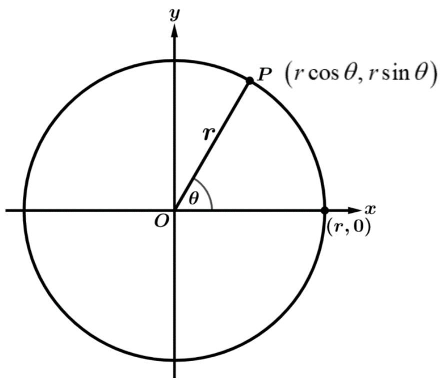
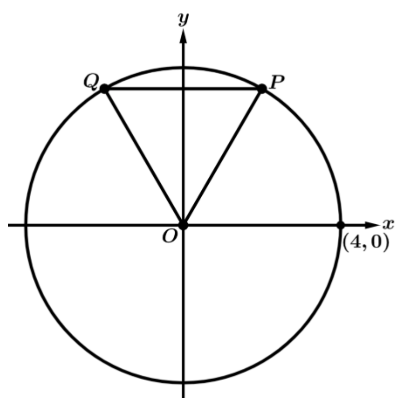
Coordinates of a Point on a Circle Centered at the Origin	
Given an angle θ in standard position on a circle with radius r centered at the origin, there is a point P, where the terminal ray intersects the circle.	
	From Topic 3.2, we know that $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$. For each of these two equations, we can multiply both sides by r and the results are: $x = r\cos\theta$ and $y = r\sin\theta$. Thus, the coordinates of point P are $(r\cos\theta,\ r\sin\theta)$.



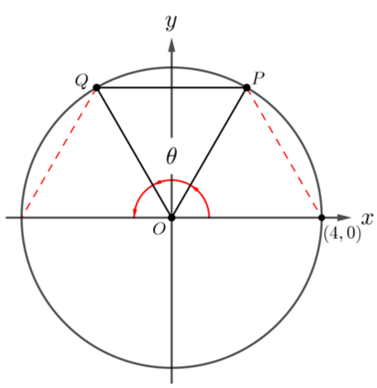
Example 1: The figure above shows a circle of radius 4 along with the equilateral triangle PQO. Which of the following gives the coordinates of point Q?

(A) $\left(4\cos\frac{\pi}{3}, 4\sin\frac{\pi}{3}\right)$

(B) $\left(4\cos\frac{5\pi}{6}, 4\sin\frac{5\pi}{6}\right)$

(C) $\left(4\cos\frac{2\pi}{3}, 4\sin\frac{2\pi}{3}\right)$

(D) $\left(-4\cos\frac{2\pi}{3}, 4\sin\frac{2\pi}{3}\right)$

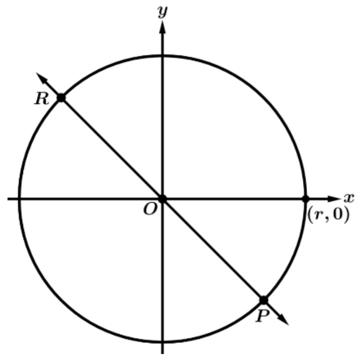


As the figure illustrates, it takes three inscribed equilateral triangles and three rotations of θ to rotate through half a full rotation of 2π.

$\theta = \frac{\pi}{3}$.

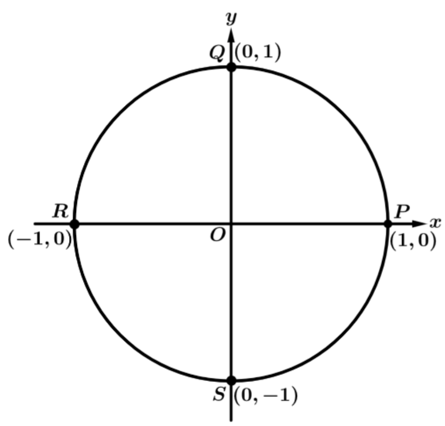
It will take 2 rotations of θ to get to Q.

$Q\ \left(4\cos\frac{2\pi}{3}, 4\sin\frac{2\pi}{3}\right)$



Example 2: The figure above shows a circle of radius r along with the line $y = -x$. The circle and the line $y = -x$ intersect at the points R, O, and P. If the coordinates of point P are $\left(6\sqrt{2}, -6\sqrt{2}\right)$, what is the value of r?

$x^2 + y^2 = r^2$ $\quad (6\sqrt{2})^2 + (-6\sqrt{2})^2 = r^2$ $\quad 72 + 72 = 144 = r^2$ $\quad r = 12$



Example 3: Consider the unit circle above with points P, Q, R, and S labeled above. Find the values of the following.

a) $\cos 0 = 1$ b) $\sin 0 = 0$

c) $\cos \frac{\pi}{2} = 0$ d) $\sin \frac{\pi}{2} = 1$

e) $\cos \pi = -1$ f) $\sin \pi = 0$

g) $\cos \frac{3\pi}{2} = 0$ h) $\sin \frac{3\pi}{2} = -1$

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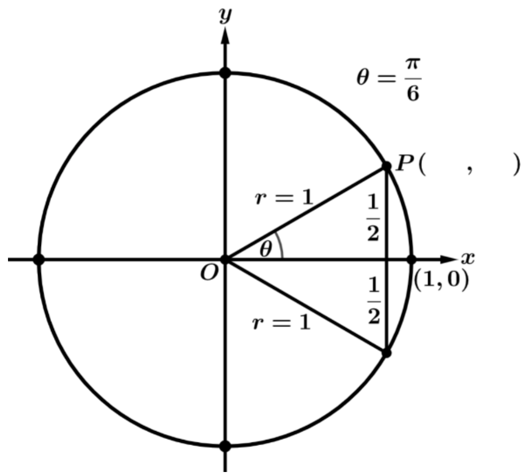
Finding Sine and Cosine Function Values

A large portion of this course requires that we can find numerical values for functions involving sine and cosine at important angle values. The angles that we have been working with on the unit circle involve multiples of $\frac{\pi}{4}$ and $\frac{\pi}{6}$. Now, we will put together several ideas from Topic 3.2 and Topic 3.3 to help us with this task!



Consider the unit circle $(r=1)$ above, with an equilateral triangle inscribed inside the circle. The angle θ, shown above, will have measure $\frac{\pi}{6}$ radians. We can see that the y-coordinate of the point P is $\frac{1}{2}$. We can use the Pythagorean theorem for find the x-coordinate.

$x^2 + y^2 = r^2 \rightarrow x^2 + \left(\frac{1}{2}\right)^2 = 1^2 \rightarrow x^2 + \frac{1}{4} = 1.$ This, leads to $x^2 = \frac{3}{4}$, and $x = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}$.

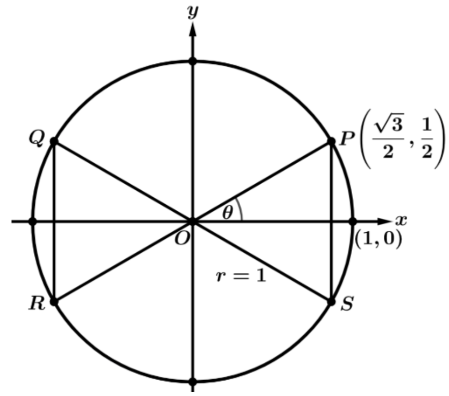
Since point P is in quadrant I, we know the x-coordinate of point P is $\frac{\sqrt{3}}{2}$. And the coordinates of P are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

Additionally, we know the coordinates of point P can be expressed as $(\cos\theta,\ \sin\theta)$, which leads us to these two important values: $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and $\sin\frac{\pi}{6} = \frac{1}{2}$.

Two Important Notes:

1. Notice, we found two values of x when solving with the Pythagorean theorem. This is because there is another point in quadrant II of the unit circle with the y-coordinate of $\frac{1}{2}$. Using symmetry, this point would be at $\frac{5\pi}{6}$ radians.

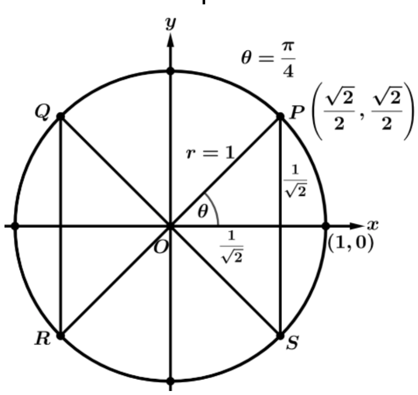
2. The Pythagorean theorem plays an important and interesting role with trigonometric functions. We will see additional ways that the Pythagorean theorem becomes useful later in Unit 3!



Example 4: The figure above shows a unit circle with the equilateral triangles QRO and PSO inscribed. Use the symmetry of the unit circle to find the coordinates and angle measures for the points Q, R, and S.

Point	P	Q	R	S
Angle Measure	$\frac{\pi}{6}$	$\pi - \frac{\pi}{6} = \frac{5\pi}{6}$	$\pi + \frac{\pi}{6} = \frac{7\pi}{6}$	$2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$
Coordinates	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$	$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$	$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$	$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

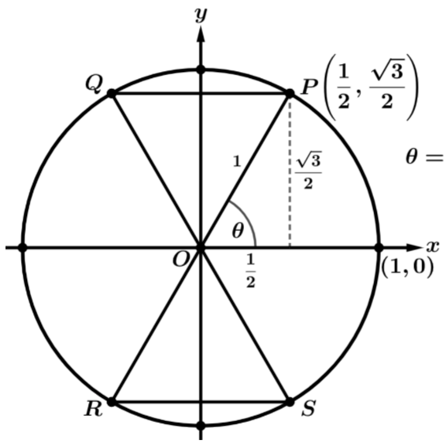
Just as we did earlier in these notes for an angle of $\frac{\pi}{6}$, we can use the unit circle, along with the Pythagorean theorem with angles that are multiples of $\frac{\pi}{4}$ by drawing four isosceles right triangles inscribed in a unit circle.



Note: Since we are using an isosceles right triangle, $x = y$.
Using Pythagorean theorem:
$x^2 + y^2 = 1 \rightarrow x^2 + x^2 = 1 \rightarrow 2x^2 = 1$
And, $x^2 = \frac{1}{2} \rightarrow x = \pm\frac{1}{\sqrt{2}}$

Example 5: The figure above shows a unit circle with four isosceles triangles inscribed. Use the symmetry of the unit circle to find the coordinates and angle measures for the points Q, R, and S.

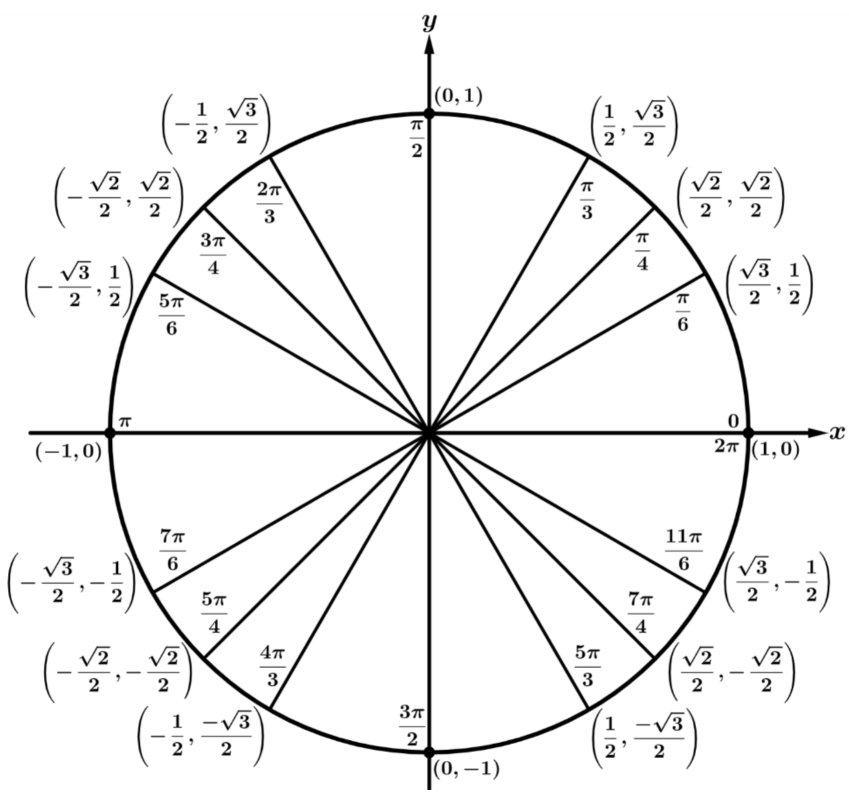
Point	P	Q	R	S
Angle Measure	$\frac{\pi}{4}$	$\pi - \frac{\pi}{4} = \frac{3\pi}{4}$	$\pi + \frac{\pi}{4} = \frac{5\pi}{4}$	$2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$
Coordinates	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$	$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$



Example 6: The figure above shows a unit circle with the equilateral triangles PQO and SRO inscribed. Use the symmetry of the unit circle to find the coordinates and angle measures for the points Q, R, and S.

Point	P	Q	R	S
Angle Measure	$\frac{\pi}{3}$	$\pi-\frac{\pi}{3}=\frac{2\pi}{3}$	$\pi+\frac{\pi}{3}=\frac{4\pi}{3}$	$2\pi-\frac{\pi}{3}=\frac{5\pi}{3}$
Coordinates	$\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$	$\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$	$\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$	$\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$

The unit circle below displays all of the angles, along with their coordinates that we will use repeatedly throughout Unit 3. It may look daunting, but as long as we know the coordinates of our important angles in quadrant I, we can use symmetry to find the coordinates of points in the other three quadrants!



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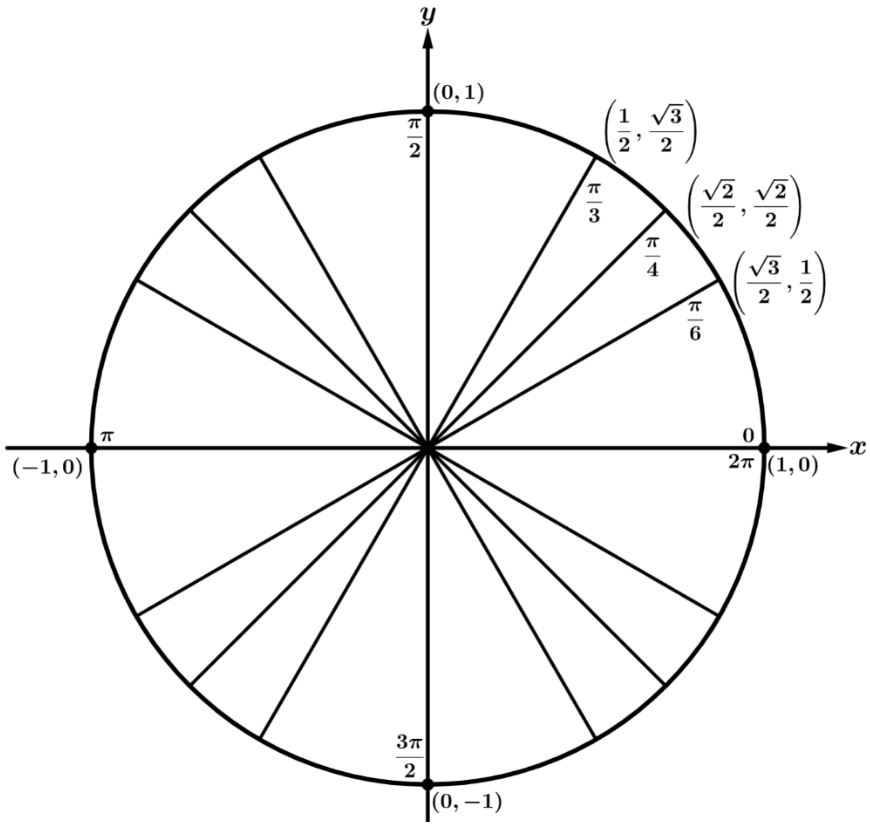

The unit circle above displays the coordinates of the points in quadrant I. These are the points that must be memorized so that we can use them for the points in the remaining quadrants.

Example 7: Find the values of the following expressions.

a) $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

b) $\cos\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$

c) $\cos\frac{11\pi}{6} = \frac{\sqrt{3}}{2}$

d) $\sin\frac{7\pi}{4} = -\frac{\sqrt{2}}{2}$

e) $\cos\frac{4\pi}{3} = -\frac{1}{2}$

f) $\sin\frac{7\pi}{6} = -\frac{1}{2}$

g) $\cos\pi = -1$

h) $\sin\frac{3\pi}{2} = -1$

i) $\cos\frac{2\pi}{3} = -\frac{1}{2}$

j) $\cos\frac{5\pi}{3} = \frac{1}{2}$

k) $\sin 0 = 0$

l) $\cos\frac{\pi}{2} = 0$