Ojlerova diferencijalna jednačina

$(ax+b)^n y^{(n)} + A_{n-1}(ax+b)^{n-1} y^{(n-1)} + \ldots + A_1(ax+b)y' + A_0 y = f(x)$

$a, b, A_0, A_1, \ldots, A_{n-1}$ − konstante,

Ako je $ax+b>0,\ a\neq 0$, smenom $ax+b=e^t \Rightarrow t=\ln(ax+b)$, tj.

$$y' = \frac{dy}{dt}\cdot\frac{dt}{dx} = y_t'\cdot\frac{a}{ax+b} = ae^{-t}y_t',$$

$$y'' = \frac{dy'}{dt}\cdot\frac{dt}{dx} = a(e^{-t}y_t'' - e^{-t}y_t')\cdot\frac{a}{ax+b} = a^2e^{-2t}(y_t''-y_t'),$$

$$y''' = \frac{dy''}{dt}\cdot\frac{dt}{dx} = a^2(-2e^{-2t}(y_t''-y_t') + e^{-2t}(y_t'''-y_t''))\cdot ae^{-t} = a^3e^{-3t}(y_t'''-3y_t''+2y_t'),\ \text{itd.}$$

data jednačina se svodi na jednačinu sa konstantnim koeficijentima.

Za $ax+b<0,\ a\neq 0$ uvodi se smena $ax+b=-e^t$.

Za $a=0,\ b\neq 0$ dobija se nehomogena linearna jednačina čiji je homogeni deo sa konstantnim koeficijentima. Za $a=0$ i $b=0$ dobija se $A_0\cdot y = f(x)$, a to nije diferencijalna jednačina.

1. Rešiti diferencijalnu jednačinu $(1+x)^3y''' + (1+x)y' - y = (1+x)^2$ za $x>-1$.

   $1+x=e^t \Rightarrow t=\ln(1+x)$

   $y' = e^{-t}y_t',\qquad y'' = (y_t''-y_t')e^{-2t},\qquad y''' = (y_t'''-3y_t''+2y_t')\cdot e^{-3t}$

   $e^{3t}\cdot e^{-3t}(y_t'''-3y_t''+2y_t') + e^t\cdot e^{-t}y_t' - y = e^{2t}$

   $y_t'''-3y_t''+3y_t'-y=e^{2t}$

- $y_t'''-3y_t''+3y_t'-y=0 \Rightarrow r^3-3r^2+3r-1=0 \Rightarrow (r-1)^3=0 \Rightarrow r_1=r_2=r_3=1$

  $y_h = c_1e^t + c_2te^t + c_3t^2e^t$

- $y_t'''-3y_t''+3y_t'-y=e^{2t}$

  $e^{2t} = e^{\alpha t}\left[P_m(t)\cos\beta t + Q_n(t)\sin\beta t\right] \Rightarrow \alpha=2,\ \beta=0,\ P_m(t)=1$

  $\alpha+\beta i = 2 \Rightarrow r=0,\ k=m=0$

  $y_p = A\cdot e^{2t},\qquad y_p' = 2A\cdot e^{2t},\qquad y_p'' = 4A\cdot e^{2t},\qquad y_p''' = 8A\cdot e^{2t}$

  $(8A-12A+6A-A)e^{2t} = e^{2t} \Rightarrow A=1$

  $y_p = e^{2t}$

$y = c_1 e^t + c_2 t e^t + c_3 t^2 e^t + e^{2t}$

$y = c_1(1+x) + c_2(1+x)\cdot\ln(1+x) + c_3(1+x)\cdot\ln^2(1+x) + (1+x)^2$

Neke metode rešavanja diferencijalnih jednačina

2. Naći dva puta diferencijabilnu funkciju $z = f(x^2+y^2)$, nad oblašću $R^2 \setminus \{(0,0)\}$ koja zadovoljava diferencijalnu jednačinu

$$\frac{\partial^2 z}{\partial y^2} + \frac{y^2 - x^2}{x(x^2+y^2)}\cdot\frac{\partial z}{\partial x} + \frac{2y^2}{(x^2+y^2)^2} z = \frac{2y^2}{x^2+y^2}\ln(x^2+y^2).$$

$x^2 + y^2 = t,\ z = f(t)$

$$\frac{\partial z}{\partial x} = f_t'\cdot 2x, \qquad \frac{\partial z}{\partial y} = f_t'\cdot 2y, \qquad \frac{\partial^2 z}{\partial y^2} = 2(f_t''\cdot y\cdot 2y + f_t') = 2f_t' + 4y^2 f_t''$$

$$2f_t' + 4y^2 f_t'' + \frac{y^2 - x^2}{x(x^2+y^2)}2x\cdot f_t' + \frac{2y^2}{(x^2+y^2)^2} f = \frac{2y^2}{x^2+y^2}\ln(x^2+y^2)$$

$$4y^2 f_t'' + 2\cdot\frac{y^2 - x^2 + x^2 + y^2}{x^2+y^2}\cdot f_t' + \frac{2y^2}{(x^2+y^2)^2} f = \frac{2y^2}{x^2+y^2}\ln(x^2+y^2)$$

$$4f_t'' + \frac{4f_t'}{t} + \frac{2f_t}{t^2} = \frac{2}{t}\ln(t)$$

$2t^2 f_t'' + 2t f_t' + f_t = t\ln t$ (Ojlerova diferencijalna jednačina)

$t = e^s,\ s = \ln t,\ t > 0, \qquad f_t' = e^{-s} f_s', \qquad f_t'' = e^{-2s}(f_s'' - f_s')$

$2e^{2s}\cdot e^{-2s}(f_s'' - f_s') + 2e^s e^{-s} f_s' + f_s = s e^s$

$2f_s'' + f_s = s e^s$

- $2f_s'' + f_s = 0$

  $2r^2 + 1 = 0 \Leftrightarrow r_1 = \frac{i}{\sqrt{2}},\ r_2 = -\frac{i}{\sqrt{2}}$

  $$e^{r_1 s} = e^{\frac{s}{\sqrt{2}}i} = \cos\frac{s}{\sqrt{2}} + i\sin\frac{s}{\sqrt{2}}, \qquad R_e\{e^{r_1 s}\} = \cos\frac{s}{\sqrt{2}}, \qquad I_m\{e^{r_1 s}\} = \sin\frac{s}{\sqrt{2}}$$

  $$f_h = c_1\cos\frac{1}{\sqrt{2}}s + c_2\sin\frac{1}{\sqrt{2}}s$$

- $2f_s'' + f = s e^s$

$$se^{s} = e^{\alpha s}\left[P_n(s)\cos\beta s + Q_m(s)\sin\beta s\right] \Rightarrow \alpha = 1,\ \beta = 0,\ P_n(s) = s$$

$$\alpha + \beta i = 1 \Rightarrow r = 0,\ k = 1$$

$$f_p = (As+B)e^{s}, \qquad f_p' = (As+A+B)e^{s}, \qquad f_p'' = (As+2A+B)e^{s}$$

$$2As + 4A + 2B + As + B = s \ \Rightarrow A = \frac{1}{3},\ B = -\frac{4}{9}$$

$$f_p = (\frac{1}{3}s - \frac{4}{9})e^{s}$$

$$f(s) = c_1 \cos\frac{1}{\sqrt{2}}s + c_2 \sin\frac{1}{\sqrt{2}} \cdot s + (\frac{1}{3}s - \frac{4}{9})e^{s}$$

3. Prelaskom na inverznu funkciju pokazati da se diferencijalna jednačina $y'y''' - 3(y'')^2 - 4y''(y')^2 - 4(y')^4 + (y')^5 \cdot (2y+1+4e^{y}) = 0$ svodi na jednačinu $x''' - 4x'' + 4x' = 2y+1+4e^{y}$ i naći njeno opšte rešenje.

$$y' = \frac{1}{x'}$$

$$y'' = -\frac{x''}{(x')^2} \cdot \frac{1}{x'} = -\frac{x''}{(x')^3}$$

$$y''' = -\frac{x''' \cdot (x')^3 - 3x'' \cdot (x')^2 \cdot x''}{(x')^6} \cdot \frac{1}{x'} = \frac{3(x'')^2}{(x')^5} - \frac{x'''}{(x')^4}$$

$$\frac{1}{x'}(\frac{3(x'')^2}{(x')^5} - \frac{x'''}{(x')^4}) - 3\frac{(x'')^2}{(x')^6} + 4\frac{x''}{(x')^3}\frac{1}{(x')^2} - 4\frac{1}{(x')^4} + \frac{1}{(x')^5} \cdot (2y+1+4e^{y}) = 0$$

$$3(x'')^2 - x'''x' - 3(x'')^2 + 4x''x' - 4(x')^2 + x'(2y+1+4e^{y}) = 0$$

$$-x''' + 4x'' - 4x' + (2y+1+4e^{y}) = 0$$

$x''' - 4x'' + 4x' = 2y+1+4e^{y}$ (Jednačina sa konstantnim koeficijentima)

- $x''' - 4x'' + 4x' = 0$

  $r^3 - 4r^2 + 4r = 0 \ \Rightarrow\ r(r^2 - 4r + 4) = 0 \ \Rightarrow\ r(r-2)^2 = 0 \ \Rightarrow\ r_1 = 0,\ r_2 = r_3 = 2$

  $x_h = c_1 + c_2 e^{2y} + c_3 y e^{2y}$

- $x''' - 4x'' + 4x' = 2y+1$

  $2y+1 = e^{\alpha y}\left[P_m(y)\cos\beta y + \mathrm{Q}_{\mathrm{n}}(y)\sin\beta y\right] \ \Rightarrow\ \alpha = 0,\ \beta = 0,\ P_m(y) = 2y+1$

  $\alpha + \beta\, i = 0 \Rightarrow r = 1,\ k = m = 1$

$x_{p1} = y(Ay+B) = Ay^2 + By, \qquad x'_{p1} = 2Ay + B, \qquad x''_{p1} = 2A, \qquad x'''_{p1} = 0$

$-4(2A) + 4(2Ay+B) = 2y+1$

$-8A + 4B + 8Ay = 2y+1$

$8A = 2 \Rightarrow A = \dfrac{1}{4}$

$-2 + 4B = 1 \Rightarrow B = \dfrac{3}{4}$

$x_{p1} = \dfrac{1}{4}y^2 + \dfrac{3}{4}y$

- $x''' - 4x'' + 4x' = 4e^y$

$4e^y = e^{\alpha y}\left[P_m(y)\cos\beta y + \mathrm{Q_n}(y)\sin\beta y\right] \Rightarrow \alpha = 1,\ \beta = 0,\ P_m(y) = 4$

$\alpha + \beta\, i = 1 \Rightarrow r = 0,\ k = m = 0$

$x_{p2} = Ae^y, \qquad x'_{p2} = x''_{p2} = x'''_{p2} = Ae^y$

$Ae^y - 4Ae^y + 4Ae^y = 4e^y$

$Ae^y = 4e^y \Rightarrow A = 4$

$x_{p2} = 4e^y$

$x = x_h + x_{p1} + x_{p2} = c_1 + c_2e^{2y} + c_3ye^{2y} + \dfrac{1}{4}y^2 + \dfrac{3}{4}y + 4e^y$

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