Midterm

Saturday, October 17, 2020 8:16 PM

1. **5 points** True or False: When numerically solving a differential equation, it is always better to choose a smaller Δt. Explain why or why not.

Choosing a smaller delta t will increase precision however it also leads to increased computation time. With modern processors, the need to balance optimization versus utility is minimized, so I will say it is almost always better to choose a smaller delta t whenever speed is not of importance. (True)

2. **10 points** Short answer: Describe the difference between a non-adaptive and adaptive numerical method. Under what conditions are adaptive methods generally useful. Are there any disadvantages to using an adaptive method?

Adaptive numerical methods are designed to permit adaptation to information obtained during solution iterations and the overall process while other methods are not. They "continuously monitor the solution and modify the (variable) to ensure that the user-specified accuracy is maintained" (Garcia 66). In other words, adaptive algorithms change their behavior depending on user-set variables when run.

These methods are generally useful when small errors cause large deviation to the solution. A program predicting the trajectory of a comet would perform better using adaptive methods since potential deviation increases greatly when the comet is near a sun compared to far away.

Adaptive methods do require knowledge of system to determine what variable(s) should reference the ongoing process. Again, adding more steps to each iteration, (checking whether certain condition is met), will increase computation time.

3. **15 points** This problem relates to the Discrete Fourier Transform. You are welcome to use the Fast Fourier Transform if you wish, but the data set is small enough there will be no appreciable difference between the two algorithms.
   (a) Import the data in data.csv, $\{x_i\}$, Fourier transform it, and plot the resulting power spectrum.
   (b) Define the sequence $\{y_i\}$, with $y_i = (x_i + x_{i+1})/2$ and $y_0 = y_N$ (periodic boundary conditions). Fourier transform this data, and comment on similarities or differences with part (a). This averaging procedure is an example of a low-pass filter. Explain why that name is appropriate.

This is an example of a low-pass filter because as frequency increases, output goes to zero.

4. **25 points** The logarithm of an $n \times n$ matrix $\mathbf{A}$ can be determined through a Taylor expansion:

$$\log(\mathbf{A}) = -\sum_{k=1}^{\infty} \frac{(\mathbf{I}-\mathbf{A})^k}{k} \quad (1)$$

with $\mathbf{I}$ the identity matrix.

   (a) What constraints are there on the matrix $\mathbf{A}$ for this sum to converge?
   (b) What is the computational complexity of computing $\log(\mathbf{A})$ for large n using the Taylor expansion?

a) The absolute value of I-A must be less than 1 for this to converge $\boxed{\|\mathbb{1} - A\| < 1 \quad (+\ A \neq 0)}$

$$\ln(A) = -\sum_{k=1}^{\infty} \frac{(\mathbb{I}-A)^k}{k} = -\left\{(\mathbb{I}-A) + \frac{(\mathbb{I}-A)^2}{2} + \frac{(\mathbb{I}-A)^2}{2} + \dots\right\}$$

$$-\ln(A) = \sum_{k=1}^{\infty} \frac{(-1)^k(-\mathbb{1}+A)^k}{k}$$

$$e^A = \mathbb{1} + \sum_{k=1}^{\infty} \frac{A^k}{n!}$$

$$\ln A = \ln\left[\ln\left(\mathbb{1} + \sum_{k=1}^{\infty} \frac{A^k}{n!}\right)\right]$$

$$= \ln\left[\ln\left(\sum_{k=0}^{\infty} \frac{A^k}{n!}\right)\right]$$

$$\ln(a+b+c)$$

b) The computational complexity of computing log(A) for large n using the Taylor expansion is:

5. **45 points** A system of 3 stars of mass M =1 solar mass located at locations $\mathbf{R}_1, \mathbf{R}_2$ and $\mathbf{R}_3$ have the potential energy

$$U(\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3) = -GM^2\left(\frac{1}{|\mathbf{R}_1 - \mathbf{R}_2|} + \frac{1}{|\mathbf{R}_1 - \mathbf{R}_3|} + \frac{1}{|\mathbf{R}_2 - \mathbf{R}_3|}\right) \quad (2)$$

where G is the gravitational constant. Note that this three body problem has no known analytical solution for arbitrary initial conditions, but some initial conditions will give rise to periodic behavior (as you will show below). In this problem, you may use any programming language you like.

   (a) Show that if we choose an arbitrary length scale, the system can be nondimensionalized in the form

$$\ddot{\mathbf{r}}_1 = -\frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} - \frac{\mathbf{r}_1 - \mathbf{r}_3}{|\mathbf{r}_1 - \mathbf{r}_3|^3} \quad \ddot{\mathbf{r}}_2 = -\frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_1 - \mathbf{r}_2|^3} - \frac{\mathbf{r}_2 - \mathbf{r}_3}{|\mathbf{r}_2 - \mathbf{r}_3|^3} \quad \ddot{\mathbf{r}}_3 = -\frac{\mathbf{r}_3 - \mathbf{r}_1}{|\mathbf{r}_1 - \mathbf{r}_3|^3} - \frac{\mathbf{r}_3 - \mathbf{r}_2}{|\mathbf{r}_2 - \mathbf{r}_3|^3}$$

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What is a velocity of 1km/s in your units?

$[U] = N$ ✓, $[G] = \frac{N \cdot m^2}{kg^2}$ ✓, $[M] = kg$ (solar mass? or)

$[R_{1,2,3}] = m$ $\quad 1\,N = 1\,kg \cdot m/s^2$ $\quad [G] = \frac{m^3}{kg \cdot s^2}$

$$N \overset{?}{=} \cancel{\frac{N \cdot m^2}{kg^2}}\, kg^2 \frac{1}{\cancel{m}} = \frac{kg \cdot m}{s^2} \checkmark$$

Let $R_i \Rightarrow a_i r_i$ and $u \to \gamma u$

so $\gamma u(a_i r_i) = -GM^2\left(\frac{1}{|ar_i - br_j|} + \frac{1}{|ar_i - cr_k|} + \frac{1}{|br_j - cr_k|}\right)$

$$3M\frac{d^2}{dt^2}\sqrt{(a_i r_i)^2 + (br_j)^2 + (cr_k)^2} = -GM^2(\quad \uparrow \quad)$$

$$3M \cdot \frac{1}{2}(\quad)^{-1/2} = -GM^2(\quad)$$

$$-\frac{3M}{4}(\quad)^{-3/2} = -GM^2(\quad)$$

$$\frac{d}{dt}\sum_1^3 2(a_i r_i)\dot{r}_i = 2\sum_1^3 (a_i \dot{r}_i + a_i r_i \ddot{r}_i)$$

$$\frac{2GM}{3}\left(\frac{1}{|\ |} + \frac{1}{|\ |} + \frac{1}{|\ |}\right) = \left[\sum_1^3 (\quad)\right]^{-3/2}$$

$$= \left(\frac{2GM}{3}\right)\left(\frac{|a-c||b-c| + |a-b||b-c| + |a-b||a-c|}{|a-b||a-c||b-c|}\right)$$

$$= \left(\frac{2GM}{3}\right)\left(\frac{1}{|a-b||a-c||b-c|} \right)$$

$|a||b| = |ab|$ so

$$= \frac{2GM}{3}\left(\frac{|ab-ac-bc+c^2| + |ab-ac-b^2+bc| + |a^2-ac-ab+bc|}{|a^2b-\cancel{abc}-ab^2+b^2c-a^2c+ac^2+\cancel{abc}-bc^2|} \right)$$

$$a^2(b-c)+b^2(c-a)+c^2(a-b)$$

Let $t = \tau t$

$$M\frac{d^2}{dt^2} = m\frac{1}{(t')^2}\frac{d^2}{d\tau^2}$$

$$\vec{r}_{ij} = \vec{r}_j - \vec{r}_i \qquad r_{ij} = |\vec{r}_j - \vec{r}_i|$$

$$r = |\vec{r}| = \sqrt{r_i^2 + r_j^2 + r_k^2}$$

$$m_i a_i = G\sum_{\substack{j=1\\ j\neq i}}^{N} \frac{m_i m_j}{r_{ij}^2}\hat{r}_{ij} \xleftarrow{\text{plug}} \hat{r}_{ij} = \frac{\vec{r}_j - \vec{r}_i}{r_{ij}}$$

$\downarrow$ Get

$$m_i a_i = G\sum_j \frac{m_i m_j}{r_{ij}^3}\vec{r}_{ij}$$

$$r_{ij}^3 = \left[(x_j - x_i)^2 + (y_j + y_i)^2 + (z_j - z_i)^2\right]^{3/2}$$

Let $G = 1$; $\left([G] = M^{-1}L^3T^{-2}\right)$

$$\boxed{\begin{gathered} \cancel{m_i} a_i = \cancel{m_i}\frac{d^2\vec{r}_i}{dt^2} = \sum_j \frac{m_i m_j}{r_{ij}^3}\vec{r}_{ij} \\ \frac{d^2}{dt^2}x_i(t) = \sum_j \frac{m_j}{r_{ij}^3}(x_j - x_i) \end{gathered}}$$

$[U] = N$, $[G] = \frac{m^3}{kg\, s^2}$, $[M] = kg$ (or solar mass?)

$[R_{1,2,3}] = m$ $\quad 1\, N = 1\, kg \cdot m/s^2$

$[G] = M^{-1} L^3 T^{-2} \rightarrow [U] = M L T^{-2}$

$[M] = M$ & $[R_i] = L$

$G \cdot G^{-1} = I$

$[G]^{-1} = M L^{-3} T^2$

$[u] = U G^{-1} = M^2 L^{-2}$, $[r_i] = M L^{-2} T^2$,

$[u] = M^2 L^{-3} T^2$

$$\frac{M^2}{L^2} = -\left(M^4 L^{-6} T^4\right) \quad \frac{1}{M L^{-2} T^2}$$

$$1 = -\frac{M^2 L^{-8} T^4}{M L^{-2} T^2} = -M L^{-6} T^2$$

Let $M^2 G = 1$?

$[M^2 G] = M L^3 T^{-2} \equiv u \qquad u^{-1} = M^{-1} L^{-3} T^2$

$$\frac{[U]}{u} = \left(M L T^{-2}\right)\left(M^{-1} L^{-3} T^2\right) = L^{-2}$$

$$\ddot{\vec{r}}_i = -\left(\frac{r_i - r_j}{|r_i - r_j|^3}\right) - \frac{r_i - r_k}{|r_i - r_k|^3}$$

$$\ddot{\vec{r}}_i = -\left(\frac{\vec{r}_{ij}}{r_{ij}^3}\right) - \frac{\vec{r}_{ik}}{r_{ik}^3} = -\frac{1}{r_{ij}^2}\hat{r}_{ij} - \frac{1}{r_{ik}^2}\hat{r}_{ij}$$

$$M^2(M^{-1}L^3T^{-2}) = ML^3T^{-2} = [M^2G]$$

$$\boxed{\frac{d^2\vec{r}_i}{dt^2} = -\sum_j \frac{1}{r_{ij}^2}\hat{r}_{ij}}$$

where $r_{ij} = |r_i - r_j|$

& $\hat{r}_{ij}$ = unit vector

& $[G = 1]$?

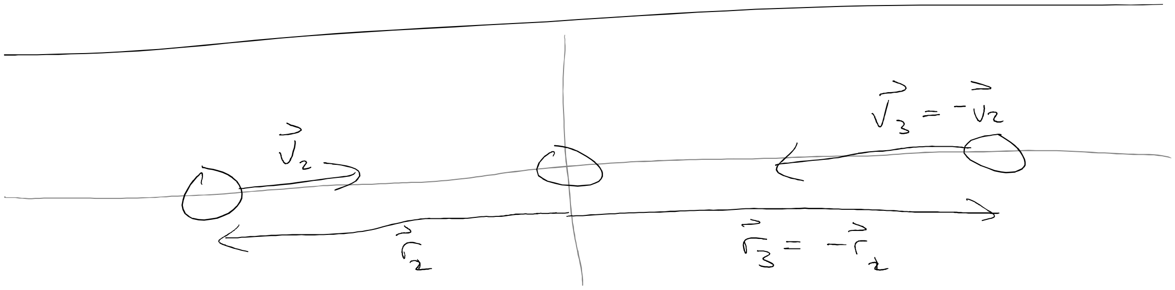
$M^{-1}L^3T^{-2}$

$$1\ \frac{km}{s} = 1000\ \frac{m}{s}$$

$$= 1000\left[\frac{L}{T}\right]$$

$$= [(1000)^{-3}]^2 \text{ or } [(1000)^3]^{-2} \text{ ?}$$

$$= (10^3)^{-6} = \boxed{10^{-18}\ m}$$



(b) Analytically show that if the first star is initially stationary at the origin, and if the second and third stars satisfy $\mathbf{r}_2 = -\mathbf{r}_3$ and with velocities $\mathbf{v}_2 = -\mathbf{v}_3$, that the first star will remain stationary for all time. Is this solution stable if $\mathbf{r}_1$ is perturbed?

or ↗ etc ↙

$$\ddot{\vec{r}}_i = \frac{1}{r_{ij}^2}\hat{r}_{ij} - \frac{1}{r_{ik}^2}\hat{r}_{ik}$$

If $\vec{r}_{ij} = -\vec{r}_{ik}$ then $r_{ij} = r_{ik}$

and $\hat{r}_{ij} = -\hat{r}_{ik}$.

$$\therefore \ddot{r}_i = r_{ij}^{-3} - r_{ik}^{-3} = 0$$ so no

acceleration. Velocity also applies.

$$\ddot{r}_i = \frac{dv_i}{dt} = \frac{dv_i}{dx_{ij}}\frac{dx_{ij}}{dt}$$

$\boxed{\text{No, the solution isn't stable.}}$

$$dG = d(M^{-1}L^3T^{-2}) = -M^{-2}dM\,L^3T^{-2} + M^{-1}\left(3L^2dL\,T^{-2} + L^3(-2T^{-3})dT\right)$$

$$= -M^{-2}L^3T^{-2}dM + 3M^{-1}L^2T^{-2}dL - 2M^{-1}L^3T^{-3}dT$$

$$= \underbrace{M^{-1}L^3T^{-2}}_{\text{original } G \overset{?}{=} 1}\left(-M^{-1}dM + 3L^{-1}dL - 2T^{-1}dT\right)$$

$$dG = d(1) = 0 = -M^{-1}dM + 3L^{-1}dL - 2T^{-1}dT$$

$$\ddot{\vec{r}}_i = -\frac{\vec{r}_{ij}}{|r_{ij}|^3} - \frac{\vec{r}_{ik}}{|r_{ik}|^3}$$

Let $\hat{r} = \hat{r}_i = -\hat{r}_j$

$$\vec{r}_{ij} = \vec{r}_i - \vec{r}_j = \left(\sqrt{r_{x_i}^2 + r_{y_i}^2} - \sqrt{r_{x_j}^2 + r_{x_j}^2}\right)\hat{r}$$

$$d r_{ij} = dr_i - dr_j = \frac{1}{2}\left(r_{x_i}^2 - r_{x_j}^2 + r_{y_i}^2 - r_{y_j}^2\right)^{-1/2}\left(2 r_{x_i} dr_{x_i} \text{ etc}\right)$$

$$V = \int \ddot{r}_i \, dt = \int \frac{r_{ij}}{|r_{ij}|^3} - \frac{r_{ik}}{|r_{ik}|^3} \, dt + V_0$$

$$r_2 = -r_3$$

$$V_2 = -V_3$$

so $r_{12} = r_1 - r_2$ + $r_{13} = r_1 + r_2$

$$-\frac{1}{(r_1 - r_2)} - \frac{1}{(r_1 + r_2)}$$

$$-\frac{(r_1 + r_2) - (r_1 - r_2)}{r_1^2 - r_2^2}$$

$$\ddot{r}_1 = \frac{-2 r_1}{r_1^2 - r_2^2}$$

$\dddot{r}$. $\rightarrow$ (0) $= 0$

$$
\hat{Y}\begin{cases}
\ddot{r}_1(0) = \dfrac{-2(0)}{0^2 - 1^2} = 0 \\[2ex]
\ddot{r}_2 = -\dfrac{(r_2 - r_3)}{|r_2 - r_3|^3} - \dfrac{(r_2 - r_1)}{|r_2 - r_1|^3} = \dfrac{-2r_2}{(2r_2)^3} \\[2ex]
\ddot{r}_2(0) = -\dfrac{2}{2^3} - \dfrac{1}{1^3} = \dfrac{7}{8} \\[2ex]
\ddot{r}_3(0) = -\dfrac{1}{2^2} + \dfrac{2}{2^3} = \dfrac{1}{4}
\end{cases}
$$

$$
\frac{d^2 r_i}{dt^2} = -\left(\frac{r_i - r_j}{|r_i - r_j|^2}\right) - \left(\frac{r_i - r_k}{|r_i - r_k|^2}\right)
$$

$$
\frac{dr_i}{dt} = -\int \left(\frac{r_i - r_j}{|r_i - r_j|^2} + \frac{r_i - r_k}{|r_i - r_k|^2}\right) dt
$$

$$
v_i = A + \frac{dt}{6}\left(\underbrace{\frac{dA}{dt}}_{B} + \frac{d(x + 0.5dB)}{d(\underbrace{t + 0.5dt}_{c})} + \frac{d(x + 0.5dC)}{d(t + 0.5dt)} + \frac{d(x + 0.5dD)}{d(t + dt)}\right)
$$

where $A = \dfrac{r_i - r_j}{|r_i - r_j|^2} + \dfrac{r_i - r_k}{|r_i - r_j|^2}$