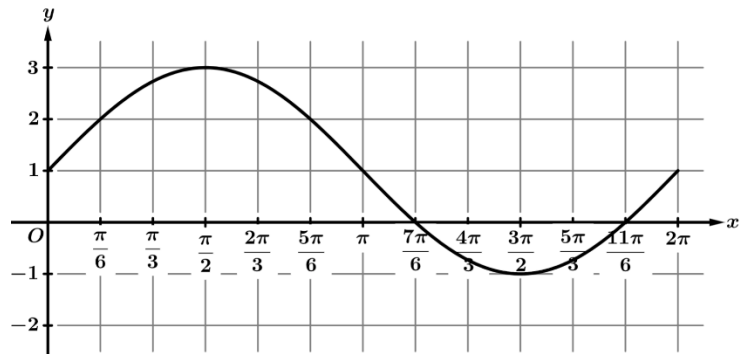In this section, we will learn ways that we can describe characteristics of the graph of a polar function.

Given a polar function $r = f(\theta)$, we know that r represents the "signed radius" of the function. We use the phrase "signed radius" because r can be a positive or negative value.

As we trace the graph of a polar function, we are interested in whether the graph of $r = f(\theta)$ is getting closer to the origin or further from the origin over a given interval.

Changes in the Distance from $r = f(\theta)$ to the Origin		
$r = f(\theta)$ is **positive** and **increasing** $r = f(\theta)$ is **negative** and **decreasing**	The distance between $r = f(\theta)$ and the origin is **increasing**	$f(\theta) = 1 + 2\sin\theta$
$r = f(\theta)$ is **positive** and **decreasing** $r = f(\theta)$ is **negative** and **increasing**	The distance between $r = f(\theta)$ and the origin is **decreasing**	

Example 1: The graph of $f(x) = 1 + 2\sin x$ is shown below for $0 \le x \le 2\pi$. Use the graph below to complete the given table with the appropriate intervals.

Description of $f(x)$	Interval(s)
f is **positive** and **increasing**	
f is **positive** and **decreasing**	
f is **negative** and **increasing**	
f is **negative** and **decreasing**	

In the polar coordinate system, the graph of $f(x) = 1 + 2\sin x$ above becomes $f(\theta) = 1 + 2\sin\theta$, as shown in the table above. The labeled points A, B, C, and D correspond to the intervals found in **Example 1**.
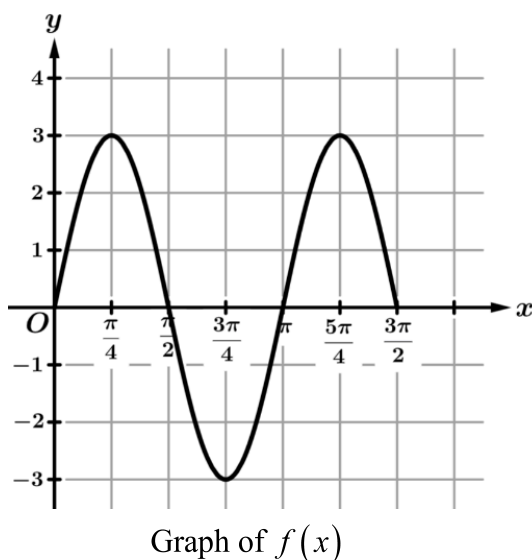
Points on $f(\theta)$	From A to B	From B to C	From C to D	From D to C	From C to A
Interval	$0 < \theta < \dfrac{\pi}{2}$	$\dfrac{\pi}{2} < \theta < \dfrac{7\pi}{6}$	$\dfrac{7\pi}{6} < \theta < \dfrac{3\pi}{2}$	$\dfrac{3\pi}{2} < \theta < \dfrac{11\pi}{6}$	$\dfrac{11\pi}{6} < \theta < 2\pi$
$r = f(\theta)$ is	**positive** and **increasing**	**positive** and **decreasing**	**negative** and **decreasing**	**negative** and **increasing**	**positive** and **increasing**

Distance between $f(\theta)$ and the origin is	increasing	decreasing	increasing	decreasing	increasing

AP Exam Tip: It is often helpful to sketch the graph of a given function in rectangular coordinates when attempting to describe the behavior of a polar function.

Example 2: Consider the graph of the polar function $r = f(\theta)$, where $f(\theta) = 2 - 4\cos\theta$, in the polar coordinate system for $0 \le \theta \le 2\pi$. Which of the following statements is true about the distance between the point with polar coordinates $(f(\theta), \theta)$ and the origin?

(A) The distance is increasing for $\pi < \theta < \dfrac{5\pi}{3}$, because $f(\theta)$ is positive and increasing on the interval.

(B) The distance is increasing for $\dfrac{5\pi}{3} < \theta < 2\pi$, because $f(\theta)$ is negative and increasing on the interval.

(C) The distance is decreasing for $\pi < \theta < \dfrac{5\pi}{3}$, because $f(\theta)$ is positive and decreasing on the interval.

(D) The distance is decreasing for $\dfrac{5\pi}{3} < \theta < 2\pi$, because $f(\theta)$ is negative and decreasing on the interval.

Graph of $f(x)$

Example 3: The graph of $f(x) = 3\sin(2x)$, where $0 \le x \le \dfrac{3\pi}{2}$ is shown above in the rectangular coordinate system.

If the polar function $r = f(\theta)$, where $f(\theta) = 3\sin(2\theta)$, is graphed in the polar coordinate system for $0 \le \theta \le \dfrac{3\pi}{2}$,

on which of the following intervals is the distance between the point with polar coordinates $(f(\theta), \theta)$ and the origin decreasing?

(A) $0 < \theta < \dfrac{\pi}{4}$ (B) $\dfrac{\pi}{2} < \theta < \dfrac{3\pi}{4}$ (C) $\dfrac{3\pi}{4} < \theta < \pi$ (D) $\pi < \theta < \dfrac{5\pi}{4}$

Relative Extrema and Polar Functions

Another characteristic that arises when we study polar functions are relative extrema (minima and maxima). For polar functions, if $r = f(\theta)$ changes from increasing to decreasing (or from decreasing to increasing), then the function has a relative extremum on the interval corresponding to a point relatively closest to or farthest from the origin.

Example 4: Consider the graph of the polar function $r = f(\theta)$, where $f(\theta) = 1 - 2\sin(2\theta)$, in the polar coordinate system for $0 \le \theta \le \pi$. Which of the following statements is true about the graph of $r = f(\theta)$?

(A) The graph of $r = f(\theta)$ has a relative minimum on the interval $\dfrac{\pi}{3} < \theta < \dfrac{2\pi}{3}$, because $r = f(\theta)$ changes from negative to positive.

(B) The graph of $r = f(\theta)$ has a relative minimum on the interval $\dfrac{2\pi}{3} < \theta < \pi$, because $r = f(\theta)$ changes from decreasing to increasing.

(C) The graph of $r = f(\theta)$ has a relative maximum on the interval $\dfrac{\pi}{3} < \theta < \dfrac{2\pi}{3}$, because $r = f(\theta)$ changes from positive to negative.

(D) The graph of $r = f(\theta)$ has a relative maximum on the interval $\dfrac{2\pi}{3} < \theta < \pi$, because $r = f(\theta)$ changes from increasing to decreasing.
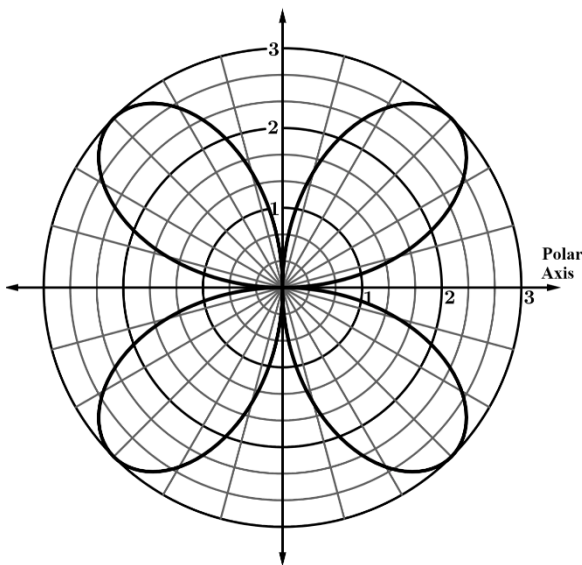
Average Rate of Change

In previous units, we learned about the average rate of change of a function in rectangular coordinates. In the polar coordinate system, we will find the average rate of change of r with respect to θ over a given interval of θ.

Average Rate of Change of a Polar Function

For the polar function $r = f(\theta)$, the average rate of change of $r = f(\theta)$ over the interval $a \le \theta \le b$ is given by the expression $\dfrac{f(b) - f(a)}{b - a}$.	Geometrically, the average rate of change indicates the rate at which the radius is changing per radian.

Example 5: Consider the graph of the polar function $r = f(\theta)$, where $f(\theta) = 3 - 3\cos\theta$, in the polar coordinate system. What is the average rate of change of $r = f(\theta)$ over the interval $\dfrac{\pi}{2} \le \theta \le \pi$?
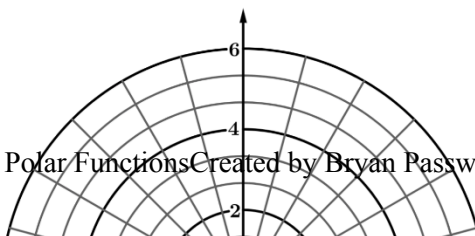

Example 6: The figure shows the graph of the polar function $r = f(\theta)$, where $f(\theta) = 3\sin(2\theta)$ for $0 \le \theta \le 2\pi$, in the polar coordinate system. On which of the following intervals is the average rate of change of $f(\theta)$ equal to zero?

(A) $\dfrac{\pi}{4} \le \theta \le \dfrac{3\pi}{4}$ (B) $\dfrac{3\pi}{4} \le \theta \le \dfrac{5\pi}{4}$ (C) $\dfrac{3\pi}{4} \le \theta \le \dfrac{7\pi}{4}$ (D) $\dfrac{\pi}{4} \le \theta \le \dfrac{7\pi}{4}$

Notes: Topic 3.15Rates of Change in Polar FunctionsCreated by Bryan Passwater

Example 7: In the polar coordinate system, the graph of a polar function $r = f(\theta)$ is shown with a domain of all real values of θ for $0 \le \theta \le 2\pi$. On this interval of θ, the graph has no holes, passes through each point exactly one time, and as θ increases, the graph passes through the labeled points A, B, C, and D, in that order. On which of the following intervals is the average rate of change of r with respect to θ least?

(A) From A to B

(B) From B to C

(C) From C to D

(D) From D to A

Estimating Values of $r = f(\theta)$ Using the Average Rate of Change

For a given interval, we can use the average rate of change of $r = f(\theta)$ over the interval to estimate other values of $r = f(\theta)$ inside the given interval.

Recall: The point-slope form of a line is given by $y - y_1 = m(x - x_1)$, where (x_1, y_1) is a known point on the line and m is the slope of the line.

We can utilize this concept and create a linear function that will help us approximate a given polar function. We will use the average rate of change of r with respect to θ as our slope, and we can use a point at either end of the given interval as our given point. This leads us to the following

$$f(\theta) \approx f(\theta_1) + \frac{f(b) - f(a)}{b - a}(\theta - \theta_1)$$

where $(f(\theta_1), \theta_1)$ is a known point on the graph of $r = f(\theta)$, and the average rate of change of $r = f(\theta)$ with respect to θ over the interval $a \le \theta \le b$ is given by $\dfrac{f(b) - f(a)}{b - a}$.

Notes: Topic 3.15 Rates of Change in Polar Functions Created by Bryan Passwater

θ	$\dfrac{\pi}{6}$	$\dfrac{7\pi}{6}$
$f(\theta)$	2	-1

Example 8: The table above gives values of the polar function $r = f(\theta)$ at selected values of θ. Use the average rate

of change of $r = f(\theta)$ over the interval $\dfrac{\pi}{6} \le \theta \le \dfrac{7\pi}{6}$ to approximate $f\left(\dfrac{5\pi}{6}\right)$.