HW7

Sunday, November 22, 2020 8:59 PM

PHYS 6350 HW7. Due Monday Nov 23 at 9am.

1. **7 points**. The dimensionless Navier-Stokes (NS) equations are $\nabla \cdot \mathbf{u} = 0$ and $\dot{\mathbf{u}} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + Re^{-1}\nabla^2\mathbf{u}$. This problem will consider this equation *in two dimensions only*, with $\mathbf{u} = (u_x, u_y, 0)$. The fluid is trapped between two walls (with the boundary conditions $\mathbf{u}(y=1) = \mathbf{u}(y=-1) = 0$, and periodic boundary conditions on $\mathbf{u}$ in the x direction).

   (a) Show analytically that *a* solution to the NS equations are $p = p_0 - \alpha x$, $u_x = \alpha Re(1-y^2)/2$, and $u_y = 0$, for any α. This parabolic flow profile is referred to as Poiseuille flow.

   (b) The vorticity of a flow is defined as $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, with $\boldsymbol{\omega} = (0, 0, \omega)$ for a 2D system. For an arbitrary 2D flow (*not* the Poiseuille flow in (a)), show that $\dot{\omega} + (\mathbf{u} \cdot \nabla)\omega = Re^{-1}\nabla^2\omega$. What are the boundary conditions on ω?

   (c) Can the PDE in (b) be solved using the methods for solving PDE's we described in class? If so, explain how. If not, describe a predictor-corrector method (like the SIMPLE algorithm described in class) that could be used?

1) $\vec{\nabla} \cdot \vec{u} = 0 \qquad \dot{\vec{u}} + (\vec{u} \cdot \vec{\nabla})\vec{u} = -\vec{\nabla} p + \frac{1}{Re}\nabla^2 \vec{u}$

$\vec{u} = (u_x, u_y, 0)\ ; \qquad u(y = \pm 1) = 0$

a) $p = p_0 - \alpha x \qquad u_x = \dfrac{\alpha Re (1 - y^2)}{2} \qquad u_y = 0$

$\vec{\nabla} \cdot \vec{u} = 0 \quad + \quad (\vec{u} \cdot \vec{\nabla})\vec{u} = -\vec{\nabla} p + \frac{1}{Re}\nabla^2 \vec{u}$

$(\vec{u} \cdot \vec{\nabla})\vec{u} + \vec{\nabla} p - \frac{1}{Re}\nabla^2 \vec{u} = 0$

$\vec{\nabla} \cdot \vec{u} = \frac{\partial}{\partial x} u_x + \frac{\partial}{\partial y} u_x = \frac{\partial}{\partial x}\left[\frac{\alpha Re (1-y^2)}{2}\right] + \cancel{\frac{\partial}{\partial y}(0)}$

$$\frac{\partial}{\partial x} u_x + \frac{\partial}{\partial y} u_x - \frac{\partial}{\partial x}\left[\frac{\quad}{2} \right] + \cancel{\frac{\partial}{\partial y}}(0)$$

$$\frac{\alpha Re}{2} \frac{\partial}{\partial x}(1-y^2) = 0 \quad \therefore \quad y = \pm 1.$$

b) $\dot{\vec{u}} + (\vec{u} \cdot \vec{\nabla}) \vec{u} = -\vec{\nabla} p + \frac{1}{Re} \nabla^2 \vec{u}$

$$\vec{\nabla} \times \left[(\dot{\vec{u}} \cdot \vec{\nabla}) \vec{u} \right] = \vec{\nabla} \times \left(-\vec{\nabla} p + \frac{1}{Re} \nabla^2 \vec{u} \right)$$

← had to start over after miswriting

$$\vec{\nabla} \times \left[\dot{\vec{u}} + (\vec{u} \cdot \vec{\nabla}) \vec{u} \right] = \vec{\nabla} \times \left(-\vec{\nabla} p + \frac{1}{Re} \nabla^2 \vec{u} \right)$$

$$= \frac{\partial}{\partial t} \vec{\nabla} \times \vec{u} + (\vec{u} \cdot \vec{\nabla}) \vec{\nabla} \times \vec{u} = \underbrace{-\vec{\nabla} \times \vec{\nabla} p}_{\equiv 0} + \frac{1}{Re} \vec{\nabla} \times \nabla^2 \vec{u}$$

$$\therefore \quad \frac{\partial}{\partial t} \left[\vec{\nabla} \times \vec{u} \right] + (\vec{u} \cdot \vec{\nabla}) \left[\vec{\nabla} \times \vec{u} \right] = \frac{1}{Re} \vec{\nabla} \times (\vec{\nabla} \cdot \vec{\nabla}) \vec{u}$$

$$= \frac{1}{Re} \nabla^2 (\vec{\nabla} \times \vec{u})$$

c) I have no idea how to solve this.

What do I do with the boundary conditions?