P8130: Biostatistical Methods I

Lecture 5: Continuous Probability Distributions

Cody Chiuzan, PhD
Department of Biostatistics
Mailman School of Public Health (MSPH)

Lecture 4: Recap

- Randomness and random variables
- Binomial distribution: definition and statistical properties
- Poisson distribution: definition and statistical properties

Lecture 5: Outline

- Continuous random variables and probability distributions
- Uniform distribution: definition and statistical properties
- Normal distribution: definition and statistical properties

Continuous Random Variables

Compared to discrete r. v., continuous random variables can assume <u>any value over an entire interval.</u>

Continuous distributions are typically represented by a probability density function (pdf) or density curve.

A density curve is a representation of the underlying population distribution (not a description of the sample data)

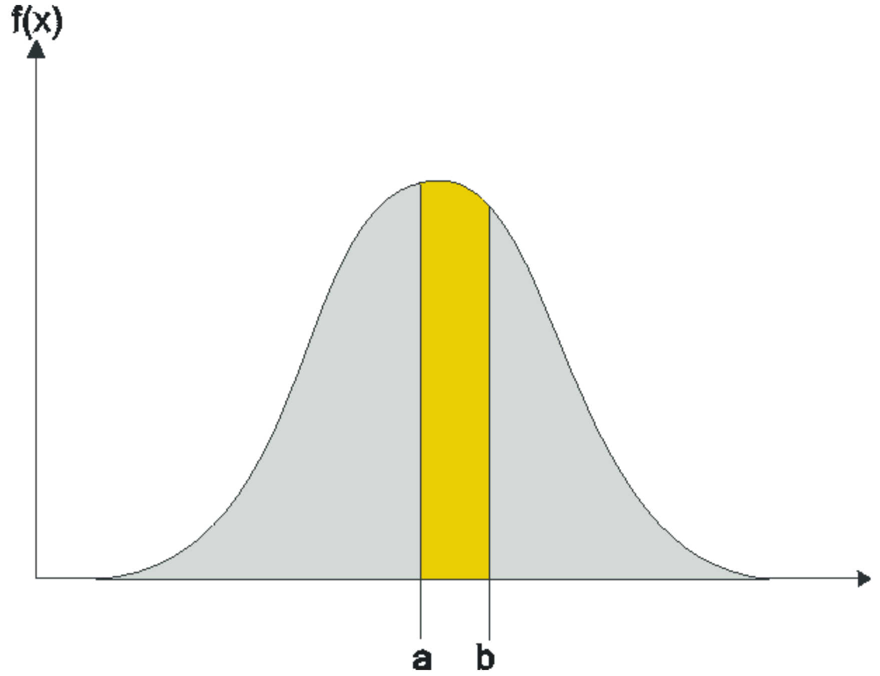
Properties of Density Functions (PDF)

The *pdf* of a continuous random variable X is a function defined for all real numbers *x* such that:

1. $f_X(x) = f(x) \geq 0$, for all values of x
2. The density function is always above or on the horizontal axis (curve cannot have a negative value)
3. The total region/area between the curve and horizontal axis is exactly 1.
4. For any real numbers *a* and *b*, $P(a \leq X \leq b)$ is given by the area bounded by the graph of f, the vertical lines $x = a \; and \; x = b$, and the x axis.

Properties of Density Functions (PDF)

For any $a < b$, $P(a \leq X \leq b) = \int_a^b f(x)dx$



Also, all the following probabilities are equal:

$$P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b)$$

Cumulative Distribution Function

The cumulative distribution of a continuous r. v. is denoted by:

$$F(x) = P(X \leq t) = \int_{-\infty}^{t} f(x)dx\,, -\infty < t < \infty$$

Properties:

1. $F(x)$ is a non-decreasing function, $t_1 \leq t_2$ implies that $F(t_1) \leq F(t_2)$
2. $F(x)$ is continuous
   - For discrete random variables, $F(x)$ is only right continuous
3. $F(x)$'= $f(x)$, if $F(x)$' exists
4. $P(X > a) = 1 - F(a)$

$$P(a < X < b) = F(b) - F(a)$$

Expected Value of a Continuous Random Variable

The <u>expected value</u> of a continuous random variable is defined as:

$$\mu = E(X) = \int_{-\infty}^{-\infty} x \cdot f(x)dx$$

The <u>variance</u> of a continuous random variable is given by:

$$\sigma^2 = var(X) = E(x^2) - \mu^2 = \int_{-\infty}^{-\infty} (x - \mu)^2 \cdot f(x)dx$$

PDF: Example

Let the continuous random variable X have the following PDF:

$$f(x) = 4x^3, 0 \leq x \leq 1$$

Let us compute the following:

1. $\int_0^1 f(x)dx =$
2. $P(0.2 < X < 0.5) =$
3. $E(X) = \int_0^1 xf(x)dx =$
4. $var(x) = \int_0^1 x^2 f(x)dx - [E(x)]^2 =$

Uniform Distribution

The probability density function of a uniform r. v. X, $X \sim Unif(a, b)$, is given by:

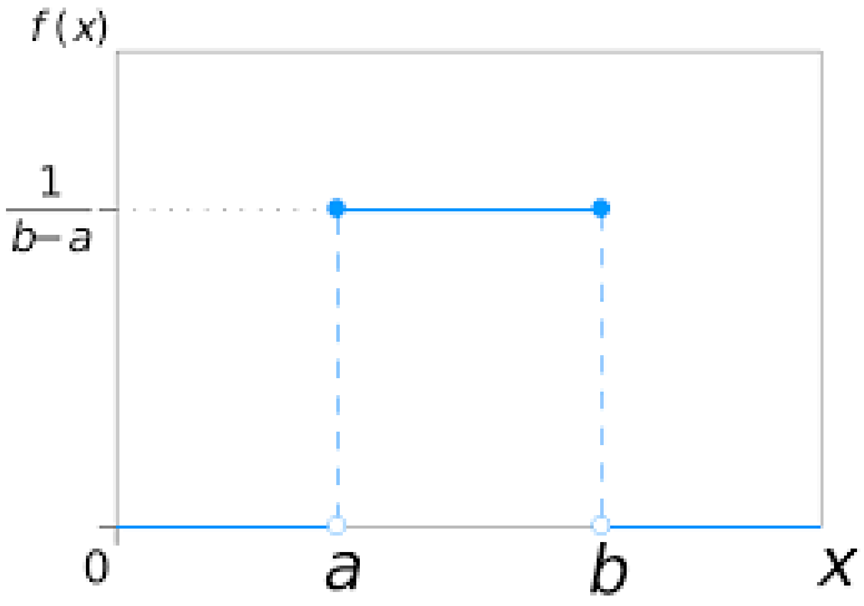
$$f(x) = \begin{cases} \frac{1}{b-a}, & \mathrm{a} < x < b \\ 0, & otherwise \end{cases}$$

The <u>expected value</u> of the uniform distribution is given by:

$$\mu = E(X) = \int_a^b xf(x)dx = \frac{b+a}{2}$$

The <u>variance</u> of the uniform distribution is given by:

$$\sigma^2 = var(X) = \int_a^b x^2 f(x)dx - \mu^2 = \frac{(b-a)^2}{12}$$



Uniform Distribution: Examples

An operator just announced a maximum 30min delay for your subway. What is the probability that the subway will arrive between 15 and 20 min?

$$X \sim Unif(0,30)$$

The probability density function is given by:

$$f(x) = \frac{1}{30-0} = \frac{1}{30}, 0 < x < 30$$

$$P(15 < x < 20) = \int_{15}^{20} f(x)dx = \; \ldots = \frac{5}{30} = \frac{1}{6}$$

In class derivation

Normal Distribution

Probably the most common distribution used in statistics, the normal distribution is also called the *Gaussian* or the *'bell-shaped'* distribution.

-> Some variables are normal

-> Some are approximately normal

-> Some can be transformed to be approximately normal

-> The sampling error of the means tends towards normality even for non-normal populations

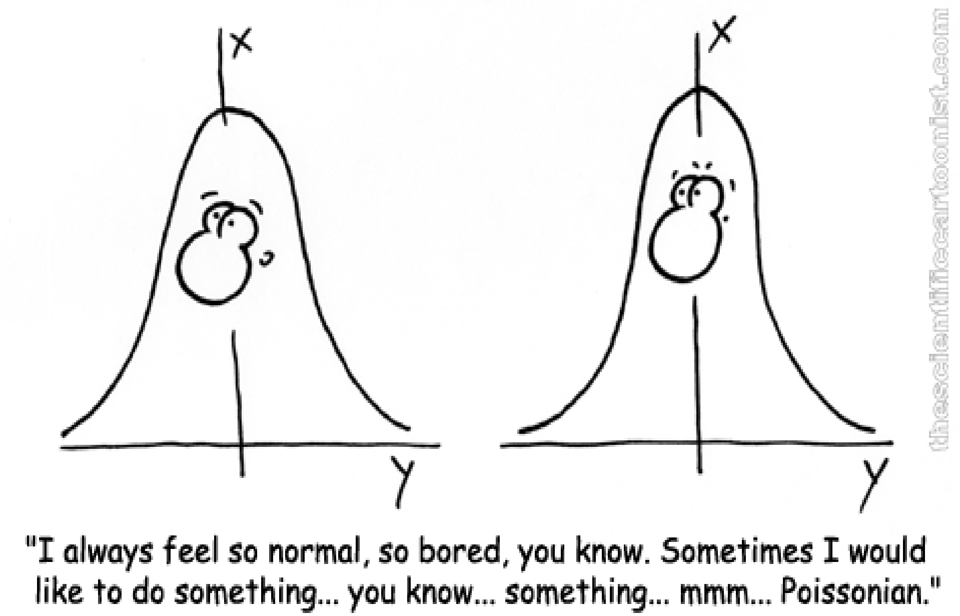


Image taken from the Scientific Cartoonist

Normal Distribution

The probability distribution function of a normal r. v. X, $X \sim N(\mu, \sigma^2)$, is given by:

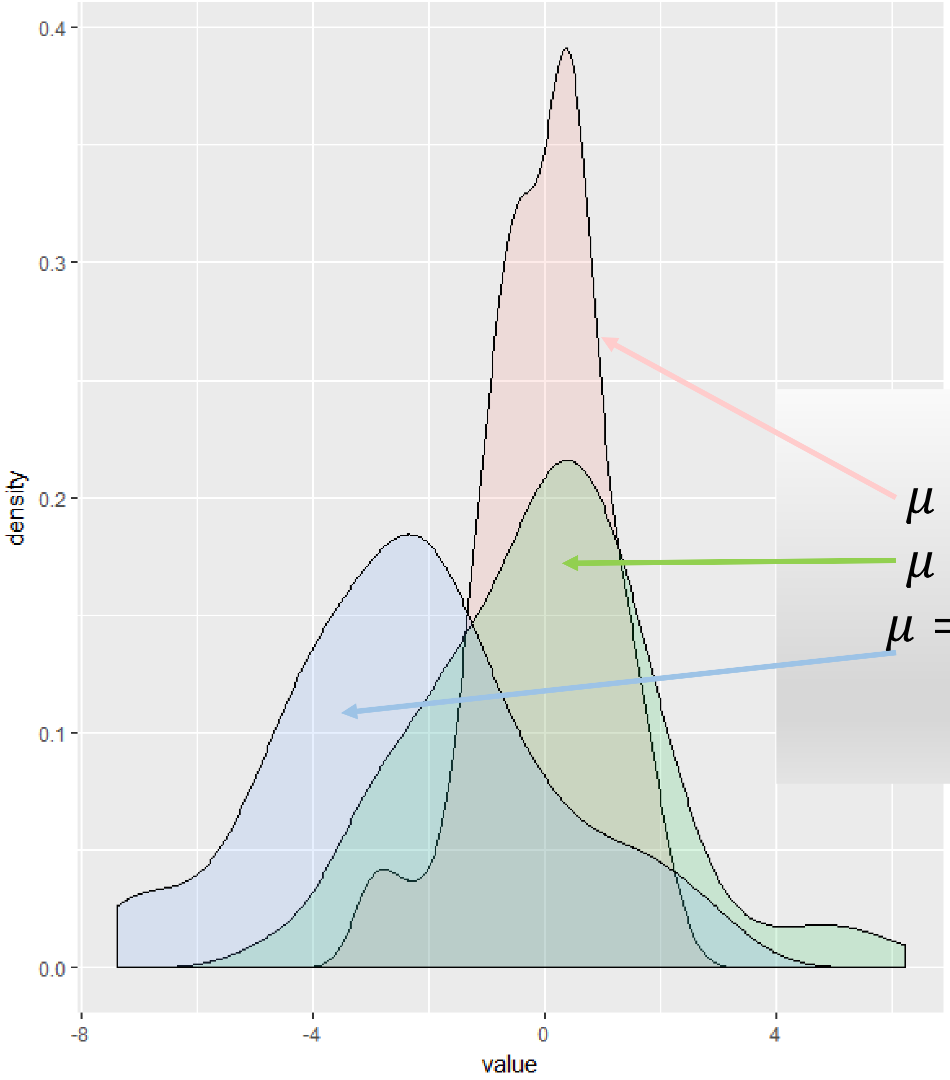
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}, -\infty < x < \infty,$$

With parameters μ and σ, where $-\infty < \mu < \infty$, and $\sigma > 0$.

Properties:

1. The mean μ describes the center of the distribution
2. The standard deviation σ describes how much the curve is spread around the center
3. The normal distribution is symmetric around the mean

Normal Distribution



$\mu = 0, \sigma = 1$

$\mu = 0, \sigma = 2$

$\mu = -2, \sigma = 2$

Standard Normal Distribution

A normal distribution with mean 0 and variance 1 is referred to as a 'standard normal' or 'unit normal distribution'.

The probability density function of a standard normal denoted by $N(0,1\ )$, is given by:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}, -\infty < x < \infty \qquad (1)$$

If $X \sim N(\mu, \sigma)$ then $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$. Thus, an alternative notation for (1) is:

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}}, -\infty < z < \infty \qquad (2)$$

Why do we even need the standard normal distribution?

Standard Normal Distribution: Empirical Rule

Based on the Empirical Rule, for the standard normal distribution:

$$P(-1 < z < 1) = 0.68$$

$$P(-2 < z < 2) = 0.95$$

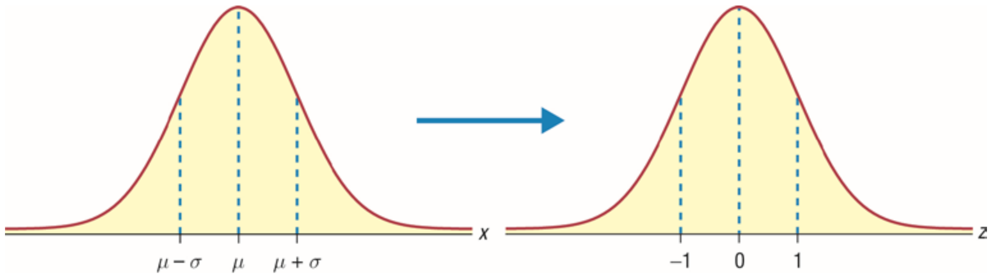
$$P(-3 < z < 3) = 0.99$$

This rules applies to any *approximately* bell-shaped distribution and it is usually interpreted as:

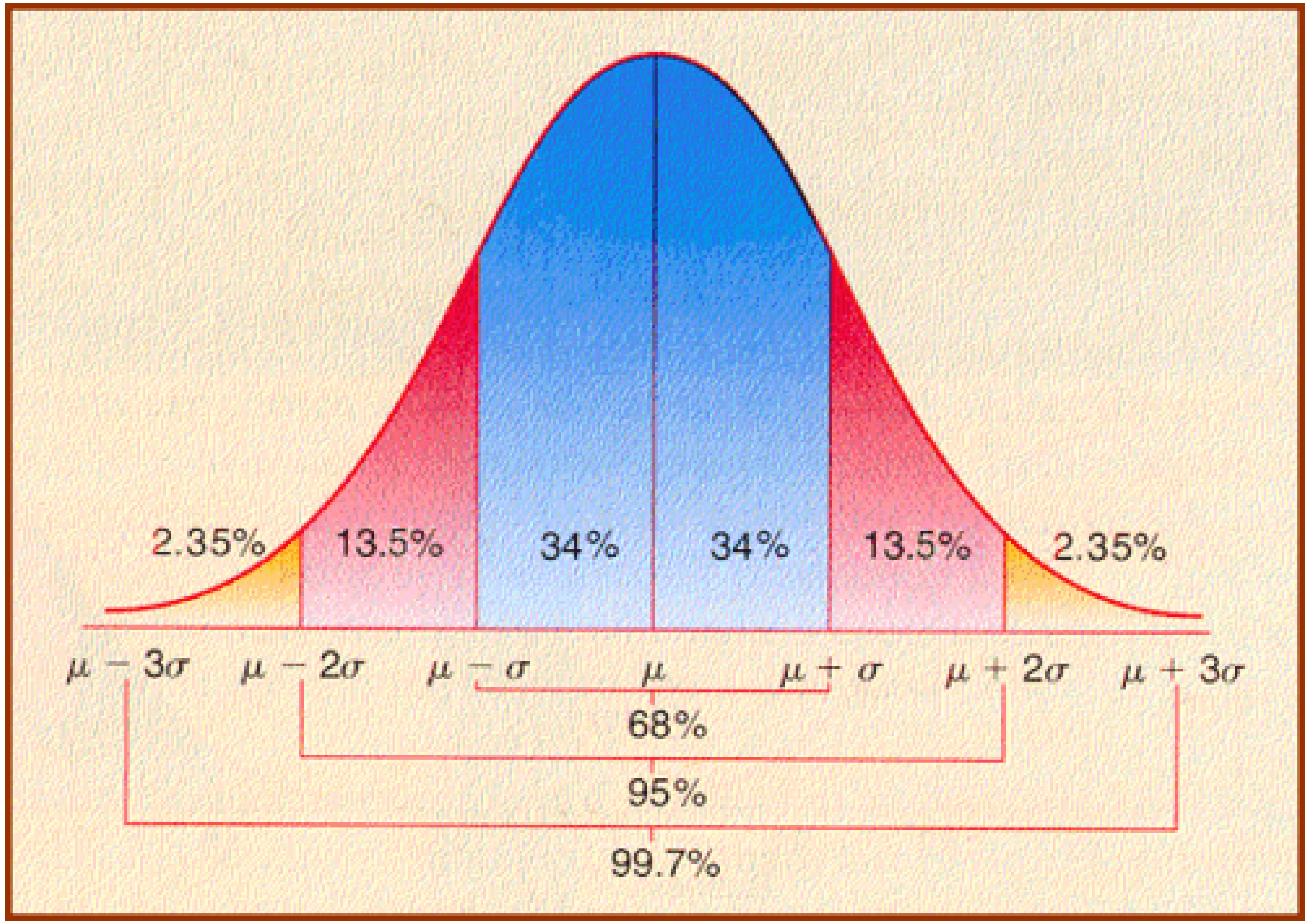
Approximately 68% of all values fall within one standard deviation from the mean.

Approximately 95% of all values fall within two standard deviations from the mean.

Approximately 99% of all values fall within three standard deviations from the mean.



Standard Normal Distribution: Empirical Rule



Standard Normal: Cumulative Distribution Function

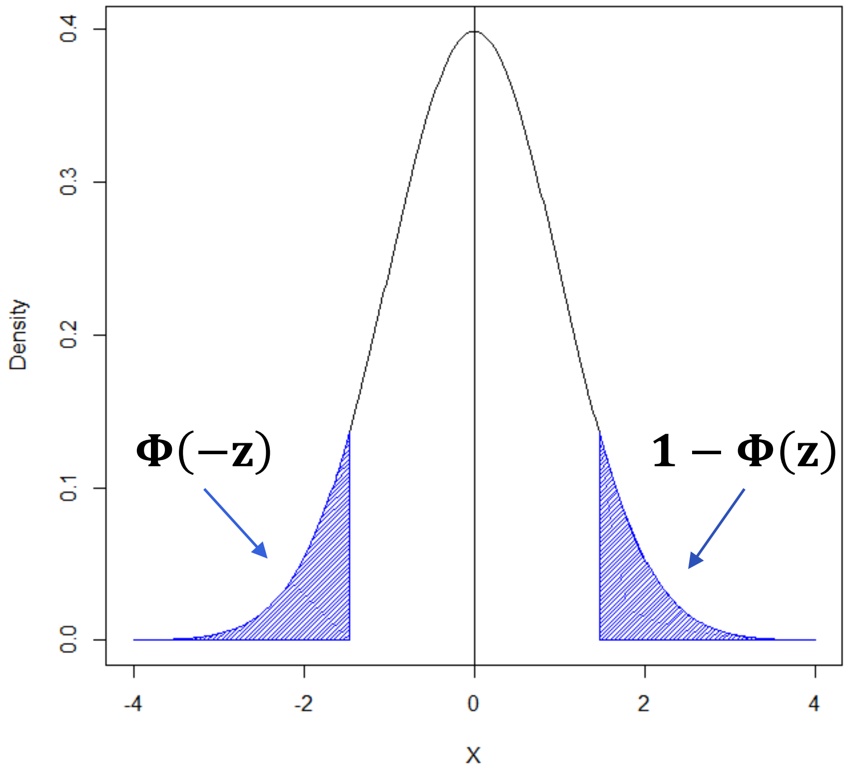
The cumulative distribution function (CDF) of a standard normal is denoted by:

$$\Phi(z) = P(Z \leq z) = \int_{-\infty}^{t} f(z)dz\,, -\infty < t < \infty$$

From the symmetry of standard normal:

$$\Phi(-z) = P(Z \leq -z) = P(Z \geq z)$$

$$P(Z \geq z) = 1 - P(Z \leq z) = 1 - \Phi(z)$$



Z-transformation

It is used to transform any normal r. v. to a standard normal r. v. We have already stated that:

$$\text{If } X \sim N(\mu, \sigma) \text{ then } Z = \frac{X-\mu}{\sigma} \sim N(0,1)$$

The 'z-score' measures how many standard deviations your observation is from the mean.

<u>Example:</u> Let X be a variable normally distributed with mean $\mu = 70$ and standard deviation $\sigma = 10$.

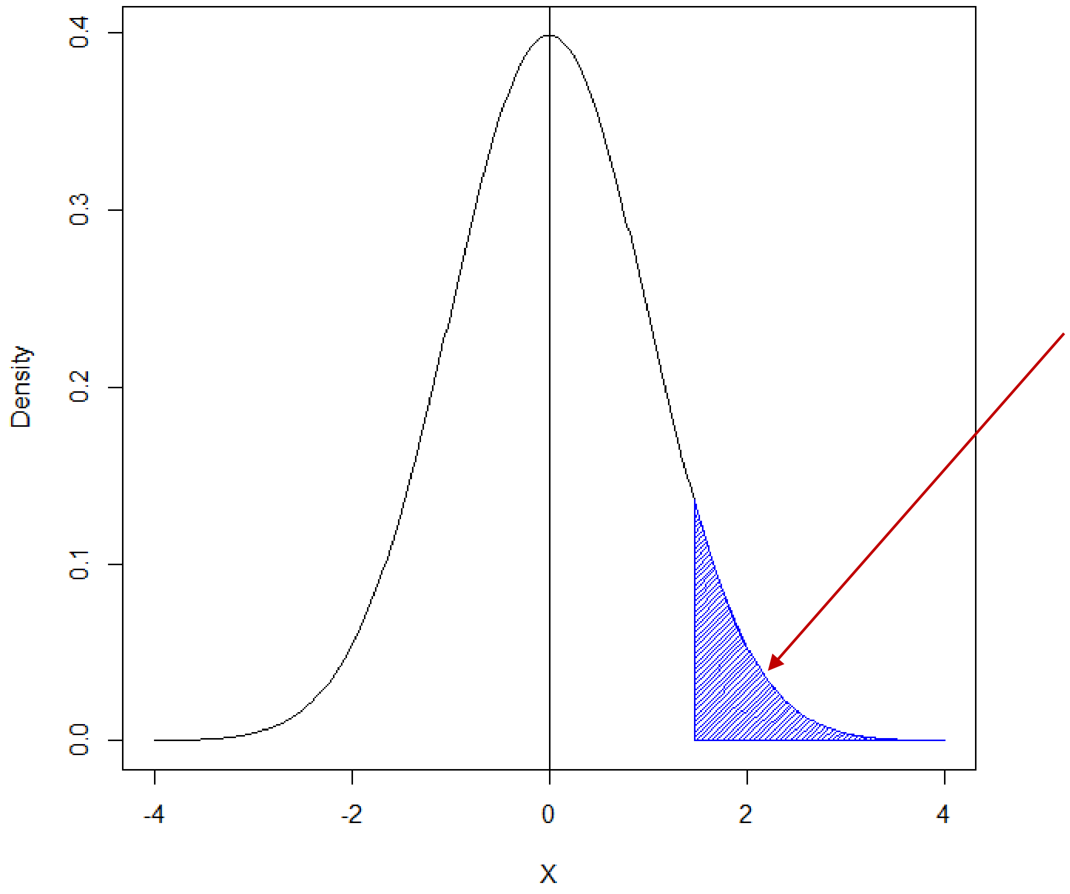
Find $P(X \leq 65)$ and $P(40 \leq X \leq 60)$.

$$P\left(Z \leq \frac{65-70}{10}\right) = P(Z \leq -0.5) = 1 - P(Z \leq 0.5) =$$

$$P(40 \leq X \leq 60) = P\left(\frac{40-70}{10} \leq Z \leq \frac{60-70}{10}\right) = P(-3 \leq Z \leq -1) =$$

z_α- notation

z_α - value on the x-axis for which the area under the curve lies to the right of z_α



Shaded area:
$P(Z \geq z_\alpha) = \alpha$

It follows that:
$P(Z < z_\alpha) = 1 - P(Z \geq z_\alpha) = 1 - \alpha$

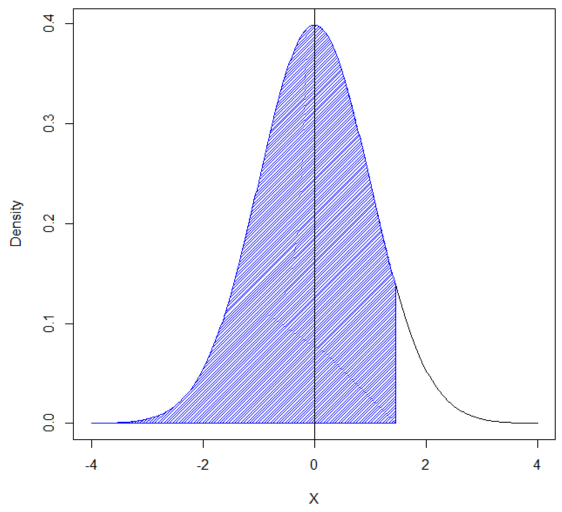
z_α values can be found in normal tables (see Rosner, page 818) or can be computed using statistical software

Percentiles: Examples

Let $Z \sim N(0,1)$. Find z_α for the following situations:

1. $P(Z < z_\alpha) = 0.9278$

From the normal tables (column A) => z_α=1.46

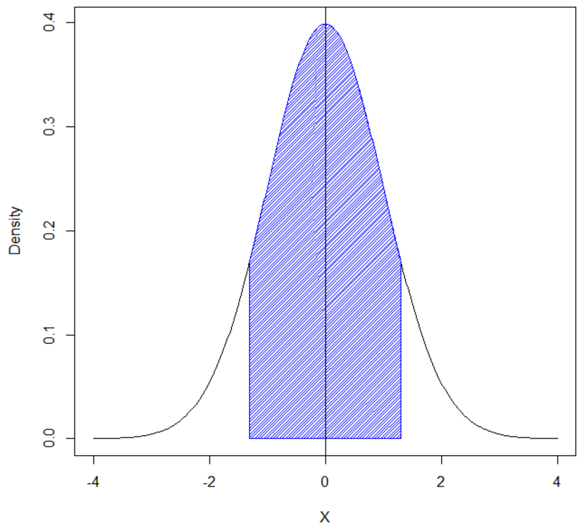


2. $P(-z_\alpha < Z < z_\alpha) = 0.8132$

$$P(-z_\alpha < Z < z_\alpha) = 2 \cdot P(0 < Z < z_\alpha) = 0.8132$$

$$P(0 < Z < z_\alpha) = \frac{0.8132}{2} = 0.4066$$

From the normal tables (column C) => z_α=1.32



Normal Approximation to Binomial

Let X be a binomial r. v. based on *n* trials and probability of success *p.* If the binomial distribution is not too skewed, *X* may be approximated by a normal distribution under the following two conditions:

$$np \geq 10$$

$$n(1-p) \geq 10$$

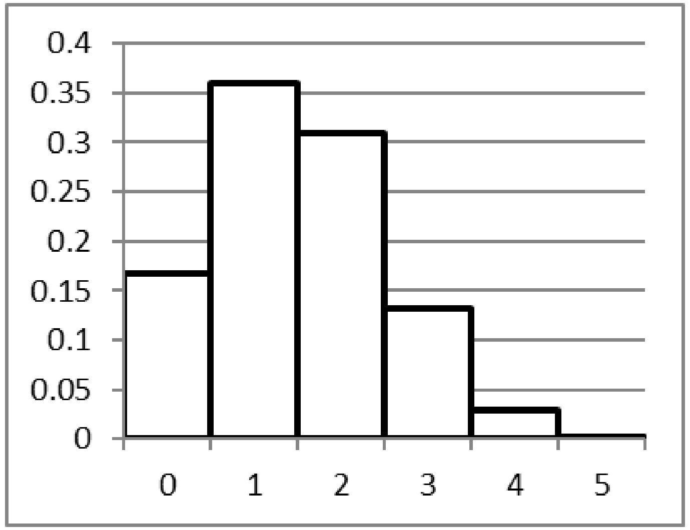
<u>Notes:</u>

- The normal approximation is easier to apply, especially if n is quite large
- Because we are approximating discrete probabilities using a continuous distribution, a continuity correction needs to be applied:

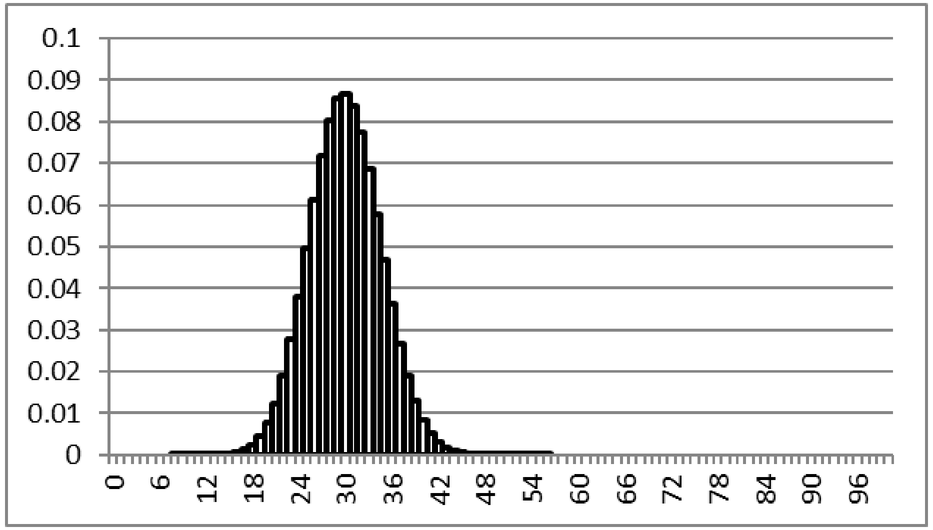
$$P\left(a - \frac{1}{2} \leq X \leq b + \frac{1}{2}\right)$$

Normal Approximation to Binomial: Examples

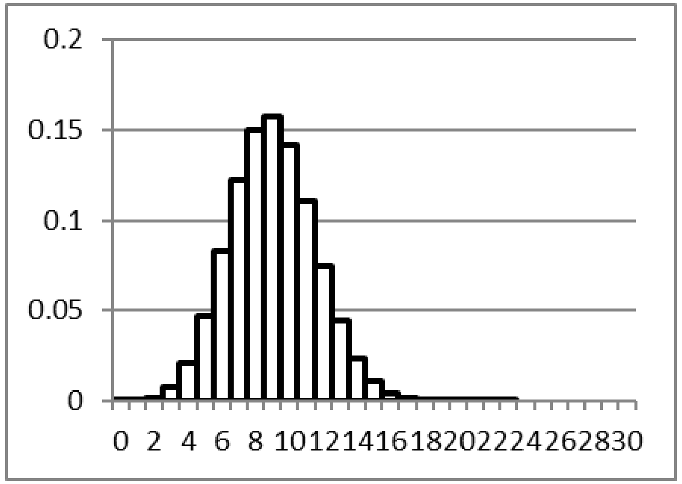
P=0.30, n=5



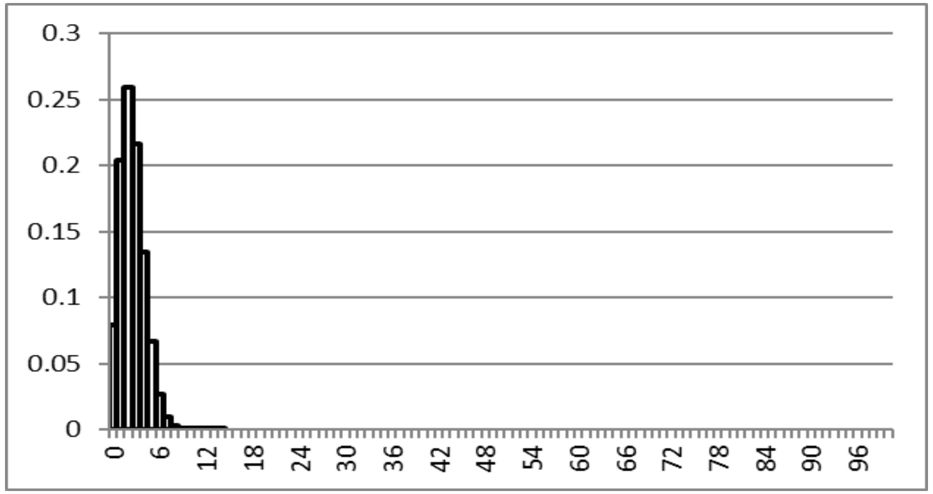
P=0.30, n=100



P=0.30, n=50



P=0.03, n=100



Normal Approximation to Poisson

A Poisson distribution with parameter λ may be approximated by a normal distribution with mean and variance both equal to λ (approximation recommended for $\lambda \geq 10$).

$P(X = x)$ is approximated by:

1. The area under $N(\lambda, \lambda)$ from $x - \frac{1}{2}$ to $x + \frac{1}{2}$, for $x > 0$
2. The area to the left of $\frac{1}{2}$ for $x = 0$.

All normal approximations are based on Central Limit Theorem (CLT). More details about CLT will be provided in Recitation 2.

Readings

Rosner, *Fundamentals of Biostatistics*, Chapter 5

- Sections: 5.2 – 5.5, 5.7 - 5.8
- More details on 5.6 and Central Limit Theorem (CTL) in Recitation 2