DOMAĆI 4

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1. Zamislimo da iz dva skupa A i B ($|A| = m$ i $|B| = n$) biramo ukupno k elemenata. To možemo odraditi na $\binom{m+n}{k}$ načina. Sada posmatrajmo ponaosob biranje elemenata iz datih skupova:

$$\binom{m}{0}\binom{n}{k} + \binom{m}{1}\binom{n}{k-1} + \binom{m}{2}\binom{n}{k-2} + \dots + \binom{m}{k}\binom{n}{0} =$$
$$= \binom{n}{0}\binom{m}{k} + \dots + \binom{n}{k-2}\binom{m}{2} + \binom{n}{k-1}\binom{m}{1} + \binom{n}{k}\binom{m}{0} =$$
$$= \sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i} = \binom{m+n}{k}$$

2. $$k \cdot \binom{n+1}{k} = k \cdot \frac{(n+1)!}{k!(n+1-k)!} = \frac{(n+1) \cdot n!}{(k-1)!(n-(k-1))!} =$$
$$= (n+1) \cdot \binom{n}{k-1}$$

3. $$\binom{2n}{2} = \frac{2n!}{2!(2n-2)!} = \frac{2n \cdot (2n-1)}{2!} = \frac{4n^2 - 2n}{2} =$$
$$= \frac{2n^2 - 2n + 2n^2}{2} = n(n-1) + n^2 = \frac{n!}{(n-2)!} + n^2 =$$
$$= 2 \cdot \frac{n!}{2(n-2)!} + n^2 = 2\binom{n}{2} + n^2$$

6. $$(x+y+z)^9 = \sum_{m_1+m_2+m_3=9} \binom{n}{m_1, m_2, m_3} x^{m_1} y^{m_2} z^{m_3}$$

Koeficijent uz $x^3 y^2 z^4$ je $\binom{9}{3,2,4} = \frac{9!}{3!2!4!} =$

$$= \frac{9 \cdot \cancel{8} \cdot 7 \cdot \cancel{6} \cdot 5}{\cancel{3 \cdot 2} \cdot \cancel{2}} = 9 \cdot 7 \cdot 5 \cdot 4 =$$
$$= 63 \cdot 20 = 1260$$

2. $(x+y+z)^{2020} = \sum\limits_{\substack{m_1+m_2+m_3=2020 \\ 0\le m_1,m_2,m_3\le 2020}} \binom{m}{m_1,m_2,m_3} x^{m_1} y^{m_2} z^{m_3}$

Sabiraka ima onoliko koliko ima i cjelobrojnih pozitivnih rješenja jednačine $m_1+m_2+m_3=2020$, a to je $\binom{2020+3-1}{3-1} = \binom{2022}{2}$

4. $\binom{m+n+1}{n} = \binom{m+n}{n} + \binom{m+n}{n-1} =$

$= \binom{m+n}{n} + \left(\binom{m+n-1}{n-1} + \binom{m+n-1}{n-2}\right) =$

$= \binom{m+n}{n} + \binom{m+n-1}{n-1} + \left(\binom{m+n-2}{n-2} + \binom{m+n-2}{n-3}\right) =$

$= \dots = \binom{m+n}{n} + \binom{m+n-1}{n-1} + \binom{m+n-2}{n-2} + \dots + \binom{m+n-n}{n-n} =$

$= \sum\limits_{i=0}^{n} \binom{m+i}{i}$

5. Zamislimo dva skupa A i B, $|A| = n$ i $|B| = m+1$. Treba da izaberemo ukupno n elemenata iz unije ta dva skupa (a sami skupovi su disjunktni). To možemo odraditi na $\binom{m+1+n}{n}$ načina, ali i raspisati kao

$\binom{n}{0}\binom{m+1}{n} + \binom{n}{1}\binom{m+1}{n-1} + \dots \binom{n}{n}\binom{m+1}{0} = \sum\limits_{i=0}^{n} \binom{n}{i}\binom{m+1}{n-i}$

Sada, premjestimo jedan element iz B u A i tada

3° je $|A| = n+1$ i $|B| = m$. Na isti način, n elemenata iz unije skupova se bira na $\binom{n+1+m}{n}$ načina, odnosno

$\binom{n+1}{0}\binom{m}{n} + \binom{n+1}{1}\binom{m}{n-1} + \dots + \binom{n+1}{n}\binom{m}{0} = \sum\limits_{i=0}^{n} \binom{n+1}{i}\binom{m}{n-i}$

$\Rightarrow \sum\limits_{i=0}^{n} \binom{n}{i}\binom{m+1}{n-i} = \sum\limits_{i=0}^{n} \binom{n+1}{i}\binom{m}{n-i} = \binom{m+n+1}{n}$