14.02 Principles of Macroeconomics
Solutions to Problem Set # 3
Due: October 23, 2009

October 15, 2009

1 True/False/Uncertain [30 points]

1. "In the Solow model, an economy that starts with a higher stock of capital per capita will reach a higher steady state level of capital per capita" [7 points]

   FALSE. The steady state level of capital per capita is independent of the initial condition for the capital stock.

2. " In the money market, a decrease in real income will tend to decrease the equilibrium interest rate (abstracting from the goods market)" [7 points]

   TRUE. It will shift money demand inwards (decrease money demand for every interest rate). Thus the interest rate needs to decrease, so as to make money more attractive.

3. "If prices adjust inmediately, then monetary policy will have no effect whatsover" [8 points]

   ANSWER. True. See class notes.

4. "Fiscal policy cannot affect output in the long run" [8 points]

   FALSE. Fiscal policy can affect Y^*. See class notes page 20.

2 Solow Model [40 points]

Consider the basic solow model. Assume total labor is fixed at $L = 1$. Time is discrete, and indexed by $t = 0, 1, 2, ...$ The production function is Cobb Douglas:

$$y_t = f(k_t) = Ak_t^{\alpha}$$

where y_t is per capita income at time t and k_t is capital per capita at time t. The law of motion for the capital stock is

$$k_{t+1} = (1 - \delta) k_t + i_t$$

where i_t is per capita investment. The economy consumes a constant fraction of output:

$$c_t = (1 - s)\, y_t \tag{1}$$

where $s \in (0, 1)$. Assume that the economy starts with initial capital $k_0 > 0$.

1. Show that equation (1) implies that per capita investment is also a constant fraction of per capita income. Which equation, which was not stated in the set up of this question, do you need to show this? [4 points]

   ANSWER. You need the basic macroeconomic equation

   $$y_t = c_t + i_t$$

   This equation says that, in this economy, production is used for either consumption or investment. Combining this equation together with eq. (1) you get

   $$i_t = sy_t$$

2. Using the previous result, state a law of motion for capital per capita. This is, an equation relating k_{t+1} to k_t. [4 points]

   ANSWER.

   $$k_{t+1} = (1 - \delta)\, k_t + sAk_t^{\alpha}$$

3. Compute the steady state capital per capita in this economy. Call it k. Also, compute the level of investment per capita (i) and income per capita (y) in the steady state. [6 points]

   ANSWER. We are looking for a level of capital such that if the economy reaches such level, then it remains there forever. Hence we plug $k = k_{t+1} = k_t$ in the equation from the previous point, and solve for k

   $$\begin{aligned} k &= (1 - \delta)\, k + sAk^{\alpha} \\ k &= \left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}} \end{aligned}$$

   Then, per capita income and investment in the steady state are

   $$\begin{aligned} y &= A\left(\frac{sA}{\delta}\right)^{\frac{\alpha}{1-\alpha}} \\ i &= sA\left(\frac{sA}{\delta}\right)^{\frac{\alpha}{1-\alpha}} \end{aligned}$$

4. We will now derive an *analytical expression for the evolution of the per capita capital stock in this economy.* Assume from now on that $\boldsymbol{\delta = 1}$, that is, that the capital stock fully depreciates in one period.

(a) Write k_1 as a function of k_0. [2 points]

ANSWER. $k_1 = sAk_0^\alpha$

(b) Write k_2 as a function of k_1, and, using (a), as a function of k_0. [2 points]

ANSWER. $k_2 = sA\left[sAk_0^\alpha\right]^\alpha = (sA)^{1+\alpha} k_0^{\alpha^2}$

(c) Write k_3 as a function of k_0. [2 points]

ANSWER. $k_3 = sAk_2^\alpha = sA\left[(sA)^{1+\alpha} k_0^{\alpha^2}\right]^\alpha = (sA)^{1+\alpha+\alpha^2} k_0^{\alpha^3}$

(d) Now, in light of the 3 previous answers, write an equation that generalizes to any t. That is, write k_t as a function of k_0 for any $t \geq 0$[1]. [7 points]

ANSWER.

$$k_t = (sA)^{\sum_{i=0}^{t-1} \alpha^i} k_0^{\alpha^t}$$

(e) Show that the limit (as $t \to \infty$) of the expression you just got is exactly k (the steady state level of capital derived in point 4). That is, show $lim_{t\to\infty} k_t = k$. [Hint: Recall that for any constant $x \in (0,1)$: $\sum_{i=0}^\infty x^i = \frac{1}{1-x}$] [3 points]

ANSWER.

$$\lim k_t = (sA)^{\lim_{t\to\infty} \sum_{i=0}^{t-1} \alpha^i} \lim_{t\to\infty} k_0^{\alpha^t}$$

Since $\alpha < 1$, we have that $\lim_{t\to\infty} \alpha^t = 0$, and thus $\lim_{t\to\infty} k_0^{\alpha^t} = 1$. Also

$$\lim_{t\to\infty} \sum_{i=0}^{t-1} \alpha^i = \sum_{i=0}^{\infty} \alpha^i = \frac{1}{1-\alpha}$$

and thus

$$\lim k_t = (sA)^{\frac{1}{1-\alpha}}$$

which is exactly the formula from equation 4, with $\delta = 1$.

5. Using the analytical expression derived in point 5 (d), plot the evolution of the per capita capital stock in this economy. Take $s = 0.2$, $A = 1$, $\alpha = 0.5$ (and of course, $\delta = 1$). That is, plot k_t as a function of t. Do this graph for both the case in which $k_0 = 1$ and $k_0 = 0.01$.[10 points]

ANSWER. Using the formula for a geometric sum we have that

$$\sum_{i=0}^{t-1} \alpha^i = \frac{1-\alpha^t}{1-\alpha}$$

and thus the formula from 5(d) becomes

$$k_t = (sA)^{\frac{1-\alpha^t}{1-\alpha}} k_0^{\alpha^t}$$

[1] This procedure is called recursive substitution.

For $k_0 = 1$ we need to plot

$$k_t = (0.2)^{2\left(1-\frac{1}{2^t}\right)}$$

while for $k_0 = 0.01$ we have to plot

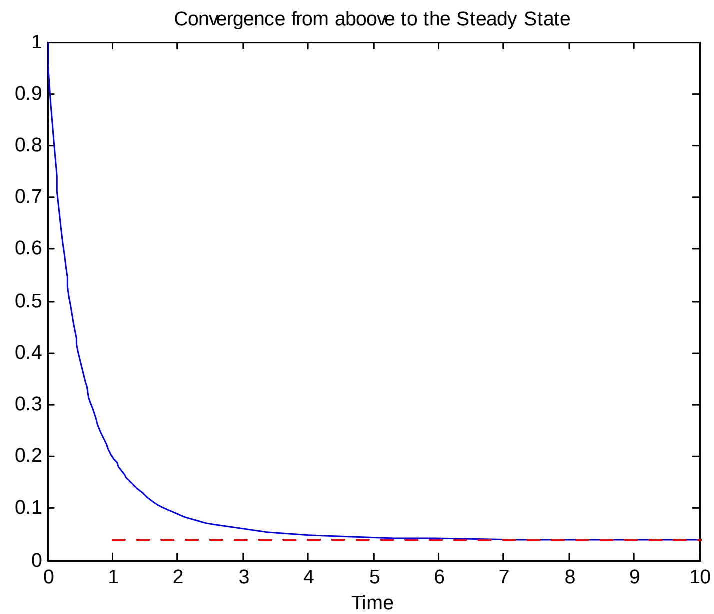
$$k_t = (0.2)^{2\left(1-\frac{1}{2^t}\right)} (0.01)^{\frac{1}{2^t}}$$

Note that the steady state level of capital for these parameters is

$$k = 0.04$$

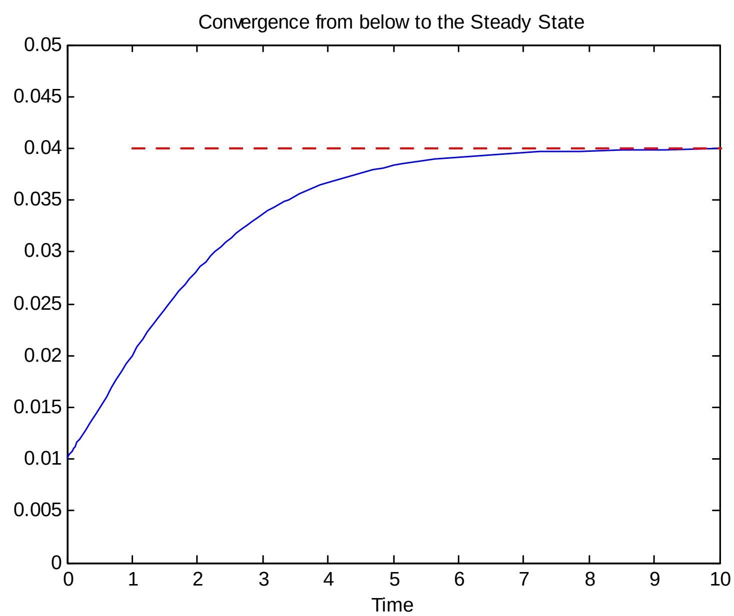
(make sure you understand why this is the same for both levels of k_0)

For the case in which $k_0 = 1 > 0.04 = k$ the graph is



The blue line represents the capital stock, while the dashed red line represents the steady state level of capital. We see that when the economy starts with a level of capital that is higher than the steady state level ($k_0 > k$), the capital stock decreases until convergence.

For the case $k_0 = 0.01 < 0.04 = k$ the graph is



When the initial level of capital is below the steady state level, the capital stock increases until convergence. Note that as time increases the graph becomes flatter. This reflects the fact that the growth rate of capital is becoming smaller.

3 IS-LM [30 points]

Consider the following version of the IS-LM model, where expected inflation is zero ($\pi^e = 0$). Assume net exports (NX) are zero.

$$\begin{aligned} C &= c_0 + c_1 (Y - T) \\ I &= b_0 + b_1 Y - b_2 i \\ M^d/P &= d_1 Y - d_2 i \\ M^s &= M \end{aligned}$$

where all the variables are as defined in class, and T represents taxes.

1. Find the IS relation. What assumption on the parameters do we have to impose to ensure that the goods market reaches an equilibrium? Show graphically happens to this relation when taxes increase. [6 points]

ANSWER.

$$
\begin{aligned}
Y &= c_0 + c_1 (Y - T) + b_0 + b_1 Y - b_2 i + G \\
Y &= \frac{1}{1 - b_1 - c_1} [c_0 - c_1 T + b_0 - b_2 i + G]
\end{aligned}
$$

Clearly, we need to assume $b_1 + c_1 < 1$. If T increases, Y goes down for every interest rate, so the IS curve shifts inwards.

2. Derive the LM relation. What happens when M increases? Explain the intuition. [6 points]

   ANSWER.

$$
\begin{aligned}
M^s &= M^d \\
M/P &= d_1 Y - d_2 i \\
Y &= \frac{1}{d_1} \left[\frac{M}{P} + d_2 i \right]
\end{aligned}
$$

   If M increases (money supply increases), then for every Y the interest rate must go down to make people willing to hold more money. This will shift the LM curve down and to the right. See page 15 of class notes.

3. Find the short-run equilibrium (i.e. find the pair (Y, i) that makes the goods and money markets both be in equilibrium at the same time) [6 points]

   ANSWER. Call $m \equiv 1/(1 - b_1 - c_1)$.

$$
\begin{aligned}
IS &= LM \\
m [c_0 - c_1 T + b_0 - b_2 i + G] &= \frac{1}{d_1} \left[\frac{M}{P} + d_2 i \right]
\end{aligned}
$$

$$
\begin{aligned}
i^* &= \frac{m d_1 [c_0 - c_1 T + b_0 + G] - \frac{M}{P}}{b_2 m d_1 + d_2} \\
Y^* &= \frac{1}{d_1} \left[\frac{M}{P} + d_2 \frac{m d_1 [c_0 - c_1 T + b_0 + G] - \frac{M}{P}}{b_2 m d_1 + d_2} \right]
\end{aligned}
$$

4. What happens to equilibrium output and the interest rate as taxes increase? Use your results from the previous part, as well as the IS-LM graph to illustrate your answer. [6 points]

   ANSWER. From the above formulas we have that the interest rate goes down and output goes down. The LM curve does not move. The IS curve shifts inwards, so both i and Y go down.

5. What happens to equilibrium output and the interest rate as money supply increases? [6 points]

   ANSWER. If M increases the LM shifts down and to the right. Hence output increases and the interest rate decreases. To see this in algebra, note that the above expression for Y can be written as

$$Y^* = \frac{1}{d_1}\left[\frac{d_2 m d_1 \left[c_0 - c_1 T + b_0 + G\right]}{b_2 m d_1 + d_2} + \left(1 - \frac{d_2}{b_2 m d_1 + d_2}\right)\frac{M}{P}\right]$$

   so that Y* is increasing in M.