Ojlerova diferencijalna jednačina

$$(ax+b)^n y^{(n)} + A_{n-1}(ax+b)^{n-1} y^{(n-1)} + \ldots + A_1(ax+b)y' + A_0 y = f(x)$$

$a, b, A_0, A_1, \ldots, A_{n-1}$ – konstante,

Ako je $ax+b>0,\ a\neq 0$, smenom $\boxed{ax+b=e^t} \Rightarrow t=\ln(ax+b)$, tj.

$$y' = \frac{dy}{dt}\cdot\boxed{\frac{dt}{dx}} = y_t'\cdot\boxed{\frac{a}{ax+b}} = ae^{-t}y_t',$$

$$y'' = \frac{dy'}{dt}\cdot\boxed{\frac{dt}{dx}} = a(e^{-t}y_t'' - e^{-t}y_t')\cdot\boxed{\frac{a}{ax+b}} = a^2e^{-2t}(y_t''-y_t'),$$

$$y''' = \frac{dy''}{dt}\cdot\boxed{\frac{dt}{dx}} = a^2(-2e^{-2t}(y_t''-y_t') + e^{-2t}(y_t'''-y_t''))\cdot\boxed{ae^{-t}} = a^3e^{-3t}(y_t'''-3y_t''+2y_t'),\ \text{itd.}$$

data jednačina se svodi na jednačinu sa konstantnim koeficijentima.

Za $ax+b<0,\ a\neq 0$ uvodi se smena $\boxed{ax+b=-e^t}$.

Za $a=0,\ b\neq 0$ dobija se nehomogena linearna jednačina čiji je homogeni deo sa konstantnim koeficijentima. Za $a=0$ i $b=0$ dobija se $A_0\cdot y = f(x)$, a to nije diferencijalna jednačina.

1. Rešiti diferencijalnu jednačinu $(1+x)^3y''' + (1+x)y' - y = (1+x)^2$ za $x>-1$.

$1+x=e^t \Rightarrow t=\ln(1+x)$

$y' = e^{-t}y_t',\qquad y'' = (y_t''-y_t')e^{-2t},\qquad y''' = (y_t'''-3y_t''+2y_t')\cdot e^{-3t}$

$\cancel{e^{3t}}\cdot\cancel{e^{-3t}}(y_t'''-3y_t''+2y_t') + \cancel{e^{t}}\cdot\cancel{e^{-t}}y_t' - y = e^{2t}$

$y_t'''-3y_t''+3y_t'-y=e^{2t}$

y_h
- $y_t'''-3y_t''+3y_t'-y=0 \Rightarrow r^3-3r^2+3r-1=0 \Rightarrow (r-1)^3=0 \Rightarrow r_1=r_2=r_3=1$

  $y_h = c_1e^t + c_2te^t + c_3t^2e^t$

y_p
- $y_t'''-3y_t''+3y_t'-y=e^{2t}$ $\qquad e^{2t} = e^{\overset{\alpha}{\underline{2}}t}\left(\underbrace{1}_{0.\ \text{stepen}}\cos(\overset{\beta}{0}t) + \underline{\ 0\ }\sin(\overset{\beta}{0}t)\right)$

  $e^{2t} = e^{\alpha t}\left[P_m(t)\cos\beta t + Q_n(t)\sin\beta t\right] \Rightarrow \alpha=2,\ \beta=0,\ P_m(t)=1$

  $\alpha+\beta i = 2 \Rightarrow r=0,\ k=m=0$ $\qquad y_p = t^0\cdot e^{2t}(A\cos(0t)+B\sin(0t))$

  $y_p = A\cdot e^{2t},\qquad y_p' = 2A\cdot e^{2t},\qquad y_p'' = 4A\cdot e^{2t},\qquad y_p''' = 8A\cdot e^{2t}$

  $(8A-12A+6A-A)e^{2t} = e^{2t} \Rightarrow A=1$

  $y_p = e^{2t}$

$$y = c_1 e^t + c_2 t e^t + c_3 t^2 e^t + e^{2t}$$

$$\boxed{y = c_1(1+x) + c_2(1+x)\cdot\ln(1+x) + c_3(1+x)\cdot\ln^2(1+x) + (1+x)^2}$$ Opšte rešenje

Neke metode rešavanja diferencijalnih jednačina

2. Naći dva puta diferencijabilnu funkciju $z = f(x^2+y^2)$, nad oblašću $R^2 \setminus \{(0,0)\}$ koja zadovoljava diferencijalnu jednačinu

$$\frac{\partial^2 z}{\partial y^2} + \frac{y^2 - x^2}{x(x^2+y^2)}\cdot\frac{\partial z}{\partial x} + \frac{2y^2}{(x^2+y^2)^2} z = \frac{2y^2}{x^2+y^2}\ln(x^2+y^2).$$

$$x^2 + y^2 = t,\; z = f(t)$$

$$\frac{\partial z}{\partial x} = f_t'\cdot 2x, \qquad \frac{\partial z}{\partial y} = f_t'\cdot 2y, \qquad \frac{\partial^2 z}{\partial y^2} = 2(f_t''\cdot y\cdot 2y + f_t') = 2f_t' + 4y^2 f_t''$$

$$\frac{x^2+y^2}{x^2+y^2}\; 2f_t' + 4y^2 f_t'' + \frac{y^2-x^2}{\cancel{x}(x^2+y^2)} 2\cancel{x}\cdot f_t' + \frac{2y^2}{(x^2+y^2)^2} f = \frac{2y^2}{x^2+y^2}\ln(x^2+y^2)$$

$$4y^2 f_t'' + 2\cdot\frac{y^2 - \cancel{x^2} + \cancel{x^2} + y^2}{x^2+y^2}\cdot f_t' + \frac{2y^2}{(x^2+y^2)^2} f = \frac{2y^2}{x^2+y^2}\ln(x^2+y^2) \quad \Big|: y^2$$

$$4f_t'' + \frac{4f_t'}{t} + \frac{2f_t}{t^2} = \frac{2}{t}\ln(t) \quad \Big|\cdot\frac{t^2}{2}$$

$$2t^2 f_t'' + 2t f_t' + f_t = t\ln t \text{ (Ojlerova diferencijalna jednačina)}$$

Smena $\quad t = e^s,\; s = \ln t,\; t > 0, \qquad f_t' = e^{-s} f_s', \qquad f_t'' = e^{-2s}(f_s'' - f_s')$

$$2\cancel{e^{2s}}\cdot\cancel{e^{-2s}}(f_s'' - f_s') + 2\cancel{e^{s}}\cancel{e^{-s}} f_s' + f_s = se^s$$

$$2f_s'' + f_s = se^s$$

f_h

- $2f_s'' + f_s = 0$

$$2r^2 + 1 = 0 \Leftrightarrow r_1 = \frac{i}{\sqrt{2}},\; r_2 = -\frac{i}{\sqrt{2}}$$

$$e^{r_1 s} = e^{\frac{s}{\sqrt{2}}i} = \cos\frac{s}{\sqrt{2}} + i\sin\frac{s}{\sqrt{2}}, \qquad R_e\{e^{r_1 s}\} = \cos\frac{s}{\sqrt{2}}, \qquad I_m\{e^{r_1 s}\} = \sin\frac{s}{\sqrt{2}}$$

$$f_h = c_1\cos\left(\frac{1}{\sqrt{2}}s\right) + c_2\sin\left(\frac{1}{\sqrt{2}}s\right)$$

f_p

- $2f_s'' + f = se^s$

$$se^{s} = e^{\overset{\alpha}{1}\cdot s}\Big(\underbrace{s}_{1.\ \text{stepen}}\cdot\cos(\overset{\beta}{0}\cdot s) + 0\cdot\sin(\overset{\beta}{0}\cdot s)\Big)$$

$$se^{s} = e^{\alpha s}\left[P_n(s)\cos\beta s + Q_m(s)\sin\beta s\right] \Rightarrow \alpha = 1,\ \ \beta = 0,\ \ P_n(s) = s$$

$$\alpha + \beta i = 1 \Rightarrow r = 0,\ \ k = 1 \qquad f_p = s^{0}\cdot e^{1\cdot s}\cdot\Big(\underbrace{(As+B)}_{1.\ \text{stepen}}\cos(0\cdot s) + \underbrace{(Cs+D)}_{1.\ \text{stepen}}\sin(0\cdot s)\Big)$$

$$f_p = (As+B)e^{s}, \qquad f_p' = (As+A+B)e^{s}, \qquad f_p'' = (As+2A+B)e^{s}$$

$$2As + 4A + 2B + As + B = s \ \Rightarrow A = \frac{1}{3}\ ,\ \ B = -\frac{4}{9}$$

$$f_p = \left(\frac{1}{3}s - \frac{4}{9}\right)e^{s}$$

$$f(\underset{\overset{\shortparallel}{\ln(x^2+y^2)}}{s}) = c_1\cos\frac{1}{\sqrt{2}}s + c_2\sin\frac{1}{\sqrt{2}}\cdot s + \left(\frac{1}{3}s - \frac{4}{9}\right)e^{s} = C_1\cdot\cos\left(\frac{1}{\sqrt{2}}\ln(x^2+y^2)\right) + C_2\sin\left(\frac{1}{\sqrt{2}}\ln(x^2+y^2)\right) + \left(\frac{1}{3}\ln(x^2+y^2) - \frac{4}{9}\right)(x^2+y^2)$$

3. Prelaskom na inverznu funkciju pokazati da se diferencijalna jednačina $y'y''' - 3(y'')^2 - 4y''(y')^2 - 4(y')^4 + (y')^5\cdot(2y+1+4e^{y}) = 0$ svodi na jednačinu $x''' - 4x'' + 4x' = 2y+1+4e^{y}$ i naći njeno opšte rešenje.

$$y' = \frac{1}{x'}$$

$$y'' = -\frac{x''}{(x')^2}\cdot\boxed{\frac{1}{x'}} = -\frac{x''}{(x')^3}$$

$$y''' = -\frac{x'''\cdot(x')^3 - 3x''\cdot(x')^2\cdot x''}{(x')^6}\cdot\boxed{\frac{1}{x'}} = \frac{3(x'')^2}{(x')^5} - \frac{x'''}{(x')^4}$$

$$\frac{1}{x'}\left(\frac{3(x'')^2}{(x')^5} - \frac{x'''}{(x')^4}\right) - 3\frac{(x'')^2}{(x')^6} + 4\frac{x''}{(x')^3}\frac{1}{(x')^2} - 4\frac{1}{(x')^4} + \frac{1}{(x')^5}\cdot(2y+1+4e^{y}) = 0 \quad \Big\backslash\cdot(x')^6$$

$$3(\cancel{x''})^2 - x'''x' - 3(\cancel{x''})^2 + 4x''x' - 4(x')^2 + x'(2y+1+4e^{y}) = 0 \quad \Big|:x'$$

$$-x''' + 4x'' - 4x' + (2y+1+4e^{y}) = 0$$

$x''' - 4x'' + 4x' = 2y+1+4e^{y}$ (Jednačina sa konstantnim koeficijentima)

x zavisna, y nezavisna promenljiva

x_h • $x''' - 4x'' + 4x' = 0$

$$r^3 - 4r^2 + 4r = 0 \ \Rightarrow\ r(r^2 - 4r + 4) = 0 \ \Rightarrow\ r(r-2)^2 = 0 \ \Rightarrow\ r_1 = 0,\ \ r_2 = r_3 = 2$$

$$x_h = c_1 + c_2e^{2y} + c_3ye^{2y}$$

x_{p_1} • $x''' - 4x'' + 4x' = 2y+1$ $\qquad 2y+1 = e^{\overset{\alpha}{0}\cdot y}\Big(\overbrace{(2y+1)}^{1.\ \text{stepen}}\cos(\overset{\beta}{0}\cdot y) + 0\cdot\sin(\overset{\beta}{0}\cdot y)\Big)$

$$2y+1 = e^{\alpha y}\left[P_m(y)\cos\beta y + Q_n(y)\sin\beta y\right] \ \Rightarrow\ \alpha = 0,\ \ \beta = 0,\ \ P_m(y) = 2y+1$$

$$\alpha + \beta i = 0 \Rightarrow r = 1,\ \ k = m = 1$$

$$x_{p_1} = y^{1}e^{0\cdot y}\Big(\overbrace{(Ay+B)}^{1.\ \text{stepen}}\cos(0\cdot y) + \overbrace{(Cy+D)}^{1.\ \text{stepen}}\sin(0\cdot y)\Big)$$

$$x_{p1} = y(Ay+B) = Ay^2 + By, \qquad x'_{p1} = 2Ay + B, \qquad x''_{p1} = 2A, \qquad x'''_{p1} = 0$$

$$-4(2A) + 4(2Ay+B) = 2y+1$$

$$-8A + 4B + 8Ay = 2y+1$$

$$8A = 2 \Rightarrow A = \frac{1}{4}$$

$$-2 + 4B = 1 \Rightarrow B = \frac{3}{4}$$

$$x_{p1} = \frac{1}{4}y^2 + \frac{3}{4}y$$

0. stepen

x_{p_2} • $x''' - 4x'' + 4x' = 4e^y$ $\qquad 4e^y = e^{1\cdot y}\left(4\cdot\cos(0\cdot y) + 0\cdot\sin(0\cdot y)\right)$

$$4e^y = e^{\alpha y}\left[P_m(y)\cos\beta y + Q_n(y)\sin\beta y\right] \Rightarrow \alpha = 1,\ \beta = 0,\ P_m(y) = 4$$

$$\alpha + \beta i = 1 \Rightarrow r = 0,\ k = m = 0$$

$$x_{p_2} = y^0 \cdot e^{1\cdot y}\left(A\cos(0\cdot y) + B\sin(0\cdot y)\right)$$

$$x_{p2} = Ae^y, \qquad x'_{p2} = x''_{p2} = x'''_{p2} = Ae^y$$

$$Ae^y \cancel{-4Ae^y} \cancel{+4Ae^y} = 4e^y$$

$$Ae^y = 4e^y \Rightarrow A = 4$$

$$x_{p2} = 4e^y$$

$$\boxed{x = x_h + x_{p1} + x_{p2} = c_1 + c_2e^{2y} + c_3ye^{2y} + \frac{1}{4}y^2 + \frac{3}{4}y + 4e^y}$$

Opšte rešenje

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