DIFERENCIJALNE JEDNAČINE VIŠEG REDA

Snižavanje reda diferencijalne jednačine

a) $y^{(n)}(x) = f(x)$ (direktna integracija)

1. Naći rešenje diferencijalne jednačine $y''\sin^4 x = \sin 2x$.

$$y'' = \frac{\sin 2x}{\sin^4 x} = \frac{2\sin x\cos x}{\sin^4 x} = 2\frac{\cos x}{\sin^3 x}$$

$$y' = \int y''dx = 2\int \frac{\cos x}{\sin^3 x}dx = \begin{pmatrix} \sin x = t \\ \cos x dx = dt \end{pmatrix} = 2\int \frac{dt}{t^3} = 2\int t^{-3}dt = -\frac{1}{t^2} + c_1 = -\frac{1}{\sin^2 x} + c_1$$

$$y = \int y'dx = -\int \frac{dx}{\sin^2 x} + c_1\int dx = ctgx + c_1 x + c_2$$

b) $F(x, y^{(k)}, y^{(k+1)}, \ldots, y^{(n)}) = 0$, $1 \le k \le n$ (diferencijalna jednačina koja ne sadrži y)
smena: $y^{(k)} = z$, $z = z(x)$

$$y^{(k+1)} = z',$$

$$y^{(k+2)} = z'', \text{ itd.}$$

2. Naći rešenje diferencijalne jednačine $xy''' + y'' = x^2$.

Uvedimo smenu $y'' = z$, $y''' = z'$.

$$xz' + z = x^2 \Rightarrow z' + \frac{1}{x}z = x$$ (linearna diferencijalna jednačina)

$$z = u \cdot v, \; z' = u'v + uv'$$

$$vu' + uv' + \frac{1}{x}uv = x \Rightarrow vu' + (v' + \frac{v}{x})u = x$$

$$v' + \frac{v}{x} = 0 \Rightarrow \frac{dv}{v} = -\frac{dx}{x} \Rightarrow \ln|v| = -\ln|x| \Rightarrow v = \frac{1}{x}$$

$$\frac{1}{x}u' = x \Rightarrow u' = x^2 \Rightarrow du = x^2dx \Rightarrow u = \frac{x^3}{3} + c_1$$

$$z = uv = \frac{x^2}{3} + \frac{c_1}{x} \Rightarrow y'' = z = \frac{x^2}{3} + \frac{c_1}{x}$$

$$y' = \int y'' dx = \int \left(\frac{x^2}{3} + \frac{c_1}{x}\right) dx = \frac{x^3}{9} + c_1 \ln|x| + c_2$$

$$y = \int y' dx = \int \left(\frac{x^3}{9} + c_1 \ln|x| + c_2\right) dx = \frac{x^4}{36} + c_1 (x \ln|x| - x) + c_2 x + c_3$$

c) $F(y, y', \ldots, y^{(n)}) = 0$, $n \geq 1$, (diferencijalna jednačina koja ne sadrži x)

$y' = z$, $z = z(y)$,

$$y'' = \frac{dy'}{dx} = \frac{dy'}{dy} \cdot \frac{dy}{dx} = \frac{dz}{dy} \cdot y' = z' \cdot z,$$

$$y''' = \frac{dy''}{dx} = \frac{dy''}{dy} \cdot \frac{dy}{dx} = \frac{d(z' \cdot z)}{dy} \cdot y' = (z \cdot z'' + (z')^2) \cdot z = z^2 z'' + z \cdot (z')^2, \text{ itd.}$$

3. Naći rešenje diferencijalne jednačine $3yy'' - 5(y')^2 = 0$.

   Uvedimo smenu $y' = z$, $y'' = zz'$.

$$3yzz' - 5z^2 = 0 \Rightarrow 3yz' - 5z = 0 \Rightarrow z' = \frac{5z}{3y}$$

$$\frac{z'}{z} = \frac{5}{3y} \Rightarrow \frac{dz}{z} = \frac{5dy}{3y} \Rightarrow \int \frac{dz}{z} = \frac{5}{3} \int \frac{dy}{y} \Rightarrow \ln|z| = \frac{5}{3} \ln|y| + c \Rightarrow z = c_1 \sqrt[3]{y^5}, \ c = \ln|c_1|$$

$$y' = z = c_1 y^{\frac{5}{3}} \Rightarrow y^{-\frac{5}{3}} dy = c_1 dx$$

$$\int y^{-\frac{5}{3}} dy = c_1 \int dx \Rightarrow -\frac{3}{2} y^{-\frac{2}{3}} = c_1 x + c_2 \Rightarrow \sqrt[3]{y^2}(c_1 x + c_2) = -\frac{3}{2}$$

(Ne)homogena linearna diferencijalna jednačina

Homogena diferencijalna jednačina je jednačina oblika

$$a_n(x) y^{(n)} + a_{n-1}(x) y^{(n-1)} + \ldots + a_1(x) y' + a_0(x) y = 0,$$

gde su $a_0(x), a_1(x), \ldots, a_n(x)$ neke neprekidne funkcije i $a_n(x) \neq 0$.

Ako je poznato jedno partikularno rešenje $y_1(x)$ jednačine $y'' + a_1(x) y' + a_0(x) y = 0$, tada se smenom $y = z \cdot y_1$, gde je $z = z(x)$, dobija jednačina kojoj se može sniziti red (ne sadrži z).

Ako znamo dva partikularna rešenja $y_1(x)$ i $y_2(x)$ nehomogene jednačine $y'' + a_1(x) y' + a_0(x) y = f(x)$, tada je funkcija $y_3(x) = y_2(x) - y_1(x)$ jedno partikularno rešenje homogenog dela jednačine, pa se taj deo jednačine rešava smenom $y_h = y_3 z$.

Opšte rešenje polazne nehomogene jednačine je $y(x) = y_h(x) + y_1(x)$ ili $y(x) = y_h(x) + y_2(x)$.

4. Naći opšte rešenje diferencijalne jednačine $y'' + \frac{x}{1-x}y' - \frac{1}{1-x}y = 0$, ako se zna da je njeno partikularno rešenje oblika e^x.

Uvedimo smenu $y = z \cdot e^x$, $y' = (z' + z)e^x$, $y'' = (z' + z + z'' + z')e^x = (z'' + 2z' + z)e^x$.

$$(z'' + 2z' + z)e^x + \frac{x}{1-x}(z' + z)e^x - \frac{1}{1-x}z \cdot e^x = 0$$

$$z'' + (2 + \frac{x}{1-x})z' + (1 + \frac{x}{1-x} - \frac{1}{1-x})z = 0 \;\Rightarrow\; z'' + (2 + \frac{x}{1-x})z' = 0 \text{ (ne sadrži } z\text{)}$$

Da bi snizili red uvedimo smenu $z' = u,\ z'' = u'$.

$$u' = -(2 + \frac{x}{1-x})u = -\frac{2 - 2x + x}{1-x}u = -\frac{2-x}{1-x}u = -\frac{1+1-x}{1-x}u = (\frac{1}{x-1} - 1)u$$

$$\frac{du}{u} = (\frac{1}{x-1} - 1)dx \;\Rightarrow\; \int \frac{du}{u} = \int (\frac{1}{x-1} - 1)dx$$

$$\ln|u| = \ln|x-1| - x + c \;\Rightarrow\; u = e^{c - x + \ln|x-1|} = c_1(x-1) \cdot e^{-x},\ c_1 = e^c$$

$$z' = u = c_1(x-1) \cdot e^{-x}$$

$$z = \int z'dx = c_1\int (x-1) \cdot e^{-x}dx = c_1\int xe^{-x}dx - c_1\int e^{-x}dx = \begin{pmatrix} u = x & dv = e^{-x}dx \\ du = dx & v = -e^{-x} \end{pmatrix} =$$

$$= c_1(-xe^{-x} + \int e^{-x}dx) + c_1e^{-x} = c_1(-xe^{-x} - e^{-x}) + c_1e^{-x} + c_2$$

$$= -c_1(x \cdot e^{-x} + e^{-x} - e^{-x}) + c_2 = -c_1xe^{-x} + c_2$$

$$y = z \cdot e^x = (-c_1x \cdot e^{-x} + c_2) \cdot e^x = -c_1x + c_2e^x$$

5. Naći opšte rešenje jednačine $(3x^3 + x)y'' + 2y' - 6xy = 4 - 12x^2$ ako su $y_1 = ax + b$ i $y_2 = Ax^2 + Bx + C$ njena dva partikularna rešenja.

$$y_1 = ax + b,\ y_1' = a,\ y_1'' = 0$$

$$2a - 6x(ax + b) = 4 - 12x^2 \;\Rightarrow\; -6ax^2 - 6bx + 2a = 4 - 12x^2$$

$$-6a = -12, \qquad -6b = 0, \qquad 2a = 4$$

Iz sistema jednačina dobija se $a = 2$ i $b = 0 \;\Rightarrow\; y_1 = 2x$.

$$y_2 = Ax^2 + Bx + C,\ y_2' = 2Ax + B,\ y_2'' = 2A$$

$$(3x^3 + x) \cdot 2A + 4Ax + 2B - 6x \cdot (Ax^2 + Bx + C) = 4 - 12x^2$$

$$-6Bx^2+(6A-6C)\cdot x+2B=4-12x^2$$

$-6B=-12$, $\quad 6A-6C=0$, $\quad 2B=4$.

Iz sistema jednačina se dobija $B=2$ i $A=C=1 \Rightarrow y_2=x^2+2x+1$.

$y_3=y_2-y_1=x^2+1$ je partikularno rešenje homogenog dela jednačine.

Smena: $\quad y_h=z\cdot(x^2+1)$,

$$y_h'=z'\cdot(x^2+1)+2xz$$

$$y_h''=z''(x^2+1)+2xz'+2z+2xz'=(x^2+1)\cdot z''+4xz'+2z$$

$$(3x^3+x)(x^2+1)z''+(3x^3+x)4xz'+2(3x^3+x)z+2(x^2+1)z'+4xz-6x(x^2+1)z=0$$

$x\cdot(3x^2+1)(x^2+1)\cdot z''+(12x^4+6x^2+2)\cdot z'=0$ (ne sadrži z)

Smena: $\quad z'=p$, $z''=p'$, $p=p(x)$

$$x\cdot(3x^2+1)(x^2+1)\cdot p'+(12x^4+6x^2+2)\cdot p=0$$

$$\frac{dp}{p}=-\frac{12x^4+6x^2+2}{x\cdot(3x^4+4x^2+1)}dx=-2\cdot\frac{3x^4+4x^2+1-x^2+3x^4}{x\cdot(3x^4+4x^2+1)}dx=$$

$$=-2\cdot\frac{dx}{x}+2\cdot\frac{x-3x^3}{3x^4+4x^2+1}=-2\cdot\frac{dx}{x}+\frac{2x-6x^3}{(x^2+1)(3x^2+1)}dx$$

$$\frac{2x-6x^3}{(x^2+1)(3x^2+1)}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{3x^2+1} \Rightarrow A=-4,\ B=D=0 \text{ i } C=6$$

$$\frac{dp}{p}=-2\cdot\frac{dx}{x}-2\cdot\frac{2x}{x^2+1}dx+\frac{6x}{3x^2+1}dx$$

$$\ln|p|=-2\ln|x|-2\ln\left|x^2+1\right|+\ln\left|3x^2+1\right|+c \Rightarrow p=c_1\frac{3x^2+1}{x^2(x^2+1)^2},\ c=\ln|c_1|$$

$$z'=p\Rightarrow z=\int p\,dx=c_1\int\frac{x^2+1+2x^2}{x^2(x^2+1)^2}dx=c_1\int\frac{dx}{x^2(x^2+1)}+c_1\cdot\int\frac{2dx}{(x^2+1)^2}$$

$$=c_1\int\frac{dx}{x^2}-c_1\int\frac{dx}{x^2+1}+2c_1\int\frac{dx}{(x^2+1)^2}$$

$$=-c_1\frac{1}{x}-c_1 arctgx+2c_1\frac{1}{2}arctgx+2c_1\frac{x}{2(x^2+1)}+c_2$$

$z = c_1 \cdot \dfrac{1}{x(x^2+1)} + c_2 \quad \Rightarrow \quad y_h = z \cdot (x^2+1) = \dfrac{c_1}{x} + c_2 \cdot (x^2+1)$ je rešenje homogenog dela jednačine, pa je $y = y_h + y_1 = \dfrac{c_1}{x} + c_2 \cdot (x^2+1) + 2x$ rešenje polazne jednačine.

6. Naći opšte rešenje jednačine $(1-x^2)y'' + 2y = 2$ ako su $y_1 = 1$ i $y_2 = x^2$ njena dva partikularna rešenja.

$y_3 = y_2 - y_1 = x^2 - 1$ je partikularno rešenje homogenog dela.

$y_h = z(x^2-1)$, $y_h' = z'(x^2-1) + 2xz$

$y_h'' = z''(x^2-1) + 2xz' + 2z + 2xz' = (x^2-1)z'' + 4xz' + 2z$

$(1-x^2) \cdot \left[(x^2-1)z'' + 4xz' + 2z \right] + 2(x^2-1)z = 0$

$(x^2-1)^2 z'' + 4x(x^2-1)z' + 2(x^2-1-x^2+1)z = 0$

$(x^2-1)z'' + 4xz' = 0 \;\Rightarrow\; z'' + \dfrac{4x}{x^2-1} z' = 0$ (ne sadrži z)

Uvedimo smenu $z' = u,\; z'' = u'$.

$$u' = -\frac{4x}{x^2-1} u \;\Rightarrow\; \frac{du}{u} = -\frac{4x}{x^2-1} dx \;\Rightarrow\; \int \frac{du}{u} = -2\int \frac{2x\,dx}{x^2-1}$$

$$\ln|u| = -2\ln|x^2-1| + c = \ln\left|\frac{c_1}{(x^2-1)^2}\right|,\; c = \ln|c_1| \;\Rightarrow\; u = \frac{c_1}{(x^2-1)^2}$$

$$z' = u = \frac{c_1}{(x^2-1)^2} \;\Rightarrow\; z = \int \frac{c_1}{(x^2-1)^2} dx = -c_1 \int \frac{x^2-1-x^2}{(x^2-1)^2} dx = -c_1 \int \frac{dx}{x^2-1} + c_1 \int \frac{x^2 dx}{(x^2-1)^2} =$$

$$= (u = x \Rightarrow du = dx,\; dv = \frac{x\,dx}{(x^2-1)^2} \Rightarrow v = -\frac{1}{2} \cdot \frac{1}{x^2-1}) =$$

$$= c_1(-\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| - \frac{1}{2} \cdot \frac{x}{x^2-1} + \frac{1}{2}\int \frac{dx}{x^2-1}) = c_1(-\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| - \frac{x}{2(x^2-1)} + \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|) + c_2 =$$

$$= -\frac{c_1}{4}\ln\left|\frac{x-1}{x+1}\right| - \frac{c_1}{2} \cdot \frac{x}{x^2-1} + c_2 = c_3(\ln\left|\frac{x-1}{x+1}\right| + \frac{2x}{x^2-1}) + c_2,\; c_3 = -\frac{c_1}{4}$$

$$y_h = (x^2-1) \cdot z = c_3(x^2-1)(\ln\left|\frac{x-1}{x+1}\right| + \frac{2x}{x^2-1}) + c_2(x^2-1)$$

$$y = y_h + y_1 = c_3(x^2-1)(\ln\left|\frac{x-1}{x+1}\right| + \frac{2x}{x^2-1}) + c_2(x^2-1) + 1$$

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