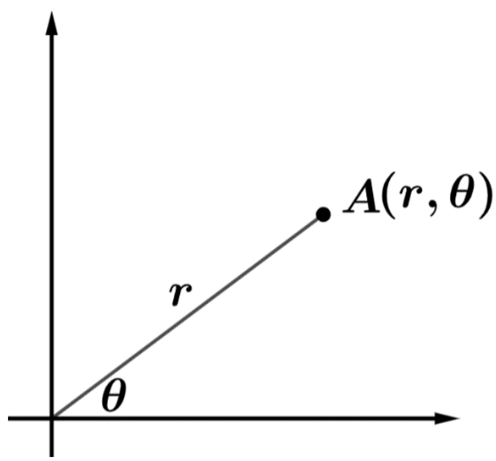
If we wanted to travel from one location to another location, there are several different methods we could use to travel: walking, biking, driving, flying, boating, etc… Each method has its own advantages and disadvantages, depending on the particular situation. Additionally, each method has specific constraints and restrictions.

If we were driving from one location to another, we would have to stay on the roads, which are generally formed by a rectangular grid system. If we were flying, we would be able to fly in a direct path to our destination, provided there was an airport.

Plotting points in space and defining functions is not much different than the transportation examples above. There are several ways to define points and functions in space, each with their advantages and disadvantages. Up until this point, the only system you have learned has been the **rectangular coordinate system**. This system uses an x-axis and y-axis, and points are described using an ordered pair (x, y). This rectangular coordinate system (also called the Cartesian plane) is very similar to a road grid, where we have to go over and then up to define a point in space.

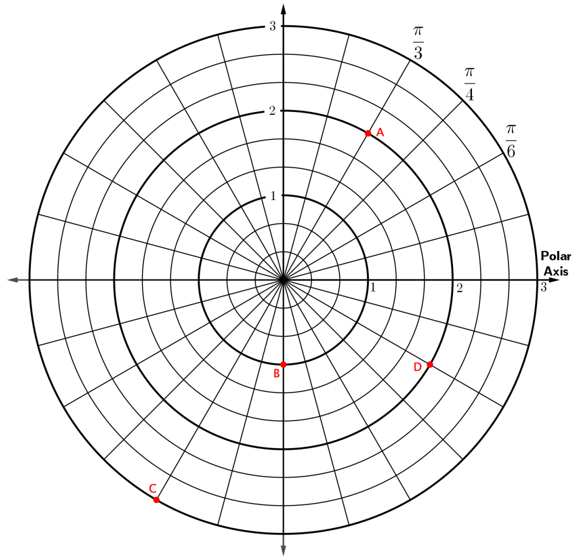
An alternate way to describe a point in space is the **polar coordinate system**. With polar coordinates, we do not use x and y values to define points. The polar coordinate system is much more like flying an airplane. To define a point, we only need to state the angle of the point in standard position and the signed distance of the point from the origin. In other words, we simply rotate until we are facing the point and then draw a line directly to the point from the origin. The polar coordinate system can be very useful, especially when working with circles, trig functions, and conics.

Polar Coordinates



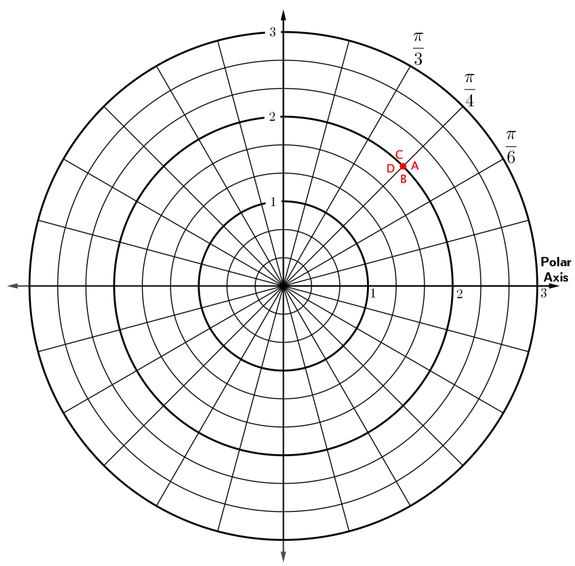
In polar coordinates, the point A is defined by (r, θ), where $|r|$ represents the radius of the circle on which point A lies, and θ represents the measure of an angle in standard position whose terminal ray includes point A.

One really cool result of using polar coordinates is that **the same point can be represented many different ways**!



Example 1: Plot (and label) the following polar coordinates on the polar coordinate grid above.

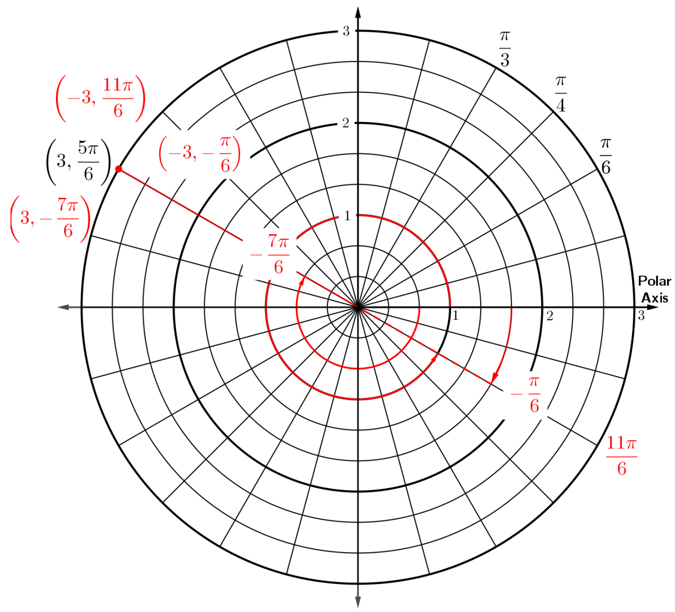
$A\left(2, \frac{\pi}{3}\right)$ $B\left(1, \frac{3\pi}{2}\right)$ $C\left(3, \frac{4\pi}{3}\right)$ $D\left(2, -\frac{\pi}{6}\right)$



Example 2: Plot (and label) the following polar coordinates on the polar coordinate grid above.

$A\left(2, \frac{\pi}{4}\right)$ $B\left(2, -\frac{7\pi}{4}\right)$ $C\left(2, \frac{9\pi}{4}\right)$ $D\left(-2, \frac{5\pi}{4}\right)$

Example 2 illustrates how a single point in polar coordinates can be expressed in many different ways. In fact, there are infinitely many ways that we could express any given point using polar coordinates!

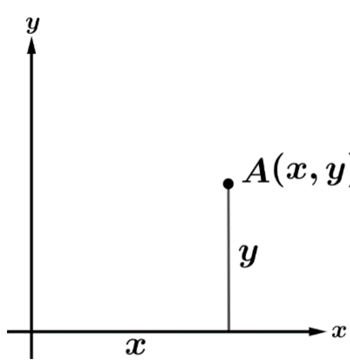
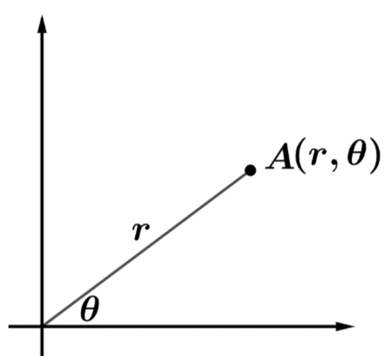
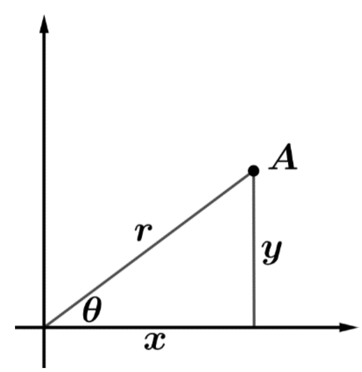


Example 3: Plot the point with polar coordinates $\left(3, \frac{5\pi}{6}\right)$ on the polar grid above. Then, write the polar coordinates for the point **two** different ways.

Rotate a full circle in the opposite direction then r stays the same. Rotate a half circle in either direction then r is the opposite.

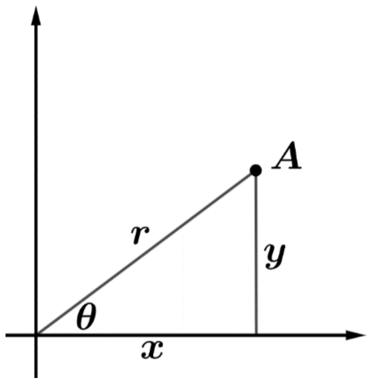
$$\text{Any two of these } \underbrace{\left(3, -\frac{7\pi}{6}\right)}_{\frac{5\pi}{6}-2\pi}, \underbrace{\left(-3, -\frac{\pi}{6}\right)}_{\frac{5\pi}{6}-\pi}, \text{ or } \underbrace{\left(-3, \frac{11\pi}{6}\right)}_{\frac{5\pi}{6}+\pi}$$

Connecting Rectangular and Polar Coordinate Systems

Rectangular Coordinates	Polar Coordinates	Connecting Polar and Rectangular
		

The image on the far right above allows us to see connections between polar and rectangular coordinates. Using this, we can quickly find relationships that allow us to convert between polar coordinates and rectangular coordinates.

Converting from Polar to Rectangular Coordinates



Using trigonometry, we know that $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$.

This leads to the following two identities:

$x = r\cos\theta$ and $y = r\sin\theta$

Tip: Whenever you are converting coordinates from one system to another, always sketch out the point first and check that your answer makes sense!

Example 4: Convert the following points from polar coordinates to rectangular coordinates.

a) $(2, \pi)$

$x = 2\cos(\pi) = 2(-1) = -2$

$y = 2\sin(\pi) = 2(0) = 0$

b) $\left(4, \frac{5\pi}{3}\right)$

$x = 4\cos\left(\frac{5\pi}{3}\right) = 4\left(\frac{1}{2}\right) = 2$

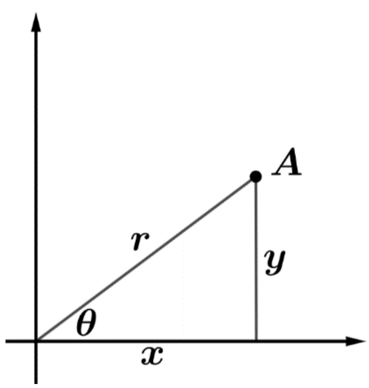
$y = 4\sin\left(\frac{5\pi}{3}\right) = 4\left(-\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}$

c) $\left(-3, \frac{7\pi}{6}\right)$

$x = -3\cos\left(\frac{7\pi}{6}\right) = -3\left(-\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{2}$

$y = -3\sin\left(\frac{7\pi}{6}\right) = -3\left(-\frac{1}{2}\right) = \frac{3}{2}$

Converting from Rectangular to Polar Coordinates



Using the Pythagorean Theorem, we have $x^2 + y^2 = r^2$.

Using trigonometry, we know that $\tan\theta = \frac{y}{x}$, which can be used to find the value of θ.

Tip: Whenever you are converting coordinates from one system to another, always sketch out the point first and check that your answer makes sense!

Example 5: Convert the following points from rectangular coordinates to polar coordinates.

a) $(-2, 2)$

$(-2)^2 + (2)^2 = 8 \quad r = \sqrt{8}$

$\tan\theta = \frac{2}{-2} = -1 \quad \theta = \frac{3\pi}{4}$

$\left(\sqrt{8}, \frac{3\pi}{4}\right)$

b) $(1, -\sqrt{3})$

$(1)^2 + \left(-\sqrt{3}\right)^2 = 4 \quad r = 2$

$\tan\theta = \frac{-\sqrt{3}}{1} = -\sqrt{3} \quad \theta = \frac{2\pi}{3}$

$\left(2, \frac{5\pi}{3}\right)$

c) $(0, -5)$

$(0)^2 + (-5)^2 = 25 \quad r = 5$

$\tan\theta = \frac{-5}{0} \to \text{DNE} \quad \theta = \frac{3\pi}{2}$

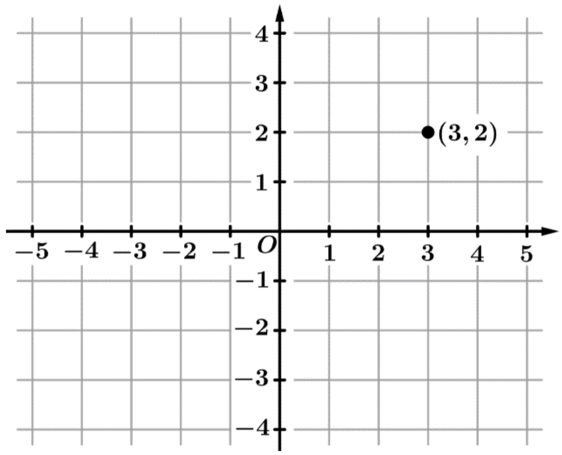
$\left(5, \frac{3\pi}{2}\right)$

Complex Numbers

Previously, we have written complex numbers using the form $a+bi$, where a and b are constants.

A complex number can be understood as a point in the complex plane and can be determined by its corresponding rectangular or polar coordinates.

When the complex number has the rectangular coordinates (a, b), it can be expressed as $a+bi$.



The complex number $3+2i$ can be represented as the point $(3, 2)$ in the rectangular coordinate system.

Earlier in these notes, we learned that $x = r\cos\theta$ and $y = r\sin\theta$. Using these identities, we can also express a complex number in the polar coordinate system.

When the complex number has polar coordinates (r, θ), it can be expressed as $(r\cos\theta)+i(r\sin\theta)$.

Example 6: A complex number is represented by a point in the complex plane. The complex number has the rectangular coordinates $(-2, -2)$. What is one way to express the complex number using its polar coordinates (r, θ)?

$(-2)^2 + (-2)^2 = 8 \qquad r = \sqrt{8} \qquad \tan\dfrac{-2}{-2} = 1 \qquad \theta = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$

$$\sqrt{8}\cos\frac{5\pi}{4} + i\left(\sqrt{8}\sin\frac{5\pi}{4}\right) = \sqrt{8}\left(-\frac{\sqrt{2}}{2}\right) + i\left(\sqrt{8}\left(-\frac{\sqrt{2}}{2}\right)\right) = 2\sqrt{2}\left(-\frac{\sqrt{2}}{2}\right) + i\left(2\sqrt{2}\left(-\frac{\sqrt{2}}{2}\right)\right) = -2 - 2i$$

Example 7: A complex number is represented by a point in the complex plane. Using polar coordinates, the complex number can be expressed as $\left(4\cos\dfrac{7\pi}{6}\right)+i\left(4\sin\dfrac{7\pi}{6}\right)$. Express the complex number using its rectangular coordinates (x, y).

$x = 4\cos\dfrac{7\pi}{6} = 4\left(-\dfrac{\sqrt{3}}{2}\right) = -2\sqrt{3} \qquad y = 4\sin\dfrac{7\pi}{6} = 4\left(-\dfrac{1}{2}\right) = -2 \qquad (-2\sqrt{3}, -2)$