HW4.2

Wednesday, October 7, 2020 1:50 AM

M is an $n \times n$ matrix
with n orthogonal eigenvectors $\vec{v}_i$

$M v_i = \lambda_i v_i$ with all eigenvalues λ_i distinct

Show $M = V L V^{-1}$ where:
$\stackrel{?}{=} V^{-1} L V$

$$V = \left(\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots \vec{v}_n \right) \quad \& \quad L = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & & 0 \\ \vdots & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & & \lambda_n \end{pmatrix}$$

If $M = VLV^{-1}$:

$$M^{-1} = (V L V^{-1})^{-1} = (V L^{-1})^{-1} V^{-1} = V L V^{-1} = M$$

$\therefore$ If true, M is symmetric. *Can I use this somehow?

Real eigenvalues assured since assuming M is symmetric.

$$M \vec{v} = \lambda \vec{v}$$

$$(M \vec{v})^{-1} = (\lambda \vec{v})^{-1} = \lambda \vec{v}^{-1}$$

I know $(AB)^{-1} = B^{-1} A^{-1}$ but may only apply to square matrices.

$$v_i^{-1} (M \vec{v}_i) = v_i^{-1} (\lambda \vec{v}_i)$$

$$\vec{v}_1^{-1} (M \vec{v}_1) = \vec{v}_1^{-1} (\lambda_1 \vec{v}_1)$$ Does inverse of vector ever make sense? ...

$$\vec{v}_2^{-1} (M \vec{v}_2) = \vec{v}_2^{-1} (\lambda_2 \vec{v}_2)$$

vectors

Long story short

$$\vec{V}_2^{-1}(M\vec{V}_2) = \vec{V}_2^{-1}(\lambda_2 \vec{V}_2)$$

Row x column, so if we place $\vec{V}_{1,2\ldots n}$ vectors into V matrix should do each λ_i, $\vec{V}_i$ respectively.

~~$(1,2,3)\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix} =$~~

~~transpose. No such thing as inverse vector...~~

$$\begin{pmatrix} \cancel{a\ b} \\ c\ d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$ae+gb \quad ce+dg$

$af+bh \quad cf+dh$

$$M\vec{v}_1 = \lambda_1 \vec{v}_1$$

$$MV = M\begin{pmatrix} v_1, v_2, v_3 \ldots v_n \end{pmatrix} = M_{11}V_{11} + M_{12}V_{21} + M_{13}V_{31}$$

$$M_{11}V_{11} + M_{12}V_{21} + M_{13}V_{31}, \quad M_{11}V_{12} + M_{12}V_{22} + M_{13}V_{32}\ldots$$

$$\therefore MV = \begin{pmatrix} \sum_{k=1}^{3} M_{1k}V_{k1}, & \sum_{k=1}^{3} M_{1k}V_{k2}, & \sum_{k=1}^{3} M_{1k}V_{k3} \\ \sum M_{2k}V_{k1}, & \sum M_{2k}, V_{k2}, & \sum M_{2k}, V_{k3} \\ \sum M_{3k}V_{k1}, & \sum M_{3k}, V_{k2}, & \sum M_{3k}, V_{k3} \end{pmatrix} = \lambda V$$

$$A \equiv$$

$$\left(\sum M_{3k} V_{k1}, \quad \sum M_{3k} V_{k2}, \quad \sum M_{3k} V_{k3} \right)$$

$$\left[V^{-1}(L\,V)\right]_{11} = \sum_{n=1}^{3} V^{-1}_{1n} \left[\sum_{k=1}^{3} \left(M_{1k}V_{k1} + M_{2k}V_{k1} + M_{3k}V_{k1} \right) \right]$$

1R, 1C ↗

$$V^{-1} L V = \begin{pmatrix} \sum_{n=1}^{3} V^{-1}_{1n} \sum_{l=1}^{3}\sum_{k=1}^{3} M_{lk} V_{k1}, & \sum_{n=1}^{3} V^{-1}_{1n} A_{n2}, & \sum_{n=1}^{3} V^{-1}_{1n} A_{n3} \\ 2n \quad n1 & 2n \quad n2 & 2n \quad n3 \\ 3n \quad n1 & 3n \quad n2 & 3n \quad n3 \end{pmatrix}$$

$$\sum_{n=1}^{3} A_{n2} = \sum_k M_{1k} V_{k2} + \sum_k M_{2k} V_{k2} + \sum_k M_{3k} V_{k2}$$

$$= \sum_l \sum_k M_{lk} V_{k2}$$

$$\sum_{n=1}^{3} A_{n3} = \sum_l \sum_k M_{lk} V_{k3}$$

$$\therefore \; V^{-1} L V = \begin{pmatrix} \sum_{n=1}^{3} V^{-1}_{1n} \sum_l \sum_k L_{lk} V_{k1}, & \sum_{n=1}^{3} V^{-1}_{1n} \sum_l \sum_k L_{lk} V_{k2}, & \sum_{n=1}^{3} V^{-1}_{1n} \sum_l \sum_k L_{lk} V_{k3} \\ \sum_{n=1}^{3} V^{-1}_{2n} \sum_l \sum_k L_{lk} V_{k1}, & \sum_{n=1}^{3} V^{-1}_{2n} \sum_l \sum_k L_{lk} V_{k2}, & \sum_{n=1}^{3} V^{-1}_{2n} \sum_l \sum_k L_{lk} V_{k3} \\ \sum_{n=1}^{3} V^{-1}_{3n} \sum_l \sum_k L_{lk} V_{k1}, & \sum_{n=1}^{3} V^{-1}_{3n} \sum_l \sum_k L_{lk} V_{k2}, & \sum_{n=1}^{3} V^{-1}_{3n} \sum_l \sum L_{lk} \sum V_{k3} \end{pmatrix}$$

'f' for list notation ↓

$$\text{Define} \quad \mathop{f}_{n=1}^{3} a_n \equiv (a_1, a_2, a_3) \quad \text{and} \quad \mathop{F}_{n=1}^{3} \equiv \begin{pmatrix} a_1, \\ a_2, \\ a_3 \end{pmatrix}$$

↖ transpose version

$$\therefore \quad V^{-1} L V = \mathop{F}_{a=1}^{3} \mathop{F}_{b=1}^{3} \sum_{n=1}^{3} V^{-1}_{an} \sum_{e=1}^{3} \sum_{k=1}^{3} L_{ek} V_{kb} \quad \text{version}$$

I'm hoping all this work helps later with for loops.

$$M = \mathop{F}_{a=1}^{3} \mathop{F}_{b=1}^{3} \sum_{n=1}^{3} V^{-1}_{an} \sum_{e=1}^{3} \sum_{k=1}^{3} L_{ek} V_{kb}$$

$L =$ diagonal matrix, so can multiply by δ_{ek} which eliminates $\sum_{e=1}^{3}$ sum

$$\boxed{\therefore M = \mathop{F}_{a=1}^{3} \mathop{F}_{b=1}^{3} \sum_{n=1}^{3} V^{-1}_{an} \sum_{k=1}^{3} L_{kk} V_{kb}} \quad ✓$$

Probably would have been faster to append V_i into $n \times n$ O matrix.

b) M^n has eigenvalues λ_i^n w/ same eigenvectors?

Implies
$$L^n = \begin{pmatrix} \lambda_1^n & & & \\ & \lambda_2^n & & \\ & & \ddots & \\ & & & \lambda_n^n \end{pmatrix}$$
so only L part changes in above expression

$$\therefore M^n \left(\mathop{F}^{3} \mathop{F}^{3} \sum^{3} V^{-1} \sum^{3} L \; V \right)^n = \mathop{F}^{3} \mathop{F}^{3} \sum^{3} V^{-1}_{an} \sum L^n_{kk} V_{kb}$$

$$M^n = \left(\prod_{a=1}^{3} \prod_{b=1}^{3} \sum_{n=1}^{3} V^{-1}_{an} \sum_{k=1}^{3} L_{kk} V_{kb} \right)^n = \prod_{a=1}^{3} \prod_{b=1}^{3} \sum_{n=1}^{3} V^{-1}_{an} \sum_{k=1}^{3} L^n_{kk} V_{kb}$$

each term L^n_{kk} is λ^n_k. only part that changes.

$$M^n = \left(\prod_{a=1}^{3} \prod_{b=1}^{3} M_{ab} \right)^n$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & b(a+d) \\ c(a+d) & bc+d^2 \end{pmatrix}$$

since M is an $n \times n$ matrix, increasing $3 \to n'$ for terms.

$$M^n = \left(\prod_{a=1}^{n'} \prod_{b=1}^{n'} M_{ab} \right)^n \neq \prod_{a=1}^{n'} \prod_{b=1}^{n'} \left(M_{ab}^n \right) \quad \text{since } a^2 \neq a^2+bc$$

$$= \prod_{a=1}^{n'} \prod_{b=1}^{n'} \sum_{m=1}^{n'} V^{-1}_{am} \sum_{k=1}^{n'} L_{kk} V_{kb}$$

$$\overleftrightarrow{M}^n \vec{b} \stackrel{?}{=} \sum_{i}^{n'} c_i \lambda_i^n \vec{v}_i$$

basis

$$\overleftrightarrow{M}^n \vec{v} = \lambda^n \vec{v}$$

n' $\underline{n'}$

$$M \; V = \lambda \; V$$

$$\vec{b} = \sum_i^{n'} c_i \vec{V}_i = \sum_i^{n'} c_i \prod_{a=1}^{n'} (V_i)_a$$

How do I solve for c?

$$\overleftrightarrow{M}^{\hat{}} = \lambda_1^{\hat{}} \vec{V}_1$$

$$\overleftrightarrow{M}^{\hat{}} \cdot \vec{b} = c_1 \lambda_1^{\hat{}} \vec{V}_1 + c_2 \lambda_2^{\hat{}} \vec{V}_2 + \dots c_{n'} \lambda_{n'}^{\hat{}} \vec{V}_{n'}$$