Notes: (Topic 1.11) Equivalent Representations of Polynomial and Rational Functions

Recall: Let $h(x)=\dfrac{f(x)}{g(x)}$. If $g(c)=0$ then $h(x)$ has a vertical asymptote or a hole at $x=c$.

Example 1: Let $h(x)=\dfrac{x^2-4}{x^2+7x+10}$. Write an equation for $h(x)$ in factored form and find any values of x where $h(x)$ has a hole or a vertical asymptote.

Factored form: $h(x)=\dfrac{(x-2)(x+2)}{(x+2)(x+5)}=\dfrac{(x-2)}{(x+5)}$

Hole: $x=-2$ because there is a common zero in the numerator and denominator.

Vertical asymptote: $x=-5$ because there is a zero in the denominator that is not in the numerator.

Example 2: Find the domain of $h(x)$ from **Example 1**. $(-\infty,-5)\cup(-5,-2)\cup(-2,\infty)$

Example 3: Let $k(x)=\dfrac{x^2-x-12}{x^3+x^2-20x}$.

a) Write an equation for $k(x)$ in factored form: $\dfrac{(x-4)(x+3)}{x(x-4)(x+5)}=\dfrac{(x+3)}{x(x+5)}$

b) Find any zeros of the function $k(x)$: $x=-3$ because there is a zero in the numerator that is not in the denominator.

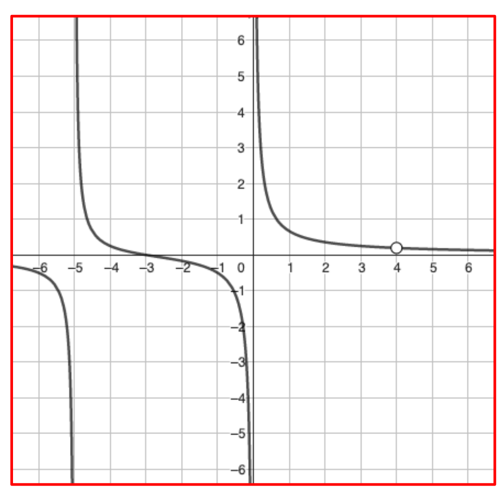
c) Find any values of x where $k(x)$ has a hole: $x=4$ because there is a common zero in the numerator and denominator.

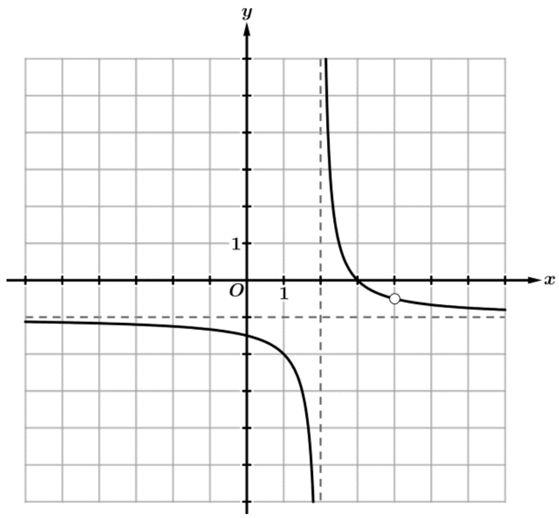
d) Find any vertical asymptotes of $k(x)$: $x=-5,0$ because there are zeros in the denominator that are not in the numerator.

e) Find any horizontal asymptotes of $k(x)$: $y=0$ because the degree in the denominator is greater than the degree in the numerator.

f) Find the domain of $k(x)$: $(-\infty,-5)\cup(-5,0)\cup(0,4)\cup(4,\infty)$

g) Use a graphing calculator to help sketch the graph of $k(x)$.





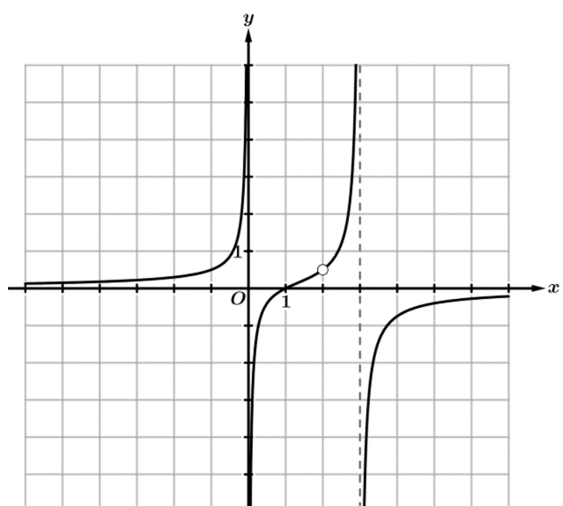
Example 4: The graph of the rational function $f(x)$ is shown above. Write an equation, in factored form, for $f(x)$.

Hole: $x = 4 \Rightarrow$ numerator common factor $\Rightarrow (x-4)$ Zero: $x = 3 \Rightarrow$ numerator factor $(x-3)$

Vertical asymptote: $x = 2 \Rightarrow$ denominator factor $\Rightarrow (x-2)$

Horizontal asymptote: $y = -1 \Rightarrow$ numerator and denominator are same degree and leading coefficient ratio -1.

$$f(x) = -\frac{(x-4)(x-3)}{(x-4)(x-2)}$$



Example 5: The graph of the rational function $g(x)$ is shown above. Write an equation, in factored form, for $g(x)$.

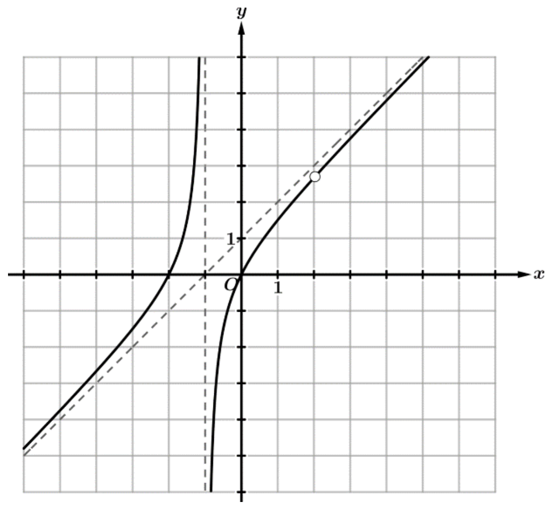
Hole: $x = 2 \Rightarrow$ numerator common factor $\Rightarrow (x-2)$ Zero: $x = 1 \Rightarrow$ numerator factor $(x-1)$

Vertical asymptote: $x = 0{,}3 \Rightarrow$ denominator factors $\Rightarrow x(x-3)$

Horizontal asymptote: $y = 0 \Rightarrow$ denominator is higher degree. $\lim_{x \to -\infty} g(x) = 0^+ \Rightarrow$ leading coefficient of the ratio is -1.

$$g(x) = -\frac{(x-2)(x-1)}{x(x-2)(x-3)}$$

Recall: Given a rational function, if the degree of the numerator is one greater than the degree of the denominator, then the rational function has a **slant asymptote**.



The function $k(x)=\dfrac{x^3-4x}{x^2-x-2}=\dfrac{x(x-2)(x+2)}{(x-2)(x+1)}$, graphed above, has a hole at $x=2$ and a vertical asymptote of $x=-1$. The graph of $k(x)$ does not have a horizontal asymptote, but $k(x)$ does have the slant asymptote $y=x+1$.

Since the degree of the numerator (3) is one greater than the degree of the denominator (2), we knew that $k(x)$ would have a slant asymptote. To find the equation of a slant asymptote, we will use **long division**.

Long Division

If $f(x)$ and $g(x)$ are polynomials, then $\dfrac{f(x)}{g(x)}=q(x)+r(x)$,
where q is the quotient, r is the remainder, and the degree of r is less than the degree of g.

Old Method	**New Method**
Numerical Long Division $676 \div 21$	Polynomial Long Division $(6x^2+7x+6)\div(2x+1)$
Work 32 R4 21)676 - 63 46 - 42 4	**Work** $3x \quad +2$ $2x+1 \,)\, 6x^2 \quad +7x \quad +6$ $-$ $6x^2 \quad +3x$ $4x \quad +6$ $-$ $4x \quad +2$ 4
Quotient: 32 Remainder: 4 Answer: $32+\frac{4}{21}=32\frac{4}{21}$	Quotient: $3x+2$ Remainder: 4 Answer: $(3x+2)+\frac{4}{2x+1}$

Example 6: Let $h(x)=\dfrac{6x^2+x+5}{2x+1}$. Use long division to find an equation for the slant asymptote of $h(x)$.

$$\begin{array}{r l}
 & \;\;3x-1 \\
2x+1 & \overline{)\,6x^2+x+5} \\
 & \underline{6x^2+3x} \\
 & \quad -2x+5 \\
 & \quad \underline{-2x-1} \\
 & \qquad\quad 6
\end{array}$$

$h(x)=(3x-1)+\frac{6}{2x+1} \Rightarrow$ slant asymptote $y=3x-1$

Example 7: Let $g(x)=\dfrac{x^3+4x^2-12x}{x^2+7x+6}$. Find the equations for any asymptotes to the graph of $g(x)$.

Factored Form: $g(x)=\dfrac{x(x-2)(x+6)}{(x+1)(x+6)}=\dfrac{x(x-2)}{(x+1)}$

Vertical Asymptotes: $x=-1$ because there is a zero in the denominator that is not in the numerator.

Horizontal Asymptotes: none because the degree of the numerator is greater than the degree of the denominator.

Slant Asymptotes: $y=x-3$

$$\begin{array}{r l}
 & \;\;x-3 \\
x+1 & \overline{)\,x^2-2x} \\
 & \underline{x^2+x} \\
 & \quad -3x-3 \\
 & \quad \underline{-3x} \\
 & \qquad\quad 3
\end{array}$$

Example 8: Let $d(x)=\dfrac{2x(x+4)(x-1)}{x^2-2x+1}$. Which of the following about the graph of $d(x)$ is correct?

a) The graph of $d(x)$ has one horizontal asymptote, one hole, and one vertical asymptote.

b) The graph of $d(x)$ has one horizontal asymptote and two vertical asymptotes.

c) The graph of $d(x)$ has one slant asymptote, one hole, and one vertical asymptote.

d) The graph of $d(x)$ has one slant asymptote no holes, and one vertical asymptote.

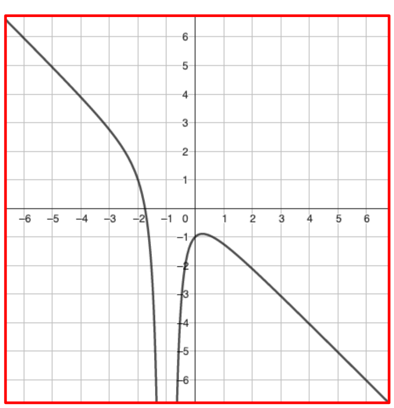
$d(x)=\dfrac{2x(x+4)(x-1)}{(x-1)^2}=\dfrac{2x(x+4)}{(x-1)}$

Slant asymptote: ratio of leading coefficients is $2x$
Vertical asymptote: $x=1$ zero of multiplicity 2 in denominator, but only multiplicity of 1 in numerator.

Example 9: Sketch a picture of the rational function $f(x)$ with the following properties:

1. $\lim\limits_{x\to\infty} f(x)=-\infty$ 2. $\lim\limits_{x\to-\infty} f(x)=\infty$ 3. $\lim\limits_{x\to-1^-} f(x)=-\infty$ 4. $\lim\limits_{x\to-1^+} f(x)=-\infty$

Sketches may vary.



Pascal's Triangle and the Binomial Theorem

Expanding expressions of the form $(a+b)^n$ can be tedious. Let's expand the expressions $(x+3)^2$ and $(x+3)^3$.

$(x+3)^2 = (x+3)(x+3) = x^2 + 6x + 9$

$(x+3)^3 = (x+3)(x+3)^2 = (x+3)(x^2+6x+9) = x^3 + 6x^2 + 9x + 3x^2 + 18x + 27 = x^3 + 9x^2 + 27x + 27$

What if we wanted to expand the expression $(x+3)^7$? As you can imagine, this would be very tedious and time-consuming using the method from above. Fortunately, there is a great mathematical property and tool to make this process much easier for us!

The Binomial Theorem

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n-1}a^1b^{n-1} + \binom{n}{n}b^n$$

The first formulations of the binomial theorem were by the mathematicians Al-Karaji and Jia Xian (separately) around the year 1000 AD.

At first glance, the binomial theorem may look intimidating and complicated. But, in reality, it is easy to use and understand if you can remember a few important details:

1. When expanding a binomial expression, the "a" begins with degree n and "b" term begins with a degree of 0.

2. As we expand the binomial expression, the degree of "a" decreases by one each term while the degree of "b" increases by one each term (until the "b" term has degree n).

3. We can use **Pascal's Triangle** to find the values of the coefficients $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots$ in our expansion.

Row 0 → 1
Row 1 → 1 1
Row 2 → 1 2 1
Row 3 → 1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Note: The value of $\binom{n}{r}$ corresponds with the rth element of the nth row in Pascal's triangle.

The first number in each row is the 0th element of that row!

Row 0 $\rightarrow$ 1

Row 1 $\rightarrow$ 1 1

Row 2 $\rightarrow$ 1 2 1

Row 3 $\rightarrow$ 1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

In the Pascal's triangle above, the 5th Row is circled. In the 5th row, the (first) number 10 has also been circled. The circled 10 is the 2nd element of row 5 and $\binom{5}{2} = 10$.

The notation $\binom{n}{r}$ represents a "**combination**", where $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$.

You will **NOT** need to know this formula for the AP Precalculus Exam!

Additional notation: $\binom{n}{r} = {}_nC_r = C(n, r)$

Example 10: Use Pascal's Triangle to expand $(x+2)^5$.

$(x+2)^5 = x^5 + 5x^4 2^1 + 10x^3 2^2 + 10x^2 2^3 + 5x^1 2^4 + 2^5 = x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$

Example 11: Use Pascal's Triangle to expand $(2x-1)^4$.

$$\begin{aligned}(2x-1)^4 &= (2x)^4 + 4(2x)^3(-1)^1 + 6(2x)^2(-1)^2 + 4(2x)^1(-1)^3 + (-1)^4 \\ &= 16x^4 - 32x^3 + 24x^2 - 8x + 1\end{aligned}$$

Example 12: What is the coefficient of the term containing x^4 when the expression $(x+5)^6$ is expanded?

$15x^4 5^2 = 375x^4 \Rightarrow$ coefficient is 375

Example 13: What is the coefficient of the term containing x^3 when the expression $(x-3)^8$ is expanded?

$\binom{8}{5} x^3(-3)^5 \Rightarrow \dfrac{8!}{5!\,3!}(-243) = \dfrac{8 \cdot 7 \cdot 6}{6}(-243) = 56(-243) = -13608$