In this section, we will explore several ways to write equivalent representations of trigonometric expressions. We already have several tools in our math toolbox that we can use to write trigonometric expressions in different forms.

Trigonometric Identities			
$\sin x = \dfrac{1}{\csc x}$	$\cos x = \dfrac{1}{\sec x}$	$\tan x = \dfrac{1}{\cot x}$	$\cot x = \dfrac{1}{\tan x}$
$\csc x = \dfrac{1}{\sin x}$	$\sec x = \dfrac{1}{\cos x}$	$\tan x = \dfrac{\sin x}{\cos x}$	$\cot x = \dfrac{\cos x}{\sin x}$

These identities allow us to manipulate trigonometric expressions and rewrite these expressions in equivalent forms using various trig functions.

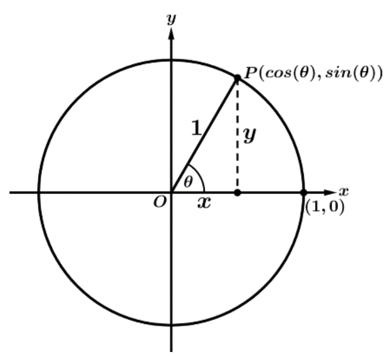
Example 1: Let $f(x) = \dfrac{(\sin x)(\sec^2 x)}{\csc x}$. Rewrite $f(x)$ as an expression involving $\tan x$ and no other trigonometric functions.

$$= \frac{(\sin x)\left(\frac{1}{\cos x}\right)^2}{\frac{1}{\sin x}} = \frac{(\sin x)^2}{(\cos x)^2} = (\tan x)^2 = \tan^2 x$$

Example 2: Let $g(x) = (\csc x)(\tan x)$. Rewrite $g(x)$ as an expression involving $\sec x$ and no other trigonometric functions.

$$= \left(\frac{1}{\sin x}\right)\left(\frac{\sin x}{\cos x}\right) = \frac{1}{\cos x} = \sec x$$

Additionally, we can discover several interesting properties and identities that arise from our understanding of the unit circle and trigonometric functions. First, let's start by relooking at the unit circle.



For the unit circle, any point (x, y) can also be expressed as $(\cos\theta, \sin\theta)$, where $x = \cos\theta$ and $y = \sin\theta$. We can apply the Pythagorean Theorem to this right triangle, resulting in the identity $\sin^2\theta + \cos^2\theta = 1$! This identity is pretty remarkable: for any angle, the square of the sine of the angle plus the square of the cosine of the angle always equals 1. We call this the **Pythagorean identity** for trigonometric functions.

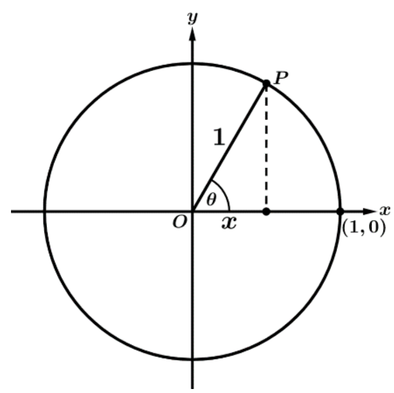
The Pythagorean identity can be manipulated into other forms as well that give us additional identities connecting trigonometric functions.

For example, we can show that $1+\tan^2\theta=\sec^2\theta$ by dividing each term in the identity $\sin^2\theta+\cos^2\theta=1$ by $\cos^2\theta$.

$$\sin^2\theta+\cos^2\theta=1\Rightarrow\frac{\sin^2\theta}{\cos^2\theta}+\frac{\cos^2\theta}{\cos^2\theta}=\frac{1}{\cos^2\theta}\Rightarrow\tan^2\theta+1=\sec^2\theta.$$

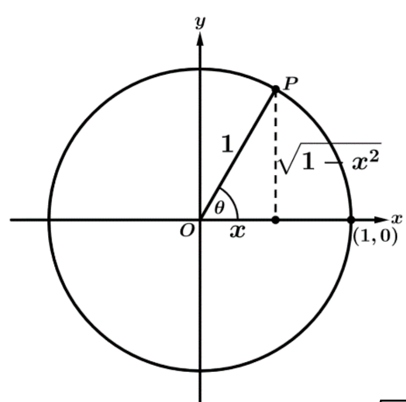
Additionally, we can show that $1+\cot^2\theta=\csc^2\theta$ by dividing each term in the Pythagorean identity by $\sin^2\theta$.

The Pythagorean Identity and Equivalent Forms		
$\sin^2\theta+\cos^2\theta=1$	$1+\tan^2\theta=\sec^2\theta$	$1+\cot^2\theta=\csc^2\theta$



Let's see if we can find another identity by using the unit circle above. When we inscribe a right triangle inside the unit circle, we can express the height of the triangle using the Pythagorean Theorem: $x^2+h^2=1\Rightarrow h=\sqrt{1-x^2}$.

This leads us to the triangle shown below.



With this triangle, we can see that $\cos\theta=\frac{x}{1}\Rightarrow\theta=\cos^{-1}x$ and $\sin\theta=\frac{\sqrt{1-x^2}}{1}\Rightarrow\theta=\sin^{-1}\sqrt{1-x^2}$, which leads us to the identity $\cos^{-1}x=\sin^{-1}\sqrt{1-x^2}$. Similarly, we can show that $\sin^{-1}x=\cos^{-1}\sqrt{1-x^2}$.

Inverse Trigonometric Identities	
$\cos^{-1}x=\sin^{-1}\sqrt{1-x^2}$ $\arccos x=\arcsin\sqrt{1-x^2}$	$\sin^{-1}x=\cos^{-1}\sqrt{1-x^2}$ $\arcsin x=\arccos\sqrt{1-x^2}$

Example 3: The function f is given by $f(x)=\dfrac{\tan x}{\csc x}$. Rewrite $f(x)$ as a fraction involving powers of $\cos x$ and no other trigonometric functions.

$$=\left(\frac{\left(\frac{\sin x}{\cos x}\right)}{\frac{1}{\sin x}}\right)=\frac{(\sin x)^2}{\cos x}=\frac{(1-\cos^2 x)}{\cos x}$$

Sum Identities for Sine and Cosine

Two additional identities that are important are the sum identities for sine and cosine. Given any two angles α and β, $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$ and $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$.

Using these two identities, we can also derive identities for $\sin(\alpha-\beta), \cos(\alpha-\beta), \sin(2\alpha), \cos(2\alpha)$.

Sum and Difference Identities	
$\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta$	$\cos(\alpha\pm\beta)=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta$
Double Angle Identities	
$\sin(2\theta)=2\sin\theta\cos\theta$	$\cos(2\theta)=\cos^2\theta-\sin^2\theta=1-2\sin^2\theta=2\cos^2\theta-1$

Example 4: Which of the following is equivalent to the expression $2\sin\dfrac{\pi}{14}\cos\dfrac{\pi}{14}$?

(A) $\sin\dfrac{\pi}{28}$ (B) $\sin\dfrac{\pi}{7}\cos\dfrac{\pi}{7}$ (C) $\sin\dfrac{\pi}{7}$ (D) $\cos\dfrac{\pi}{7}$

$\theta=\dfrac{\pi}{14}\Rightarrow 2\sin\theta\cos\theta=\sin\left(2\cdot\dfrac{\pi}{14}\right)=\sin\dfrac{\pi}{7}$

Example 5: The function k is given by $k(x)=4\cos(2x)$. Which of the following is an equivalent form for $k(x)$?

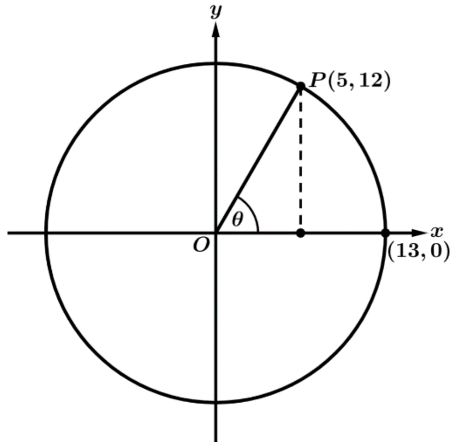
(A) $8\sin x\cos x$ (B) $4-8\cos^2 x$ (C) $8\cos^2 x-4$ (D) $4\cos^2 x+4\sin^2 x$

$4\cos(2x)=4(2\cos^2 x-1)=8\cos^2 x-4$

Example 6: Which of the following is equivalent to the expression $\cos\dfrac{\pi}{8}\cos\dfrac{\pi}{16}-\sin\dfrac{\pi}{8}\sin\dfrac{\pi}{16}$?

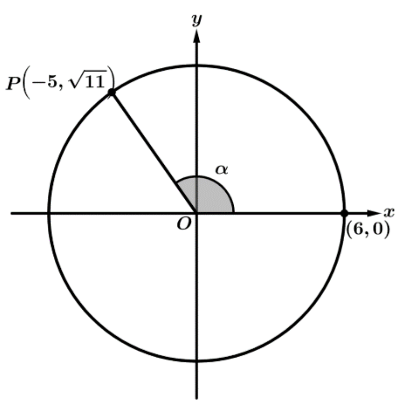
(A) $\cos\dfrac{\pi}{16}$ (B) $\sin\dfrac{\pi}{16}$ (C) $\cos\dfrac{3\pi}{16}$ (D) $\sin\dfrac{3\pi}{16}$

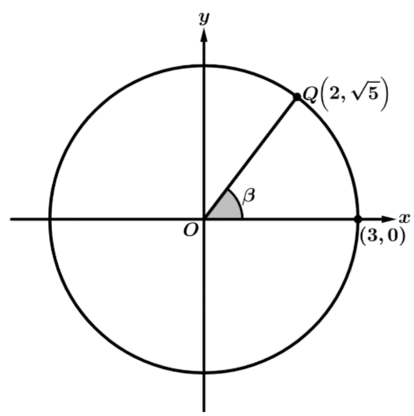
$\alpha=\dfrac{\pi}{8}$ $\beta=\dfrac{\pi}{16}$ $=\cos(\alpha+\beta)=\cos\left(\dfrac{\pi}{8}+\dfrac{\pi}{16}\right)=\cos\left(\dfrac{2\pi}{16}+\dfrac{\pi}{16}\right)=\cos\left(\dfrac{3\pi}{16}\right)$



Example 7: The figure shows a circle centered at the origin with an angle of measure θ in standard position. The terminal ray of the angle intersects the circle at point P. The coordinates of P are $(5, 12)$. Which of the value of $\sin 2\theta$?

$\sin\theta = \dfrac{12}{13}$ $\quad\cos\theta = \dfrac{5}{13}$ $\quad\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\dfrac{12}{13}\right)\left(\dfrac{5}{13}\right) = \dfrac{120}{169}$





Example 8: The figures show two circles centered at the origin with angle measures of α and β, respectively, in standard position. The terminal ray of angle α intersects the circle at point P, and the terminal ray of angle β intersects the circle at point Q. The coordinates of P are $\left(-5, \sqrt{11}\right)$ and the coordinates of Q are $\left(2, \sqrt{5}\right)$.

a) Find $\cos(2\alpha)$.

b) Find $\sin(\alpha+\beta)$.

c) Find $\cos(\alpha-\beta)$.

$\sin\alpha = \dfrac{\sqrt{11}}{6}$ $\quad\cos\alpha = \dfrac{-5}{6}$ $\quad\sin\beta = \dfrac{\sqrt{5}}{3}$ $\quad\cos\beta = \dfrac{2}{3}$

$$\cos(2\alpha) = 2\cos^2\alpha - 1$$
$$= 2\left(\frac{-5}{6}\right)^2 - 1 = 2\left(\frac{25}{36}\right) - 1$$
$$= \frac{50}{36} - \frac{36}{36} = \frac{14}{36} = \frac{7}{18}$$

$$\sin(\alpha+\beta)$$
$$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$= \left(\frac{\sqrt{11}}{6}\right)\left(\frac{2}{3}\right) + \left(\frac{-5}{6}\right)\left(\frac{\sqrt{5}}{3}\right)$$
$$= \frac{2\sqrt{11}}{18} - \frac{5\sqrt{5}}{18} = \frac{2\sqrt{11} - 5\sqrt{5}}{18}$$

$$\cos(\alpha-\beta)$$
$$= \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$= \left(\frac{-5}{6}\right)\left(\frac{2}{3}\right) + \left(\frac{\sqrt{11}}{6}\right)\left(\frac{\sqrt{5}}{3}\right)$$
$$= -\frac{10}{18} + \frac{\sqrt{55}}{18} = \frac{\sqrt{55} - 10}{18}$$