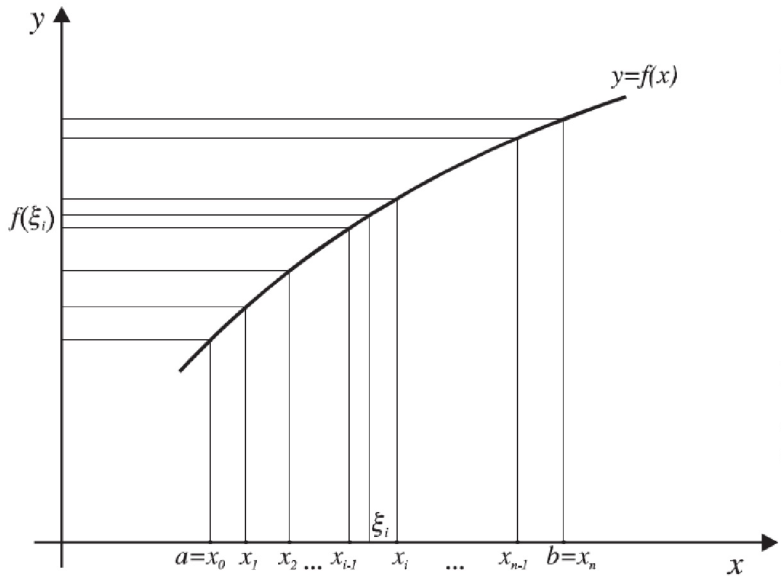
Određeni integral i njegova primena



Na intervalu $[a, b] \subset \mathbb{R}$ izvršena je podela na podintervale konačnim brojem tačaka iz skupa

$$P = \{x_0, x_1, \dots, x_n\}$$

tako da je

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

Iz svakog podintervala odabrana je tačka $\xi_i \in [x_{i-1}, x_i]$ i izračunata suma

$$I(f, P, \xi) = \sum_{i=1}^{n} f(\xi_i)\Delta x_i,$$

gde je $\Delta x_i = x_i - x_{i-1}$ dužina podintervala. Ovako dobijena suma se naziva integralna ili Rimanova suma.

Neka je sa $\lambda(P) = \max\limits_{1\le i\le n} \Delta x_i$ označena maksimalna dužina svih podintervala. Ukoliko postoji granična vrednost

$$\lim_{\lambda(P)\to 0} I(f, P, \xi_i) = I$$

nezavisno od podele P i izbora tačaka iz ξ_i, tada se broj I naziva Rimanov ili određeni integral funkcije f nad intervalom $[a, b]$ i označava se sa

$$I = \int_a^b f(x)\, dx.$$

Osobine određenog integrala koje se koriste prilikom njegovog izračunavanja su:

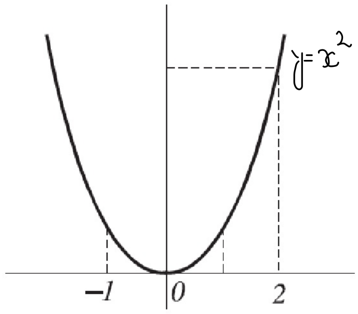
1. $\displaystyle\int_a^b c\, dx = c(b - a), \quad c \in \mathbb{R},$

2. $\displaystyle\int_a^b (f(x) \pm g(x))\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx,$

3. $\displaystyle\int_a^b c\, f(x)\, dx = c\int_a^b f(x)\, dx, \quad c \in \mathbb{R}.$

1. Izračunati po definiciji $I = \displaystyle\int_0^1 x^2\, dx$ i $I = \displaystyle\int_{-1}^{5} (1 + 3x)\, dx$.

Rešenje:



Ako je interval $[0,1]$ podeljen na n jednakih delova, tada je

$$\Delta x_i = \frac{1}{n}.$$

Neka su odabrane tačke ξ_i desni krajevi intervala $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$, tj.

$$\xi_i = \frac{i}{n}, = 0 + i \cdot \frac{1}{n}$$

$i = 1, 2, \ldots, n$. Sada je Rimanova suma

$$\sum_{i=1}^{n} f(\xi_i)\Delta x_i = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \cdot \frac{1}{n} = \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6},$$

pa je

$$I = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} = \lim_{n\to\infty} \frac{2n^3+3n^2+n}{6n^3} = \frac{1}{3}.$$

Slično, podelom intervala $[-1, 5]$ na n jednakih delova, dužina svakog intervala podele $[x_{i-1}, x_i]$ je

$$\Delta x_i = \frac{5-(-1)}{n} = \frac{6}{n}.$$

Neka su odabrane tačke ξ_i levi krajevi intervala $[x_{i-1}, x_i]$, tj.

$$\xi_i = -1 + (i-1)\frac{6}{n},$$

$i = 1, 2, \ldots, n$. Rimanova suma je sada

$$\begin{aligned}
\sum_{i=1}^{n} f(\xi_i)\Delta x_i &= \sum_{i=1}^{n} f(-1+(i-1)\frac{6}{n}) \cdot \frac{6}{n} \\
&= \sum_{i=1}^{n} \left(1 + 3(-1+(i-1)\frac{6}{n})\right) \cdot \frac{6}{n} \\
&= \frac{6}{n}\left(\sum_{i=1}^{n} 1 + \sum_{i=1}^{n} 3(-1+(i-1)\frac{6}{n})\right) \\
&= \frac{6}{n}\left(n + 3\sum_{i=1}^{n}(-1+(i-1)\frac{6}{n})\right) \\
&= \frac{6}{n}\left(n - 3\sum_{i=1}^{n} 1 + \frac{18}{n}\sum_{i=1}^{n}(i-1)\right) \\
&= \frac{6}{n}\left(n - 3n + \frac{18}{n}\sum_{i=1}^{n} i - \frac{18}{n}\sum_{i=1}^{n} 1\right) \\
&= \frac{6}{n}\left(-2n + \frac{\cancel{18}^{9}}{\cancel{n}} \cdot \frac{\cancel{n}(n+1)}{\cancel{2}} - \frac{18}{\cancel{n}} \cdot \cancel{n}\right) \\
&= -12 + 54 \cdot \frac{n+1}{n} - \frac{108}{n},
\end{aligned}$$

pa je

$$I = \lim_{n\to\infty}\left(-12 + 54\cdot \overset{1}{\frac{n+1}{n}} - \overset{0}{\frac{108}{n}}\right) = -12 + 54 + 0 = 42.$$

2. Izračunati $I = \int\limits_{-1}^{1} (1 - x^2)\,dx$.

Rešenje:

$$I = \int\limits_{-1}^{1} dx - \int\limits_{-1}^{1} x^2\,dx = x\Big|_{-1}^{1} - \frac{x^3}{3}\Big|_{-1}^{1} = 1 - (-1) - \frac{1}{3} - \frac{1}{3} = \frac{4}{3}.$$

3. Izračunati $I = \int\limits_{0}^{\frac{\pi}{4}} \cos^2 x\,dx$.

Rešenje:

$$I = \int\limits_0^{\frac{\pi}{4}} \frac{1+\cos 2x}{2}\,dx = \int\limits_0^{\frac{\pi}{4}} \frac{dx}{2} + \int\limits_0^{\frac{\pi}{4}} \frac{\cos 2x}{2}\,dx = \frac{1}{2}x\Big|_0^{\frac{\pi}{4}} + \frac{1}{2}\int_0^{\frac{\pi}{4}} \cos 2x\,dx$$

$$\left[\begin{array}{c} 2x = t \Rightarrow dx = \frac{1}{2}dt \\ x = 0 \Rightarrow t = 0 \\ x = \frac{\pi}{4} \Rightarrow t = \frac{\pi}{2} \end{array}\right] = \frac{\pi}{8} + \frac{1}{2}\int\limits_0^{\frac{\pi}{2}} \frac{1}{2}\cos t\,dt = \frac{\pi}{8} + \frac{1}{4}\sin t\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{8} + \frac{1}{4}.$$

4. Izračunati $I = \int\limits_{1}^{2} x\,e^{2x}\,dx$.

Rešenje: Integral se rešava primenom parcijalne integracije

$$\left[\begin{array}{c} u = x \Rightarrow du = dx \\ dv = e^{2x}\,dx \Rightarrow v = \frac{1}{2}e^{2x} \end{array}\right].$$

$$I = \frac{1}{2}x\,e^{2x}\Big|_1^2 - \frac{1}{2}\int\limits_1^2 e^{2x}\,dx = \left[\begin{array}{c} 2x = t \Rightarrow dx = \frac{1}{2}dt \\ x = 1 \Rightarrow t = 2 \\ x = 2 \Rightarrow t = 4 \end{array}\right] = \frac{1}{2}x\,e^{2x}\Big|_1^2 - \frac{1}{2}\int\limits_2^4 \frac{e^t}{2}\,dt$$

$$= \frac{1}{2}x\,e^{2x}\Big|_1^2 - \frac{1}{4}e^t\Big|_2^4 = \frac{1}{2}(2e^4 - e^2) - \frac{1}{4}(e^4 - e^2) = \frac{3}{4}e^4 - \frac{1}{4}e^2.$$

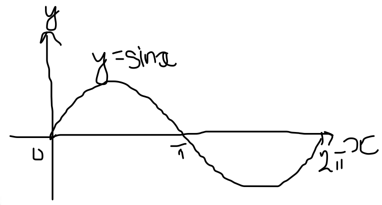
5. Izračunati $I = \int\limits_{-2}^{2} f(x)\,dx,\ f(x) = \begin{cases} 2, & x < 0 \\ 4 - x^2, & x \geq 0 \end{cases}$.

Rešenje: Iz $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} 2 = 2$ i $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+}(4 - x^2) = 4$ sledi da podintegralna funkcija nad $[-2, 2]$ ima prekid prve vrste, tj. skok.

Podintegralna funkcija je ograničena i ima jedan prekid na intervalu $[-2, 2]$, pa je data funkcija integrabilna na intervalu $[-2, 2]$, tj. postoji integral I. Dalje je

$$I = \int\limits_{-2}^{0} 2\,dx + \int\limits_{0}^{2} (4 - x^2)\,dx = 2\int\limits_{-2}^{0} dx + \int\limits_{0}^{2}(4 - x^2)\,dx = \frac{28}{3}.$$

6. Izračunati $\int\limits_0^{2\pi} \sin x dx$.



Rešenje: $\int\limits_0^{2\pi} \sin x dx = -\cos x\Big|_0^{2\pi} = -\cos(2\pi) + \cos 0 = -1 + 1 = 0.$

7. Izračunati $I = \int\limits_{-1}^{2} |x-1| dx$.

Rešenje: Iz $|x-1| = \begin{cases} x-1, & x \geq 1 \\ -x+1, & x < 1 \end{cases}$ sledi

$$I = \int\limits_{-1}^{1}(-x+1)dx + \int\limits_{1}^{2}(x-1)dx = -\frac{x^2}{2}\Big|_{-1}^{1} + x\Big|_{-1}^{1} + \frac{x^2}{2}\Big|_{1}^{2} - x\Big|_{1}^{2}$$
$$= -(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{2}}) + (1+1) + (2 - \tfrac{1}{2}) - (2-1) = 2 + \tfrac{3}{2} - 1 = \tfrac{5}{2}.$$

8. Odrediti $\lim\limits_{n\to\infty} a_n$ ako je $a_n = \frac{n}{n^2+1} + \frac{n}{n^2+2^2} + \ldots + \frac{n}{n^2+n^2}$.

Rešenje: Iz

$$a_n = \frac{n}{n^2+1} + \frac{n}{n^2+2^2} + \ldots + \frac{n}{n^2+n^2} = \frac{1}{n}\left(\frac{1}{1+\frac{1}{n^2}} + \frac{1}{1+\frac{2^2}{n^2}} + \ldots + \frac{1}{1+\frac{n^2}{n^2}}\right)$$
$$= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+\frac{i^2}{n^2}} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+(\frac{i}{n})^2}$$

$\Delta x_i = \frac{1}{n}$, $\xi_i = \frac{i}{n} = 0 + i \cdot \frac{1}{n}$

donja granica početnog intervala

sledi da je a_n gornja Darbuova suma funkcije $f(x) = \frac{1}{1+x^2}$ na intervalu $[0,1]$. Dakle,

$$\lim_{n\to\infty} a_n = \int\limits_0^1 \frac{1}{1+x^2}dx = \text{arctg}\, x\Big|_0^1 = \text{arctg}\, 1 - \text{arctg}\, 0 = \frac{\pi}{4}.$$

9. Odrediti $\lim\limits_{n\to\infty} a_n$ ako je $a_n = n^2\left(\frac{1}{(n+1)(n^2+1)} + \frac{1}{(n+2)(n^2+2^2)} + \ldots + \frac{1}{4n^3}\right)$.

Rešenje: Primetimo da je $4n^3 = (n+n)(n^2+n^2)$.

Iz

$$a_n = n^2\left(\frac{1}{(n+1)(n^2+1^2)} + \frac{1}{(n+2)(n^2+2^2)} + \ldots + \frac{1}{(n+n)(n^2+n^2)}\right)$$
$$= \sum_{i=1}^{n}\frac{n^2}{(n+i)(n^2+i^2)} = \sum_{i=1}^{n}\frac{\cancel{n^2}}{n\left(1+\frac{i}{n}\right)\cancel{n^2}\left(1+(\frac{i}{n})^2\right)} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\left(1+\frac{i}{n}\right)\left(1+(\frac{i}{n})^2\right)}$$

$\Delta x_i = \frac{1}{n}$

$\xi_i = \frac{i}{n} = 0 + i \cdot \frac{1}{n}$

sledi da je a_n Rimanova suma funkcije $f(x) = \frac{1}{(1+x)(1+x^2)}$ na intervalu $[0,1]$, pa je

$$\lim_{n\to\infty} a_n = \int\limits_0^1 \frac{1}{(1+x)(1+x^2)}dx.$$

Rastavljanjem na zbir parcijalnih razlomaka dobija se

$$\frac{1}{(1+x)(1+x^2)} = \frac{A}{1+x} + \frac{Bx+C}{1+x^2},$$

tj. $1 = A + Ax^2 + Bx + Bx^2 + C + Cx$,

odakle se dobija sistem jednačina $A+B=0, \quad B+C=0, \quad A+C=1$, čije rešenje je $A=\frac{1}{2}, B=-\frac{1}{2}, C=\frac{1}{2}$,

pa je

$$\lim_{n\to\infty} a_n = \int_0^1 \frac{1}{(1+x)(1+x^2)}dx = \frac{1}{2}\int_0^1 \frac{1}{1+x}dx + \frac{1}{2}\int_0^1 \frac{1-x}{1+x^2}dx = ...(\text{domaći})$$

10. Odrediti $\lim_{n\to\infty} a_n$ ako je

$$a_n = 2n\left(\frac{1}{(2+n)(2+2n)} + \frac{1}{(4+n)(4+2n)} + \frac{1}{(6+n)(6+2n)} + \ldots + \frac{1}{12n^2}\right).$$

Rešenje: Primetimo da je $12n^2 = (2n+n)(2n+2n)$.

Iz

$$a_n = 2n\left(\frac{1}{(2\cdot1+n)(2\cdot1+2n)} + \frac{1}{(2\cdot2+n)(2\cdot2+2n)} + \frac{1}{(2\cdot3+n)(2\cdot3+2n)} + \ldots + \frac{1}{(2n+n)(2n+2n)}\right)$$

$$= \sum_{i=1}^{n} \frac{2n}{(2i+n)(2i+2n)} = \sum_{i=1}^{n} \frac{2\cancel{n}}{n\left(1+\frac{2i}{n}\right)\cancel{n}\left(2+\frac{2i}{n}\right)} = \frac{2}{n}\sum_{i=1}^{n}\frac{1}{\left(1+\frac{2i}{n}\right)\left(2+\frac{2i}{n}\right)}$$

$\Delta x_i = \frac{2}{n}$

$\xi_i = \frac{2i}{n} = 0 + i\cdot\frac{2}{n}$

sledi da je a_n Rimanova suma funkcije $f(x) = \frac{1}{(1+x)(2+x)}$ na intervalu $[0,2]$, pa je

$$\lim_{n\to\infty} a_n = \int_0^2 \frac{1}{(1+x)(2+x)}dx = ...(\text{domaći})$$

1) $\Delta x_i = \frac{2}{n}$, $\xi_i = \frac{2i}{n} = 0 + i\cdot\frac{2}{n} \to f(x) = \frac{1}{(1+x)(2+x)}$ $[0,2]$

2) $\Delta x_i = \frac{2}{n}$, $\xi_i = 1 + \frac{2i}{n} = 1 + i\cdot\frac{2}{n} \to f(x) = \frac{1}{x(x+1)}$ $[1,3]$

3) $\Delta x_i = \frac{1}{n}$, $\xi_i = \frac{i}{n} = 0 + i\cdot\frac{1}{n} \to f(x) = \frac{2}{(1+2x)(2+2x)}$ $[0,1]$

$$a_n = \frac{1}{n}\sum_{i=1}^{n}\frac{2}{\left(1+\frac{2i}{n}\right)\left(2+\frac{2i}{n}\right)}$$