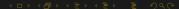
# A multi-linear geometric estimate

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#### Introduction

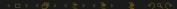
- Erdős and Szemerédi made the following significant conjecture in additive number theory: If A is a finite set of integers with |A| = n then either A + A or  $A \cdot A$  must have size at least  $C_{\epsilon} n^{2-\epsilon}$  for any  $\epsilon > 0$ .
- The sum-project conjecture naturally leads one to the consideration of the size of sets of the form  $A \cdot A + A \cdot A$ .

#### Introduction

- Hart and losevich previously showed that if  $E \subset \mathbb{F}_q^d$  with  $|E| > q^{\frac{d+1}{2}}$  then  $\mathbb{F}_q^* \subset \varpi(E^2)$  where  $\varpi$  is any non-degenerate bilinear form.
- lacksquare This estimate can be used to show that if  $A\subset \mathbb{F}_q^*$  is sufficiently large then

$$\mathbb{F}_q^* \subset dA^2 = A \cdot A + \cdots + A \cdot A.$$

• We generalize this estimate to the case that  $\varpi$  is a multi-linear form.



#### Talk outline

- Preliminaries on forms
- The main result
- Some caveats
- Applications
- Proof sketch

- Given two vector spaces V and W let  $\boxtimes$ :  $V \times W \rightarrow V \otimes W$  denote the canonical map taking (v, w) to  $v \otimes w$ .
- Note that when E is a subspace of V both  $E^{\boxtimes n}$  and  $E^{\otimes n}$  are defined and are in general distinct.
- An *n*-linear form is a linear transformation  $\varpi: V^{\otimes n} \to \mathbb{F}$  where V is an  $\mathbb{F}$ -vector space.
- For example, the usual dot product over  $\mathbb{F}_q^d$  is a bilinear (n=2) form.

#### Definition (Space of multi-linear forms)

Given a vector space V over a field  $\mathbb F$  and some  $n\in\mathbb N$  we denote by  $\mathsf{Form}(V,n)$  the dual  $\mathbb F$ -vector space to  $V^{\otimes n}$ . That is,  $\mathsf{Form}(V,n) := \mathsf{Hom}(V^{\otimes n},\mathbb F)$ .

#### Definition (Level set)

Given an  $\mathbb{F}$ -vector space V, a form  $\varpi \in \text{Form}(V, n)$ ,  $E \subset V$ ,  $t \in \mathbb{F}$  we define the t-level set of  $\varpi$  (with respect to E) to be

$$L_t \coloneqq \left\{ (z, w) \in E^{\boxtimes (n-1)} \times E \mid \varpi(z, w) = t \right\}$$

and we define  $\nu(t) \coloneqq |L_t|$ .

#### Definition (Evaluation map)

Given a vector space V, some  $n \in \mathbb{N}$ , some  $k \in [n]$ , and subspaces  $A \leq V^{\otimes (n-1)}$  and  $B \leq V$  the  $k^{th}$  evaluation map on (A, B) is

$$\mathsf{eval}_{k,A,B}$$
:  $\mathsf{Form}(V,n)\otimes B \to \mathsf{Hom}(A,\mathbb{F})$ 

is given by

$$(\mathsf{eval}_{k,A,B}(\varpi \otimes y))(x_1,\ldots,x_{n-1}) \coloneqq \varpi(x_1,\ldots,x_{k-1},y,x_{k+1},\ldots,x_{n-1}).$$

# Definition ((A, B)-non-degenerate form)

Given a form  $\varpi \in \text{Form}(V, n)$  and subspaces  $A \leq V^{\otimes (n-1)}$  and  $B \leq V$  we say that  $\varpi$  is (A, B)-non-degenerate in the  $k^{\text{th}}$  coordinate when  $\text{Ker}(\text{eval}_{k|A|B}^{\infty}) = 0$ .

Let 
$$V=\mathbb{F}_q^d$$
,  $A=V^{\otimes (n-1)}$ ,  $B=V$ ,  $n=3$ , and  $\varpi(x,y,z)=x_1y_1z_1+x_2y_2z_2+\cdots+x_dy_dz_d$ .

It is not difficult to see that this form is (A, B)-non-degenerate.

If we keep B the same, change A to  $W^{\otimes (n-1)}$ , where

$$W = \left\{ x \in \mathbb{F}_q^d \mid x_1 = 0 \right\},\,$$

and use the same form as above, we get an (A, B)-degenerate form.

#### Definition (Projective index)

Given  $E \subset \mathbb{F}_q^d$  we say that E has projective index  $\alpha$  when

$$\frac{\left|\left\{\,\left(\,\mathsf{a},\,\mathsf{w}\right)\in\left(\mathbb{F}_q^*\setminus\{1\}\right)\times\mathbb{F}_q^d\;\middle|\;\mathsf{w},\,\mathsf{aw}\in E\,\right\}\right|}{\left(\,q-2\right)|E|}\geq\alpha.$$

# The main result

# Theorem (A., Iosevich (2021))

Suppose that  $\varpi \in \text{Form}(q,d,n)$  for some  $n \geq 2$ , that  $E \subset \mathbb{F}_q^d$ , and that E has projective index  $\alpha$ . If there exists an r-dimensional subspace A of  $(\mathbb{F}_q^d)^{\otimes (n-1)}$  and a subspace B of  $\mathbb{F}_q^d$  such that

- $E^{\boxtimes (n-1)} \subset A$ ,
- $E \subset B$
- $\square$  is (A, B)-non-degenerate, and
- $\begin{array}{c|c} & |E| > q^{\frac{r+n-1}{n}} \left(1 \alpha \left(1 \frac{2}{q}\right)\right)^{\frac{1}{n}} \\ \hline \text{then } \mathbb{F}_q^* \subset \varpi(E^n). \end{array}$

### Some caveats

- This result extends the 2008 Hart-losevich result by incorporating the projective index  $\alpha$ .
- This is only a constant improvement, and we're not in the business of constants.

#### Some caveats

- This result extends the 2008 Hart-losevich result for d = 2 by incorporating the projective index  $\alpha$ .
- If I recall correctly, trying to exploit this to get a better exponent for d = 2 is a losing battle.

#### Some caveats

■ When  $dim(Span(E)) = \ell$  we need

$$\ell^{n-1} - n\ell + n < 2 - \log_q(q - \alpha(q-2))$$

in order for our exponent to be nontrivial.

■ There are only two cases where this occurs: n = 2 and d is anything, or n = 3 and d = 2.

■ When I spoke about this for Virginia Tech, I gave the following kind of example applications when n = 3 and d = 2.

- Let q = 160001.
- lacksquare Every nonzero member of  $\mathbb{F}_q^*$  may be written as

$$\varpi(h_1\gamma_1(1,h_2\gamma_2),h_3\gamma_3(1,h_4\gamma_4),h_5\gamma_5(1,h_6\gamma_6))$$

where the  $h_i$  are from a fixed set H consisting of 16 coset representatives of the subgroup  $\Gamma$  of  $\mathbb{F}_q^*$  of order  $\frac{q-1}{20}=8000$  and the  $\gamma_i$  are members of  $\Gamma$ .

lacksquare Each member of  $\mathbb{F}_q^*$  is of the form

$$h_1h_3h_5\psi_1(1+h_2h_4h_6\psi_2)$$

where  $\psi_1$  and  $\psi_2$  are  $20^{\text{th}}$  powers in  $\mathbb{F}_q$  and the  $h_i$  belong to H.

Moreover,

$$A \cdot A \cdot A + A \cdot A \cdot A \cdot A \cdot A \cdot A \supset \mathbb{F}_q^*$$

when  $A = H\Gamma$ .

#### Theorem (Cauchy-Davenport)

Given  $A, B \subset \mathbb{F}_p$  we have that

$$|A + B| \ge \min(|A| + |B| - 1, p).$$

In our example we had

$$A \cdot A \cdot A + A \cdot A \cdot A \cdot A \cdot A \cdot A \supset \mathbb{F}_{a}^{*}$$

when  $A = H\Gamma$ .

■ Note that |A| = 16(8000) = 128000,

$$|A \cdot A \cdot A| \ge |A|,$$

and

$$|A \cdot A \cdot A \cdot A \cdot A \cdot A| \ge |A|$$
.



Cauchy-Davenport yields

$$\begin{aligned} |A \cdot A \cdot A + A \cdot A \cdot A \cdot A \cdot A \cdot A| &\geq |A + A| \\ &\geq \min(|A| + |A| - 1, 160001) \\ &= \min(255999, 160001) \\ &= 160001. \end{aligned}$$

Cauchy-Davenport is stronger here.

# Theorem (A., losevich (2021))

Suppose that  $\varpi \in \text{Form}(q, d, n)$  for some  $n \geq 2$ , that  $E \subset \mathbb{F}_q^d$ , and that E has projective index  $\alpha$ . If there exists an r-dimensional subspace A of  $(\mathbb{F}_q^d)^{\otimes (n-1)}$  and a subspace B of  $\mathbb{F}_q^d$  such that

- $E \subset B$
- $\varpi$  is (A,B)-non-degenerate, and
- $\begin{array}{c|c} & |E|>q^{\frac{r+n-1}{n}}\left(1-\alpha\left(1-\frac{2}{q}\right)\right)^{\frac{1}{n}} \\ \hline then \ \mathbb{F}_q^*\subset \varpi(E^n). \end{array}$

Write

$$\nu(t) = \sum_{\substack{z \in E^{\boxtimes (n-1)} \\ w \in E}} q^{-1} \sum_{s \in \mathbb{F}_q} \chi(s(\varpi(z, w) - t)).$$

Thus,

$$\nu(t) = q^{-1} \left| E^{\boxtimes (n-1)} \right| |E| + R$$

where

$$R := \sum_{\substack{z \in E^{\boxtimes (n-1)} \\ w \in E}} q^{-1} \sum_{s \in \mathbb{F}_q^*} \chi(s(\varpi(z, w) - t)).$$

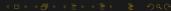
- View R as a sum in z and apply Cauchy-Schwarz.
- We find that  $R^2 < U + V$  where

$$U = \left| E^{\boxtimes (n-1)} \right| q^{r-2} \sum_{\substack{s,s' \in \mathbb{F}_q^* \\ w,w' \in \mathbb{F}_q^d \\ sw = s'w'}} \chi(t(s'-s)) E(w) E(w')$$

and

$$V = \left| E^{\boxtimes (n-1)} \right| q^{-2} \sum_{\substack{s,s' \in \mathbb{F}_q^* \\ w,w' \in E \\ sw \neq s'w'}} \chi(t(s'-s)) \sum_{z \in A} \chi(\varpi(z,sw-s'w'))$$

■ The (A, B)-nondegeneracy of  $\varpi$  and orthogonality of  $\chi$  gives V = 0.



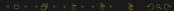
■ We have  $R^2 \le U = C + D$  where

$$C := \left| E^{\boxtimes (n-1)} \right| q^{r-2} \sum_{\substack{s,s' \in \mathbb{F}_q^* \\ w,w' \in \mathbb{F}_q^d \\ sw = s'w' \\ s \neq s'}} \chi(t(s'-s)) E(w) E(w')$$

and

$$D := \left| E^{\boxtimes (n-1)} \right| q^{r-2} \sum_{\substack{s,s' \in \mathbb{F}_q^* \\ w,w' \in \mathbb{F}_q^d \\ sw = s'w'}} \chi(t(s'-s)) E(w) E(w').$$

■ Without using the projective index  $\alpha$  we can just note that C < 0, but in general this is not enough.



Since

$$C \le -\left|E^{\boxtimes(n-1)}\right| |E| q^{r-1} \alpha \left(1 - \frac{2}{q}\right)$$

and

$$D = \left| E^{\boxtimes (n-1)} \right| |E| \, q^{r-1}$$

we have  $\nu(t) > 0$  and the result follows.

#### References

Derrick Hart and Alex Iosevich. "Sums and products in finite fields: an integral geometric viewpoint." In: Radon transforms, geometry, and wavelets. Vol. 464. Contemp. Math. Providence, RI: Amer. Math. Soc., 2008, pp. 129–135