# DIFERENCIJALNE JEDNAČINE VIŠEG REDA

## Sniživanje reda diferencijalne jednačine

a)  $y^{(n)}(x) = f(x)$ (direktna integracija)

1.  Naći rešenje diferencijalne jednačine $y'' \sin^4 x = \sin 2x$.

$$y'' = \frac{\sin 2x}{\sin^4 x} = \frac{2\sin x \cos x}{\sin^4 x} = 2\frac{\cos x}{\sin^3 x}$$

$$y' = \int y'' dx = 2\int \frac{\cos x}{\sin^3 x} dx = \left( \begin{array}{c} \sin x = t \\ \cos x\, dx = dt \end{array} \right) = 2\int \frac{dt}{t^3} = 2\int t^{-3} dt = -\frac{1}{t^2} + c_1 = -\frac{1}{\sin^2 x} + c_1$$

$$y = \int y' dx = -\int \frac{dx}{\sin^2 x} + c_1 \int dx = \operatorname{ctg} x + c_1 x + c_2$$

b)  $F(x, y^{(k)}, y^{(k+1)}, \dots, y^{(n)}) = 0$, $1 \le k \le n$ (diferencijalna jednačina koja ne sadrži $y$)
    smena: $y^{(k)} = z$, $z = z(x)$

$$y^{(k+1)} = z',$$

$$y^{(k+2)} = z'', \text{ itd.}$$

2.  Naći rešenje diferencijalne jednačine $xy''' + y'' = x^2$.

Uvedimo smenu $y'' = z$, $y''' = z'$, $z = z(x)$

$$xz' + z = x^2 \Big|:x \Rightarrow z' + \frac{1}{x}z = x \text{ (linearna diferencijalna jednačina)}$$

$$z = u \cdot v, \ z' = u'v + uv' \ , \ u = u(x), \ v = v(x)$$

$$vu' + uv' + \frac{1}{x}uv = x \Rightarrow vu' + \left( v' + \frac{v}{x} \right)u = x \qquad = 0 \quad \text{biramo } v$$

$$v' + \frac{v}{x} = 0 \Rightarrow \int \frac{dv}{v} = -\int \frac{dx}{x} \Rightarrow \ln|v| = -\ln|x| \Rightarrow v = \frac{1}{x}$$

$$\frac{1}{x}u' = x \Rightarrow u' = x^2 \Rightarrow \int du = \int x^2 dx \Rightarrow u = \frac{x^3}{3} + c_1$$

$$z = uv = \frac{x^2}{3} + \frac{c_1}{x} \Rightarrow y'' = z = \frac{x^2}{3} + \frac{c_1}{x}$$

$$y' = \int y'' dx = \int (\frac{x^2}{3} + \frac{c_1}{x}) dx = \frac{x^3}{9} + c_1 \ln|x| + c_2$$

parcijalnom integracijom

$$y = \int y' dx = \int (\frac{x^3}{9} + c_1 \ln|x| + c_2) dx = \frac{x^4}{36} + c_1(x\ln|x| - x) + c_2 x + c_3$$

c)  $F(y, y', ..., y^{(n)}) = 0$, $n \geq 1$, (diferencijalna jednačina koja ne sadrži $x$)

$y' = z$, $z = z(y)$,

$$y'' = \frac{dy'}{dx} = \frac{dy'}{dy} \cdot \frac{dy}{dx} = \frac{dz}{dy} \cdot y' = z' \cdot z,$$

$$y''' = \frac{dy''}{dx} = \frac{dy''}{dy} \cdot \frac{dy}{dx} = \frac{d(z' \cdot z)}{dy} \cdot y' = (z \cdot z'' + (z')^2) \cdot z = z^2 z'' + z \cdot (z')^2, \text{ itd.}$$

3.  Naći rešenje diferencijalne jednačine $3yy'' - 5(y')^2 = 0$.

Uvedimo smenu $y' = z$, $y'' = zz'$, $z = z(y)$

$$3yzz' - 5z^2 = 0 \overset{|: \frac{1}{z}}{\Rightarrow} 3yz' - 5z = 0 \Rightarrow z' = \frac{5z}{3y}$$

$$\frac{z'}{z} = \frac{5}{3y} \Rightarrow \frac{dz}{z} = \frac{5dy}{3y} \Rightarrow \int \frac{dz}{z} = \frac{5}{3} \int \frac{dy}{y} \Rightarrow \ln|z| = \frac{5}{3}\ln|y| + c \Rightarrow z = c_1 \sqrt[3]{y^5}, \ c = \ln|c_1|$$

$$y' = z = c_1 y^{\frac{5}{3}} \Rightarrow y^{-\frac{5}{3}} dy = c_1 dx$$

$$\int y^{-\frac{5}{3}} dy = c_1 \int dx \Rightarrow -\frac{3}{2} y^{-\frac{2}{3}} = c_1 x + c_2 \Rightarrow \boxed{\sqrt[3]{y^2}(c_1 x + c_2) = -\frac{3}{2}} \quad \text{Opšte rešenje}$$

## (Ne)homogena linearna diferencijalna jednačina

Homogena diferencijalna jednačina je jednačina oblika

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_1(x)y' + a_0(x)y = 0,$$

gde su $a_0(x), a_1(x), ..., a_n(x)$ neke neprekidne funkcije i $a_n(x) \neq 0$.

Ako je poznato jedno partikularno rešenje $y_1(x)$ jednačine $y'' + a_1(x)y' + a_0(x)y = 0$, tada se smenom $y = z \cdot y_1$, gde je $z = z(x)$, dobija jednačina kojoj se može sniziti red (ne sadrži $z$).

Ako znamo dva partikularna rešenja $y_1(x)$ i $y_2(x)$ nehomogene jednačine $y'' + a_1(x)y' + a_0(x)y = f(x)$, tada je funkcija $y_3(x) = y_2(x) - y_1(x)$ jedno partikularno rešenje homogenog dela jednačine, pa se taj deo jednačine rešava smenom $y_h = y_3 z$.

Opšte rešenje polazne nehomogene jednačine je $y(x) = y_h(x) + y_1(x)$ ili $y(x) = y_h(x) + y_2(x)$.

4. Naći opšte rešenje diferencijalne jednačine $y'' + \dfrac{x}{1-x} y' - \dfrac{1}{1-x} y = 0$, ako se zna da je njeno partikularno rešenje oblika $e^x$.

Uvedimo smenu $y = z \cdot e^x$, $y' = (z' + z)e^x$, $y'' = (z' + z + z'' + z')e^x = (z'' + 2z' + z)e^x$.

$z = z(x)$

$(z'' + 2z' + z)e^x + \dfrac{x}{1-x}(z' + z)e^x - \dfrac{1}{1-x} z \cdot e^x = 0 \quad | \cdot e^x$

mora nestati

$z'' + (2 + \dfrac{x}{1-x})z' + (1 + \dfrac{x}{1-x} - \dfrac{1}{1-x})z = 0 \implies z'' + (2 + \dfrac{x}{1-x})z' = 0$ (ne sadrži $z$)

Da bi snizili red uvedimo smenu $z' = u$, $z'' = u'$. $u = u(x)$

$u' = -(2 + \dfrac{x}{1-x})u = -\dfrac{2 - 2x + x}{1-x}u = -\dfrac{2-x}{1-x}u = -\dfrac{1 + 1 - x}{1-x}u = (\dfrac{1}{x-1} - 1)u$

$\dfrac{du}{u} = (\dfrac{1}{x-1} - 1)dx \implies \int \dfrac{du}{u} = \int (\dfrac{1}{x-1} - 1)dx$

$\ln|u| = \ln|x-1| - x + c \implies u = e^{c - x + \ln|x-1|} = c_1(x-1)\cdot e^{-x}$, $c_1 = e^c$

$z' = u = c_1(x-1)\cdot e^{-x}$

$z = \int z'dx = c_1 \int (x-1)\cdot e^{-x}dx = c_1 \int xe^{-x}dx - c_1 \int e^{-x}dx = \begin{pmatrix} u_1 = x & dv_1 = e^{-x}dx \\ du_1 = dx & v_1 = -e^{-x} \end{pmatrix} =$

$= c_1(-xe^{-x} + \int e^{-x}dx) + c_1 e^{-x} = c_1(-xe^{-x} - e^{-x}) + c_1 e^{-x} + c_2$

$= -c_1(x\cdot e^{-x} + e^{-x} - e^{-x}) + c_2 = -c_1 xe^{-x} + c_2$

$y = z \cdot e^x = (-c_1 x \cdot e^{-x} + c_2)\cdot e^x = -c_1 x + c_2 e^x$    Opšte rešenje

5. Naći opšte rešenje jednačine $(3x^3 + x)y'' + 2y' - 6xy = 4 - 12x^2$ ako su $y_1 = ax + b$ i $y_2 = Ax^2 + Bx + C$ njena dva partikularna rešenja.

$y_1 = ax + b$, $y_1' = a$, $y_1'' = 0$

$2a - 6x(ax + b) = 4 - 12x^2 \implies -6ax^2 - 6bx + 2a = 4 - 12x^2$

$-6a = -12$,      $-6b = 0$,      $2a = 4$

Iz sistema jednačina dobija se $a = 2$ i $b = 0 \implies y_1 = 2x$.

$y_2 = Ax^2 + Bx + C$, $y_2' = 2Ax + B$, $y_2'' = 2A$

$(3x^3 + x)\cdot 2A + 4Ax + 2B - 6x \cdot (Ax^2 + Bx + C) = 4 - 12x^2$

$$-6Bx^2 + (6A - 6C) \cdot x + 2B = 4 - 12x^2$$

$$-6B = -12, \qquad 6A - 6C = 0, \qquad 2B = 4. \qquad \begin{aligned} A &= C \\ B &= 2 \end{aligned}$$

Iz sistema jednačina se dobija $B = 2$ i $A = C = 1 \Rightarrow y_2 = x^2 + 2x + 1$.

$y_3 = y_2 - y_1 = x^2 + 1$ je partikularno rešenje homogenog dela jednačine.
$\qquad\qquad\qquad\qquad\qquad$ birali

Smena: $\qquad y_h = z \cdot (x^2 + 1), \qquad z = z(x)$

$$y_h' = z' \cdot (x^2 + 1) + 2xz$$

$$y_h'' = z''(x^2 + 1) + 2xz' + 2z + 2xz' = (x^2 + 1) \cdot z'' + 4xz' + 2z$$

$$(3x^3 + x)(x^2 + 1)z'' + (3x^3 + x)4xz' + 2(3x^3 + x)z + 2(x^2 + 1)z' + 4xz - 6x(x^2 + 1)z = 0$$

$$x \cdot (3x^2 + 1)(x^2 + 1) \cdot z'' + (12x^4 + 6x^2 + 2) \cdot z' = 0 \quad \text{(ne sadrži } z\text{)}$$

Smena: $\qquad z' = p, \; z'' = p', \; p = p(x)$

$$x \cdot (3x^2 + 1)(x^2 + 1) \cdot p' + (12x^4 + 6x^2 + 2) \cdot p = 0$$

$$\frac{dp}{p} = -\frac{12x^4 + 6x^2 + 2}{x \cdot (3x^4 + 4x^2 + 1)} dx = -2 \cdot \frac{3x^4 + 4x^2 + 1 - x^2 + 3x^4}{x \cdot (3x^4 + 4x^2 + 1)} dx =$$

$$= -2 \cdot \frac{dx}{x} + 2 \cdot \frac{x - 3x^3}{3x^4 + 4x^2 + 1} = -2 \cdot \frac{dx}{x} + \frac{2x - 6x^3}{(x^2 + 1)(3x^2 + 1)} dx$$

$$\frac{2x - 6x^3}{(x^2 + 1)(3x^2 + 1)} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{3x^2 + 1} \Rightarrow A = -4, \; B = D = 0 \text{ i } C = 6$$

$$\frac{dp}{p} = -2 \cdot \frac{dx}{x} - 2 \cdot \frac{2x}{x^2 + 1} dx + \frac{6x}{3x^2 + 1} dx$$

$$\ln|p| = -2\ln|x| - 2\ln|x^2 + 1| + \ln|3x^2 + 1| + c \Rightarrow p = c_1 \frac{3x^2 + 1}{x^2(x^2 + 1)^2}, \; c = \ln|c_1|$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x^2 + 1) - x^2$

$$z' = p \Rightarrow z = \int p\, dx = c_1 \int \frac{x^2 + 1 + 2x^2}{x^2(x^2 + 1)^2} dx = c_1 \int \frac{dx}{x^2(x^2 + 1)} + c_1 \cdot \int \frac{2\,dx}{(x^2 + 1)^2}$$

$$= c_1 \int \frac{dx}{x^2} - c_1 \int \frac{dx}{x^2 + 1} + 2c_1 \int \frac{dx}{(x^2 + 1)^2} \quad \text{Ilijin integral}$$

$$= -c_1 \frac{1}{x} - c_1 \operatorname{arctg} x + 2c_1 \frac{1}{2} \operatorname{arctg} x + 2c_1 \frac{x}{2(x^2 + 1)} + c_2$$

$z = c_1 \cdot \dfrac{1}{x(x^2+1)} + c_2 \quad \Rightarrow \quad y_h = z \cdot (x^2+1) = \dfrac{c_1}{x} + c_2 \cdot (x^2+1)$ je rešenje homogenog dela

jednačine, pa je $\boxed{y = y_h + y_1 = \dfrac{c_1}{x} + c_2 \cdot (x^2+1) + 2x}$ rešenje polazne jednačine.

6. Naći opšte rešenje jednačine $(1-x^2)y'' + 2y = 2$ ako su $y_1 = 1$ i $y_2 = x^2$ njena dva partikularna rešenja.

$y_3 = y_2 - y_1 = x^2 - 1$ je partikularno rešenje homogenog dela.

$y_h = z(x^2-1), \quad y_h' = z'(x^2-1) + 2xz \qquad\qquad z = z(x)$

$y_h'' = z''(x^2-1) + 2xz' + 2z + 2xz' = (x^2-1)z'' + 4xz' + 2z$

$(1-x^2) \cdot \left[ (x^2-1)z'' + 4xz' + 2z \right] + 2(x^2-1)z = 0 \quad / (-1)$

$\left(\frac{1}{x^2-1}\right) /\ (x^2-1)^2 z'' + 4x(x^2-1)z' + 2(\underbrace{x^2-1-x^2+1}_{0})z = 0 \qquad$ z mora nestati

$(x^2-1) /\ (x^2-1)z'' + 4xz' = 0 \Rightarrow z'' + \dfrac{4x}{x^2-1} z' = 0$ (ne sadrži $z$)

Uvedimo smenu $z' = u, \ z'' = u', \quad u = u(x)$

$u' = -\dfrac{4x}{x^2-1} u \Rightarrow \dfrac{du}{u} = -\dfrac{4x}{x^2-1} dx \Rightarrow \int \dfrac{du}{u} = -2\int \dfrac{2x\,dx}{x^2-1}$

$\ln|u| = -2\ln|x^2-1| + c = \ln\left|\dfrac{c_1}{(x^2-1)^2}\right|, \quad c = \ln|c_1| \Rightarrow u = \dfrac{c_1}{(x^2-1)^2}$

$z' = u = \dfrac{c_1}{(x^2-1)^2} \Rightarrow z = \int \dfrac{c_1}{(x^2-1)^2} dx = -c_1 \int \dfrac{x^2-1-x^2}{(x^2-1)^2} dx = -c_1 \int \dfrac{dx}{x^2-1} + c_1 \int \dfrac{x^2 dx}{(x^2-1)^2} =$

$= \left(u_1 = x \Rightarrow du_1 = dx, \ dv_1 = \dfrac{x\,dx}{(x^2-1)^2} \Rightarrow v_1 = -\dfrac{1}{2} \cdot \dfrac{1}{x^2-1}\right) =$

$= c_1\left(-\dfrac{1}{2}\ln\left|\dfrac{x-1}{x+1}\right| - \dfrac{1}{2}\cdot\dfrac{x}{x^2-1} + \dfrac{1}{2}\int\dfrac{dx}{x^2-1}\right) = c_1\left(-\dfrac{1}{2}\ln\left|\dfrac{x-1}{x+1}\right| - \dfrac{x}{2(x^2-1)} + \dfrac{1}{4}\ln\left|\dfrac{x-1}{x+1}\right|\right) + c_2 =$

$= -\dfrac{c_1}{4}\ln\left|\dfrac{x-1}{x+1}\right| - \dfrac{c_1}{2}\cdot\dfrac{x}{x^2-1} + c_2 = c_3\left(\ln\left|\dfrac{x-1}{x+1}\right| + \dfrac{2x}{x^2-1}\right) + c_2, \ c_3 = -\dfrac{c_1}{4}$

$y_h = (x^2-1) \cdot z = c_3(x^2-1)\left(\ln\left|\dfrac{x-1}{x+1}\right| + \dfrac{2x}{x^2-1}\right) + c_2(x^2-1)$

$\boxed{y = y_h + y_1 = c_3(x^2-1)\left(\ln\left|\dfrac{x-1}{x+1}\right| + \dfrac{2x}{x^2-1}\right) + c_2(x^2-1) + 1}$

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