Notes: (Topic 2.8) Inverse Functions Solutions

An inverse relation will "**undo**" a given relation. **Every** inverse relation can be found by switching **each** x and y value.

In some situations, this process is intuitive, but other times this may not be as obvious. In these cases, it is important that we understand the concept of inverse relations as they will help us tackle a variety of problems.

Let's look at this **numerically** (tables), **graphically** (pictures) and **analytically** (equations):

Numerical (Tables)

Example 1: Find the inverse relation of the given table.

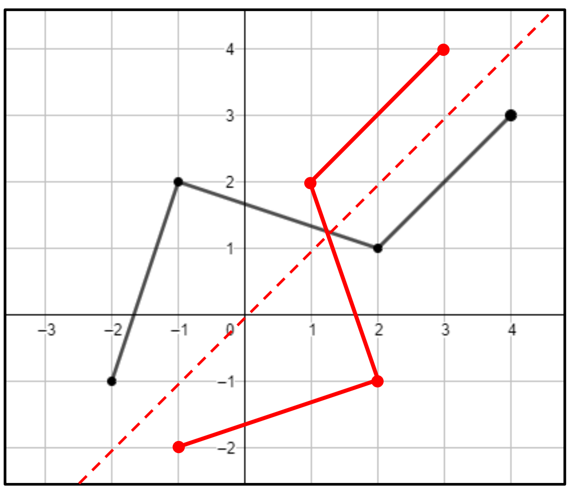
x	1	3	4	6
y	-1	2	0	2

x	-1	2	0	2
y	1	3	4	6

Note: The original table is a function because each x value has exactly one y value. However, the inverse is not a function because the x value 2 has two different y values 3 and 6.

Graphical (Pictures)

Example 2: Sketch the inverse relation of the given graph.



Steps to Sketch an Inverse Graph
(Linear Pieces)

1. List the **key** points in a **table**
2. Create a **new** table for the inverse by **switching** the x and y values.
3. Plot the **new** points for the inverse and **sketch** the inverse graph.

Key points table:

x	-2	-1	2	4
y	-1	2	1	3

Inverse table:

x	-1	2	1	3
y	-2	-1	2	4

Example 2B: Is the original relation a function? Is the inverse relation a function?

The original relation **is** a function because each x value has exactly one y value. The inverse relation **is not** a function because each x value between 1 and 2 have two y values.

Graphical Property of Inverses: The graphs of inverses are reflections over the line $y = x$.

Analytical (Equations)

Example 3: Find the inverse for $f(x) = 3x - 7$

$y = 3x - 7 \Rightarrow x = 3y - 7 \Rightarrow y = \frac{x+7}{3}$ $\quad f^{-1}(x) = \frac{x+7}{3}$

Notation: The inverse function of $f(x)$ is written as $f^{-1}(x)$.

(**Reminder**: $f(x) = y$)

Steps to Find an Inverse Equation
1. **Switch** the x and y values.
2. **Solve** for y (Get y by itself!)
3. If original equation was in $f(x) =$ form, write the inverse equation as $f^{-1}(x) =$

Finding Inverse Functions of Rational Functions (with multiple x's)	
Step 1	Switch the x and y values.
Step 2	Multiply/Distribute both sides by the denominator of the rational expression to eliminate the fraction.
Step 3	Move all terms that include the variable y to the left side of the equation and move all terms that do not include the variable y to the right side of the equation.
Step 4	Factor out an y from the terms on the left side of the equation.
Step 5	Divide both sides by the terms remaining on the left side after y was factored out.
Step 6	Rewrite the equation with proper inverse notation $\left(f^{-1}(x)\right)$

Example 4: Find the inverse functions for each of the following.

a) $f(x) = \frac{x-2}{x+3}$

$x = \frac{y-2}{y+3} \Rightarrow x(y+3) = y - 2$

$xy + 3x = y - 2$

$xy - y = -3x - 2$

$y(x-1) = -(3x+2)$

$y = -\frac{3x+2}{x-1}$

$f^{-1}(x) = -\frac{3x+2}{x-1}$

b) $g(x) = \frac{2x+1}{x-3}$

$x = \frac{2y+1}{y-3} \Rightarrow x(y-3) = 2y + 1$

$xy - 3x = 2y + 1$

$xy - 2y = 3x + 1$

$y(x-2) = 3x + 1$

$y = \frac{3x+1}{x-2}$

$g^{-1}(x) = \frac{3x+1}{x-2}$

More on Graphing Inverses

We know that to find an inverse, we simply switch the x and y values.

Graphically, this means that a function and its inverse will be reflections over the line $y = x$.

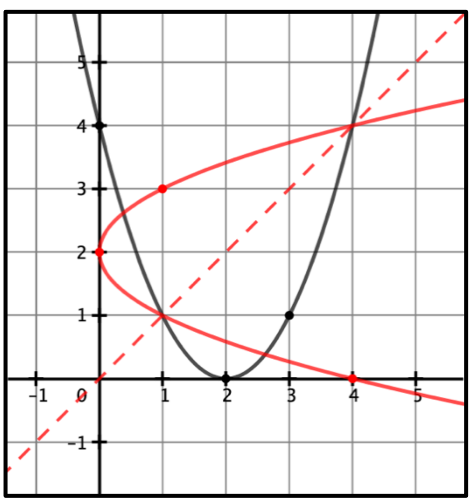
If a graph is not made up of simply line segments, sketching the inverse can be challenging. To do this, we use the fact that a function and its inverse are reflections over the line $y = x$ to help us sketch the inverse graph.

Steps to Sketch an Inverse Graph (Nonlinear pieces)

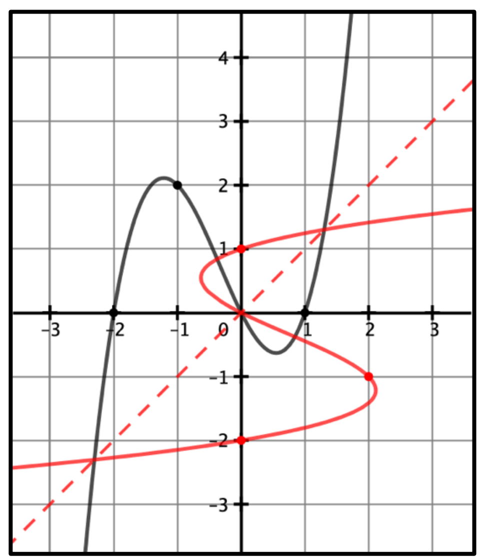
1. Sketch the line $y = x$ on the graph.
2. Mark any points from the original graph that are already on the line. These points will stay the same!
3. Select a few additional points on the original graph and find their inverse points.
   - You can find the inverse points by switching the x and y values, **OR**
   - Graphically reflecting each point over the line $y = x$ by drawing a line perpendicular from the point to the line $y = x$ and extending it an equal distance on the other side of the line.
4. Sketch the inverse graph by connecting the new points in a similar pattern to the original function.

Example 5: Sketch the inverse of the functions below.

a)



b)



What do Inverse Functions do?

If you plug a number into a function, it generally will output a new (different) number. For example, if we plug in $x = 2$ into the function $f(x) = 3x - 1$, we get 5.

Now, If we plug that **answer** into the inverse function, the answer should be 2.

This is because an inverse function essentially will "undo" a function. So, if we plug x into a function, and then plug the output into the inverse function, we should end up with plug x again.

Inverse Functions

Two functions $f(x)$ and $g(x)$ are inverses if and only if $f(g(x)) = x$ and $g(f(x)) = x$.

To show that two functions are inverses, we must show that the $f^{-1}(f(x))$ and $f(f^{-1}(x))$ **BOTH** equal x .

Example 6: Let $f(x)$ and $g(x)$ be the functions below. Determine if $f(x)$ and $g(x)$ are inverses.

$f(x) = 2x - 3$ $\qquad$ $g(x) = \frac{1}{2}x + \frac{3}{2}$

Step 1: Find $f(g(x)) = f\left(\frac{1}{2}x + \frac{3}{2}\right) = 2\left(\frac{1}{2}x + \frac{3}{2}\right) - 3 = (x + 3) - 3 = x$

Step 2: Find $g(f(x)) = g(2x - 3) = \frac{1}{2}(2x - 3) + \frac{3}{2} = \left(x - \frac{3}{2}\right) + \frac{3}{2} = x$

Note: Sometimes, we must use a restricted domain to ensure two functions are inverses.

Example 7: Show that $h(x)$ and $k(x)$ below are inverses where $x \geq 10$.

$h(x) = x^2 + 10$ $\qquad$ $k(x) = \sqrt{x - 10}$ Domain of $k(x)$ is $x \geq 10$

$$h(k(x)) = h(\sqrt{x-10}) = \left(\sqrt{x-10}\right)^2 + 10 = |x - 10| + 10 = \underbrace{(x - 10)}_{x \geq 10} + 10 = x$$

$$k(h(x)) = k(x^2 + 10) = \sqrt{(x^2 + 10) - 10} = \sqrt{x^2} = |x| = x \text{ since } x \geq 10$$

Example 8: Show that $n(x)$ and $p(x)$ below are inverses where $x \neq 0$ and $x \neq 4$.

$n(x) = \frac{6}{x - 4}$ Domain: $x \neq 4$ $\qquad$ $p(x) = \frac{6}{x} + 4$ Domain: $x \neq 0$

$$n(p(x)) = n\underbrace{\left(\frac{6}{x} + 4\right)}_{x \neq 0} = \frac{6}{\left(\frac{6}{x} + 4\right) - 4} = \frac{6}{\frac{6}{x}} = \frac{6x}{6} = x$$

$$p(n(x)) = p\underbrace{\left(\frac{6}{x - 4}\right)}_{x \neq 4} = \frac{6}{\left(\frac{6}{x - 4}\right)} + 4 = \frac{6(x - 4)}{6} + 4 = x - 4 + 4 = x$$