**Notes**: (Topic 2.1)  Change in Arithmetic and Geometric Sequences  Solutions

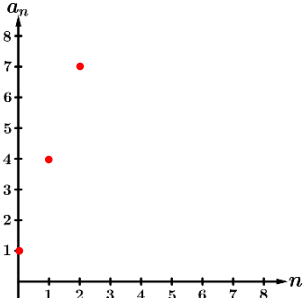A  _sequence_  is a function from the  _whole_   numbers to the _real_ numbers.

This means that we are only able to "plug" in whole numbers (0, 1, 2, 3, …) into a sequence but we can get any real number as the output.

As a result, when we graph a sequence, we will have points but we cannot "connect" them together to form a line or curve.

**Example 1**:  Consider the sequence defined by $a_n = 4n - 3$.  Find $a_1$ and $a_7$.

$$a_1 = 4(1) - 3 = 1 \qquad a_7 = 4(7) - 3 = 28 - 3 = 25$$

In this course, we will study two important types of sequences:  **arithmetic sequences** and **geometric sequences**.

| Arithmetic Sequences | | |
|---|---|---|
| **Property of Successive Terms** | **Formulas/Equations** | **Notes** |
| Successive terms have a **common difference**, or constant rate of change. | $a_n = a_0 + dn$ <br> **or** <br> $a_n = a_k + d(n - k)$ <br><br> where $a_0$ = initial value <br> $d$ = common difference <br> $a_k$ = $k$th term of the sequence | Arithmetic sequences behave like **linear functions**, except they are not continuous. <br><br> Increasing arithmetic sequences increase equally each step. (slope always stays the same!) |

**Example**

$a_n = 3n + 1$



| $n$ | $a_n$ |
|---|---|
| 0 | 1 |
| 1 | 4 |
| 2 | 7 |
| 3 | 10 |
| ⋮ | ⋮ |

**Example 2:**  For each of the following, determine if the sequence could be arithmetic.  If yes, identify the common difference.

a)  $S_n = n^2 - 3$

| $n$ | $S_n$ |
|---|---|
| 0 | -3 |
| 1 | -2 |
| 2 | 1 |
| 3 | 6 |
| ⋮ | ⋮ |

not arithmetic
no common difference

b)  $S_n = 6 - 2n$

| $n$ | $S_n$ |
|---|---|
| 0 | 6 |
| 1 | 4 |
| 2 | 2 |
| 3 | 0 |
| ⋮ | ⋮ |

arithmetic
$d = -2$

c)  $-7, -2, 3, 8, 13, ...$

arithmetic, $d = 5$

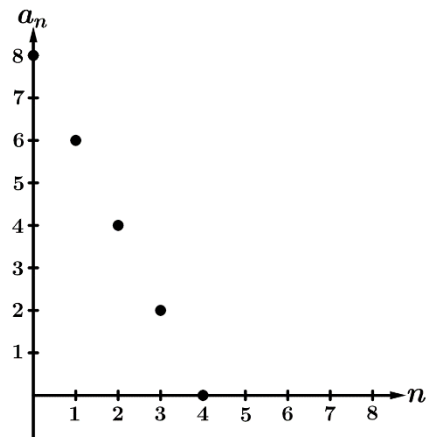d)  $1, -2, 3, -4, 5, ...$

not arithmetic, no common difference

**Example 3:** Let $a_n$ be an arithmetic sequence with $a_3 = 8$ and $d = -3$. Find an expression for $a_n$, and use the expression to find $a_{12}$.

$$a_n = a_3 + (-3)(n-3) = 8 - 3n + 9 = 17 - 3n \qquad a_{12} = 17 - 3(12) = 17 - 36 = -19$$

**Example 4:** Let $a_n$ be an arithmetic sequence with $a_2 = 7$ and $a_6 = 9$. Find an expression for $a_n$, and use the expression to find $a_{24}$.

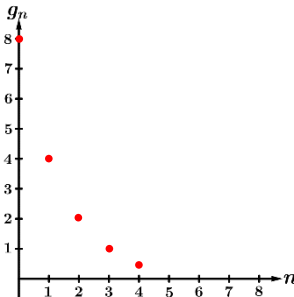$$a_6 = a_2 + d(6-2) \Rightarrow 9 = 7 + 4d \Rightarrow 4d = 2 \Rightarrow d = \frac{4}{2} = \frac{1}{2}$$

$$a_{24} = a_6 + \frac{1}{2}(24-6) = 9 + \frac{18}{2} = 18$$



**Example 5:** Several terms of the arithmetic sequence $a_n$ are shown above. Find an expression for $a_n$ and use the expression to find $a_{17}$.

$$a_1 - a_0 = 6 - 8 = -2 = d \qquad a_n = a_0 + (-2)n = 8 - 2n$$

$$a_{17} = 8 - 2(17) = 8 - 34 = -26$$

<table>
<tr><td colspan="3" align="center">**Geometric Sequences**</td></tr>
<tr><td align="center">**Property of Successive Terms**</td><td align="center">**Formulas/Equations**</td><td align="center">**Notes**</td></tr>
<tr><td>Successive terms have a **common ratio**, or constant proportional change.</td><td>$$g_n = g_0 r^n$$ **or** $$g_n = g_k r^{(n-k)}$$ where $g_0 =$ initial value $r =$ common ratio $g_k = k$th term of the sequence</td><td>Geometric sequences behave like **exponential functions**, except they are not continuous.

Increasing geometric sequences increase by a larger amount each step. (% increase always stays the same!)</td></tr>
</table>

**Example**

$$g_n = 8\left(\frac{1}{2}\right)^n$$



| $n$ | $g_n$ |
|---|---|
| 0 | 8 |
| 1 | 4 |
| 2 | 2 |
| 3 | 1 |
| 4 | $\frac{1}{2}$ |
| ⋮ | ⋮ |

**Example 6:** For each of the following, determine if the sequence could be geometric. If yes, identify the common ratio.

a) $S_n = 3n^2$

| $n$ | $S_n$ |
|---|---|
| 0 | 0 |
| 1 | 3 |
| 2 | 12 |
| 3 | 27 |
| ⋮ | ⋮ |

not geometric
no common ratio

b) $S_n = 4(2)^{n-1}$

| $n$ | $S_n$ |
|---|---|
| 0 | 2 |
| 1 | 4 |
| 2 | 8 |
| 3 | 16 |
| ⋮ | ⋮ |

geometric
$r = 2$

c) $1, 3, 2, 6, 4, 12, 8, 24, \ldots$

| $S_n$ | $r$ |
|---|---|
| 1 | |
| 3 | 3/1 |
| 2 | 2/3 |
| 6 | 6/2 |
| ⋮ | ⋮ |

not geometric
no common ratio

d) $16, -8, 4, -2, 1, \ldots$

| $S_n$ | $r$ |
|---|---|
| 16 | |
| −8 | $-\frac{1}{2}$ |
| 4 | $-\frac{1}{2}$ |
| −2 | $-\frac{1}{2}$ |
| ⋮ | ⋮ |

geometric $r = -\frac{1}{2}$

**Example 7:** Let $g_n$ be a geometric sequence with $g_1 = 12$ and $r = 2$. Find an expression for $g_n$, and use the expression to find $g_4$. $g_n = g_1 r^{(n-1)} = 12(2)^{(n-1)} \Rightarrow g_4 = 12(2)^{(4-1)} = 12(8) = 96$
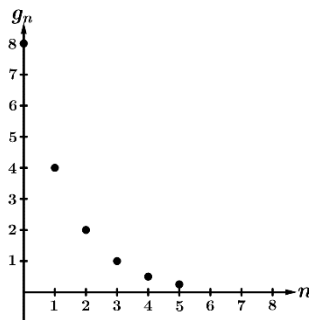


**Example 8:** Several terms of the geometric sequence $g_n$ are shown above. Find an expression for $g_n$ and use the expression to find $g_{10}$. $\frac{g_1}{g_0} = \frac{4}{8} = \frac{1}{2} \Rightarrow g_n = 8\left(\frac{1}{2}\right)^n$ $g_{10} = 8\left(\frac{1}{2}\right)^{10} = \frac{8}{1024} = \frac{1}{128}$