of the factors $(x+1)$ does not cancels out of the denominator with a factor in the numerator.

c) $g(x) = \dfrac{1}{x^3 + 4x} = \dfrac{1}{x(x^2+4)}$

Vertical asymptote at $x = 0$ because the factor (x) does not cancels out of the denominator with a factor in the numerator. The factor (x^2+4) is never 0.

Example 2: Write a left and a right limit statement as x approaches 2 for each of the following functions.

a) $f(x) = \dfrac{(x-1)(x+3)}{(x-2)}$

Left: $\lim\limits_{x \to 2^-} f(x) = -\infty$

Right: $\lim\limits_{x \to 2^+} f(x) = \infty$

Vertical asymptote at $x = 2$

near $x = 2$: $\overbrace{\_\ \_\ \_}^{\frac{(+)(+)}{(-)} = -}\ \underset{2}{+}\ \overbrace{\_\ \_\ \_}^{\frac{(+)(+)}{(+)} = +}$

b) $g(x) = \dfrac{(x-2)(x+4)}{(x-2)(x-3)}$

Left: $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} \frac{x+4}{x-3} = \frac{6}{-1} = -6$

Right: $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^-} \frac{x+4}{x-3} = \frac{6}{-1} = -6$

Hole at $x = 2$

c) $h(x) = \dfrac{(x-4)(x-2)}{(x-2)^2(x-1)}$

Left: $\lim\limits_{x \to 2^-} f(x) = \infty$

Right: $\lim\limits_{x \to 2^+} f(x) = -\infty$

Vertical asymptote at $x = 2$

near $x = 2$: $\overbrace{\_\ \_\ \_}^{\frac{(-)(-)}{(+)(+)} = +}\ \underset{2}{+}\ \overbrace{\_\ \_\ \_}^{\frac{(-)(+)}{(+)(+)} = -}$

Example 3: Write an equation of a rational function with the following limit properties. Answers can vary

a) $\lim\limits_{x \to 3^-} f(x) = 5 \qquad \lim\limits_{x \to 3^+} f(x) = 5$

$\lim\limits_{x \to 1^-} f(x) = -\infty \qquad \lim\limits_{x \to 1^+} f(x) = +\infty$

$f(x) = \dfrac{k(x-3)}{(x-3)(x-1)} \qquad \lim\limits_{x \to 3} \dfrac{k}{x-1} = \dfrac{k}{2} = 5 \Rightarrow k = 10$

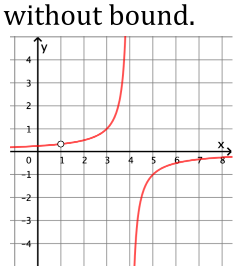
b) $\lim\limits_{x \to -2^-} f(x) = 4 \qquad \lim\limits_{x \to -2^+} f(x) = 4$

$\lim\limits_{x \to -1^-} f(x) = +\infty \qquad \lim\limits_{x \to -1^+} f(x) = +\infty$

$f(x) = \dfrac{k(x+2)}{(x+2)(x+1)^2} \qquad \lim\limits_{x \to -2} \dfrac{k}{(x+1)^2} = \dfrac{k}{1} = 4 \Rightarrow k = 4$

Example 4: Sketch a picture of a rational function that has the following properties. Sketches may vary

a) $f(x)$ has a hole at $x = 1$
As x approaches 4 from the left, $f(x)$ increases without bound.
As x approaches 4 from the right, $f(x)$ decreases without bound.



b) $g(x)$ has holes at $x = -2$ and $x = 3$
As x approaches -1 from the left, $g(x)$ decreases without bound.
As x approaches -1 from the right, $g(x)$ decreases without bound.